

# Investments

## Topic 8: Options

UCLA | Fall 2024

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# Overview of Topic 8

- ① Option Contracts
- ② Option quotes and trading
- ③ Basic Trading Strategies
- ④ Static vs. Dynamic Replication
- ⑤ The Black-Scholes formula
- ⑥ Hedging and Market-Making
- ⑦ The Black-Scholes in the Real World
- ⑧ The Volatility Smile
- ⑨ Corporate Bond Valuation

## **1. Option Contracts**

# Options and their main types

- Options, as their name indicates, give the holder the right, but not the obligation, to do something in the future
  - A real option: the right to cut down a forest in some point in the future, or the right to invest in a fund or start-up
  - A stock option: the right, but not the obligation, to buy or sell a share of stock at a time in the future for a given price
- Types of options by payoff:
  - Call options: the right, but not the obligation, to buy a share of stock at a given price
  - Put Options: the right, but not the obligation, to sell a share of stock at a given price
  - Exotic options:
    - Knock-out (or in) options: a call or put option that ceases to exist if the underlying price crosses a certain barrier
    - Digital option: pays off only if underlying is in a certain range.
    - Options on options: compound or chooser options
- Types of options by exercise: European/American

## Option contract specifications

- Underlying asset
  - Usually equity, could be a bond, an interest rate or a currency
- Exercise price
  - Price at which the financial security can be bought or sold
- Expiration date: date on which the option expires
- Exercise style: when can you exercise your option?
- Payoff/settlement
  - What is received upon exercise? If an individual equity options is exercised, shares of stock are actually exchanged.
  - Other contracts would settle in cash
    - Index (Nasdaq 100 and S&P 500) options
    - Interest rate options
    - Commodity options

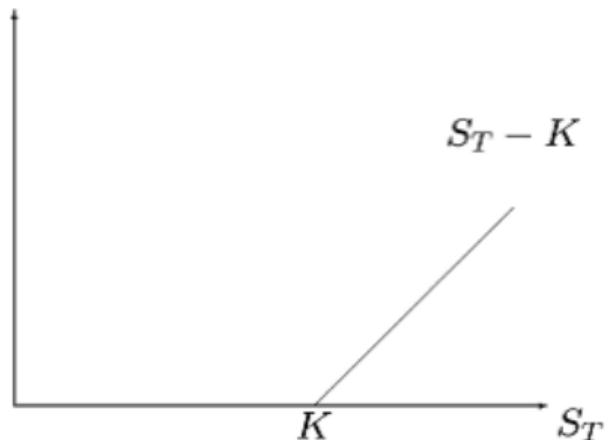
## Payoffs from a Call Option

- Consider a call with a strike price of  $K = 100$
- Let  $S_T$  denote the price of the underlying at maturity
- Gross payoffs to long and short positions at maturity:

|                  | $S_T$ | Long Call Payoffs | Short Call Payoffs |
|------------------|-------|-------------------|--------------------|
| Out-of-the-money | 70    | 0                 | 0                  |
|                  | 80    | 0                 | 0                  |
|                  | 90    | 0                 | 0                  |
| At-the-money     | 100   | 0                 | 0                  |
| In-the-money     | 110   | 10                | -10                |
|                  | 120   | 20                | -20                |
|                  | 130   | 30                | -30                |

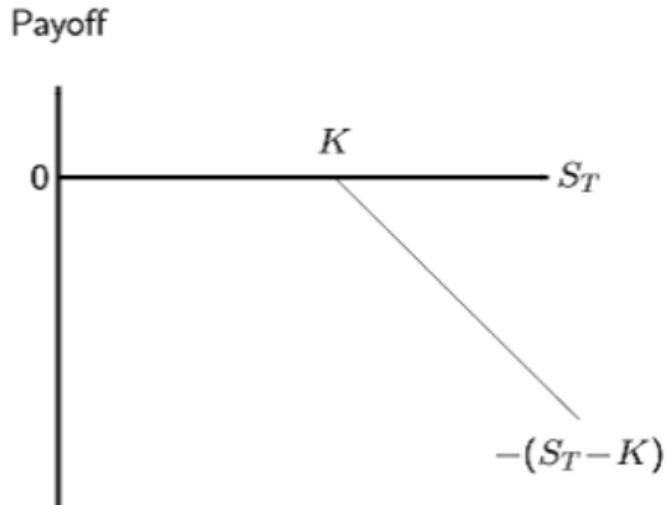
## Long Call with Strike K

Payoff



$$\text{Payoff} = \begin{cases} 0, & \text{if } S_T < K \\ S_T - K, & \text{if } S_T \geq K \end{cases}$$

## Short Call with Strike K



$$\text{Payoff} = \begin{cases} 0, & \text{if } S_T < K \\ -(S_T - K), & \text{if } S_T \geq K \end{cases}$$

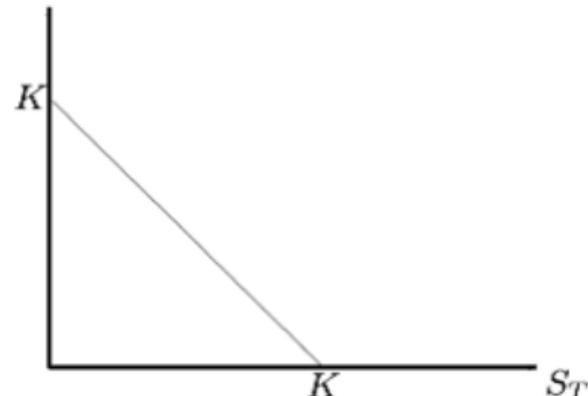
## Payoffs from a Put Option

- Consider a call with a strike price of  $K = 100$
- Let  $S_T$  denote the price of the underlying at maturity
- Gross payoffs to long and short positions at maturity:

|                  | $S_T$ | Long Put Payoffs | Short Put Payoffs |
|------------------|-------|------------------|-------------------|
| In-the-money     | 70    | 30               | -30               |
|                  | 80    | 20               | -20               |
|                  | 90    | 10               | -10               |
| At-the-money     | 100   | 0                | 0                 |
| Out-of-the-money | 110   | 0                | 0                 |
|                  | 120   | 0                | 0                 |
|                  | 130   | 0                | 0                 |

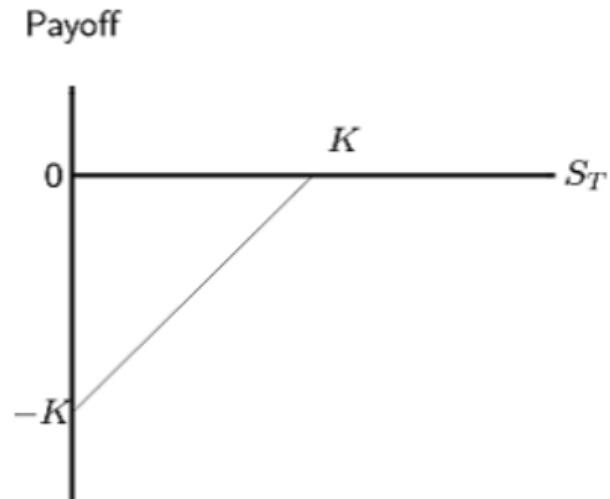
## Long Put with Strike K

Payoff



$$\text{Payoff} = \begin{cases} K - S_T, & \text{if } S_T \leq K \\ 0, & \text{if } S_T > K \end{cases}$$

## Short Put with Strike K



$$\text{Payoff} = \begin{cases} S_T - K, & \text{if } S_T < K \\ 0, & \text{if } S_T \geq K \end{cases}$$

## Determinants of an option value

- The underlying value,  $S_t$
- The strike price  $K$
- An option's moneyness,  $K/S_t$ , captures the relation
  - “a 6% OTM put” means a put with  $K/S_t = 0.94$
- If you are negative on the underlying, would you
  - Sell a call
  - Buy a put

## Option Value and Volatility

**Example:** Consider a call option with  $K = 100$ .

Compare two distributions for  $S_T$ :

$$S_T = \begin{cases} 110, & \text{with prob } 1/2 \\ 90, & \text{with prob } 1/2 \end{cases}$$

$$S_T = \begin{cases} 120, & \text{with prob } 1/2 \\ 80, & \text{with prob } 1/2 \end{cases}$$

- Second distribution is more volatile
- It also yields superior payoffs to the option holder because of the asymmetry in call payoffs
- Without the asymmetry, the increase in volatility may not be attractive

## Options and Views on Volatility

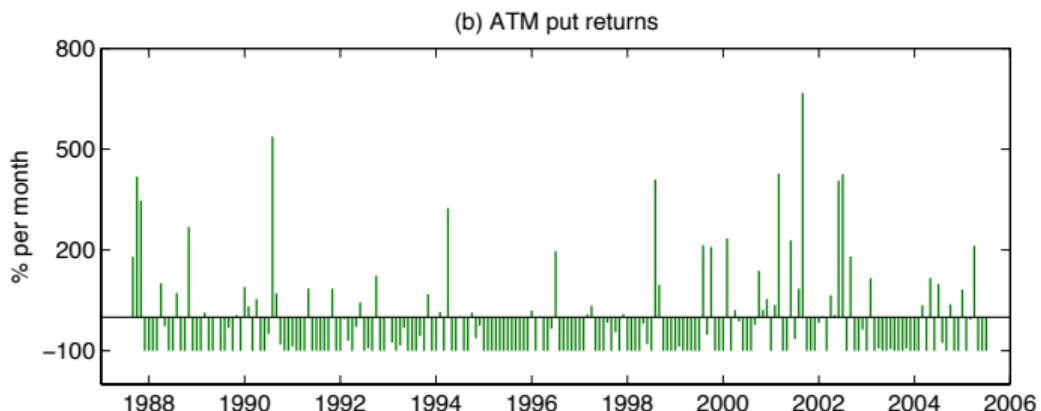
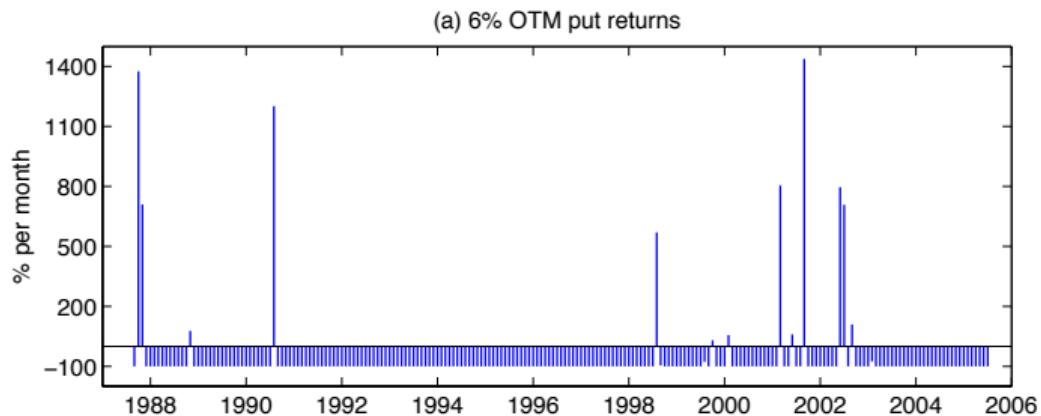
- Every option position (long/short put/call) embodies a view on direction
- From the options – volatility relationship, it can also be seen that every option also embodies a view on volatility
- Long option position (calls or puts): bullish view on volatility
  - Such a position increases in value when volatility increases, and decreases in value when volatility decreases.
- Short option position (calls or puts): bearish on volatility
  - Such a position increases in value when volatility decreases, and vice versa.

## Leverage

- Out-of-the-money options are cheap and have a lot of embedded leverage
- Consider a stock of Chernov Educational Services (CES) trading at \$666 today
- The cost of a call struck at \$680 with four months to expiration is \$39
- Should I buy the stock or a call?
  - Buy a stock. Suppose the price goes to \$730 in 4 months. My return is 9.6%
  - Buy a call. The return is:

$$\frac{730 - 680 - 39}{39} = 28\%$$

# SPX 1-month option returns



## **2. Option quotes and trading**

# Option Listing

- Exchange-traded options:
  - Stocks (American)
  - Futures (American)
  - Indices (European and American)
  - Currencies (European and American)
- OTC options
  - Vanilla (standard calls/puts as defined above)
  - Exotic (everything else - e.g., Asians, barriers)
- Others (e.g., embedded options)

## Exchanges (Stocks/Indexes)

- 100 shares of the underlying equity/\$100 times the index
- Premiums stated in points. One point equals \$100
- Strike intervals: \$2.50 increments for strikes below \$25, \$5 increments between \$25 and \$200, \$10 above \$200
- American/European or American
- Delivery is  $T + 2$  business days following exercise/Cash
- Expiration
  - monthly: two near-term months plus two additional months in the January, February or March quarterly cycle; expiration is on the third Friday of the expiration month
  - weekly: expire at the end of the trading week
  - daily: expire at the end of the trading day when they were bought (0DTE)

## Types of Options

- A dizzying array of exchange-traded contracts even if you limit to SPX as underlying
- LEAPS (**L**ong **T**erm **E**quity **A**ntici**P**ation **S**ecurity), maturity > 12 months, American
- Binary options, European
- **FLEX** (**F**lexible **E**Xchange) Options offer a choice of
  - The underlying Index (e.g, S&P 100, S&P 500, Nasdaq 100, Russell 2000, or DJIA 30)
  - Call or Put.
  - Expiration date – up to 15 years from creation.
  - Strike price – may be specified as an index level, as a percentage, or any other readily understood method for deriving an index level
  - American or European
  - Settlement Value – settlement may be based on either the opening settlement value or closing settlement value
- Jumbo/Mini options (1000/10 shares per contract)

## Options clearing

- Options Clearing Corporation (OCC) acts as guarantor between clearing parties ensuring that the obligations of the contracts are fulfilled
- Central counterparty clearing is a process mitigating counterparty risk
  - novating trades between counterparties, e.g., a trade between member firm A and firm B becomes two trades: A-OCC and OCC-B
  - netting offsetting transactions between multiple counterparties
  - requiring collateral (margin)
  - providing a guarantee fund that can be used to cover losses that exceed a defaulting member's collateral

## Closing positions

- You may exercise an option
- If you cannot or do not want to exercise, take an offsetting position
  - Standardized times of maturity help
- When would you want to offset instead of exercising?

# Volume and open interest

## Quotes

- Volume
  - number of option contracts being exchanged between buyers and sellers
- Open interest
  - number of option contracts that are still open and held by traders and investors
  - to close a position one must take an offsetting position (buy or sell “to close”) or exercise
  - a measure of liquidity
- Most trading is OTM/ATM

## The option market size

- The Chicago Board Options Exchange (CBOE) is the largest market for stock options.
- CBOE's market share is 30.0% for options on equities and exchange-traded products and 99.2% for index options.

CBOE Annual Contract Volume, mil

|                       | 2024 (Oct) | 2023 | 2022 | 2021 | 2020 | 2019 | 2018  | 2017  | 2016  |
|-----------------------|------------|------|------|------|------|------|-------|-------|-------|
| Option Contracts      |            |      |      |      |      |      |       |       |       |
| Equities & ETPs       | 2239       | 2704 | 2707 | 2541 | 2104 | 1375 | 1,415 | 1,165 | 691   |
| Indexes               | 864        | 950  | 715  | 497  | 459  | 474  | 560   | 496   | 433   |
| Total Options Volume  | 3103       | 3653 | 3421 | 3038 | 2563 | 1849 | 1,975 | 1,662 | 1,124 |
| Futures               | 51         | 56   | 55   | 58   | 51   | 63   | 75    | 74    | 60    |
| Total Contract Volume | 3154       | 3709 | 3476 | 3096 | 2614 | 1912 | 2,051 | 1,736 | 1,185 |

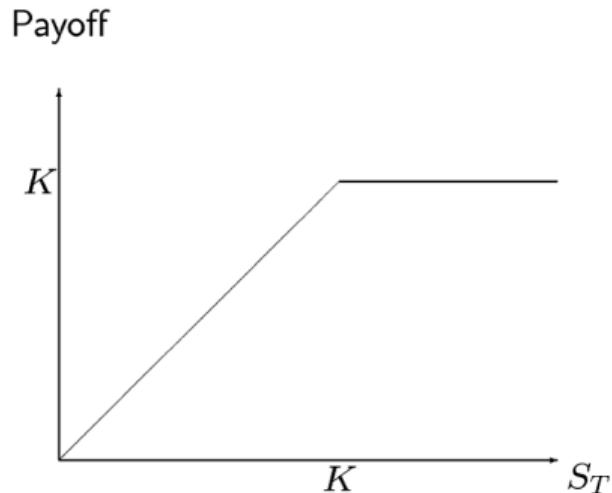
### **3. Basic Trading Strategies**

# Basic Trading Strategies

- A “trading strategy” refers to a portfolio that consists of options on a given asset and possibly the asset itself
  - ① Covered calls & Protective puts
  - ② Spreads
    - Bull spreads
    - Bear spreads
  - ③ Combinations
    - Straddles
    - Strangles

## Covered Call with Strike $K$

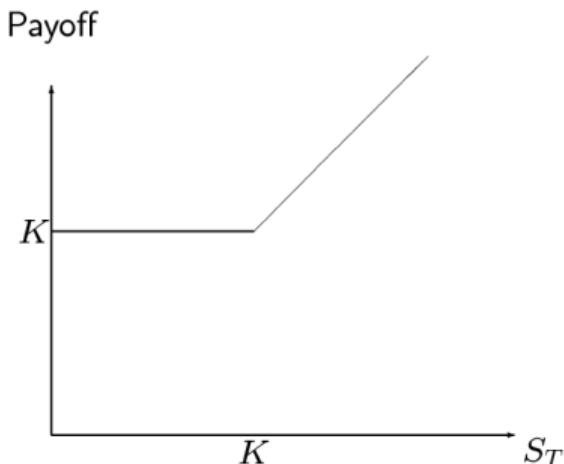
- Portfolio: Long stock, short call with strike  $K$



$$\text{Payoff} = \begin{cases} S_T, & \text{if } S_T < K \\ K, & \text{if } S_T \geq K \end{cases}$$

## Protective Put with Strike K

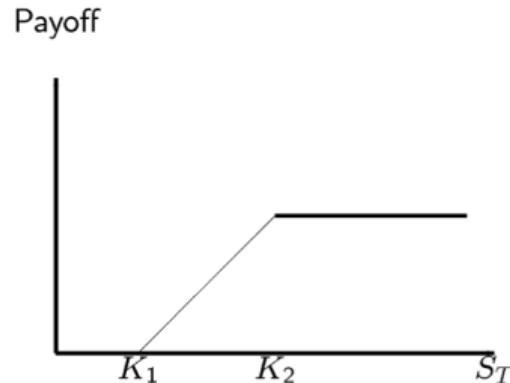
- Portfolio: Long stock, long put with strike  $K$



$$\text{Payoff} = \begin{cases} K, & \text{if } S_T < K \\ S_T, & \text{if } S_T \geq K \end{cases}$$

## Bullish Spread:

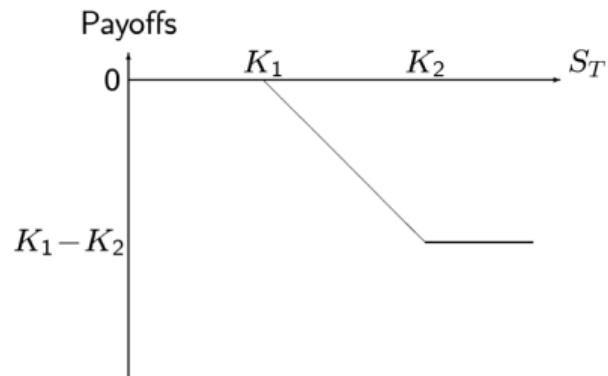
- Portfolio: Long call with strike  $K_1$ , short call with strike  $K_2$ ,  $K_1 < K_2$



$$\text{Payoff} = \begin{cases} 0, & \text{if } S_T \leq K_1 \\ S_T - K_1, & \text{if } K_1 < S_T \leq K_2 \\ K_2 - K_1, & \text{if } S_T > K_2 \end{cases}$$

## Bearish Spread:

- Short call with strike  $K_1$ , long call with strike  $K_2$

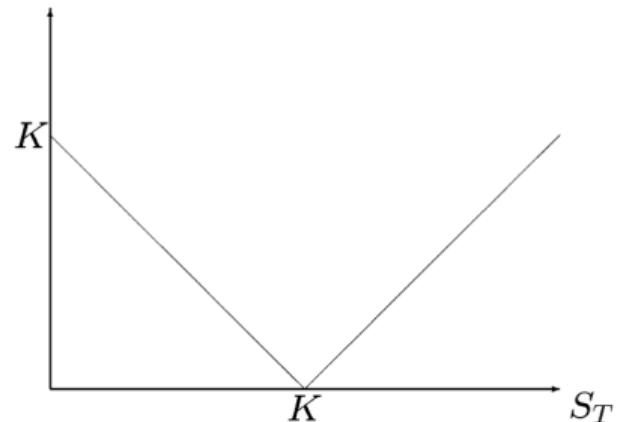


$$\text{Payoff} = \begin{cases} 0, & \text{if } S_T \leq K_1 \\ -(S_T - K_1), & \text{if } K_1 < S_T \leq K_2 \\ -(K_2 - K_1), & \text{if } S_T > K_2 \end{cases}$$

## Straddle

- Portfolio: Long call with strike  $K$ , long put with strike  $K$

Payoff

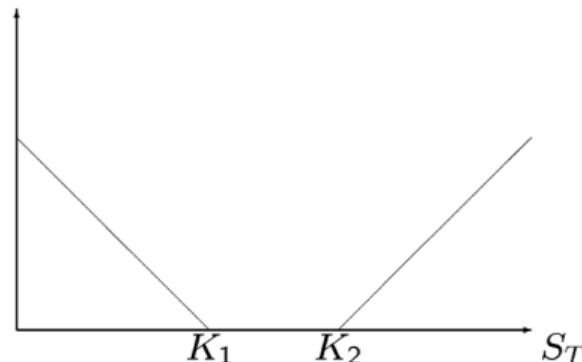


$$\text{Payoff} = \begin{cases} K - S_T, & \text{if } S_T < K \\ S_T - K, & \text{if } S_T \geq K \end{cases}$$

## Strangle

- Portfolio: Long put with strike  $K_1$ , long call with strike  $K_2$

Payoff



$$\text{Payoff} = \begin{cases} K_1 - S_T, & \text{if } S_T \leq K_1 \\ 0, & \text{if } K_1 < S_T \leq K_2 \\ S_T - K_2, & \text{if } S_T > K_2 \end{cases}$$

## **4. Static vs. Dynamic Replication**

## Static vs. Dynamic Replication

- Why are we interested in understanding how to replicate derivatives?
  - That is how we find their value!
- For forwards and swaps, we found static replicating trading strategies
  - Benefit: “Model free”, except assuming no trading frictions for large investors
- Do static replicating strategies exist for options?
  - Put-Call parity for European options
  - Otherwise, none
  - But, static considerations can give us no-arbitrage bounds
- For pricing of options, we rely on dynamic replicating trading strategies
  - The Binomial Model
  - Black-Scholes

## Static No-Arbitrage Bounds for Option Prices

- First, note that a stock “payoff” is  $S_T$  (ex-dividend), then ...
- ... a call
  - payoff:  $\max(S_T - K, 0)$
  - $C_t \geq \max(S_t - Ke^{-r(T-t)}, 0)$
  - $C_t \leq S_t$
- ... a put
  - payoff:  $\max(K - S_T, 0)$
  - $P_t \geq \max(Ke^{-r(T-t)} - S_t, 0)$
  - $P_t \leq Ke^{-r(T-t)}$

## European Put-Call Parity

- Note the similarity between the protective put and long call payoffs
- The difference in the payoffs is equal to  $K$
- Let's buy a call (struck at  $K$ ) and an asset which pays  $\$K$  at maturity for sure, i.e., a ZCB with the face value of  $\$K$
- By no-arbitrage, price (protective put) = price (call + bond):

$$\begin{aligned} P_t(S_t, K, T, r) + S_t &= C_t(S_t, K, T, r) + PV_t(K, T, r) \\ &= C_t(S_t, K, T, r) + Ke^{-r(T-t)} \end{aligned}$$

## Using Put-Call Parity for Arbitrage

- If the parity is violated, then there is an arbitrage opportunity
- Example:  $S_0 = \$100$ ,  $K = \$100$ ,  $T = 1$  year,  $r = 7.69\%$ ,  $C_0 = \$18$ , and  $P_0 = \$10$
- The parity implies

$$C_0 = P_0 + S_0 - PV_0(K, T, r) = 10 + 100 - 100e^{-0.0769 \cdot 1} = 17.40$$

- Sell the (relatively expensive) call at \$18, and replicate it for \$17.40. Make \$0.60

| Action             | Cashflow<br>at $t = 0$ | Cashflow at $t = T$ |                |
|--------------------|------------------------|---------------------|----------------|
|                    |                        | if $S_T \leq 100$   | if $S_T > 100$ |
| Sell a call        | 18                     | 0                   | $-(S_T - 100)$ |
| Buy stock          | -100                   | $S_T$               | $S_T$          |
| Borrow PV of \$100 | 92.60                  | -100                | -100           |
| Buy a put          | -10                    | $100 - S_T$         | 0              |
| Net                | 0.60                   | 0                   | 0              |

## **5. The Black-Scholes formula**

## Main Assumptions of the Model

- Main assumption: Underlying asset price follows a *Geometric Brownian Motion*.
- In words, this effectively means two conditions must be satisfied:
  - ① Stock prices are lognormally distributed (log returns are normally distributed).
  - ② Asset prices are continuous and cannot “jump.”
- In addition, the model’s key assumptions are:
  - ① Asset prices and trading strategies can change continuously, i.e., at each instant of time. (This requires techniques from stochastic calculus.)
  - ② Volatility of asset returns is constant.
  - ③ Interest rates are constant.

## Are These Assumptions Reasonable?

- The assumption of continuous trading and continuous changes in asset prices is clearly unreasonable but is in fact a *close approximation* to reality for many markets.
- The assumption of constant interest rates and constant volatility of returns is unrealistic and violated in practice.
- Also importantly, the fact that prices cannot “gap” is objectionable.
  - Price formation and news arrival process are inherently discrete (not continuous).
  - Prices often experience “jumps.”
  - Prices often jump simply due to dividend payments, etc.
- To be revisited

## Are These Assumptions Reasonable? (Cont'd)

- It turns out that interest rates and volatility can be made functions of time and current asset price. While the Black–Scholes formula will no longer be a valid valuation tool, the fundamental approach will be the same.
- Predictable dividend payments can be tackled easily.
- However, allowing volatility to have its own randomness (“stochastic volatility”) and introducing jumps in asset prices leads to non-trivial complications.
  - Evidence suggests that both are required to generate the violations of normality observed in (log) returns: “fat tails” or “leptokurtosis” of asset returns.
  - Either in itself is unsatisfactory to explain the richness of option pricing dynamics.
  - Is it better to use a simpler model and be aware of its pitfalls than more complex ones which are also unsatisfactory? Many people think so.

## To What End?

- Option prices in the Black–Scholes model can be expressed in *closed-form*, i.e., as explicit, readily calculated functions of underlying parameters.
- This makes computing option prices very easy.
- More importantly, it makes computing and understanding option price *sensitivities* to underlying parameters very easy.
- These sensitivities are very difficult to compute in the binomial model.
- Note, however, that binomial model is robust for calculation of variety of options including American options.

## The Black-Scholes Formula

- The argument behind Black–Scholes is one of replication and no–arbitrage.
- To replicate a call, we must
  - ① Take a long position in  $\Delta > 0$  units of the underlying, and
  - ② Borrow  $-B > 0$  units at the risk-free rate.
- Then, the price of the call can be expressed as:

$$C_0 = \Delta \cdot S_0 + B$$

- Alternatively, the price can be expressed via risk-neutral expectation:

$$C_0 = e^{-rT} E_0^* C_T$$

## The Black–Scholes Formula (Cont'd)

- The quantities  $\Delta$  and  $B$  depend on many factors including:
  - ① Depth-in-the-money of the option.
  - ② Volatility.
  - ③ Time-to-maturity.
  - ④ Interest rate.
- The unique feature of the Black–Scholes formula is that it provides precise expressions for  $\Delta$  and  $B$  in terms of the above parameters in the continuous limit case.

## The Black–Scholes Formula (Cont'd)

- No-arbitrage price of a European call option in the Black–Scholes model:

$$C_0 = S_0 \cdot N(d_1) - PV_0(K) \cdot N(d_1 - \sigma\sqrt{T})$$
$$d_1 = \frac{1}{\sigma\sqrt{T}} [\ln(S_0/K) + (r + \sigma^2/2)T]$$

- $S_0$  : current asset price.
- $K$  : strike price of the option.
- $T$  : calendar time of option's maturity.
- $0$  : current calendar time.
- $\sigma$  : volatility, i.e., standard deviation of annualized asset returns.
- $r$  : risk-free interest rate.
- $N(\cdot)$  : cumulative standard normal distribution:

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

## The Black–Scholes Formula (Cont'd)

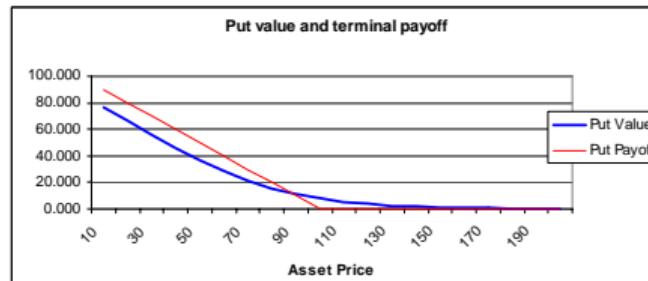
- Comparing the formula to the general expression, we see that
  - $\Delta = N(d_1)$
  - $-B = PV_0(K) \cdot N(d_1 - \sigma\sqrt{T}) \equiv PV_0(K) \cdot N(d_2).$
- Thus, the Black–Scholes formula provides us with a precise composition of the replicating portfolio for the European call option.
- We also know that the call can be evaluated by the risk-neutral approach:

$$\begin{aligned}C_0 &= e^{-rT} E_0^* C_T = e^{-rT} E_0^* \max(S_T - K, 0) \\&= e^{-rT} E_0^*(S_T - K | S_T > K) \\&= e^{-rT} E_0^*(S_T | S_T > K) - PV_0(K) Q\{S_T > K\}\end{aligned}$$

- Thus,  $N(d_2) = Q\{S_T > K\}$ , and  $N(d_1) = e^{-rT} E_0^*(R_T | R_T > K / S_0)$
- The put can be valued by put-call parity

# The Black–Scholes Formula: Examples

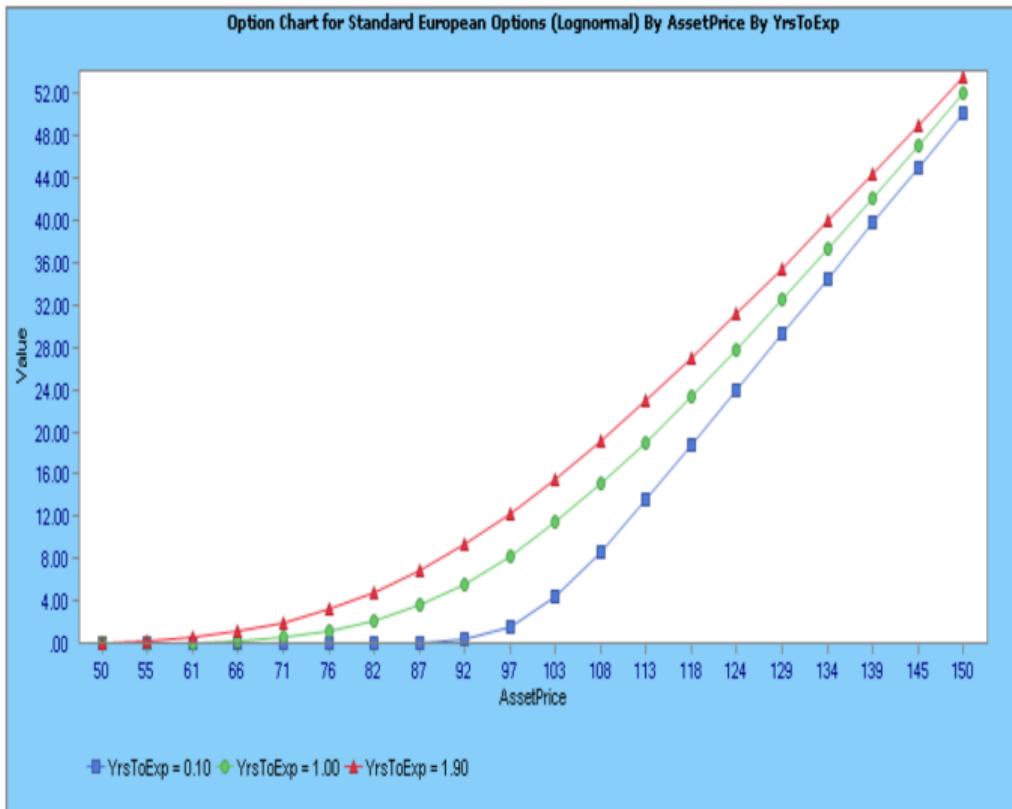
- Let us first do an example of this calculation. If  $K = 100, \sigma = 0.3, r = 0.1, T = 1.5$  :
- Next, let us see a graph of this price function. Observe the non-linearity of the option price as a function of underlying spot price and the effect of moneyness of the option.



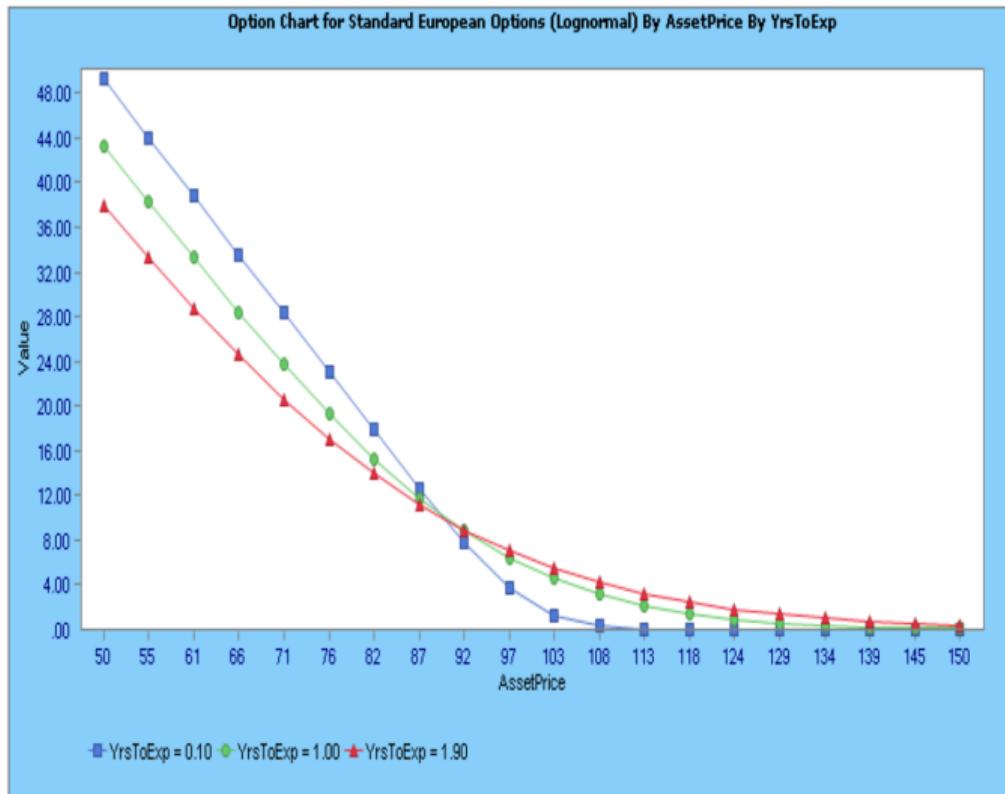
## The Black–Scholes Formula (Cont'd)

- There are several remarkable features of these expressions:
  - ① Option prices depend only on five variables:  $S, K, r, T, \sigma$ .
  - ② Of these, only volatility  $\sigma$  is not directly observable.
  - ③ This makes the model relatively easy to implement in practice.
  - ④ They have been derived from principles of no–arbitrage and replication. If the model assumptions are correct, then deviation from these prices leads to arbitrage opportunity using the replicating portfolios that mimic perfectly the option payoffs at all points of time.
  - ⑤ They tell us the hedging strategy for option exposures: Delta hedge, i.e., if long the option, then sell  $\Delta$  of the underlying.

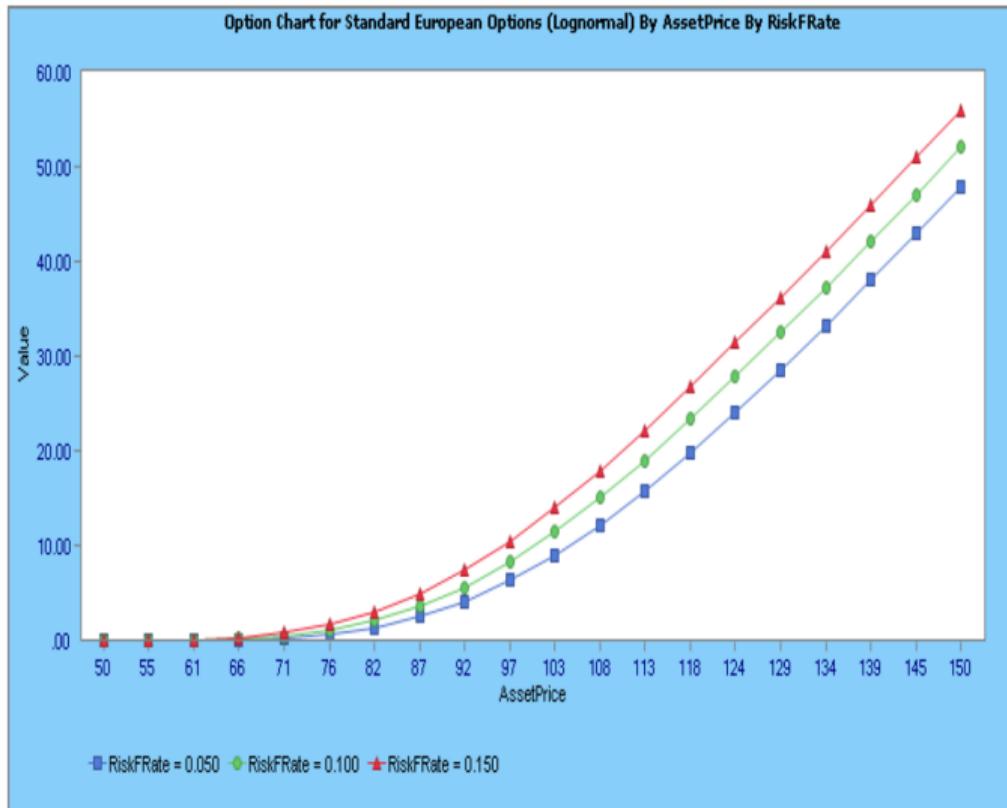
# Call Value vs. $T$ , $K = 100$



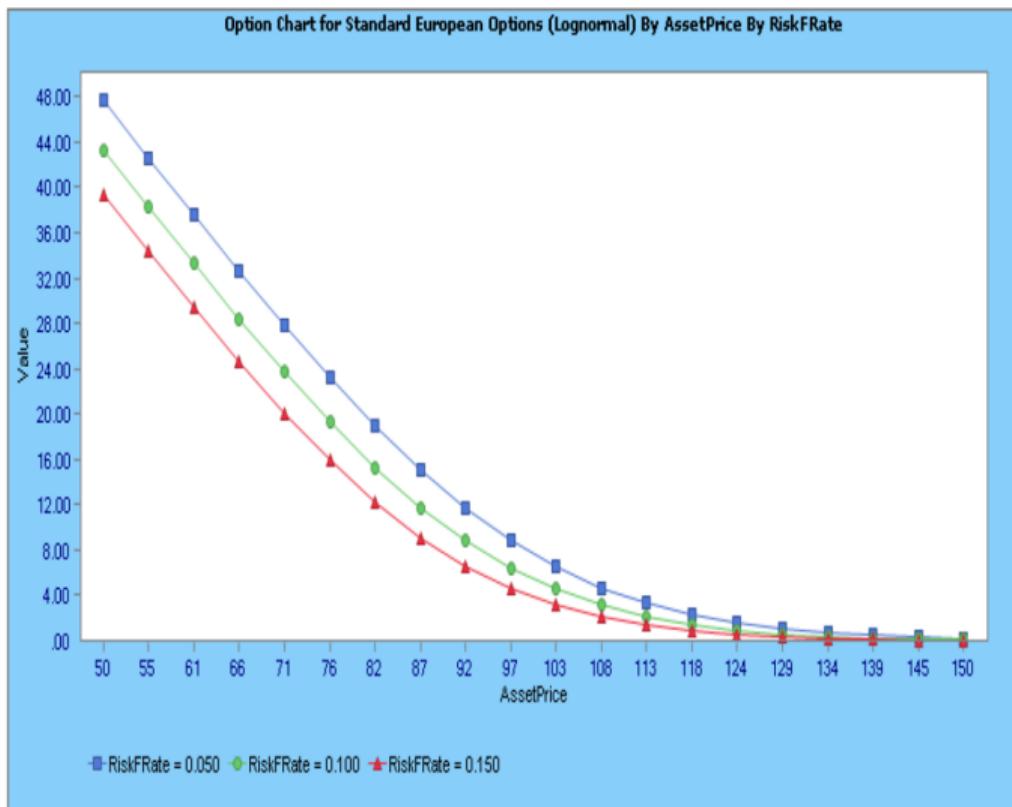
# Put Value vs. T, K = 100



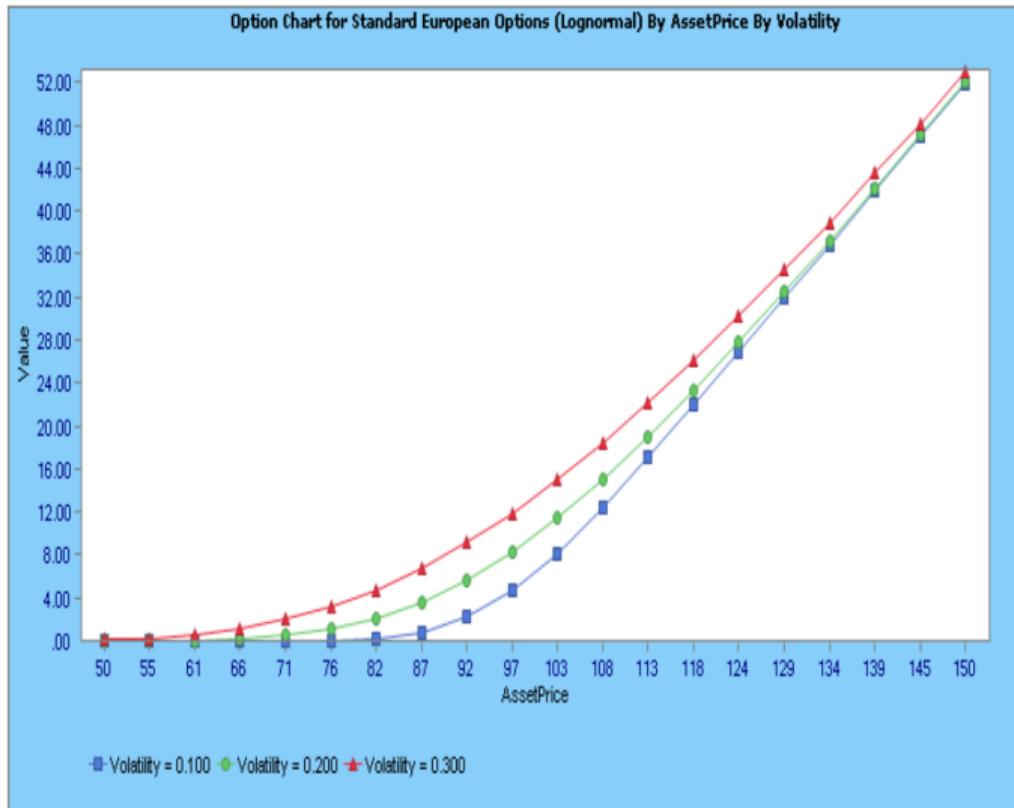
# Call Value vs. r, K = 100



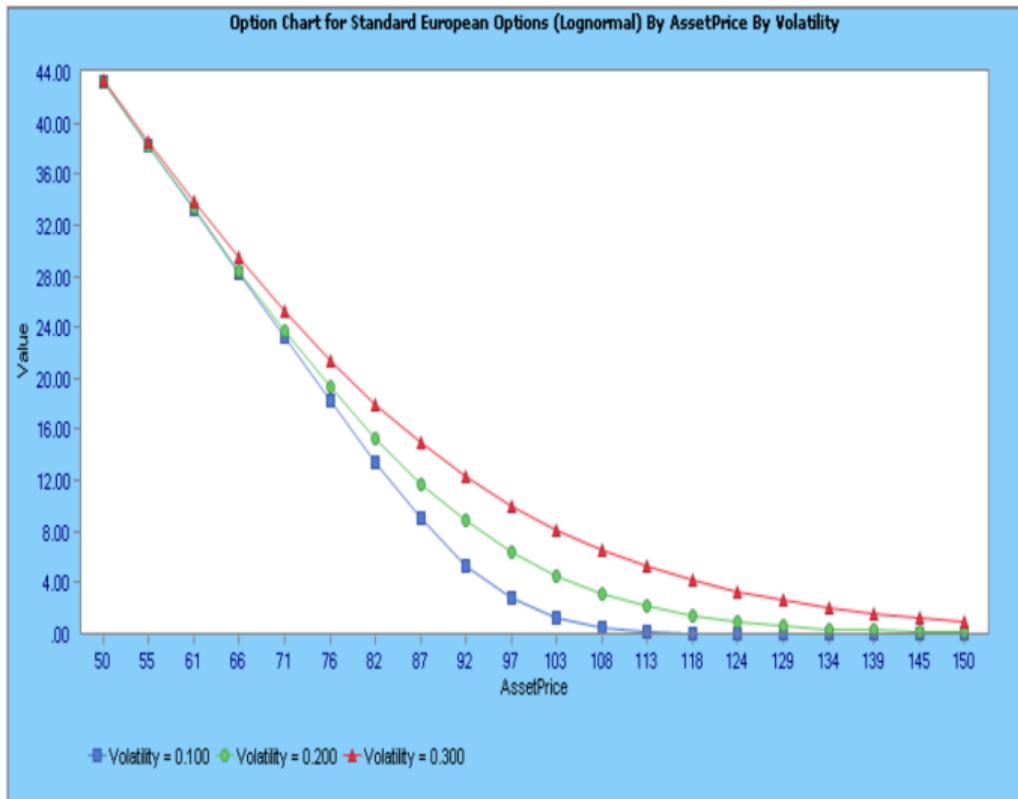
## Put Value vs. r, K = 100



# Call Value vs. Vol, $K = 100$



# Put Value vs. Vol, K = 100



## **6. Hedging and Market-Making**

# The Need for Delta-Hedging

- Market-makers in options markets have a natural need for hedging option exposure
  - Provide immediacy: ready to sell or buy options at quoted prices.
  - Make their profits from the bid-ask spread.
  - Let's say a large number of buy orders on calls come through:
    - Market-maker is then short call options
    - Increase in price of underlying could bankrupt market maker
    - Market-maker delta-hedges by buying required units of underlying so as to not be exposed to movements in underlying
- Conceptually different from Proprietary trading
  - Often bet on view or take advantage of arbitrage opportunities
  - The market-maker's positions, on the other hand, are the result of customer order flow

## The Need for Delta-Hedging (Cont'd)

- The Street is on average short options, so hedging in the underlying is in aggregate from the perspective of a short option position
  - i.e., buying underlying if a call and selling underlying if a put.
- Since investors are net buyers of insurance, hedging with other options is generally not feasible in the aggregate
- Another reason why market makers do not use other options to hedge is high transaction costs
  - Usually, the bid-ask spreads on the underlying are so much smaller that even with infrequent trading it is desirable to delta-hedge
- This aggregate short volatility has been a reason for the success of volatility markets
  - E.g., variance swaps and, more recently, VIX futures and options

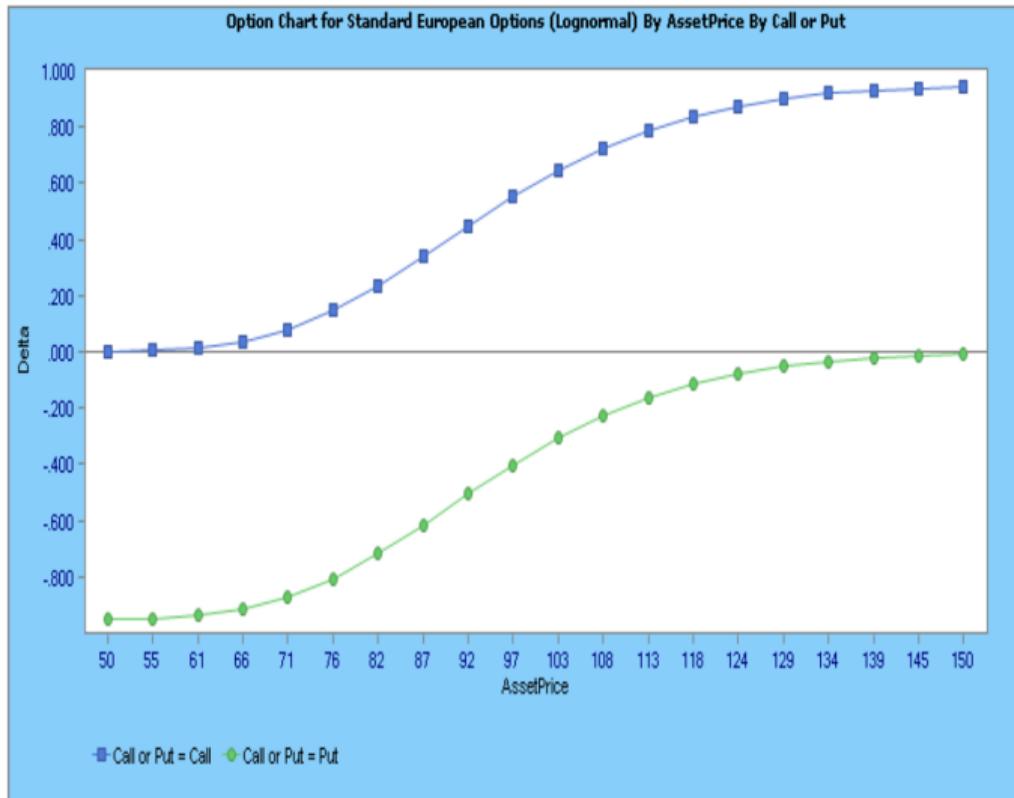
## Our Perspective

- We will consider the case: Buy a European call
- We will assume you delta hedge the call, but only revise your hedge at discrete intervals (here we assume that's weekly)
- We will think about the Profit and Loss between each time-interval from the hedging strategy:
  - Buy the call ( $C_t$ ), and delta hedge with the stock (short  $\Delta_t$  units)
  - The change in the value of your position between days is then:

$$P\&L_{t,t+\Delta t} = -\Delta_t (S_{t+\Delta t} - S_t) + (C_{t+\Delta t} - C_t)$$

- Is this a perfect hedge?

# Black-Scholes Put and Call Delta's



## P&L of Delta Hedge if No Time Passes

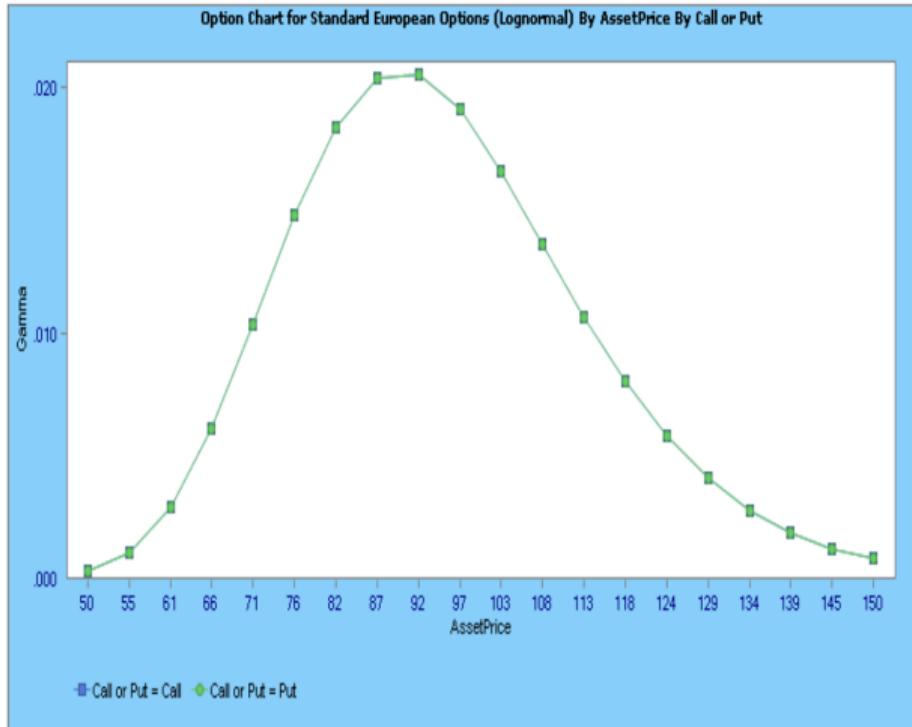
- Let's, for the sake of argument, assume the stock price changes but no time passes.
  - Thus, no interest loss and no time-value loss on call.
- The net position makes money unless the stock price stays constant
- Why? By Taylor formula,

$$C_{t+\Delta t} - C_t = \Delta_t (S_{t+\Delta t} - S_t) + \frac{1}{2} \Gamma_t (S_{t+\Delta t} - S_t)^2,$$

where  $\Gamma = \frac{\partial^2 C}{\partial S^2}$  captures convexity of the option price

- The position is long convexity
  - Higher  $\Gamma$  means more convexity and larger average gain

# Black-Scholes Call and Put Gammas



- Call's  $\Gamma = \frac{1}{\sigma S_0 \sqrt{T}} N'(d_1)$
- They are the same! Not surprising since  $\Delta_c = 1 + \Delta_p$

## P&L of Delta Hedge: Effect of Theta

- Why would anyone consider delta hedging as a viable hedging strategy?
  - We assumed no time passes.
  - In reality, call must lose value in time...
- Theta,  $\frac{\partial C}{\partial t} = \Theta$ , measures the change in the value of the option when a unit of time passes, all else equal
- For calls,  $\Theta < 0$ , so the call will decrease in value.
  - From P&L: Assume the stock price stays constant

$$P&L_{t,t+\Delta t} = \Theta_t \Delta t.$$

- The net position loses due to the time-decay on the option
- The BS theta is for call/put

$$\Theta_0 = -\frac{\sigma S_0 N'(d_1)}{2\sqrt{T}} \mp rKe^{-rT} N(\pm d_2) \approx -\frac{1}{2} \Gamma_0 \sigma^2 S_0^2$$

(exact at  $r = 0$ )

## Gamma versus Theta

- The net effect of time-decay and changes in the price of the underlying:

$$\begin{aligned} P\&L_{t,t+\Delta t} &= -\Delta_t (S_{t+\Delta t} - S_t) + (C_{t+\Delta t} - C_t) \\ &= \Theta_t \Delta t + \frac{1}{2} \Gamma_t (S_{t+\Delta t} - S_t)^2 \end{aligned}$$

- So, when does the  $\Gamma$ -effect cancel with the  $\Theta$ -effect?

$$\begin{aligned} P\&L_{t,t+\Delta t} &\approx -\frac{1}{2} \Gamma_t \sigma^2 S_t^2 \Delta t + \frac{1}{2} \Gamma_t (S_{t+\Delta t} - S_t)^2 \\ &= \frac{1}{2} \Gamma_t S_t^2 \left[ \left( \frac{S_{t+\Delta t} - S_t}{S_t} \right)^2 - \sigma^2 \Delta t \right] \end{aligned}$$

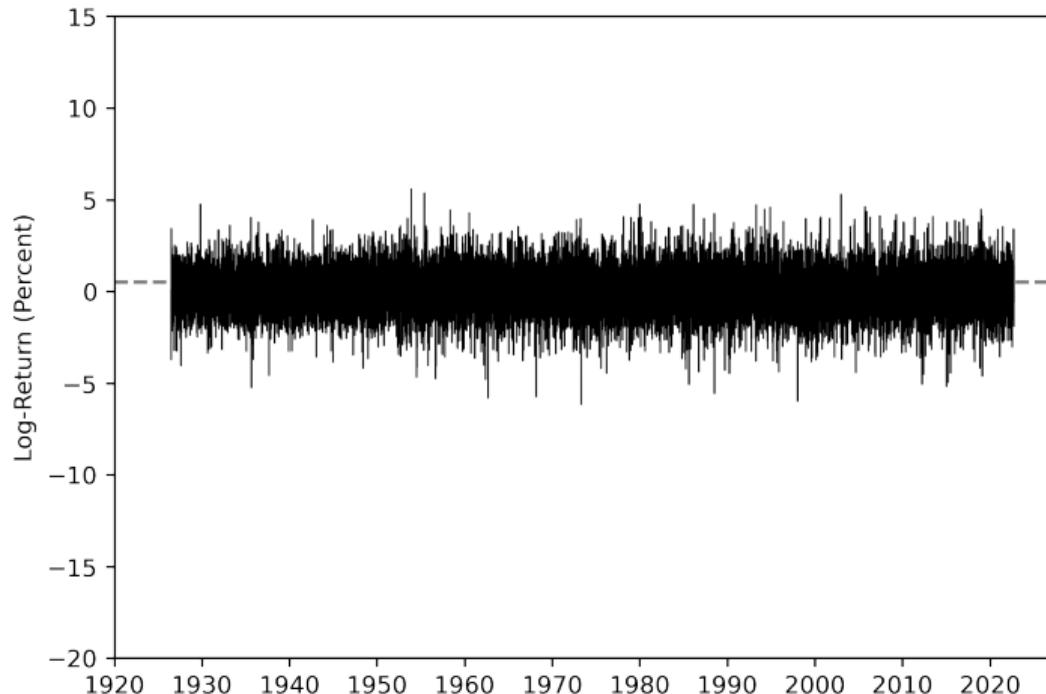
- Thus, the  $\Gamma$ -effect and the  $\Theta$ -effect cancel when the stock prices move by about 1 standard deviation.
- The hedged P&L is driven by the spread between realized and implied variance  $\Rightarrow$  *volatility trading*

## **7. The Black-Scholes in the Real World**

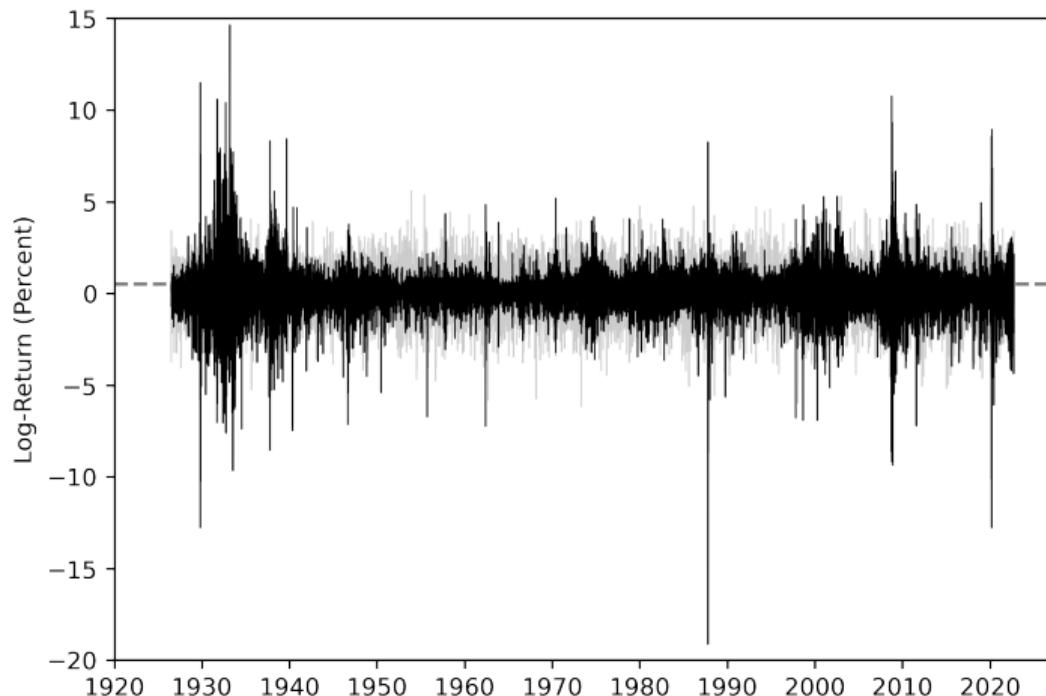
## The Black-Scholes in the Real World

- If stock prices follow the Random Walk model, then we can accurately estimate  $\mu$  and  $\sigma$  using a very long dataset
- Take the series of CRSP value-weighted market return beginning in 1927 and estimate the parameters via sample mean and volatility
- Simulate the returns given these parameters
- Compare with the actual returns
- Consider rolling estimates of volatility

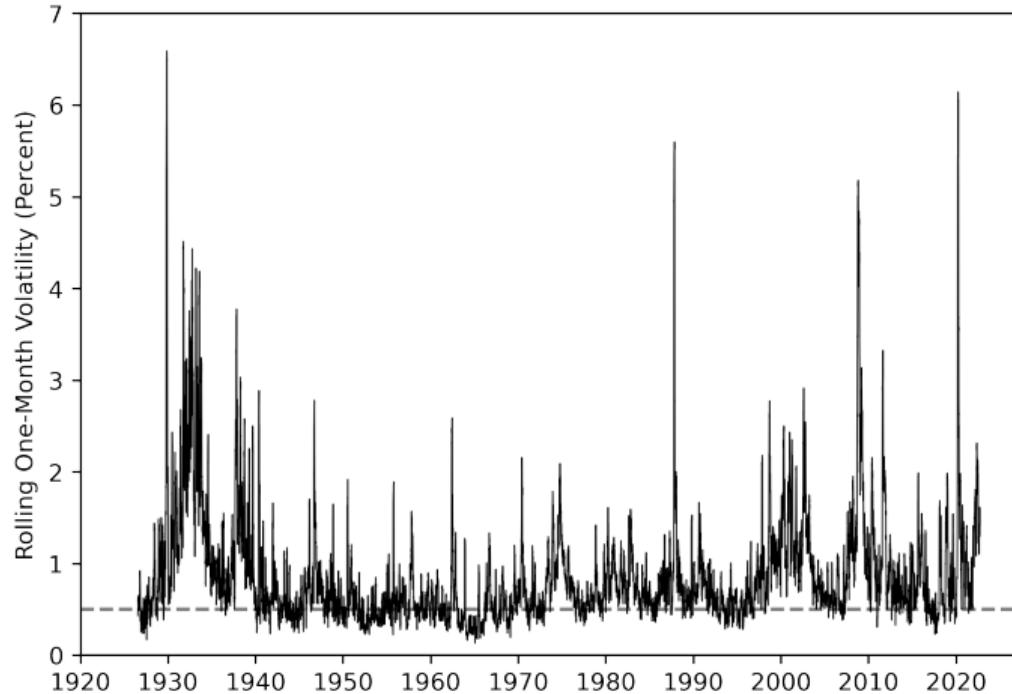
## Simulated Log-Returns



## Actual CRSP Market Log-Returns



## Realized CRSP Market Volatility



## Stock Return Distribution

- The easiest assumption to relax is the evolution of returns
- Add time-varying volatility and jumps:

$$d \log S_t = \mu_t dt + \sigma_t dW_t + dZ_t$$

- Details on jumps,  $Z_t$  :
  - $Z_t = N_t \cdot z$
  - $N_t$  counts the number of events (jumps)
  - $z$  is the jump magnitude
- How does this affect the usage of the Black-Scholes model?

## Delta Hedging in Practice

- How do traders use the Black-Scholes formula?
- When writing a call, a trader wants to reduce the exposure to changes in the stock price
  - It is enough to make money on the huge bid-ask spread
- Use delta-hedging
- Let's see how well the delta-hedging strategy works

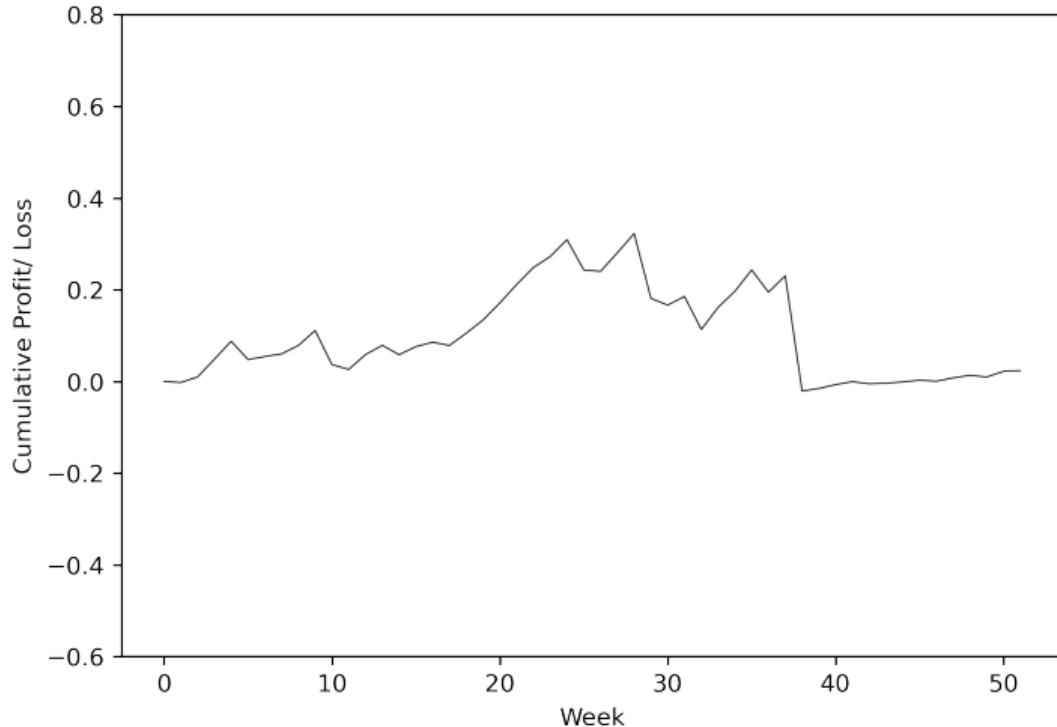
## Dynamic Hedging

- Assume the trader sold a European call with 1 year to maturity and the strike  $K = \$55$  with the current stock price of  $\$50$
- Because rebalancing can be potentially costly, she rebalances her delta-hedged portfolio once a week
- What is her portfolio?

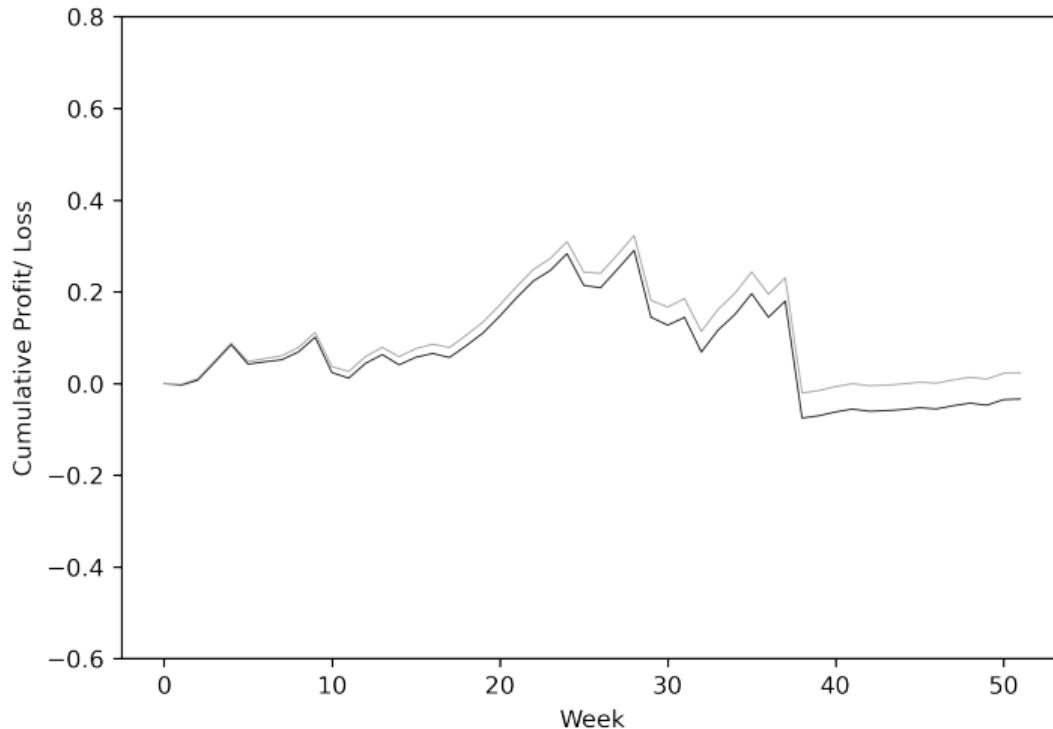
$$H_t = \Delta_t \cdot S_t - C_t + B_t$$

- Every week, she recomputes  $\Delta$  and  $B$  according to the Black-Scholes formula
- Consider three scenarios:
  - ① Stock price evolves according to the Black- Scholes Model
  - ② In addition to case 1, allow for transaction costs,  $\$0.03$  per share
  - ③ Market crashes at a rate of 2 per year,  $z_j \sim N(-0.03, 0.05^2)$

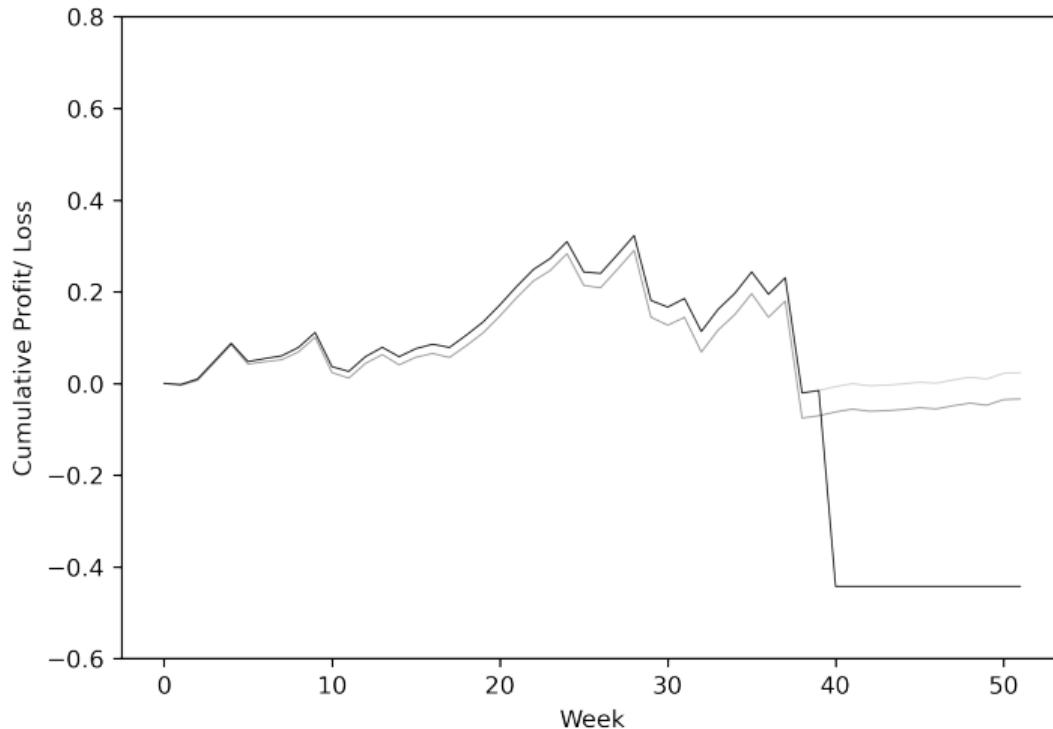
## P&L 1: Black-Scholes



## P&L 2: Black-Scholes with T-costs



## P&L 3: Black-Scholes + Crashes



## **8. The Volatility Smile**

## The Volatility Smile

- The Black-Scholes model assumes constant volatility of returns
- What happens if you value options using historical volatility?
  - There is going to be a pricing error
  - Hard to characterize systematically because the magnitude of error depends on the stock price
  - Could be high or low even for the same stock
- Consider the following thought experiment:
  - Consider the observed market price of an option,  $O_t^M$
  - Which value of volatility produces correct BS price?
  - Holding the option price fixed and solving for  $\sigma$  gives an “implied volatility”.

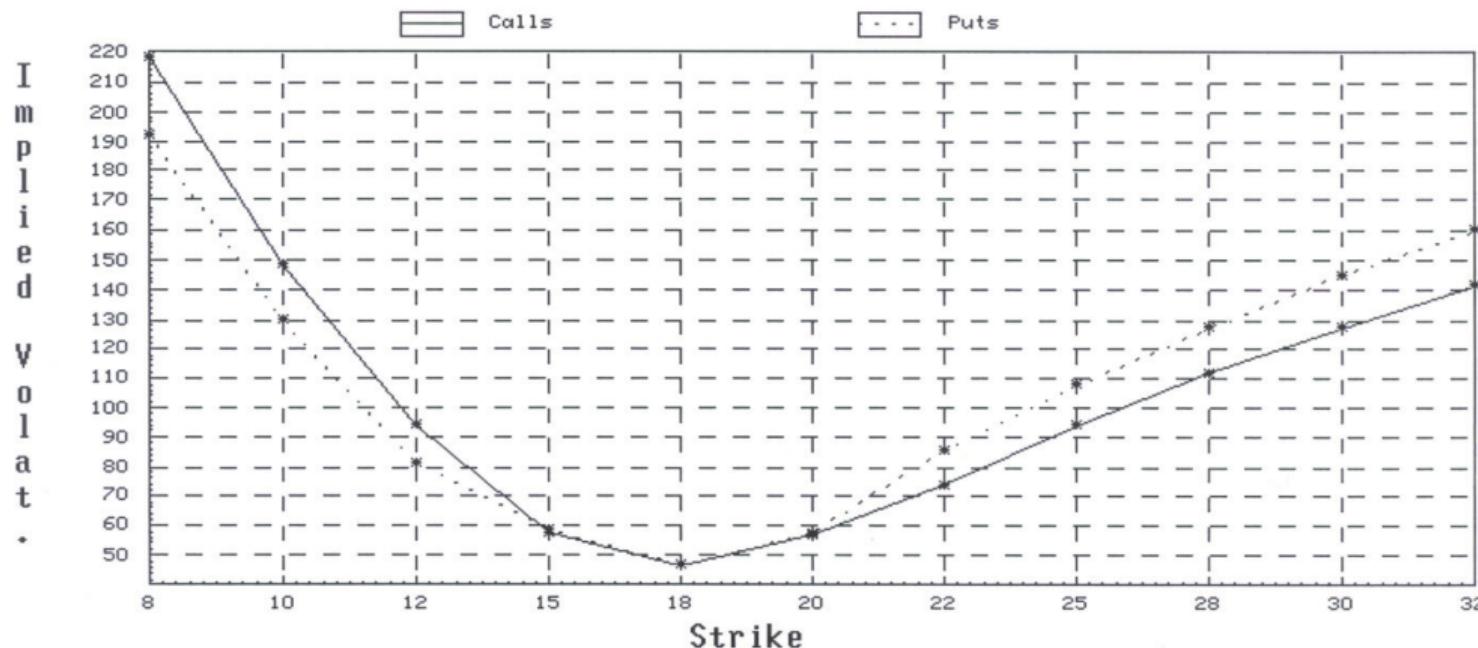
$$O_t^M = BS(S_t, K, T, r, \sigma_t^{IV})$$

- The relationship is one-to-one

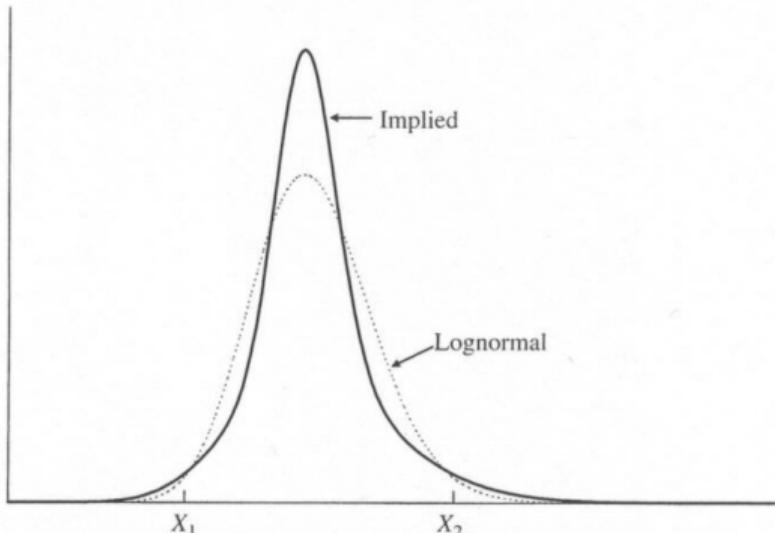
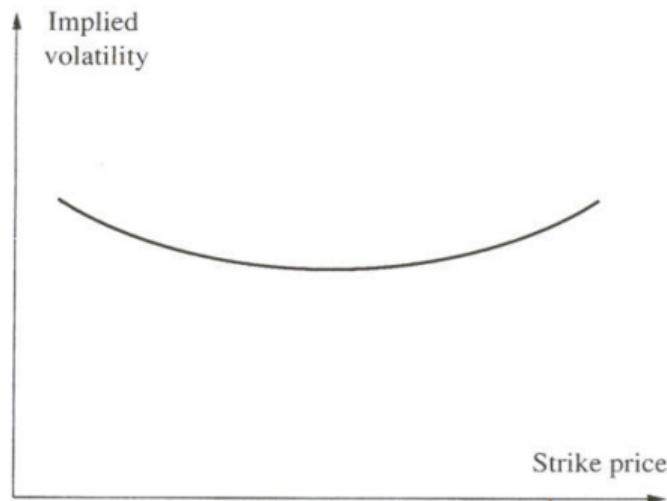
# Implied Volatility Smile

DG21 Equity SKEW

**CYQ** Option Volatility Skew Graph  
Expiration Date Prices Ask Center 17 Show: Imp Vol  
4/20/02 Graph or Table G Number of Strikes 18 Premium Delta  
Page 2 of 10 Und. Price: 16.89 Fin Rate: 1.79 Gamma Vega



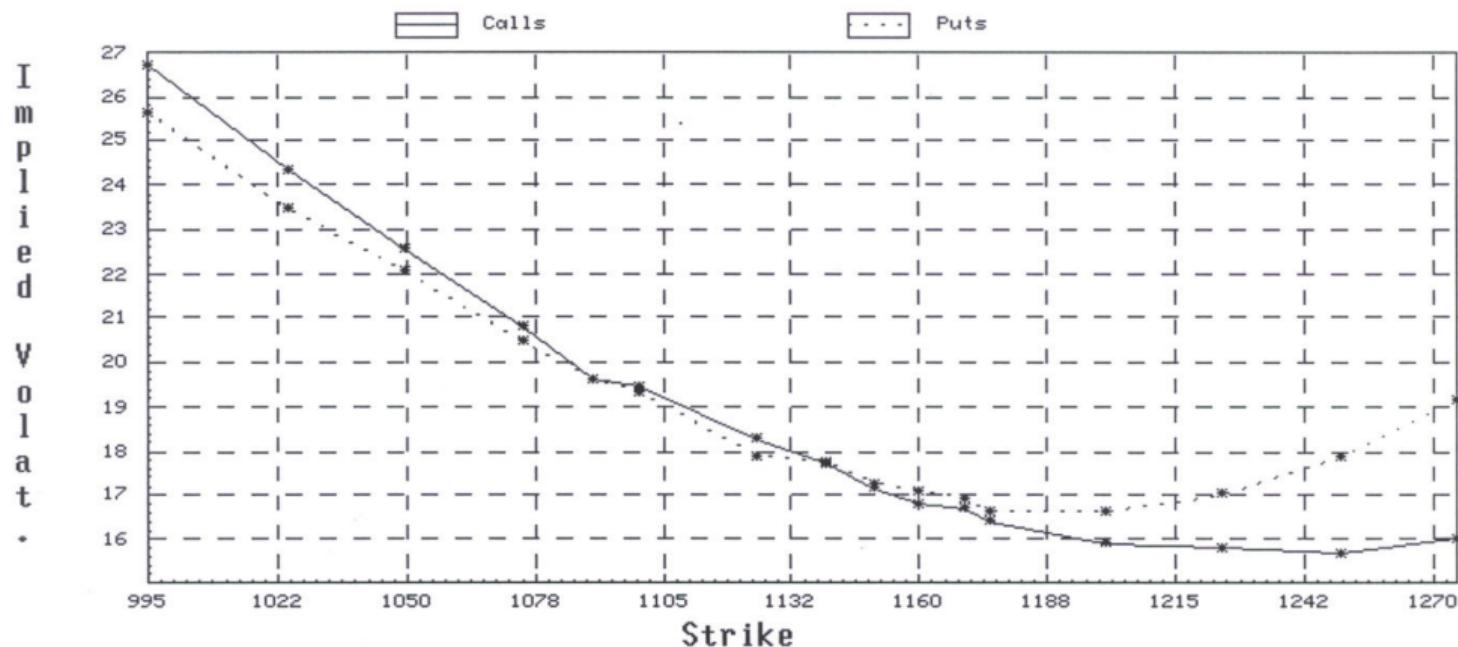
## Smile Implied Distribution



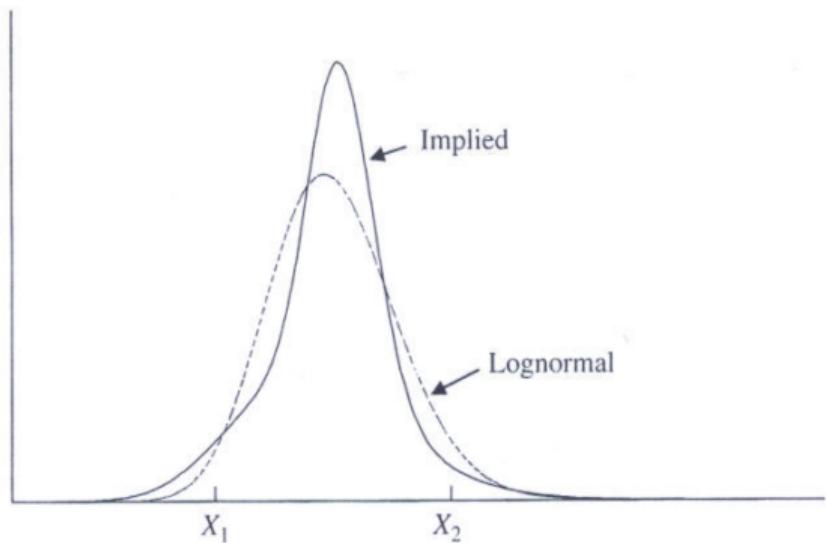
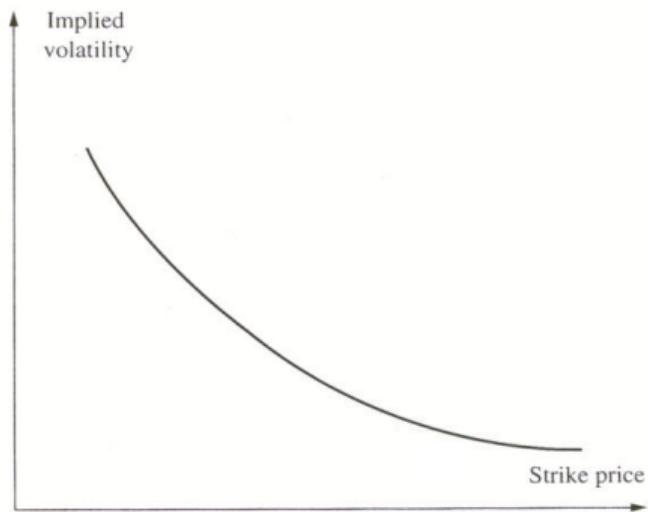
# Implied Volatility Smirk

DG21 Index SKEW

**SPX** Option Volatility Skew Graph  
Expiration Date Prices Ask Center 1139  
5/18/02 Graph or Table G Number of Strikes 18  
Page 2 of 8 Und. Price: 1138.52 Fin Rate: 1.79  
Show: Imp Vol Premium Delta Gamma Vega

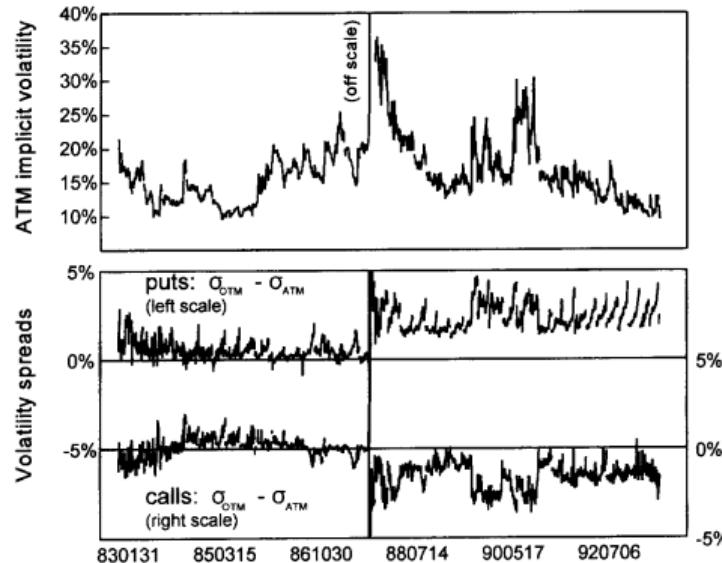


## Smirk Implied Distribution



# Reasons for Smiles and Smirks

- Leverage effect:
  - The Stock Price goes down
  - Leverage goes up
  - Volatility goes up
- Crashophobia
- Traders incorporate the possibility of Oct.87-like crash



## **9. Corporate Bond Valuation**

## Corporate Bond Valuation

- Merton proposed in 1974 to view shares and bonds as options on the firm's assets value
- Such a view allows for a unified approach to the firm's capital structure
  - It is known as structural approach, or
  - contingent claims approach (CCA)
- We will discuss the simplest structural model:
  - A firm has one zero bond
  - A firm issued one share (claim) on its equity

## A firm's balance sheet

- $V$  = the value of firm's assets is shared by debtholders and shareholders:
  - $S$  = shareholder's value, or equity value
  - $P$  = bondholder's value, or debt with face value  $F$
- $S$  can be viewed as a price of a call written on  $V$  with strike  $F$  :
  - Downside is limited to zero because of limited liability
  - Payoffs start after the debt of value  $F$  is paid out to bondholders, i.e., positive payoffs when  $V > F$
- $P$  resembles a price of a put on  $V$  struck at  $F$ . Why?

## Valuation

- Suppose the bond matures at time  $T$
- Debt payoff at maturity is

$$P(V_T) = \min(V_T, F) = F + \min(V_T - F, 0) = F - \max(F - V_T, 0)$$

- Therefore, the value of debt at  $t < T$  is

$$P(V_t) = PV(F) - P(V_t, F, T)$$

- Equity payoff at maturity is

$$S(V_T) = \max(V_T - F, 0)$$

- Therefore, the value of equity at  $t < T$  is

$$S(V_t) = C(V_t, F, T)$$

- Alternatively, we split the value of firms assets,  $V$ , between bondholders and shareholders

$$S(V_t) = V_t - P(V_t) = V_t - PV(F) + P(V_t, F, T)$$

- Are these two answers the same?

## Black-Scholes Value of Debt and Credit Spread

- Debt's value is  $P(V_t) = PV(F) - P(V_t, F, T)$
- Black-Scholes formula gives us:

$$P(V_t) = Fe^{-r(T-t)} - [Fe^{-r(T-t)} N(-d_2) - V_t N(-d_1)]$$

- Re-write:

$$P(V_t) = Fe^{-r(T-t)} \left( 1 - N(-d_2) \cdot \left[ 1 - \frac{V_t}{Fe^{-r(T-t)}} \frac{N(-d_1)}{N(-d_2)} \right] \right)$$

- Interpretation:
  - $N(-d_2)$  is default probability, **DP**
  - [...] is expected discounted loss given default per \$1, **LGD**
- Yield to maturity solves  $P(V_t) = Fe^{-y(T-t)}$ , that is,

$$y = -\frac{1}{T-t} \ln \frac{P(V_t)}{F}$$

- The credit spread  $s = y - r$  reflects the key objects:

$$s = -\frac{1}{T-t} \ln(1 - DP \cdot LGD) \approx \frac{DP \cdot LGD}{T-t}$$

## Ratings and Spreads

- We can use the Black-Scholes model to relate ratings to spreads
- $F$  = total liabilities
- $V$  = equity value + book value
- Use Ito's lemma to measure volatility,  $\sigma_V$ :

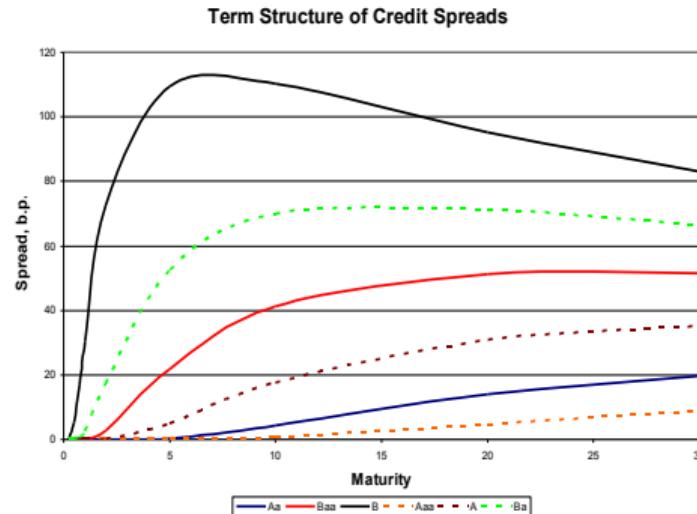
$$\sigma_S(t) = \sigma_V \cdot \frac{\partial S(V_t)}{\partial V_t} \cdot \frac{V_t}{S(V_t)} = \sigma_V N(d_1) \frac{V_t}{S_t}$$

# Ratings and Spreads

- Typical numbers

- $r = 2\%$
- $\sigma_V = 23\%$
- Leverage ratio  $F/V_t, \%$

| Aaa | Aa | A  | Baa | Ba | B  |
|-----|----|----|-----|----|----|
| 13  | 21 | 32 | 43  | 53 | 66 |



Observations:

- Spreads are zero at short maturity
- Lower rated companies have a hump-shaped term structure
- Spreads are underestimated

| 10-yr spread, bps | Aa | Baa | B   |
|-------------------|----|-----|-----|
| Data              | 91 | 194 | 408 |
| Model             | 4  | 41  | 110 |

## KMV

- Alternative implementation of the Merton models based on DP is offered by KMV
  - KMV was founded in 1989 by Stephen Kealhofer, John Andrew McQuown, and Oldrich Vasicek
  - KMV was acquired by Moody's in 2002 for \$70 mil and became Moody's KMV
  - Moody's Analytics is formed in 2008
- KMV compute their proprietary measure of default probability, known as EDF - expected default frequency
- The measure is based on a similar formula and a comprehensive database, which tracks companies

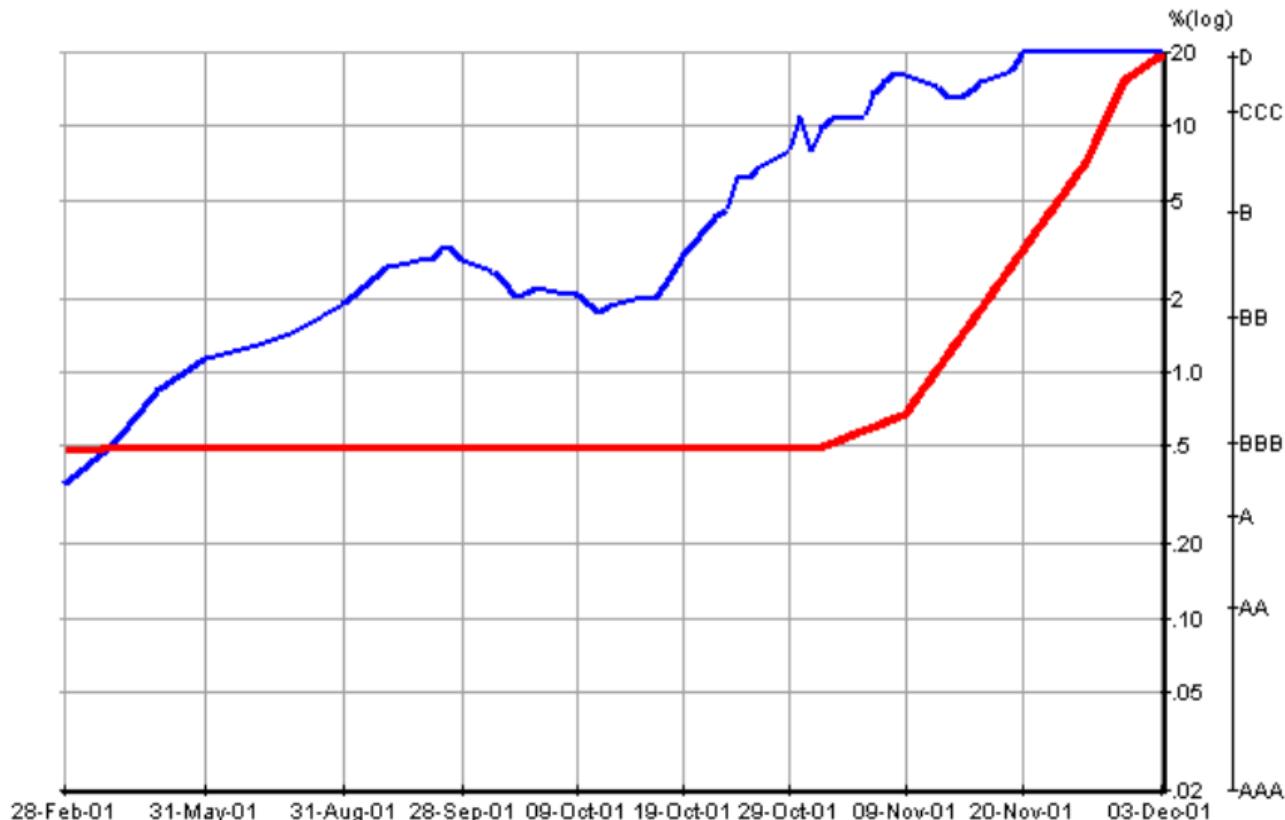
## KMV - details

- Use two equations:

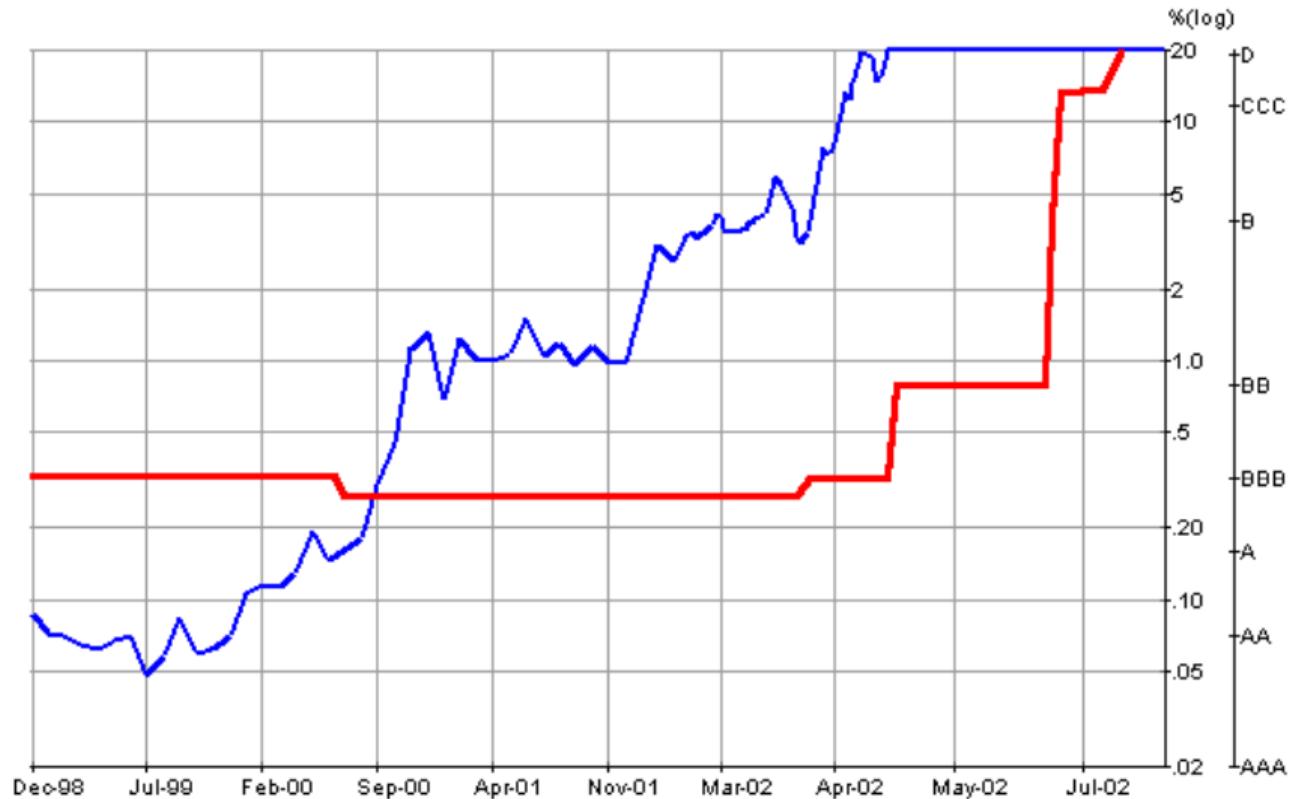
$$\begin{aligned}S(V_t) &= V_t N(d_1) - F e^{-r(T-t)} N(d_2) \\ \sigma_S(t) &= \sigma_V N(d_1) \frac{V_t}{S_t}\end{aligned}$$

- We know:
  - $S_t$  – stock price
  - $\sigma_S$  – can estimate
  - $r$  – risk-free rate
  - $T$  – bond's maturity
- Estimate  $F$  as debt in current liabilities + 1/2 of long-term debt
  - short-term debt requires a repayment of the principal soon
  - long-term debt requires only coupon payments
- Solve the two equations for the two unknowns,  $\sigma_V$  and  $V_t$

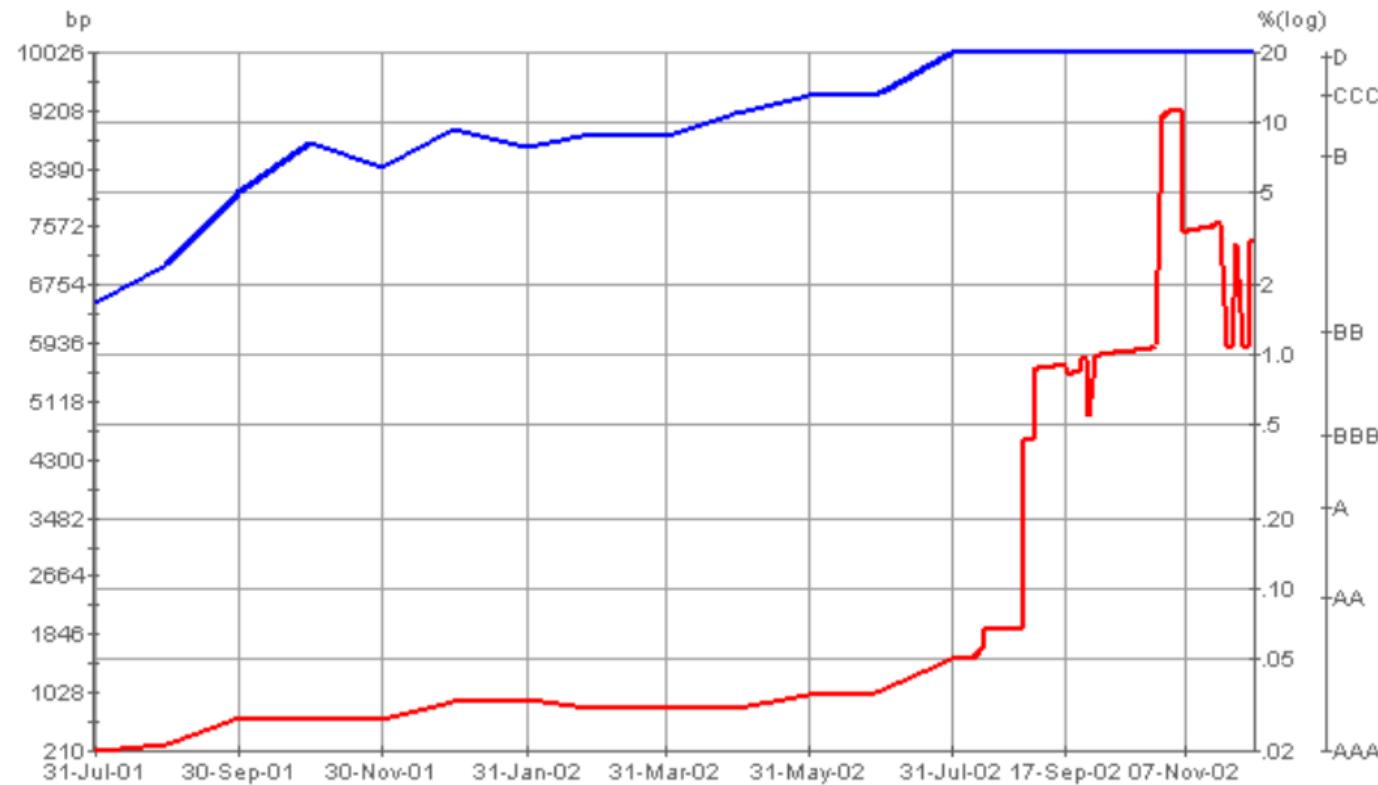
## Enron EDF



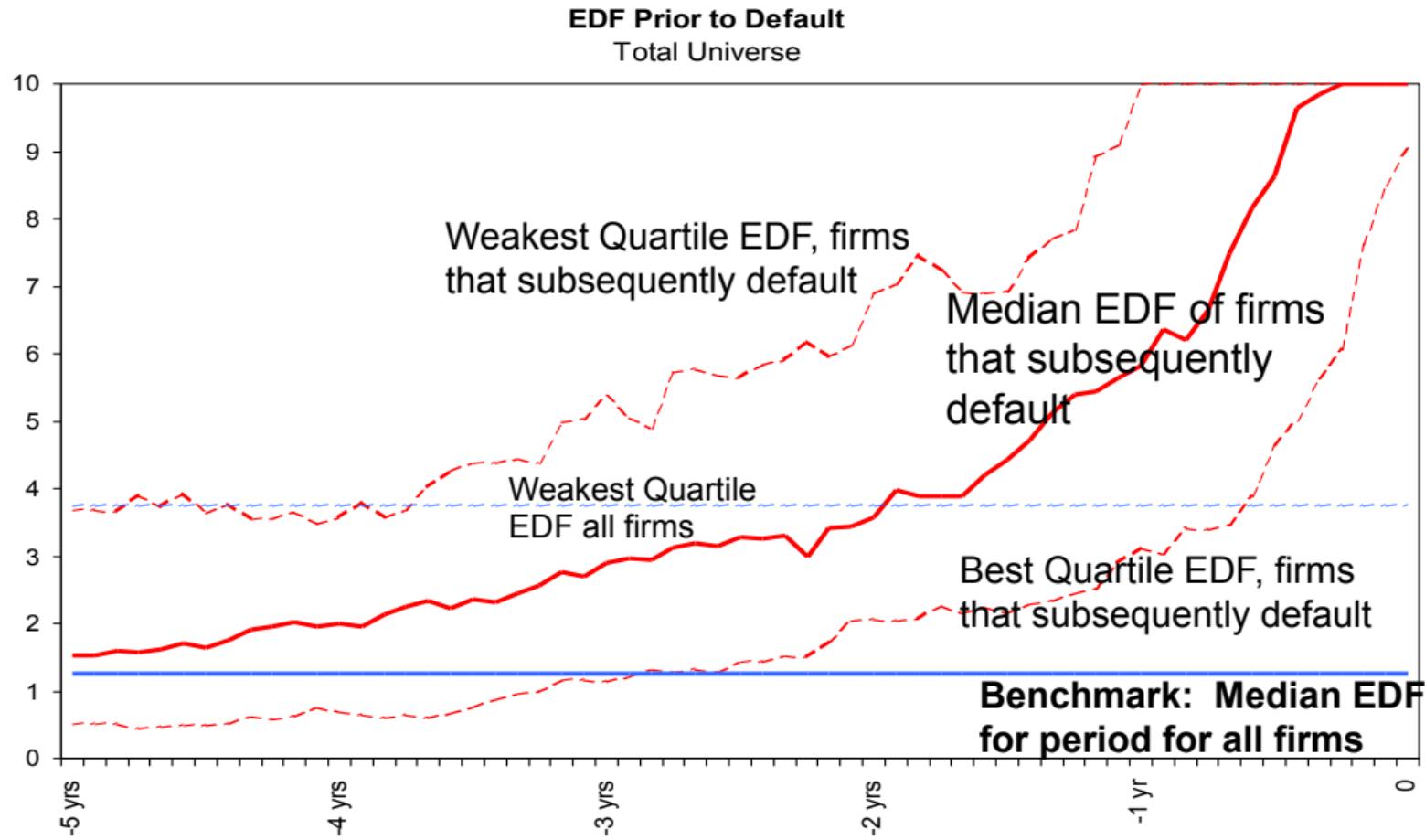
## WorldCom EDF



# United EDF



# Moody's Analytics EDF



## 10. Trading Strategies

## Dynamic Replication

- Familiar from the Black-Scholes option valuation argument
- Replicate a position in a call with  $\Delta$  shares of the stock and a risk-free bond
- If a call appears cheap, buy it, and sell  $\Delta$  shares of the stock
- The position is riskless as long as the  $\Delta$  does not change
- $\Delta$  increases when the stock goes up, so have to sell more shares; and vice versa
- Risks of dynamic replication:
  - Cannot re-balance every instant – costly
  - If a model is wrong, the delta is wrong
- **NB** Similar to bond immunization

# Capital Structure Arbitrage

- This is an example of a model-based dynamic replication
- Idea:
  - compare the value of debt and equity
  - buy the cheap asset, sell the expensive one – in the right proportions
  - wait for convergence
- How?
  - Use a model to establish whether debt is cheap or expensive, e.g., a “structural model”, like the Merton model
  - The model also tells us how debt value changes with equity value – debt’s delta w.r.t. equity
  - Hedge debt with delta-weighted equity

## A Joint Model of Debt and Equity

- We know that debt and equity could be viewed as options on value of unlevered assets,  $V_0$ 
  - Equity

$$S_0 = S(V_0) = C(V_0, F, T) = \Delta_C \cdot V_0 + B_C$$

- Debt

$$D_0 = D(V_0) = PV(F) - P(V_0, F, T) = -\Delta_P \cdot V_0 - PV(F) - B_P$$

- Put-call parity implies:

$$V_0 = S_0 + D_0$$

- So, a short position in debt can be offset by a long position in equity

## Example

- In November 2001 CDS on Hanson (a building materials group) was trading at 95 bps
- The Merton-like model was implying 160 bps – debt is expensive relative to the model
- So want to sell debt, or buy protection, and delta-hedge with long equity position
- On 11/5, a trader bought protection at 95 bps and using a model-based delta of 0.12 bought stock at £2.91
- On 11/17, CDS spread was at 140 bps, and stock at £2.95
- Both legs went into the trader's favor

## Model-based delta

- Debt

$$D_0 = D(V_0) = PV(F) - P(V_0, F, T)$$

- Definition of delta [combined with the chain rule and a derivative of an inverse function] implies:

$$\Delta_D(V_0) = \frac{\partial D_0}{\partial S_0} = \frac{\partial D(V_0)}{\partial V_0} \left\{ \frac{\partial S(V_0)}{\partial V_0} \right\}^{-1} = -\frac{\Delta_P}{\Delta_C} = \frac{N(-d_1)}{N(d_1)}$$

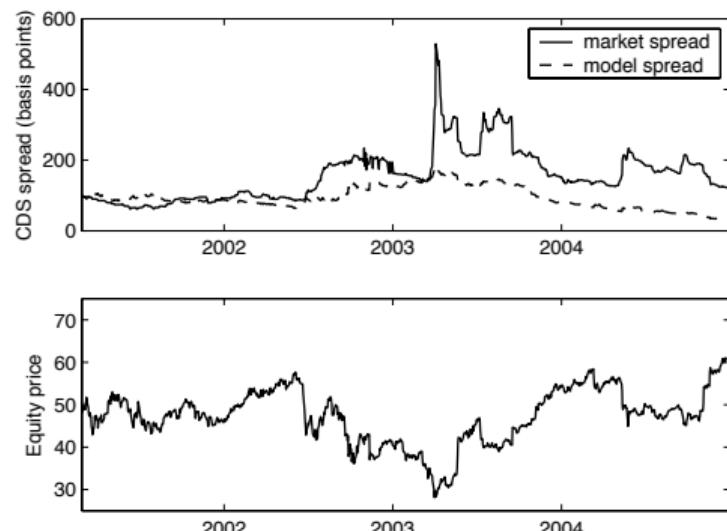
- Last step: construct  $\Delta_D(S_0)$  by replacing  $V_0$  in by  $S_0 + D_0$  in the expression for  $d_1$

## Implicit volatility bet

- One can compare the model and market value of debt
- Because debt can be viewed as an option, implied volatility will always equalize the two
- Therefore, a view that the market value of debt is too low, or too high is implicitly a view that implied volatility is too high, or too low relative to historical volatility
  - You may track the spread between implied and historical vol and see if it is at historical highs or lows
  - This view helps in establishing and closing the position

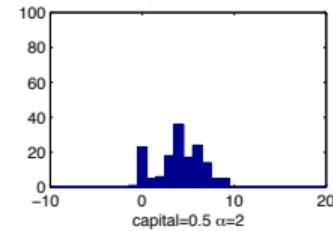
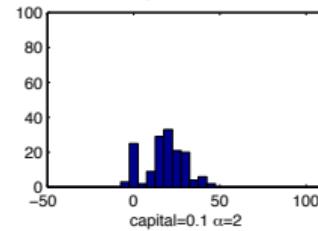
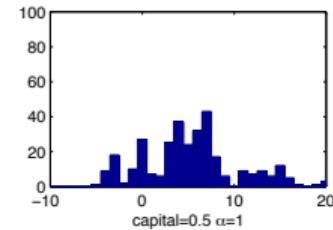
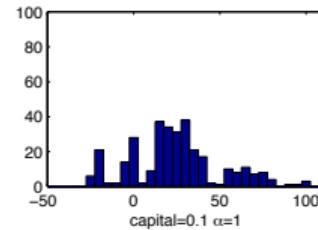
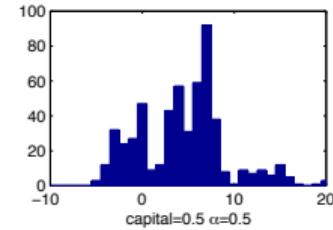
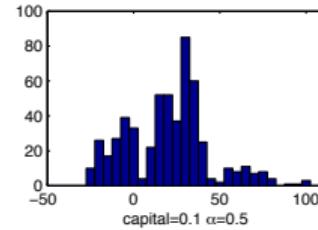
## A Case of Altria

- Altria, previously Philip Morris, was involved in tobacco-related litigation since the early 1990s
- In March 2003, a judge in IL ordered Altria to post a \$12bn bond to appeal a class-action lawsuit
- In response, Moody's downgraded Altria from A2 to Baa1



## Strategy: Trade if ...

- market CDS >  $(1 + \alpha) \times$  model CDS, or
- model CDS >  $(1 + \alpha) \times$  market CDS
- hedge with equity using  $\Delta_D(S_0)$  per \$1 of the CDS position



## Static Replication

- It would be nice to find assets, or portfolios of assets, that would have the same cash flows period by period and different prices
- We've seen an example in credit:
  - CDS basis
- One more example:
  - Convertible bonds

## Convertible Bonds

- The bonds give their holders the right to convert them into a pre-specified number of shares
  - A risky corporate bond ...
  - ... and a(n American) call option on the stock struck at the value of debt
- In contrast to a regular bond, gives an opportunity to participate in the upside
- Allows the issuer to pay lower coupon
  - Option adds extra value
  - Need a lower coupon to keep the issue priced at par
  - Smaller liability “on paper”
- Effectively, there is a spread between coupons on convertible and regular debt
  - There is a scope to make convertible coupons higher than they should be to attract investors

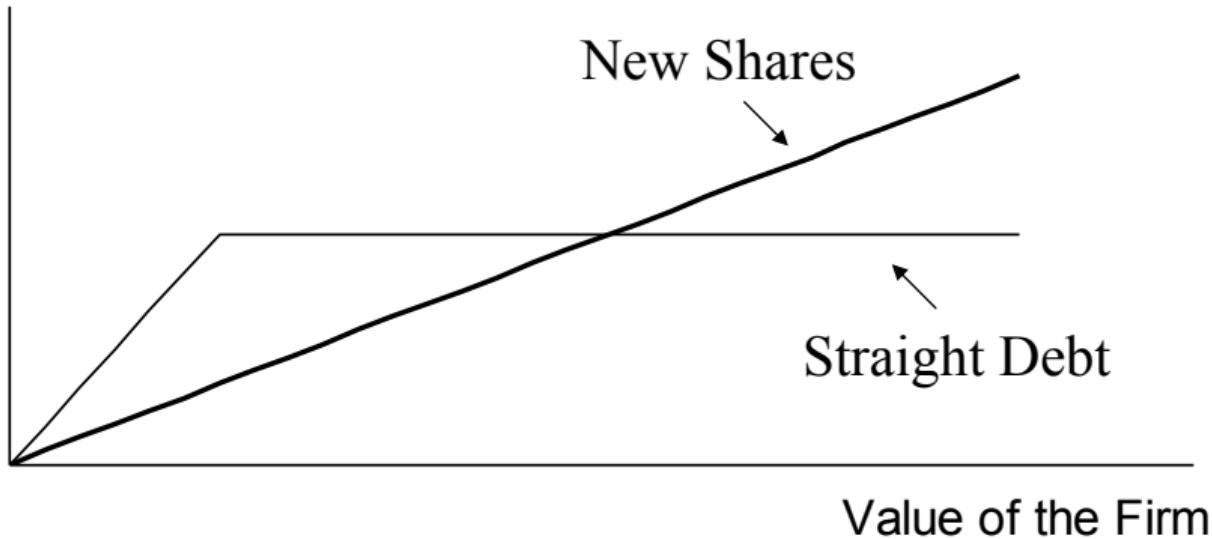
## Optionality

- Higher coupon on a convertible that is still sold at par is equivalent to lower volatility in the embedded option
- One can think of the spread between implied volatility of a convert and implied volatility on a regular call option
- The case for making volatility more low is better during the periods of higher volatility
- Implication 1: We can predict convert issuance waves
- Implication 2: We can take advantage of cheapness of the embedded option by buying the cheap bond and replicating with a short position in more expensive securities

## Who issues converts?

- More than 95% of convertible issuers are large and mid-cap companies
- 35% of issuers are rated investment-grade
- Examples:
  - Bank of America
  - Intel
  - Microsoft
  - Citigroup

## Static replication

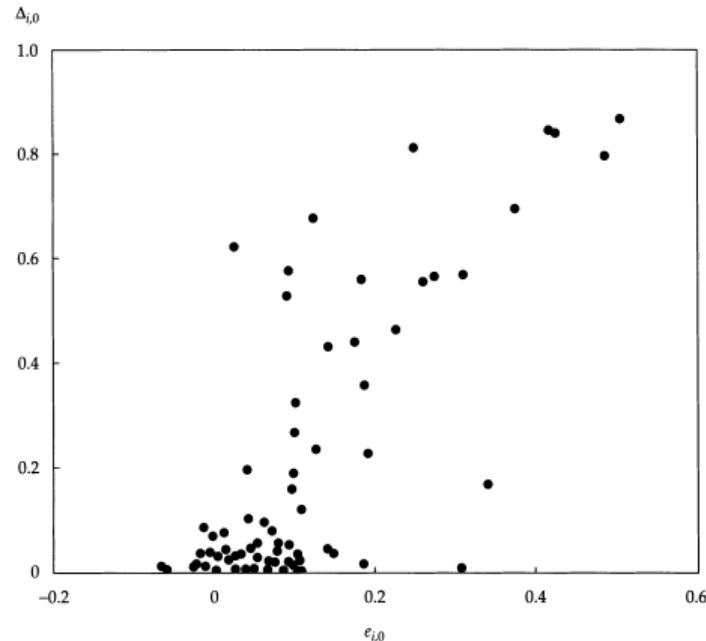


## Trading

- Buy the cheap convert
- Its payoff is the same as that of a position in a regular corporate bond and a call option struck at face value of debt [when no dividends]
- Sell a call and a bond, or buy a protection via CDS + pay fixed on an interest rate swap
- But this is too boring/expensive/infeasible
  - Some drop the swap leg
  - A short call position would have to pay out if in the money, so some replace a call with a delta-weighted short position in equity
  - Back to dynamic replication
- Hedge funds participate in 80% of all convertible issues
  - Big players: Pine River, Silverback

# Mispricing

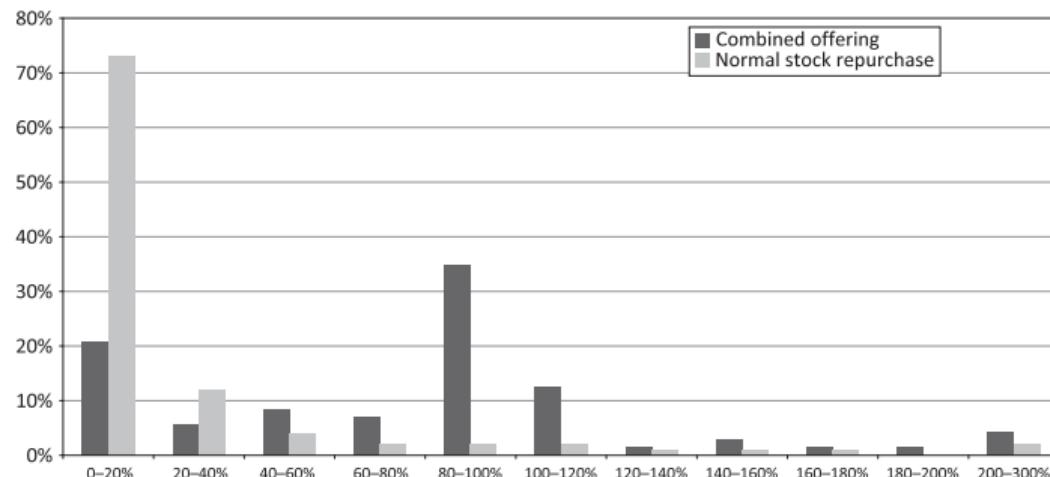
- Rank securities by their delta
- The higher the delta, the more the convert resembles equity



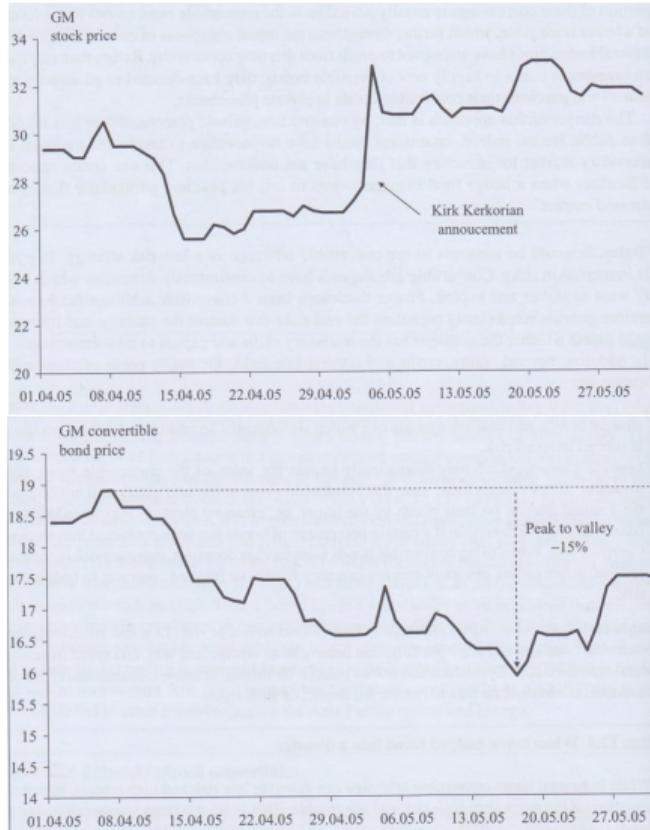
Notes: This figure depicts values of delta and the degree of mispricing at the issuance date. The delta measure ( $\Delta_{i,0}$ ) is computed as in Equation 2. Mispricing is defined as  $\epsilon_{i,0} = (M_{i,0} - B_{i,0})/B_{i,0}$ , where  $M_{i,0}$  represents the model and  $B_{i,0}$  represents the trading price at the issuance date.

## Stock repurchase

- Companies repurchase shares simultaneously with issuing converts
- This facilitates short-selling associated with delta-hedging to the convert issuer via an underwriter
- Percentages of actual repurchases in the 1st qtr after the convert issuance announcement:

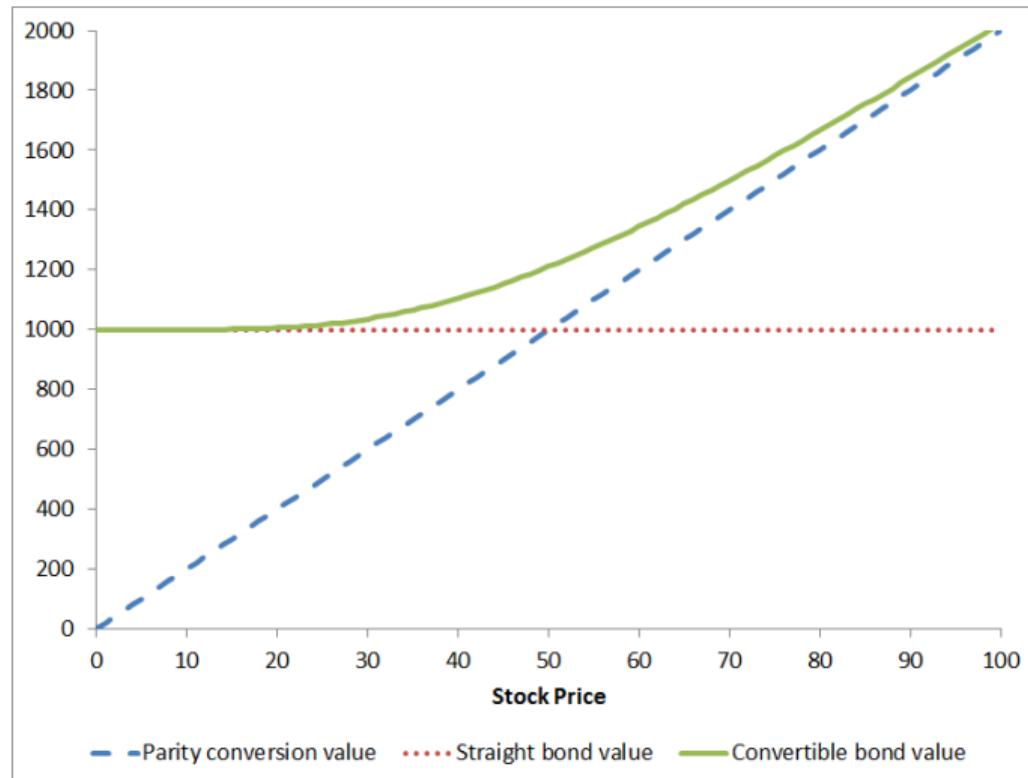


# What can go wrong?

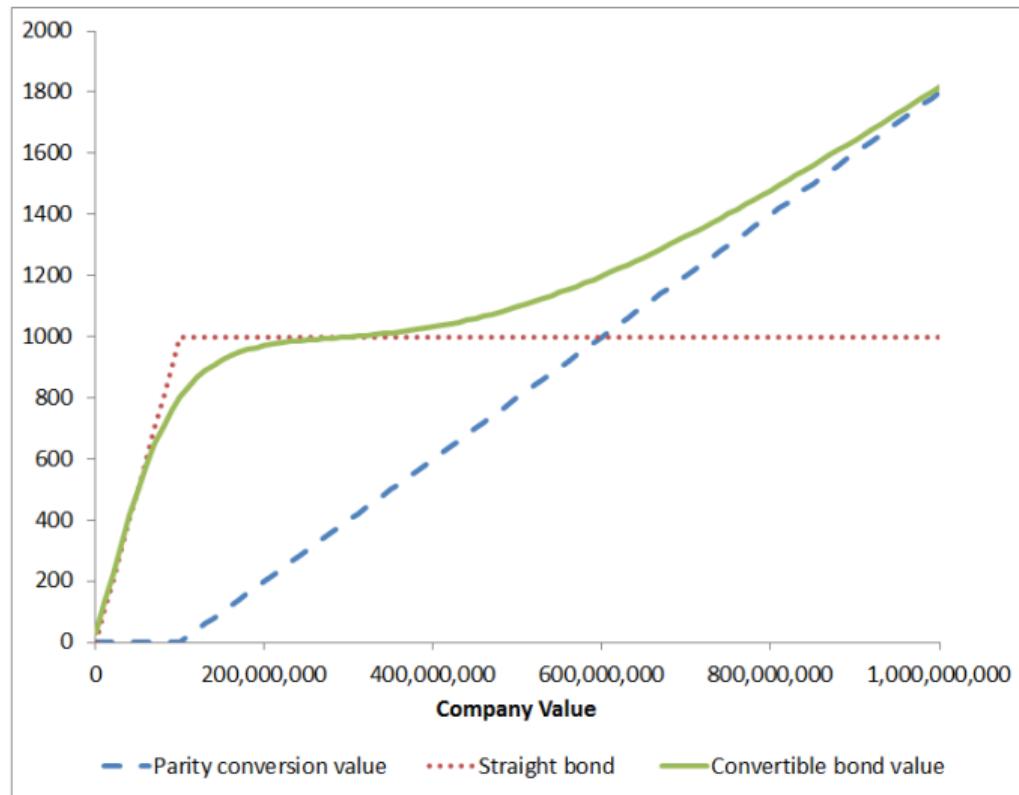


Need dynamic hedging

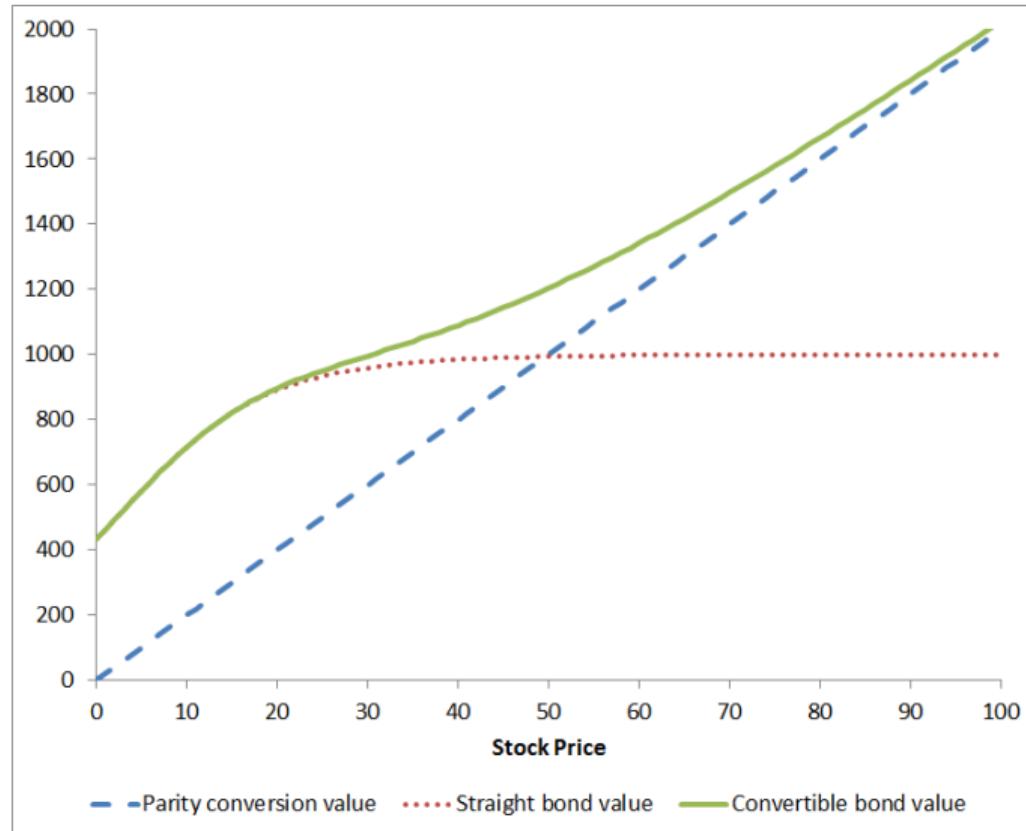
## Valuation (A: No Default)



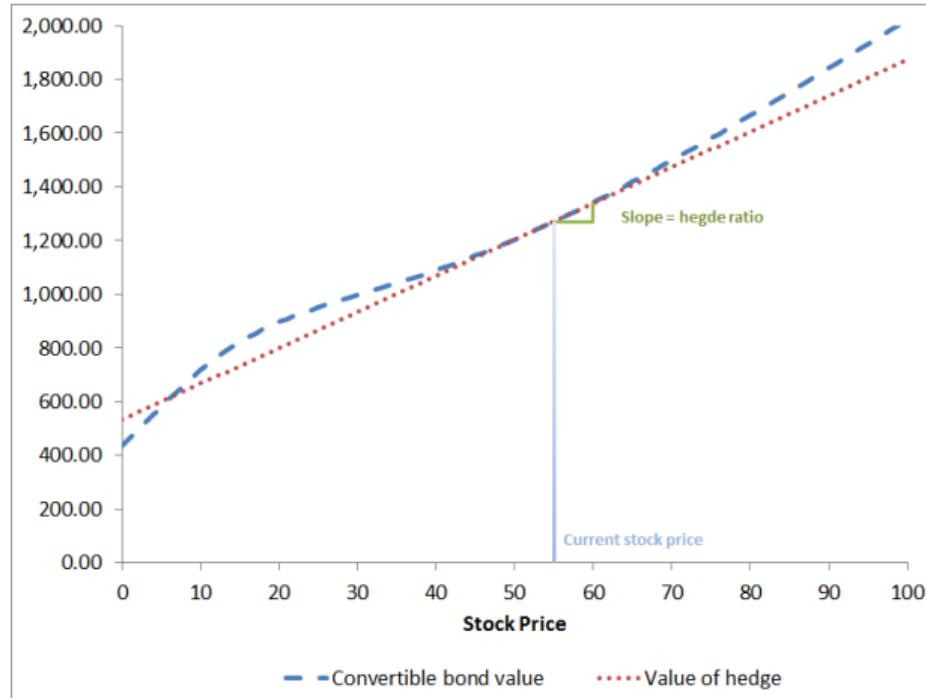
## Valuation (B: as a function of V)



## Valuation ( $C$ : as a function of $S$ )



# Hedging



- When would one convert a bond?
- Convexity (outside of the low stock price region)