Week 7: Extensions of Arbitrage Pricing

Professor Stavros Panageas

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I. Multi-dimensional version of Black Scholes

- There are n risky assets and one riskless asset.
- Under the objective probability measure P the dynamics of the risky assets are given by

$$dS_i(t) = \mu_i S_i(t)\,dt + S_i(t)\sum_{j=1}^{n}\sigma_{ij}d\overline{W}_j(t)$$

with $\overline{W}_1(t), ..., \overline{W}_n(t)$ independent Brownian motions.

- We will denote the $n \times n$ matrix σ

$$\sigma = \{\sigma_{ij}\}$$

and assume throughout that it is non-singular and the $n \times 1$ vector $\mu = \{\mu_i\}$.

- Finally suppose there is a risk-free asset

$$\frac{dB(t)}{B(t)} = rdt$$

- More compact notation: Let

$$\overline{W}(t) = \begin{bmatrix} \overline{W}_1(t) \\ \ldots \\ \overline{W}_n(t) \end{bmatrix}$$

and σ_i the $i'th$ row vector of σ.

- Then we can write more compactly

$$dS_i(t) = \mu_i S_i(t)\, dt + S_i(t)\, \sigma_i d\overline{W}(t)$$

- Once again we are interested in pricing a claim whose payoff takes the form

$$\Phi(S_T).$$

- Specifically we are looking for the arbitrage-free price $F(t, S_t)$ where $F : R_+ \times R^n \to R$.

- The strategy:
- Form a self-financing portfolio based on $S_1, .., S_n$, B and F.
- Since we have $n+2$ assets and the portfolio weights must add to one, we have $n+1$ degrees of freedom in forming that portfolio.
- Choose the weights such that the driving Wiener processes are cancelled in the portfolio, thus leaving us with a value process of the form

$$dV(t) = V(t)\, k(t)\, dt.$$

- We will then derive the arbitrage free conditions from the requirement that $k(t) = r$.

- To put these ideas into action we start by observing that the multi-dimensional Ito Formula implies that

$$dF = F\mu_F dt + F\sigma_F d\overline{W}_t$$

where

$$\mu_F = \frac{1}{F}\left[\frac{\partial F}{\partial t} + \sum_{i=1}^{n} \mu_i S_i F_{S_i} + \frac{1}{2} tr\left\{\sigma' S F_{SS} S \sigma\right\}\right] \qquad (1)$$

where

$$S = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & S_n \end{bmatrix}$$

and

$$\sigma_F = \frac{1}{F}\sum_{i=1}^{n} S_i F_{S_i} \sigma_i$$

Step 1: Portfolio Formation

- We next form a riskless portfolio consisting of $S_1, ..., S_n, B, F$.

- The requirements on that (relative) portfolio is that i) the weights sum up to one, and ii) the weights are such that the portfolio has no diffusion term.

- To achieve the first requirement

$$u_B = 1 - \left(\sum_{i=1}^{n} u_i + u_F\right) \qquad (2)$$

and to satisfy the latter

$$\sum_{i=1}^{n} u_i \sigma_i + u_F \sigma_F = 0 \qquad (3)$$

- Supposing that this can be done, the value dynamcis are given by

$$\begin{aligned} dV &= V\left[\sum_{i=1}^{n} u_i \frac{dS_i}{S_i} + u_F \frac{dF}{F} + u_B \frac{dB}{B}\right] \\ &= V\left[\sum_{i=1}^{n} u_i\left(\mu_i - r\right) + u_F\left(\mu_F - r\right) + r\right] dt \\ &\quad \underbrace{+V\left[\sum_{i=1}^{n} u_i\sigma_i + u_F\sigma_F\right]}_{=0} d\overline{W}_t \end{aligned}$$

- Now note that we have $n+2$ assets. The requirements (2) and (3) are $n+1$ linear requirements. Suppose that we use the remaining degree of freedom so that for some positive β,

$$\sum_{i=1}^{n} u_i\left(\mu_i - r\right) + u_F\left(\mu_F - r\right) + r = r + \beta \tag{4}$$

Step 2: Excluding Arbitrage

- If it is possible to find $n+2$ weights $u_1, ..., u_n, u_F, u_B$ such that the $n+2$ linear conditions (2) , (3), and (4) are satisfied, then we have an arbitrage.

- To formulate this observation in equivalent matrix notation, an arbitrage will exist if the following system of equations has a solution:

$$\underbrace{\begin{bmatrix} \mu_1 - r....\mu_n - r & \mu_F - r \\ \sigma_1'.....\sigma_n' & \sigma_F' \end{bmatrix}}_{H} \begin{bmatrix} u_S \\ u_F \end{bmatrix} = \begin{bmatrix} \beta \\ 0 \end{bmatrix}$$

- If H is invertible then the above system would have a solution, and there would exist an arbitrage.

- Hence if the market is aribtrage free, it must mean that H is singular.

- This means that we can write the fist row as a linear combination of the remaining rows, i.e., that there exist $\lambda_1 ... \lambda_n$ such that

$$\begin{aligned} \mu_i - r &= \sum_{j=1}^{n} \sigma_{ij} \lambda_j \\ \mu_F - r &= \sum_{j=1}^{n} \sigma_{Fj} \lambda_j \end{aligned} \tag{5}$$

- We will refer to λ_j as the "market price of risk" for the brownian risk j.

- Letting $\lambda = [\lambda_1, ..., \lambda_n]'$, we can write more in vector form:

$$\mu - r1_n = \sigma\lambda$$

or

$$\lambda = \sigma^{-1} [\mu - r1_n] \tag{6}$$

and also

$$\sigma_F = \frac{1}{F} \left[S_1 F_{S_1}, ..., S_n F_{S_n} \right] \sigma \tag{7}$$

Step 3: From market prices of risk to arbitrage pricing

- Combining (5), (6), and (7) leads to

$$\mu_F - r = \frac{1}{F}\left[S_1 F_{S_1}, ..., S_n F_{S_n}\right]\left[\mu - r1_n\right] \tag{8}$$

- Finally, combining (8) with (1) gives after some re-arranging

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} rS_i F_{S_i} + \frac{1}{2} tr\left\{\sigma' S F_{SS} S \sigma\right\} - rF = 0$$

subject to the boundary condition

$$F\left(T, S_T\right) = \Phi\left(S_T\right)$$

- Using the Feynman Kac Theorem, we can give a probabilistic interpretation to the solution of the above PDE as follows

$$F\left(t, S_t\right) = e^{-r(T-t)} E_t^Q\left[\Phi\left(S_T\right)|S\left(t\right) = S_t\right]$$

where

$$dS_i = rS_i dt + S_i \sigma dW_i$$

under the fictitious, risk neutral probability Q, for all $i = 1, ..., n$.

- As in the one-dimensional case, it is still the case that

$$d\Pi\left(t\right) = r\Pi\left(t\right) dt + \Pi\left(t\right) \sigma_\Pi\left(t\right) dW_t$$

where the volatility vector σ_Π is the same under the probability measure P and Q.

II. An Example: The Exchange Option

- An example: Suppose there are two assets (1 and 2) following the processes

$$\begin{aligned} dS_{1,t} &= S_{1,t}\mu_1 dt + S_{1,t}\sigma_1 d\overline{W}_{1,t} \\ dS_{2,t} &= S_{2,t}\mu_2 dt + S_{2,t}\sigma_2 d\overline{W}_{2,t} \end{aligned}$$

- Price an exchange option, i.e., a claim that pays

$$\max\left[S_{1,T} - S_{2,T}\ ,0\right]$$

- Solution: The claim $F(S_1, S_2, t)$ must satisfy the equation

$$F_t + rS_1F_1 + rS_2F_2 + \frac{1}{2}S_1^2\sigma_1^2F_{11} + \frac{1}{2}S_2^2\sigma_2^2F_{22} - rF = 0 \tag{9}$$

subject to the boundary condition

$$F(S_1, S_2, T) = \max[S_1 - S_2, 0]$$

- Trick: Make an educated guess that $F(S_1, S_2, t)$ is multiplicatively separable:

$$F(S_1, S_2, t) = S_2 G(t, z_t)$$

where $z_t = \frac{S_1}{S_2}$.

- Plugging this conjecture for F into (9) shows that the conjectured functional form satisfies (9) as long as the function G satisfies the PDE

$$\begin{aligned} G_t(t,z) + \frac{1}{2}z^2G_{zz}(t,z)\left(\sigma_1^2 + \sigma_2^2\right) &= 0, \\ G(T,z) &= \max[z-1, 0] \end{aligned}$$

- But this is just a special case of the Black Scholes equation with $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$, $r = 0$, and $K = 1$.

- Hence

$$
\begin{aligned}
F(t, s_1, s_2) &= S_2 \{zN[d_1(t,z)] - N[d_2(t,z)]\} \\
&= S_1 N[d_1(t,z)] - S_2 N[d_2(t,z)]
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{1}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}} \left\{ \ln\left(\frac{S_1}{S_2}\right) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)(T-t) \right\} \\
d_2 &= d_1 - \sqrt{\left(\sigma_1^2 + \sigma_2^2\right)(T-t)}
\end{aligned}
$$

III. Dividends

- Introducing dividends is not hard at all. We start with modeling dividends that accrue at discrete time

$$0 < T_n < T_{n-1} < ... < T_2 < T_1 < T$$

- The stock price follows the process

$$dS = \mu S dt + \sigma S d\overline{W}_t$$

- The dividend process is a function $\delta : R \to R$, where the dividend at time t has the form

$$\delta = \delta\left(S_{t-}\right)$$

- The notation S_{t^-} reflects the notion that we are capturing the price of the stock an instant before the stock gets paid.

- We then have the following obvious condition to exclude arbitrage

$$S_t = S_{t^-} - \delta\left(S_{t^-}\right)$$

- To price a claim in the presence of dividends let's start backwards. Specifically, let's focus on the interval $T_1 \leq t \leq T$.

- Given the assumption that prices are ex-dividend in this last interval, we are really faced with a no-dividend claim. So the claim satisfies

$$\begin{aligned} \frac{\partial F^0}{\partial t} + rS_t\frac{\partial F^0}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F^0}{(\partial S)^2} - rF^0 &= 0 \\ F^0(T, S_T) &= \Phi(S_T) \end{aligned}$$

- Suppose we solve this PDE and obtain some solution $F^0\left(T_1, S_{T_1}\right)$ at time T_1.

- Then in the period $T_2 \leq t \leq T_1$ we have that F^1 must satisfy

$$\begin{aligned}\frac{\partial F^1}{\partial t} + rS_t \frac{\partial F^1}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F^1}{(\partial S)^2} - rF^1 &= 0 \\ F^1(T_1^-, S_{T_1^-}) &= F^0\left(T_1, S_{T_1^-} - \delta\left(S_{T_1^-}\right)\right)\end{aligned}$$

- Using the Feynman Kac Theorem, we can express F^1 as

$$\begin{aligned}F^1(t, S_t) &= e^{-r(T_1 - t)} E_t^Q \left[F^0\left(T_1^-, S_{T_1^-} - \delta\left(S_{T_1^-}\right)\right)\right] \\ &= e^{-r(T_1 - t)} E_t^Q \left[e^{-r(T - T_1)} E_{T_1}^Q \left[\Phi\left(S_T\right) | S_{T_1} = S_{T_1^-} - \delta\left(S_{T_1^-}\right)\right]\right]\end{aligned}$$

- Using the law of the iterated expectation we have thus the following result:
- Consider at $T-$Claim of the form $\Phi(S_T)$. Then the arbitrage free pricing function $F(t, S_t)$ has the representation

$$F(t, S_t) = e^{-r(T-t)} E_t^Q [\Phi(S_T)]$$

where the Q dynamics of S between dividend payments are given by

$$dS_t = rS_t dt + \sigma S_t dW_t$$

with the jump conditions

$$S_t = S_{t^-} - \delta(S_{t^-})$$

at each dividend point $t = T_1, T_2 ...$

- A very important special case: $\delta(S_{t^-}) = \delta S_{t^-}$, where δ is constant. Then

$$F_\delta(t, S_t) = F(t, (1-\delta)^n S_t)$$

where F is the price of the derivative on a non-dividend paying stock, and F_δ is the price of the derivative on a dividend paying stock.

- Let's formally push this down to the limit where dividends are paid in N equidistant intervals from t to T. Then

$$F_\delta(t, S_t) = F\left(t, \left(1 - \delta\frac{(T-t)}{N}\right)^N S_t\right)$$

- As $N \to \infty$ we would expect that

$$F_{\delta}(t, S_t) = F\left(t, e^{-\delta(T-t)} S_t\right)$$

- Indeed this can be verified by a direct argument (see section 16.2 in ATC)

- We have the following result in the case of continuously accruing dividends

$$F(t, S_t) = e^{-r(T-t)} E_t^Q [\Phi(S_T)]$$

where

$$dS_t = (r - \delta) S_t dt + S_t \sigma dW_t$$

IV. Examples

- Example 1: Pricing index futures. Suppose that an index pays a continuous dividend δ. Find the price of a forward maturing at T.

- Solution: The payoff of a forward contract at time T is given by

$$\Phi(S_T; K) = S_T - K$$

The price of a forward at time $t = 0$ is zero, so K needs to be determined so that

$$\begin{aligned} 0 &= e^{-rT} E^Q (S_T - K) \\ &= e^{-rT} S_0 e^{(r-\delta)T} - e^{-rT} K \\ &\Rightarrow K = S_0 e^{(r-\delta)T} \end{aligned}$$

- Example 2: Pricing currency derivatives. Let's assume that the exchange rate follows some dynamics

$$dX_t = X_t \mu dt + X_t \sigma_X d\overline{W}_t$$

and that the bonds follow the dynamics

$$\begin{aligned} dB_t^d &= r^d B_t^d dt \\ dB_t^f &= r^f B_t^f dt \end{aligned}$$

- Let's price two claims: a) A forward on the exchange rate and b) an Option on the exchange rate.

- To price such claims the key thing is to figure out the dynamics of X_t under the probability measure Q.

- To derive the dynamics of the exchange rate under Q we make the following observation: Investing in the foreign bond and converting the funds back into the home currency has a payoff equal to

$$L_t = X_t B_t^f.$$

- Under Q, the dynamics of L_t are given by

$$dL_t = r^d L_t dt + L_t \sigma_X dW_t$$

- This implies that under Q, the exchange rate must obey the dynamics

$$dX_t = X_t \left(r^d - r^f \right) dt + X_t \sigma_X dW_t$$

- Note that this is similar to assuming that X_t is a dividend-paying asset.

- Hence, the price of a $T-$forward is given by

$$K = X_0 e^{\left(r^d - r^f\right)T}$$

This relation is sometimes called covered interest parity.

- Similarly, using the identity

$$F_\delta\left(t, S_t\right) = F\left(t, e^{-\delta(T-t)} S_t\right)$$

and noting that in our race $\delta = r^f$, we have that a call option on the exchange rate is given by

$$F\left(T, X_t\right) = X_t e^{-r^f(T-t)} N\left[d_1\right] - e^{-r^d(T-t)} K N\left[d_2\right]$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma_X \sqrt{T-t}} \left\{ \ln\left(\frac{x}{K}\right) + \left(r^d - r^f + \frac{1}{2}\sigma_X^2\right)(T-t) \right\}, \\ d_2 &= d_1 - \sigma_X \sqrt{T-t} \end{aligned}$$