MGMT MFE 406 – Derivative Markets (4 units)

Part 6: Multi-Asset Payoffs and Their Simulation

Prof. Eric S. Reiner

UCLA Anderson

Winter 2025

Outline

1. Recap: A Classification of Exotic Payoffs
2. Currency Options
   - Thought Experiment 1
   - Thought Experiment 2
3. General Approach to Valuing Multi-Asset Payoffs
   - Monte Carlo for Multi-Asset Payoffs
4. Some Common Multi-Asset Payoffs

1. Recap: A Classification of Exotic Payoffs

- Two characteristics allow us to subdivide the set of exotic payoffs:
  - Path dependence/independence
  - Single/multiple underlying assets
- Table listing examples of each type:

	Path Independent	**Path Dependent**
Single Asset	Standard Binary Power	Asian Barrier Lookback
Multiple Asset	Outperformance Basket Quanto	Multi-asset Barrier Mountain Range(s)

- In this section of the course, focus is on multi-asset payoffs (and their combination with time-dependent features) and the analytical & simulation tools that are useful in valuing them.
- Multi-asset options give us a window into the richness of the derivatives pricing theory, particularly change-of-numéraire techniques.

2. Currency Options

- Return to our $1+1$ element (vector) stochastic process framework, with a slight change in notation:

$$\frac{dB}{B} = \frac{dS_0}{S_0} = r\,dt \qquad \text{(riskless/numéraire asset, money market account)}$$

$$\frac{dX}{X} = \frac{dS_1}{S_1} = (\mu_1 - y_1)\,dt + \sigma_1\,dW_1$$

$$= (\mu_x - y_x)\,dt + \sigma_x\,dW_x \qquad \text{(risky asset = currency)}$$

  - r, μ, y, σ in principle are functions of X, t (more generally, are *adapted* processes).

- Alternative process for numéraire (0^{th}) asset (USD):

$$\frac{d1}{1} = \frac{d\$}{\$} = (r - r)\,dt + 0\,dW_0$$

  - Cash is an asset that pays a yield equal to the risk-free rate.

- What yield does a currency pay?
  - By analogy, foreign cash pays a yield equal to the (foreign) risk-free rate.
  - $y_x = r_f = r_x$

2. Currency Options (2)

- We can apply the Black-Scholes-Merton machinery directly, without any need for modifications
- Valuation is therefore a discounted expectation with respect to the "risk-neutral" process ($\mathbf{Q}$ measure):

$$\frac{dX}{X} = (r - r_f)\, dt + \sigma_x\, dW_x^{\mathbf{Q}}$$

  in which all discounting is done at the risk-free rate r.

- Garman-Kohlhagen[-Grabbe-Biger-Hull] (1983) formula: if $C_T(X_T) = [X_T - K]^+$;

$$C_t(X_t) = X_t\, e^{-r_f(T-t)} \mathcal{N}[z_+] - K\, e^{-r(T-t)} \mathcal{N}[z_-]$$

$$\text{with } z_\pm = \frac{\ln(X_t/K) + (r - r_f)(T-t)}{\sigma_x\sqrt{T-t}} \pm \frac{\sigma_x\sqrt{T-t}}{2}$$

- Rubinstein's symmetric valuation formula:

$$\begin{Bmatrix} C \\ P \end{Bmatrix}(X_t, t) = \phi\left[X_t\, e^{-r_f(T-t)} \mathcal{N}(\phi\, z_+) - K\, e^{-r(T-t)} \mathcal{N}(\phi\, z_-)\right]$$

  with $\phi = 1$ for calls and $\phi = -1$ for puts

- What are the (spot) deltas?

$$\begin{aligned} \Delta_X &= \phi\, e^{-r_f(T-t)} \mathcal{N}[\phi\, z_+] \\ \Delta_\$ &= -\phi\, K e^{-r(T-t)} \mathcal{N}[\phi\, z_-] \end{aligned}$$

2. Currency Options (3)

Currency markets have their own deeply-rooted conventions and terminology

- Quotation conventions:
  - GBP/USD = £/\$ = $X_{£/\$}$ = USD price of 1 GBP (ca. 1.21-1.34) “cable"
  - USD/GBP = \$/£= $X_{\$/£}$ = GBP price of 1 USD (ca. 0.74-0.83)
  - EUR/USD = €/\$ = $X_{€/\$}$ = USD price of 1 EUR (ca. 1.02-1.12)
  - GBP/EUR = £/€ = $X_{£/€}$ = EUR price of 1 GBP = (GBP/USD)/(EUR/USD) (ca. 1.16-1.23)
  - For each currency pair, there is usually a preferred direction (“way”) for quoting.
- ATMF convention:
  - Whereas equity markets tend to quote option prices relative to spot (ATM = ATMS), currency markets quote relative to the forward price (ATMF).
  - E.g., if GBP/USD spot = $X_{£/\$,0}$ = 1.2900 and the 1-year continuously-compounded interest rates are: $r_{\$} = 400bp$ and $r_{£} = 415bp$,

    then a 1-year ATMF GBP/USD option is struck at:

    $F_T = 1.2900 \exp[(0.0400 - 0.0415)(1)] = 1.2881$ USD.
  - Forward quoting conventions also extend to basis for Deltas (vs. spot basis for equities) and implied vols (usually expressed in “normalized” log-moneyness or “Delta” terms, vs. $\frac{\ln(F/K)}{\sigma_{atmf}\sqrt{T}}$ or $\mathcal{N}\left[\frac{\ln(F/K)}{\sigma_{atmf}\sqrt{T}} + \frac{\sigma_{atmf}\sqrt{T}}{2}\right]$), (usually) with preferable theoretical properties.

2. Currency Options (4)

- To make things tangible, consider the parameters above together with $\sigma_x = 8.5\%$. Then a 1-year ATMF cable call (priced in USD) is worth:

$$\begin{aligned} C_0 &= e^{-rT} F_T (\mathcal{N}[z_+] - \mathcal{N}[z_-]) \ = e^{-(0.0400)(1)} (1.2881) \, (\mathcal{N}[0.0425] - \mathcal{N}[-0.0425]) \\ &= 1.2376 \cdot 0.03390 = 0.04195 \text{ USD} = \frac{0.04195}{1.2900} \text{ GBP} = 0.03252 \text{ GBP} \\ &= \frac{0.04195}{1.2881} F_T = 0.03257 \, F_T = 0.03257 \, K \end{aligned}$$

- The (spot) deltas are:

$$\begin{aligned} \Delta_X = \Delta_{\pounds} &= \quad e^{-r_f T} \mathcal{N}[z_+] = \qquad e^{-0.0415} \, \mathcal{N}[+0.0425] = \text{GBP} \ +0.4959 \\ \Delta_{\$} &= -K \, e^{-rT} \mathcal{N}[z_-] = -(1.2881) e^{-0.0400} \, \mathcal{N}[-0.0425] = \text{USD} \ -0.5978 \end{aligned}$$

- How much is a 1-year ATMF cable put worth?

$$\begin{aligned} P_0 &= -\left(X_0 \, e^{-r_f T} \mathcal{N}[-z_+] - K \, e^{-rT} \mathcal{N}[-z_-]\right), \text{ using Rubinstein's symmetric formula} \\ &= e^{-rT} F_T \, (\mathcal{N}[-z_-] - \mathcal{N}[-z_+]) = ??? \end{aligned}$$

  - Use put-call parity: $C_0 - P_0 = X_0 \, e^{-r_f T} - K \, e^{-rT}$
  - But if $K = F_T = X_0 \, e^{(r - r_f)T}$:

$$C_0 - P_0 = X_0 \, e^{-r_f T} - K \, e^{-rT} = X_0 \, e^{-r_f T} - X_0 \, e^{(r-r_f)T} e^{-rT} = 0$$

$\Rightarrow$ $P_0 = C_0$ for ATMF European options.

2. Currency Options (5)

- For the purposes of currency option illustrations, we will use the following inputs (based on data as of 2025-Mar-10, all corresponding to a 1-year term):

Currency	symbol	rate
USD	\$	400bp
GBP	£	415bp
EUR	€	225bp
JPY	¥	60bp

Pair	Spot	σ_{imp}
GBP/USD	1.2900	8.5%
EUR/USD	1.0900	9.5%
USD/JPY	147.00	13.0%

- E.g., we can represent the value of a GBP/USD call struck at K as:

$$C_0[X_{£/\$,0}, K] = X_{£/\$,0}\, e^{-r_£ T} \mathcal{N}[z_+] - K\, e^{-r_\$ T} \mathcal{N}[z_-]$$

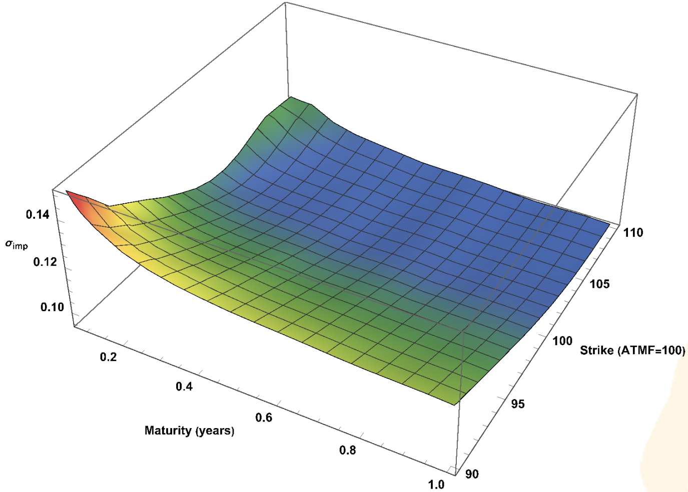
$$\text{or:}\quad C_0[1\,£_0, K\,\$_0] = 1\,£_0\, e^{-r_£ T} \mathcal{N}[z_+] - K\,\$_0\, e^{-r_\$ T} \mathcal{N}[z_-]$$

$$\text{with } z_\pm = \frac{\ln(1£_0/K\$_0) + (r_\$ - r_£)T}{\sigma_{£/\$}\sqrt{T}} \pm \frac{\sigma_{£/\$}\sqrt{T}}{2}$$

2. Currency Options (6)

GBP/USD Implied Volatility Surface

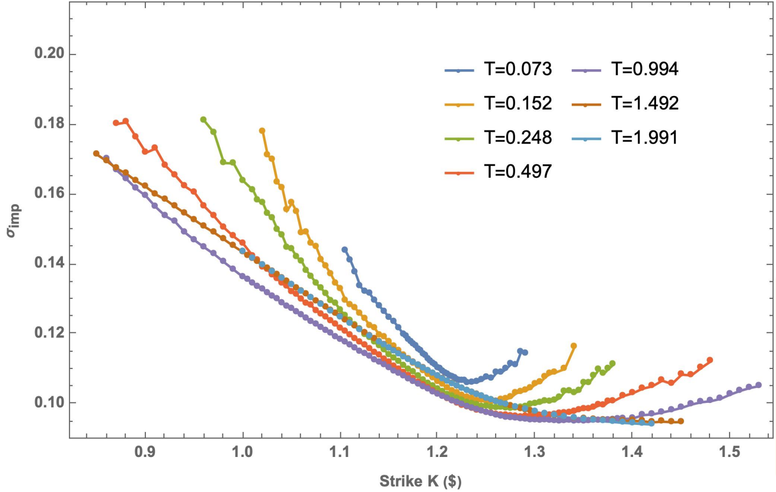
- CME monthly futures options settlement data, 2023-03-10



2. Currency Options (7)

GBP/USD Implied Volatility vs. Strike

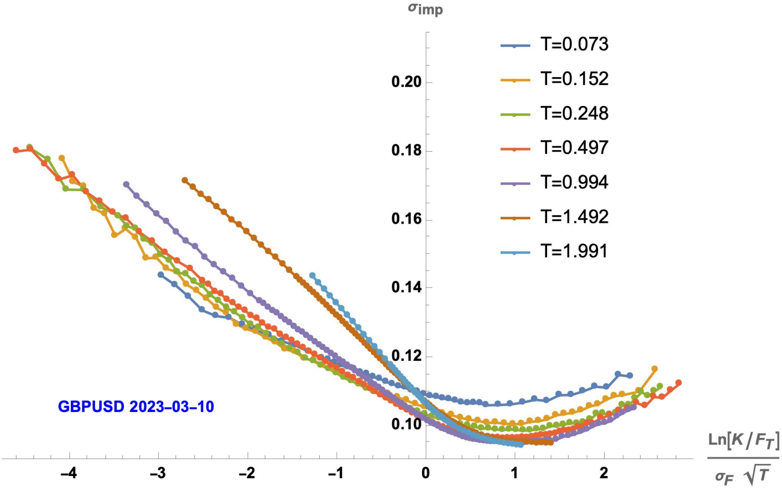
- CME monthly futures options settlement data, 2023-03-10



2. Currency Options (8)

GBP/USD Implied Volatility vs. Normalized Strike

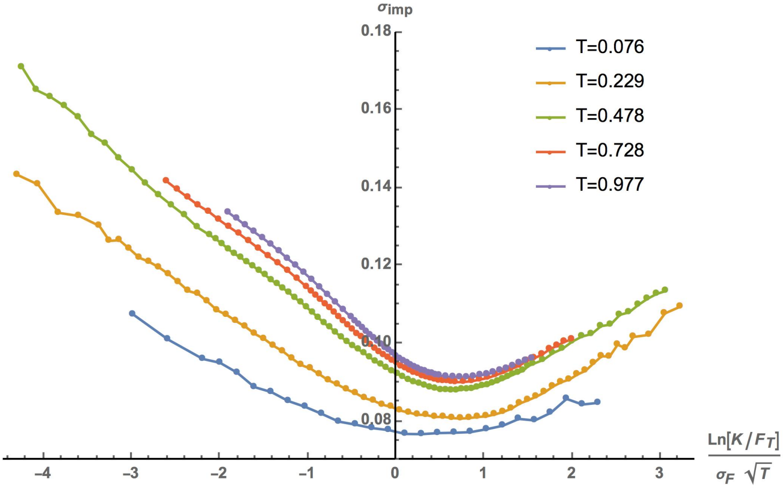
- CME monthly futures options settlement data, 2023-03-10



2. Currency Options (9)

GBP/USD Implied Volatility vs. Normalized Strike

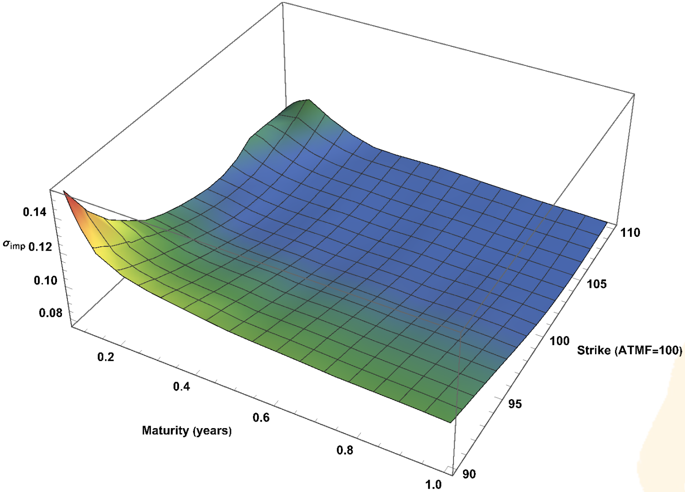
- CME futures options settlement data, 2020-07-17



2. Currency Options (10)

EUR/USD Implied Volatility Surface

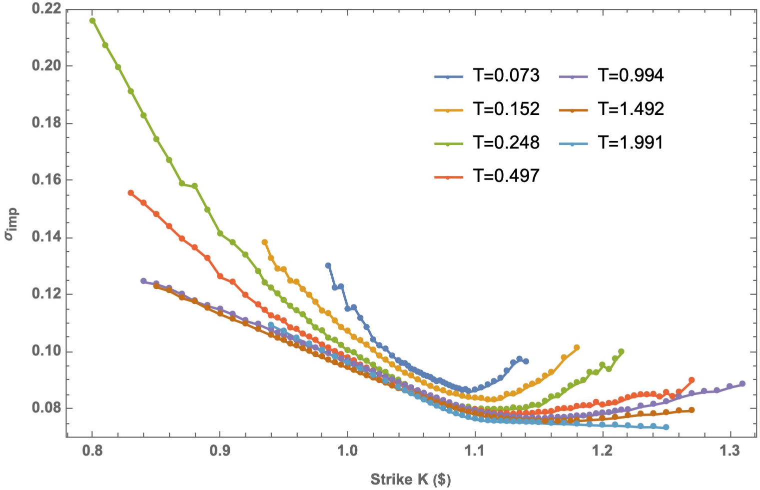
- CME monthly futures options settlement data, 2023-03-10



2. Currency Options (11)

EUR/USD Implied Volatility vs. Strike

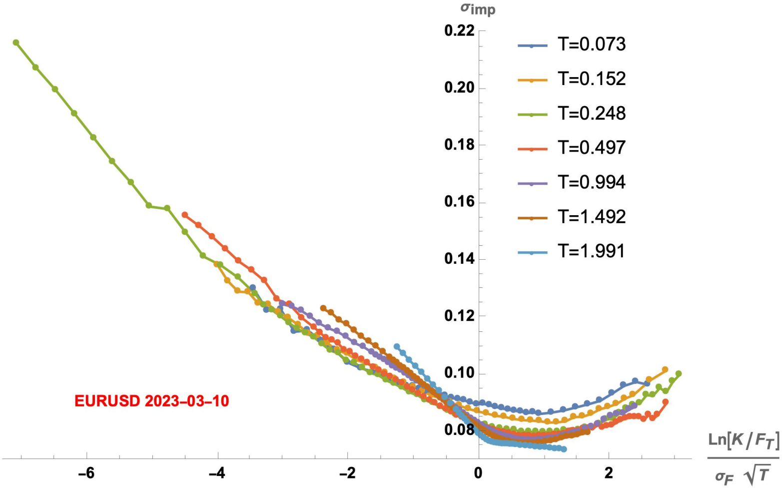
- CME monthly futures options settlement data, 2023-03-10



2. Currency Options (12)

EUR/USD Implied Volatility vs. Normalized Strike

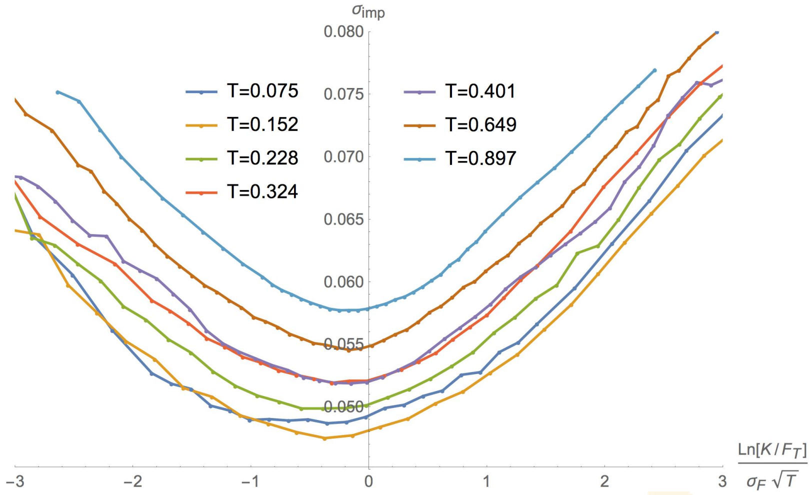
- CME monthly futures options settlement data, 2023-03-10



2. Currency Options (13)

EUR/USD Implied Volatility vs. Normalized Strike

- CME futures options settlement data, 2019-07-12



2.1. Thought Experiment 1

- Consider the pricing problem (for the same option) from the viewpoint of a London-based trader.
- Again work with our $1+1$ element (vector) stochastic process framework:

$$\frac{dB'}{B'} = \frac{dS_0'}{S_0'} = r_f\,dt = r_£\,dt \qquad \text{(riskless/numéraire asset, GBP money market account)}$$

$$\begin{aligned}\frac{dX'}{X'} = \frac{dS_1'}{S_1'} = \frac{dX_{\$/£}}{X_{\$/£}} &= (\mu_1' - y_1')\,dt + \sigma_1'\,dW_1' \\ &= (\mu_x' - y_x')\,dt + \sigma_x'\,dW_x' \\ &= (\mu_{\$/£} - y_\$)dt + \sigma_{\$/£}dW_{\$/£} \qquad \text{(risky asset = USD currency = \$/£)}\end{aligned}$$

- Alternative process for numéraire (0^{th}) asset (GBP):

$$\frac{d1'}{1'} = \frac{d£}{£} = (r_£ - r_£)\,dt + 0\,dW_0'$$

  - GBP cash is an asset that pays a yield equal to the (sterling) risk-free rate.
- What yield does a currency pay?
  - USD cash pays a yield equal to the USD risk-free rate.
  - $y_x' = y_\$ = r = r_\$$

2.1. Thought Experiment 1 (2)

- Consider the nominal (observed, **P**-measure) price processes for X_t and X'_t:

$$\frac{dX}{X} = \hat{\mu}_{£/\$}\,dt + \sigma_{£/\$}\,dW_{£/\$} = \hat{\mu}_x\,dt + \sigma_x\,dW_x$$
$$\frac{dX'}{X'} = \hat{\mu}_{\$/£}\,dt + \sigma_{\$/£}\,dW_{\$/£} = \hat{\mu}'_x\,dt + \sigma'_x\,dW'_x$$

- We should be able to derive the real-world process for $X' = 1/X$ from that for X using Ito's lemma:

$$\begin{aligned} dX' &= \frac{\partial X'}{\partial t}dt + \frac{\partial X'}{\partial X}dX + \frac{1}{2}\frac{\partial^2 X'}{\partial X^2}(dX)^2 \\ &= 0\,dt + \frac{-1}{X^2}\left[\hat{\mu}_x X\,dt + \sigma_x X\,dW_x\right] + \frac{1}{2}\cdot\frac{2}{X^3}\cdot\left(\sigma_x^2 X^2 dt\right) \\ &= (-\hat{\mu}_x + \sigma_x^2)X'\,dt - \sigma_x X'\,dW_x \\ &= (\hat{\mu}'_x)X'\,dt + \sigma'_x X'\,dW'_x \end{aligned}$$

$$\Rightarrow \quad \hat{\mu}'_x = -\hat{\mu}_x + \sigma_x^2\,,$$
$$\sigma'_x = \sigma_x\ (i.e.\ \sigma_{\$/£} = \sigma_{£/\$}),\ \text{and}$$
$$dW'_x = -dW_x\ (i.e.\ dW_{\$/£} = -dW_{£/\$})$$

2.1. Thought Experiment 1 (3)

- We can again apply the Black-Scholes-Merton machinery directly
- Present value in GBP of an option $C'_t(X'_t)$ is a discounted expectation of its payoff $C'_T(X'_T)$ with respect to the risk-neutral process:

$$\begin{aligned}\frac{dX'}{X'} &= (r_£ - r_\$)dt + \sigma_x\, dW'^{\mathbf{Q}}_x \\ &= (r_£ - r_\$)dt + \sigma_{\$/£}\, dW^{\mathbf{Q}}_{\$/£} = (r_£ - r_\$)dt + \sigma_{£/\$}\, dW^{\mathbf{Q}}_{\$/£}\end{aligned}$$

  in which discounting is done at the risk-free sterling rate $r_£$, i.e. solving the BS PDE:

$$\frac{\partial C'}{\partial t} + (r_£ - r_\$)X' \frac{\partial C'}{\partial X'} + \frac{\sigma^2_{\$/£}}{2} X'^2 \frac{\partial^2 C'}{\partial X'^2} = r_£\, C'$$

- Compare to present value in USD of an option $C_t(X_t)$ as discounted expectation of its payoff $C_T(X_T)$ with respect to:

$$\frac{dX}{X} = (r_\$ - r_£)dt + \sigma_x\, dW_x{}^{\mathbf{Q}} = (r_\$ - r_£)dt + \sigma_{£/\$}\, dW^{\mathbf{Q}}_{£/\$}$$

  in which discounting is done at the risk-free dollar rate $r_\$$, i.e. solving the BS PDE:

$$\frac{\partial C}{\partial t} + (r_\$ - r_£)X \frac{\partial C}{\partial X} + \frac{\sigma^2_{\$/£}}{2} X^2 \frac{\partial^2 C}{\partial X^2} = r_\$\, C$$

  - Technical note: even though $dW'_x = -dW_x$, $dW'^{\mathbf{Q}}_x \neq -dW^{\mathbf{Q}}_x$

2.1. Thought Experiment 1 (4)

- Price the same option as before:

$$C_T = [1\,GBP_T - F_T\,USD_T]^+ = F_T\left[\frac{1}{F_T}GBP_T - 1\,USD_T\right]^+$$

$$C'_T = F_T[F'_T\,GBP_T - 1\,USD_T]^+$$

with the USD/GBP forward price $F'_T = \frac{1}{F_T} = \frac{1}{1.2881} = 0.7764$ GBP

- But this is K puts on USD (struck in GBP)!

$$C_T(1\,GBP_T, K\,USD_T) = K\,P'_T\left(1\,USD_T, \frac{1}{K}\,GBP_T\right) = K\,P'_T\,(1\,USD_T, K'\,GBP_T)$$

- Use Garman-Kohlhagen (with GBP-numéraire parameters) to price the put:

$$K\,P'_0[(USD/GBP)_0, K'] = K\left(K'e^{-r_£ T}\mathcal{N}[-z'_-] - (USD/GBP)_0\,e^{-r_\$ T}\mathcal{N}[-z'_+]\right)$$

$$K\,P'_0[1\,\$_0, K'\,£_0] = K\left(K'\,£_0\,e^{-r_£ T}\mathcal{N}[-z'_-] - 1\,\$_0\,e^{-r_\$ T}\mathcal{N}[-z'_+]\right)$$

$$P'_0[K\,\$_0, 1\,£_0] = 1\,£_0\,e^{-r_£ T}\mathcal{N}[-z'_-] - K\,\$_0\,e^{-r_\$ T}\mathcal{N}[-z'_+]$$

$$\text{with } z'_\pm = \frac{\ln(1\$_0/K'£_0) + (r_£ - r_\$)T}{\sigma_{£/\$}\sqrt{T}} \pm \frac{\sigma_{£/\$}\sqrt{T}}{2}$$

$$= \frac{\ln(K\$_0/1£_0) + (r_£ - r_\$)T}{\sigma_{£/\$}\sqrt{T}} \pm \frac{\sigma_{£/\$}\sqrt{T}}{2}$$

2.1. Thought Experiment 1 (5)

- What are the Garman-Kohlhagen (spot) deltas?

$$\begin{aligned} \Delta_{X'} = \Delta_{\$} &= K\left(-e^{-r_{\$}T}\mathcal{N}\left[-z'_+\right]\right) = -K\,e^{-r_{\$}T}\mathcal{N}\left[-z'_+\right] \\ \Delta_{£} &= K\left(K' e^{-r_{£}T}\mathcal{N}\left[-z'_-\right]\right) = \quad e^{-r_{£}T}\mathcal{N}\left[-z'_-\right] \end{aligned}$$

- Making this tangible for $K = F_T = 1.2881$ and $(USD/GBP)_0 = 1/1.2900 = 0.7752$:

$$\begin{aligned} P'_0[K\,\$_0,\,1\,£_0] &= 1\,£_0\,e^{-r_{£}T}\mathcal{N}\left[-z'_-\right] - K\,\$_0\,e^{-r_{\$}T}\mathcal{N}\left[-z'_+\right] \\ &= e^{-r_{£}T}\mathcal{N}\left[-z'_-\right] - K\,(USD/GBP)_0\,e^{-r_{\$}T}\mathcal{N}\left[-z'_+\right] \quad GBP \\ &= e^{-0.0415(1)}\mathcal{N}[-(-0.0425)] - (1.2881)(0.7752)e^{-0.0400(1)}\mathcal{N}[-(0.0425)] \\ &= 0.03252\,GBP \\ &= \frac{0.03252}{0.7764}\,F'_T = 0.04189\,F'_T = 0.04189\,K' \\ &= (0.03252)(1.2900)\text{ USD} = 0.04195\text{ USD! (same price as before)} \end{aligned}$$

- Calculating the deltas:

$$\begin{aligned} \Delta_{\$} &= -K\,e^{-r_{\$}T}\mathcal{N}\left[-z'_+\right] = -(1.2881)e^{-0.0400}\mathcal{N}[-0.0425] = \text{USD} \; -0.5978 \\ \Delta_{£} &= \quad e^{-r_{£}T}\mathcal{N}\left[-z'_-\right] = \quad e^{-0.0415}\mathcal{N}[+0.0425] = \text{GBP} \; +0.4959 \end{aligned}$$

  - Also same as before!

2.1. Thought Experiment 1 (6)

Put-Call Symmetry

- It's reassuring that the value is the same (law of one price) and the deltas are the same (numéraire invariance), but we can conclude even more.
- Put-call symmetry: a call on one asset (struck at K units of a second asset) is equivalent to K puts on the 2nd asset (struck at $1/K$ units of the 1st), or more simply: $C(S_1, S_2) = P(S_2, S_1)$.
- Symmetry in the normal integral values (and their arguments): $-z'_- = z_+$ and $-z'_+ = z_-$:

$$
\begin{aligned}
-z'_\pm &= -\frac{\ln(K\$_0/1£_0) + (r_£ - r_\$)T}{\sigma_{£/\$}\sqrt{T}} \mp \frac{\sigma_{£/\$}\sqrt{T}}{2} \\
&= \frac{\ln(1£_0/K\$_0) + (r_\$ - r_£)T}{\sigma_{£/\$}\sqrt{T}} \mp \frac{\sigma_{£/\$}\sqrt{T}}{2} = z_\mp
\end{aligned}
$$

  Remember the comment that we could interpret $\mathcal{N}(z_+)$ as the probability of finishing in the money in a measure in which the risky asset price is the numéraire?

- Put-call symmetry is one of the simplest examples of numéraire invariance and of the power, elegance, and simplicity of change-of-numéraire techniques.

2.2. Thought Experiment 2

- Now consider the same option pricing problem from the viewpoint of a Eurozone-based trader.
- Need a bit more structure: $2+1$ element (vector) stochastic process framework:

$$\frac{dB''}{B''} = \frac{dS''_0}{S''_0} = r_€\, dt \qquad \text{(riskless/numéraire asset, EUR money market account)}$$

$$\begin{aligned}
\frac{dX''_1}{X''_1} = \frac{dS''_1}{S''_1} &= \frac{dX_{£/€}}{X_{£/€}} \\
&= (\mu''_1 - r_1)\, dt + \sigma''_1\, dW''_1 \\
&= (\mu_{£/€} - y_£) dt + \sigma_{£/€}\, dW_{£/€} \qquad \text{(risky asset 1 = GBP currency = £/€)}
\end{aligned}$$

$$\begin{aligned}
\frac{dX''_2}{X''_2} = \frac{dS''_2}{S''_2} &= \frac{dX_{\$/€}}{X_{\$/€}} \\
&= (\mu''_2 - r_2)\, dt + \sigma''_2\, dW''_2 \\
&= (\mu_{\$/€} - y_\$) dt + \sigma_{\$/€}\, dW_{\$/€} \qquad \text{(risky asset 2 = USD currency = \$/€)}
\end{aligned}$$

with: $dW''_1 dW''_2 = dW_{£/€}\, dW_{\$/€} = \rho_{1,2}\, dt = \rho_{£/€,\$/€}\, dt$

- Alternative process for numéraire (0^{th}) asset (EUR):

$$\frac{d1''}{1''} = \frac{d€}{€} = (r_€ - r_€)\, dt + 0\, dW''_0$$

2.2. Thought Experiment 2 (2)

- Our payoff looks like: $C''_T = [S''_{1,T} - K\, S''_{2,T}]^+ = [X_{£/€,T} - K_{(£/\$)} X_{\$/€,T}]^+$
- If we only had one risky currency (GBP/EUR or USD/EUR), we could just use the Garman-Kohlhagen formula.
- What is the valuation approach with 2 risky assets?
  Start with multi- (l)- dimensional version of Ito's formula (and drop $''$):

$$
\begin{aligned}
dC &= \frac{\partial C}{\partial t}dt + \sum_{k=1}^{l} \frac{\partial C}{\partial S_k} dS_k + \frac{1}{2}\sum_{k,k'=1}^{l} \frac{\partial^2 C}{\partial S_k \partial S_{k'}} dS_k dS_{k'} \\
&= \frac{\partial C}{\partial t}dt + \sum_{k=1}^{l} \left[(\mu_k - r_k)dt + \sigma_k dW_k\right] S_k \frac{\partial C}{\partial S_k} + \frac{1}{2}\sum_{k,k'=1}^{l} \rho_{k,k'}\sigma_k\sigma_{k'} S_k S_{k'} \frac{\partial^2 C}{\partial S_k \partial S_{k'}} dt \\
&= \left[\frac{\partial C}{\partial t} + \sum_{k=1}^{l}(\mu_k - r_k) S_k \frac{\partial C}{\partial S_k} + \frac{1}{2}\sum_{k,k'=1}^{l} \rho_{k,k'}\sigma_k\sigma_{k'} S_k S_{k'} \frac{\partial^2 C}{\partial S_k \partial S_{k'}}\right] dt + \sum_{k=1}^{l} \sigma_k S_k \frac{\partial C}{\partial S_k} dW_k
\end{aligned}
$$

- Compare to the single risky-asset version with $r_k = y$ (Notes Part 2, p. 29):

$$
dC = \left[\frac{\partial C}{\partial t} + (\mu - y) S \frac{\partial C}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2}\right] dt + \sigma S \frac{\partial C}{\partial S} dW
$$

- How many deltas do we have?

2.2. Thought Experiment 2 (3)

- Form the portfolio Π = long C and short $\left[\sum_k \Delta_k S_k = \sum_k \frac{\partial C}{\partial S_k} S_k\right]$.

$$d\Pi = \left[\frac{\partial C}{\partial t} + \sum_{k=1}^{l}(\mu_k - r_k) S_k \frac{\partial C}{\partial S_k} + \frac{1}{2}\sum_{k,k'=1}^{l} \rho_{k,k'}\sigma_k\sigma_{k'} S_k S_{k'} \frac{\partial^2 C}{\partial S_k \partial S_{k'}}\right] dt$$
$$+ \sum_{k=1}^{l} \sigma_k S_k \frac{\partial C}{\partial S_k} dW_k - \left[\sum_{k=1}^{l} \frac{\partial C}{\partial S_k} S_k (\mu_k\, dt + \sigma_k\, dW_k)\right]$$

  - Why is there no $-r_k$ in the term proportional to dt in the last sum (on the second line)?

- Subtracting offsetting terms in μ_k and $\sigma_k\, dW_k$:

$$d\Pi = \left[\frac{\partial C}{\partial t} + \sum_{k=1}^{l}(-r_k) S_k \frac{\partial C}{\partial S_k} + \frac{1}{2}\sum_{k,k'=1}^{l} \rho_{k,k'}\sigma_k\sigma_{k'} S_k S_{k'} \frac{\partial^2 C}{\partial S_k \partial S_{k'}}\right] dt$$

- Just as in the single risky-asset case, Π is locally riskless (and independent of the drifts μ_k), so it must earn a riskless return (in €) over dt:

$$d\Pi = r_€\, \Pi\, dt = r_€ \left[C - \sum_k S_k \frac{\partial C}{\partial S_k}\right] dt = r_€\, C\, dt - r_€ \left[\sum_k S_k \frac{\partial C}{\partial S_k}\right] dt$$
$$\Rightarrow \frac{\partial C}{\partial t} + \sum_{k=1}^{l}(-r_k) S_k \frac{\partial C}{\partial S_k} + \frac{1}{2}\sum_{k,k'=1}^{l} \rho_{k,k'}\sigma_k\sigma_{k'} S_k S_{k'} \frac{\partial^2 C}{\partial S_k \partial S_{k'}} = r_€ \left[C - \sum_k S_k \frac{\partial C}{\partial S_k}\right]$$

2.2. Thought Experiment 2 (4)

- Rearrange to obtain the multi-asset BS PDE (with € as numéraire and r_k as the k-th asset's yield):

$$\frac{\partial C}{\partial t}+\sum_{k=1}^{l}(r_{€}-r_k)S_k\frac{\partial C}{\partial S_k}+\frac{1}{2}\sum_{k,k'=1}^{l}\rho_{k,k'}\sigma_k\sigma_{k'}S_kS_{k'}\frac{\partial^2 C}{\partial S_k\partial S_{k'}}=r_{€}\,C$$

- Compare to the single risky-asset BS PDE (Notes Part 2, p. 30):

$$\frac{\partial C}{\partial t}+(r-y)S\frac{\partial C}{\partial S}+\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}=r\,C$$

- For two risky assets:

$$S_1=X_1=\text{GBP/EUR, with } r_1=r_{£},\ \sigma_1=\sigma_{£/€};$$
$$S_2=X_2=\text{USD/EUR, with } r_2=r_{\$},\ \sigma_2=\sigma_{\$/€},\text{ and } \rho_{1,2}=\rho_{£/€,\$/€}$$

$$\frac{\partial C}{\partial t}+(r_{€}-r_1)X_1\frac{\partial C}{\partial X_1}+(r_{€}-r_2)X_2\frac{\partial C}{\partial X_2}+\frac{\sigma_1^2}{2}X_1^2\frac{\partial^2 C}{\partial X_1^2}+\rho_{1,2}\sigma_1\sigma_2X_1X_2\frac{\partial^2 C}{\partial X_1\partial X_2}+\frac{\sigma_2^2}{2}X_2^2\frac{\partial^2 C}{\partial X_2^2}=r_{€}\,C$$

2.2. Thought Experiment 2 (5)

Two usual lines of attack for European option payoffs

1. Feynman-Kac approach.
   The solution to the multi-asset PDE for the option's present value in EUR $C_t(X_{1,t}, X_{2,t})$ is a discounted expectation of its payoff $C_T(X_{1,T}, X_{2,T})$ with respect to the risk-neutral processes:

$$\frac{dX_1}{X_1} = (r_{€} - r_{£})dt + \sigma_1\, dW_1^{\mathbf{Q}} = (r_{€} - r_{£})dt + \sigma_{£/€}\, dW_{£/€}^{\mathbf{Q}}$$

$$\frac{dX_2}{X_2} = (r_{€} - r_{\$})dt + \sigma_2\, dW_2^{\mathbf{Q}} = (r_{€} - r_{\$})dt + \sigma_{\$/€}\, dW_{\$/€}^{\mathbf{Q}},$$

$$\text{with } \frac{dX_1}{X_1}\frac{dX_2}{X_2} = \rho_{1,2}\,\sigma_1\,\sigma_2\, dt = \rho_{£/€,\$/€}\,\sigma_{£/€}\,\sigma_{\$/€}\, dt$$

   in which discounting is done at the risk-free Euro rate $r_{€}$
   - Both X_1 and X_2 can be represented in terms of exponential martingales:

$$X_{1,T} = X_{1,0}\, e^{\tilde{\mu}_1^{\mathbf{Q}} T + \sigma_1 W_{1,T}^{\mathbf{Q}}} = X_{1,0}\, e^{\tilde{\mu}_1^{\mathbf{Q}} T + \sigma_1 \sqrt{T}\, z_1}$$

$$X_{2,T} = X_{2,0}\, e^{\tilde{\mu}_2^{\mathbf{Q}} T + \sigma_2 W_{2,T}^{\mathbf{Q}}} = X_{2,0}\, e^{\tilde{\mu}_2^{\mathbf{Q}} T + \sigma_2 \sqrt{T}\, z_2}$$

$$\text{with: } \tilde{\mu}_1^{\mathbf{Q}} = r_{€} - r_{£} - \sigma_1^2/2,\ \tilde{\mu}_2^{\mathbf{Q}} = r_{€} - r_{\$} - \sigma_2^2/2;\ \mathbb{E}^{\mathbf{Q}}[z_1\, z_2] = \rho_{1,2}$$

$$\Rightarrow q(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho_{1,2}^2}} \exp\left[\frac{-1}{2(1-\rho_{1,2}^2)}\left(z_1^2 - 2\rho_{1,2}\, z_1 z_2 + z_2^2\right)\right]$$

2.2. Thought Experiment 2 (6)

1. Feynman-Kac (continued)
   - Calculate risk-neutral expectation of our payoff: $C_T = [X_{1,T} - K\,X_{2,T}]^+$

$$\begin{aligned}\mathbb{E}^{\mathbf{Q}}[C_T] &= \int_0^\infty dX_{1,T} \int_0^\infty dX_{2,T}\, q(X_{1,T}, X_{2,T})\, [X_{1,T} - K\,X_{2,T}]^+ \\ &= \int_0^\infty dX_{1,T} \int_0^{X_{1,T}/K} dX_{2,T}\, q(X_{1,T}, X_{2,T})\, [X_{1,T} - K\,X_{2,T}] \\ &= \int_{-\infty}^\infty dz_1 \int_{-\infty}^{z_2^*(z_1)} dz_2\, q(z_1, z_2)\, [X_{1,0}\, e^{\tilde{\mu}_1^{\mathbf{Q}} T + \sigma_1 \sqrt{T}\, z_1} - K\,X_{2,0}\, e^{\tilde{\mu}_2^{\mathbf{Q}} T + \sigma_2 \sqrt{T}\, z_2}]\end{aligned}$$

$$\text{with: } z_2^*(z_1) = \frac{1}{\sigma_2\sqrt{T}} \ln\left(\frac{X_{1,0}\, e^{\tilde{\mu}_1^{\mathbf{Q}} T + \sigma_1\sqrt{T}\, z_1}}{K\,X_{2,0}\, e^{\tilde{\mu}_2^{\mathbf{Q}} T}}\right) = \frac{\ln(X_{1,0}/K\,X_{2,0}) + (\tilde{\mu}_1^{\mathbf{Q}} - \tilde{\mu}_2^{\mathbf{Q}})T}{\sigma_2\sqrt{T}} + \frac{\sigma_1}{\sigma_2} z_1$$

   - Writing the inner integral as a conditional expectation over z_2:

$$\mathbb{E}^{\mathbf{Q}}[C_T] = \int_{-\infty}^\infty dz_1\, q(z_1) \int_{-\infty}^{z_2^*(z_1)} dz_2\, q(z_2|z_1)\, [X_{1,0}\, e^{\tilde{\mu}_1^{\mathbf{Q}} T + \sigma_1 \sqrt{T}\, z_1} - K\,X_{2,0}\, e^{\tilde{\mu}_2^{\mathbf{Q}} T + \sigma_2 \sqrt{T}\, z_2}]$$

$$\begin{aligned}\text{with: } q(z_2|z_1) = \frac{q(z_1, z_2)}{q(z_1)} &= \frac{1}{\sqrt{2\pi(1-\rho_{1,2}^2)}} \exp\left[\frac{-1}{2(1-\rho_{1,2}^2)}\left(\rho_{1,2}^2\, z_1^2 - 2\rho_{1,2}\, z_1 z_2 + z_2^2\right)\right] \\ &= \frac{1}{\sqrt{2\pi(1-\rho_{1,2}^2)}} \exp\left[\frac{-\,(z_2 - \rho_{1,2}\, z_1)^2}{2(1-\rho_{1,2}^2)}\right]\end{aligned}$$

2.2. Thought Experiment 2 (7)

1. Feynman-Kac (continued)
   - Evaluating the inner integral (and completing the square in the second term):

$$\mathbb{E}^{\mathbf{Q}}[C_T] = \int_{-\infty}^{\infty} dz_1\, q(z_1) \Bigg[X_{1,0}\, e^{\tilde{\mu}_1^{\mathbf{Q}} T + \sigma_1 \sqrt{T}\, z_1}\, N\left(\frac{z_2^*(z_1) - \rho_{1,2}\, z_1}{\sqrt{1-\rho_{1,2}^2}} \right) - K\, X_{2,0}\, e^{(\tilde{\mu}_2^{\mathbf{Q}} + (1-\rho_{1,2}^2)\sigma_2^2) T + \rho_{1,2}\sigma_2\sqrt{T}\, z_1}\, N\left(\frac{z_2^*(z_1) - (1-\rho_{1,2}^2)\,\sigma_2\sqrt{T}}{\sqrt{1-\rho_{1,2}^2}} \right) \Bigg]$$

   - (After much messy algebra):

$$\mathbb{E}^{\mathbf{Q}}[C_T] = X_{1,0}\, e^{(r_€ - r_£)T}\, N\left(\frac{\ln(X_{1,0}/K\, X_{2,0}) + (r_\$ - r_£)T}{\sqrt{(\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2)T}} + \frac{\sqrt{(\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2)T}}{2} \right) - K\, X_{2,0}\, e^{(r_€ - r_\$)T}\, N\left(\frac{\ln(X_{1,0}/K\, X_{2,0}) + (r_\$ - r_£)T}{\sqrt{(\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2)T}} - \frac{\sqrt{(\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2)T}}{2} \right)$$

   - Hence: $C_t = X_{1,0}\, e^{-r_£ T}\, N(z_+) - K\, X_{2,0}\, e^{-r_\$ T}\, N(z_-)$, with:

$$z_\pm = \frac{\ln(X_{1,0}/K\, X_{2,0}) + (r_\$ - r_£)T}{\sigma_{\mathit{eff}}\sqrt{T}} \pm \frac{\sigma_{\mathit{eff}}\sqrt{T}}{2} \text{ and } \sigma_{\mathit{eff}} = \sqrt{\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2}$$

   - Surely there must be an easier way!

2.2. Thought Experiment 2 (8)

2. Dimensional reduction

- The payoff is homogeneous of degree 1 in X_1 and X_2:

$$C_T(\lambda X_{1,T}, \lambda X_{2,T}) = [\lambda X_{1,T} - K\,\lambda X_{2,T}]^+ = \lambda[X_{1,T} - K\,X_{2,T}]^+$$

while the multi- risky-asset BS PDE is invariant under scaling of X_1 and X_2:

$$\begin{aligned}
&\frac{\partial C}{\partial t} + (r_€ - r_1)\lambda X_1 \frac{\partial C}{\partial \lambda X_1} + (r_€ - r_2)\lambda X_2 \frac{\partial C}{\partial \lambda X_2} \\
&\quad + \frac{\sigma_1^2}{2}(\lambda X_1)^2 \frac{\partial^2 C}{\partial(\lambda X_1)^2} + \rho_{1,2}\sigma_1\sigma_2\,\lambda X_1 \lambda X_2 \frac{\partial^2 C}{\partial \lambda X_1 \partial \lambda X_2} + \frac{\sigma_2^2}{2}(\lambda X_2)^2 \frac{\partial^2 C}{\partial(\lambda X_2)^2} = r_€\, C \\
\Rightarrow\ &\frac{\partial C}{\partial t} + (r_€ - r_1)X_1 \frac{\partial C}{\partial X_1} + (r_€ - r_2)X_2 \frac{\partial C}{\partial X_2} \\
&\quad + \frac{\sigma_1^2}{2}X_1^2 \frac{\partial^2 C}{\partial X_1^2} + \rho_{1,2}\sigma_1\sigma_2\,X_1 X_2 \frac{\partial^2 C}{\partial X_1 \partial X_2} + \frac{\sigma_2^2}{2}X_2^2 \frac{\partial^2 C}{\partial X_2^2} = r_€\, C
\end{aligned}$$

- The combination of these properties implies that: $C_t(\lambda X_{1,t}, \lambda X_{2,t}) = \lambda C_t(X_{1,t}, X_{2,t})$ and suggests a solution of the form:

$$C_t = X_2\, D_t(R = X_1/X_2) \text{ or alternatively: } C_t = X_1\, D'_t(R' = X_2/X_1)$$

2.2. Thought Experiment 2 (9)

2. Dimensional reduction (continued)

- Try the $C_t = X_2\, D_t(R = X_1/X_2)$ version of this *ansatz* and collect results using the chain rule:

$$
\begin{aligned}
\frac{\partial C}{\partial t} &= X_2 \frac{\partial D}{\partial t} \\
\frac{\partial C}{\partial X_1} &= X_2 \frac{\partial R}{\partial X_1} \frac{\partial D}{\partial R} = \frac{\partial D}{\partial R} \\
\frac{\partial C}{\partial X_2} &= D + X_2 \frac{\partial R}{\partial X_2} \frac{\partial D}{\partial R} = D + X_2 \left(\frac{-X_1}{X_2^2} \right) \frac{\partial D}{\partial R} = D - R \frac{\partial D}{\partial R} \\
\frac{\partial^2 C}{\partial X_1^2} &= X_2 \left(\frac{\partial R}{\partial X_1} \right)^2 \frac{\partial^2 D}{\partial R^2} = \frac{1}{X_2} \frac{\partial^2 D}{\partial R^2} \\
\frac{\partial^2 C}{\partial X_1 \partial X_2} &= \partial_{X_2} \frac{\partial C}{\partial X_1} = \frac{\partial R}{\partial X_2} \frac{\partial^2 D}{\partial R^2} = -\frac{X_1}{X_2^2} \frac{\partial^2 D}{\partial R^2} = -\frac{R}{X_2} \frac{\partial^2 D}{\partial R^2} \\
\frac{\partial^2 C}{\partial X_2^2} &= \partial_{X_2} \frac{\partial C}{\partial X_2} = \frac{\partial R}{\partial X_2} \partial_R \left(D - R \frac{\partial D}{\partial R} \right) = -\frac{X_1}{X_2^2} \left(-R \frac{\partial^2 D}{\partial R^2} \right) = \frac{R^2}{X_2} \frac{\partial^2 D}{\partial R^2}
\end{aligned}
$$

- We can see where this is going.

2.2. Thought Experiment 2 (10)

2. Dimensional reduction (continued)

- Substitute these results into the PDE:

$$\frac{\partial C}{\partial t} + (r_€ - r_1) X_1 \frac{\partial C}{\partial X_1} + (r_€ - r_2) X_2 \frac{\partial C}{\partial X_2} + \frac{\sigma_1^2}{2} X_1^2 \frac{\partial^2 C}{\partial X_1^2} + \rho_{1,2}\sigma_1\sigma_2 X_1 X_2 \frac{\partial^2 C}{\partial X_1 \partial X_2} + \frac{\sigma_2^2}{2} X_2^2 \frac{\partial^2 C}{\partial X_2^2} = r_€ C$$

$$\Rightarrow X_2 \frac{\partial D}{\partial t} + (r_€ - r_1) X_1 \frac{\partial D}{\partial R} + (r_€ - r_2) X_2 \left(D - R \frac{\partial D}{\partial R} \right) + \frac{\sigma_1^2}{2} \frac{X_1^2}{X_2} \frac{\partial^2 D}{\partial R^2} - \rho_{1,2}\sigma_1\sigma_2 X_1 R \frac{\partial^2 D}{\partial R^2} + \frac{\sigma_2^2}{2} X_2 R^2 \frac{\partial^2 D}{\partial R^2} = r_€ X_2 D$$

$$\Rightarrow \frac{\partial D}{\partial t} + (r_€ - r_1) R \frac{\partial D}{\partial R} + (r_€ - r_2) \left(D - R \frac{\partial D}{\partial R} \right) + \frac{\sigma_1^2}{2} R^2 \frac{\partial^2 D}{\partial R^2} - \rho_{1,2}\sigma_1\sigma_2 R^2 \frac{\partial^2 D}{\partial R^2} + \frac{\sigma_2^2}{2} R^2 \frac{\partial^2 D}{\partial R^2} = r_€ D$$

$$\Rightarrow \frac{\partial D}{\partial t} + (r_2 - r_1) R \frac{\partial D}{\partial R} + \frac{\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2}{2} R^2 \frac{\partial^2 D}{\partial R^2} = r_2 D$$

$$\Rightarrow \frac{\partial D}{\partial t} + (r_\$ - r_£) R \frac{\partial D}{\partial R} + \frac{\sigma_{eff}^2}{2} R^2 \frac{\partial^2 D}{\partial R^2} = r_\$ D$$

2.2. Thought Experiment 2 (11)

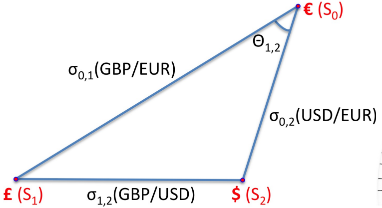
2. Dimensional reduction (continued)

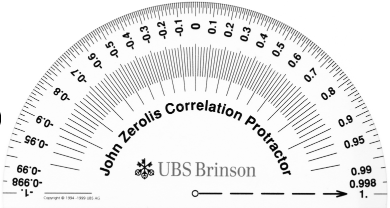
- The payoff becomes: $D_T = C_T/X_{2,T} = [X_{1,T}/X_{2,T} - K]^+ = [R-K]^+$
- But: $R = X_1/X_2 = X_{£/€}/X_{\$/€} = X_{£/\$}$
  and $D_t = C_t/X_{2,t}$ is just the EUR value of the option divided by the USD/EUR spot price, i.e. it is the USD value of the option.
- By changing the units of measurement, we've simply converted the EUR pricing problem back into the (original) USD setting.
  - Change of numéraire from EUR to USD
  - What is this σ_{eff}?
    Remember that in the EUR numéraire, σ_1 is the (log-) volatility of GBP/EUR and σ_2 is the (log-) volatility of USD/EUR.
    $$\sigma_{eff} = \sigma_R = \sigma_{X_1/X_2} = \sqrt{\frac{1}{dt}\left(\frac{dX_1}{X_1} - \frac{dX_2}{X_2}\right)^2} = \sqrt{\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2}$$
    But $\sigma_{X_1/X_2} = \sigma_{(GBP/EUR)/(USD/EUR)} = \sigma_{GBP/USD}$!
  - The law of one price (absence of arbitrage) requires that $\sigma_{eff} = \sigma_{GBP/USD}$.

2.2. Thought Experiment 2 (12)

2. Dimensional reduction (continued)

- Currency volatility arbitrage illustration:





  - Law of cosines: $\sigma_{eff}^2 = \sigma_{1/2}^2 = \sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2$ with $\rho_{1,2} = \cos(\theta_{1,2})$
  - Constraints on relative volatility: $|\sigma_1 - \sigma_2| \leq \sigma_{eff} \leq \sigma_1 + \sigma_2$.
    What would (could) we do if one of these is violated?
- This is an outperformance option payoff: $C_T = [X_{1,T} - K\, X_{2,T}]^+$
- What are the deltas? Borrow from the dimensional reduction results:

$$\frac{\partial C}{\partial X_1} = X_2 \frac{\partial R}{\partial X_1}\frac{\partial D}{\partial R} = \frac{\partial D}{\partial R}; \qquad \frac{\partial C}{\partial X_2} = D + X_2 \frac{\partial R}{\partial X_2}\frac{\partial D}{\partial R} = D + X_2\left(\frac{-X_1}{X_2^2}\right)\frac{\partial D}{\partial R} = D - R\frac{\partial D}{\partial R}$$

  - I.e., the same as in the original measure(s).
  - What about $\Delta_€$?

2.2. Thought Experiment 2 (13)

2. Dimensional reduction (continued)

- General properties of change of numéraire from (implicitly) asset 0 to asset k:
  - All prices are divided by S_k; $S_k \to 1/S_k$ (switch 0^{th} asset with k^{th}).
  - Drifts for asset k' change from $r_0 - r_{k'}$ to $r_k - r_{k'}$ (transform linearly)
  - Discounting changes from r_0 to r_k (transforms linearly)
  - Volatility changes from $\sigma_{k'} \to \sqrt{\sigma_{k'}^2 + \sigma_k^2 - 2\rho_{k',k}\sigma_{k'}\sigma_k}$ (transforms quadratically)

    Strictly speaking, we probably should write volatilities with respect to numéraire, e.g. as $\sigma_{0,k'} \to \sigma_{k,k'} = \sqrt{\sigma_{0,k'}^2 + \sigma_{0,k}^2 - 2\rho_{k',k}\sigma_{0,k'}\sigma_{0,k}}$
  - Correlations change accordingly: $\rho_{k',k''} \to \frac{\rho_{k',k''}\sigma_{k'}\sigma_{k''} - \rho_{k',k}\sigma_{k'}\sigma_k - \rho_{k'',k}\sigma_{k''}\sigma_k + \sigma_k^2}{\sqrt{\sigma_{k'}^2 + \sigma_k^2 - 2\rho_{k',k}\sigma_{k'}\sigma_k}\sqrt{\sigma_{k''}^2 + \sigma_k^2 - 2\rho_{k'',k}\sigma_{k''}\sigma_k}}$

    Again, strictly speaking, we should probably reflect numéraires (even if inconsistent) in correlations:

$$\rho_{\{0,k'\},\{0,k''\}} \to \rho_{\{k,k'\},\{k,k''\}} = \frac{\rho_{\{0,k'\},\{0,k''\}}\sigma_{0,k'}\sigma_{0,k''} - \rho_{\{0,k'\},\{0,k\}}\sigma_{0,k'}\sigma_{0,k} - \rho_{\{0,k''\},\{0,k\}}\sigma_{0,k''}\sigma_{0,k} + \sigma_{0,k}^2}{\sqrt{\sigma_{0,k'}^2 + \sigma_{0,k}^2 - 2\rho_{\{0,k'\},\{0,k\}}\sigma_{0,k'}\sigma_{0,k}}\sqrt{\sigma_{0,k''}^2 + \sigma_{0,k}^2 - 2\rho_{\{0,k''\},\{0,k\}}\sigma_{0,k''}\sigma_{0,k}}}$$

  - Deltas are unchanged (invariant)
  - Discrete group property (isomorphic to permutation group)!

3. General Approach to Valuing Multi-Asset Payoffs

- For the European case, we want to value a payoff that is some general function $G_T(S_{1,T}, S_{2,T}, \ldots, S_{l,T})$
  - For the American case, the payoff might be some (endogenous) function $G_t(S_{1,t}, S_{2,t}, \ldots, S_{l,t})$
- $l + 1$ element (vector) stochastic process framework:

$$\frac{dB}{B} = \frac{dS_0}{S_0} = r\,dt \qquad \text{(riskless/numéraire asset, money market account)}$$

$$\frac{dS_k}{S_k} = (\mu_k - y_k)\,dt + \sigma_k\,dW_k \qquad (l \text{ risky assets})$$

  with: $dW_k\,dW_{k'} = \rho_{k,k'}\,dt$

- We derived the l-dimensional BS PDE in the currency setting:
  - Just as in the single-asset case, drifts μ_k are replaced by r.
  - Invoking Feynman-Kac, valuation is therefore a discounted expectation with respect to the set of "risk-neutral" processes (**Q** measure):

$$\frac{dS_k}{S_k} = (r - y_k)\,dt + \sigma_k\,dW_k^{\mathbf{Q}}$$

    in which all discounting is done at the risk-free rate r.

3. General Approach to Valuing Multi-Asset Payoffs (2)

- The vector of S_k can be represented in terms of exponential martingales:

$$S_{k,T} = S_{k,0}\, e^{\tilde{\mu}_k^{\mathbf{Q}} T + \sigma_k W_{k,T}^{\mathbf{Q}}} = S_{k,0}\, e^{x_k} = S_{k,0}\, e^{\tilde{\mu}_k^{\mathbf{Q}} T + \sigma_k \sqrt{T}\, z_k}$$
$$\text{with: } \tilde{\mu}_k^{\mathbf{Q}} = r - y_k - \sigma_k^2/2,\, \mathbb{E}^{\mathbf{Q}}[z_k\, z_{k'}] = \rho_{k,k'}$$

- Starting with the general formula for the joint density $q[\underline{\mathbf{x}}]$ for l normal variates $\underline{\mathbf{x}} \sim N_l(\underline{\boldsymbol{\mu}}, \underline{\underline{\boldsymbol{\Sigma}}} = \underline{\underline{\boldsymbol{\sigma}}} \cdot \underline{\underline{\boldsymbol{\rho}}} \cdot \underline{\underline{\boldsymbol{\sigma}}})$:

$$q[\underline{\mathbf{x}}] = (2\pi)^{-l/2} |\underline{\underline{\boldsymbol{\Sigma}}}|^{-1/2} \exp[-(\underline{\mathbf{x}} - \underline{\boldsymbol{\mu}})^T \cdot \underline{\underline{\boldsymbol{\Sigma}}}^{-1} \cdot (\underline{\mathbf{x}} - \underline{\boldsymbol{\mu}})/2]$$
$$\leftrightarrow \quad q[\underline{\mathbf{z}}] = (2\pi)^{-l/2} |\underline{\underline{\boldsymbol{\rho}}}|^{-1/2} \exp[-\underline{\mathbf{z}}^T \cdot \underline{\underline{\boldsymbol{\rho}}}^{-1} \cdot \underline{\mathbf{z}}/2]$$

  - E.g., two-asset case considered in currency context:

$$q(z_1, z_2) = \frac{1}{2\pi\sqrt{1-\rho_{1,2}^2}} \exp\left[\frac{-1}{2(1-\rho_{1,2}^2)} \left(z_1^2 - 2\rho_{1,2}\, z_1 z_2 + z_2^2\right)\right]$$

- In principle, integrate payoff over l-dimensional space and discount
  - Take advantage of any opportunities to reduce the dimensionality of the integration space!

3.1. Monte Carlo for Multi-Asset Payoffs

- Consider simulation of scenarios: $1 \leq i \leq n$.
- For each scenario, generate l standard *uncorrelated* normal variates z'_k, from which we need to construct l standard normal variates z_k with correlations $\rho_{k,k'}$
- How do we go about this in general?
  - Idea (1): $\underline{\underline{\rho}}$ is (should be) symmetric and positive definite, so $\underline{\underline{\rho}}^{1/2}$ should exist (and also be symmetric and positive definite).
    - If $\underline{\mathbf{z}} = \underline{\underline{\rho}}^{1/2} \cdot \underline{\mathbf{z}}'$, then:
$$\mathbb{E}^{\mathbf{Q}}[\underline{\mathbf{z}}\,\underline{\mathbf{z}}^T] = \mathbb{E}^{\mathbf{Q}}[\underline{\underline{\rho}}^{1/2} \cdot \underline{\mathbf{z}}'\,\underline{\mathbf{z}}'^T \cdot \underline{\underline{\rho}}^{1/2}] = \underline{\underline{\rho}}^{1/2} \cdot \underline{\underline{\mathbf{I}}} \cdot \underline{\underline{\rho}}^{1/2} = \underline{\underline{\rho}}^{1/2} \cdot \underline{\underline{\rho}}^{1/2} = \underline{\underline{\rho}}$$
    - E.g., for $l = 2$: $\underline{\underline{\rho}}^{1/2} = \begin{pmatrix} \frac{\sqrt{1-\rho}}{2} + \frac{\sqrt{1+\rho}}{2} & \frac{\sqrt{1+\rho}}{2} - \frac{\sqrt{1-\rho}}{2} \\ \frac{\sqrt{1+\rho}}{2} - \frac{\sqrt{1-\rho}}{2} & \frac{\sqrt{1-\rho}}{2} + \frac{\sqrt{1+\rho}}{2} \end{pmatrix}$
  - In general, we don't want to be computing powers of correlation matrices and (ideally) we don't want to have to use a full l^2 terms to represent l correlated variates.
  - Idea (2): Any orthogonal (unitary) transformation ($\sim$ l-dimensional rotation) $\underline{\underline{\mathbf{Q}}}$ of $\underline{\underline{\rho}}^{1/2}$ has the same properties since:
$$\left(\underline{\underline{\rho}}^{1/2} \cdot \underline{\underline{\mathbf{Q}}}\right) \cdot \left(\underline{\underline{\mathbf{Q}}}^{\mathbf{T}} \cdot \underline{\underline{\rho}}^{1/2}\right) = \underline{\underline{\rho}}^{1/2} \cdot \underline{\underline{\mathbf{I}}} \cdot \underline{\underline{\rho}}^{1/2} = \underline{\underline{\rho}}$$
    - One such transformation is the $\underline{\underline{\mathbf{L}}}\,\underline{\underline{\mathbf{L}}}^T$ factorization of $\underline{\underline{\rho}}$

3.1. Monte Carlo for Multi-Asset Payoffs (2)

Cholesky (*not* "Choleski") Decomposition

- E.g.: $\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{pmatrix} = \begin{pmatrix} L_{11}^2 & & \text{(symmetric)} \\ L_{21}L_{11} & L_{21}^2 + L_{22}^2 & \\ L_{31}L_{11} & L_{31}L_{21} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{pmatrix}$
- Reducible to algorithm:
$$L_{k,k'} = \frac{1}{L_{k',k'}} \left(\rho_{k,k'} - \sum_{k''=1}^{k'-1} L_{k,k''} L_{k',k''} \right), \forall k' < k$$
$$L_{k,k} = \sqrt{\rho_{k,k} - \sum_{k''=1}^{k-1} L_{k,k''}^2}$$
- Then the composition of z_k is just $\sum_{k'=1}^{k} L_{k,k'}\, z'_{k'}$, e.g.:
$$z_1 = z'_1$$
$$z_2 = \rho_{12}\, z'_1 + \sqrt{1 - \rho_{12}^2}\; z'_2$$
$$z_3 = \rho_{13}\, z'_1 + \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}} z'_2 + \sqrt{\frac{1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{13}\rho_{23}\rho_{12}}{1 - \rho_{12}^2}}\, z'_3$$

3.1. Monte Carlo for Multi-Asset Payoffs (3)

Bare-Bones Simulation Strategy

1. Collect asset/simulation parameters, including number of scenarios n and number of assets l; pre-compute frequently used parameters
2. Perform Cholesky decomposition: $\rho_{k,k'} \to L_{k,k'}$
3. For paths $i = 1$ to n:
   1. Generate l standard normal samples $z'_{k'}$
   2. For assets $k = 1$ to l:
      1. $z_k = 0$
      2. For samples $k' = 1$ to k: $z_k \mathrel{+}= L_{k,k'} z'_{k'}$
      3. $S_{k,T} = S_{k,0} \exp(\tilde{\mu}_k^{\mathbf{Q}} T + \sigma_k \sqrt{T} z_k)$
   3. Compute payoff $C_{T,i}$
4. Calculate estimators/statistics (e.g. $\langle C_T \rangle_n$, $\mathrm{var}[C_T]_n$), discount, return

- This may not be the most efficient way to organize the calculations, e.g.:
  - Reverse order of loops to take advantage of vectorization
  - Pre-compute normal samples
- The bottleneck is the inner (Cholesky-based) loop over k' (3.2.2).
  - Any way to optimize or completely avoid this by taking advantage of special correlation (i.e. factor) structures should be done!

3.1. Monte Carlo for Multi-Asset Payoffs (4)

Variance Reduction

- Antithetic scenarios
  - Usually, generate only one antithetic sample per scenario with all asset z_k signs negated, unless payoff structure suggests asymmetric form may be more anti-correlated with base scenario(s)
- Moment matching
  - First moments are relatively easy, at least for log-matching
    - Exponential matching isn't *too* much trickier
  - Second moments, however, *are* trickier
    - Not only must we match sample marginal variances, we must also match co-variances
    - It may be helpful to pre-compute all the normal samples, then Gram-Schmidt orthogonalise them (conceptually related to Cholesky)
- Control variates, control variates, control variates...
  - Take some time to understand the payoff structure and understand (guess) what might be closely correlated

4. Some Common Multi-Asset Payoffs

Two Risky Assets

- Outperformance Option: $C_T = [S_{1,T} - S_{2,T}]^+$
  - Valuation by change of numéraire (reduction of dimensionality) $\Rightarrow$ pre-wash
- Outperformance with Strike (Spread Option): $C_T = [S_{1,T} - S_{2,T} - K]^+$
  - No reduction possible; not expressible in closed-form as sum of normal CDFs
  - Numerical integration of conditional BS formula required.
- Two-Asset Basket Option: $C_T = [S_{1,T} + S_{2,T} - K]^+$ (or $C_T = [w_1 S_{1,T} + w_2 S_{2,T} - K]^+$)
  - Reducible to spread option by change of numéraire; not expressible as sum of normal CDFs
  - Specialized approximations or numerical integration of conditional BS formula required.
- Better of 2: $C_T = \max[S_{1,T}, S_{2,T}]$
  - Reducible to $S_{2,T}$ forward $+$ outperformance option
- Worse of 2: $C_T = \min[S_{1,T}, S_{2,T}]$
  - Use composition ("better-of/worse-of parity"): $\max[S_{1,T}, S_{2,T}] + \min[S_{1,T}, S_{2,T}] = S_{1,T} + S_{2,T}$

4. Some Common Multi-Asset Payoffs (2)

Two Risky Assets (continued)

- Product Option: $C_T = [S_{1,T}\, S_{2,T} - K]^+$ (or $C_T = [S_{1,T}^{w_1} S_{2,T}^{w_2} - K]^+$)
  - Use change of variables to reduce dimensionality $\Rightarrow$ pre-wash
  - Note that forward $\mathbb{E}^{\mathbf{Q}}[S_{1,T}\, S_{2,T}]$ will contain a covariance term!
- Ratio Option: $C_T = [S_{1,T}/S_{2,T} - K]^+$
  - Change of variables $\Rightarrow$ pre-wash; forward $\mathbb{E}^{\mathbf{Q}}[S_{1,T}/S_{2,T}]$ will also contain a covariance term!
- Options on the Max or Min of Two Assets: $C_T = [\phi\, \eta \max(\eta S_{1,T}, \eta S_{2,T}) - \phi K]^+$
  - Closed-form valuation in terms of $\mathcal{N}_2(\bullet, \bullet; \bullet)$
- Best, Worst, or Median of Two Assets and Cash:
  - Relatable by closure and addition transformations to Options on the Max or Min of Two
  - Closed-form valuation in terms of $\mathcal{N}_2(\bullet, \bullet; \bullet)$, e.g. Max ($\phi = 1$) or Min ($\phi = -1$):

$$C_{t=0} = S_0 e^{-y_0 T} \mathcal{N}_2(\phi z_{01+}, \phi z_{02+}; \rho_{12}) + S_1 e^{-y_1 T} \mathcal{N}_2(\phi z_{12+}, \phi z_{10+}; \rho_{20}) + S_2 e^{-y_2 T} \mathcal{N}_2(\phi z_{20+}, \phi z_{21+}; \rho_{01})$$

with $S_0 \doteq K$, $y_0 \doteq r$, $z_{ij}+$ defined as the usual z_+ with i as risky asset and j as numéraire, and ρ_{ij} the correlation between assets i and j with the third asset as numéraire.

4. Some Common Multi-Asset Payoffs (3)

Two Risky Assets (continued)

- Dual Strike Options: $C_T = [\phi \max[\phi(S_{1,T} - K_1), \phi(S_{2,T} - K_2)]]^+$
  - No closed form possible; numerical integration required
- Binary on 1 / Option on 2: $C_T = I_{S_{1,T}-K_1}[S_{2,T} - K_2]^+$
  - Closed-form valuation in terms of $\mathcal{N}_2(\bullet, \bullet; \bullet)$
- Product of Option Payoffs: $C_T = [S_{1,T} - K_1]^+[S_{2,T} - K_2]^+$
  - Closed-form valuation in terms of $\mathcal{N}_2(\bullet, \bullet; \bullet)$
- Quantity-Adjusting Option: $C_T = S_{1,T}[S_{2,T} - K_2]^+$
  - Valuation by change of numéraire (reduction of dimensionality) $\Rightarrow$ pre-wash

4. Some Common Multi-Asset Payoffs (4)

Two Risky Assets: {Foreign Equity S^f + Currency X} Special Cases

- Option on a foreign stock struck in foreign currency: $C_T = [S_T^f X_T - K^f X_T]^+$
- Option on a foreign stock struck in domestic currency: $C_T = [S_T^f X_T - K]^+$
- Quantity-Adjusting (Quanto) Option: $C_T = [S_T^f \bar{X} - K^f \bar{X}]^+$ (with constant $\bar{X}$)
- Equity-linked FX (ELFX) Option: $C_T = S_T^f [X_T - \bar{X}]^+$ (with FX strike $\bar{X}$)

- All can be valued by change of numéraire (reduction of dimensionality), leading to BS-like formulae with "pre-washed" parameters:

Black-Scholes (domestic equity)	S_0	K	r	y	σ_S
Garman-Kohlhagen (FX)	X_0	K	r	r_f	σ_X
Foreign equity/foreign strike	$S_0^f X_0$	$K^f X_0$	r_f	y	σ_{S^f}
Foreign equity/domestic strike	$S_0^f X_0$	K	r	y	$\sigma_{S^f X}$
Fixed FX foreign equity (Quanto)	$S_0^f \bar{X}$	$K^f \bar{X}$	r	$y + r - r_f + \rho_{S^f,X} \sigma_{S^f} \sigma_X$	σ_{S^f}
Equity-linked FX	$S_0^f X_0$	$S_0^f \bar{X}$	$r + y - r_f + \rho_{S^f,X} \sigma_{S^f} \sigma_X$	y	σ_X

with $\sigma_{S^f X} = \sqrt{\sigma_{S^f}^2 + \sigma_X^2 + 2\rho_{S^f,X}\sigma_{S^f}\sigma_X}$ and $\rho_{S^f,X}$ is the (numéraire-inconsistent) correlation between the foreign stock price S_f and the FX rate X.

4. Some Common Multi-Asset Payoffs (5)

Three or More Risky Assets

- Best of 3: $C_T = \max[S_{1,T}, S_{2,T}, S_{3,T}]$
- Worst of 3: $C_T = \min[S_{1,T}, S_{2,T}, S_{3,T}]$
- Middle of 3: $C_T = S_{1,T} + S_{2,T} + S_{3,T} - \max[S_{1,T}, S_{2,T}, S_{3,T}] - \min[S_{1,T}, S_{2,T}, S_{3,T}]$
  - Using change of numéraire $\rightarrow$ any one of the assets, all have closed-form valuation in terms of $\mathcal{N}_2(\bullet, \bullet; \bullet)$
  - Also options on any of the above, e.g.:
    - Call on Best of 3: $C_T = \max[\max[S_{1,T}, S_{2,T}, S_{3,T}] - K, 0]$
    - Introducing strike K breaks homogeneity of the payoff, hence no longer permits dimensional reduction.
    - Solvable in closed-form in terms of $\mathcal{N}_3(\bullet_1, \bullet_2, \bullet_3; \underline{\underline{\bullet}})$
- Basket Option: $C_T = \left[\sum_{k=1}^{l} w_k S_{k,T} - K\right]^+$
  - Most common type of multi-asset payoff
  - Generally amenable to same techniques as Asian options
    - In fact, Asian options can be considered special cases of basket options (with special correlation structures)