

3. Brownian Motion Representation & Black-Scholes-Merton Formula

- Lognormal transformation. If:

$$\frac{dS}{S} = (r - y) dt + \sigma dW^Q$$

then:

$$d \ln(S) = \left(r - y - \frac{\sigma^2}{2} \right) dt + \sigma dW^Q = \tilde{\mu}^Q dt + \sigma dW^Q$$

with: $\tilde{\mu}^Q \doteq r - y - \sigma^2/2$

- Hence,

$$\ln(S_T) - \ln(S_t) = \ln\left(\frac{S_T}{S_t}\right) \sim N[\tilde{\mu}^Q(T-t), \sigma^2(T-t)]$$

and the probability density q for $\ln(S_T/S_t)$ is given by:

$$q[\ln(S_T/S_t), T-t] = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \exp\left[-\frac{(\ln(S_T/S_t) - \tilde{\mu}^Q(T-t))^2}{2\sigma^2(T-t)}\right]$$

3. Brownian Motion Representation & Black-Scholes-Merton Formula (2)

- Alternatively, we can write:

$$\frac{S_T}{S_t} = \exp(\tilde{\mu}^Q(T-t) + \sigma\sqrt{T-t}z)$$

with $z \sim N(0, 1)$

- Valuation of the payoff $C_T(S_T)$:

$$C(S_t, t) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} C_T\left(S_t e^{\tilde{\mu}^Q(T-t) + \sigma\sqrt{T-t}z}\right)$$

- If $C_T(S_T, T) = \max[S_T - K, 0]$:

$$\begin{aligned} C(S_t, t) &= e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \max\left[S_t e^{\tilde{\mu}^Q(T-t) + \sigma\sqrt{T-t}z} - K, 0\right] \\ &= e^{-r(T-t)} \int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[S_t e^{\tilde{\mu}^Q(T-t) + \sigma\sqrt{T-t}z} - K\right] \\ &\text{with } z^* = \frac{\ln(K/S_t) - \tilde{\mu}^Q(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

3. Brownian Motion Representation & Black-Scholes-Merton Formula (3)

- Expanding terms in the integral:

$$\begin{aligned}
 C(S_t, t) &= e^{-r(T-t)} \int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} \left[S_t e^{\tilde{\mu}^Q(T-t)} e^{-z^2/2 + \sigma\sqrt{T-t}z} - K e^{-z^2/2} \right] \\
 &= S_t e^{-y(T-t)} \int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2 + \sigma\sqrt{T-t}z - \sigma^2(T-t)/2} - K e^{-r(T-t)} \int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \\
 &= S_t e^{-y(T-t)} \int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-(z - \sigma\sqrt{T-t})^2/2} - K e^{-r(T-t)} \int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}
 \end{aligned}$$

- Using $\int_{z^*}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} = 1 - \mathcal{N}[z^*] = \mathcal{N}[-z^*]$ and substituting for z^* :

$$\begin{aligned}
 C(S_t, t) &= S_t e^{-y(T-t)} \mathcal{N}[-z^* + \sigma\sqrt{T-t}] - K e^{-r(T-t)} \mathcal{N}[-z^*] \\
 &= S_t e^{-y(T-t)} \mathcal{N} \left[\frac{\ln(S_t/K) + \left(r - y - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} + \sigma\sqrt{T-t} \right] \\
 &\quad - K e^{-r(T-t)} \mathcal{N} \left[\frac{\ln(S_t/K) + \left(r - y - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right]
 \end{aligned}$$

3. Brownian Motion Representation & Black-Scholes-Merton Formula (4)

- Simplifying:

$$\begin{aligned}
 C(S_t, t) &= S_t e^{-y(T-t)} \mathcal{N} \left[\frac{\ln(S_t/K) + (r-y)(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2} \right] \\
 &\quad - K e^{-r(T-t)} \mathcal{N} \left[\frac{\ln(S_t/K) + (r-y)(T-t)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2} \right] \\
 &= S_t e^{-y(T-t)} \mathcal{N}[z_+] - K e^{-r(T-t)} \mathcal{N}[z_-] \\
 \text{with } z_{\pm} &= \frac{\ln(S_t/K) + (r-y)(T-t)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}
 \end{aligned}$$

- Put valuation: we must change two things:

- ① Change sign of payoff function:

$$\max[S_T - K, 0] = [S_T - K]^+ \rightarrow \max[K - S_T, 0] = [K - S_T]^+$$

- ② Change limits on integral: $\int_{z^*}^{\infty} dz \rightarrow \int_{-\infty}^{z^*} dz$

- Result:

$$P(S_t, t) = K e^{-r(T-t)} \mathcal{N}[-z_-] - S_t e^{-y(T-t)} \mathcal{N}[-z_+]$$

3. Brownian Motion Representation & Black-Scholes-Merton Formula (5)

- Alternatively, we can use put-call parity:

$$\begin{aligned} P(S_t, t) &= C(S_t, t) - S_t e^{-y(T-t)} + K e^{-r(T-t)} \\ &= S_t e^{-y(T-t)} \mathcal{N}[z_+] - K e^{-r(T-t)} \mathcal{N}[z_-] - S_t e^{-y(T-t)} + K e^{-r(T-t)} \\ &= S_t e^{-y(T-t)} (\mathcal{N}[z_+] - 1) + K e^{-r(T-t)} (1 - \mathcal{N}[z_-]) \\ &= K e^{-r(T-t)} \mathcal{N}[-z_-] - S_t e^{-y(T-t)} \mathcal{N}[-z_+] \end{aligned}$$

- Symmetric formula (Rubinstein):

$$\begin{Bmatrix} C \\ P \end{Bmatrix}(S_t, t) = \phi \left[S_t e^{-y(T-t)} \mathcal{N}(\phi z_+) - K e^{-r(T-t)} \mathcal{N}(\phi z_-) \right]$$

with $\phi = 1$ for calls and $\phi = -1$ for puts

- Primary references:

- Fischer Black and Myron Scholes, “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy* 81(3), (May-Jun, 1973), pp. 637-654. www.jstor.org/stable/1831029
- Robert C. Merton, “Theory of Rational Option Pricing,” *The Bell Journal of Economics and Management Science* 4(1), (Spring, 1973), pp. 141-183. www.jstor.org/stable/3003143

3. Brownian Motion Representation & Black-Scholes-Merton Formula (6)

Options on Futures/Forwards

- FX trading is based in the forward market, so we re-write the BSM formula in terms of $F_{t,T}$:

$$F_{t,T} = S_t e^{(r-y)(T-t)} \implies S_t e^{-y(T-t)} = F_{t,T} e^{-r(T-t)}$$

- Re-write Rubinstein's symmetric formula as:

$$\begin{aligned}\{C, P\}(F_{t,T}, t) &= \phi \left[F_{t,T} e^{-r(T-t)} \mathcal{N}(\phi z_+) - K e^{-r(T-t)} \mathcal{N}(\phi z_-) \right] \\ &= \phi e^{-r(T-t)} [F_{t,T} \mathcal{N}(\phi z_+) - K \mathcal{N}(\phi z_-)]\end{aligned}$$

with $\{\phi = 1 \text{ for calls, } \phi = -1 \text{ for puts}\}$ and $z_\pm = \frac{\ln(F_{t,T}/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$.

- Since $F_{t,T}$ converges to S_T as $t \nearrow T$, this is identical in content to the BSM formula.
 - This is theoretically a slightly preferable formulation to BSM because all inputs have the same tenor T .
 - Technically, the volatility input σ really should be that of the forward (or futures) price $F_{t,T}$.
- Extension: instead of the forward/futures contract terminating at T , let it mature at $T' > T$.
 - This is how futures options are set up in practice: $T' - T$ is typically a few days to a few months.
 - Black's Model for futures options: use the formula above with $F_{t,T'}$ in place of $F_{t,T}$

4. Inputs & Sensitivities

- Inputs

- ① Risky asset price, S
- ② Strike price, K
- ③ Time to maturity, T (or $T-t$)
- ④ Riskless interest rate, r
- ⑤ (Dividend or convenience) yield, y
- ⑥ Volatility, σ

- Sensitivities (to all of the above)



4.1. Inputs

- Risky asset (e.g. Stock) price: $S = S_t = S(t)$
 - Units: (quantity of numéraire asset / unit of risky asset) e.g. \$/share
 - For currencies, sometimes use X (“eXchange rate”) for clarity
 - Units: (quantity of numéraire or domestic currency / unit of foreign currency) e.g. \$/£, \$/¥, \$/€...
 - When we study currency options, we'll see that quotation conventions appear to be the opposite of this (but are actually completely consistent)
- Strike price: K
 - Units: same as risky asset
 - Some older texts use X (for eXercise price)
- Time to maturity: T
 - Units: time (years, months, days...).
 - N.B. conventions associated with volatility and rate inputs (trading days, day counts)
 - We may also use $T - t$ when we want to fix T at a specific point in the future and vary the current time t

4.1. Inputs (2)

- Riskless rate r :
 - Units: time $^{-1}$ (e.g. %/year).
 - N.B. day count and settlement conventions vary between different interest bearing assets, e.g.:
US Treasury bonds (ACT/ACT), but Treasury bills are ACT/360
LIBOR and similar indices (usually ACT/360, but also ACT/365 and other variants), etc.
- Academic literature traditionally references US Treasury (or, for other countries, the corresponding sovereign / central bank) domestic currency yield curve as (most) riskless
 - Closest philosophically to B-S-M assumption of riskless borrowing, but is the US government a.s. default-free?
What about other countries?
 - Evidence from rating agencies, credit default swaps (CDS), history: perhaps not.
 - Traditional argument: a government can always print money to inflate its way out of debt in its own currency, so domestically-denominated debt should be riskless while foreign-denominated debt is not
 - Historical studies show that sovereigns default on own-currency debt surprisingly often
 - US Government debt crises (e.g. Oct 2013) have threatened USD treasury defaults
(and markets responded demonstrating credibility of threat)
 - What about sovereigns that don't control their own currency (e.g., Euro-zone countries)?
 - For credit-unworthy sovereigns' currencies, could there be another (sovereign, financial, or corporate) counterparty that borrows at a lower (less risky) rate?

4.1. Inputs (3)

- Riskless rate r (continued): **Sovereign CDS Spreads**

| Country | Rating | 5 Years Credit Default Swaps | | | | |
|----------------|--------|------------------------------|----------|----------|----------|--------|
| | | S&P | 5Y CDS* | Var 1m | Var 6m | PD (*) |
| Switzerland | AAA | 8.01 | +0.38 % | -0.00 % | 0.10 % | 5 Feb |
| Australia | AAA | 10.05 | -0.99 % | -23.69 % | 0.17 % | 5 Feb |
| Sweden | AAA | 10.22 | +4.39 % | -16.80 % | 0.17 % | 5 Feb |
| Denmark | AAA | 10.38 | +18.19 % | +7.14 % | 0.17 % | 5 Feb |
| Netherlands | AAA | 12.58 | +6.70 % | +21.66 % | 0.21 % | 5 Feb |
| Germany | AAA | 13.52 | -2.73 % | +68.00 % | 0.23 % | 5 Feb |
| Austria | AA+ | 15.41 | +8.42 % | +18.48 % | 0.28 % | 5 Feb |
| Ireland | AA | 16.38 | +1.05 % | -19.67 % | 0.27 % | 5 Feb |
| Japan | A+ | 19.82 | -3.11 % | -18.08 % | 0.35 % | 5 Feb |
| Finland | AA+ | 19.93 | +1.94 % | +8.28 % | 0.35 % | 5 Feb |
| United Kingdom | AA | 23.17 | +6.14 % | +1.66 % | 0.36 % | 5 Feb |
| Belgium | AA | 25.86 | -1.46 % | +84.41 % | 0.45 % | 5 Feb |
| Portugal | A- | 27.09 | -8.80 % | -29.08 % | 0.46 % | 5 Feb |
| United States | AA+ | 31.47 | +6.05 % | -1.41 % | 0.62 % | 5 Feb |
| Spain | A- | 33.76 | -4.42 % | -8.88 % | 0.65 % | 5 Feb |
| South Korea | AA | 34.11 | -10.82 % | -4.88 % | 0.67 % | 5 Feb |
| France | AA- | 36.45 | -8.69 % | +17.69 % | 0.81 % | 5 Feb |
| Canada | AAA | 39.60 | 0.00 % | 0.00 % | 0.66 % | 5 Feb |
| China | A+ | 55.74 | -12.77 % | -18.70 % | 0.93 % | 5 Feb |
| Italy | BBB | 57.27 | -8.81 % | -18.11 % | 0.85 % | 5 Feb |
| Greece | BBB- | 57.78 | -3.88 % | -14.83 % | 0.96 % | 5 Feb |
| Indonesia | BBB | 76.87 | -2.16 % | -4.22 % | 1.28 % | 5 Feb |
| India | BBB- | 84.08 | 0.00 % | 0.00 % | 1.40 % | 5 Feb |
| Israel | A | 87.08 | -8.80 % | -30.34 % | 1.45 % | 5 Feb |
| Mexico | BBB | 130.87 | -4.99 % | +16.81 % | 2.18 % | 5 Feb |
| Brazil | BB | 177.28 | -16.10 % | +6.83 % | 2.06 % | 5 Feb |
| South Africa | BB- | 200.19 | +6.27 % | +1.62 % | 3.34 % | 5 Feb |
| Turkey | BB | 255.13 | -1.02 % | -2.47 % | 4.25 % | 5 Feb |
| Egypt | B- | 521.34 | -7.11 % | -31.11 % | 8.00 % | 5 Feb |
| Russia | NR | 13775.17 | 0.00 % | 0.00 % | 100.00 % | 5 Feb |

(*) Implied probability of default, calculated on the hypothesis of a 40% recovery rate.

- Source: www.worldgovernmentbonds.com/sovereign-cds/ as of 05 Feb 2024

4.1. Inputs (4)

- Riskless rate r (continued):

- ② Up to the financial crisis, the over-the-counter (OTC) market traditionally used LIBOR or some other, usually **unsecured**, inter-bank lending rate as the “riskless” interest rate
 - Always questionable, but during the “great moderation” average rating of LIBOR-poll banks was typically a strong AA and TED spreads were in the tens of basis points
 - Some oddities for firms with stronger (or weaker) credit; artifacts in pricing vs. Treasuries
 - Systematic downgrades of banking sector before and during the crisis
 - Explosion of TED spread during the crisis and appearance of significant basis spreads between e.g. different tenors reflecting perceptions of credit (un-)worthiness under stress
 - LIBOR scandals calling into question whether poll actually reflects inter-bank lending rates
- ③ Post-crisis: evolution to OIS (Overnight Index Swap) market as benchmark “riskless” rate
 - Based on e.g. (floating) Fed Funds overnight lending rate \Rightarrow like a money-market account
 - Rolling daily settlement and amount at risk equal to (difference between fixed and floating rates) limits credit exposure

4.1. Inputs (5)

- Riskless rate r (continued):

④ Current state of play:

- Central bank-calculated reference rates based on either:
 - (secured) overnight (treasury) repo market (e.g.: SOFR for USD, SARON for CHF)
 - (unsecured) overnight interbank market (e.g.: SONIA for GBP, EONIA/ESTR for EUR, TONR for JPY, ...)
- Reference swap rates published by exchanges (e.g., ICE/NYSE: www.theice.com/iba/ice-swap-rate)
- As of end-2021, new swap issuance based on LIBOR was “strongly discouraged”, e.g., US:
www.federalreserve.gov/supervisionreg/srletters/SR2117a1.pdf
and even outright banned in some jurisdictions, e.g., UK:
www.fca.org.uk/publication/libor-notices/article-21a-benchmarks-regulation-prohibition-notice.pdf
- Publication of LIBOR has been phased out:
 - The final (USD) LIBOR panel ceased to exist as of end-June 2023.
 - A few “synthetic” GBP & USD fall-back rates (SOFR or SONIA + a spread) were updated until Mar/Sep 2024.
www.fca.org.uk/news/news-stories/fca-announces-decision-synthetic-us-dollar-libor
www.theice.com/iba/libor
- Eurodollar futures and options were transitioned/converted to SOFR in April 2023 and delisted in June 2023.
- Lots of issues (e.g., backward-looking rates), mostly beyond our scope! See:
 - BIS paper: *Beyond LIBOR: a primer on the new reference rates*
 - Gregory S. Faranello, CFA commentary: *The Path to SOFR and Alternatives*
 - NY Fed Alternative Reference Rates Committee: *Transition from LIBOR*
 - J.P. Morgan Solutions: *Leaving LIBOR: A Landmark Transition*

4.1. Inputs (6)

- Riskless rate r (continued):
 - Although it wreaks havoc on the theory, a “multi-curve” framework embedding:
 - ① a (riskless) money-market-like rate to generate forward prices and
 - ② one or more risky rates, perhaps with a benchmark analogous to LIBOR, for discounting is becoming standard.
 - The most sophisticated broker-dealers were already moving in this direction before the crisis
 - Accounting standards: Counterparty credit (CVA) and own-credit (DVA) adjustments
 - Active area of research, but results are very messy because discounting rates depend on:
 - collateralization mechanics,
 - correlation between payoff and credit spread, and even
 - entire structures / risk profiles of your and counterparty's books (and full sets of counterparty relationships)
 - And then there is the fact that interest rates are actually stochastic...



4.1. Inputs (7)

- yield y (q in some notations):
 - Units: time $^{-1}$ (e.g. %/year).
 - Various interpretations:
 - Dividend yield (also denoted d or δ) for stocks & indices, but also stock borrow/lending rate;
 - Foreign interest rate r_f for currencies;
 - Convenience yield and/or storage rate for commodities and real assets.
- Stock/index dividend yields:
 - Strong evidence (at least for index aggregates) that dividend **amounts** D and asset prices S are co-integrated (M. Lettau and S. Ludvigson, “Expected Returns and Expected Dividend Growth,” *JFE* 76(3), 2005, pp. 583-626)
⇒ assumption of fixed dividend yields appears a reasonable place to start, at least for long-maturity options;
 - For single stocks, can get to this assumption by assuming that (in the long run) companies seek to pay out a fixed fraction of their earnings;
 - Most financial firms maintain a dividend database and many analysts attempt to forecast dividends for at least the next year or two.
 - Problem 1: Dividends are paid **discretely** (are lumpy) and aren't distributed evenly throughout the year
⇒ sometimes need to model $d \sim \sum_i d_i \delta(t - t_i)$
 - Problem 2: In the short term, dividend **amounts** D are sticky (signaling, declaration, accruals, etc).
 - Traders often conflate these two issues, but they are both of real importance for both option valuation **and** hedging (e.g. UK's 1997 repeal of dividend withholding exemption for broker-dealers)
 - We briefly glanced at some of these issues in Part 1 of the course notes and the first problem set.

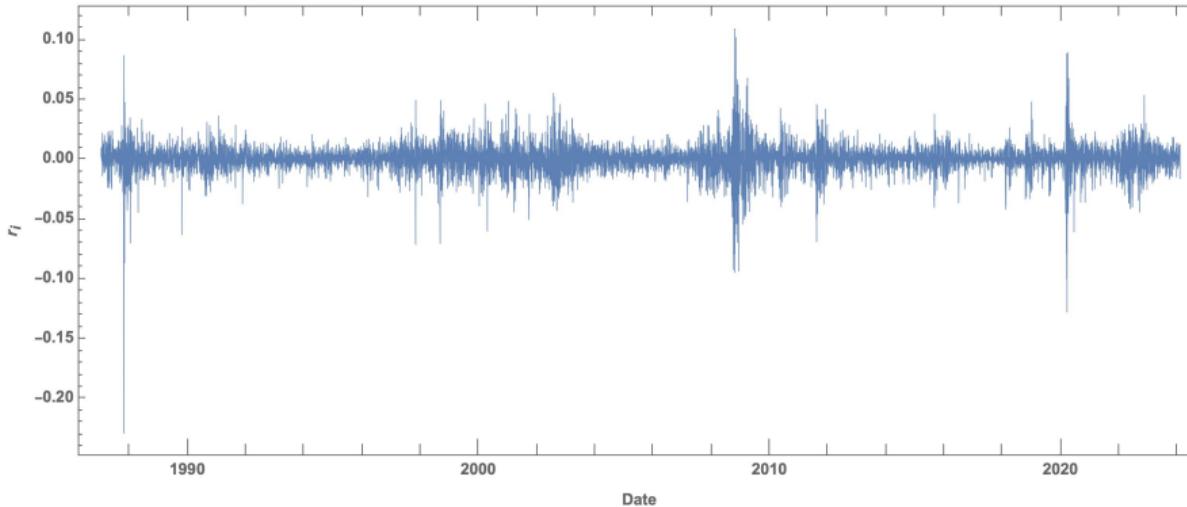
4.1. Inputs (8)

- Volatility σ :
 - Units: $\text{time}^{-1/2}$ (e.g. %/year $^{1/2}$). Why?
Because **variance** σ^2 is a **rate** and therefore scales as time $^{-1}$
 - σ represents the characteristic size or scale of asset price movements over a period of time:
 - $\ln(S_{t+\Delta t}/S_t) \sim \sigma\sqrt{\Delta t}$
 - Traders' rule: daily vol \approx (annual vol)/16,
e.g., annual volatility of 16% corresponds approximately to a daily move of $\pm 1\%$
 - What does the word “volatility” mean in the context of option pricing and risk management?
Many (inter-related) answers:
 - ① Historical / Realized vol
 - ② Forecast vol
 - ③ Implied vol

4.1. Inputs (9)

Volatility Properties

- SPX daily (log) returns, 1987-present:



- Asset price process volatility (in the **P**-world) isn't directly observable
 \Rightarrow volatility must be *estimated* or *forecast*
- Short-term persistence: volatility “*clusters*” then displays mean reversion-like behavior
- Is volatility a stationary process? If so, over what time horizon?

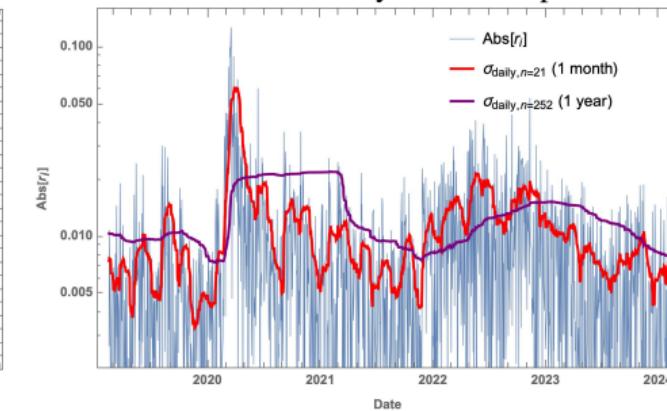
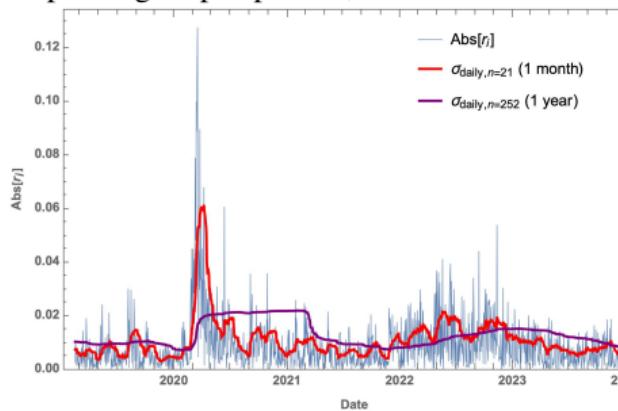
4.1. Inputs (10)

① Historical / Realized Volatility

- Looking backward, calculate estimator of historical vol (over some interval, with some frequency) from e.g. the root-mean-square formula:

$$\sigma_{\text{daily}}^2 \sim \sigma_{\text{daily};n}^2 \doteq \frac{1}{n-1} \sum_{i=0}^{n-1} (r_{-i} - \langle r \rangle_{-n})^2 \text{ with } r_{-i} = \ln[S_{-i}/S_{-(i+1)}] \text{ and } \langle r \rangle_{-n} = \frac{1}{n} \sum_{i=0}^{n-1} r_{-i}$$

- Depending on perspective, estimator is either unconditional or conditional only on the sampled time range



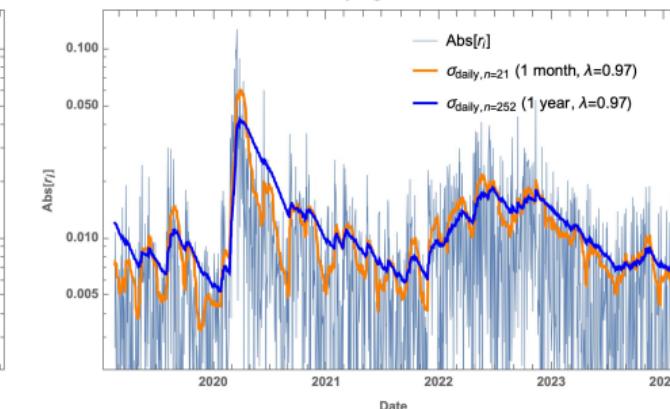
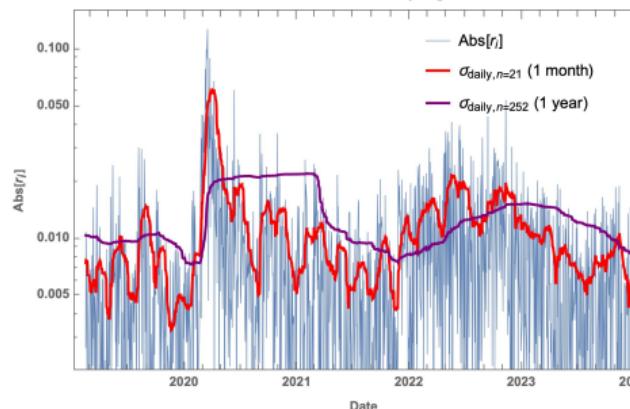
- Trade-off: shorter vs. longer time series of returns
 - Longer time series: better confidence in estimator but may be less responsive to changes, less relevant to future
 - Shorter time series: responsive and more current, but poorer confidence and more volatile
- Possible solutions/extensions: use more than just close-close returns (OHLC, hi-frequency data)

4.1. Inputs (11)

➊ Historical / Realized Volatility (continued)

- What we really want is an estimator conditional on a particular (future) time or range of times.
 - Looking forward, seek to predict the future realized volatility (driving the cost of dynamically hedging an option)
 - Backward-looking estimator isn't necessarily a good predictor of future volatility, particularly around times of stress...
- Perhaps weight returns (e.g., exponential decay) so that more recent price moves (with presumably greater value in estimating future volatility) are weighted more heavily:

$$\sigma_{\text{daily}}^2 \sim \sigma_{\text{daily};n,\lambda}^2 \doteq \frac{1-\lambda^2}{2(\lambda-\lambda^n)} \sum_{i=0}^{n-1} \lambda^i (r_{-i} - \langle r \rangle_{-n,\lambda})^2 \text{ with } \langle r \rangle_{-n,\lambda} = \frac{1-\lambda}{1-\lambda^n} \sum_{i=0}^{n-1} \lambda^i r_i; n_{\text{eff}} = \frac{(1+\lambda)(1-\lambda^n)}{(1-\lambda)(1+\lambda^n)}$$

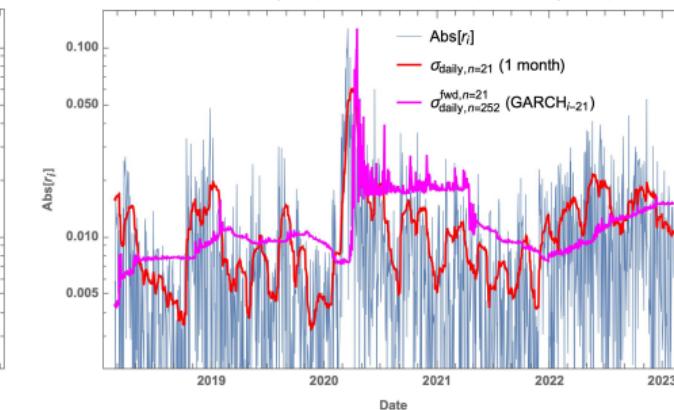
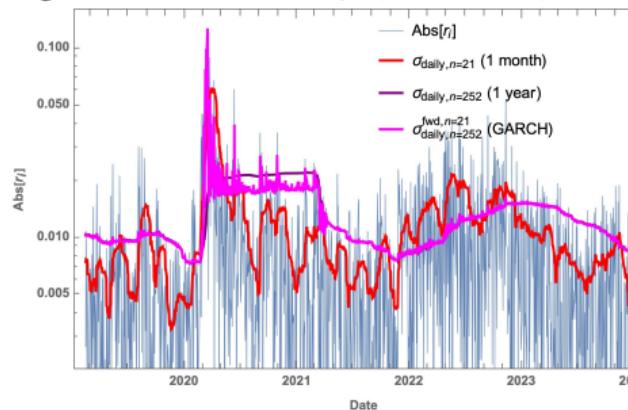


EWMA vol estimator or “poor man’s GARCH” (above: $\lambda = 0.97$, $n_{\text{eff}} = \{20.3, 65.6\}$)

4.1. Inputs (12)

② Forecast Volatility

- Try to model stylized properties of volatility (persistence/mean reversion) as conditional heteroskedasticity:
E.g., ARCH(1): $r_i = \sigma_i z_i$; $\sigma_i^2 = k + \alpha_1 r_{i-1}^2$; GARCH(1,1): $r_i = \sigma_i z_i$; $\sigma_i^2 = k + \alpha_1 r_{i-1}^2 + \beta_1 \sigma_{i-1}^2$



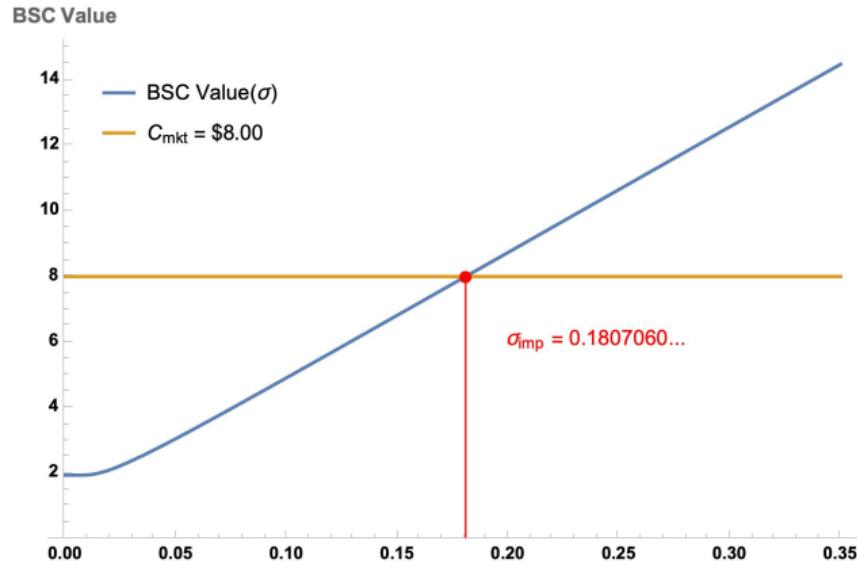
- Empirically has *some* predictive value, particularly for shorter time horizons, particularly using high-frequency data as input (but recognizing that some issues arise at very high frequencies)
- Popular in academic circles, but never really developed much industry traction, at least on the sell side.
- Should we use $\mathbb{E}[\sigma_T]$, $\mathbb{E}[\sigma_T^2]$, or, more generally, some form of term structure involving $\sigma_t : 0 \leq t \leq T$, e.g., $\mathbb{E}[\int_0^T dt \sigma_t^2]$ as input to our pricing model?
 - Problems of convexity of option prices vs. σ and correlation of σ with S
 - It would be better to consider this in a stochastic volatility pricing framework

4.1. Inputs (13)

⑤ Implied Volatility

Definition of implied vol: $\sigma_{imp} \doteq \sigma : C(S, K, T; r, y, \sigma) = C_{mkt}$

- Example: “Base Case” parameters: $S=K=100$, $T-t = 1$ yr, $r = 4\%$ p.a., $y = 2\%$ p.a., with $C_{mkt} = 8$:



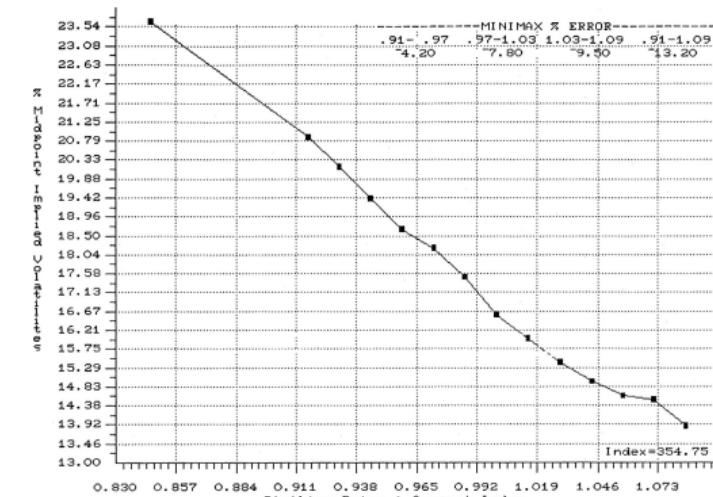
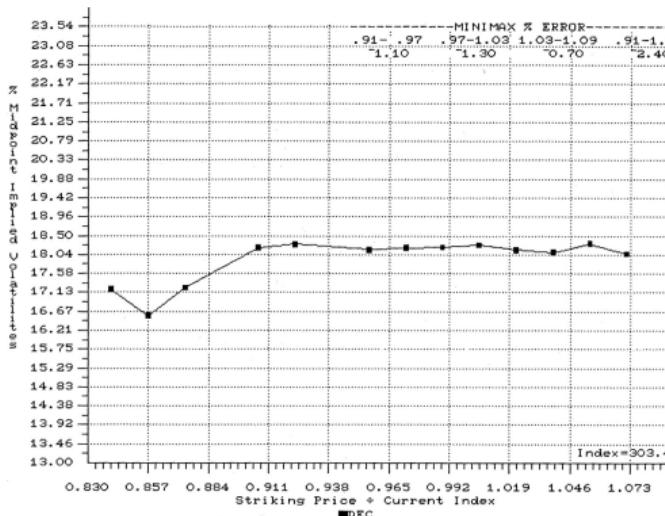
- In practice, this is a non-linear (numerical) root-finding problem, requiring a good initial guess for σ_{imp} .

4.1. Inputs (14)

③ Implied Volatility (continued):

- For index options, significant volatility “skew” emerged rapidly following the 1987 market crash.

SPX σ_{imp} vs. K/S : Dec '87 expiry as of 1-Jul-87 vs. Jun '90 expiry as of 2-Jan-90 (both $T \approx 5$ months)



Source: Mark Rubinstein, *Journal of Finance* 49(3), (Jul 1994), pp. 771-818.

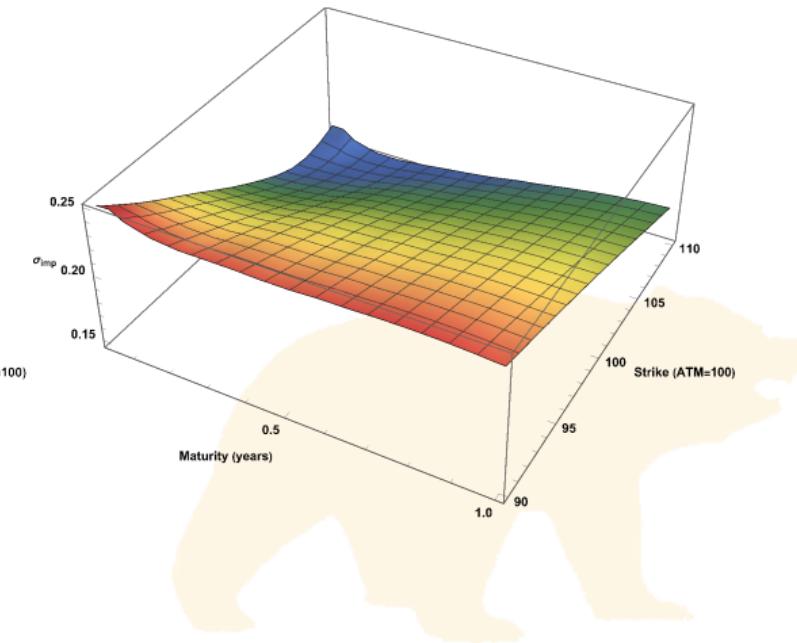
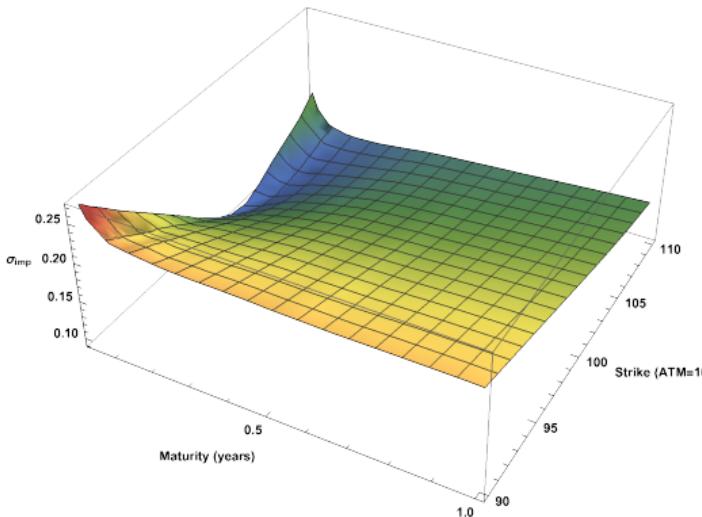
- In general, $\sigma_{imp} = \sigma_{imp}(T, K)$.

In addition, σ_{imp} changes (with unpredictable components) as $\{t, S_t\}$ evolve

4.1. Inputs (15)

⑤ Implied Volatility (continued):

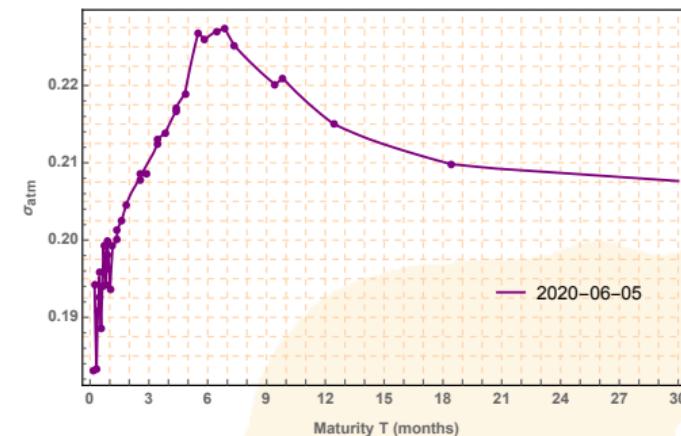
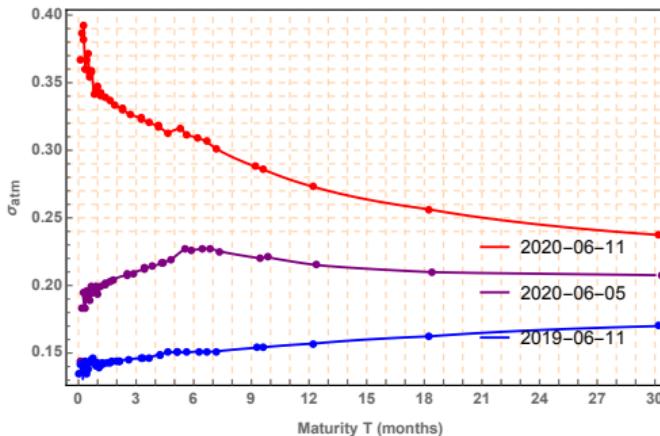
- SPX Implied Volatility Surfaces as of 2021-06-04 and 2023-01-20



4.1. Inputs (16)

③ Implied Volatility (continued):

- SPX ATM Implied Volatility Curves as of 2019-06-11, 2020-06-05, and 2020-06-11:



4.1. Inputs (17)

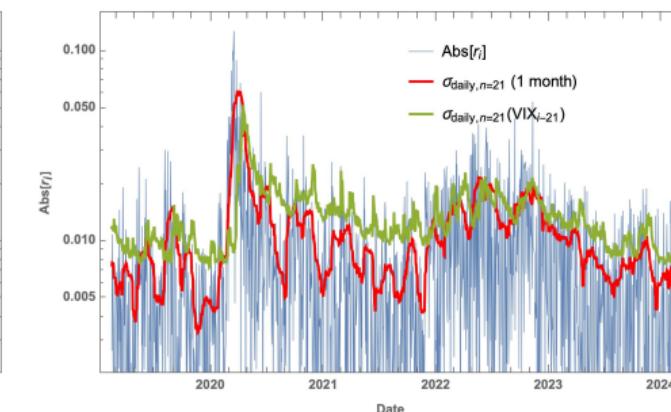
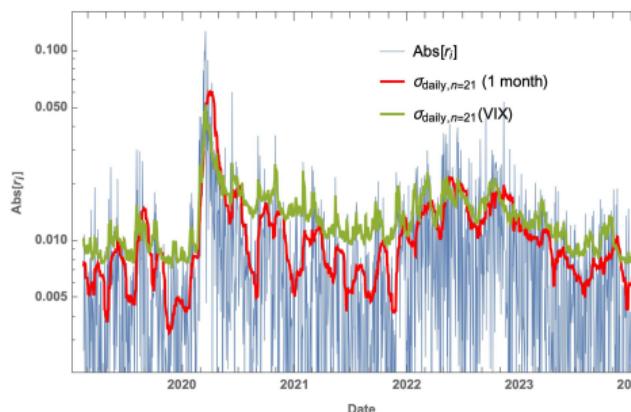
⑤ Implied Volatility (continued):

- Standard aphorism: σ_{imp} is the market's forecast of future asset price (realized) volatility
- How should we trade if we want to take a position that the “market's forecast” is too high or too low?
- If so, which implied volatility/instruments?
 - $\sigma_{imp,ATM}$ via, e.g., ATM straddles (where liquidity is usually concentrated)?
 - We've already raised some of the issues with this approach when we discussed straddles in Part 1.
- Is there, in some sense, an *unconditional* market volatility forecast that doesn't depend on assumptions like remaining at/near the money?
 - Remarkably, the answer is YES! Start with our log-process: $d \ln(S_t) = (r - y - \sigma^2/2)dt + \sigma dW_t^Q$
 - Drift is inconvenient, but is neutralized if we consider the process for $F_{t,T} = e^{(r-y)(T-t)}S_t$:
$$d \ln(F_{t,T}) = -\sigma_t^2/2 dt + \sigma_t dW_t^Q$$
 - Integrating and taking expectations:
$$\ln\left(\frac{F_{T,T}}{F_{0,T}}\right) = -\frac{1}{2} \int_0^T dt \sigma_t^2 + \int_0^T \sigma_t dW_t^Q \implies \mathbb{E}^Q\left[\ln\left(\frac{S_T}{F_{0,T}}\right)\right] = -\frac{1}{2} \mathbb{E}^Q\left[\int_0^T dt \sigma_t^2\right] = -\frac{1}{2} \int_0^T dt \mathbb{E}^Q[\sigma_t^2]$$
- Hence: $\mathbb{E}^Q\left[\int_0^T dt \sigma_t^2\right] = -2 \mathbb{E}^Q\left[\ln\left(\frac{S_T}{F_{0,T}}\right)\right]$... a remarkable result!
 - The LHS is the market's expectation of the total variance to be experienced between now and T . It forms the basis of *realized variance* contracts and *variance swaps*.
 - The RHS contains the Q -expectation of the payoff of a *log contract*. We will see how to replicate (and value) such payoffs in the next set of notes.
 - Together, these form the basis for calculating the VIX index, corresponding to the square-root of the LHS expectation over the next calendar month, times an annualization factor.

4.1. Inputs (18)

⑤ Implied Volatility (continued):

- VIX calculated & published as of 1990, originally based on ATM implied vols for the S&P 100 (OEX)
 - Updated to the variance contract methodology applied to the SPX in 2003
 - Subsequently, a plethora of listed VIX futures and options have appeared



- Is σ_{imp} actually predictive of future realized vol?
- Does it contain information beyond that contained in measures of historical vol?
- Evidence is equivocal.
 - Early work (Canina & Figlewski, 1993), using ATM vols, says not really
 - Later efforts (e.g., Jiang & Tian, 2005), using measures akin to the VIX, suggest yes, albeit with positive bias.

4.1. Inputs (19)

⑤ Implied Volatility (continued):

- OTC markets operate almost entirely on the basis of σ_{imp} .
- Although reference is made to historical and forecast volatilities, and traders (in a proprietary book) may lean one way or another if implied vols are out of line with historical vols, option books are marked (to market) at implied vols.
- Especially for short maturities, split in roles between:
 - providers of liquidity in volatility/vanilla options (e.g. “locals”) and
 - consumers of volatility (vanilla options) to produce more complex products (e.g. broker-dealers)
- Notion of volatility as building block in manufacture of structured, OTC products
- (Sign of) spread between realized and implied vols typically reflects who’s providing the liquidity in the underlying volatility (vanilla options) market
- Examples of reversals:
 - Japanese insurance firms selling (naked) Nikkei puts to investment banks in the late 1980s;
 - retail single-stock structured bond market in Europe incorporating embedded (short) puts (GOAL = Geld oder Aktien Lieferung), 2000s - present.

4.1. Inputs (20)

Measurable vs. forecast vs. implied inputs

- The use of implied volatility (and our inability to perfectly forecast future realized volatility) introduces some nuance into our notions of real-world (**P**) and risk-neutral (**Q**) measures.
 - Absolute vs. relative pricing: fair (forecast) value of dynamically replicating an option vs. pricing one option relative to another;
 - Arguably, reflects market incompleteness or inability to perfectly hedge
⇒ introduction of vanilla options (helps) complete the market
 - Pricing in implied vol space usually involves some form of assumption analogous to having identified a unique EMM (equivalent martingale measure)
 - More subtly, the use of implied vols to parameterize the valuation of other (more complex) options also implies the use of the options underlying those parameters to hedge the manufactured book against changes in implied vols.
 - Does implied vol surface imply completeness of the market? Are there any unhedged risks?
- To some extent, the same notions hold true for interest rate and dividend inputs:
 - “Stripping” a series of swap rates and/or bond values to create a “riskless” yield curve of (actually stochastic) rates
 - Inferring dividends/yields from futures prices
 - To “lock in” (or hedge) rates and yields, what do we need to do?
 - Are there any residual risks?

4.1. Inputs (21)

Classification of Parameter Dependence

- Black-Scholes- (Merton-) world BS(M):
 - constant rates r , yields y , and volatilities σ
 - Technically, this is a bit of a misnomer since Merton '73 already provided for σ and r as functions of time (and a whole lot more).
 - Merton, R. C., "A Rational Theory of Option Pricing," *Bell Journal of Economics and Management Science* 4(1) (Spring, 1973), 141-183.
- BS(M)+:
 - $r(t)$, $y(t)$, and $\sigma(t)$ are known, explicit functions of time $t \leq T$ (**P** measure)
 - Equivalently, *implied* values $r(t, T)$, $y(t, T)$, $\sigma(t, T)$ are functions of (maturity) time T observed at time t (**Q** measure)
 - We will spend much of the rest of the course in this world.
- BS(M)++:
 - $r(t, \dots)$, $y(t, \dots)$, and $\sigma(t, \dots)$ are potentially stochastic functions of time t and may depend on S_t , its path, or more generally on the realization of the filtration (**P** measure)
 - Alternatively, *implied* values of r , y , and in particular $\sigma(t, T; S_t, K)$ are functions of (maturity) time T and strike K observed at time t and spot price S_t (**Q** measure)

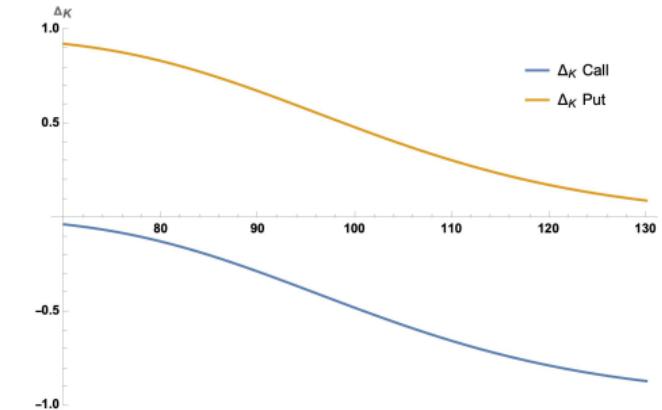
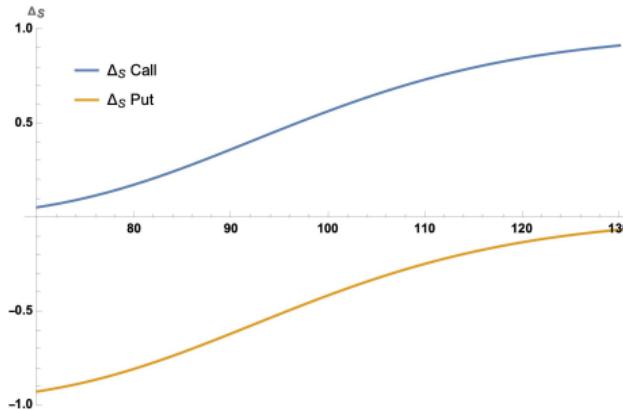
4.2. Sensitivities

- ➊ First-order in risky asset price, $S : \Delta_S$, or Strike price, $K : \Delta_K$
- ➋ First-order in riskless interest rate, $r : \rho$ or ρ_r (or ρ_B)
- ➌ First-order in (dividend or convenience) yield, $y : \rho_y$ or ρ_S
- ➍ Second-order in risky asset price, $S : \Gamma_{SS}$, or Strike price, $K : \Gamma_{KK}$
- ➎ First-order in calendar time $t : \Theta$, or time to maturity, T
- ➏ First-order in volatility, $\sigma : \nu$
- ➐ Other second (and higher-order) sensitivities...
- ➑ Base-case scenario: $S = K = 100$, $T - t = 1$ yr, $r = 4\%$ p.a., $y = 2\%$ p.a., $\sigma = 20\%$ p.a.
- ➒ (Usually) vary S for calls and puts, holding other scenario parameters constant, to examine moneyness effects.

4.2. Sensitivities (2)

- **Delta:** $\Delta = \Delta_S \doteq \partial\{C, P\}/\partial S$ and “**Dollar Delta**” = $S\Delta_S$ (dropping t for brevity):

$$\Delta_S = \partial_S \{ \phi [Se^{-yT}\mathcal{N}(\phi z_+) - Ke^{-rT}\mathcal{N}(\phi z_-)] \} = \phi e^{-yT}\mathcal{N}(\phi z_+)$$



- Also, “**Strike Delta**” = $\Delta_K \doteq \partial\{C, P\}/\partial K$ and “**Cash Delta**” $\Delta_{\$} = K\Delta_K$:

$$\Delta_K = \partial_K \{ \phi [Se^{-yT}\mathcal{N}(\phi z_+) - Ke^{-rT}\mathcal{N}(\phi z_-)] \} = -\phi e^{-rT}\mathcal{N}(\phi z_-)$$

- Equity spot convention ($\Delta = \Delta_S \doteq \partial_S$) vs. FX forward convention: $\Delta = \Delta_F \doteq \partial_{[e^{-r(T-t)}F_{t,T}]}$

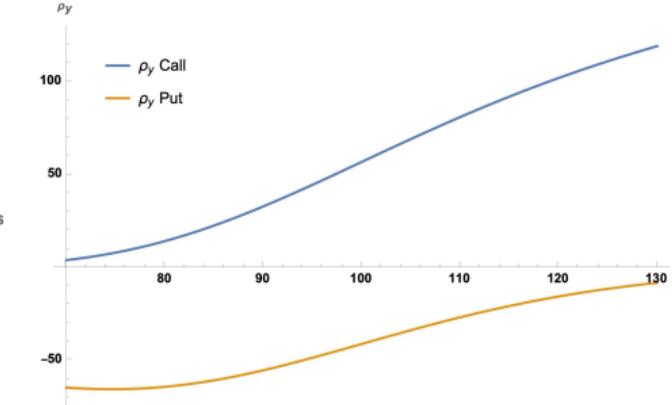
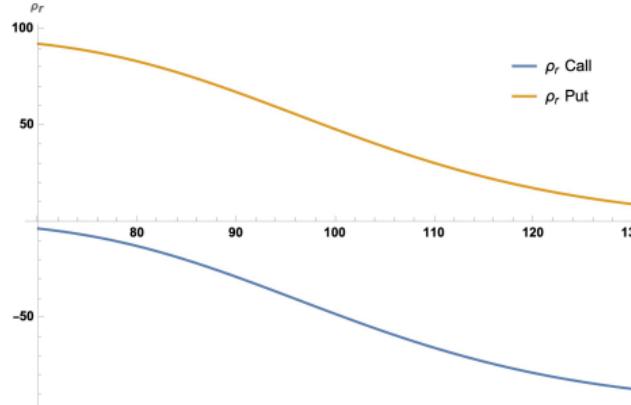
- Note that $S\Delta_S + K\Delta_K = S\Delta_S + \$\Delta_{\$} = \{C, P\}$

- Suggests vector notation \underline{S} for assets, gradient notation $\nabla_{\underline{S}}$ for Deltas, dot product for value: $C = \underline{S} \cdot \nabla_{\underline{S}} C$

4.2. Sensitivities (3)

- Ordinarily, one proceeds from Δ_S to Γ_{SS} , but there is a close relationship between deltas and rhos.
- **Interest rate rho:**

$$\rho = \rho_r \text{ (or } \rho_B) \doteq \frac{\partial \{C, P\}}{\partial r} = \partial_r \{ \phi [Se^{-yT}\mathcal{N}(\phi z_+) - Ke^{-rT}\mathcal{N}(\phi z_-)] \} = \phi K T e^{-rT} \mathcal{N}(\phi z_-)$$



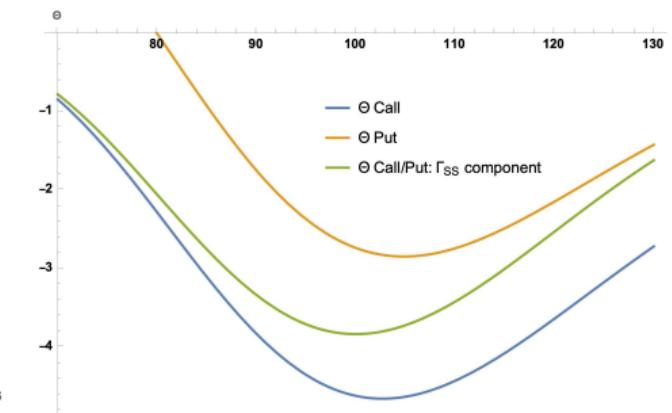
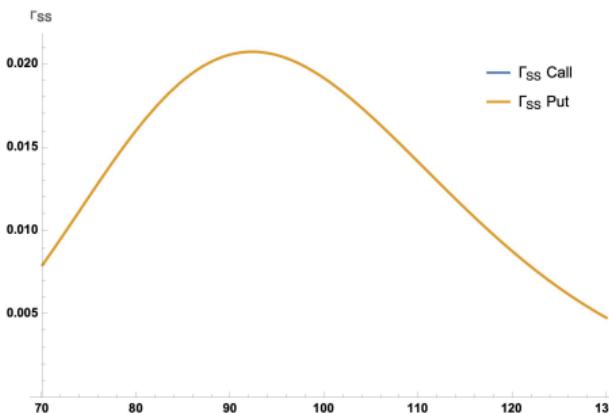
- **Yield rho** (formerly sometimes nu = ν):

$$\rho_y \text{ (or } \rho_S) \doteq \frac{\partial \{C, P\}}{\partial y} = \partial_y \{ \phi [Se^{-yT}\mathcal{N}(\phi z_+) - Ke^{-rT}\mathcal{N}(\phi z_-)] \} = -\phi S T e^{-yT} \mathcal{N}(\phi z_+)$$

- Alternatively, if we define rate/yield vector \underline{r} with, e.g., $r_0 = r$ and $r_1 = y$ (corresponding to assets in \underline{S}), then rho vector can be defined as: $\underline{\rho} \doteq \nabla_{\underline{r}} \{C, P\}$ with $\rho_i \doteq \partial \{C, P\} / \partial r_i$
- Some firms use scaled rho, e.g. $0.01 \frac{\partial \{C, P\}}{\partial r}$ or $0.0001 \frac{\partial \{C, P\}}{\partial r}$. Conventions vary...

4.2. Sensitivities (4)

- **Gamma:** $\Gamma = \Gamma_{SS} \doteq \frac{\partial^2 \{C, P\}}{\partial S^2} = \partial_S \left\{ \phi e^{-yT} \mathcal{N}(\phi z_+) \right\} = \frac{e^{-yT}}{\sigma \sqrt{T} S} n(z_+)$
 - Sometimes there are “Dollar Gammas” = $S \Gamma_{SS}$ or $0.01S \Gamma_{SS}$ (why?). Conventions vary...
 - Similarly, “**Cross Gammas**” $\Gamma_{SK} \doteq \frac{\partial^2 \{C, P\}}{\partial S \partial K} = \Gamma_{KS} \doteq \frac{\partial^2 \{C, P\}}{\partial K \partial S}$ and $\Gamma_{KK} \doteq \frac{\partial^2 \{C, P\}}{\partial K^2}$
 - This framework is most useful in a multi-asset setting. Nevertheless, we will identify an interpretation for Γ_{KK} !

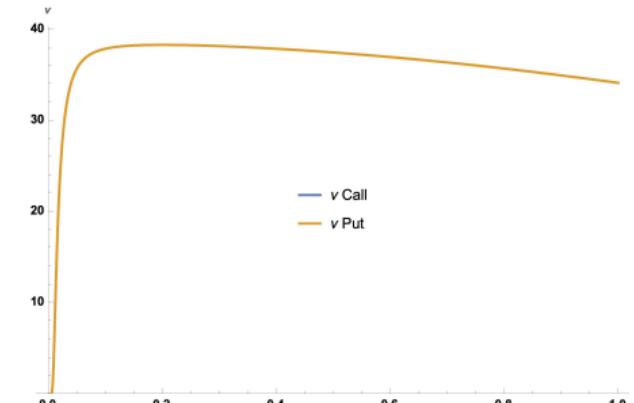
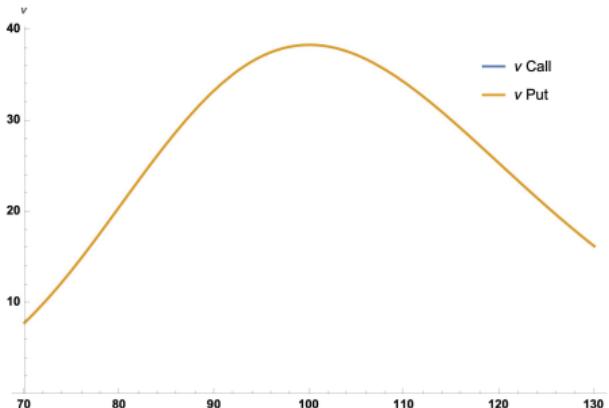


- **Theta:** θ (or Θ) $\doteq \frac{\partial\{C,P\}}{\partial t} = rK\Delta_K + yS\Delta_S - \frac{\sigma^2}{2} S^2 \Gamma_{SS}$
 - Time decay: the trader's obsession
 - Similar to (but not necessarily the same as) (-) maturity sensitivity $-\frac{\partial\{C,P\}}{\partial T}$ (why)
 - Sometimes normalized to passage of one (trading) day.

4.2. Sensitivities (5)

- Vega:**

$$\nu = \nu \doteq \frac{\partial \{C, P\}}{\partial \sigma} = \partial_\sigma \left\{ \phi [Se^{-yT}\mathcal{N}(\phi z_+) - Ke^{-rT}\mathcal{N}(\phi z_-)] \right\} = S\sqrt{T}e^{-yT}n(z_+) = K\sqrt{T}e^{-rT}n(z_-)$$



- Other terminology: tau (τ) or kappa (κ)
- Some firms use scaled vega, e.g. $0.01 \frac{\partial \{C, P\}}{\partial \sigma}$ (why?). Conventions vary...

4.2. Sensitivities (6)

From: MARK RUBINSTEIN

To: Eric Reiner

Sent: Saturday, June 17, 2017 2:42 PM

Subject: Vega

Nothing that creative here. Delta, Gamma, Beta were taken. Needed to use a Greek letter, and Vega was free. I don't think Vega is actually Greek but it sounds as if it should be. And being the name of a bright star, it had that going for it.

Fischer Black, best to my knowledge, gets full credit for delta and gamma. Perhaps I get credit for ~~vanna~~. But I wouldn't assume that was true.

Mark Rubinstein

- Other second (& higher) order sensitivities: define as needed, e.g.:

- vanna $\doteq \frac{\partial \Delta}{\partial \sigma} = \frac{\partial^2 \{C, P\}}{\partial S \partial \sigma} = \frac{\partial^2 \{C, P\}}{\partial \sigma \partial S} = \frac{\partial v}{\partial S}$

- volga = vomma $\doteq \frac{\partial v}{\partial \sigma} = \frac{\partial^2 \{C, P\}}{\partial \sigma^2}$