Weeks 4 and 5: Portfolio dynamics and Arbitrage pricing

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Anderson School of Management, MGMTMFE403, Fall 2023

I. Self financing portfolios - motivation

- Definitions
  - $N =$ number of assets
  - $h_i(t) =$ number of shares of type i held during period $[t, t + \Delta t)$
  - $h(t) = [h_1(t), ..., h_N(t)]$
  - $c(t) =$ the amount of money spent for consumption per unit of time during the period $[t, t + dt)$
  - $S_i(t) =$ price per share of type i during the period $[t, t+dt)$
  - $V(t) =$ value of the portfolio h at time t

- The value process can be expressed as

$$V(t) = \sum_{i=1}^{N} h_i(t - \Delta t) S_i(t) = h(t - \Delta t) S(t)$$

where we have used the notation $xy = \sum_{i=1}^{N} x_i y_i$

- Budget equation

$$h(t - \Delta t) S(t) = h(t) S(t) + c(t) \Delta t$$

- Using the notation $\Delta X(t) = X(t) - X(t - \Delta t)$ we can re-write this equation more compactly as

$$S(t) \Delta h(t) + c(t) \Delta(t) = 0$$

- Add and subtract $S(t - \Delta t) \Delta h(t)$ to both sides of the above equation to obtain

$$S(t - \Delta t) \Delta h(t) + \Delta S(t) \Delta h(t) + c(t) \Delta(t) = 0$$

- Taking the limit as Δt goes to zero results in

$$S(t)\, dh(t) + dS(t)\, dh(t) + c(t)\, dt = 0$$

- Since $V(t) = h(t) S(t)$, we also have by Ito's Lemma

$$dV(t) = h(t)\, dS(t) + S(t)\, dh(t) + dS(t)\, dh(t)$$

- Combining the two equations above gives

$$dV(t) = h(t)\, dS(t) - c(t)\, dt$$

- Henceforth we will call a portfolio, consumption pair (h, c) **self-financing** if the value process

$$V^h = \sum_{i=1}^{N} h_i(t) S_i(t)$$

satisfies the equation

$$dV^h = \sum_{i=1}^{N} h_i(t) \, dS_i(t) - c(t) \, dt$$

- The corresponding **relative portfolio** is defined as

$$u_i(t) = \frac{h_i(t) S_i(t)}{V(t)}$$

- By construction,

$$\sum_{i=1}^{N} u_i(t) = 1$$

- Implications:
  - A portfolio-consumption pair is self-financing if and only if

$$dV^h(t) = V^h(t) \sum_{i=1}^{N} u_i(t) \frac{dS_i(t)}{S_i(t)} - c(t) \, dt$$

  - Let c be a consumption process, and assume that there exist a scalar process Z and a vector process $q = (q_1, ..., q_N)$ such that

$$\begin{aligned} dZ(t) &= Z(t) \sum_{i=1}^{N} q_i(t) \frac{dS_i(t)}{S_i(t)} - c(t) \, dt \\ \sum_{i=1}^{N} q_i(t) &= 1 \end{aligned}$$

    Defining a portfolio h by $h_i(t) = \frac{q_i(t) Z(t)}{S_i(t)}$, the value process is given by $V^h = Z$, the pair (h, c) is self-financing, and the corresponding relative portfolio u is given by $u = q$.

- Dividends:
  - Let $D_1(t), ..., D_N(t)$ denote cumulative dividends

$$dD_i(t) = \delta_i(t)\,dt$$

  - Let the gains process be defined as

$$G(t) = S(t) + D(t)$$

  - Then the portfolio consumption pair (c, h) is self-financing if

$$dV^h(t) = \sum_{i=1}^{N} h_i(t)\,dG_i(t) - c(t)\,dt$$

II. Arbitrage Pricing

- From this point on, we specialize to a financial market that includes a single stock and a bond

- Bond

$$dB(t) = rB(t)\,dt$$

  Note that the evolution of $B(t)$ is locally deterministic. The process r is the **instantaneous interest rate**.

- Stock

$$dS(t) = \alpha(t, S(t))\,S(t)\,dt + \sigma(t, S(t))\,S(t)d\overline{W}_t$$

  Note that $S(t)$ is stochastic.

- Special case: the Black Sholes model $\alpha(t, S(t)) = \alpha$, and $\sigma(t, S(t)) = \sigma$.

- We will be pricing contingent claims in this slidework.
- Simplest contingent claim: A European Call option: $\Phi(S_T) = \max[0, S_T - K]$
- Exercise: Draw the payoff diagram for a European call option, i.e., draw $\Phi(S_T)$ as a function of S_T.
- Clearly, the price of an option at time T must equal $\Phi(S_T)$. We will write this as

$$\Pi(T) = \Phi(S_T)$$

- But how do we determine $\Pi(t)$ for $t < T$?

- We will derive the price that eliminates arbitrage opportunities. To do that we need to define an arbitrage
- An **arbitrage** possibility is a self-financed portfolio h such that

$$\begin{aligned} V^h(0) &= 0 \\ P\left(V^h(T) \geq 0\right) &= 1 \\ P\left(V^h(T) > 0\right) &> 0 \end{aligned}$$

- We will say that the market is arbitrage free if there are no arbitrage possibilities.
- We will determine the (unique) price process $\Pi(t)$ that renders the market arbitrage free.

- Here is an important result
- Suppose that there exists a self-financed portfolio h such that the value process V^h has the dynamics

$$dV^h(t) = k(t) V^h(t)dt$$

where $k(t)$ is an adapted process. Then it must be the case that $k(t) = r(t)$, or there exist arbitrage possibilities.

III. Black Scholes pricing

- Assumptions
  - The derivative security (the call option) can be bought and sold in the market
  - The market is free of arbitrage
  - The price process for the derivative security is a smooth function $F(t, S(t))$

$$\Pi(t) = F(t, S(t))$$

- Strategy
  - Form a portfolio of the stock and the derivative that is locally deterministic
  - By the absence of arbitrage, that portfolio must be yielding the instantaneous riskless rate
  - Use this observation to derive a partial differential equation for $F(t, S(t))$

- Analysis
  - Applying the Ito Formula to the price dynamics of the derivative asset,

$$dF(t) = \alpha_\pi(t) F(t) dt + \sigma_\pi F(t) d\overline{W}(t)$$

where

$$\begin{aligned}\alpha_\pi(t) &= \frac{\frac{\partial F}{\partial t} + \alpha S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS}}{F(t)} \\ \sigma_\pi(t) &= \frac{\sigma S F_S}{F(t)}\end{aligned}$$

  - Note that we used α, σ, F_S as shorthand for the more appropriate notation $\alpha(t, S_t)$, $\sigma(t, S_t)$, $F_S(t, S(t))$, etc.

- Form a portfolio of the stock and the derivative that is locally riskless
  - The value process of a portfolio investing a fraction u_s in the stock and u_π in the derivative is

$$dV = V\left\{u_s\left[\alpha dt + \sigma d\overline{W}_t\right] + u_\pi\left[\alpha_\pi dt + \sigma_\pi d\overline{W}_t\right]\right\}$$

    or

$$dV = V\left\{\left[u_s\alpha + u_\pi\alpha_\pi\right]dt + \left[u_s\sigma + u_\pi\sigma_\pi\right]d\overline{W}_t\right\}$$

  - To ensure that the portfolio is locally determinisic, we want that

$$u_s\sigma + u_\pi\sigma_\pi = 0$$

  - Moreover, by construction

$$u_s + u_\pi = 1$$

- Solving for u_s,u_π gives

$$u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma}, u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma}$$

- Using the expression for σ_π allows us to write more explicitly

$$\begin{aligned} u_s(t) &= \frac{S(t)\,F_S(t, S(t))}{S(t)\,F_S(t, S(t)) - F(t, S(t))} \\ u_\pi(t) &= \frac{-F(t, S(t))}{S(t)\,F_S(t, S(t)) - F(t, S(t))} \end{aligned}$$

- The critical step: Since the portfolio is locally riskless and has a drift equal to $u_s\alpha + u_\pi\alpha_\pi$, it must be the case

$$u_s\alpha + u_\pi\alpha_\pi = r$$

- Using the two previous expressions for u_s and u_π and α_π and re-arranging leads to the famous Black Scholes equation

$$\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2(t, S(t))\, S^2(t)\frac{\partial^2 F}{(\partial S)^2} - rF(t, S(t)) = 0$$

subject to the boundary condition

$$F(T, S_T) = \Phi(S_T)$$

IV. Risk neutral valuation

- How do we solve the Black Sholes PDE?

- Recall the Feynman Kac Theorem from last class. The solution of the partial differential equation

$$\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2(t, S(t))\, S^2(t)\frac{\partial^2 F}{(\partial S)^2} - rF(t, S(t)) = 0$$

subject to the boundary condition

$$F(T, S_T) = \Phi(S_T)$$

can be represented as a conditional expectation by specifying appropriate dynamics for $S(t)$

- To obtain that conditional expectation introduce the **auxiliary** stock process

$$dS\left(t\right)=rS\left(t\right)dt+S\left(t\right)\sigma\left(t,S\left(t\right)\right)dW\left(t\right) \tag{1}$$

Note that we introduced a new Wiener measure $W\left(t\right)$ to be clear that $S\left(t\right)$ denotes some fictitious dynamics for the process $S\left(t\right)$, not its actual dynamics

- For this auxiliary process we can express the arbitrage free price as

$$F\left(t,s\right)=e^{-r(T-t)}E_t^Q\left[\Phi\left(S\left(T\right)\right)|S_t=s\right] \tag{2}$$

  where the notation "Q''" in the expectation is meant to capture that we are not computing expectations under the actual measure, but rather using the dynamics (1) for the stock price process

- Mechanically speaking, (1) and (2) provide a very simple way to derive the price of any derivative security:
  1. Pretend that the stock price follows the dynamics (1)
  2. Using these dynamics to compute transitions densities and expectations, compute the price of any derivative security according to (2)

- As an example, let's derive the celebrated Black Scholes formula for a call option.
- Assume that σ is constant.
- Under the fictitious (sometimes called **risk-neutral measure**) Q, the dynamics of the stock are given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t)$$

- Hence $S(t)$ is log-normally distributed

$$S(T) = S_t \exp\left\{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(W_T - W_t\right)\right\}$$

- Accordingly,

$$F(t, S_t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \Phi\left(S_t e^z\right) f(z)\,dz$$

where f is the density of a random normal variable

$$N\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma\sqrt{T-t}\right]$$

- For $\Phi = \max[0, S_T - K]$ the integral becomes

$$F(t, S_t) = e^{-r(T-t)}\left(\int_{-\infty}^{\ln\left(\frac{K}{S_t}\right)} 0 \times f(z)\,dz + \int_{\ln\left(\frac{K}{S_t}\right)}^{+\infty} \left(S_t e^z - K\right) f(z)\,dz\right)$$

- After some standard calculations, we are left with the expression

$$F(t, S_t) = S_t N\left[d_1(t, S_t)\right] - e^{-r(T-t)} K N\left[d_2(t, S_t)\right]$$

where $N()$

$$\begin{aligned} d_1(t, S_t) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\} \\ d_2(t, S_t) &= d_1(t, S_t) - \sigma\sqrt{T-t} \end{aligned}$$

- This is the seminal Black Scholes Formula for a European Call option

V. Black-Scholes Pricing: Further examples

- Note that the PDE for any contingent claim Φ does not depend on the specific Φ that we choose. Only the boundary condition of the PDE changes
- Example: Derive the PDE that characterizes the price of the claims

$$\begin{aligned} \Phi^{(1)}(S_T) &= S_T, \\ \Phi^{(2)}(S_T) &= \frac{1}{S_T} \end{aligned}$$

- Solution: In all of the above cases the PDE has the same functional form

$$\frac{\partial F}{\partial t} + rS(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 F}{(\partial S)^2} - rF(t, S(t)) = 0$$

- The only thing that changes is the boundary condition. Let's start with the first claim

$$F(S_T, T) = S_T$$

- The Feynman Kac Formula implies that

$$F(S_t, t) = e^{-r(T-t)} E^Q [S_T | S(t) = S_t]$$

where under the (fictitious) probability measure Q the stock price follows the dynamics

$$dS(t) = rS(t)\,dt + \sigma(S(t), t)\,S(t)\,dW(t)$$

- The solution to the above SDE is given by

$$S(T) = S_t e^{\left(r - \frac{1}{2}\sigma^2\right)(T-t)} e^{\sigma(W_T - W_t)}$$

- Accordingly,

$$E^Q [S_T | S(t) = S_t] = S_t e^{r(T-t)}$$

- Therefore

$$F(S_t, t) = e^{-r(T-t)} E^Q [S_T | S(t) = S_t] = S_t$$

- Similarly, for the second claim. The Feynman Kac Formula implies that

$$F(S_t,t) = e^{-r(T-t)} E^Q \left[\frac{1}{S_T} | S(t) = S_t \right]$$

where under the (fictitious) probability measure Q the stock price follows the dynamics

$$dS(t) = rS(t)\,dt + \sigma(S(t),t)\,S(t)\,dW(t)$$

- Accordingly, Ito's Lemma implies that under the probability measure Q :

$$d\left(\frac{1}{S(t)}\right) = \left(\frac{1}{S_t}\right)\left(-r+\sigma^2\right)(T-t) - \left(\frac{1}{S_t}\right)\sigma dW(t)$$

- Letting $x(t)$ be given by $x(t) \equiv \frac{1}{S(t)}$, a further application of Ito's Lemma implies that $\log x(T) - \log x(t)$ is normal with mean

$$N\left[\left(-r+\frac{\sigma^2}{2}\right)(T-t), \sigma\sqrt{T-t}\right]$$

- Therefore

$$E^Q \left[\frac{1}{S_T} | S(t) = S_t \right] = \frac{1}{S(t)} e^{\left(-r+\sigma^2\right)(T-t)}$$

and

$$F(S_t,t) = \frac{1}{S_t} e^{-2r(T-t)} e^{\sigma^2(T-t)}$$

- Further examples:
  - Pricing of a forward contract
  - Pricing of an option on a forward contract

- A (long) forward contract is an aggrement to purchase the stock at time T at a price that is agreed upon at time $0 < T$. At time 0 no money changes hands.

- Question, what is the arbitrage-free price K for forward delivery?

- The payoff of a forward contract at time T is given by

$$S_T - K$$

where K is the agreed-upon price.

- To determine K, we note that the initial price of a forward contract is equal to zero:

$$F(S_0, 0) = 0$$

- Accordingly

$$0 = F(S_0, 0) = e^{-rT} E^Q (S_T - K | S(0) = S_0)$$

- We have already shown earlier that

$$E^Q (S_T | S(0) = S_0) = e^{rT} S_0$$

Therefore

$$0 = e^{-rT} E^Q (S_T - K | S(0) = S_0) = S_0 - e^{-rT} K$$

or uppon re-arranging

$$K = S_0 e^{rT}$$

- An option on a forward contract
  - forward matures at time T_1
  - European Call option on a forward contract. Option maturity date: $T < T_1$. Exercise price: K

- Payoff function at time T :

$$\begin{aligned}\Phi\left(S_T\right) &= \max\left[F\left(S_T\right) - K, 0\right] = \\ &= \max\left[S_T e^{r(T_1 - T)} - K, 0\right] \\ &= e^{r(T_1-T)} \max\left[S_T - K e^{-r(T_1-T)}, 0\right]\end{aligned}$$

- Note that this is just $e^{r(T-T_1)}$ times the payoff of a regular call option on S_T with strike price $Ke^{-r(T_1-T)}$.

- Accordingly, the price of such an option at time t is given by

$$\begin{aligned}\Pi\left(S_t\right) &= e^{-r(T-t)}\left\{e^{r(T_1-T)} E^Q \max\left[S_T - Ke^{-r(T_1-T)}, 0\right]\right\} \\ &= e^{r(T_1-T)}\left\{\underbrace{e^{-r(T-t)} E^Q \max\left[S_T - Ke^{-r(T_1-T)}, 0\right]}_{\text{Price of a European Call option with strike price } Ke^{-r(T_1-T)}}\right\}\end{aligned}$$

V. Black Sholes and Risk neutral valuation: A summary

- In this class we developed the classical approach to no-arbitrage pricing

- The basic cookbook recipe:
  - Start by postulating some dynamics for the stock of the form

$$dS_t = \alpha\left(S\left(t\right), t\right) S_t dt + \sigma\left(S\left(t\right), t\right) S_t d\overline{W}\left(t\right)$$

    and a contract function

$$\Phi\left(S_T\right)$$

  - Pretend as if the stock market follows the dynamics

$$dS_t = rS_t dt + \sigma\left(S\left(t\right), t\right) S_t dW\left(t\right)$$

  - Sometimes we call this the dynamics of S_t under the probability measure "Q"

– Then the price of the contingent claim is given by

$$\Pi\left(S_t\right) = e^{-r(T-t)} E^Q\left[\Phi\left(S_T\right)\right]$$

– It's that simple!