Lecture 4b

Deep Learning for Solving Economic Models

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Overview

A simple portfolio choice problem

- Solving for portfolio choices for investment horizon T
  - Backward recursions
  - Functional approximations and neural nets revisited

Complications

- Imposing portfolio constraints
- Many assets and the Curse of Dimensionality
  - Neural nets offer promising solution!

A Simple Portfolio Choice Problem

Motivation

We have spent a lot of time on the myopic mean-variance problem

- Solution:

$$\omega_t \propto \Sigma_t^{-1} \mu_t.$$

This assumes

1. The investor only cares about mean and variance of returns
2. The investment horizon is (effectively) one period.

Most investors do care about crash risk (beyond just variance) and have longer investment horizons

CRRA utility over terminal wealth

A common "utility function" is Constant Relative Risk Aversion

- It implies that your aversion to risk is in percentage loss terms, ie not a function of the level of your wealth
- The CRRA function is $\frac{1}{1-\gamma}W^{1-\gamma}$, where γ is risk aversion and W is wealth

Consider an investor with a T-period investment horizon

- Importantly, the investor can trade every period
- Want to maximize utility of period T wealth

$$\max E_0 \left(\frac{1}{1-\gamma} W_T^{1-\gamma} \right)$$

But, what are we maximizing over and how do we think about what W_T could be?

Asset return dynamics

First, we need the dynamics of available asset returns

- Let's make it simple – only two assets!
- Risk-free rate is constant, $R_f = 1 + r_f$
- Excess market returns are given by

$$\begin{aligned} r_t &= \mu_{t-1} + \sigma_{r\varepsilon}\varepsilon_t, \\ \mu_t &= \phi_0 + \phi_1\mu_{t-1} - \sigma_{\mu\varepsilon}\varepsilon_t + \sigma_{\mu\eta}\eta_t, \end{aligned}$$

  where ε_t, η_t are iid standard Normals.

Thus, if I have current wealth W_t, tomorrow I will have

$$W_{t+1} = W_t\left(R_f + \omega_t r_{t+1}\right),$$

where ω_t is my loading on excess market returns.

State variables and decision variables

Based on the previous slide, I at each time t have two state variables and one decision variable

State variables

- Current wealth, W_t
- Current expected excess market return, μ_t

Decision variable

- Market share in portfolio, ω_t
- Remainder, $1 - \omega_t$, invested in risk-free asset

Solving through backwards recursion: Time T

At time T, there is nothing to do but take your money and eat it

Your utility of that is

$$V_T\left(W_T, \mu_T\right) = \frac{1}{1-\gamma} W_T^{1-\gamma}$$

This is our terminal condition

- We write V_T also as a function of μ_T, although it actually is independent of μ_T as dependence on μ will enter in all earlier periods

Solving through backwards recursion: Time T-1

Consider the investor at time $T-1$:

$$V_{T-1}\left(W_{T-1},\mu_{T-1}\right)=\max_{\omega_{T-1}} E_{T-1}\left(V_T\left(W_T,\mu_T\right)\right),$$

where $V_T\left(W_T,\mu_T\right)\equiv\frac{1}{1-\gamma}W_T^{1-\gamma}=\frac{1}{1-\gamma}W_{T-1}^{1-\gamma}\left(R_f+\omega_{T-1}r_T\right)^{1-\gamma}$.

- So, V_{T-1} is your utility as a function of W_{T-1} and μ_{T-1}, conditional on making the optimal choice ω_{T-1} next period
- V is the *value function*

Solving through backwards recursion: Time T-2

Let's go back to $T-2$

- Choose ω_{T-2} to maximize next period utility under assumption we make optimal choice ω_{T-1}

$$V_{T-2}\left(W_{T-2},\mu_{T-2}\right)=\max_{\omega_{T-2}} E_{T-2}\left(V_{T-1}\left(W_{T-1},\mu_{T-1}\right)\right),$$

where $W_{T-1}=W_{T-2}\left(R_f+\omega_{T-2}r_{T-1}\right)$,
$\mu_{T-1}=\phi_0+\phi_1\mu_{T-2}-\sigma_{\mu\varepsilon}\varepsilon_{T-1}+\sigma_{\mu\eta}\eta_{T-1}$, and $r_{T-1}=\mu_{T-2}+\sigma_r\varepsilon_{T-1}$

And so on until we get to today's time $t=0$...

$$V_0=\max_{\omega_0} E_0\left(V_1\right)$$

subject to the transition equations for wealth and μ

- Note, we now have optimal portfolio choice today!

Getting rid of wealth

Since risk aversion is constant for returns (not $'s), wealth can be dropped from the optimization problem.

- Thus, without loss of generality we ignore wealth when it comes to the first order condition for finding optimal $\{\omega_t\}_{t=1}^{T}$

We can write:

$$V_t(W,\mu) = \frac{W^{1-\gamma}}{1-\gamma} J_t(\mu),$$

where $J(\mu_t)$ is not a function of wealth.

- The first-order condition:

$$\frac{\partial E_t(V_{t+1})}{\partial \omega_t} \propto E_t\left(r_{t+1}(R_f + \omega_t r_{t+1})^{-\gamma} J_{t+1}(\mu_{t+1})\right) = 0$$

  where we have used $W_{t+1} = W_t(R_f + \omega_t r_{t+1})$.

- Wealth W_t features multiplicatively here (in the proportionality $\propto$), so doesn't affect solution for ω_t setting FOC to zero

Final backwards recursion (1 out of 3)

1. Choose a grid for μ_t
   - We have $Var\left(\mu_t\right) = \frac{\sigma^2_{\mu\varepsilon} + \sigma^2_{\eta\mu}}{1 - \phi_1^2}$.
   - Thus, ± 3 st.dev. bounds are good, perhaps with $N = 20$ grid points

We will solve for optional ω_t at each time t recursively

- Save these optimal values on the same grid as μ

Final backwards recursion (2 out of 3)

2. Start at $T-1$. For each value of μ_{T-1} find optimal ω^*_{T-1} using

$$E_{T-1}\left(r_T\left(R_f+\omega_{T-1}r_T\right)^{-\gamma}J\left(\mu_T\right)\right)=0,$$

where $J\left(\mu_T\right)=1$ and $r_T=\mu_{T-1}+\sigma_r\varepsilon_T$

- Solve expectation numerically (e.g., Monte-Carlo) for each value on the grid for μ using a guess for ω_{T-1}
- Use an optimizer over ω_{T-1} to find where expectation equals zero

3. Find $J_{T-1}\left(\mu\right)$ as value function if wealth $=1$ (arbitrary number)

$$J_{T-1}\left(\mu\right)=E_{T-1}\left(V^*_T\right),$$

where $V^*_T=\frac{1}{1-\gamma}\left(R_f+\omega_{T-1}r_T\right)^{1-\gamma}$

- Create a neural net $f_{T-1}\left(\mu\right)$ to fit $J_{T-1}\left(\mu\right)$ using the points solved for on the μ-grid

Final backwards recursion (3 out of 3)

4. Go to $T-2$. Solve for ω^*_{T-2} for each value of the μ-grid

$$E_{T-2}\left(r_{T-1}\left(R_f+\omega_{T-2}r_{T-1}\right)^{-\gamma}f_{T-1}\left(\mu_{T-1}\right)\right)=0,$$

   using $r_{T-1}=\mu_{T-2}+\sigma_{r\varepsilon}\varepsilon_{T-1}$ and
   $\mu_{T-1}=\phi_0+\phi_1\mu_{T-2}-\sigma_{\mu\varepsilon}\varepsilon_{T-1}+\sigma_{\mu\eta}\eta_{T-1}$

5. Find $J_{T-2}(\mu)$ using for each value on μ-grid

$$J_{T-2}(\mu)=E_{T-2}\left(\frac{1}{1-\gamma}\left(R_f+\omega^*_{T-2}r_{T-1}\right)^{1-\gamma}f_{T-1}\left(\mu_{T-1}\right)\right).$$

6. And so on until $t=0$.
   Now we have $\{\omega^*_t(\mu)\}_{t=0}^{T-1}$ and $\{f_t(\mu)\}_{t=0}^{T}$.

Homework 4

In this homework, you will solve such portfolio problems

We will add constraints on ω^* e.g., no short-sales, leverage constraints, VaR constraint

Here: Neural net is a bit of overkill for function fitting

- However, if you have a cross-section of assets, all with their own μ_i-dynamics, the state space becomes very large and deep learning does very well in approximating such functions

Practical use:

- Any investment decision for long-horizon investors
- Creating financial advisor AI-agents