Weeks 5 and 6: Completeness and Hedging

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I. Reachable Payoffs

II. Replicating Portfolio

III. Delta-Hedging and the Greeks

IV. Volatility Smiles and Skew

V. Put-Call Parity

I. Reachable payoffs

- Sofar, we have behaved as if the contingent claim is already tradable
- In many situations the contingent claim may not be tradeable and we need to come up with a price
  - Suppose for instance that a client approaches an investment bank and requests an unusual contingent claim
  - The investment bank wants to know the minimum price it should charge

- The approach we take in this section
  - Look for a (dynamic) portfolio of the bond and the stock that will exactly replicate the contingent claim's payoffs in all contingencies
  - The initial setup cost of that portfolio should equal the price of that contingent claim in a frictionless market

- One additional benefit of this lecture is that we will re-derive Black-Scholes pricing in a way that provides a link to our very first lecture (binomial pricing).

- We say that a claim with payoff χ can be **replicated** if there exists a self-financing portfolio h such that

$$V^h(T) = \chi$$

- In this case we say that h is the **replicating portfolio** of claim χ.

- If every claim can be replicated, then we will say that the **market is complete**.

- If a claim can be hedged by some portfolio h, then the only price which is consistent with an arbitrage-free market is

$$\Pi(t;\chi) = V^h(t)$$

- Why?

- Fact: The Black Scholes model, i.e., the assumptions on bond and stock dynamics

$$\begin{aligned} dB(t) &= rB(t)\,dt \\ dS(t) &= \alpha(S(t),t)\,S(t)\,dt + \sigma(S(t),t)\,S(t)\,d\overline{W}_t \end{aligned}$$

where $\sigma(S(t),t) > 0$, is complete.

- We will postpone proving this very re-assuring in its full generality until a later class
- Today, we will only show this result for claims that take the form

$$\chi = \Phi(S_T)$$

- Specifically, we will take an arbitrary function $\Phi(S_T)$ and show how to construct a self-financing portfolio $\left[h^0(t), h^*(t)\right]$ with the property

$$V^h(T) = h^0(T)\,B(T) + h^*(T)\,S_T = \Phi(S_T)$$

- Mostly for notational convenience, let's recall a result from a previous lecture
- Suppose that there exists an adapted process V and an adapted process $u = \left[u^0(t), u^*(t)\right]$ with

$$u^0(t) + u^*(t) = 1,$$

such that

$$\begin{aligned} dV(t) &= V(t)\left\{u^0(t)\,r + u^*(t)\,\alpha(t, S(t))\right\}dt + V(t)\,u^*(t)\,\sigma(S(t),t)\,d\overline{W}(t) \\ V(T) &= \phi(S(T)) \end{aligned}$$

Then the claim $\chi = \Phi(S_T)$ can be replicated using u as the relative portfolio, and the corresponding absolute portfolio is

$$\begin{aligned} h^0(t) &= \frac{u^0(t)\,V(t)}{B(t)} \\ h^*(t) &= \frac{u^*(t)\,V(t)}{S(t)} \end{aligned}$$

II. Replicating portfolio

- Suppose that we can write the value process as a function of time and S_t

$$V(t) = F(t, S(t))$$

where $F()$ is unknown at this point.

- Applying Ito's Lemma gives

$$dV_t = \left\{ \frac{\partial F}{\partial t} + \alpha S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \right\} dt + \sigma S F_S d\overline{W}_t$$

where we have used the short-hand notation α for $\alpha(t, S(t))$, σ for $\sigma(t, S(t))$, F_S for $\frac{\partial F}{\partial S}$ etc.

- Re-writing

$$dV_t = V_t \frac{\left\{ \frac{\partial F}{\partial t} + \alpha S F_S + \frac{1}{2}\sigma^2 S^2 F_{SS} \right\}}{V_t} dt + \sigma V_t \frac{S F_S}{V_t} d\overline{W}_t$$

Recall that to identify the reprlicating portfolio, we want to find u^0 and u^* such that $u^0 + u^* = 1$ and

$$dV_t = V_t (u_0 r + u^* \alpha) dt + \sigma V_t u_t^* d\overline{W}_t$$

- where once again we have used the shorthand notation u_0 instead of $u_0(t, S(t))$, u^* for $u^*(t, S(t))$.
- Matching the diffusion terms of the above two equations gives

$$u^* = \frac{SF_S}{V_t},$$

- Matching drift terms of the above two equations gives (after some obvious cancellations)

$$u_0 = \frac{\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 F_{SS}}{rV_t}$$

- Finally, imposing $u_0 + u^* = 1$ along with $V_t = F(t, S(t))$ leads to

$$\frac{\partial F}{\partial t} + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} - rF_t = 0 \qquad (1)$$

- This is just the familiar Black-Sholes equation.

- So we have proved the following result: Suppose that F satisfies the PDE (1) subject to the boundary condition $F(S_T, T) = \Phi(S_T)$. Then the relative portfolio u_0, u^*, or equivalently the self-financing absolute portfolio

$$h^* = F_S(t, S(t)), h = \frac{F(t, S(t)) - S(t) F_S(t, S(t))}{B(t)}$$

leads to a value process that satisfies $V^h(t, S(t)) = F(t, S(t))$ and by implication $V^h(T, S(T)) = \Phi(S_T)$

- What have we learned?
  - We re-derived the Black Sholes formula, but without assuming that the underlying derivative is traded
  - Much more importantly: We now have a specific prescription of how to compute the replicating portfolio of an aribitrary claim.

III. Delta-Hedging and the Greeks

- Let C_t denote the price of a European Call and P_t the price of a put.
- The **"Greeks"** indicate how the option price changes when the underlying price (S_t) or other parameters (τ, σ, r) change:

Change in	**Name**	**Call**	**Put**
S_t	Δ (Delta)	$\Delta^C = \frac{\partial C}{\partial S_t} \in [0,1]$	$\Delta^P = \frac{\partial P}{\partial S_t} \in [-1,0]$
τ	Θ (Theta)	$\Theta^C = \frac{\partial C}{\partial \tau} > 0$	$\Theta^P = \frac{\partial P}{\partial \tau} > 0$
σ	ν (Vega)	$\nu^C = \frac{\partial C}{\partial \sigma} > 0$	$\nu^P = \frac{\partial P}{\partial \sigma} > 0$
r	ρ (Rho)	$\rho^C = \frac{\partial C}{\partial r} > 0$	$\rho^P = \frac{\partial P}{\partial r} < 0$
S_t	Γ (Gamma)	$\Gamma^C = \frac{\partial \Delta^C}{\partial S_t}$	$\Gamma^P = \frac{\partial \Delta^P}{\partial S_t}$

- E.g. if time until expiration τ increases by 0.1 (years), then C_t increases by approximately $\Theta^C \times 0.1$ (everything else equal)
  - $\Theta^C > 0$: an option with longer maturity is more valuable because it has more time to expire in the money

- **Delta** is the best known
  - It measures the dollar change in the value of the option for a \$1 change in the value of the stock
  - The same delta as in the binomial pricing model!
- Using the Black–Scholes formula, we obtain the formula

$$\boxed{\Delta = \frac{\partial C}{\partial S} = N(d_1)}$$

- Delta always lies between 0 and 1
- **Delta Hedging:** You can hedge away the risk of a call option: For every call option you are long, sell Δ shares of stock. Now if S_t goes up by \$1, C_t also goes up by $\Delta \times \$1$, and your portfolio value stays approximately constant.

- **Example:** Suppose you hold a call option on IBM, but you don't want to bear stock price risk. Suppose that $N(d_1) = 0.635$.
  - How much of the stock should you buy/sell?
- **Answer:** Suppose you are long 1000 calls (10 lots). Then you should short $1000 \times N(d_1) = 635$ shares of IBM
- Suppose IBM goes up by \$1. Then C_t goes up *approximately* by $\$\Delta$. How does your portfolio change?
  - Short position in IBM: lose $-635 \times \$1 = -\$635$
  - Long position in call: gain $1000 \times \$\Delta = \$635$
- You are therefore immune to changes in the price of IBM: you are **delta-hedged!**

- The same idea works for the put option. The put option delta is

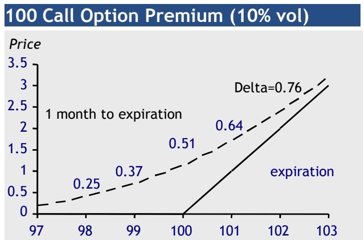
$$\begin{aligned} \Delta^P &= \partial P/\partial S = N(d_1) - 1 = \Delta - 1 \\ &= 0.635 - 1 = -0.365 \end{aligned}$$

  - Since the call option Δ is always between 0 and 1, Δ^P always lies between -1 and 0

Properties of Delta

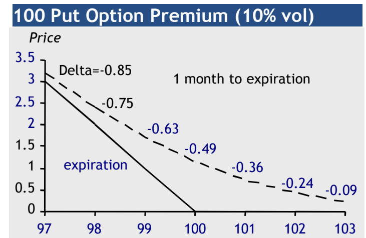
Call Deltas

- increase as the underlying price increases
- can be interpreted as the probability that the option will finish in the money



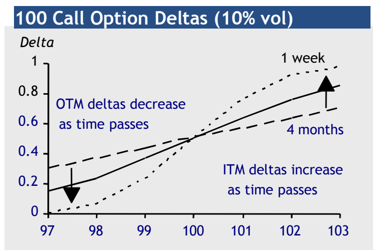
Put Deltas

- increase as the underlying price increases
- can be interpreted as -1 times the probability that the option will finish in the money



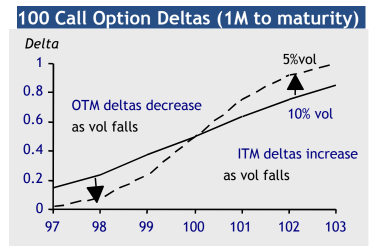
As time passes,

- the delta of ITM options increases
- the delta of OTM options decreases



As volatility falls,

- the delta of ITM options increases
- the delta of OTM options decreases



Understanding Gamma

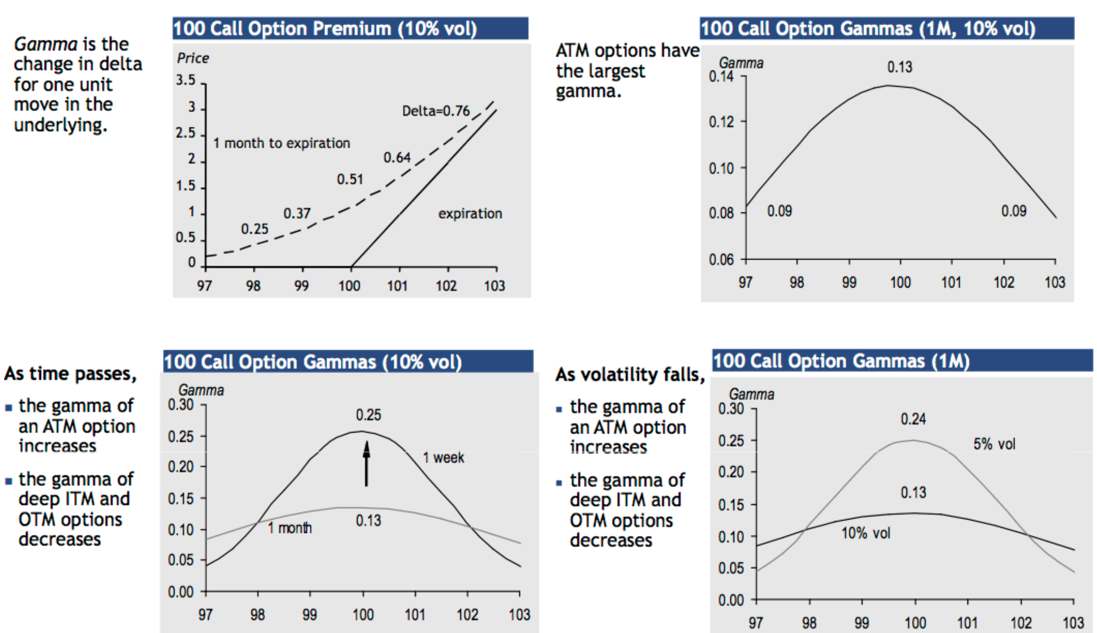
- Gamma is the change in delta for a $1 move in the underlying
  - Hence it is a *second* rather than a first derivative
- If you delta-hedge a long option position, you benefit from changes in delta, in both directions
  - Consider an ATM call

		t=1 (dP < 0)		t=1 (dP > 0)	
	t=0	dP = large	dP = small	dP = small	dP = large
Stock price	100	98	99	101	102
Option delta	0.51	0.25	0.37	0.64	0.76
Delta hedge:	short 0.51 stocks				
Chg in option price		-0.88	-0.51	0.51	1.15
Chg in delta hedge		1.02	0.51	-0.51	-1.02
Total chg in portfolio		0.14	0.00	0.00	0.13
Chg in option price (alt*)		-0.76	-0.44	0.58	1.27
Total chg in portfolio (alt*)		0.26	0.07	0.06	0.25

*Alt: use average (δ_0, δ_1) for option price change

- Because the changes in delta always benefit a long options position, gamma is sometimes called positive convexity, or right-way risk
- You make gamma gains whenever: *1)* you delta-hedge at non-continuous increments (i.e. moves in the underlying are large enough to cause delta to change), and *2)* transactions costs to hedging are small
- Gamma gains are larger when realized volatility is high (large move in underlying $\Longrightarrow$ large move in δ)

Properties of Gamma



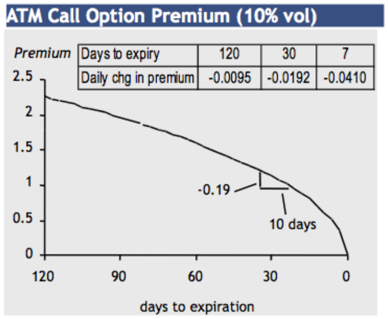
Theta = Gamma rent

- Gamma sounds great. But it doesn't come free
- Theta = time decay = the cost you pay to enjoy gamma
- Every day that passes, the value of an option declines a little bit
  - If the move in the underlying is large enough, it will generate enough gamma gains to outweigh theta losses, and you will profit on that day
  - If the move in the underlying is not large enough, it will not generate enough gamma gains to outweigh theta losses, and you will lose money on that day

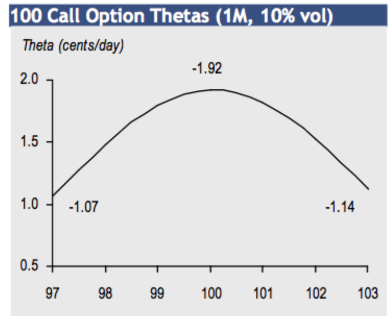
- Another way to look at option prices:
  - Intrinsic value = the amount by which the option is in-the-money
  - Time value = the difference between the price of an option and the intrinsic value of an option
    - For example, suppose the underlying price = 102. Then a 100 strike call with a price of 2.42 has an intrinsic value of 2 and time value of 0.42
  - The higher the time value, the higher the theta and the faster the option price will decay (towards intrinsic value) over time
  - High theta $\Longleftrightarrow$ high gamma $\Longrightarrow$ extreme sensitivity to realized volatility over the remaining life of the option
- If you are short options (long options), you hope for quiet markets (volatile markets) so you can earn theta (earn gamma)

Properties of Theta

Theta is the change in the value of an option with one day's passage of time.

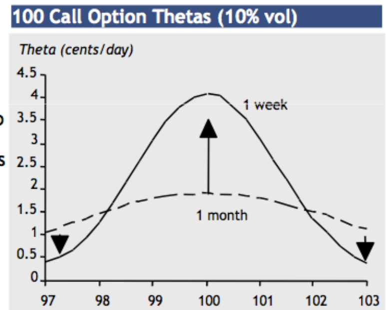


ATM options have the largest time value and therefore the largest theta.



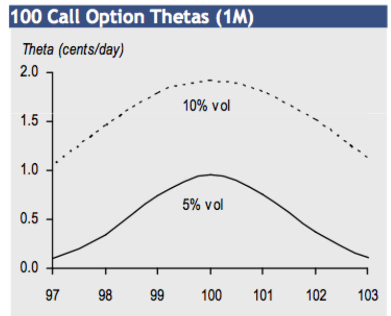
As time passes,

- the theta of an ATM option increases
- the theta of deep ITM and OTM options decreases



As volatility falls,

- time value declines
- theta declines



Implied Volatility

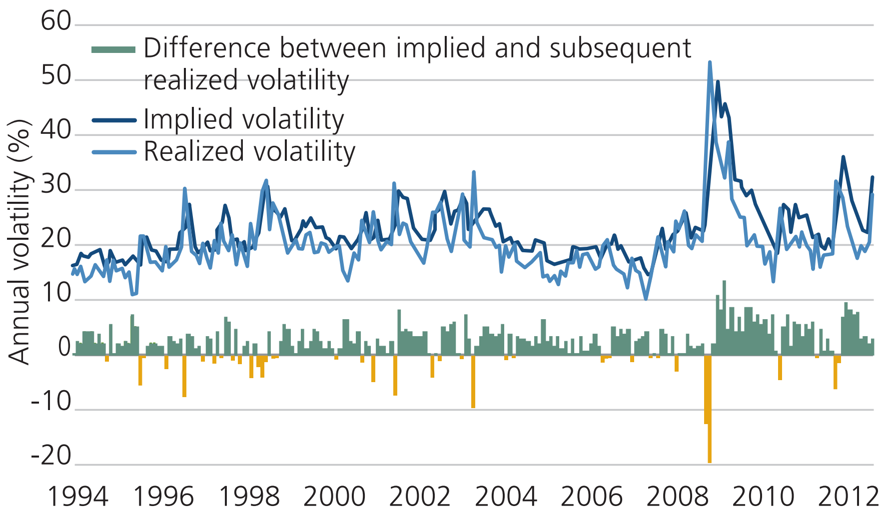
- The Black–Scholes formula shows that option price depends on five parameters
  - S, K, r, τ, and σ
- The first four of these are directly observable
  - And so is the option's price
- Then we can use the market price to "back out" the volatility σ that the market is using to price the option
  - The **implied volatility** is the value of σ that solves

$$\begin{aligned}\text{Market call price} &= N(d_1)\, S_t - N(d_2)\, Ke^{-r\tau} \\ &= C_{\mathrm{B-S}}(S_t, K, r, \tau, \sigma)\end{aligned}$$

- Implied volatilities are widely used, e.g., VIX is a traded index of implied volatility on the S&P 500 using 30-day options

- Realized and implied volatility:
  - Implied volatilities are on average higher than subsequent realized volatilities (the **volatility premium**)
  - This is because, if you are short options, you will lose much more on a bad day than you will make on a good day. Investors demand risk premium (higher option prices) to be short options
  - Implied volatility generally tracks realized volatility quite closely, and spikes during a crisis
  - Note on gamma and theta: the implied volatility is the level of volatility which, if realized over the life of the option, ensures that delta-hedged profits equal 0, or in other words, gamma gains exactly equal theta losses

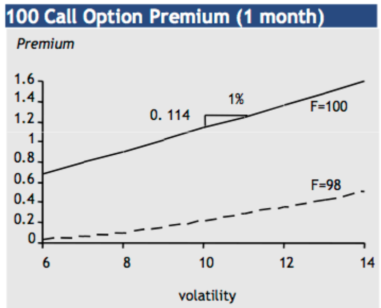
FIGURE 1: IMPLIED VOLATILITY IS TYPICALLY HIGHER THAN SUBSEQUENT REALIZED VOLATILITY (AS SHOWN BY GLOBAL EQUITY, INTEREST RATE, COMMODITY AND CURRENCY MARKETS), (MAY 1994 TO JUNE 2012)



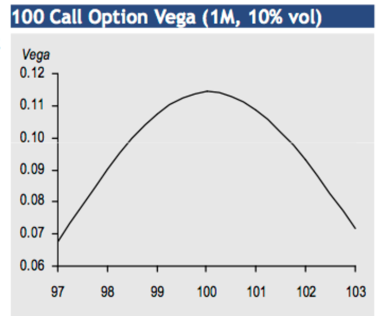
Source PIMCO, Viewpoint, Sept. 2012

Properties of Vega

Vega is the change in the price of an option for a one percentage point increase in implied volatility.

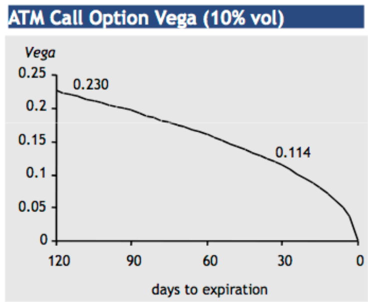


ATM options have the largest vega.



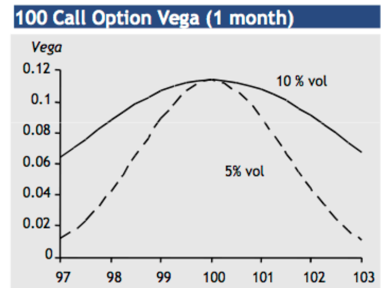
As time passes,

- vega decreases



As volatility falls,

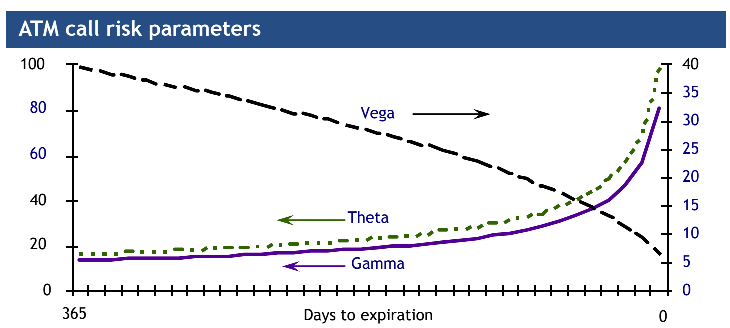
- vega is unchanged for ATM options
- vega decreases for ITM and OTM options



The Greeks over time

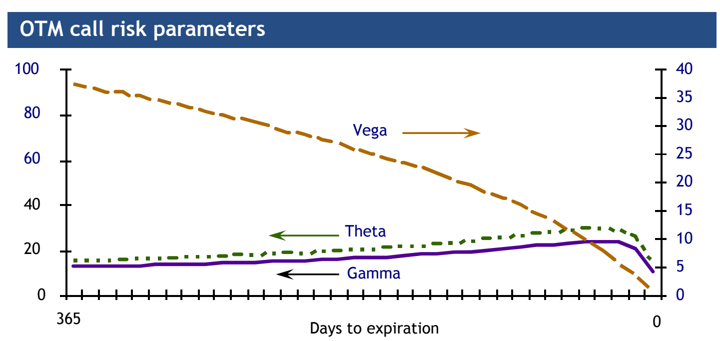
Short-dated options have higher gamma (and theta)

Short-dated options have greater gamma, which produces profit primarily from the difference between the time decay on the options and the profit or loss from moves in the underlying market. Of course, along with greater gamma comes greater theta.



Long-dated options have higher vega

Long-dated options have greater vega, which produces profit from changes in option prices due to implied volatility changes.



IV. Volatility Smiles and Skew

- **Problem:** In the previous example find the implied volatilities corresponding to all the call options on IBM expiring in July

Strike	65	70	75	80	85	90	95	100	105	110	115	120
Price	58.85	53.65	49.05	43.80	39.05	33.95	29.15	24.35	19.80	15.30	11.25	7.55
Strike	125	130	135	140	145	150	155	160	165	170	175	180
Price	4.55	2.40	1.14	0.51	0.24	0.13	0.09	0.08	0.06	0.07	0.05	0.05

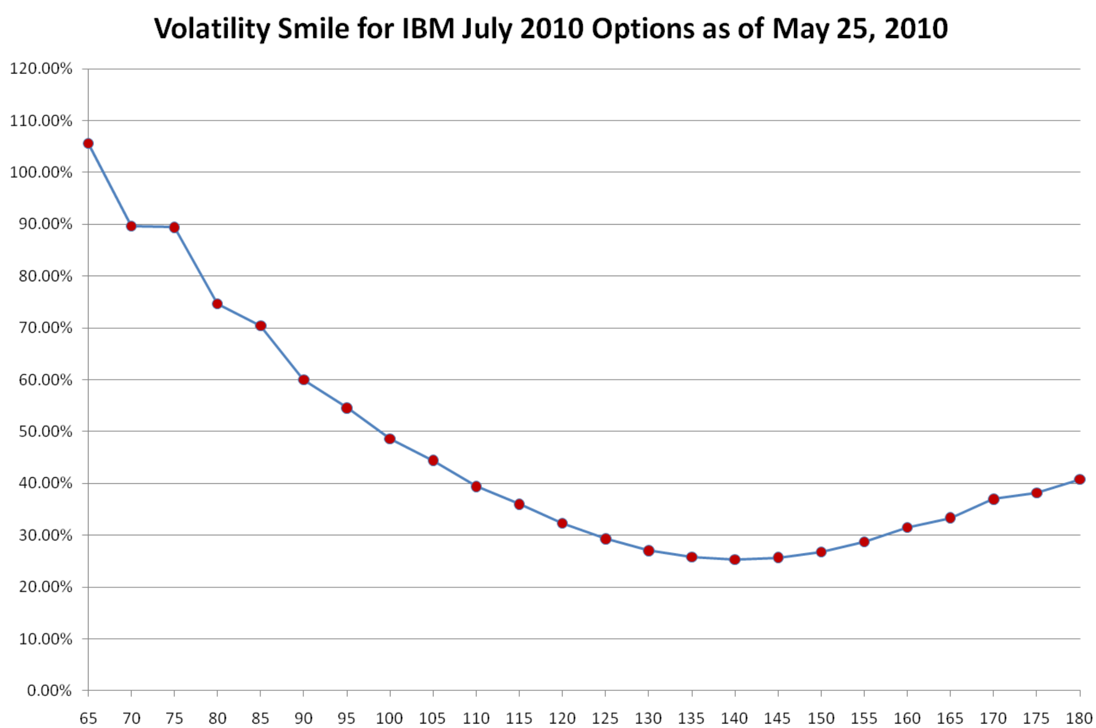
- **Solution:** E.g. to find the σ_{implied} for $K = 120$, solve

$$\begin{aligned} 7.55 &= C_{\text{B-S}}(S_t\ ,\ K\ ,\ r\ ,\ \tau\ ,\ \sigma_{\text{implied}}) \\ &= C_{\text{B-S}}(123.02\ ,\ 120\ ,\ 0.00165\ ,\ 0.1425\ ,\ \sigma_{\text{implied}}) \end{aligned}$$

(can use the MATLAB function `blsimpv`(S, K, r, τ, C))

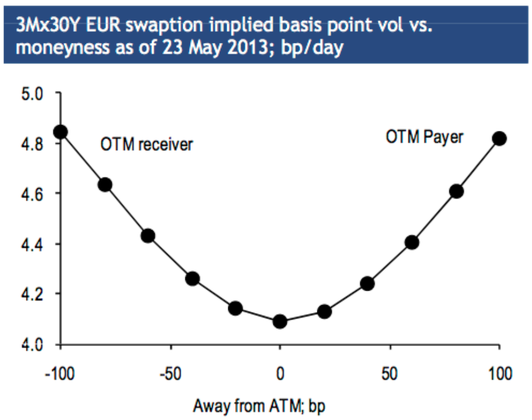
- We get these implied volatilities $\Longrightarrow$ **volatility smile/smirk**

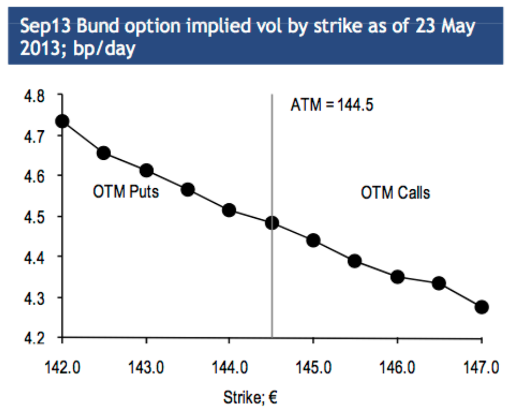
K	65	70	75	80	85	90	95	100	105	110	115	120
C_{market}	58.85	53.65	49.05	43.80	39.05	33.95	29.15	24.35	19.80	15.30	11.25	7.55
σ_{implied} (%)	105.62	89.60	89.40	74.63	70.45	60.04	54.57	48.57	44.44	39.43	36.00	32.28
K	125	130	135	140	145	150	155	160	165	170	175	180
C_{market}	4.55	2.40	1.14	0.51	0.24	0.13	0.09	0.08	0.06	0.07	0.05	0.05
σ_{implied} (%)	29.32	27.04	25.77	25.30	25.68	26.79	28.77	31.54	33.36	36.96	38.16	40.78



- In a **smile**, the lowest implied volatility typically occurs close to the at-the-money strike.

- In a **skew**, OTM calls trade at lower (or higher) implied volatility than OTM puts.





V. Put-Call Parity

- A relationship between the price of a *European* put and a *European* call, with the *same* strike price
- Compare the following two portfolios

  1: Buy a European call option with strike price K and invest $e^{-rT}K$ in T-Bills. Cost $= C_0 + e^{-rT}K$

  2: Buy the underlying stock at its current price $S = S_0$ and buy a put option on the stock with strike K. Cost $= P_0 + S_0$

- Now hold both portfolios until expiration
  - What are the final payoffs of these two portfolios?

- The initial costs and the payoffs at T are

Portfolio	**1**	**2**	$\mathbf{1-2}$
Initial cost	$C_0 + e^{-rT}K$	$S_0 + P_0$	?
Payoff if $S_T > K$	$(S_T - K) + K$	S_T	0
Payoff if $S_T = K$	K	K	0
Payoff if $S_T < K$	K	$S_T + (K - S_T)$	0

- Future cash flows are the same $\Longrightarrow$ *Today's* cost must be identical. We get the **Put–Call Parity** formula

$$\boxed{C + e^{-rT}K \;=\; P + S}$$

- Caution: Put-call parity only holds for European calls and puts with the same strike price K