MGMT MFE 406 – Derivative Markets (4 units)

Part 1: Derivatives Markets Overview

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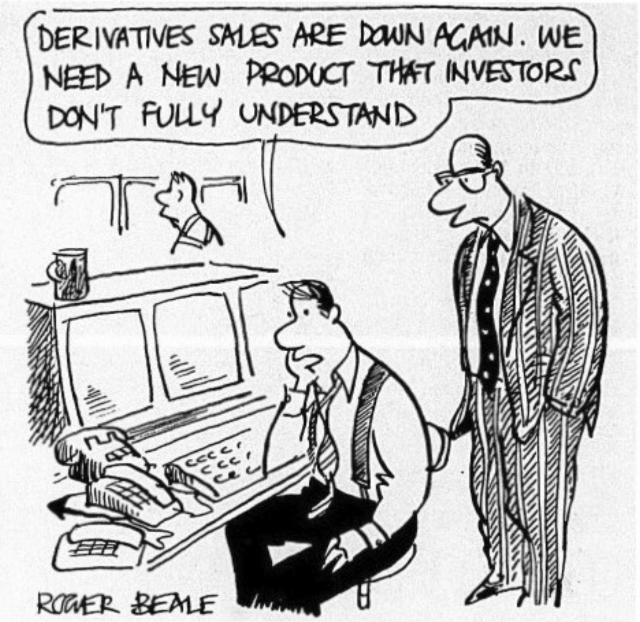
Outline

1. Definitions, History, Classification, and Statistics



1. Definitions, History, Classification, and Statistics (2)

What is a derivative security?

- Derivatives *derive* their value from the value of some other security, commodity, or entity
- In the most basic sense, a derivative is a contract whose payoff depends on the market value of another asset, e.g.:
  - A call option on AAPL, maturing in February of 2025, with a strike of $240
  - A futures contract on gold, expiring in April of 2026
  - A two-year swap with 6-month legs (resets), exchanging a specified fixed interest rate (e.g., 6%) for a floating rate (e.g., LIBOR)
- In a broader sense, a derivative (or *contingent claim*) is a contract whose payoff depends on the realization of some random variable, e.g.:
  - A convertible bond, allowing the holder to exchange a given set of risky bond cash flows for a number of the issuing company's shares
  - A weather derivative, paying off based on the realized temperature profile at a given location
  - An MBS tranche, providing a claim to some portion of the cashflow waterfall of a pool of housing loans
  - A structured note, with interest payments and principal risk linked to the performance of the SPX
  - The decision to operate a gold mine or invest in a project
  - The value of a firm's equity, considered relative to the firm's assets and strategy
- One can have much fun with the definition's wording, e.g., Peter Carr's "Deriving Derivatives of Derivative Securities"

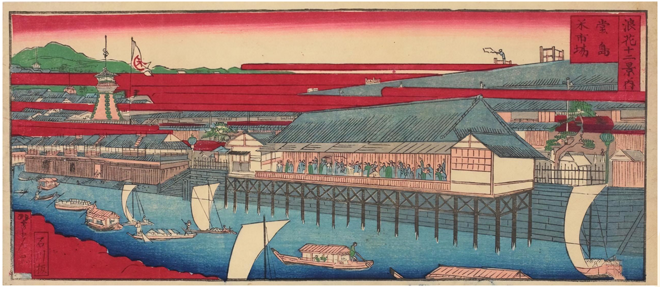
1. Definitions, History, Classification, and Statistics (3)

Why Derivatives?

- By allowing us to access portions of assets' returns distributions (in an Arrow-Debreu sense), derivatives allow us to achieve a richer set of risk profiles
  - Upside, downside, or other aspects of an asset's price/return at some particular time (options)
  - Inter-temporally (forwards and futures)
  - Derivatives help make markets "more complete"
- These characteristics lead to a variety of potential uses and abuses
  - Hedging
  - Investment
  - Speculation
- Derivatives concepts have come to pervade much of modern finance and financial markets, from fixed income, to ABS, to corporate finance, to asset management.

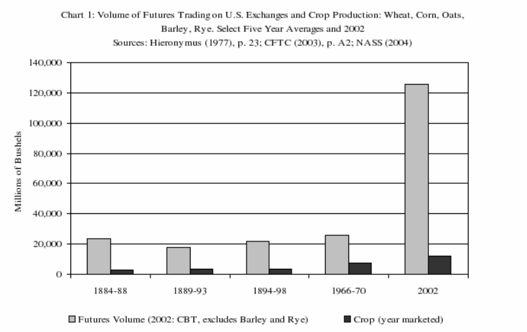
1.1. Derivatives History

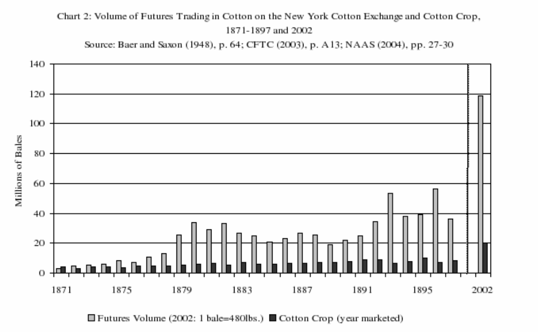
- Contracts resembling forwards and futures can be traced back to ancient history
  - Evidence for forward contracting of olive presses, grain mills, etc., traceable to ancient Mesopotamian, Greek and Roman periods
  - Repurchase agreements for *chametz* around Passover, dating to Talmudic times
  - Commodity and trade goods (grain, spices, ..., herrings!) – and eventually share – forwards began to appear in Europe in the middle ages
    - Antwerp (16th century), Amsterdam (17th century), and London (18th century).
    - Mark-to-market and collateralization features also emerged.
  - Dôjima exchange rice futures during the Tokugawa period (early-mid 1700s)



1.1. Derivatives History (2)

- In the US, organized trading of forwards can be traced to grain contracts on the CBOT (Chicago Board of Trade) as of ca. 1851.
  - By 1865, these had taken on characteristics of futures contracts (standardized contract terms; margin and settlement procedures; dispute resolution)
  - Extended to cotton contracts in New York and New Orleans during the 1850s to 1880s





1.1. Derivatives History (3)

- Futures trading expanded beyond physical commodities in the 1970s and 1980s: currency futures in 1972; interest rate futures in 1975; and stock index futures in 1982
- Arguably driven by deregulation and increased volatility in key markets, e.g.:
  - Floating exchange rates since Bretton Woods collapse in March 1973
  - Creation of OPEC in 1973, leading to higher variability in oil prices
  - Greater volatility in US interest rates following 1970s stagflation
  - Natural gas market gradually deregulated since 1978

1.1. Derivatives History (4)

- A similarly long history exists for options contracts
  - Thales of Miletus (c. 624–620 BCE∼548–545 BCE) supposedly paid the owners of olive presses a sum of money each in order to secure the rights to use them at harvest time.
  - Options on commodities and shares grew out of the organized markets in Amsterdam and London
    - Calls and puts played a role in the tulip mania of the 1630s and subsequent market collapse
    - In 1720, subscription purchase agreements for South Seas company stock were essentially installment (compound) options or mortgages to purchase the shares
  - Informal (OTC) stock option markets began shortly after the opening of the NYSE in 1791
  - Equity puts and calls with 30, 60, and 90 day maturities were listed in newspapers in the 1870s. The contracts read like the following examples:

    The bearer may **call** *on the undersigned for one hundred shares of Atlantic and Pacific Telegraph Company capital stock, at 19 1/2, at any time within 30 days from date.*

    The bearer may **put** *or deliver to the undersigned one hundred shares of Toledo and Wabash common stock, at 22, any time within thirty days from date.*
  - The transformational event in option history was arguably CBOT's spin-off of the CBOE (in a smoking lounge) and the establishment of the Options Clearing Corp (OCC), both in 1973
    - The nearly simultaneous publication of the Black and Scholes paper was, perhaps, a fortuitous coincidence
    - In 1983, the first index options were traded on the CBOE 100 index – later renamed the S&P 100 (OEX).
    - Four months later, options began trading on the S&P 500 index (SPX).

1.2. Derivatives Classification

Derivatives can be classified across a number of different dimensions

- By underlying
  - Equities
    - Single stocks, (e.g., AAPL, NVDA, ...)
    - Baskets, (e.g., an equally-weighted portfolio of the "Magnificent 7"" stocks)
    - Indices (e.g, SPX, DJIA, NDX...)
  - Fixed Income
    - Interest rates (e.g., LIBOR, SOFR, 10-year CMT yield...)
    - Bonds (e.g., 10-year Treasury or a specific bond issued by a corporation)
    - Credit-sensitive instruments (e.g., CDS, credit indices, ...)
  - Currencies
    - Single exchange rates (e.g., GBP/USD, EUR/USD)
    - Cross-rates (e.g., GBP/EUR)
    - Baskets & Indices (e.g., USDX)
  - Commodities
    - Precious (Ag, Au, Pt, Pd) and base metals (e.g., Cu)
    - Agriculturals (softs), e.g., grains, coffee, pork bellies...
    - Energy (oil, natural gas, electricity)
  - ...and many others

1.2. Derivatives Classification (2)

- By trading mechanism / venue
  - Listed/exchange traded (e.g., CBOT, CME, ...)
  - Over-the-counter/off-exchange (OTC)
- By linearity / non-linearity (convexity)
  - Linear (or effectively linear) "Delta-One" products:
    - Forwards
    - Futures
    - Swaps
  - Non-linear products (essentially everything else, including calls and puts)
- Vanilla vs. Exotic
  - European (and possibly American) Puts & Calls
  - Vanilla Portfolios/Strategies
  - Everything else (e.g., power-law calls, paying off $F_T = \max[S_T^\alpha - K, 0]$)

1.2. Derivatives Classification (3)

- Path Dependence
  - European payoffs and portfolios
  - American options
  - Path-dependent structures
    - Barrier options
    - Lookbacks
    - Average-dependent (Asian) options
    - Others, e.g., Parisian options
- Single vs. Multiple Assets
  - Single-asset
  - Baskets (e.g., $F_T = \max[0.6S_{1,T} + 0.4S_{2,T} - K, 0]$)
  - Two- and Multi-asset “rainbow” options (e.g., “Best-of” calls: $F_T = [\max[S_{1,T}, S_{2,T}, ...] - K]^+$
- Others, including settlement mechanism (cash vs. stock/“in-kind”)

1.3. Derivatives Statistics

Exchange-traded futures and options, by location of exchange

Notional principal, in billions of US dollars

« ‹ Q3 2024 › »	Open interest			Daily average turnover						
Level: 1 2 3 4	2023-Q4	2024-Q2	2024-Q3	2022	2023	2024-05	2024-06	2024-07	2024-08	2024-09
Futures										
All markets	37,343	37,320	44,845	8,045	9,248	8,747	9,294	9,406	12,263	12,831
Interest rate	37,002	36,947	44,475	7,892	9,107	8,622	9,110	9,254	12,088	12,646
Foreign exchange	342	373	370	153	140	124	184	152	176	185
North America	26,680	26,841	33,294	5,898	6,787	5,562	5,841	6,438	9,050	9,120
Interest rate	26,485	26,633	33,090	5,820	6,711	5,494	5,734	6,364	8,964	9,016
Foreign exchange	195	208	203	79	76	68	107	75	87	104
Europe	7,861	7,724	8,808	1,848	2,116	2,798	2,995	2,591	2,776	3,257
Interest rate	7,852	7,711	8,796	1,846	2,115	2,798	2,993	2,591	2,775	3,256
Foreign exchange	8	13	12	1	1	0	2	0	1	1
Asia and Pacific	1,610	1,622	1,541	198	239	285	320	257	300	335
Interest rate	1,569	1,567	1,497	174	214	255	290	225	262	297
Foreign exchange	42	55	44	24	25	30	30	31	37	38
Other Markets	1,192	1,133	1,202	101	106	101	138	120	138	119
Interest rate	1,096	1,036	1,092	52	68	75	93	74	86	77
Foreign exchange	96	97	110	49	38	26	45	46	52	42
Options										
All markets	49,836	50,718	56,948	1,805	2,237	1,915	1,587	1,869	2,626	2,780
Interest rate	49,723	50,596	56,814	1,786	2,216	1,897	1,578	1,863	2,619	2,774
Foreign exchange	113	121	134	19	21	18	8	6	7	5
North America	34,871	33,506	38,774	1,316	1,813	1,214	1,132	1,417	2,071	2,159
Interest rate	34,805	33,425	38,684	1,311	1,809	1,211	1,127	1,412	2,066	2,155
Foreign exchange	66	81	90	4	5	4	5	5	5	4
Europe	13,504	15,673	16,806	466	385	513	450	450	553	619
Interest rate	13,504	15,673	16,805	466	385	513	450	450	553	619
Foreign exchange	0	0	0	0	0	0	0	0	0	0
Asia and Pacific	13	1	0	15	15	0	0	0	0	0
Interest rate	1	0	0	2	0	0	0	0	0	0
Foreign exchange	12	0	0	13	15	0	0	0	0	0
Other Markets	1,448	1,538	1,368	8	23	188	4	2	2	2
Interest rate	1,413	1,498	1,325	7	22	173	1	1	0	0
Foreign exchange	35	40	43	1	1	14	3	1	1	1

- Source: BIS (https://data.bis.org/topics/XTD_DER/tables-and-dashboards/BIS,XTD_D1,1.0)

1.3. Derivatives Statistics (2)

Exchange-traded futures and options, by currency
Notional principal, in billions of US dollars

Q3 2024	Open interest			Daily average turnover						
Level: 1 2 3	2023-Q4	2024-Q2	2024-Q3	2022	2023	2024-05	2024-06	2024-07	2024-08	2024-09
Interest rate	86,724	87,544	101,289	9,678	11,323	10,519	10,688	11,117	14,707	15,420
Australian Dollar	1,211	1,170	1,069	101	135	167	181	147	163	165
Brazilian Real	2,184	2,182	2,055	58	88	248	93	74	86	76
Canadian Dollar	794	753	979	70	96	108	120	104	123	138
Swiss Franc	1	0	0	0	1	0	0	0	0	0
Renminbi	72	75	79	27	27	38	25	26	44	41
Danish Krone	0	-	0	0	0	-	-	-	-	-
EUR	16,282	15,777	17,808	1,835	1,910	2,341	2,629	2,306	2,360	2,847
Pound Sterling	5,087	7,638	7,845	477	596	984	830	748	982	1,048
Hong Kong Dollar	0	0	0	0	0	-	-	-	0	-
Forint	-	-	-	-	-	-	-	-	-	-
Indian Rupee	0	0	0	0	0	0	0	0	0	0
Yen	148	145	182	26	28	30	58	28	31	60
Won	42	43	53	17	17	14	22	17	18	25
Mexican Peso	-	-	-	0	-	-	-	-	-	-
Norwegian Krone	-	-	-	-	-	-	-	-	-	-
New Zealand Dollar	97	133	113	5	6	6	5	7	6	6
Zloty	-	-	-	-	-	-	-	-	-	-
Russian rouble	-	-	-	-	-	-	-	-	-	-
Swedish Krona	21	15	16	1	0	1	1	0	0	0
Singapore Dollar	-	-	-	-	-	-	-	-	-	-
New Turkish Lira	-	-	-	-	-	-	-	-	-	-
New Taiwan Dollar	-	-	-	-	-	-	-	-	-	-
US Dollar	60,777	59,601	71,079	7,061	8,419	6,583	6,726	7,660	10,893	11,012
Rand	9	9	10	0	0	0	0	1	0	0
Other currencies	0	0	0	0	0	0	0	-	-	0
Foreign exchange	455	494	504	172	161	142	192	158	183	190

- Source: BIS (https://data.bis.org/topics/XTD_DER/tables-and-dashboards/BIS,XTD_D2,1.0)

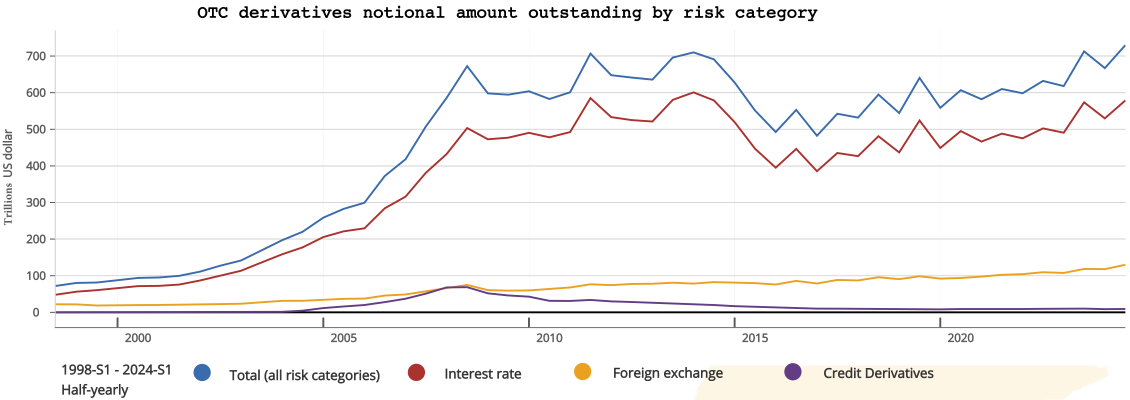
1.3. Derivatives Statistics (3)

Exchange-traded futures and options, by currency
Notional principal, in billions of US dollars

Q3 2024	Open interest			Daily average turnover						
Level: 1 2 3	2023-Q4	2024-Q2	2024-Q3	2022	2023	2024-05	2024-06	2024-07	2024-08	2024-09
Interest rate	86,724	87,544	101,289	9,678	11,323	10,519	10,688	11,117	14,707	15,420
Foreign exchange	455	494	504	172	161	142	192	158	183	190
Australian Dollar	15	19	20	7	7	7	9	8	8	10
Brazilian Real	126	124	144	49	39	40	47	46	52	42
Canadian Dollar	19	23	21	7	6	7	11	6	8	9
Swiss Franc	15	20	19	4	3	5	8	5	6	6
Renminbi	16	27	20	8	14	21	21	23	27	28
Danish Krone	-	0	0	0	0	-	0	-	-	0
EUR	156	161	166	37	35	28	49	30	36	45
Pound Sterling	27	31	39	11	9	9	12	9	11	13
Hong Kong Dollar	0	0	0	-	0	0	0	0	0	0
Forint	0	0	0	0	0	0	0	0	0	0
Indian Rupee	20	9	6	23	22	3	3	2	3	2
Yen	39	50	41	16	17	14	20	19	20	22
Won	10	12	11	5	5	5	6	6	8	7
Mexican Peso	16	13	11	2	2	2	4	2	4	3
Norwegian Krone	1	1	1	0	0	0	0	0	0	0
New Zealand Dollar	3	5	4	2	2	2	3	3	2	3
Zloty	0	1	1	0	0	0	0	0	0	0
Russian rouble	-	-	-	1	-	-	-	-	-	-
Swedish Krona	0	1	1	0	0	0	0	0	0	0
Singapore Dollar	0	0	0	0	0	0	0	0	0	0
New Turkish Lira	2	3	3	0	0	0	0	0	0	0
New Taiwan Dollar	0	0	0	0	0	0	0	0	0	0
US Dollar	436	472	487	172	160	141	189	156	181	187
Rand	3	6	5	0	0	0	1	0	0	1
Other currencies	4	10	6	1	1	1	1	1	1	1

- Source: BIS (https://data.bis.org/topics/XTD_DER/tables-and-dashboards/BIS,XTD_D2,1.0)

1.3. Derivatives Statistics (4) – OTC Notionals Outstanding



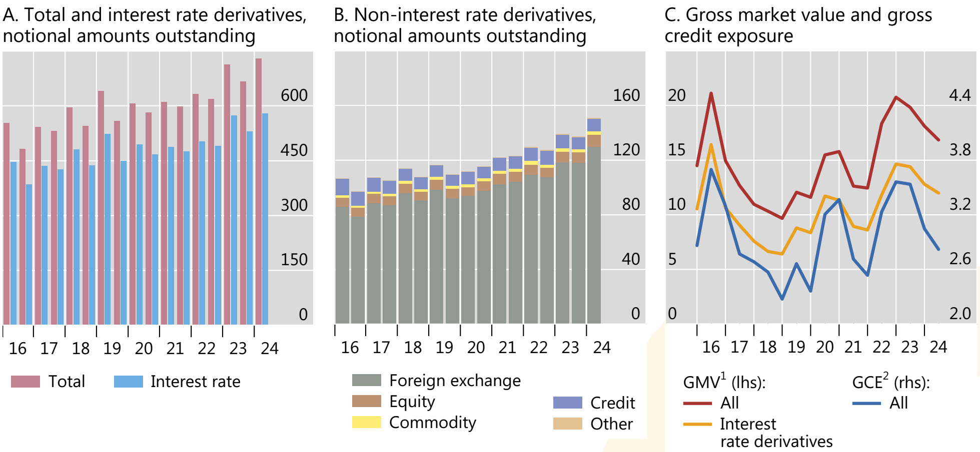
- Source: BIS (OTC Derivatives Statistics, Tables D5.1 and D5.2)

1.3. Derivatives Statistics (5)

Outstanding OTC derivatives

In trillions of US dollars

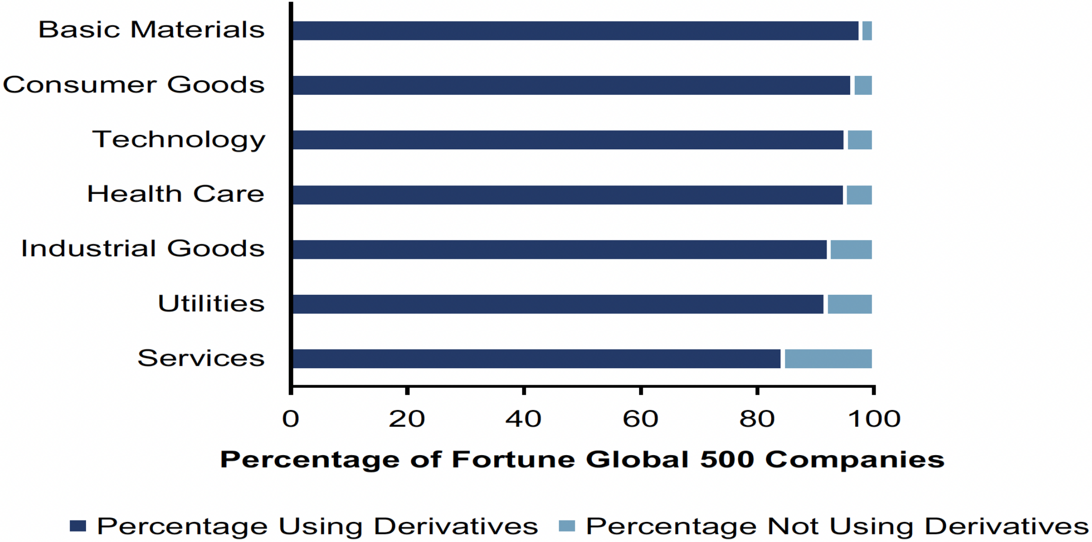
Graph 1



[1] Gross market value. [2] Gross credit exposure.

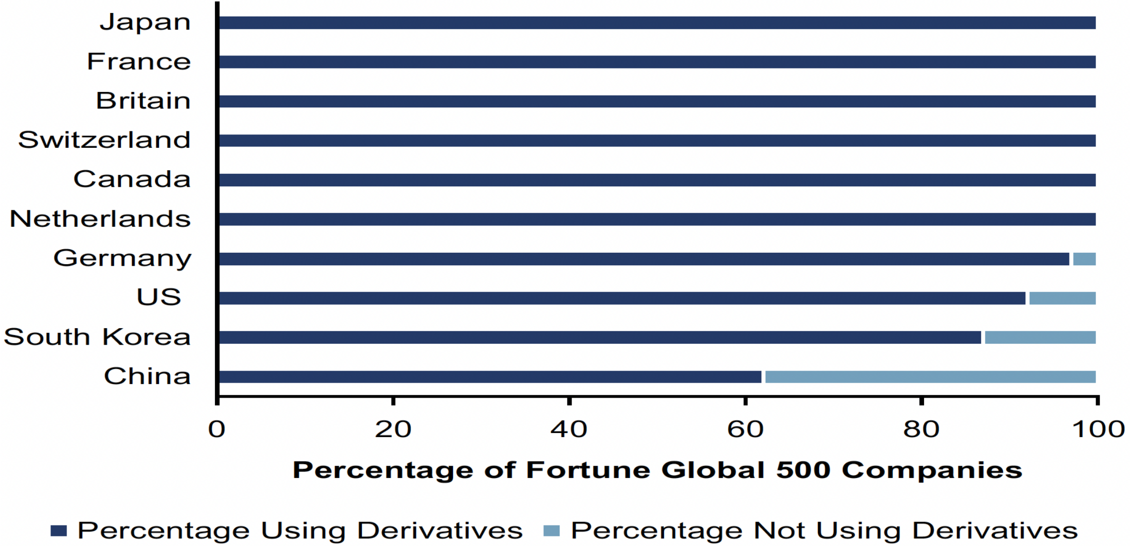
Source: BIS OTC derivatives statistics (Tables D5.1 and D5.2). https://www.bis.org/publ/otc_hy2411.pdf

1.3. Derivatives Statistics (6) – Global Fortune 500 Derivs Usage by Sector



- Source: 2009 ISDA Derivatives Usage Survey (www.isda.org/a/SSiDE/isda-research-notes2.pdf)

1.3. Derivatives Statistics (7) – Global Fortune 500 Usage by Country



- Source: 2009 ISDA Derivatives Usage Survey (www.isda.org/a/SSiDE/isda-research-notes2.pdf)

2. Linear Payoffs: Forwards, Futures, and Swaps

Forward Contracts

- A forward is a contract to buy
  - a specified quantity of a specific asset
  - on a specified future date (the delivery or maturity date)
  - at a specified price (the delivery or **forward price**)
- The future asset buyer is said to have a **long** position, while the asset seller has a **short** position
- The (fair) forward price is determined so that the contract's value is zero at the time it is written.
  - That is, it costs nothing to take either a long or short position.
- The underlying is typically a bond, a stock, a stock index, a currency, or a commodity
- Forward contracts are usually traded in the OTC market

2. Linear Payoffs: Forwards, Futures, and Swaps (2)

Forward Contracts – Some Notation

- Definitions
  - Delivery date: T
  - Delivery price: K
  - Price of underlying (spot price) at time t is S_t
  - Payoff to long position at time T: $S_T - K$
  - Payoff to short position at time T: $K - S_T$
  - (Fair) forward price for a contract written at t is $F_{t,T}$
    - When t is implicit/understood, we will sometimes abbreviate this as F_T
- Convergence property: on the delivery date T, the forward price should be $F_{T,T} = S_T$
  - In some (rough) sense, $\lim_{t \nearrow T} F_{t,T} = \lim_{t \nearrow T} S_t = S_T$
- The gain at time T on a long forward position opened at time 0 should therefore be:

$$S_T - K = F_{T,T} - F_{0,T}$$

2. Linear Payoffs: Forwards, Futures, and Swaps (3)

Futures Contracts

- Similar to forward contracts, with two primary differences:
  - While forward contracts are arranged by dealers in the over-the-counter (OTC) market, futures contracts are traded on exchanges
    - Futures contracts are standardized and (arguably) more liquid
  - Futures contracts are *marked to market*:

    The following discussion is a simplification/idealization that ignores many details about collateralization, margining, and settlement for both futures and forwards
    - While the forward contract gain $F_{T,T} - F_{0,T}$ is only settled on the delivery date, futures contracts are settled daily.
    - Fixing notation: let $\hat{F}_{t,T}$ be the futures price at time t for a contract maturing at T
    - At the end of every trading day, e.g., day t, the holder of a long future position receives the amount $\hat{F}_{t,T} - \hat{F}_{t-1,T}$ from the holder of the short position.
    - Designed to limit exposure to credit risk
- Most actively traded futures underlyings:
  - Interest rates: T-Notes/Bonds, ~~Eurodollar~~ $\to$ SOFR, Fed Funds
  - Stock Indices: S&P 500, Russell 2000, Nasdaq 100
  - Commodities: Crude Oil (WTI), Natural Gas, Corn, Soybeans, Gold
  - Foreign Exchange: EUR, GBP, CAD, JPY, AUD, MXP

2. Linear Payoffs: Forwards, Futures, and Swaps (4)

Most Active US Futures Chains by Open Interest

TRADE DATE: *FRIDAY 10 JAN 2025 - FINAL* Help Download data

PRODUCT NAME	CLEARING	GLOBEX	FLOOR	CLEARPORT	EXCH	ASSET CLASS	PRODUCT GROUP	CATEGORY	SUB-CATEGORY	CLEARED AS	VOLUME	OPEN INTEREST
Three-Month SOFR Futures	SR3	SR3	-	SR3	CME	Interest Rate	Stirs	-	-	Futures	5,679,979	10,117,476
5-Year T-Note Futures	25	ZF	-	25	CBOT	Interest Rate	US Treasury	-	-	Futures	1,783,871	6,154,065
10-Year T-Note Futures	21	ZN	-	21	CBOT	Interest Rate	US Treasury	-	-	Futures	2,709,398	4,680,647
2-Year T-Note Futures	26	ZT	-	26	CBOT	Interest Rate	US Treasury	-	-	Futures	1,200,506	4,296,115
Ultra 10-Year U.S. Treasury Note Futures	TN	TN	-	TN	CBOT	Interest Rate	US Treasury	-	-	Futures	786,781	2,220,461
E-mini S&P 500 Futures	ES	ES	-	ES	CME	Equities	S&P	-	-	Futures	2,133,120	2,060,827
Crude Oil Futures	CL	CL	-	CL	NYMEX	Energy	Crude Oil	North American	Outrights	Futures	2,064,345	2,005,880
U.S. Treasury Bond Futures	17	ZB	-	17	CBOT	Interest Rate	US Treasury	-	-	Futures	596,881	1,932,554
Ultra U.S. Treasury Bond Futures	UBE	UB	-	UBE	CBOT	Interest Rate	US Treasury	-	-	Futures	412,287	1,783,520
30 Day Federal Funds Futures	41	ZQ	-	41	CBOT	Interest Rate	Stirs	-	-	Futures	460,576	1,768,655
Corn Futures	C	ZC	-	C	CBOT	Agriculture	Grains	-	-	Futures	974,964	1,758,964
Henry Hub Natural Gas Futures	NG	NG	-	NG	NYMEX	Energy	Natural Gas	-	Outrights	Futures	790,234	1,553,642
One-Month SOFR Futures	SR1	SR1	-	SR1	CME	Interest Rate	Stirs	-	-	Futures	217,747	1,354,279
Soybean Futures	S	ZS	-	S	CBOT	Agriculture	Oilseeds	-	-	Futures	512,559	824,107
Adjusted Interest Rate S&P 500 Total Return (EFFR) Futures	ASR	ASR	-	ASR	CME	Equities	S&P	-	-	Futures	43,243	738,651
Euro FX Futures	EC	6E	-	EC	CME	FX	G10	-	-	Futures	246,501	602,509

Sources: CME Group (www.cmegroup.com/markets/products.html)

2. Linear Payoffs: Forwards, Futures, and Swaps (5)

Most Liquid (front month) US Futures Contracts, ranked by daily volume

	Symbol	Contract Name	Last	Change	%Chg	Volume	Time
+	ZNH25	10-Year T-Note (Mar '25)	107-125s	-0-265	-0.77%	2,517,737	01/10/25
+	NMH25	Nasdaq 100 Micro (Mar '25)	21,016.00s	-344.75	-1.61%	2,151,541	01/10/25
+	ESH25	S&P 500 E-Mini (Mar '25)	5,866.25s	-93.00	-1.56%	2,006,438	01/10/25
+	ZFH25	5-Year T-Note (Mar '25)	105-150s	-0-195	-0.57%	1,698,129	01/10/25
+	ETH25	S&P 500 Micro (Mar '25)	5,866.25s	-93.00	-1.56%	1,276,010	01/10/25
+	ZTH25	2-Year T-Note (Mar '25)	102-163	-0-086	-0.26%	1,132,039	01/10/25
+	SQZ25	3-Month SOFR (Dec '25)	95.8850s	-0.1650	-0.17%	901,078	01/10/25
+	TNH25	Ultra 10-Year T-Note (Mar '25)	109-180s	-0-260	-0.74%	765,000	01/10/25
+	NQH25	Nasdaq 100 E-Mini (Mar '25)	21,016.00s	-344.75	-1.61%	691,846	01/10/25
+	ZBH25	30-Year T-Bond (Mar '25)	111-07s	-0-29	-0.81%	542,340	01/10/25
+	ZCH25	Corn (Mar '25)	470-4s	+14-4	+3.18%	437,510	01/10/25
+	CLH25	Crude Oil WTI (Mar '25)	75.75s	+2.51	+3.43%	415,658	01/10/25
+	UDH25	Ultra T-Bond (Mar '25)	115-18s	-0-24	-0.64%	396,831	01/10/25
+	GCG25	Gold (Feb '25)	2,715.0s	+24.2	+0.90%	271,894	01/10/25
+	ZSH25	Soybean (Mar '25)	1025-2s	+26-2	+2.63%	254,169	01/10/25
+	E6H25	Euro FX (Mar '25)	1.02740s	-0.00535	-0.52%	242,393	01/10/25
+	QRH25	Russell 2000 E-Mini (Mar '25)	2,201.70s	-52.30	-2.32%	238,896	01/10/25
+	J6H25	Japanese Yen (Mar '25)	0.0063810s	+0.0000090	+0.14%	202,904	01/10/25
+	ZLH25	Soybean Oil (Mar '25)	45.58s	+2.82	+6.59%	176,771	01/10/25
+	YMH25	Dow Futures Mini (Mar '25)	42,160s	-718	-1.67%	157,436	01/10/25

Sources: CME Group, Barchart.com (www.barchart.com/futures/most-active/all)

2. Linear Payoffs: Forwards, Futures, and Swaps (6)

Some US Equity Futures Chains

TRADE DATE: FRIDAY 10 JAN 2025 - FINAL   Help   Download data

PRODUCT NAME	CLEARING	GLOBEX	FLOOR	CLEARPORT	EXCH	ASSET CLASS	PRODUCT GROUP	CATEGORY	SUB-CATEGORY	CLEARED AS	VOLUME	OPEN INTEREST
E-mini S&P 500 Futures	ES	ES	-	ES	CME	Equities	S&P	-	-	Futures	2,133,120	2,060,827
Adjusted Interest Rate S&P 500 Total Return (EFFR) Futures	ASR	ASR	-	ASR	CME	Equities	S&P	-	-	Futures	43,243	738,651
E-mini Russell 2000 Index Futures	RTY	RTY	-	RTY	CME	Equities	Russell	-	-	Futures	251,890	457,742
E-mini Nasdaq-100 Futures	NQ	NQ	-	NQ	CME	Equities	Nasdaq	-	-	Futures	705,405	243,223
S&P 500 Annual Dividend Index Futures	SDA	SDA	-	SDA	CME	Equities	S&P	-	-	Futures	4,371	236,551
Micro E-mini S&P 500 Index Futures	MES	MES	-	MES	CME	Equities	S&P	-	-	Futures	1,278,543	130,684
E-mini Dow Jones Industrial Average Index Futures	YM	YM	-	YM	CBOT	Equities	Dow Jones	-	-	Futures	161,351	85,703
E-mini Financial Select Sector Futures	XAF	XAF	-	XAF	CME	Equities	Select Sectors	-	-	Futures	1,910	66,324
Micro E-mini Nasdaq-100 Index Futures	MNQ	MNQ	-	MNQ	CME	Equities	Nasdaq	-	-	Futures	2,157,545	64,859
Dow Jones Real Estate Futures	JR	RX	-	JR	CBOT	Equities	Dow Jones	-	-	Futures	3,290	58,221
E-mini S&P MidCap 400 Futures	ME	EMD	-	ME	CME	Equities	S&P	-	-	Futures	8,892	46,070
S&P 500 Quarterly Dividend Index Futures	SDI	SDI	-	SDI	CME	Equities	S&P	-	-	Futures	400	30,652

Source: CME Group (www.cmegroup.com/trading/equity-index/us-index.html)

2. Linear Payoffs: Forwards, Futures, and Swaps (7)

E-MINI S&P 500 FUTURES - CONTRACT SPECS

CONTRACT UNIT	\$50 x S&P 500 Index
PRICE QUOTATION	U.S. dollars and cents per index point
TRADING HOURS	**CME Globex:** Sunday 6:00 p.m. - Friday - 5:00 p.m. ET (5:00 p.m. - 4:00 p.m. CT) with a daily maintenance period from 5:00 p.m. - 6:00 p.m. ET (4:00 p.m. - 5:00 p.m. CT) TACO: Sunday - Friday 6:00 p.m. - 9:30 a.m. ET. Monday - Friday 11:00 a.m. - 5:00 p.m. ET with a daily maintenance period 5:00 p.m. - 6:00 p.m. ET BTIC: Sunday - Friday 6:00 p.m. - 4:00 p.m. ET TMAC: Sunday - Friday 6:00 p.m. - 4:00 p.m. ET **CME ClearPort:** Sunday 6:00 p.m. - Friday 6:45 p.m. ET (Sun 5:00 - Fri 5:45 p.m. CT) with no reporting Monday - Thursday 6:45 p.m. - 7:00 p.m. ET (5:45 p.m. - 6:00 p.m. CT) TACO: Sunday 6:00 p.m. - Monday 9:30 a.m. ET. Monday - Thursday 11:00 a.m. - 5:00 p.m. ET and 6:00 p.m. - 9:30 a.m. ET BTIC: Sunday - Friday 6:00 p.m. - 4:00 p.m. ET. BTICs for the following trading day can be submitted after 7:00 p.m. ET (6:00 p.m. CT) TMAC: Sunday - Friday 6:00 p.m. - 4:00 p.m. ET. TMACs for the following trading day can be submitted after 7:00 p.m. ET (6:00 p.m. CT)
MINIMUM PRICE FLUCTUATION	**Outright:** 0.25 index points = \$12.50 TACO, BTIC, TMAC: 0.05 index points = \$2.50 Zero or +/- 20 ticks (4 futures ticks) around the Market at Close price of the outright **CALENDAR SPREAD** 0.05 index points = \$2.50
PRODUCT CODE	**CME Globex:** ES **CME ClearPort:** ES **Clearing:** ES **BTIC:** EST **TACO:** ESQ **TMAC:** ESX
LISTED CONTRACTS	Futures, BTIC, TACO: Quarterly contracts (Mar, Jun, Sep, Dec) listed for 21 consecutive quarters TMAC: One TMAC available on the contract month nearest to expiry
SETTLEMENT METHOD	Financially Settled
TERMINATION OF TRADING	Trading terminates at 9:30 a.m. ET on the 3rd Friday of the contract month. TACO trading terminates at 9:30 a.m. ET on the Thursday before the 3rd Friday of the contract month. BTIC trading terminates at 4:00 p.m. ET on the Thursday before the 3rd Friday of contract month. TMAC trading terminates at 4:00 p.m. ET on the Thursday before the 3rd Friday of the contract month.

Source: CME Group (www.cmegroup.com/markets/equities/sp/e-mini-sandp500.contractSpecs.html)

2. Linear Payoffs: Forwards, Futures, and Swaps (8)

E-MINI S&P 500 FUTURES - CALENDAR

CONTRACT MONTH	PRODUCT CODE	FIRST TRADE LAST TRADE	SETTLEMENT
MAR 2025	ESH25	16 DEC 2022 21 MAR 2025	21 MAR 2025
JUN 2025	ESM25	17 MAR 2023 20 JUN 2025	20 JUN 2025
SEP 2025	ESU25	16 JUN 2023 19 SEP 2025	19 SEP 2025
DEC 2025	ESZ25	07 JUN 2021 19 DEC 2025	19 DEC 2025
MAR 2026	ESH26	21 AUG 2023 20 MAR 2026	20 MAR 2026
JUN 2026	ESM26	21 AUG 2023 18 JUN 2026	18 JUN 2026
SEP 2026	ESU26	21 AUG 2023 18 SEP 2026	18 SEP 2026
DEC 2026	ESZ26	17 SEP 2021 18 DEC 2026	18 DEC 2026
MAR 2027	ESH27	21 AUG 2023 19 MAR 2027	19 MAR 2027
....	.....	....	....
SEP 2029	ESU29	21 JUN 2024 21 SEP 2029	21 SEP 2029
DEC 2029	ESZ29	20 SEP 2024 21 DEC 2029	21 DEC 2029
MAR 2030	ESH30	20 DEC 2024 15 MAR 2030	15 MAR 2030

Source: CME Group (www.cmegroup.com/markets/equities/sp/e-mini-sandp500.calendar.html)

2. Linear Payoffs: Forwards, Futures, and Swaps (9)

Where's the Liquidity in the SPX futures chain?

E-MINI S&P 500 FUTURES - VOLUME & OPEN INTEREST

FINAL DATA

Last Updated 13 Jan 2025 08:50:07 AM CT

TRADE DATE Friday, 10 Jan 2025

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TOTALS	GLOBEX	OPEN OUTCRY	PNT / CLEARPORT	TOTAL VOLUME	BLOCK TRADES	EFP	EFR	TAS	DELIVERIES	AT CLOSE	CHANGE
	2,118,524	0	14,596	2,133,120	2,624	10,250	1,722	0	0	2,060,827	2,901

MONTH	VOLUME								DELIVERIES	OPEN INTEREST	
	VENUE DETAIL				TRADE TYPE DETAIL						
	GLOBEX	OPEN OUTCRY	PNT CLEARPORT	TOTAL VOLUME	BLOCK TRADES	EFP	EFR	TAS		AT CLOSE	CHANGE
MAR 2025	2,115,471	0	14,596	2,130,067	2,624	10,250	1,722	0	0	2,049,564	+2,553
JUN 2025	3,023	0	0	3,023	0	0	0	0	0	8,852	+351
SEP 2025	30	0	0	30	0	0	0	0	0	777	-3
DEC 2025	0	0	0	0	0	0	0	0	0	1,251	0
DEC 2026	0	0	0	0	0	0	0	0	0	383	0

Source: CME Group (www.cmegroup.com/markets/equities/sp/e-mini-sandp500.volume.html)

2. Linear Payoffs: Forwards, Futures, and Swaps (10)

Swaps

- A *swap* is an agreement to exchange – over a given period of time – two series of cash flows depending on the market value(s) of one or more assets.
  - Typically, contract terms are chosen so that the initial value is zero.
- Examples:
  - *Plain "vanilla" interest rate swap:* interest payments at fixed rate exchanged for interest payments at floating rate (e.g., LIBOR, SOFR...) on same notional principal
  - *Currency swap:* series of principal and interest payments exchanged in one currency (e.g., USD) for a series of principal and interest payments in another currency (e.g., GBP)
  - *Equity swap:* dividends and capital gains on an equity index (e.g., SPX) exchanged for a series of interest payments
  - *Commodity swap:* cash flows based on future values of a commodity (e.g., gold) exchanged for series of interest payments

2.1. Determining the Fair Forward Price

- For simplicity, assume initially that we are pricing a forward contract on a dividend-free stock S.
- The basic idea is to find a set of time$-t$ cashflows that *replicate* the time-T cashflows $S_T - K$.
- The easy part of this is the stock piece: if we buy S at time t (paying S_t), because there are no intermediate cashflows, we will have exactly S_T at time T. ✓
- How about K?
  - We need to introduce time value of money, i.e., the time t value of borrowing or lending \$$K$ until T.
  - Assuming there is some "riskless" borrowing and lending (continuously compounded) rate r, the time t value of \$1 will be $\mathcal{R}(t,T)$, with the *discount factor* $\mathcal{R}(t,T) \doteq e^{-r(T-t)}$.
    - More generally, if r is a known, deterministic function of time:
    $$\mathcal{R}(t,T) \doteq \exp[-R(t,T)] \doteq \exp\left[-\int_t^T dt'\, r(t')\right]$$
    and $r(t')$ is the (deterministic) instantaneous spot interest rate at time t'.
    - We might as well introduce the complement to $\mathcal{R}(t,T)$, the money-market account or *accumulation factor*:
    $$\mathcal{M}(t,T) \doteq \exp[R(t,T)] \doteq \exp\left[\int_t^T dt'\, r(t')\right] .$$
    - While $\mathcal{R}(t,T) = 1/\mathcal{M}(t,T)$ when rates are deterministic, if rates are stochastic then $\mathbb{E}[\mathcal{R}(t,T)] \cdot \mathbb{E}[\mathcal{M}(t,T)] \neq 1$
    - Questions: can we say whether $\mathbb{E}[\mathcal{R}(t,T)] \cdot \mathbb{E}[\mathcal{M}(t,T)] \geq 1$ or ≤ 1? When does equality hold?
  - Again, because there are no intermediate cashflows, the time t value of borrowing or lending K until time T is $\mathcal{R}(t,T)K = e^{-r(T-t)}K$. ✓

2.1. Determining the Fair Forward Price (2)

- The time t value of going long the stock and borrowing K until T is $S_t - e^{-r(T-t)}K$.
- The $S_t - e^{-r(T-t)}K$ portfolio perfectly replicates the time T cashflows of the forward contract in all states of nature.
- If we want the forward contract to have a fair value of 0 at T, since there are no intermediate cashflows it should also have a fair value of 0 today.
  - But we know the values of all the inputs (except $K = F_{t,T}$ to be determined) at t:

$$S_t - e^{-r(T-t)}K = 0 \;\Rightarrow\; K = e^{r(T-t)}S_t \;\Rightarrow\; F_{t,T} = e^{r(T-t)}S_t$$

- This is a remarkable (and remarkably robust) result.
  - In particular, independent of distributional assumptions about S_T.
  - Simplest example of pricing by *replication* (finding a portfolio or trading strategy that replicates a desired set of cashflows in all states of nature).
  - Simplest example of pricing by *static replication* (finding a “buy-and-hold” portfolio that replicates a desired set of cashflows in all states of nature).
  - Simplest example of *risk-neutral pricing*: we’ve managed to transform the pricing problem into one in which all assets grow at the risk-free rate r and all cashflows are discounted at the same risk-free rate r.

2.1. Determining the Fair Forward Price (3) – **P**-measure vs. **Q**-measure

- In particular, we've circumvented the problem of predicting the distribution (set of probabilities) $P(S_T)$ of future asset prices – or, say, their expectation $E_t^P[S_T]$ – and finding a way of consistently discounting that set of risky cashflows so that today's value is S_t.
- Abstracting a bit, although we don't have to say much about it right now, our ability to replicate the cashflows implies (induces) an imaginary distribution (set of probabilities) $Q(S_T)$ of future asset prices – right now, their expectation $E_t^Q[S_T] = e^{r(T-t)}S_t$ – with a built-in, consistent discounting rule (using $\mathcal{R}(t,T) \doteq e^{-r(T-t)}$) so that today's value is S_t.
- This transformation applies to all cashflows for all assets, as long as we can risklessly replicate time-T cashflows.
- In quantitative finance parlance, the real-world probability distribution or *measure* (or our own beliefs about or estimates of that distribution/measure) is called the **P**-measure.
  - It may not be unique.
  - It might not even be observable.
  - We often actually have to know very little about it to value a security.
- A risk-neutral p.d. or *measure* is called a **Q**-measure or *Equivalent Martingale Measure (EMM)*.
  - If a market is *arbitrage-free*, there exists at least one such *equivalent* measure (First Fundamental Theorem).
  - If a market is *complete*, the measure is unique* (Second Fundamental Theorem), in the sense that any such measure leads to the same set of prices.

2.1. Determining the Fair Forward Price (4)

Reconciling P vs. Q

- In modern portfolio theory, **P**-world risky asset prices grow at a rate μ ($\mu > r$ in equilibrium).
- Risky cashflows are discounted at a rate $r + \lambda\sigma$, where λ is the market price of risk and σ is a measure of asset price risk.
  - In equilibrium, $\mu = r + \lambda\sigma \;\Rightarrow\; \lambda = (\mu - r)/\sigma$
  - In a CAPM world, the only μ and σ that matter are those of the market portfolio – the *systematic* risk.
- In the risk-neutral (**Q**) world, we don't know or care about any of this (though it's awfully pretty).
- In some sense, the two worlds are orthogonal
  - The **Q** measure has nothing to say about portfolio choice or optimization. That's the **P** measure's job.
  - Once you've built the **P** world portfolio, the **P** measure has nothing to say about pricing a forward.
    - That's simply an overlay based on replication, so it's the **Q** measure's job.
    - It doesn't change your underlying risk exposure, so the **P** measure is irrelevant.

2.1. Determining the Fair Forward Price (5)

Forward Prices with Dividends

- Many/most stocks pay dividends.
- One model: discrete cash dividends
  - Set of dividends of known amounts D_j, paid at times (dividend dates) $t_j : t < t_j < T$.
  - Modified replication argument:
    - If I go long the stock by paying S_t at t, at time T I have S_T *plus* the dividends paid to me *plus* any interest earned by depositing the dividends at the riskless rate at times t_j (until T).
    - Hence, I only need to borrow $K - \sum_j e^{r(T-t_j)} D_j$.
  - Rearranging and completing the argument:

$$F_{t,T} = e^{r(T-t)} S_t - \sum_j e^{r(T-t_j)} D_j$$

  - Equivalently:

$$S_t \Leftrightarrow e^{-r(T-t)} \left(F_{t,T} + \sum_j e^{r(T-t_j)} D_j \right) = e^{-r(T-t)} F_{t,T} + \sum_j e^{-r(t_j-t)} D_j$$

    in a present value sense.
  - Good in the short term and/or for single stocks or concentrated portfolios.

2.1. Determining the Fair Forward Price (6)

Forward Prices with Dividends, continued

- Alternative model: dividend yield
  - Think of S as being like a money-market account, paying out at a rate $y \cdot S_t \cdot dt$ over a short interval dt.
  - Modified replication argument:
    - Idea is that I continuously re-invest the dividends in the stock (like a money-market account), so that S_t becomes $e^{\int_t^T dt' \, y} S_T = e^{y(T-t)} S_T$ by time T.
    - So, I end up with $e^{y(T-t)}$ shares of stock.
    - Hence, I only need to buy $e^{-y(T-t)}$ shares at time t, costing me $e^{-y(T-t)} S_t$.
  - Rearranging and completing the argument:

$$F_{t,T} = e^{r(T-t)}\left(e^{-y(T-t)} S_t\right) = e^{(r-y)(T-t)} S_t$$

  - Equivalently:

$$S_t \Leftrightarrow e^{-r(T-t)}\left(e^{y(T-t)} F_{t,T}\right) = e^{-(r-y)(T-t)} F_{t,T}$$

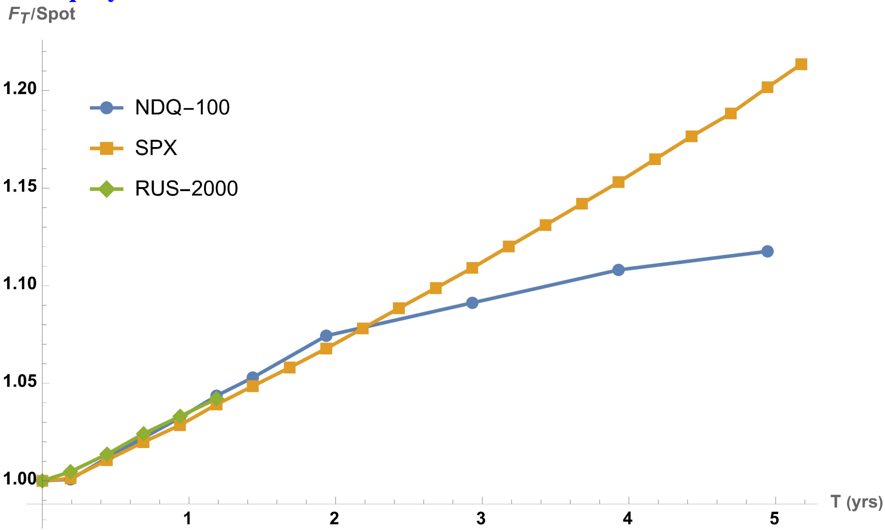
    in a present value sense.
  - Better in the long term and/or for indices (or for currencies!)

2.2. Forward vs. Futures Prices

- To account for the differences between the futures price $\hat{F}_{t,T}$ and the forward price $F_{t,T}$, we need to consider the impact of the mark-to-market process.
  - At the end of each trading day t_j, the holder of a long future position receives $\hat{F}_{t_j,T} - \hat{F}_{t_{j-1},T}$ from the holder of the short position.
  - More precisely, this amount moves from the short's margin account to the long's margin account.
  - If we simply sum (telescope) the cashflows between initiation and delivery, the long will experience a net gain of $\hat{F}_{T,T} - \hat{F}_{0,T} = S_T - \hat{F}_{0,T}$.
- But there's a bit more to the story than this:
  - Margin accounts earn interest!
  - So, the daily cashflow to the margin account will be $(\hat{F}_{t_j,T} - \hat{F}_{t_{j-1},T}) + (e^{r(t_j - t_{j-1})} - 1)\hat{F}_{t_{j-1},T}$
  - Now what?
    - We can either proceed by induction or consider a strategy in which we adjust the position by a factor of $e^{r(t_j - t_{j-1})}$ from day-to-day (called "tailing").
    - Prof. Longstaff will analyze this strategy in his Lecture 4.
    - See also John Hull's *Technical Note No. 24*: www-2.rotman.utoronto.ca/∼hull/TechnicalNotes/TechnicalNote24.pdf
    - Executive summary: In general, $\hat{F}_{t,T} = F_{t,T}$ only when rates are known and deterministic (i.e., either constant or a known function of time).

2.3. Forward/Futures Prices: Further Properties & Examples

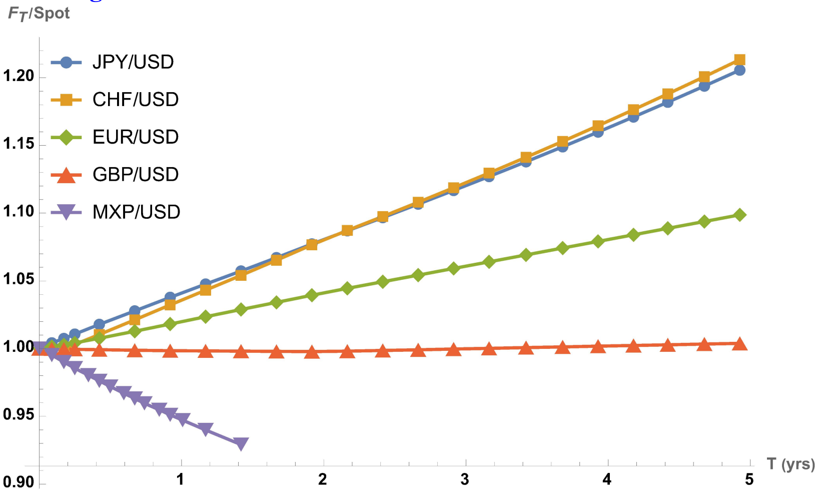
US Equity Index Futures Term Structures



- Source: CME, 2025-01-10 settlements (via: www.cmegroup.com/markets/equities.html#products)
- What do the differences in curve levels/slopes tell us?

2.3. Forward/Futures Prices: Further Properties & Examples (2)

Exchange Rate Futures Term Structures



- Source: CME, 2025-01-10 settlements (via: www.cmegroup.com/markets/fx.html#products)
- Replication strategy isn't purchase of spot currency: it's investment in a security or account that returns the foreign riskless rate, r_f (in the foreign currency) $\Rightarrow$ $r_f \mapsto y$.

2.3. Forward/Futures Prices: Further Properties & Examples (3)

Other Determinants of Futures/Forward Prices

- Storage costs (usually *de minimis* for financial assets, but potentially meaningful for commodities held for investment, e.g., precious metals).
- Convenience yield: generalization/abstraction of the idea of dividends/income received by spot holding but foregone by future/forward:
  - Inverse of Opportunity Cost ("A bird in the hand is worth two in the bush...")
  - E.g., borrowing costs (financial assets, precious metals)
  - Preference to hold consumption commodities rather than rely on uncertain markets in the future
  - Other benefits accruing to asset owners, e.g., rent in housing markets
- Supply & Demand effects in commodity markets (supply disruptions, seasonal demand patterns)

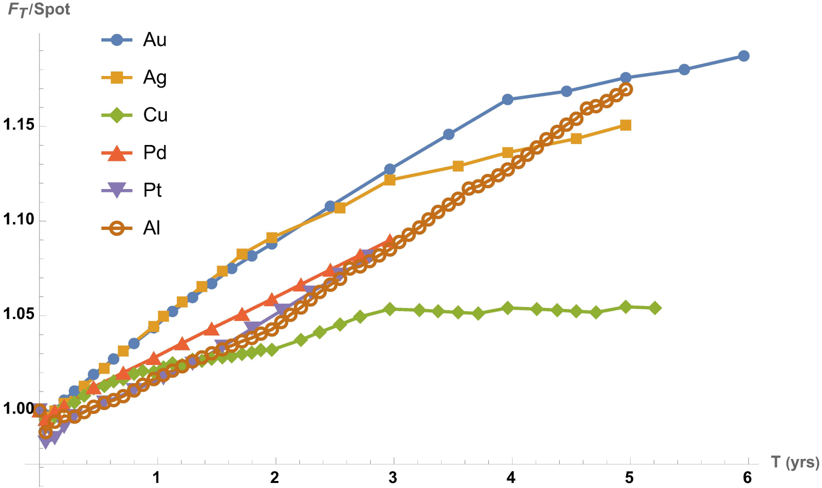
2.3. Forward/Futures Prices: Further Properties & Examples (4)

Other Determinants of Futures/Forward Prices, continued

- Abstract generalization of forward price formula:
  $$F_{t,T} = e^{r(T-t)}\Big(S_t - \text{PV(foregone income/dividends)} + \text{PV(storage costs)} - \text{PV(convenience yield)} \pm \text{(supply/demand effects)}\Big)$$
- Often, we simplify/approximate this by: $F_{t,T} \simeq e^{(r-y_{eff})(T-t)} S_t$
- Futures basis: $F_{t,T} - S_t$
- Cost of carry (or *carry rate*): $r - y_{eff}$
- For investment commodities, the "normal" situation is that $F_{t,T} > S_t$, i.e., the futures curve is rising. This is known as *contango*.
  - Also considered the "normal" situation for financial assets
  - But beware of low (or even negative) interest rates!
- For many consumption commodities, convenience yield effects dominate, leading to *(normal) backwardation*, i.e., the futures curve is falling.

2.3. Forward/Futures Prices: Further Properties & Examples (5)

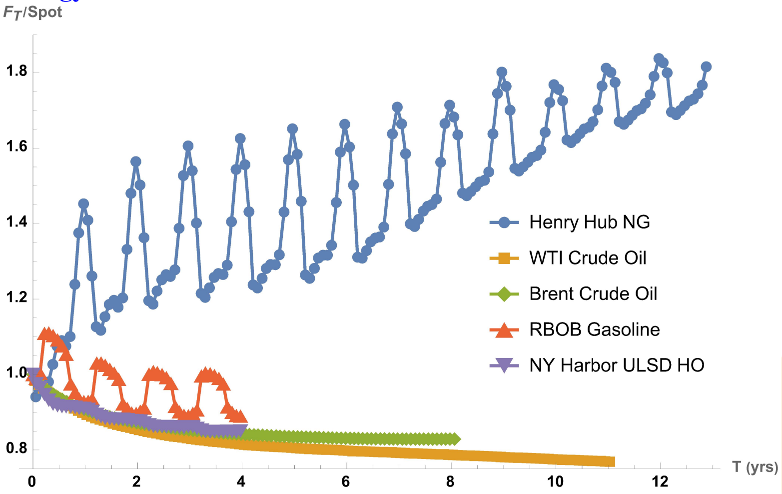
Metals Futures Term Structures



- Source: CME, 2025-01-10 settlements (via: www.cmegroup.com/markets/metals.html#products)
- Platinum & Palladium behavior is intermediate between investment and consumption commodities
- Remarkably, Aluminum appears most similar to precious metals

2.3. Forward/Futures Prices: Further Properties & Examples (6)

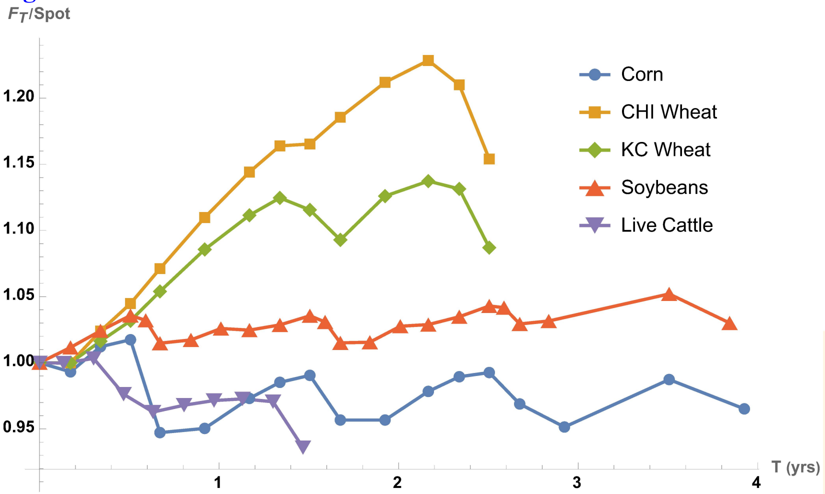
Energy Futures Term Structures



- Source: CME, 2025-01-10 settlements (via: www.cmegroup.com/markets/energy.html#products)
- Seasonal supply-and-demand effects are apparent, especially for natural gas.

2.3. Forward/Futures Prices: Further Properties & Examples (7)

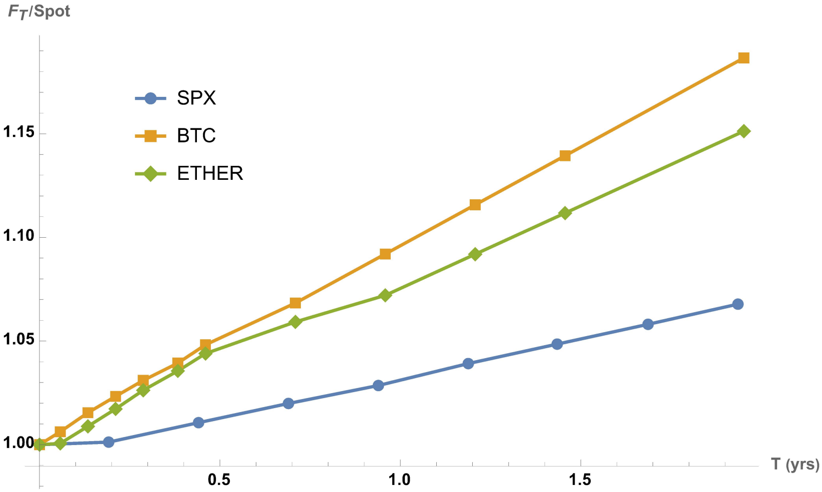
Agriculturals/Softs Futures Term Structures



- Source: CME, 2025-01-10 settlements (www.cmegroup.com/markets/agriculture.html#products)

2.3. Forward/Futures Prices: Further Properties & Examples (8)

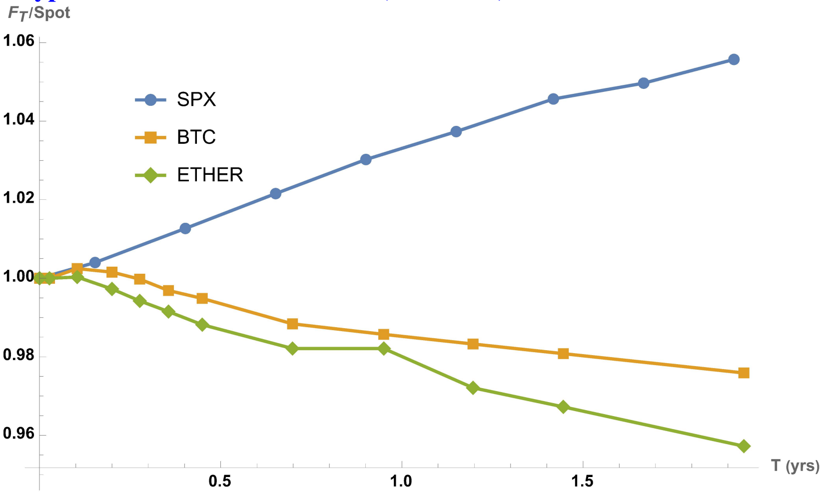
Crypto Futures Term Structures



- Source: CME, 2025-01-10 (www.cmegroup.com/markets/cryptocurrencies.html#products)
- Are these curves believable?

2.3. Forward/Futures Prices: Further Properties & Examples (9)

Crypto Futures Term Structures (2023 data)



- Source: CME, 2023-01-20 (settlement data)
- How should we interpret Bitcoin and Ethereum backwardation?

3. Non-linear Payoffs: Options Markets

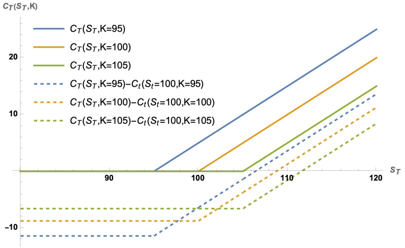
Prologue

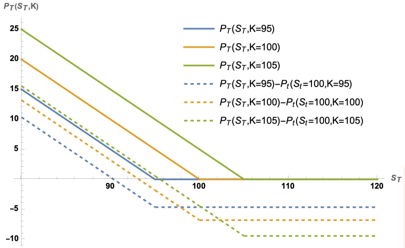
- In this section, we review the payoffs (both outright and P&L net of initial premium) of calls, puts, and portfolios/strategies built by combining them
- This seems as good a place as any to say a bit about the valuation model inputs we'll be using during the remainder of the course: both for in-class examples and for assignments (which serve slightly different purposes)
  - In-class examples: our objective is to use consistent, simplified inputs to "get a feel" for the relative values of different structures
    - It makes the most sense to use rounded, generic values that are easy to compare (for prices) and of typical magnitudes (for rates, yields, and vols)
    - Hence, we will base prices around $S = K = 100$ and take $r = 4\%, y = 2\%, \sigma = 20\%$
  - Homework: we'll use figures as close as possible to real market values, e.g., SPX $\simeq$ 5827 (as of Friday, 10-Jan-2025) and as consistently as possible thereafter – unless market conditions change significantly [$\rightarrow$ 5996.66 $\simeq$ 6000 close as of F 17 Jan].
- In discussing portfolios/strategies in this section of the notes, we can only hope to cover those that are most relevant to the narrative later in the course.
  - Even though it's more than 20 years old, the LIFFE (now ICE Futures Europe) document is still a great reference for the menagerie of vanilla option strategies: www.globalriskguard.com/resources/deriv/liffe_options.pdf

3. Non-linear Payoffs: Options Markets (2)

Calls and Puts

- A call option represents the right (but not the obligation) to purchase an asset S on [or before] the maturity date T for a given (*strike*) price K.
  - Payoff at time T : $C_T = \max[S_T - K, 0] = [S_T - K]^+$
  - Think of it as the upside of a forward contract – hence the famous "hockey stick" payoff.
  - At-the-money (ATM): $K = S_t$; In-the-money (ITM): $K < S_t$; Out-of-the-money (OTM): $K > S_t$





- A put option represents the right (but not the obligation) to sell an asset S on [or before] the maturity date T for the strike price K.
  - Payoff at time T : $P_T = \max[K - S_T, 0] = [K - S_T]^+$
  - Think of it as the (negative) downside of a forward contract.
  - At-the-money (ATM): $K = S_t$; In-the-money (ITM): $K > S_t$; Out-of-the-money (OTM): $K < S_t$

3. Non-linear Payoffs: Options Markets (3)

S&P 500® Index Options

Symbol
SPX/SPXW

Underlying
The Standard & Poor's 500 Index is a capitalization-weighted index of 500 stocks from a broad range of industries. The component stocks are weighted according to the total market value of their outstanding shares. The impact of a component's price change is proportional to the issue's total market value, which is the share price times the number of shares outstanding. These are summed for all 500 stocks and divided by a predetermined base value. The base value for the S&P 500 Index is adjusted to reflect changes in capitalization resulting from mergers, acquisitions, stock rights, substitutions, etc.

Multiplier
$100.

Strike Price Intervals
Generally, $5, $10, $25, $50, $100, and $200 strike prices are available. Newly added and longer-term expirations have less granularity. Granularity is added over the life of the expiration with nearer term options having the most granularity. Standard 3rd Friday AM and PM options always have the most granularity and widest ranges.

Strike (Exercise) Prices
In-, at- and out-of-the-money strike prices are initially listed. New strikes can be added as the index moves up or down.

Premium Quote
Stated in decimals. One-point equals $100. Minimum tick for options trading below 3.00 is 0.05 ($5.00) and for all other series, 0.10 ($10.00). For complex orders legs may trade in .01 ($1.00) increments, but net package price must be in .05 ($5.00) increments except for Boxes and Box Swaps which are eligible to trade in .01 ($1.00) increments.

Exercise Style
European - SPX/SPXW options generally may be exercised only on the expiration date.

Last Trading Day
Trading in SPX options will ordinarily cease on the business day (usually a Thursday) preceding the day on which the exercise-settlement value (i.e., the expiration date) is calculated, 3:15pm CT.
Trading in SPXW options will ordinarily cease on the day of expiration, 3:00 pm CT.

Expiration Date for Standard, Weekly and EOM Option
SPX AM options expire on the third Friday of the expiration month or the immediately preceding business day if the Exchange is not open on that Friday.
SPXW PM expiring options including Monday thru Friday, EOM and EOQ settle on their expiration date. If the Exchange is not open on that Friday. SPXW options expire on a Monday, Tuesday, Wednesday, Thursday, or a Friday. If the Exchange is not open on a Tuesday, Wednesday, Thursday, or Friday, the normally expiring Tuesday, Wednesday, Thursday, or Friday Weekly will expire on the immediately preceding business day. If the Exchange is not open on a Monday, the normally expiring Monday XSP Weekly will expire on the first business day immediately following that Monday. EOM XSP options expire on the last business day of the expiration month.

- Source: CBOE (www.cboe.com/tradable_products/sp_500/spx_options/specifications/)

3. Non-linear Payoffs: Options Markets (4)

Expiration Months

Cboe may list up to twelve standard monthly expirations. Cboe may list up to ten (10) SPX LEAPS® monthly expirations at one time that expire from 12 to 180 months from the date of issuance.

For SPXW, the exchange lists 5 weeks of daily expirations for each Monday-Thursday expirations, if an EOM or EOQ falls on a Monday-Thursday it is counted towards that total. The Exchange lists 5 weeks of EOW expiring options, if a 3rd Friday, EOM or EOQ expiration falls on the EOW it is not counted towards the total. The Exchange may list series that expire at the end of the next consecutive four calendar quarters (EOQ), as well as the fourth quarter of the next calendar year. The Exchange may list 12 expirations for Monthly Options Series (EOM). Monthly Option Series expirations need not be for consecutive months; however, the expiration date of a nonconsecutive expiration may not be beyond what would be considered the last expiration date if the maximum number of expirations were listed consecutively.

Settlement of Options Exercise

Exercise will result in delivery of cash on the business day following expiration. The exercise-settlement value, SET, is calculated using the opening sales price in the primary market of each component security on the expiration date. The exercise-settlement amount is equal to the difference between the exercise-settlement value and the exercise price of the option, multiplied by $100.

SPXW exercise will result in delivery of cash on the business day following expiration. The exercise-settlement value is calculated using the closing sales price in the primary market of each component security on the expiration date. The exercise-settlement amount is equal to the difference between the exercise-settlement value and the exercise price of the option, multiplied by $100.

Position and Exercise Limits

No position and exercise limits are in effect.

Margin

Purchases of puts or calls with 9 months or less until expiration must be paid for in full. Writers of uncovered puts or calls must deposit / maintain 100% of the option proceeds* plus 15% of the aggregate contract value (current index level x $100) minus the amount by which the option is out-of-the-money, if any, subject to a minimum for calls of option proceeds* plus 10% of the aggregate contract value and a minimum for puts of option proceeds* plus 10% of the aggregate exercise price amount. (*For calculating maintenance margin, use option current market value instead of option proceeds.) Additional margin may be required pursuant to Exchange Rule 12.10.

Cusip Number

648815

Trading Hours

Regular Hours

8:30 a.m. to 3:15 p.m. (CT)

Note SPXW expiring options trade until 3:00 p.m. (CT) on their expirations.

Curb

3:15 p.m. to 4:00 p.m. (CT)

Global Trading Hours

7:15 p.m. to 8:25 a.m. (CT)

*Position and Exercise limits are subject to change.

- Source: CBOE (www.cboe.com/tradable_products/sp_500/spx_options/specifications/)

3. Non-linear Payoffs: Options Markets (5)

SP 500 INDEX (^spx)

Bid: **5880.9199** Ask: **6003.5601** Vol: **0** Last: **5,949.91** Change: **+107.0002 (+1.8313%)**

2025-01-15 16:14:59 ET (Delayed)

Filters By:

Volume: All | Expiration Type: All | Options Range: Near the Money | Size: 3 | Expiration: 2025 January | View Chain

Options Chain

Total Records: 72

Calls — Wed Jan 15 2025 — Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
14.7	+12.2	14.5	15.7	25,760	0	0	0	1872	SPXW 5935.000	0.05	-113.25	0	0.05	46,250	0	0	0	285
9.5	+7.4	9.1	11.1	43,081	0	0	0	2481	SPXW 5940.000	0.05	-117.7	0	0.05	46,473	0	0	0	238
4.8	+3	4.1	5.6	38,426	0	0	0	2109	SPXW 5945.000	0.05	-122.4	0	0.05	45,159	0	0	0	736
0.35	-1.125	0.2	0.75	97,344	0	0	0	5081	SPXW 5950.000	0.65	-126.5	0.55	0.7	61,433	0	0	0	2273
0.05	-1.2	0	0.05	64,869	0	0	0	1071	SPXW 5955.000	5	-126.9	4.5	5.7	27,889	0	0	0	78
0.05	-1	0	0.05	100,117	0	0	0	2419	SPXW 5960.000	9.93	-126.72	9.5	11.3	11,436	0	0	0	112

Calls — Fri Jan 17 2025 — Puts

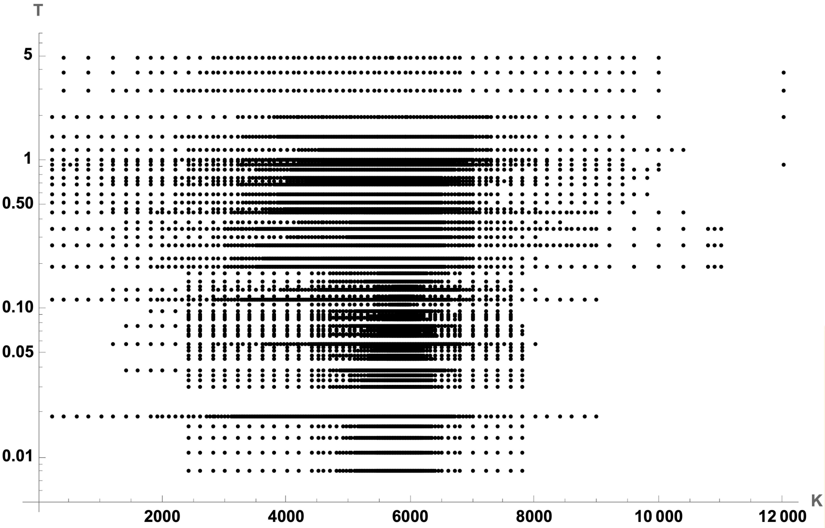
Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
35.1	+27.3	31.9	32.3	951	0.15	0.5675	0.0058	1082	SPXW 5940.000	21.99	-99.35	22.2	22.6	941	0.16	-0.4323	0.0058	573
26.38	+21.03	25.3	25.9	389	0.14	0.546	0.0068	1633	SPX 5945.000	20.89	-102.45	20.7	21.3	299	0.14	-0.4539	0.0068	497
29.8	+22.8	29	29.4	1,089	0.15	0.5387	0.0059	2646	SPXW 5945.000	24.4	-99.85	24.3	24.7	533	0.15	-0.4611	0.0059	445
23.52	+18.64	22.6	23.1	28,446	0.14	0.5118	0.0069	26624	SPX 5950.000	22.9	-105.29	22.9	23.5	28,094	0.14	-0.488	0.0069	32348
26.78	+21.05	26.3	26.7	5,467	0.15	0.5092	0.006	2975	SPXW 5950.000	26.52	-103.4	26.6	26.9	3,734	0.15	-0.4906	0.006	2571
21.4	+15.9	20	20.6	445	0.14	0.477	0.007	447	SPX 5955.000	25.3	-106.8	25.3	26	170	0.14	-0.5228	0.007	423

Jan 2025 | Feb 2025 | Mar 2025 | Apr 2025 | May 2025 | Jun 2025 | Jul 2025 | Aug 2025
Sep 2025 | Oct 2025 | Nov 2025 | Dec 2025 | Jan 2026 | Mar 2026 | Jun 2026 | Dec 2026
Dec 2027 | Dec 2028 | Dec 2029

- Source: CBOE, 2025-01-15 (www.cboe.com/delayed_quotes/spx/quote_table)

3. Non-linear Payoffs: Options Markets (6)

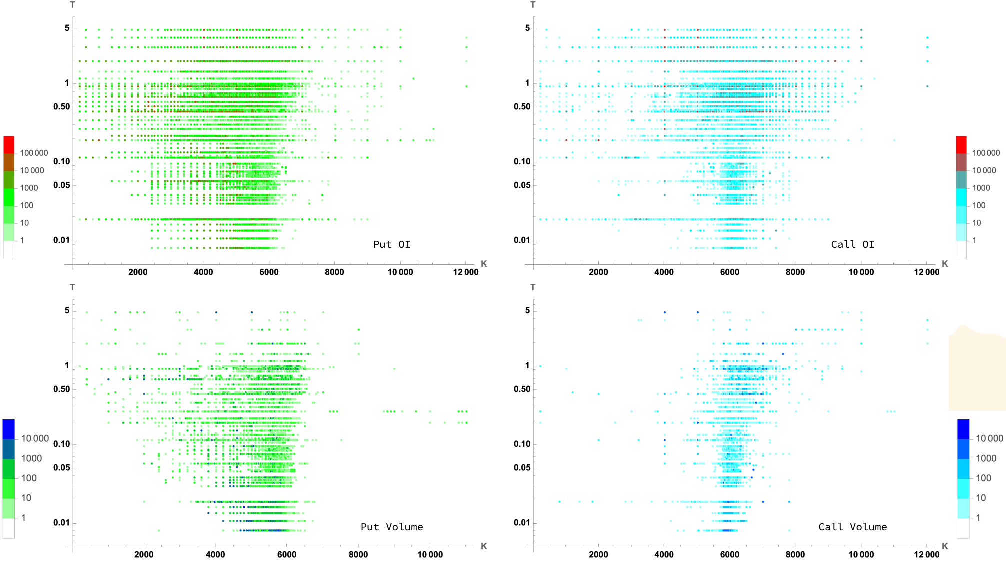
Listed SPX Options by Strike (K) & Maturity (T, years)



- Source: CBOE, 2025-01-10 (www.cboe.com/delayed_quotes/spx/quote_table)

3. Non-linear Payoffs: Options Markets (7)

Listed SPX Options Open Interest & Volume by Strike (K) & Maturity (T, years)



- Source: CBOE, 2025-01-10 (www.cboe.com/delayed_quotes/spx/quote_table)

3. Non-linear Payoffs: Options Markets (8)

Equity Options Product Specifications

Symbol

The option symbols are the same as for the underlying equity security. Visit the Cboe Symbol Directory for specific symbols.

Underlying

Generally, 100 shares of the underlying equity security.

Strike Price Intervals

Generally, 2 1/2 points when the strike price is between $5 and $25, 5 points when the strike price is between $25 and $200, and 10 points when the strike price is over $200. Strikes are adjusted for splits, re-capitalizations, etc.

Strike (Exercise) Prices

In-, at- and out-of-the-money strike prices are initially listed. New series are generally added when the underlying trades through the highest or lowest strike price available.

Premium Quotation

Stated in decimals. One point equals $100. Generally, minimum tick for options trading below $3 is $0.05 and for all other series, $0.10. For classes participating in the *Penny Pilot Program*, the minimum tick for options trading below $3 is $0.01 and $0.05 for options trading at $3 or above.

Expiration Date

The third Friday of the expiration month.

Expiration Months

Two near-term months plus two additional months from the January, February or March quarterly cycles.

Exercise Style

American - Equity options generally may be exercised on any business day up to and including on the expiration date.

Settlement of Option Exercise

Exercise notices properly tendered on any business day will result in delivery of the underlying stock on the second business day following exercise.

- Source: CBOE (www.cboe.com/exchange_traded_stock/equity_options_spec/)

3. Non-linear Payoffs: Options Markets (9)

Apple Inc. (aapl) Bid: **238.27** Ask: **238.48** Vol: **39,761,757** Last: **238.45** Change: **+4.59 (+1.9676%)**

2025-01-15 16:00:00 ET (Delayed)

Filters By:

Volume: All | Expiration Type: All | Options Range: Near the Money | Size: 3 | Expiration: 2025 January | View Chain

Options Chain

Total Records: 18

Calls — Fri Jan 17 2025 — Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
5.67	+2.875	5.6	5.85	2,366	0.26	0.8792	0.0423	4933	AAPL 232.500	0.27	-1.875	0.27	0.28	10,941	0.27	-0.1214	0.0426	5983
3.6	+2.025	3.55	3.7	23,006	0.25	0.7521	0.0715	44493	AAPL 235.000	0.64	-2.81	0.63	0.66	37,360	0.25	-0.2502	0.0727	27439
1.93	+1.155	1.88	1.94	34,249	0.24	0.5474	0.0946	10238	AAPL 237.500	1.47	-3.68	1.41	1.45	19,148	0.24	-0.4599	0.0973	5108
0.78	+0.445	0.78	0.82	63,913	0.23	0.3126	0.087	58542	AAPL 240.000	2.87	-4.43	2.78	2.89	5,816	0.23	-0.7017	0.0893	33028
0.28	+0.145	0.26	0.28	21,967	0.23	0.1343	0.0537	11625	AAPL 242.500	4.98	-4.645	4.7	5	862	0.22	-0.884	0.0547	7884
0.07	+0.005	0.08	0.09	15,685	0.23	0.0457	0.0235	49846	AAPL 245.000	7.4	-4.675	6.95	7.4	1,841	0	-0.9721	0.0216	14870

Calls — Fri Jan 24 2025 — Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
6.6	+2.55	6.6	6.85	652	0.21	0.7736	0.0388	1494	AAPL 232.500	1.05	-2.15	1.02	1.05	6,088	0.21	-0.2295	0.0394	3270
4.75	+1.99	4.75	4.9	4,342	0.2	0.6698	0.0485	5339	AAPL 235.000	1.65	-2.8	1.62	1.68	7,339	0.2	-0.3351	0.0493	7214
3.2	+1.41	3.2	3.3	6,654	0.19	0.5419	0.0547	3470	AAPL 237.500	2.63	-3.37	2.54	2.6	2,406	0.19	-0.4654	0.0556	1270
1.98	+0.88	1.98	2.03	11,152	0.19	0.404	0.0546	5185	AAPL 240.000	3.95	-3.875	3.75	3.95	3,082	0.19	-0.6059	0.0555	3272
1.15	+0.5	1.13	1.18	12,620	0.19	0.2741	0.0478	3049	AAPL 242.500	5.5	-4.45	5.35	5.65	365	0.19	-0.7386	0.049	474
0.63	+0.245	0.62	0.64	11,796	0.19	0.1697	0.0364	8204	AAPL 245.000	7.5	-4.75	7.25	7.8	607	0.19	-0.8478	0.0391	2149

Jan 2025 | Feb 2025 | Mar 2025 | Apr 2025 | May 2025 | Jun 2025 | Jul 2025 | Aug 2025
Sep 2025 | Oct 2025 | Dec 2025 | Jan 2026 | Jun 2026 | Dec 2026 | Jan 2027

- Source: CBOE, 2025-01-15 (www.cboe.com/delayed_quotes/aapl/quote_table)

3. Non-linear Payoffs: Options Markets (10)

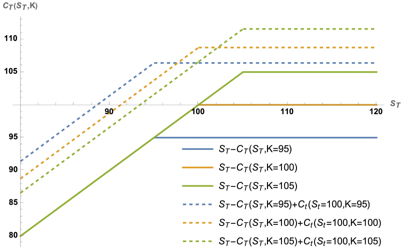
Calls and Puts (continued)

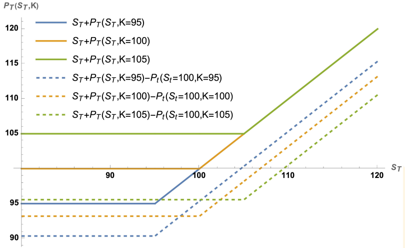
- Equity index options can usually only be exercised at maturity T (*European Style*).
- Single stock options can usually be exercised at any time up to T (*American Style*).
- For FX and commodities, exercise style varies:
  - For FX and commodities with a robust spot market, exercise is American
  - Otherwise, where primary underlying asset is a (near) futures contract, exercise is typically European (and occasionally American).
- In some OTC markets or for some corporate securities, exercise is at a discrete set of dates up to T (*Bermud[i]an Style*).
- Of interest in equity markets: Triple witching expirations
  - Days (3rd Fridays) on which single-stock options, index futures, and index options all expire.
  - Four times a year (quarterly expiries): third Friday of March, June, September, and December.
- ATM conventions
  - In equity markets, ATM refers to at-the-money relative to Spot S_t (ATMS).
  - In currency markets, ATM refers to at-the-money relative to Forward $F_{t,T}$ (ATMF).
  - ITM, OTM conventions follow those for ATM.
  - Intrinsic value at t: $[S_t - K]^+$ for calls; $[K - S_t]^+$ for puts; map $S_t \mapsto F_{t,T}$ for FX

3.1. Option Strategies

Stock $\pm$ Option Strategies: Covered Calls and Protective Puts

- Covered call strategy: long stock, short a (usually OTM) call.
  - Payoff at time T : $S_T - \max[S_T - K, 0] = S_T - [S_T - K]^+$ (plus any dividends)
  - Call over-writing strategies are very common buy-side "yield enhancement" techniques
  - Buy-side is generally short OTM calls:
    Implied volatilities for $K > S_t$ (and hence OTM call prices) are reduced relative to ATM



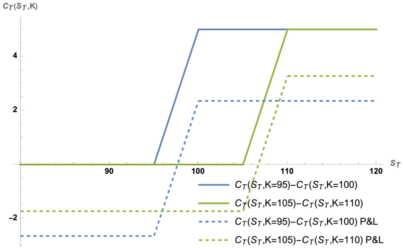


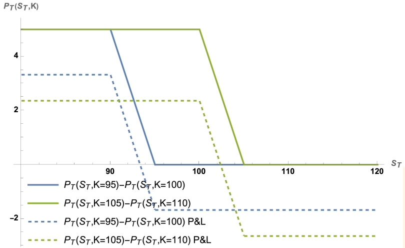
- Protective put strategy: long stock, long a (usually OTM) put.
  - Payoff at time T : $S_T + \max[K - S_T, 0] = S_T + [K - S_T]^+$ (plus any dividends)
  - Simplest form of *portfolio insurance*
  - Buy-side is generally long OTM puts:
    Implied volatilities for $K < S_t$ (and hence OTM put prices) are increased relative to ATM

3.1. Option Strategies (2)

Long/Short Option Strategies: Call and Put Spreads

- Bull spread, vertical call spread, or capped call:
  Long call with K_1 (usually near ATM), short call with $K_2 > K_1$, both with same maturity T
  - Payoff at time T : $\max[S_T - K_1, 0] - \max[S_T - K_2, 0] = [S_T - K_1]^+ - [S_T - K_2]^+$
  - Maximum payoff = $K_2 - K_1$: capture market gains over a range, at reduced cost relative to a single option
  - Valuation bounds: payoff is non-negative everywhere, so call option values must be *decreasing* in strike.



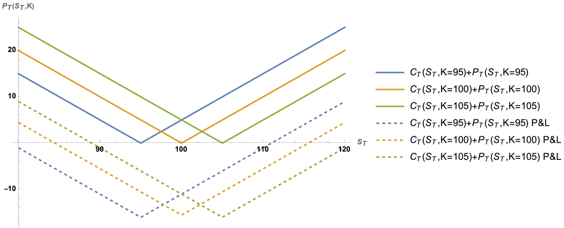


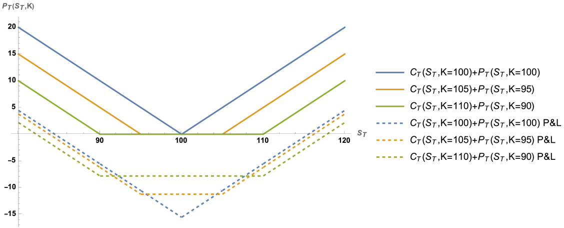
- Bear spread, vertical put spread, or capped put:
  Long put with K_1 (usually near ATM or OTM), short put with $K_2 < K_1$, both with same maturity T
  - Payoff at time T : $\max[K_1 - S_T, 0] - \max[K_2 - S_T, 0] = [K_1 - S_T]^+ - [K_2 - S_T]^+$
  - Maximum payoff = $K_1 - K_2$: capture market losses over a range, at reduced cost relative to a single option
  - Valuation bounds: payoff is non-negative everywhere, so put option values must be *increasing* in strike.

3.1. Option Strategies (3)

Both Long (or Short) Call + Put Strategies: Straddles and Strangles

- Straddle: long a call & a put, with identical strike K & maturity T, usually at (or near) the money
  - Payoff at time T : $\max[S_T - K, 0] + \max[K - S_T, 0] = [S_T - K]^+ + [K - S_T]^+$
  - Generally not a directional or "first-moment" play
    - Instead, used to express views on volatility, second-moment, or large vs. short moves. E.g., long straddle: buy vol cheaply, wait until implied volatility rises or market moves away from ATM, then close out position



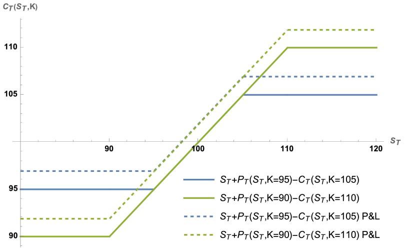


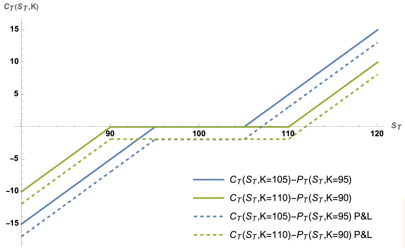
- Strangle: long call with strike K_c & put with strike $K_p < K_c$, but same maturity T, again, usually at (or near) the money
  - Payoff at time T : $\max[S_T - K_c, 0] + \max[K_p - S_T, 0] = [S_T - K_c]^+ + [K_p - S_T]^+$
  - Conceptually similar to a straddle, but requiring less premium up-front... and larger moves away from ATM to lock in profit

3.1. Option Strategies (4)

Long/Short Call/Put (Risk Reversal) Strategies: Collars and Reverse Collars

- Collar or range forward strategy: long stock, long a (usually OTM) put, short a (usually OTM) call.
  - Payoff at time T : $S_T + [K_p - S_T]^+ - [S_T - K_c]^+$ (plus any dividends)
  - Combination of protective put and covered call strategies: allow gains or losses within a limited range
  - Call and put strikes are often selected so that initial value of the collar overlay is zero: *zero-cost collar*



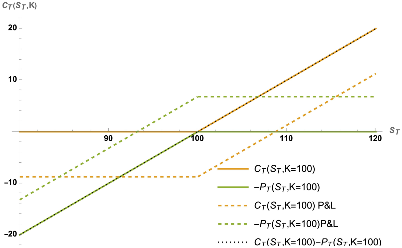


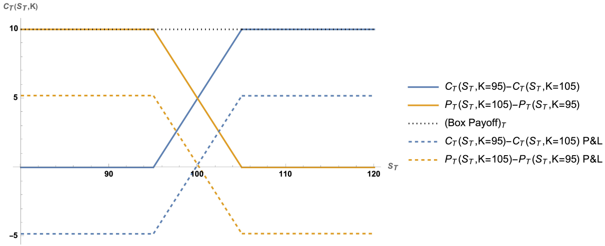
- Reverse collar or participating forward: short a (usually OTM) put, long a (usually OTM) call
  - Payoff at time T : $[S_T - K_c]^+ - [K_p - S_T]^+$
  - Lock-in a future price for small moves; participate in gains or losses for larger moves.
  - Like zero-cost collar, call and put strikes are often selected so that initial value of the position is zero.
  - Like many other strategies, can combine with a zero-coupon bond (ZCB) to create a structured note (in this case, *not* principal-protected)

3.1. Option Strategies (5)

Long Call – Short Put (Conversion) Strategies: Synthetic Forwards and Boxes

- Conversion or synthetic forward: long a call, short a put, both with the same strike K & maturity T
  - Payoff at time T : $C_T - P_T = \max[S_T - K, 0] - \max[K - S_T, 0] = S_T - K$
  - Hence (long) a European Call & (short) a European Put is a *synthetic forward* contract with forward price K.
  - Valuation by the same replication argument as for forwards: $C_t - P_t = S_t - e^{-r(T-t)}K$ (if no divs)
  - With a continuous yield y: $C_t - P_t = e^{-y(T-t)}S_t - e^{-r(T-t)}K$ **(European) Put-Call parity relationship**
  - Does Put-Call parity hold in reality?
    We'll look at some data when we study the option price (implied volatility) surface in detail.



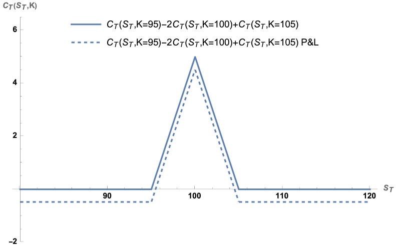


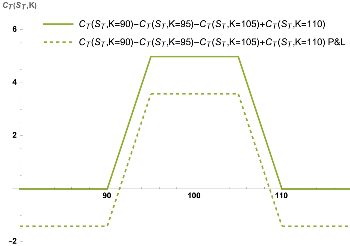
- Box: long synthetic forward of strike K_1, short synthetic forward of strike K_2 (same maturity T)
  - Equivalent to long $\{K_1, K_2\}$ call spread and long $\{K_2, K_1\}$ put spread
  - Payoff at time T : $(S_T - K_1) - (S_T - K_2) = K_2 - K_1$
  - Objective: lock-in differential mispricings of call & put spreads (or conversions)

3.1. Option Strategies (6)

Butterflies and Condors

- Butterfly Call Spread: long 1 call struck at $K-\Delta K$, short 2 calls at K, long 1 call at $K+\Delta K$.
  - Butterfly Put Spread: long 1 put struck at $K+\Delta K$, short 2 puts at K, long 1 put at $K-\Delta K$.
  - Also equivalent to long the $(K-\Delta K, K)$ call spread, short the $(K, K+\Delta K)$ call spread
  - Payoff at time T : $[S_T-(K-\Delta K)]^+ - 2[S_T-K]^+ + [S_T-(K+\Delta K)]^+$
  - Similar to argument for bull and bear spreads, payoff is everywhere non-negative $\Rightarrow$ present value of a butterfly is also non-negative ($\Rightarrow$ option prices are *convex* in strike)





- Condor: similar to Butterfly, but short 2 calls at $K \to$ short 1 at K, short another at $K' > K$
  - Final long call strike is also shifted: $K+\Delta K \mapsto K'+\Delta K$
  - Equivalent put portfolio can again be constructed by reversing the order of the strikes
  - Also equivalent to long the $(K-\Delta K, K)$ call spread, short the $(K', K'+\Delta K)$ call spread
  - Same arguments hold as for butterflies, but resulting constraints (and P&L) are not as tight.

3.1. Option Strategies (7)

Calendar (Horizontal) Spreads

- Long a call (or put) with strike K and maturity T_2, short the corresponding call (or put) (usually) with strike K and maturity $T_1 < T_2$
- Not usually a directional position, except in a divergence or mean-reversion sense.
- Generally used to gain exposure to volatility between T_1 and T_2, though more modern instruments (e.g., variance/volatility swaps) allow this to be done in a more precise way
- Under certain conditions, it's possible to say that the T_2-maturity option must have greater value than the T_1-maturity option $\Rightarrow$ potential arbitrage strategies.
- Also *diagonal spreads*: going long vs. short options of differing strikes and maturities.

3.2. Option Valuation: Bounds & Constraints

- Easy enough to say that calls and puts, as rights but not obligations to buy (or sell) the underlying asset for a given strike price K, must have non-negative values.
- Similar conclusions for values of vertical spreads and butterflies:
  - Bull/bear spreads must have non-negative value
    $\Rightarrow$ calls must be decreasing in K and puts must be increasing in K
  - Butterfly spreads must have non-negative value
    $\Rightarrow$ call and put values must be convex in K
- Another set of fairly simple relationships:
  (American option value) $\geq$ (Bermud[i]an option value) $\geq$ (European option value)
  - What arbitrage trade could we put on if any of these inequalities were violated?
- Can we say a bit more, particularly about option values *per se*?

3.2. Option Valuation: Bounds & Constraints (2)

Upper Bounds

- Calls:
  - American: $C_t(S,K,T) < S_t$ (why?)
  - European: $C_t(S,K,T) < e^{-y(T-t)}S_t$ (why?)
- Puts:
  - American: $P_t(S,K,T) < K$ (why?)
  - European: $C_t(S,K,T) < e^{-r(T-t)}K$ (why?)

Lower Bounds

- Calls:
  - American: $C_t(S,K,T) \geq \max[S_t - K, 0]$ (why?)
    - Corollary: American calls are increasing in T
  - European: $C_t(S,K,T) \geq \max[e^{-y(T-t)}S_t - e^{-r(T-t)}K, 0]$ (why?)
    - Corollary: American calls on a non-dividend paying stock should never be exercised early.
    - Corollary: European calls on a non-dividend paying stock are increasing in T
- Puts:
  - American: $P_t(S,K,T) \geq \max[K - S_t, 0]$ (why?)
    - Corollary: American puts are increasing in T
  - European: $P_t(S,K,T) \geq \max[e^{-r(T-t)}K - e^{-y(T-t)}S_t, 0]$ (why?)
    - No analogous corollaries can be deduced for put options (unless $r = 0$).

3.3. Option Valuation: Binomial Model

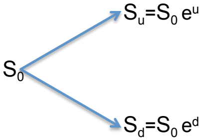
Binomial Model Overview

- Binomial Model starts from viewpoint of discrete approximation of the asset price process S_t (or, equivalently, the underlying Brownian motion). Over each (time-)step Δt:
  - Define the set of two nodes $\{S_{t+\Delta t}\} = \{S_{t+\Delta t,u}, S_{t+\Delta t,d}\}$ reachable from S_t over Δt
  - Apply replication argument to determine risk-neutral probabilities of reaching the two nodes
  - Calculate expected payoff $\mathbb{E}[\{C(S_{t+\Delta t}, t+\Delta t)\}|S_t]$ using the risk-neutral probabilities
  - Discount at risk-neutral rate over Δt to obtain $C(S_t, t)$
  - Linkage to hedging strategy clear:
    Risk-neutral probabilities of reaching $\{S_{t+\Delta t}\}$ from S_t are induced by no-arbitrage condition
- Historical background
  - Seminar on Financial Decision-making under Uncertainty, sponsored by The Israel National Council for Research and Development, Ein Bokek (Dead Sea beach resort), March 1975.
  - Two years after Black-Scholes (& Merton), discussion emphasized how difficult it was to build intuition around the continuous time theory and results.
  - Comment attributed to Bill Sharpe: essentially, "What if there are effectively just two states of nature spanned by the stock and bond?"
  - Resulting paper: John Cox, Stephen Ross, and Mark Rubinstein, "Option pricing: A simplified approach," *Journal of Financial Economics* **7**(3): 229-63, (September 1979). doi.org/10.1016/0304-405X(79)90015-1.

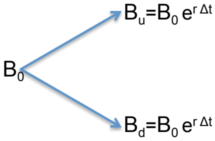
3.3. Option Valuation: Binomial Model (2)

Binomial Model Framework

- $1+1$ element (vector) stochastic process framework, with assets $\{S, B\}$
- Two possible outcomes for the risky asset price S over time Δt:



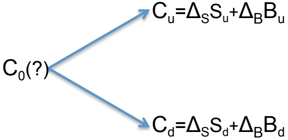
- Assume probabilities of p_u for the up state and $p_d = 1 - p_u$ for the down state
  - We shall see that we don't care about and don't even need to know $\{p_u, p_d\}$.
- Assume the riskless bond or money-market account B grows at the continuously-compounded rate r over Δt. Then for the two states of nature $\{u, d\}$:



3.3. Option Valuation: Binomial Model (3)

Binomial Model Framework, continued

- Assume a derivative C that pays off C_u in the up state and C_d in the down state. What is the value C_0 of C today?



- Basic idea of Cox-Ross-Rubinstein (CRR) model:
  - The two assets $\{S, B\}$ suffice to span all states of nature (market is complete), so the payoffs of C can be replicated by a portfolio of $\{\Delta_S, \Delta_B\}$ units of the two, respectively $\Rightarrow$ C is *redundant*.
  - By the Law Of One Price (LOOP: absence of arbitrage), the value C_0 of C today must be equal to the value of the replicating portfolio today: $C_0 = \Delta_S S_0 + \Delta_B B_0$
- What are $\{\Delta_S, \Delta_B\}$? Solve the replicating equations:

$$\begin{pmatrix} S_u & B_u \\ S_d & B_d \end{pmatrix} \cdot \begin{pmatrix} \Delta_S \\ \Delta_B \end{pmatrix} = \begin{pmatrix} C_u \\ C_d \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \Delta_S \\ \Delta_B \end{pmatrix} = \frac{1}{S_u B_d - S_d B_u} \begin{pmatrix} B_d & -B_u \\ -S_d & S_u \end{pmatrix} \cdot \begin{pmatrix} C_u \\ C_d \end{pmatrix} = \frac{1}{S_u B_d - S_d B_u} \begin{pmatrix} B_d C_u - B_u C_d \\ S_u C_d - S_d C_u \end{pmatrix}$$

$$= \frac{1}{S_0 (e^u - e^d)\, B_0 e^{r\Delta t}} \begin{pmatrix} B_0 e^{r\Delta t} (C_u - C_d) \\ S_0 e^u C_d - S_0 e^d C_u \end{pmatrix}$$

3.3. Option Valuation: Binomial Model (4)

Binomial Model Framework, continued

- Substitute to obtain C_0:

$$C_0 = \frac{C_u - C_d}{e^u - e^d} - e^{-r\Delta t}\frac{e^d C_u - e^u C_d}{e^u - e^d} = \frac{1 - e^{d-r\Delta t}}{e^u - e^d} C_u + \frac{e^{u-r\Delta t} - 1}{e^u - e^d} C_d$$
$$= e^{-r\Delta t}\left(\frac{e^{r\Delta t} - e^d}{e^u - e^d} C_u + \frac{e^u - e^{r\Delta t}}{e^u - e^d} C_d\right) = e^{-r\Delta t}(q_u C_u + q_d C_d) \text{ with } q_u + q_d = 1$$

- Comment 1: Interpretation of $\{q_u, q_d\}$ as probabilities (existence of equivalent martingale measure or EMM) holds as long as $u > r\Delta t > d$, i.e. neither asset dominates the other. (First Law: absence of arbitrage)
- Comment 2: Uniqueness of q-measure is independent of and indeed requires no knowledge (or uniqueness) of p-measure. What is required is the (unique) hedging strategy that allows us to span the set of possible payoffs. (Second Law: complete markets)
- Comment 3: To modify for a (re-invested) dividend yield y, we must recognize that Δ_S shares at time Δt are equal to $e^{-y\Delta t}\Delta_S$ shares today. Then:

$$C_0 = e^{-y\Delta t}\frac{C_u - C_d}{e^u - e^d} - e^{-r\Delta t}\frac{e^d C_u - e^u C_d}{e^u - e^d} = \frac{e^{-y\Delta t} - e^{d-r\Delta t}}{e^u - e^d} C_u + \frac{e^{u-r\Delta t} - e^{-y\Delta t}}{e^u - e^d} C_d$$
$$= e^{-r\Delta t}\left(\frac{e^{(r-y)\Delta t} - e^d}{e^u - e^d} C_u + \frac{e^u - e^{(r-y)\Delta t}}{e^u - e^d} C_d\right) = e^{-r\Delta t}(q_u C_u + q_d C_d) \text{ with } q_u + q_d = 1$$

3.3. Option Valuation: Binomial Model (5)

Binomial Model Framework, continued

- Comment 4: Existence & uniqueness hold even if B has stochastic returns as long as returns of B & S are not collinear $(S_u B_d \neq S_d B_u)$ and neither asset dominates the other. Then:

$$\begin{aligned} C_0 = S_0 \Delta_S + B_0 \Delta_B &= S_0 \frac{B_u C_d - B_d C_u}{B_u S_d - B_d S_u} + B_0 \frac{S_u C_d - S_d C_u}{S_u B_d - S_d B_u} \\ &= \frac{B_0 S_d - B_d S_0}{B_u S_d - B_d S_u} C_u + \frac{B_u S_0 - B_0 S_u}{B_u S_d - B_d S_u} C_d \\ &= \frac{\frac{B_u - B_d}{B_0} - \frac{S_u - S_d}{S_0}}{\frac{B_u}{B_0}\frac{S_d}{S_0} - \frac{B_d}{B_0}\frac{S_u}{S_0}} \left(\frac{\frac{S_d}{S_0} - \frac{B_d}{B_0}}{\frac{B_u - B_d}{B_0} - \frac{S_u - S_d}{S_0}} C_u + \frac{\frac{B_u}{B_0} - \frac{S_u}{S_0}}{\frac{B_u - B_d}{B_0} - \frac{S_u - S_d}{S_0}} C_d \right), \text{ etc.} \end{aligned}$$

  - Interpretation of probabilities is reasonably straightforward; interpretation of discount factor not so easy (it's the present value of the synthetic "riskless" security paying off $\{\$1, \$1\}$)
  - We don't always need to have a riskless numeraire asset to be able to derive option prices (and hedges). E.g.: outperformance option.

3.3. Option Valuation: Binomial Model (6)

Binomial Model $\Delta t \searrow 0$ Limit

- How do we adapt this model for use as a numerical method that represents the process $dS/S = (r-y)dt + \sigma dW$ and in particular converges to the distribution of S_T as $\Delta t \searrow 0$?
- Use idea that we can represent a Brownian motion as the limit of a Bernoulli (binomial) up/down process, with increments ΔW over Δt:
  - $\mathbb{E}[\Delta W] = 0$
  - $\mathbb{E}[(\Delta W)^2] = \mathrm{var}[\Delta W] = \Delta t \Rightarrow \mathrm{std}[\Delta W] = \sqrt{\Delta t}$
- These conditions imply that $\Delta W = \pm\sqrt{\Delta t}$ + higher order terms in $\sqrt{\Delta t}$, i.e.: $\Delta W = \pm\sqrt{\Delta t} + \alpha \cdot \Delta t + \ldots$
- Assuming equal probability weights (1/2) for up and down moves:
  1. Divide T into m increments of length $\Delta t = T/m$;
  2. Take $\Delta W_j = \pm\sqrt{\Delta t}$, each with probability $1/2, \forall j = 1, \ldots, m$;
  3. Let $W_m = \sum_{j=1}^{m} \Delta W_j$;
  4. Take the limit $m \to \infty$.

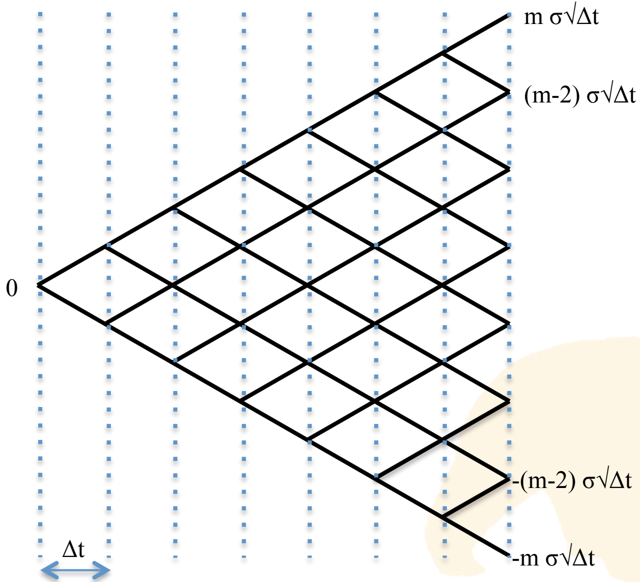
3.3. Option Valuation: Binomial Model (7)

Binomial Model $\Delta t \searrow 0$ Limit, continued

- Just as an Itô process for an asset price transforms a Brownian increment into an increment of the asset price, we (may) need to transform our Bernoulli increments to obtain a discrete approximation that converges to the desired price process as $\Delta t \searrow 0$.
- Log-normal model: asset prices look like $S_0 e^{\sigma W_t}$, which we approximate as $\{S_0 e^u, S_0 e^d\}$
- Hence, $u \sim \sigma\sqrt{\Delta t}$, $d \sim -\sigma\sqrt{\Delta t}$. Choosing these specific values leads to the simplest (and probably most widely used) version of the binomial model, with:
$$q_u = \frac{e^{(r-y)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \sim \frac{1}{2}\left[1 + \frac{(r-y-\sigma^2/2)\Delta t}{\sigma\sqrt{\Delta t}} + \mathcal{O}(\Delta t^{3/2}) + \ldots\right] = \frac{1}{2} + \mathcal{O}(\Delta t^{1/2})$$
  - At the m^{th} time step, we will have $m+1$ nodes, with the maximum and minimum nodes at $S_0 e^{\pm m\sigma\sqrt{\Delta t}} = S_0 e^{\pm\sqrt{m}\sigma\sqrt{T}}$: potential to "prune" tree beyond some large multiple of $\sigma\sqrt{T}$
  - Advantage of this method: lattice/grid points always at integer multiples of $\sigma\sqrt{\Delta t}$, so path-dependence (tracking of barriers, extrema, averages...) can be handled cleanly

3.3. Option Valuation: Binomial Model (8)

Binomial Grid in Log-Price Space



3.3. Option Valuation: Binomial Model (9)

Binomial Model $\Delta t \searrow 0$ Limit, continued

- This solution is not unique. Indeed, almost any (reasonably bounded) choice of α in: $u = \sigma\sqrt{\Delta t} + \alpha\Delta t$, $d = -\sigma\sqrt{\Delta t} + \alpha\Delta t$ (drifting lattice) will work.
  Then: $q_u = \frac{e^{(r-y)\Delta t} - e^{-\sigma\sqrt{\Delta t}+\alpha\Delta t}}{e^{\alpha\Delta t}(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}})} \sim \frac{1}{2}\left[1 + \frac{(r-y-\alpha-\sigma^2/2)\Delta t}{\sigma\sqrt{\Delta t}} + \mathcal{O}(\Delta t^{3/2}) + \dots\right]$
- Two obvious choices of drifty lattice:
  - $\alpha = r-y$ (drifting with the forward): $q_u = \frac{1}{2} - \frac{\sigma\sqrt{\Delta t}}{4} + \frac{\sigma^3\Delta t^{3/2}}{48} + \mathcal{O}(t^{5/2}) + \dots$
    - Natural interpretation of lattice evolution
  - $\alpha = r-y-\frac{\sigma^2}{2}$ (drifting with the log-forward): $q_u = \frac{1}{2} + \frac{\sigma^3\Delta t^{3/2}}{24} + \mathcal{O}(t^{5/2}) + \dots$
    - Only slightly less natural interpretation of lattice evolution.
    - Superior theoretical convergence for smooth payoffs... but that may not be the limiting factor for convergence...

3.3. Option Valuation: Binomial Model (10)

Binomial Model $\Delta t \searrow 0$ Limit, continued

- General valuation approach: backward induction of values at $j\Delta t$ from those at $(j+1)\Delta t$
  - $j\Delta t$ values determined by probability-weighting those at $(j+1)\Delta t$, then discounting by $e^{-r\Delta t}$
- Generic algorithm:
  for $(k=-m, k\leq m, k{+}{=}\,2)$ $C[m\Delta t, S_k = S_0 e^{k\sigma\sqrt{\Delta t}}]$ = payoff condition $C_T(S_k)$
  for $(j=m-1, j\geq 0, j{-}{-})$
  for $(k=-j, k\leq j, k{+}{=}\,2)$ $C[j\Delta t, S_k] = e^{-r\Delta t}\left(q_u C[(j+1)\Delta t, S_{k+1}] + q_d C[(j+1)\Delta t, S_{k-1}]\right)$
- With some careful bookkeeping and a bit of clever indexing to take advantage of vectorized functionality, this can be made pretty efficient
  - Nevertheless, the required computational effort/time is of order $\mathcal{O}(m^2)$
- For European payoffs, there may not be a reason to do full time-stepping.

  Rather, just recognize that at maturity (time step m), the (risk-neutral) probability of state: $ku+(m-k)d = (2k-m)\sigma\sqrt{\Delta t} + m\alpha\Delta t$, $0\leq k\leq m$, is:

$$\frac{m!}{k!(m-k)!}q_u^k q_d^{m-k} = \binom{m}{k} q_u^k q_d^{m-k} = q_d^m \binom{m}{k}\left(\frac{q_u}{q_d}\right)^k$$

- Hence, European option valuation can be written as:

$$C(0\Delta t, S_0) = e^{-rT} q_d^m \sum_{k=0}^{m}\binom{m}{k}\left(\frac{q_u}{q_d}\right)^k C_T\left(S_k = S_0 e^{(2k-m)\sigma\sqrt{\Delta t}+m\alpha\Delta t}\right),$$

  requiring $\mathcal{O}(m)$ computational effort.

3.3. Option Valuation: Binomial Model (11)

American/Bermudian option valuation

- Initialize lattice with terminal payoff values $C_T(S_k)$ as before
- Proceed with backward induction, calculating *continuation* values $C(j\Delta t, S_k)$ at time $j\Delta t$ by probability-weighting those at $(j+1)\Delta t$, then discounting by $e^{-r\Delta t}$
- Additional step(s) at a time $j\Delta t$ at which exercise is permitted:
  - Identify range of nodes S_k for which exercise value (e.g., $[S_k - K]^+$) > continuation value $C(j\Delta t, S_k)$
  - At each such node, replace continuation value by exercise value
- Continue iterating backward in time.
- Again, there are tweaks to make this algorithm reasonably efficient but it almost inherently requires computational effort/time of order $\mathcal{O}(m^2)$

3.3. Option Valuation: Binomial Model (12)

American Put Illustration

- $S_0 = K = 100$, $T = 1.0$, $r = 0.04$, $y = 0.02$, $\sigma = 0.2$, $m = 100$