Chapter 7. Exotic and Path Dependent Options, Variance Swaps, Jump-Diffusion Processes (v.A.6)

In this chapter, we will study exotic and path dependent options, payoffs of which depend on the whole path of the underlying asset price, and not only price at option's maturity.

Barrier Options

In general, there are two types of barrier options: knock-out and knock-in options.

Knock-out options: The option contract is cancelled if a pre-specified barrier S_b is crossed by the price of the underlying security at any time during the life of the option.

Knock-in options: The option contract is activated only if a predetermined barrier S_b is crossed by the price of the underlying security at any time during the life of the option.

Examples.

Down-and-Out Put (**do**) This option has a similar payoff to the European Put option, however, it becomes void (the contract is cancelled), if the underlying security's price crosses it and goes below a pre-specified barrier S_b (the fixed and known barrier) at any time during the life of the option. If the underlying security's price never hits the barrier S_b, then, this is simply a European Put option and its payoff is like the one of the European Put option. The barrier should be below the initial stock price and below the strike price:

$$S_b < X \quad \text{and} \quad S_b < S_0$$

Let the price of this option be P_{d0}. Then,

$$P_{d0} = e^{-rT} \cdot \boldsymbol{E}^*\big((X - S_T)^+ \ \mathbb{1}_{(S_{min} \geq S_b)}\big)$$

where $S_{min} = \min\{S_t\colon\ t \in [0,T]\}$ and $\mathbb{1}_{(S_{min} \geq S_b)} = \begin{cases} 1, & if\ S_{min} \geq S_b \\ 0, & else \end{cases}$

The risk (to the writer of this option) is lower compared to the vanilla European put option. For the option holder, the payoff is lower than the one of the vanilla European put option. Thus, it would be expected that the price of this option would be lower than the one of the vanilla European put option with the same specifications.

Down-and-In Put (**di**) This option becomes active and pays as a European Put option, **only if** a barrier S_b is crossed by the underlying security's price at any time during the life of the option. Let the price of this put option be P_{di}. Then,

$$P_{di} = e^{-rT} \cdot \boldsymbol{E}^*\big(\ (X - S_T)^+\ \mathbb{1}_{(S_{min} \leq S_b)}\big)$$

where $S_{min} = \min\{S_t\colon\ t \in [0,T]\}$ and $\mathbb{1}_{(S_{min} \leq S_b)} = \begin{cases} 1, & if\ S_{min} \leq S_b \\ 0, & else \end{cases}$.

Then, it is easy to see that

$$P_{\ di} + P_{d0} \quad = \quad P \text{ (price of European Put option)}$$

The closed-form solution for **Down-and-Out** Put option is given by:

$$P_{do} = Xe^{-rT}[N(d_4) - N(d_2) - a\{N(d_7) - N(d_5)\}] - S_0[N(d_3) - N(d_1) - b\{N(d_8) - N(d_6)\}]$$

where $a = \left(\frac{S_b}{S_0}\right)^{\frac{2r}{\sigma^2}-1}$, $\quad b = \left(\frac{S_b}{S_0}\right)^{\frac{2r}{\sigma^2}+1}$

$$d_1 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_2 = \frac{ln\left(\frac{S_0}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_3 = \frac{ln\left(\frac{S_0}{S_b}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_4 = \frac{ln\left(\frac{S_0}{S_b}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_5 = \frac{ln\left(\frac{S_0}{S_b}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_6 = \frac{ln\left(\frac{S_0}{S_b}\right) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_7 = \frac{ln(S_0X/S_b^2) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad , \quad d_8 = \frac{ln(S_0X/S_b^2) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The two cases below can be used as benchmarks for computing **do** put option prices:

1. $(S_0, X, r, T, \sigma, S_b) = \left(50, 50, 0.1, \frac{5}{12}, 0.4, 30\right)$, then $P_{do} = 3.228, \quad P = 4.076.$
2. $(S_0, X, r, T, \sigma, S_b) = \left(50, 50, 0.1, \frac{5}{12}, 0.3, 30\right)$, then $P_{do} = 2.729, \quad P = 2.845.$

<u>How to price these options numerically?</u>

Pricing the **Down-and-Out** Put option (do):

- Take a uniform partition of the time interval [0,T]: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$,

  where $t_k = \frac{T}{n} k = \Delta k$, where $\Delta = \frac{T}{n}$.

- Simulate N paths of the stock prices at times $t_0, t_1, \ldots, t_{n-1}, t_n$: $\{S_{t_1}^i, \ldots, S_{t_m}^i\}_{i=1}^N$.

- The price of the down-and-out put option is given by

$$P_{do} = e^{-rT} \cdot E^*\left((X - S_T)^+ \, \mathbb{1}_{(S_{min} \geq S_b)}\right),$$

  which can be approximated by

$$P_{do} \approx e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^N (X - S_{t_m}^i)^+ \, \mathbb{1}_{(S_{min}^i \geq S_b)} \text{ where } S_{min}^i = \min\{\{S_{t_1}^i, \ldots, S_{t_m}^i\}.$$

Pricing the **Down-and-In** Put option (di):

- Take a uniform partition of the time interval [0,T]: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$,

  where $t_k = \frac{T}{n} k = \Delta k$, where $\Delta = \frac{T}{n}$.

- Simulate N paths of the stock prices at times $t_0, t_1, \ldots, t_{n-1}, t_n$: $\{S_{t_1}^i, \ldots, S_{t_m}^i\}_{i=1}^N$.

- The price of the down-and-in put option is given by

$$P_{di} = e^{-rT} \cdot E^*\left((X - S_T)^+ \, \mathbb{1}_{(S_{min} \leq S_b)}\right),$$

  which can be approximated by

$$P_{di} \approx e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^N (X - S_{t_m}^i)^+ \, \mathbb{1}_{(S_{min}^i \leq S_b)} \text{ where } S_{min}^i = \min\{\{S_{t_1}^i, \ldots, S_{t_m}^i\}.$$

Note: Since $\boldsymbol{P(S_{min} = S_b) = 0}$, then we can put the equality sign in both down-and-in and down-and out options.

Spread Options

Here, we review a model for the European spread option which has the following payoff:

$$\max(0, S_T^2 - S_T^1 - K)$$

A special case of this is the **Exchange Option,** in which $K = 0$.

$$V = V(t, S_t^1, S_t^2) = \exp(-r(T-t))\, \mathbb{E}_t^Q[\max(0, S_T^2 - S_T^1)]$$

In the case when K = 0 there is a closed form solution which is given by the Margrabe's Formula. In general, for non-zero K, there is no closed-form solution.

Assume that the risk-neutral dynamics of the two underlying securities follow Geometric Brownian Motion processes and their stochastic differential equations are of the following form:

$dS_t^1 = rS_t^1 dt + \sigma_1 S_t^1 dW_t^1$ and $dS_t^2 = rS_t^2 dt + \sigma_2 S_t^2 dW_t^2$, where $dW_t^1 dW_t^2 = \rho dt$.

The price of the Exchange Option, V, is given by:

$$V = S_0^2 N(d_1) - S_0^1 N(d_0)$$

where $d_1 = \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) + \frac{\sigma^2}{2}\sqrt{T}}{\sigma\sqrt{T}}$ and $d_0 = \frac{\ln\left(\frac{S_0^2}{S_0^1}\right) - \frac{\sigma^2}{2}\sqrt{T}}{\sigma\sqrt{T}}$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$

In general, for non-zero *K*, there is no closed-form solution for the price of the Spread Option.

We can use the Monte Carlo simulation method to compute spread option prices (European or American.)

The price of the American Spread Option will be given by the following formula:

$$V_t = \sup_{\tau \in [t,T]} \mathbb{E}_t^*\left(e^{-(\tau - t)r}\, Payoff(\tau) | \mathcal{F}_t\right)$$

In particular, at time $t = 0$, the price of the Spread Option will be

$$V_0 = \sup_{\tau \in [0,T]} \mathbb{E}_0^*(e^{-\tau r}(S_\tau^2 - S_\tau^1 - K))$$

We can estimate the American Spread Call option price by using Least Square Monte Carlo Simulation method, as it was done to price American Put Options (in the previous Chapter.)

The valuation technique is based on the following idea: for every time step and every stock price node estimate the Expected Continuation Value (ECV) and the Exercise Value (EV) of the option and take the lager of the two.

$$V_t = \max(EV_t,\ \mathbb{E}CV_t | \mathcal{F}_t) \text{ for any } t \leq T$$

The challenge here, as it was with the case of pricing of American Put options, is to estimate the expected continuation value: $\mathbb{E}CV_t = \mathbb{E}_t^*(\text{Sum of all discounted Cash Flows after time } t|\mathcal{F}_t)$, which can be estimated by the LSMC method.

Comment: Use first 3 Monomials as Basis Functions for both factors when using the LSMC method: a constant, and first two powers of prices, which will result in a total of 9 basis functions:

$1, S_t^1, (S_t^1)^2, S_t^2, (S_t^2)^2, S_t^1 S_t^2, S_t^1 (S_t^2)^2, (S_t^1)^2 S_t^2, (S_t^1)^2 (S_t^2)^2$

Adding more basis functions has proven to have only marginal effect on the accuracy of results. In fact, removing last 3 of the cross-products of basis functions will not result in a large loss of accuracy, but will reduce the computational cost, as can be tested numerically. That is, using the following $k = 6$ functions in the LSMC method (in which we will have two independent variables - $X_1 = S_t^1 \ and \ X_2 = S_t^2$):

$L_1(S_t^1, S_t^2) = 1$; $L_2(S_t^1, S_t^2) = S_t^1$; $L_3(S_t^1, S_t^2) = (S_t^1)^2$; $L_4(S_t^1, S_t^2) = S_t^2$; $L_5(S_t^1, S_t^2) = (S_t^2)^2$; $L_6(S_t^1, S_t^2) = S_t^1 S_t^2$.

Asian Options

Asian options have a stronger degree of path dependency. The option payoff depends on the average stock price over the option's life. Define two averages: Arithmetic and Geometric:

$$\underset{(Arithmetic)}{\underline{A_a = \frac{1}{T}\int_0^T S_t dt}} \qquad\qquad \underset{(Geometric)}{\underline{A_g = e^{\frac{1}{T}\int_0^T lnS_t dt}}}$$

The following 4 payoffs are based on average stock prices, **A**. The averages are used as the strike prices or as the underlying price in the payoff of the option at maturity as follows.

Fixed strike Asian Call payoff: $max\ (\boldsymbol{A} - K, 0)$

Fixed strike Asian put payoff: $max\ (K - \boldsymbol{A}, 0)$

Floating strike Asian call payoff: $max\ (S_T - \boldsymbol{A}, 0)$

Floating strike Asian put payoff: $max\ (\boldsymbol{A} - S_T, 0)$.

For the **geometric average option prices**, **there exist closed-form solutions** but not for the arithmetic-average options. In every case, however, we can estimate the options' prices by using Monte Carlo simulation techniques.

<u>Steps to price Asian Arithmetic Average Rate Call options by simulation:</u>

STEP 1: Take a uniform partition of the time interval [0,T]: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, where $t_k = \frac{T}{n}k = \Delta k$, where $\Delta = \frac{T}{n}$.

STEP 2: Simulate N paths of stock prices on $t_0, t_1, \ldots, t_{n-1}, t_n$: $\{S_{t_1}^i, \ldots, S_{t_m}^i\}_{i=1}^N$

STEP 3: Compute the average stock price along each path: $A^i = \frac{1}{m}\sum_{j=1}^m \{S_{t_j}^i$

STEP 4: Option Price ≈ PV of the Average of $(A^j - K)^+ = e^{-rT} \cdot \frac{1}{N}\sum_{i=1}^N \left(A^i - K\right)^+$

Lookback Options

Extreme values of stocks are monitored during the life of the option and option payoffs are dependent on the lowest and highest observable stock prices during the life of the option.

<u>Lookback call options</u>:

Floating Lookback Call payoff: $(S_T - S_{min})^+ = S_T - S_{min},\ \ S_{min} = \min\{S_t, t \in [0, T]\}$

Fixed Strike Lookback Call payoff: $(S_{max} - K)^+,\ \ S_{max} = \max\{S_t, t \in [0, T]\}$

<u>Lookback put options</u>:

Floating Lookback Put payoff: $(S_{max} - S_T)^+ = S_{max} - S_T,$

Fixed Strike Lookback Put payoff: $(K - S_{min})^+$

There is an analytic formula for pricing a Floating Lookback Call (FLC) option provided below.

Assume the underlying pays dividends continuously at a rate of q.

For the case when $\boldsymbol{r \neq q}$:

$$C_{FLC} = S_0 e^{-qT} N(a_1) - S_{min} e^{-rT} N(a_2) - S_0 e^{-qT} \frac{\sigma^2}{2(r-q)} N(-a_1) + S_0 e^{-rT} \frac{\sigma^2}{2(r-q)} \left(\frac{S_0}{S_{min}}\right)^{\frac{-2(r-q)}{\sigma^2}} N(-a_3)$$

$$a_1 = \frac{\ln\left(\frac{S_0}{S_{min}}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},\ a_2 = a_1 - \sigma\sqrt{T}$$

$$a_3 = \frac{\ln\left(\frac{S_0}{S_{min}}\right) + \left(-r + q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$S_{min} = \min\{S_t, 0 \leq t \leq today\}$, and at origination, $S_{min} = S_0$.

For the case when $\boldsymbol{r = q}$, the price is given by:

$$C_{FLC} = S_0 e^{-rT} N(a_1) - S_{min} e^{-rT} N(a_2) + S_0 e^{rT} \sigma\sqrt{T}\left(n(a_1) + a_1(N(a_1) - 1)\right)$$

Volatility Swaps and Variance Swaps

The importance of Volatility and Variance have been subjects to many studies and volatility and variance have been promoted to be important assets. Volatility and Variance Swaps are examples of popular securities to trade on volatility.

A volatility swap is an agreement to exchange realized volatility of an asset (from time 0 to T) for a pre-specified fixed volatility σ_X.

The realized volatility is calculated as $\sqrt{\frac{1}{n-1}\sum_{i=1}^{n} r_i^2} \cdot \sqrt{\tau}$, where r_i is the return in period i, τ ~ frequency (τ =12 for monthly, 252 for daily data). For daily data we have

$$s = \sqrt{\frac{252}{n-1}\sum_{i=1}^{n}\left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2}$$

where S_i is the stock price on day i.

Payoff to the fixed-volatility-payer: $L_{vol}(s - \sigma_X)$, where L_{vol} is the notional principal amount, σ_X is the fixed prespecified volatility (implied volatility that is measured on the day when the contract is entered into) and s is the ex-post realized volatility.

The following Bloomberg screenshots demonstrate the uncertainty of volatility (non- constant and stochastic), compare the historical volatility to implied volatilities, and capture the changes in the volatility around earnings announcements.

Variance Swaps

An agreement to exchange the realized variance of an asset s^2 (between time 0 and T) with pre-specified variance rate σ_X^2.

The payoff of the Variance Swap to the fixed-variance payer is $L_{var}(s^2 - \sigma_X{}^2)$.

Valuation of Variance Swaps

For any S^* (value of asset price), the expected value of the variance (from time 0 to T) is given by:

$$\mathbb{E}^*(s^2) = \frac{2}{T}\ln\left(\frac{F_0}{S^*}\right) - \frac{2}{T}\left(\frac{F_0}{S^*} - 1\right) + \frac{2}{T}\int_0^{S^*} \frac{1}{k^2} e^{rT} \cdot P(k)dk + \frac{2}{T}\int_{S^*}^{\infty} \frac{1}{k^2} e^{rT} \cdot C(k)dk \qquad (*)$$

where F_0 is the forward price of the asset at maturity T, $C(K)$ is the price of a European call with strike price K, maturity T, $P(K)$ is the price of European put with strike K, maturity T.

The <u>value today of the agreement</u> to pay ${\sigma_X}^2$ and receive the realized variance s^2 between 0 and T is:

$$L_{var}(\mathbb{E}^*(s^2) - {\sigma_X}^2) \cdot e^{-rT}$$

In order to price variance swaps, one needs to implement (*) to estimate $\mathbb{E}^*(s^2)$. Belo we describe the steps to numerically estimate $\mathbb{E}^*(s^2)$.

Choose $K_1 < \cdots < K_n$ so that there are European options on the underlying asset with K_i strike prices. Set S^* in (*) as

$$S^* = \max\{K_i, \qquad \text{so that,} \qquad \text{for all i,} \qquad K_i \le F_0\}$$

Then, we can approximate the expected variance as follows:

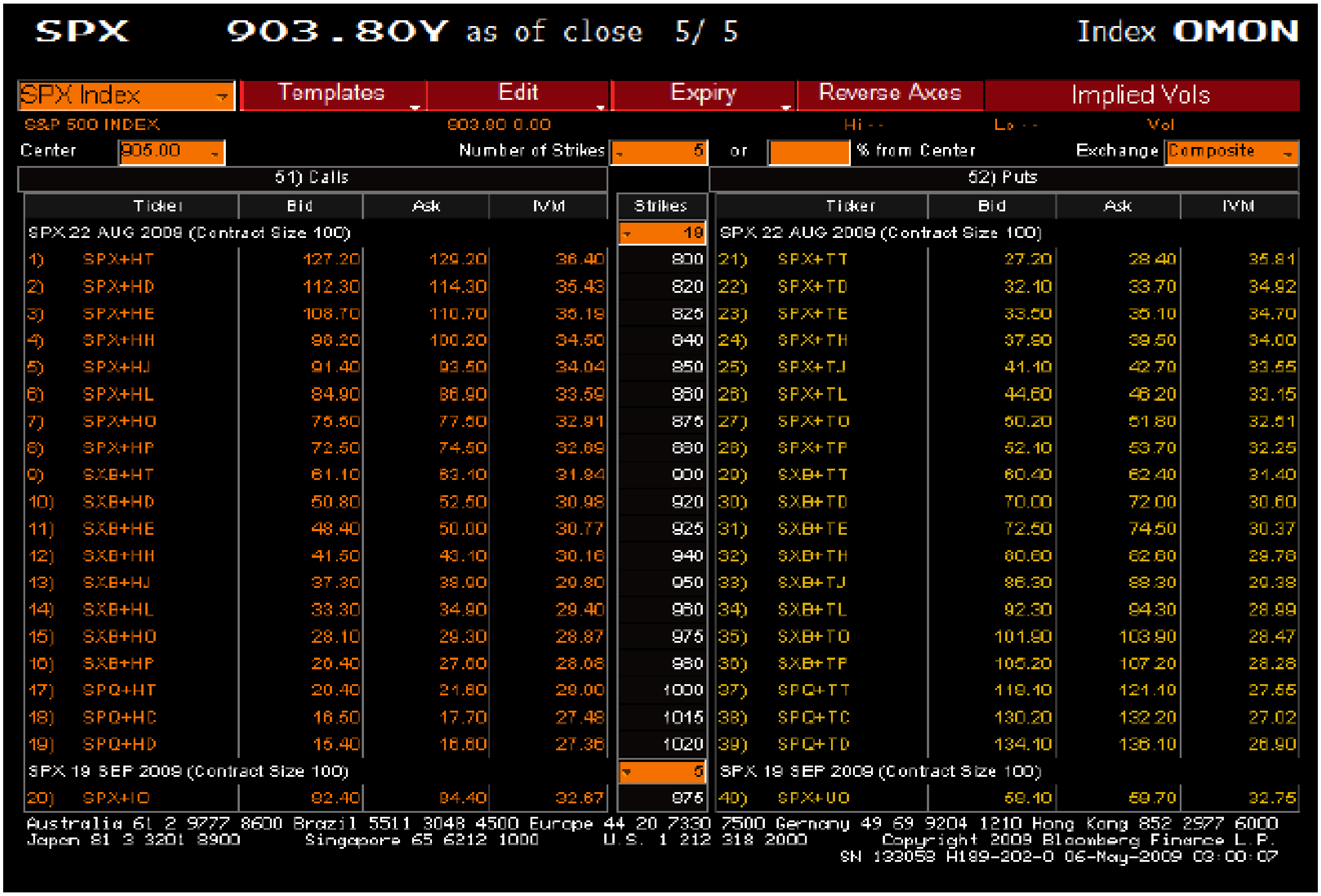
$$\mathbb{E}^*(s^2) \approx \frac{2}{T}\ln\left(\frac{F_0}{S^*}\right) - \frac{2}{T}\left(\frac{F_0}{S^*} - 1\right) + \frac{2}{T} \cdot \left\{\sum_{i=1}^{n} \frac{\triangle K_i}{{K_i}^2} \cdot e^{rT} \cdot U(K_i)\right\}$$

where $\triangle K_i = \frac{1}{2}(K_{i+1} - K_{i-1})$ for $2 \le i \le n-1$, $\triangle K_1 = K_2 - K_1$, $\triangle K_n = K_n - K_{n-1}$, and

$$U(K_i) = \begin{cases} \text{Price of European put with } K_i \text{ strike, if } K_i < S^* \\ \text{Price of European call with } K_i \text{ strike, if } K_i > S^* \\ \frac{\textit{European call+European put price}}{2}, \textit{if } K_i = S^* \end{cases}.$$

Example: Valuation of a Variance Swap. On May 6, 2009, the following information was obtained from Bloomberg to value a variance swap on S&P500 index:

S_0 = 903.80, r = 1%, q = 1%, T = 3 months,



SPX 903.80Y as of close 5/ 5 Index OMON

SPX Index | Templates | Edit | Expiry | Reverse Axes | Implied Vols

S&P 500 INDEX 903.80 0.00 Hi - - Lo - - Vol

Center 905.00 Number of Strikes 5 or % from Center Exchange Composite

51) Calls Ticker	Bid	Ask	IVM	Strikes	52) Puts Ticker	Bid	Ask	IVM
SPX 22 AUG 2009 (Contract Size 100)				19	SPX 22 AUG 2009 (Contract Size 100)			
1) SPX+HT	127.20	129.20	36.40	800	21) SPX+TT	27.20	28.40	35.81
2) SPX+HD	112.30	114.30	35.43	820	22) SPX+TD	32.10	33.70	34.92
3) SPX+HE	108.70	110.70	35.19	825	23) SPX+TE	33.50	35.10	34.70
4) SPX+HH	98.20	100.20	34.50	840	24) SPX+TH	37.90	39.50	34.00
5) SPX+HJ	91.40	93.50	34.04	850	25) SPX+TJ	41.10	42.70	33.55
6) SPX+HL	84.90	86.90	33.59	860	26) SPX+TL	44.60	46.20	33.15
7) SPX+HO	75.50	77.50	32.91	875	27) SPX+TO	50.20	51.80	32.61
8) SPX+HP	72.50	74.50	32.69	880	28) SPX+TP	52.10	53.70	32.25
9) SXB+HT	61.10	63.10	31.84	900	29) SXB+TT	60.40	62.40	31.40
10) SXB+HD	50.80	52.50	30.98	920	30) SXB+TD	70.00	72.00	30.60
11) SXB+HE	48.40	50.00	30.77	925	31) SXB+TE	72.50	74.50	30.37
12) SXB+HH	41.50	43.10	30.16	940	32) SXB+TH	80.60	82.60	29.78
13) SXB+HJ	37.30	38.90	29.80	950	33) SXB+TJ	86.30	88.30	29.38
14) SXB+HL	33.30	34.90	29.40	960	34) SXB+TL	92.30	94.30	28.99
15) SXB+HO	28.10	29.30	28.87	975	35) SXB+TO	101.90	103.90	28.47
16) SXB+HP	26.40	27.60	28.68	980	36) SXB+TP	105.20	107.20	28.28
17) SPQ+HT	20.40	21.60	28.00	1000	37) SPQ+TT	119.10	121.10	27.55
18) SPQ+HC	16.50	17.70	27.48	1015	38) SPQ+TC	130.20	132.20	27.02
19) SPQ+HD	15.40	16.60	27.36	1020	39) SPQ+TD	134.10	136.10	26.90
SPX 19 SEP 2009 (Contract Size 100)				5	SPX 19 SEP 2009 (Contract Size 100)			
20) SPX+IO	92.40	94.40	32.67	875	40) SPX+UO	58.10	59.70	32.75

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SN 133058 H189-202-0 06-May-2009 03:00:07

Figure 1: **Bloomberg Screen for Options on S&P 500 Index on May 6 2009**.

We would like to price a swap in which the investor pays the realized variance and receives ${\sigma_X}^2 = 0.12$ on \$1,000,000 notional amount. Denote $U(K_i) = U_i$. Choose K_i as follows:

$$K_1 = 800 \quad K_2 = 820 \quad K_3 = 825, K_4 = 840 \quad K_5 = 850 \quad K_6 = 860, K_7 = 875$$
$$K_8 = 880 \quad K_9 = 900, \quad K_{10} = 920 \quad K_{11} = 925 \quad K_{12} = 940, K_{13} = 950 \quad K_{14} = 960$$
$$K_{15} = 975, \; K_{16} = 980 \quad K_{17} = 1000 \quad K_{18} = 1015, K_{19} = 1020.$$

$$F_0 = S_0 e^{(r-q)T} = 903.80, \text{ then } S^* = 900.$$

The prices of call and put options are found to be as follows (we take the bid-ask midpoint of the quotes). For $K_i < 900$ we use the prices of puts, for $K_i = 900$ we use the average of the call and put, and for $K_i > 900$ we use the prices of calls:

$U_1 = 27.80$, $U_2 = 32.90$, $U_3 = 34.30$, $U_4 = 38.70$, $U_5 = 41.90$, $U_6 = 45.40$

$U_7 = 51.00$, $U_8 = 52.90$, $U_9 = 61.75$. $U_{10} = 51.65$, $U_{11} = 49.20$, $U_{12} = 42.30$

$U_{13} = 38.10$, $U_{14} = 34.10$, $U_{15} = 28.70$, $U_{16} = 27.00$, $U_{17} = 21.00$, $U_{18} = 17.10$, $U_{19} = 16.00$.

Then, $\sum_{i=1}^{19} \frac{\Delta K_i}{K_i{}^2} e^{rT} \cdot U_i = x$ (do the calculations) and $\mathbb{E}^*(s^2) \approx \frac{2}{0.25} \ln\left(\frac{903.8}{900}\right) - \frac{2}{0.25} \cdot \left(\frac{903.8}{900} - 1\right) + \frac{2}{0.25} \cdot x = y$. Value of variance swap $= \$1\, m \cdot (y - 0.12) \cdot e^{-0.01 \cdot \frac{1}{4}}$

Valuation of Volatility Swap

We need to estimate $\mathbb{E}^*(s) = \mathbb{E}^* \sqrt{\mathbb{E}^*(s^2)(1 + \frac{S^2 - \mathbb{E}^* s^2}{\mathbb{E}^* s^2}})$.

By using the following second order Taylor approximation we can write:

$$f(x) = \sqrt{1 + x} \approx f(0) + f'(0) \cdot x + \frac{1}{2} f''(0) \cdot x^2$$

which yields this approximation:

$\sqrt{(1 + x)} \approx 1 + x\left(1 + \frac{1}{2 \cdot \sqrt{1+x}} \Big|_{x=0}\right) - \frac{1}{2} x^2 \left(\frac{1/\sqrt{1+x}}{4(1+x)} \Big|_{x=0}\right) \approx 1 + \frac{x}{2} - \frac{x^2}{8}$. Then, taking $x = \frac{S^2 - \mathbb{E}^* s^2}{\mathbb{E}^* s^2}$ we can write

$$\mathbb{E}^*(s) = \sqrt{\mathbb{E}^*(s^2)}\ \mathbb{E}^* \left(1 + \frac{S^2 - \mathbb{E}^* S^2}{2 \mathbb{E}^* S^2} - \frac{1}{8} \frac{(S^2 - \mathbb{E}^* S^2)^2}{(\mathbb{E}^* S^2)^2} \right) = \sqrt{\mathbb{E}^*(s^2)} \cdot \left(1 - \frac{1}{8} \frac{Var^*(S^2)}{(\mathbb{E}^* S^2)^2}\right)$$

Therefore, the value of the volatility swap is $L_{vol} \cdot [\mathbb{E}^*(s) - \sigma_X] \cdot e^{-rT}$.

For the derivation of $Var^*(S^2)$ see the following articles.

- Brockhaus, O. and Long, D. (2000) Volatility swaps made simple, Risk, 2(1), pp. 92–95.
- Javaheri, A. et al. (2002) GARCH and volatility swaps, Wilmott Magazine, January, pp. 1–17.
- Robert J. Elliott , Tak Kuen Siu & Leunglung Chan (2007). Pricing Volatility Swaps Under Heston's Stochastic Volatility Model with Regime Switching, Applied Mathematical Finance, 14:1, 41-62, DOI: 10.1080/13504860600659222
- Swishchuk, A. (2005) Modeling and pricing of variance swaps for stochastic volatilities with delay, Wilmott Magazine.

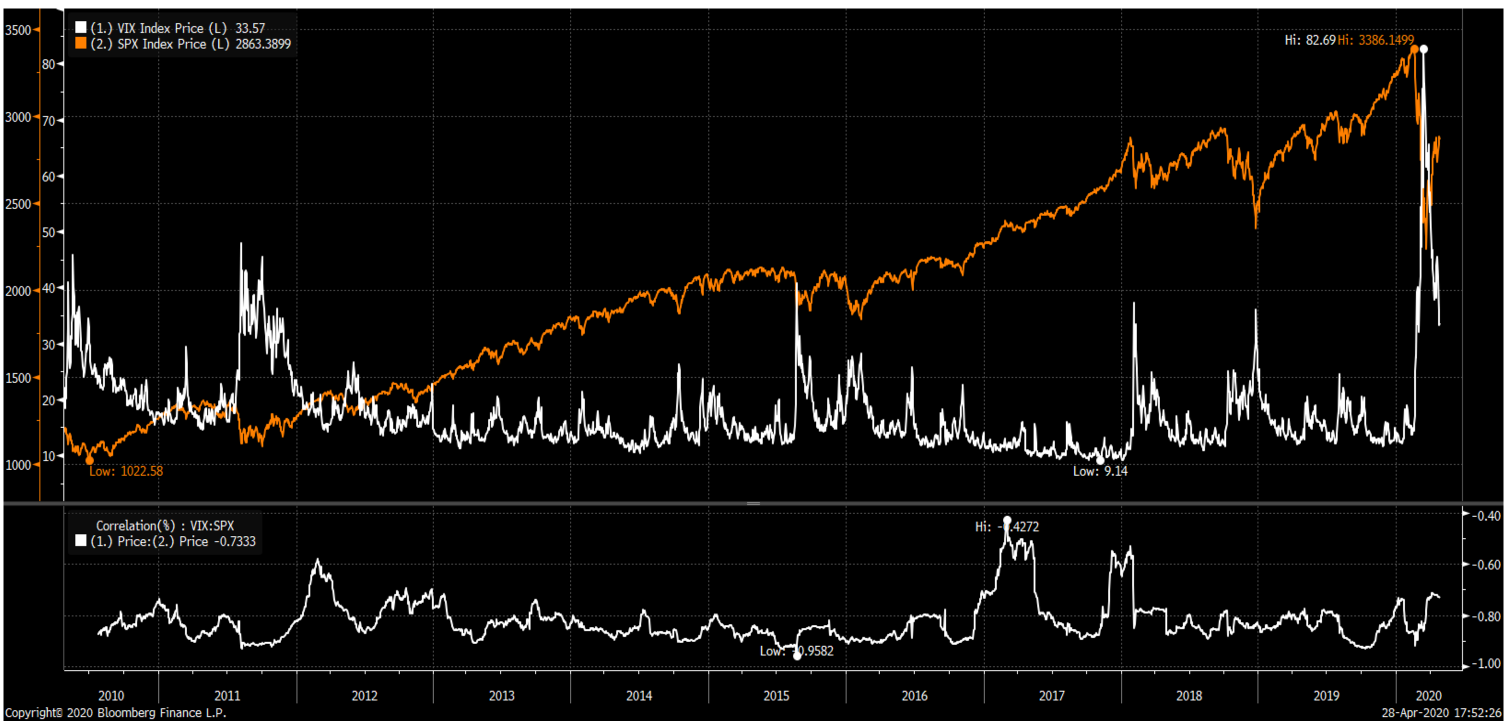
The VIX Index

The VIX index is a measure of market expectations of near-term volatility conveyed by S&P500 stock index option prices. It was introduced in 1993 and has been perceived to be a gauge of investors' sentiment and market volatility. The VIX index is negatively correlated with S&P500 index, the correlation being less than -50%. The following figures demonstrate the negative correlation between S&P500 and VIX on three different time periods.

Figure 2: Bloomberg Screen. SPX and VIX. Bloomberg Screen.

Figure 3: Bloomberg Screen. SPX and VIX.

Figure 4: Bloomberg Screen. SPX and VIX.



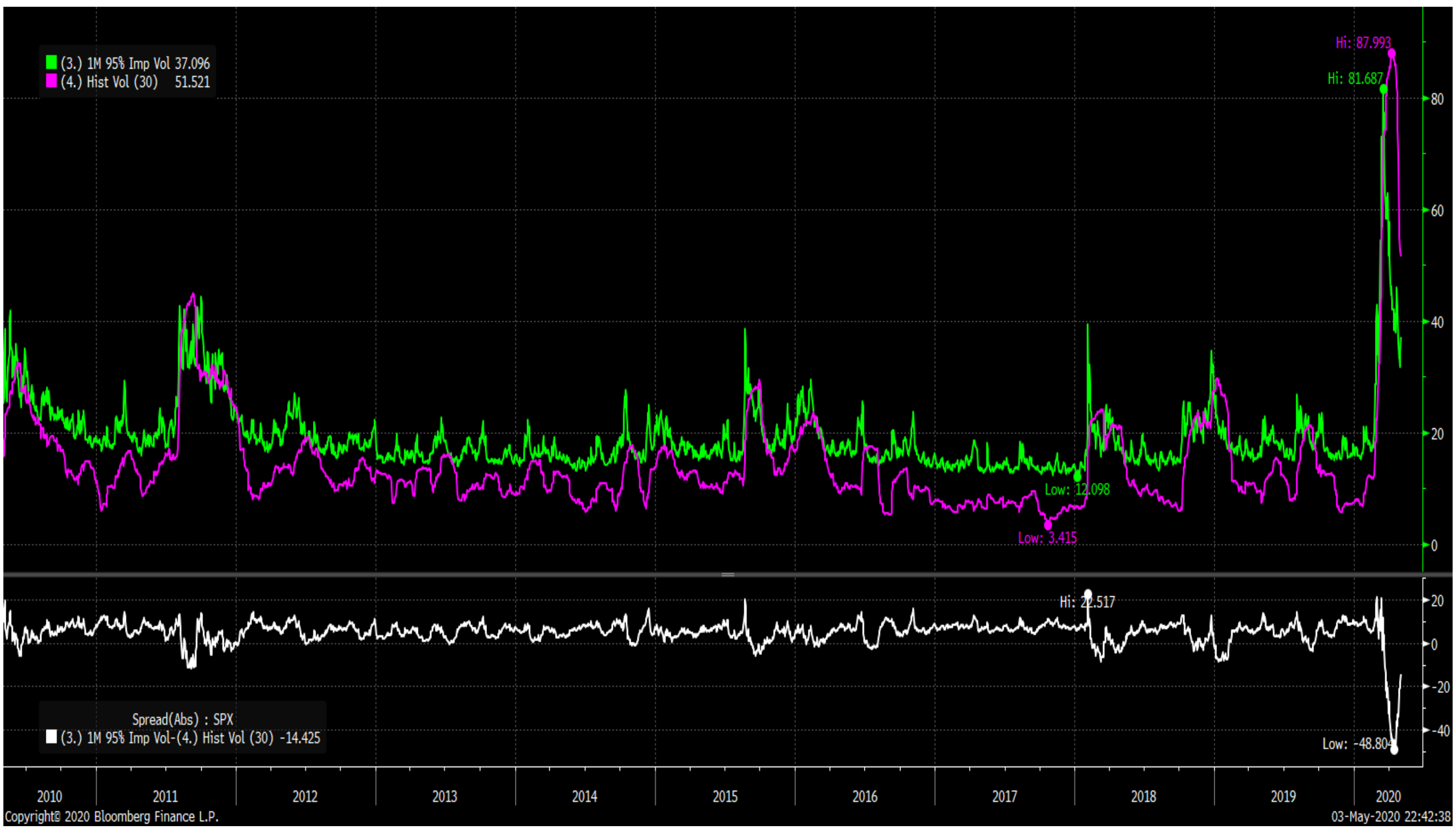
The calculation of the VIX index is based on the formula:

$$\mathbb{E}^*(s^2) = \frac{2}{T}\ln\left(\frac{F_0}{S^*}\right) - \frac{2}{T}\left(\frac{F_0}{S^*} - 1\right) + \frac{2}{T}\int_0^{S^*} \frac{1}{k^2} e^{rT} \cdot P(k)dk + \frac{2}{T}\int_{S^*}^{\infty} \frac{1}{k^2} e^{rT} \cdot C(k)dk$$

Interested readers are referred to www.cboe.com for more information and details on the VIX Index, futures and options on the index, its construction and analysis.

The following figure demonstrates the relationship between Implied and Historical volatilities of the S&P500 Index (SPX) by comparing 30 day historical volatility with 30-day implied volatility of 95% OTM Options on SPX.

Figure 5: Bloomberg Screen. S&P500 (SPX): IV (30; 95% OTM), HV (30).



The following six screenshots demonstrate the relationship between Implied Volatility (IV) of 30-day OTM Options, the Historical Volatility (HV) of stock prices on a 30-day period, and Earnings Announcements for FB, AMZN, GOOGL, and AMD.

Figure 6: Bloomberg Screen. FB: IV (30, 90% OTM), HV (30), and Earnings Announcements.

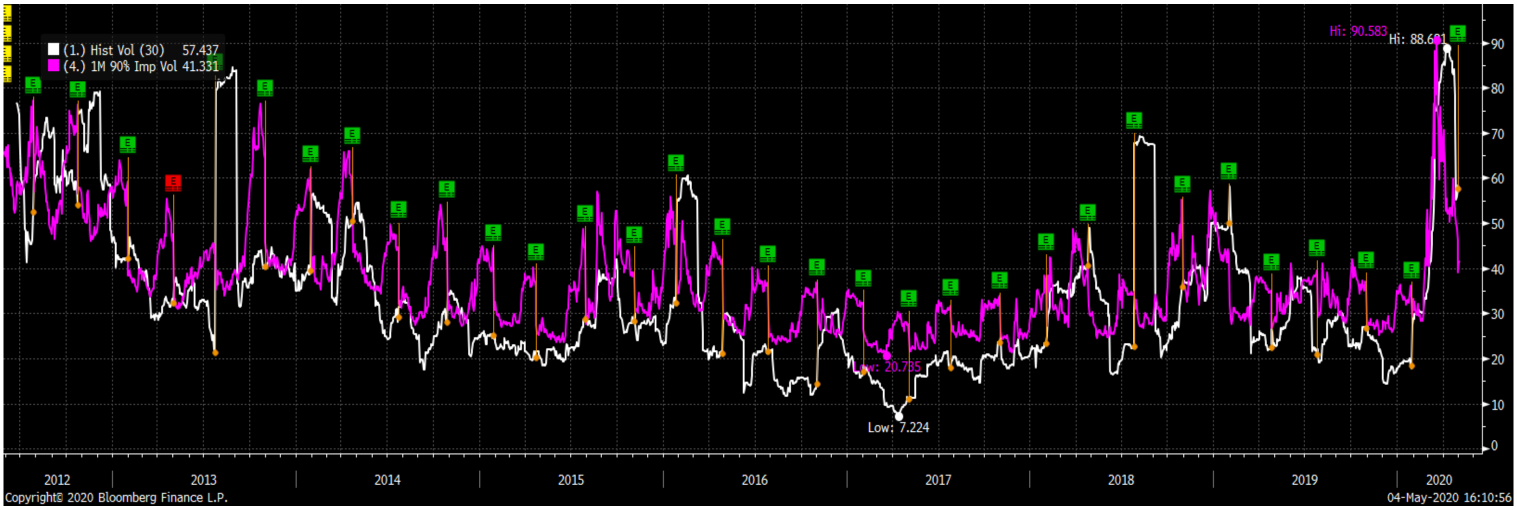


Figure 7: Bloomberg Screen. FB: IV (30, 100% ATM), and Earnings Announcements.

Figure 8: Bloomberg Screen. AMZN: IV (30, 90% OTM), HV(30), and Earnings Ann-ts.

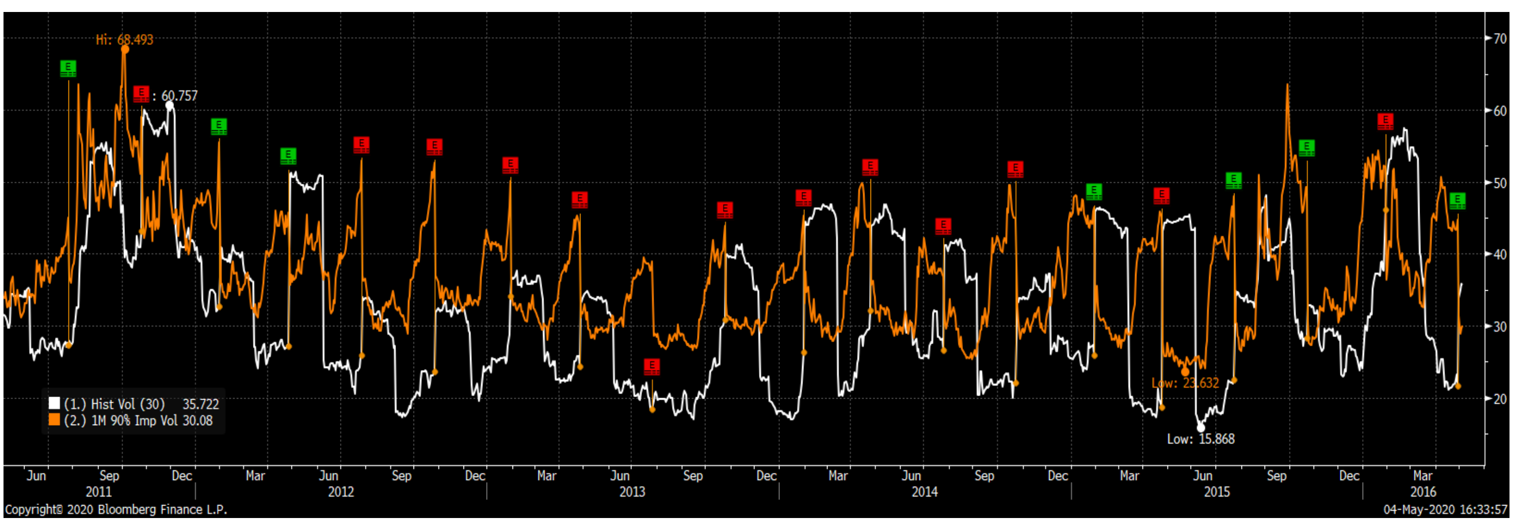


Figure 9: Bloomberg Screen. GOOGL: IV (30, 90% OTM), and Earnings Announcements.

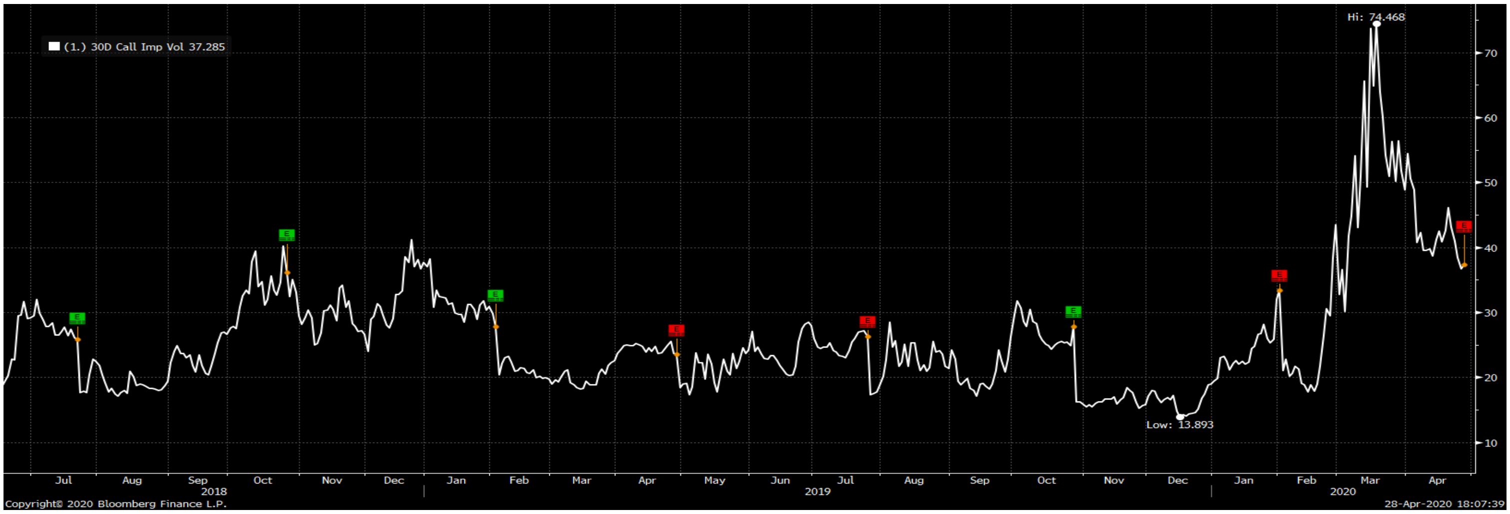
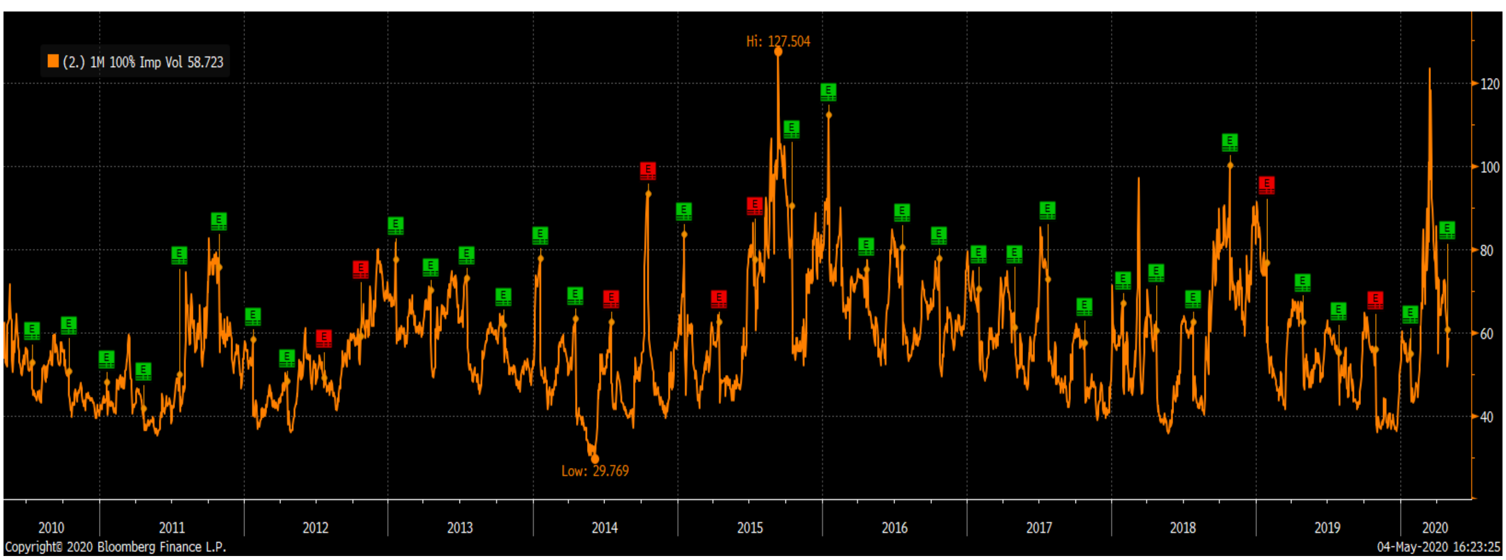


Figure 10: Bloomberg Screen. GOOGL: IV(30, 100% ATM), Stock Prices, and Earnings Ann-ts.

Figure 11: Bloomberg Screen. AMD: IV (30, 100%) and Earnings Announcements.



Exercises

1. Consider a 12-month Fixed Strike Lookback Call and Put options, when the interest rate is 3% per annum, the volatility is 30% per annum, and the strike price is \$100. Use the MC simulation method to estimate the prices of the Call and Put options. The payoff of the call is $(S_{max} - X)^+$, where $S_{max} = \max\{S_t : t \in [0, T]\}$, and the payoff of the put option is: $(X - S_{min})^+$, where $S_{min} = \min\ \{S_t : t \in [0, T]\}$.

2. Compute, via MC simulation, the prices of the following options using 50,000 simulations of paths of the stock price and dividing the time-interval into 50 equal parts:

   (a) Down-and-Out- Put: $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$, S_b=40.

   (b) Down-and-In - Put: $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$, S_b=40.

3. Compute the price of the Asian average rate and Asian average strike call options by using:

   (a) Standard MC method, where $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$.

   (b) Halton's Low-discrepancy sequences to generate the paths of the stock price, where $S(0) = 50,\ X = 50,\ r = 0.1,\ T =$ 2 months, $\sigma = 0.4$.

4. Compute the price of the Floating Strike Lookback Call and Put options by using: $S(0) = 50$, $X = 50,\ r = 0.05,\ q = 0.03, T =$ 2 months, $\sigma = 0.4$.

5. Assume the stock price follows an Arithmetic Brownian Motion. Derive the formula for a price of a European call option, using all the other Black-Scholes assumptions.

Simulation of Jump-Diffusion Processes

We have modeled the stock price dynamics using the Geometric Brownian Motion process in earlier chapters. Thus, we have implicitly assumed that stock prices do not jump, so we could use diffusion processes to model their movements. Considering a few stock market crashes (example: 2020, 2008, 1987) and also the fact that prices are only recorded at discrete points in time, it seems reasonable to allow the possibility of the stock price jumps at random times and study such models of prices.

We assume that the dynamics of the stock price are given by the following jump-diffusion equation:

$$\frac{dS_t}{S_{\bar{t}}} = \mu dt + \sigma dW_t + dJ_t, \quad W \perp J,$$

where $J_t = \sum_{j=1}^{N_t}(Y_j - 1)$ is a particular case, where Y_j are random variables and

N_t is the number of arrivals in $[0, t]$, $dJ_t = d\left(\sum_{j=1}^{N_t}(Y_j - 1)\right) = \begin{cases} Y_j - 1 & if \quad Y_i > 0 \\ 0 & else \end{cases}$

Here dJ_t is the jump at time t, and the size of the jump at that time is $Y_j - 1$. Thus, we have

$dS_t = \mu S_{t-}dt + \sigma S_{t-}dW_t + S_{t-}dJ_t$, where $S_{t-} = \lim_{u \to t-} S_u$.

In case if a jump occurs at time t_j,we can discretize the process at that time as follows:

$$S_{t_j} - S_{t_{\bar{j}}} = S_{t_{\bar{j}}}\left[J_{t_j} - J_{t_{\bar{j}}}\right] = S_{t_{\bar{j}}} \cdot (Y_j - 1)$$

Or, $S_{t_j} = S_{t_{\bar{j}}} \cdot Y_j$. If we consider $\ln S_t$, then, $\ln S_{t_j} = \ln S_{t_{\bar{j}}} + \ln Y_j$

Thus, jumps are additive when the $\ln(price)$ process is considered. Thus, the stock price can be written as

$$S_t = S_0 \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_\tau} \cdot \prod_{j=1}^{N_t} Y_j$$

Example. Assume N_t~Poisson(λ). Then, the times between any two consecutive realizations of a Poisson process are $\exp(\lambda)$ −distributed.

Assume Y_j are $i.i.d.$, $N_U \perp W_t$. We now discuss two different methods to simulate a jump-diffusion.

Method 1: Timeline

We have $\prod_{l=N_{t_j}+1}^{N_{t_{j+1}}} Y_l = 1$ if $N_{t_{j+1}} = N_{t_j}$. Since $S_{t_{j+1}} = S_{t_j} \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)(t_{j+1}-t_j)+\sigma\left(W_{t_{j+1}}-W_{t_j}\right)}$.

$\prod_{l=N_{t_j}+1}^{N_{t_{j+1}}} Y_l$, For the discretized $X_t = \ln S_t$ process:

$$X_{j+1} = X_j + \left(\mu - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j) + \sigma\left(W_{t_{j+1}} - W_{t_j}\right) + \sum_{j=N_{t_j}+1}^{N_{t_{j+1}}} \ln Y_j$$

<u>Step 1:</u> Generate $Z_i \sim N(0,1)$ at step $i(t_i \to t_{i+1})$

<u>Step 2:</u> Generate $N_i \sim \text{Poisson}\big(\lambda(t_{i+1} - t_i)\big)$ (number of jumps in $[t_i, t_{i+1}]$ interval)

<u>Step 3:</u> Generate $\ln Y_1, \ldots, \ln Y_{N_i}$, and set $M_i = \left(\ln Y_1 \cdots Y_{N_i}\right)$

<u>Step 4:</u> $X_{i+1} = X_i + \left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} \cdot Z_i + M_i$

Recall that, Poisson processes possess the following properties, that:

(1) $N_{t_{i+1}} - N_{t_i} \sim P\big(\lambda(t_{i+1} - t_i)\big)$
(2) $N_t - N_s \perp N_s - N_u, u < s < t$

Method 2: Simulating Jump-Times.

Simulate the process from one jump time τ_i to the next τ_{i+1}.

$S_{\tau_{\overline{i+1}}} = S_{\tau_i} \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)(\tau_{i+1}-\tau_i)+\sigma\left(W_{\tau_{i+1}}-W_{t_i}\right)}$ and $S_{t_{i+1}} = S_{\tilde{t}_{\overline{i+1}}} \cdot Y_{j+1}$. This implies, that

$$X_{i+1} = X_i + \left(\mu - \frac{1}{2}\sigma^2\right)(\Delta_i) + \sigma W_{\Delta_i} + \ln Y_{i+1}$$

Steps: $t_i \to t_{i+1}$

1. Simulate $Z_{i+1} \sim N(0,1)$, $T_{i+1} \sim \exp(\lambda) \perp Z_{i+1}$
2. $\tau_{i+1} = \tau_i + T_{i+1}$ (next jump-time)
3. $X_{i+1} = X_i + \left(\mu - \frac{1}{2}\sigma^2\right) T_{i+1} + \sigma\sqrt{T_{i+1}} Z_{i+1} + \ln Y_{i+1}$

Examples of $\boldsymbol{Y_i}$

1. $Y_i \sim \log N(a, b^2)$, then $\sum_{i=1}^{n} \ln Y_i \sim N(an, nb^2) = a \cdot n + b\sqrt{n} \cdot Z$
2. $\ln Y_i \overset{d}{=} \begin{cases} E^+ & w \cdot p \quad p \\ -E^- & w \cdot p \quad 1-p \end{cases}$ where $E^{\pm} \sim \exp\left(\frac{1}{\lambda}, \lambda\right) \sim \text{Do}uble\ Exponential.$

Application (Use of **Jump-Diffusion processes** and Pricing of Default Options)

Assume that the value of a collateral follows the following jump-diffusion process:

$$\frac{dV_t}{V_t^-} = \mu dt + \sigma dW_t + \gamma dJ_t$$

where $\mu, \sigma,\ \gamma < 0$, and V_0 are given, J is a Poisson process, with arrival rate λ_1, independent of the Brownian Motion process W. V_t^- is the value process before jump occurs at time t (if any).

Consider a loan on the above-collateral, with a contract rate of *r* per period, and maturity of *T* years. Assume that the value of that loan follows this process (t is time):

$$\boldsymbol{L_t = a - bc^t}$$

where $\boldsymbol{a > 0, b > 0\ , c > 1}$, **and** $\boldsymbol{L_0}$ are given. We have that $\boldsymbol{L_T = 0}$.

Notes:

1. The process *J* may have more than one realization during the life of the loan (T).
2. One may simulate the realizations of *J* using the relationship between Exponentials and Poisson processes: the time of occurrence of the first Poisson realization (and the time

between any two consecutive Poisson realizations) is exponentially distributed random variable with the same intensity as the one of the Poisson process.

3. L_t can be modeled as Brownian Bridge process: $L_0 = l$ and $L_T = 0$ are known.

The loan borrower has an embedded "default option" here, the exercise of which implies that the loan liability is released in return for surrendering the collateral.

Define the following stopping time:

$$Q = min\{t \geq 0:\ V_t \leq q_t L_t\}$$

Here, q_t is a deterministic but time-dependent function.

Q is the first time when the relative value of the collateral (with respect to the outstanding loan balance) hits a certain boundary, q_t , which is the "optimal exercise boundary of the embedded default option". The default option is exercised if this boundary is hit by the value process before time *T*.

Define another stopping time: $S = min\{t \geq 0: N_t > 0\}$.

N_t is a Poisson process with arrival rate λ_2 independent of the J and W. That is, *S* is the first time (after t=0, before time *T*) of a realization of another jump (*N*), which is independent of *J* and *W*. If no jump is realized until T, then *S* is taken to be ∞.

This can be thought of as a random time when the borrower defaults on the loan due an **external reason** such as inability or unwillingness to pay due to job loss or other events.

We assume the "default option" will be exercised if and only if $\tau = \min\{Q, S\} < T$.

That is, if the "optimal exercise boundary" is hit (i.e. $Q < T$), or a jump *(N)* occurs (i.e. $\tau < T$) during the life of the loan, **then the default option is exercised**. The option is exercised at the first time one of these two happens: either Q or τ.

Notes:

1. If $\min\{Q, S\} > T$ then, there is no default option exercise.
2. It is only the first realization of the *N* process (if it happens before *T*) that is useful here. That is, the option is exercised upon the first realization of *N* if and only if it happens before *T*.

Assume J_t is a Poisson process with arrival rate λ_1 and N_t is a Poisson process with arrival rate λ_2 , and that *N*, *J* and *W* are independent of each other.

Here, a, b, c are functions of λ_2 (λ_2 determines the default probability of the borrower, which will be related to his creditworthiness and will determine the loan contract rate).

Notes on Poisson processes:

1. J is Poisson process with an arrival rate λ means that:
   i. $J = 0$;
   ii. J has independent increments;
   iii. The number of arrivals (Poisson process realizations) in any interval of length T has Poisson (λT) distribution.
2. J is Poisson process with arrival rate λ. Then, $P(J = k) = \frac{e^{\lambda}\lambda^k}{k!}\ for\ k = 0,1,2, \ldots$

   and $E(J) = Var(J) = \lambda$.
3. J is Poisson process with arrival rate λ. Then, the **interarrival times** of this process, $X_1, X_{2,}$ ... are independent random variables, and $X_i \sim Exponential(\lambda)$ for $i = 1,2, \ldots$
4. J is Poisson process with arrival rate λ. Let X_1 be the first arrival time of this process. Then,

   $P(X_1 > t) = P\big(no\ arrival\ in\ [0, t)\big) = e^{-\lambda t}$, and $P(X_1 \leq t) = 1 - e^{-\lambda t}$. Therefore,

   $X_1 \sim Exponential(\lambda)$.
5. Now, assume X_2 is the elapsed time between the first arrival and the second arrival.

Then, $P(X_2 > t \mid X_1 = s) = P(no\ arrival\ in\ (s, s + t] \mid X_1 = s) = P(no\ arrival\ in\ (s, s + t]\,) = e^{-\lambda t}$. (due to independence of increments [0,s] and (s, s+t].)

That is, $X_1\ and\ X_2$ re independent random variables, and $X_2 \sim Exponential(\lambda)$.

Assume the APR of the loan is R and the contract rate per period is r. Assuming monthly compounding, we will have $r = R/12$. In fact, a, b, c are functions of the contract rate (r) and we model the dependence as follows: $R = r_0 + \delta\lambda_2$, where r_0 is the "prime" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate r. Assume $q_t = \alpha + \beta t$, where $\beta > 0, \alpha > L_0/V_0$.

Example 3.

Assume that the value of a collateral follows a jump-diffusion process:

$$\frac{dV_t}{V_t^-} = \mu dt + \sigma dW_t + \gamma dJ_t$$

where $\mu, \sigma,\ \gamma < 0$, and V_0 are given, J is a Poisson process, with arrival rate λ_1, independent of the Brownian Motion process W. V_t^- is the value process before jump occurs at time t (if any).

Consider a collateralized loan, with a contract rate per period r and maturity T on the above-collateral, and assume the outstanding balance of that loan follows this process:

$$\boldsymbol{L_t = a - bc^{12t}}$$

where $\boldsymbol{a > 0},\ \boldsymbol{b > 0},\ \boldsymbol{c > 1},\ \textbf{and}\ \boldsymbol{L_0}$ are given. We have that $\boldsymbol{L_T = 0}$.

Define the following stopping time:

$$Q = min\{t \geq 0:\ V_t \leq q_t L_t\}$$

This stopping time is the first time when the relative value of the collateral (with respect to the outstanding loan balance) crosses a threshold which will be viewed as the "optimal exercise boundary" of the option to default.

Define another stopping time, which is the first time an adverse event occurs:

$$S = min\{t \geq 0: N_t > 0\}$$

Assume that N_t is a Poisson process with arrival rate of λ_2. Define $\tau = \min\{Q, S\}$.

We assume the embedded default option will be exercised at time τ, if and only if $\tau < T$.

If the option is exercised at time Q then, the "payoff" to the borrower is $\left(L_Q - \epsilon V_Q\right)^+$.

If the option is exercised at time S then the "payoff" to the borrower is $abs(L_S - \epsilon V_S)$, where *abs(.)* is the absolute value function.

Notes:

1. If $\min\{Q, S\} > T$ then there is no default option exercise.
2. ϵ should be viewed as the recovery rate of the collateral, so $(1 - \epsilon)$ can be viewed as the rate of legal and administrative expenses.

Assume J_t is a Poisson process with arrival rate λ_1 and N_t is a Poisson process with arrival rate λ_2, and that *N*, *J* and *W* are independent of each other.

Assume the APR of the loan is $R = r_0 + \delta\lambda_2$ where r_0 is the "risk-free" rate, and δ is a positive parameter to measure the borrower's creditworthiness in determining the contract rate per period: *r*. We have monthly compounding here, so $r = R/12$.

Assume that $q_t = \alpha + \beta t$, where $\beta > 0$, $\alpha < V_0/L_0$ and $\beta = \frac{\epsilon - \alpha}{T}$.

Use r_0 for discounting cash flows. Use the following base-case parameter values:

$V_0 = \$20{,}000, L_0 = \$22{,}000,\ \mu = -0.1,\ \sigma = 0.2,\ \gamma = -0.4,\ \lambda_1 = 0.2,\ T = 5$ years, $r_0 = 0.02, \delta = 0.25, \lambda_2 = 0.4,\ \alpha = 0.7,\ \epsilon = 0.95$. Notice that, $PMT = \frac{L_0 . r}{\left[1 - \frac{1}{(1+r)^n)}\right]}$, where $r = R/12$, $n = T * 12$, and $a = \frac{PMT}{r}, b = \frac{PMT}{r(1+r)^n}$, $c = (1 + r)$. Notice that, $q_T = \epsilon$.

Write the code as a function Proj6_2funcion.* that takes λ_1, λ_2 and T as parameters, setting defaults if these parameters are not supplied, and outputs the default option price, the default probability and the expected exercise time.

Function specification:

```
function [D, Prob, Et] = Proj6_2funcion(lambda1, lambda2, T)
```

(a) Estimate the value of the default option for the following ranges of parameters:
- λ_1 from 0.05 to 0.4 in increments of 0.05;
- λ_2 from 0.0 to 0.8 in increments of 0.1;
- T from 3 to 8 in increments of 1;

(b) Estimate the default probability for the following ranges of parameters:
- λ_1 from 0.05 to 0.4 in increments of 0.05;
- λ_2 from 0.0 to 0.8 in increments of 0.1;
- T from 3 to 8 in increments of 1;

(c) Find the Expected Exercise Time of the default option, conditional on $\tau < T$. That is, estimate $E(\tau \mid \tau < T)$ for the following ranges of parameters:
- λ_1 from 0.05 to 0.4 in increments of 0.05;
- λ_2 from 0.0 to 0.8 in increments of 0.1;
- T from 3 to 8 in increments of 1;

Inputs: s*eed*

Outputs:

i. Values: the default option D, the default probability Prob and the expected exercise time Et for parts (a), (b) and (c) with λ_1=.2, λ_2=0.4 and T=5.
ii. Graphs: For each of (a), (b) and (c) two graphs as a function of T, first with λ_1=0.2 and λ_2 from 0.0 to 0.8 in increments of 0.1, then with λ_2 = 0.4 and λ_1 from 0.05 to 0.4 in increments of 0.05. Put the two graphs in one .png file.

(d) Make additional assumptions as necessary to estimate the IRR of the investment.

Note: The drift of the *V* process should be a function of r_0, λ_1, σ under the risk-neutral measure, to be able to price the option, but not done so in this case.

Implementation Tips for Example 3.

Take a uniform partition of the time interval:

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T, \text{ where } t_k = \frac{T}{n}k = \Delta k, \text{ where } \Delta = \frac{T}{n}.$$

Simulate M paths of the V $-$process:

{ V_j^i is the simulated value at time t_j along path i, where $j = 1, \ldots, n$ and $i = 1, \ldots, M$.}

For every path, we need to do the following:

Simulate a realization of the jump time S, and also calculate $Q = min\{t \geq 0\colon\ V_t \leq q_t L_t\}$ so that we can estimate the time of the exercise of the default-option as: $\tau = \min\{Q, S\}$.

For simulation of the jump time *S:*

All we need to do is to simulate a random variate X that has **exponential distribution** with parameter $\lambda_2 T$. That is, the CDF of X is given by $F(x) = P(X \leq x) = 1 - e^{-\lambda_2 T x}$, and the PDF of X is $f(x) = \lambda_2 T e^{-\lambda_2 T x}$. (**Note**: $E(X) = 1/(\lambda_2 T)$).

Then, set *S*=*X*.

The calculation of Q is as follows:

For all values of V_j^i along the path i, we need to find the first time (j) when $V_j^i \leq q_j L_j$.

(Notice that, q_j and L_j are not path-dependent: they have the same values at time j for all paths.

Set $\tau = \min\{Q, S\}$.

If $\tau \geq T$, then this path has no contribution to the calculation of the value of default option.

If $\tau < T$, then the payoff of the option depends on the relative values of S and Q as follows:

If $Q < S$ then the option-payoff to the borrower is: $payoff = \left(L_Q - \epsilon V_Q\right)^+$.

If $Q \geq S$ then the option-payoff to the borrower is: $payoff = abs(L_S - \epsilon V_S)$

Take $e^{-\tau r_0} payoff^i$ and add it to *SUM* that will determine the value of the option: after all *M* paths are considered and *SUM* is evaluated, we will have "Default Option Value" estimated as: *SUM/M*.

To simulate a path of the V-process (jump-diffusion process) on $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, and also at certain jump-times, follow these steps:

First, simulate the jump-times (could be multiple ones) for the process J on the [0,T] interval:

Simulate an independent random variate Y_1 that has exponential distribution with parameter $\lambda_1 T$. That is, the CDF of Y_1 is given by $F(y) = P(Y_1 \leq y) = 1 - e^{-\lambda_1 T y}$, and the PDF of Y_1 is $f(x) = \lambda_1 T e^{-\lambda_1 T y}$. (**Note**: $E(Y_1) = 1/\lambda_1 T$).

IF $Y_1 > T$ then that means that the realization of the jump *J* is NOT in [0,T], therefore it will not impact the V-process. In this case, there are no jumps in the path of V.

IF $Y_1 < T$, then Y_1 is the first jump time for *J*.

We will simulate another random variate Y_2 that has exponential distribution with parameter $\lambda_1 T$ and is independent of Y_1.

IF $Y_1 + Y_2 >$ T then, the <u>second</u> jump-realization of *J* is NOT in [0,T], therefore it will not impact the V-process. Therefore, there will be only one jump time, which is Y_1 that will be used in the calculations and in the simulation of a path of the V-process.

IF $Y_1 + Y_2 < T$ then, $Y_1 + Y_2$ is the second jump time of *J* in [0,T]. So far, we have 2 jump times for the process V in [0,T]. Those jump times are: Y_1, and $Y_1 + Y_2$.

Continue this process (of simulation exponentials) until for some integer l we have

$$Y_1 + \cdots + Y_l < T \quad \text{but} \quad Y_1 + \cdots + Y_l + Y_{l+1} \geq T.$$

Then, that means there will be exactly l jump-times for the process J in [0,T], which are:

Y_1, $Y_1 + Y_2$, ..., $Y_1 + Y_2 + \cdots + Y_l$. Call these jump-times $h_1, h_{2,\dots,} h_l$.

Now that jump-times are estimated along the path, we simulate the V-process (which incorporates the jumps) as follows. To simulate the V-process by taking the jump-times, $(h_1, h_{2,\dots,} h_l)$, into account, we do the following:

Take a uniform partition of the time interval [0,T]: $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, where $t_k = \frac{T}{n} k = \Delta k$, where $\Delta = \frac{T}{n}$.

Discretize the V-process using the Euler discretization scheme and simulate the process $\frac{dV_t}{V_t} = \mu dt + \sigma dW_t$on the set $t_0 < t_1 < \cdots$ until the first jump-time, which is h_1.

Note: If $h_1 \geq T$ then simulate the process V on the set $t_0 < t_1 < \cdots < t_{n-1} < t_n$ as there are no jumps along that path.

If $h_1 < T$, then, in addition to simulating the V-process on the set

$t_0 < t_1 < \cdots < t_{n-1} < t_n$ we need to simulate the process at time h_1.

Assume $t_j < h_1 < t_{j+1}$.

Simulate a path of the process V on the set $t_0 < t_1 < \cdots < t_j$:

$$V_{t_k} = V_{t_{k-1}} + V_{t_{k-1}} \mu\Delta + V_{t_{k-1}} \sigma\sqrt{\Delta}\, Z_k, \text{ for } k = 1, \dots, j.$$

Simulate $V_{h_1}^-$, the value of process V just before time h_1:

$$V_{h_1}^- = V_{t_j} + V_{t_j}\,\mu(h_1 - t_j) + V_{t_j}\sigma\sqrt{(h_1 - t_j)}\,Z_{j+1,-}$$

Simulate $V_{h_1}^+$, the value of process V just after the jump time h_1:

$$V_{h_1}^+ = V_{h_1}^-(1+\gamma).$$

Simulate $V_{t_{j+1}}$:

$$V_{t_{j+1}} = V_{h_1}^+ + V_{h_1}^+\,\mu(t_{j+1} - h_1) + V_{h_1}^+\sigma\sqrt{(t_{j+1} - h_1)}\,Z_{j+1,+}$$

Here, $Z_{j+1,-}$ and $Z_{j+1,+}$ are i.i.d. N(0,1).

Simulate the rest of the path of the process V on the set $t_{j+2} < t_{j+3} < \cdots < t_n$:

$$V_{t_{k+1}} = V_{t_k} + V_{t_k}\,\mu\Delta + V_{t_k}\sigma\sqrt{\Delta}\,Z_{k+1}, \text{ for } k = j+1, \ldots$$

Exactly the same way, we will continue and simulate the V-process from time h_1 to next jump-time h_2; then account for the jump at time h_2 as follows:

$$V_{h_2}^+ = V_{h_2}^-(1+\gamma).$$

Continue the process until we reach time T.

Notes:

1. For every path, we need to simulate the jump times first, then the process V.
2. From path-to-path, the number of jumps and their occurrence-times will be different.
3. Assume you have simulated a path of the process X at times $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, and an additional time step h is added to $t_0 < t_1 < \cdots < t_{n-1} < t_n$.

Assume that h falls in between time-steps t_j and t_{j+1}. Then, to estimate the value of the process X at time h, a linear interpolation can be used as follows: $X_{\boldsymbol{h}} = X_{t_j} + \frac{\boldsymbol{h}-t_j}{t_{j+1}-t_j} (X_{t_{j+1}} - X_{t_j})$