Investments

Topic 6: Portfolio Allocation for Multiple Assets

UCLA | Fall 2024

by Mikhail Chernov

This version: November 15, 2024

Overview of Topic 6

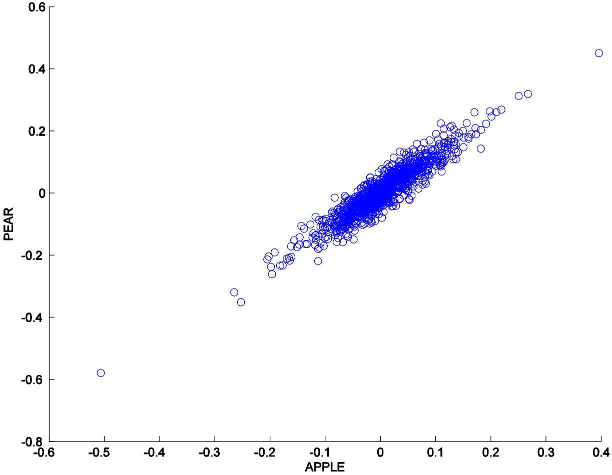
1. Two Risky Assets.
2. Benefits of Diversification.
3. Many Risky Assets.
4. Mean-Variance Optimization and the Portfolio Frontier.
5. Portfolio Frontier with Risky Assets Only.
6. Selecting a Risky Portfolio.
7. Portfolio Frontier with a Riskless Asset.

1. Two risky assets

Two risky assets

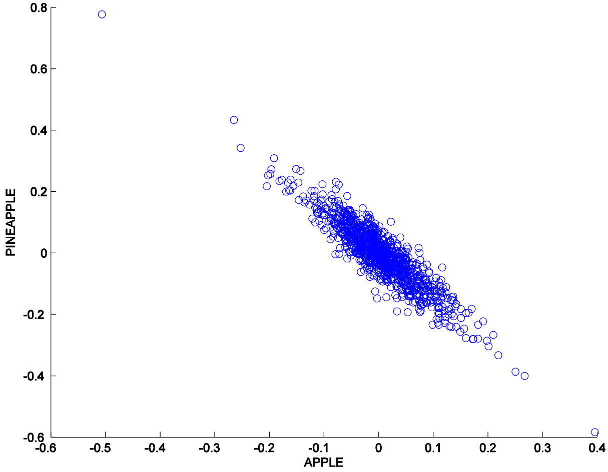
- We know how to handle the problem of the individual investor investing in the market.
- We now turn to the problem of the institutional investor: hedge or mutual fund manager
- You have chosen a number of assets in which to invest and now you must determine how to allocate capital across different positions.
- Consider the case with only 2 assets
- Statistical properties:
  - Means
  - Variance
  - Covariance / Correlation
- How does covariance affect the allocation?

Positive Correlation



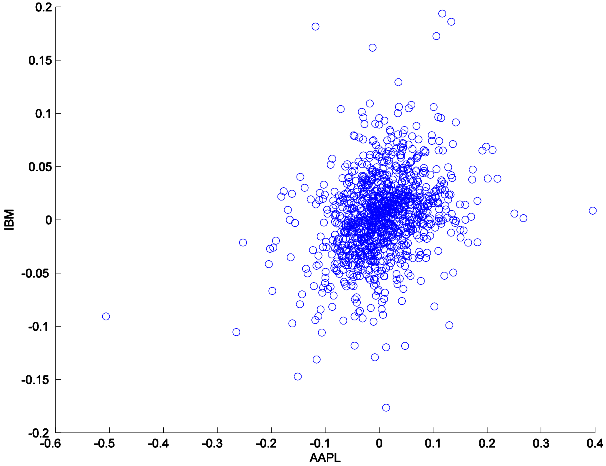
covariance is 0.0053; correlation is 0.94

Negative Correlation



covariance is -0.0063; correlation is -0.91

Realistic Correlation



correlation is 0.32

Variance

- The variance of a portfolio depends not only on the variances of the individual stocks, but also on their covariances.
- It is greater when the covariances are positive rather than negative.
- The variance of a portfolio with two stocks is

$$V(R) = w_1^2 V(R_1) + w_2^2 V(R_2) + 2 w_1 w_2 Cov(R_1, R_2).$$

Standard Deviation

- The equations for portfolio variance can be written in terms of standard deviation and correlation, rather than variance and covariance. Just replace
  - $V(R_n)$ by $\sigma(R_n)^2$
  - $Cov(R_n, R_m)$ by $\rho(R_n, R_m)\sigma(R_n)\sigma(R_m)$.

- For instance, the standard deviation of a portfolio of two stocks is given by

$$\sigma(R) = \sqrt{w_1^2\sigma(R_1)^2 + w_2^2\sigma(R_2)^2 + 2w_1w_2\rho(R_1, R_2)\sigma(R_1)\sigma(R_2)}.$$

- In practice, we do not know the standard deviations and correlations of the individual stocks, but can estimate them using sample standard deviations and correlations.

Standard Deviation: Example

Compute (an estimate of) the standard deviation of a portfolio consisting of \$300 in Disney and \$100 in IBM.

- We have
$$w_1 = \frac{300}{400} = 0.75, w_2 = \frac{100}{400} = 0.25.$$
- Estimating the standard deviations $\sigma(R_1)$ and $\sigma(R_2)$, and correlation $\rho(R_1, R_2)$, we get
$$\sigma(R_1) = 32.1\%, \sigma(R_2) = 25.1\%, \rho(R_1, R_2) = 0.38.$$
- The estimate of the standard deviation of the portfolio is given by

$$\begin{aligned} \sigma(R)^2 &= (0.75)^2\sigma(R_1)^2 + (0.25)^2\sigma(R_2)^2 \\ &\quad +2(0.75)(0.25)\rho(R_1, R_2)\sigma(R_1)\sigma(R_2) \\ \Rightarrow \sigma(R) &= 27.1\%. \end{aligned}$$

2. Benefits of Diversification

Expected Return

- Expected returns are the same as before, but both returns are uncertain now

$$E(R) = w_1 E(R_1) + w_2 E(R_2), \quad w_1 + w_2 = 1.$$

- Example: compute (an estimate of) the expected return of a portfolio consisting of \$300 in Disney and \$100 in IBM.
  - We have

    $$w_1 = \frac{300}{400} = 0.75$$

    and

    $$w_2 = \frac{100}{400} = 0.25.$$

  - Estimating the expected returns $E(R_1)$ and $E(R_2)$ using averages, we have

    $$\overline{R}_1 = 18.3\%, \quad \overline{R}_2 = 11.8\%.$$

  - The estimate of the expected return of the portfolio $E(R)$ is

    $$\overline{R} = 0.75 \times \overline{R}_1 + 0.25 \times \overline{R}_2 = 16.68\%.$$

Benefits of Diversification

The table below represents the estimates of the expected return and standard deviation of a Disney/IBM portfolio, as we vary the weight, w_1, on Disney.

w_1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$E(R)$	11.8	12.5	13.1	13.8	14.4	15.1	15.7	16.4	17.0	17.7	18.3
$\sigma(R)$	25.1	24.0	23.3	23.0	23.2	23.8	24.9	26.3	28.0	29.9	32.1

- What do you notice?
- The role of correlation
- Consider three cases:
  1. Perfect positive correlation: $\rho = 1$
  2. Perfect negative correlation: $\rho = -1$
  3. No correlation: $\rho = 0$
- What does the variance of the portfolio look like?

Case 1: $\rho = 1$

- Simplify the variance equation:

$$\begin{aligned}\sigma(R)^2 &= w^2\sigma(R_1)^2 + (1-w)^2\sigma(R_2)^2 \\ &+ 2w(1-w)\underbrace{\rho(R_1,R_2)}_{1}\sigma(R_1)\sigma(R_2) \\ &= (w\sigma(R_1) + (1-w)\sigma(R_2))^2\end{aligned}$$

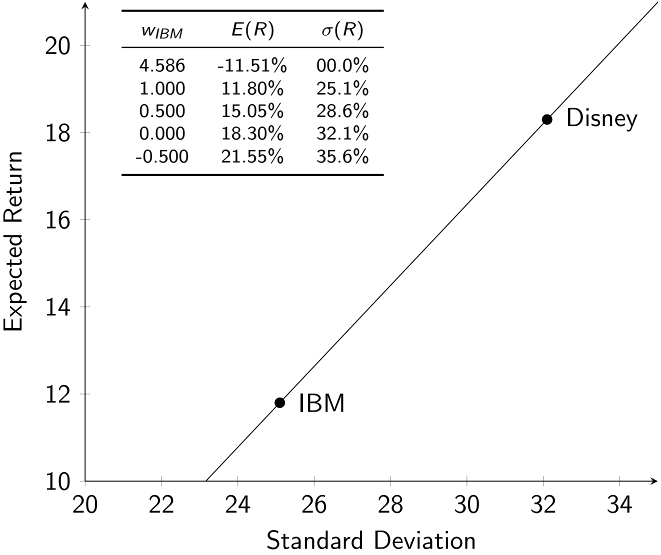
- As we change portfolio weight, expected returns and volatility of the portfolio move along:

$$\begin{aligned}E(R) &= wE(R_1) + (1-w)E(R_2), \\ \sigma(R) &= |w\sigma(R_1) + (1-w)\sigma(R_2)|\end{aligned}$$

- In the Disney/IBM case we have

$$\begin{aligned}E(R) &= 0.183w + 0.118(1-w) = 0.065w + E(R_2), \\ \sigma(R) &= |0.321w + 0.251(1-w)| = |0.07w + \sigma(R_2)|\end{aligned}$$

Case 1: $\rho = 1$



w_{IBM}	$E(R)$	$\sigma(R)$
4.586	-11.51%	00.0%
1.000	11.80%	25.1%
0.500	15.05%	28.6%
0.000	18.30%	32.1%
-0.500	21.55%	35.6%

Case 2: $\rho = -1$

- Simplify the variance equation:

$$
\begin{aligned}
\sigma(R)^2 &= w^2\sigma(R_1)^2 + (1-w)^2\sigma(R_2)^2 \\
&+ 2w(1-w)\underbrace{\rho(R_1,R_2)}_{-1}\sigma(R_1)\sigma(R_2) \\
&= (w\sigma(R_1) - (1-w)\sigma(R_2))^2
\end{aligned}
$$

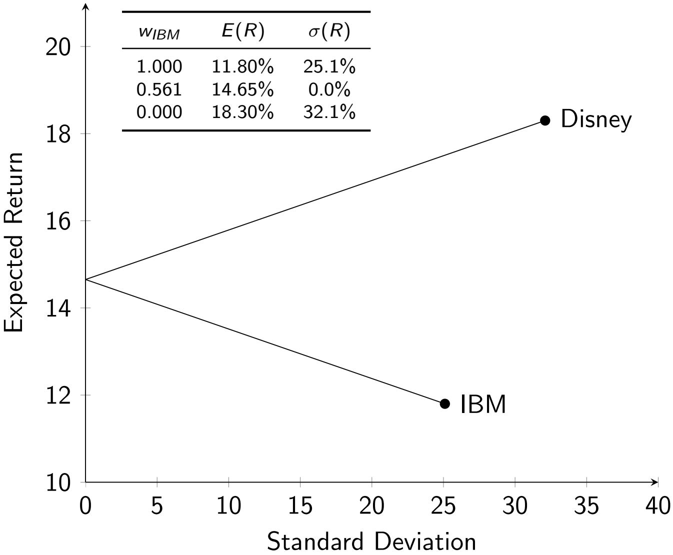
- As we change portfolio weight, expected returns and volatility of the portfolio move along:

$$
\begin{aligned}
E(R) &= wE(R_1) + (1-w)E(R_2), \\
\sigma(R) &= |w\sigma(R_1) - (1-w)\sigma(R_2)|
\end{aligned}
$$

- In the Disney/IBM case we have

$$
\begin{aligned}
E(R) &= 0.183w + 0.118(1-w) = 0.065w + E(R_2), \\
\sigma(R) &= |0.321w - 0.251(1-w)| = |0.572w - \sigma(R_2)|
\end{aligned}
$$

Case 2: $\rho = -1$



w_{IBM}	$E(R)$	$\sigma(R)$
1.000	11.80%	25.1%
0.561	14.65%	0.0%
0.000	18.30%	32.1%

The Perfect Hedge

- When $\rho = 1$ or -1, we can construct a zero risk portfolio
  - if $\rho = -1$,

$$\begin{aligned} \sigma(R) &= 0 \Rightarrow w = \frac{\sigma(R_2)}{\sigma(R_1) + \sigma(R_2)} = 0.44, \\ E(R) &= 0.14 = R_f \end{aligned}$$

  - if $\rho = 1$,

$$\begin{aligned} \sigma(R) &= 0 \Rightarrow w = -\frac{\sigma(R_2)}{\sigma(R_1) - \sigma(R_2)} = -3.58, \\ E(R) &= -0.11 = R_f \end{aligned}$$

- In practice, there is no difference between negative and positive correlation
  - If $\rho = 1$, short one of the assets, then the positions are negatively correlated
  - this ignores the costs of shorting
- Thus, the polar opposites are $|\rho| = 1$ and $\rho = 0$

Case 3: $\rho = 0$

- Simplify the variance equation:

$$\begin{aligned}\sigma(R)^2 &= w^2\sigma(R_1)^2 + (1-w)^2\sigma(R_2)^2 \\ &+ 2w(1-w)\underbrace{\rho(R_1,R_2)}_{0}\sigma(R_1)\sigma(R_2) \\ &= w^2\sigma(R_1)^2 + (1-w)^2\sigma(R_2)^2\end{aligned}$$

- As we change portfolio weight, expected returns and volatility of the portfolio move along:

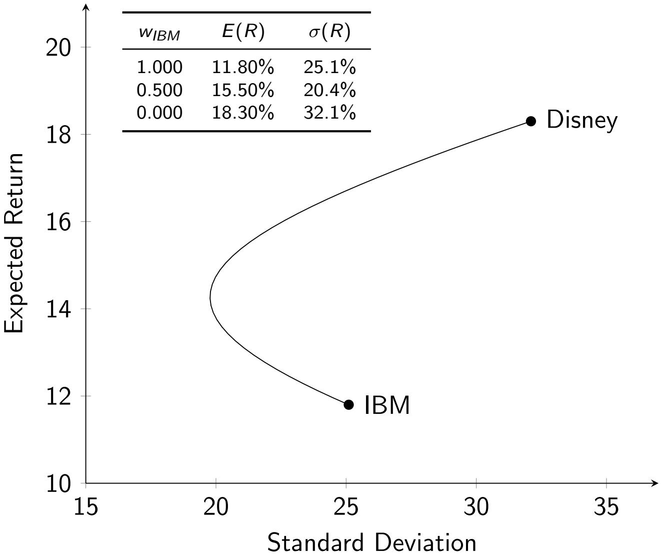
$$\begin{aligned}E(R) &= wE(R_1) + (1-w)E(R_2), \\ \sigma(R) &= \sqrt{w^2\sigma(R_1)^2 + (1-w)^2\sigma(R_2)^2}\end{aligned}$$

- In the Disney/IBM case we have

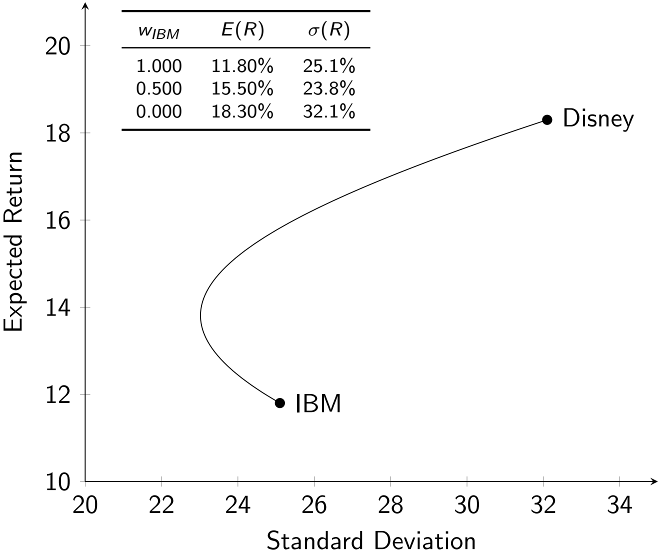
$$\begin{aligned}E(R) &= 0.183w + 0.118(1-w) = 0.065w + E(R_2), \\ \sigma(R) &= \sqrt{0.321^2w^2 + 0.251^2(1-w)^2}\end{aligned}$$

Case 3: $\rho = 0$

w_{IBM}	$E(R)$	$\sigma(R)$
1.000	11.80%	25.1%
0.500	15.50%	20.4%
0.000	18.30%	32.1%



Real data: $\rho = 0.38$



w_{IBM}	$E(R)$	$\sigma(R)$
1.000	11.80%	25.1%
0.500	15.50%	23.8%
0.000	18.30%	32.1%

Benefits of Diversification

- The key observation from the tables and the graphs is that diversification can reduce risk substantially.
  - Most portfolios have a lower standard deviation than a pure Disney or IBM portfolio.

- A diversified portfolio can have lower risk because the individual stocks do not always move together.

- A second observation from the tables and the graphs is that diversification does not necessarily reduce expected return.
  - By including Disney in a pure IBM portfolio, we can raise expected return

3. Many risky assets

Many risky assets

- We've covered a combination of riskless and risky assets
- We've covered a combination of two risky assets
- What's left to do:
  - Many risky assets
  - A combo of many risky assets with the riskless asset
- Many risky assets highlight a new challenge:
  - Many different portfolios give the same expected return
  - Which one should we pick?

Example: Technology firms

- Consider Intel, Microsoft, and Qualcomm

Asset #	Ticker	$E(R_i)$	$\sigma(R_i)$
1	INTC	0.40	0.47
2	MSFT	0.39	0.40
3	QCOM	0.76	0.71

- Suppose you want to achieve a 50% return

w_1	w_2	w_3
0.72	0.00	0.28
11.00	-10.00	0.00
0.00	0.70	0.30

- Which one should we choose?

Expected Returns

- The expected return of a portfolio with N stocks is

$$E(R) = \sum_{n=1}^{N} w_n E(R_n), \quad \sum_{n=1}^{N} w_n = 1$$

- It is a weighted average of the expected returns of the individual stocks.
- Use a vector notation:

$$\begin{aligned} w &= (w_1, w_2, \ldots, w_N)^\top, \\ R_r &= (R_1, R_2, \ldots, R_N)^\top, \\ 1_N &= (1, 1, \ldots, 1)^\top \end{aligned}$$

- Then $E(R) = w^\top E(R_r)$, $w^\top 1_N = 1$

Variance

- The variance of a portfolio with N stocks is

$$
\begin{aligned}
V(R) &= \sum_{n=1}^{N} w_n^2 V(R_n) + \sum_{n \neq m} w_n w_m Cov(R_n, R_m) \\
&= \sum_{n=1}^{N} w_n^2 V(R_n) + 2 \sum_{n < m} w_n w_m Cov(R_n, R_m) \\
&= \sum_{n,m} w_n w_m Cov(R_n, R_m)
\end{aligned}
$$

- Use a matrix notation

$$
V(R_r) = \begin{pmatrix}
V(R_1) & Cov(R_1, R_2) & \dots & Cov(R_1, R_N) \\
Cov(R_2, R_1) & V(R_2) & \dots & Cov(R_2, R_N) \\
\dots & \dots & \dots & \dots \\
Cov(R_N, R_1) & Cov(R_N, R_2) & \dots & V(R_N)
\end{pmatrix}
$$

- Then $V(R) = w^\top V(R_r) w$

Mean-Variance Optimization and the Portfolio Frontier

- Among all portfolios that have a given expected return, which is the portfolio with the smallest risk?
- This will give us a set of portfolios, one for each expected return.
- This set is known as the **portfolio frontier** (PF). Its elements are the **frontier portfolios**.
- We only need to consider portfolios on the PF. (Assuming that we care only about mean and variance.)

4. Mean-Variance Optimization and the Portfolio Frontier

Choosing a Global Portfolio

- To illustrate mean-variance optimization and the PF, we consider the problem of choosing a global portfolio using historical data.

- The "individual stocks" are indices from the largest seven stock markets. These indices are constructed by Morgan Stanley Capital International (MSCI).

	Percentage of World Market Capitalization
US	29.9%
Japan	8.2
UK	6.8
China	5.4
France	4.4
Hong Kong	4.3
Canada	3.7
Total	62.7

Statistical Properties of Returns

Using monthly returns, we can compute:

	Sample Average	Sample St. Dev.
Canada	13.2	19.0
China	6.5	38.1
France	14.4	22.0
Hong Kong	23.7	36.2
Japan	13.2	21.8
UK	14.0	22.2
US	11.4	15.1

Statistical Properties of Returns (cont'd)

Sample Correlations

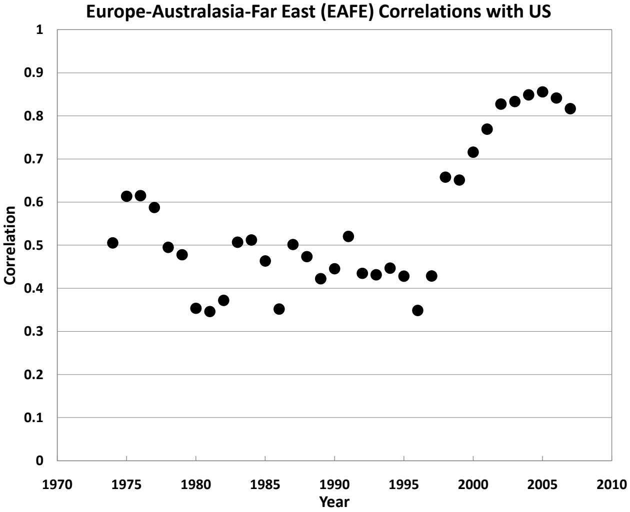
	Can	CN	Fra	HK	Jap	UK	US
Canada	1						
China	0.49	1					
France	0.48	0.31	1				
Hong Kong	0.38	0.65	0.31	1			
Japan	0.33	0.17	0.40	0.31	1		
UK	0.52	0.38	0.58	0.39	0.37	1	
US	0.71	0.43	0.50	0.36	0.31	0.54	1

A Word of Caution

- We should always keep in mind that our numbers are only estimates.
- Mean-variance optimization is only as precise as these estimates are.
- The estimates for expected returns are quite imprecise.
(Remember that even for a 75-year sample, the standard error is around 2.5%.)
- The estimates for standard deviations are generally quite precise, not so for correlations
- Can evaluate whether correlations are changing during sample period.

Are Correlations Changing?

Using monthly returns for the 5 years preceding a given year, we can compute:



5. Portfolio Frontier with Risky Assets Only

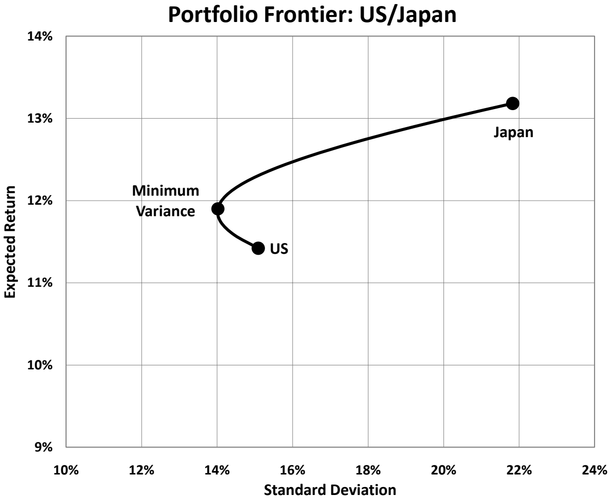
Outline

We will construct three portfolio frontiers (PF).

- PF of two countries, US and Japan.
- PF of US and Japan, but with short sales.
- PF of all countries.

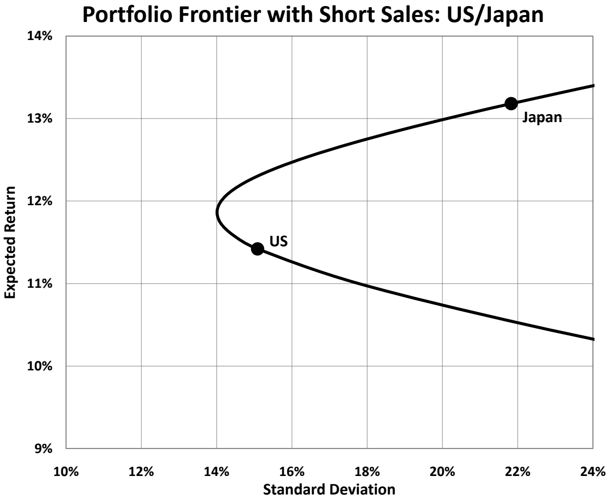
Portfolio Frontier with Two Assets

With two assets, the PF can be determined very easily. Any portfolio is a frontier portfolio, because there is no other portfolio having the same expected return.



Portfolio Frontier with Short Sales

Short sales expand the PF. Expected returns above Japan or below US can be achieved. PF becomes a hyperbola.



Portfolio Frontier with Many Risky Assets

The Optimization Problem

- Among all portfolios that have a given expected return (call it E), which is the portfolio with the minimum variance?
- Choose portfolio weights w to minimize

$$V(R) = w^\top V(R_r) w$$

subject to

$$w^\top 1_N = 1$$

and

$$E(R) = w^\top E(R_r) = E.$$

Portfolio Frontier with Many Risky Assets

Numerical implementation

- Quadratic Programming
- Objective:

$$min_x \frac{1}{2} x^\top G x - a^\top x$$

- Subject to:

$$C^\top x \geq b$$

- Call: `quadprog.solve_qp(G, a, C, b, meq)`

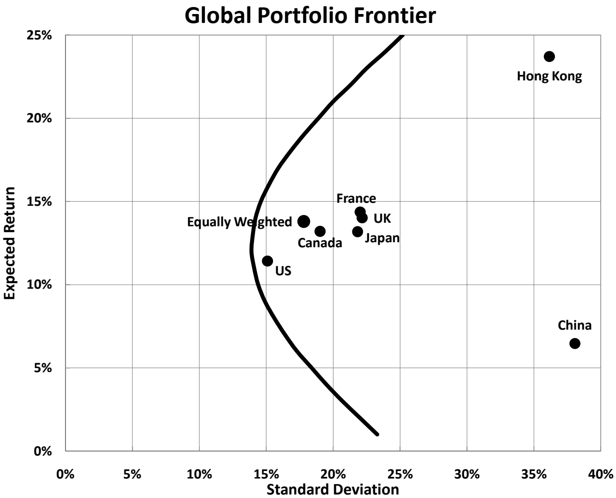
Portfolio Frontier with Many Risky Assets in Python

```
import numpy as np
from numpy.linalg import multi_dot
import quadprog as QD

n=7
err = np.array([0.132, 0.065, 0.144, 0.237, 0.132, 0.140, 0.114])
sigmarr = np.array([0.190, 0.381, 0.220, 0.362, 0.218, 0.222, 0.151])
corr = np.array([[1,0.49, 0.48, 0.38,0.33,0.52, 0.71],
                [0.49, 1, 0.31, 0.65, 0.17, 0.38, 0.43],
                [0.48, 0.31, 1, 0.31, 0.40, 0.58, 0.50],
                [0.38, 0.65, 0.31, 1, 0.31, 0.39, 0.36],
                [0.33, 0.17, 0.40, 0.31, 1, 0.37, 0.31],
                [0.52, 0.38, 0.58, 0.39, 0.37, 1, 0.54],
                [0.71, 0.43, 0.50, 0.36, 0.31, 0.54, 1]])

vrr = multi_dot([np.diag(sigmarr), corr, np.diag(sigmarr)])
vrr = np.round(vrr,6)
onen = np.repeat(1,n)
w = np.repeat(1/n,n)
er = multi_dot([w,err])
vr = multi_dot([w,vrr,w])
A = np.c_[ onen, err ]
etarget = 0.1

arg1 = (2*vrr).astype("double")
arg2 = np.repeat(0,n).astype("double")
arg3 = A.astype("double")
mrf1 = QD.solve_qp(arg1,arg2, arg3, np.array([1,etarget]), meq=2)

mrf1[0]
```

```
array([-0.02108241,  0.07754602,  0.02738722, -0.12887136,  0.26110659,
        0.01957236,  0.76434159])
```

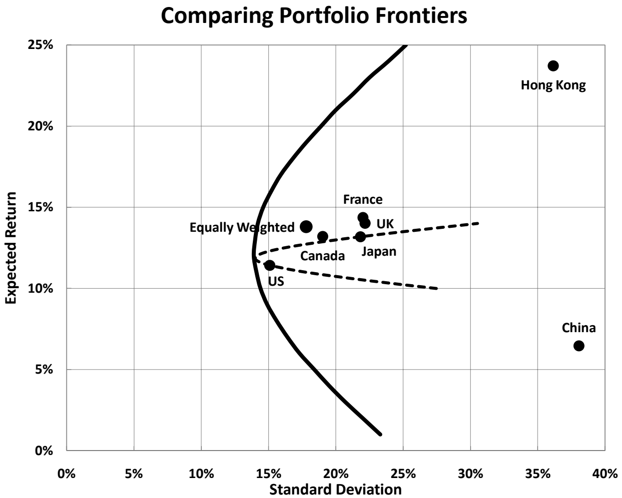
Portfolio Frontier of All Countries

Not all portfolios are frontier portfolios. In particular, none of the countries, nor the equally weighted portfolio, are on the PF.



Portfolio Frontier of All Countries (cont'd)

Adding assets shifts the PF to the left. The variance that can be achieved for a given expected return decreases.

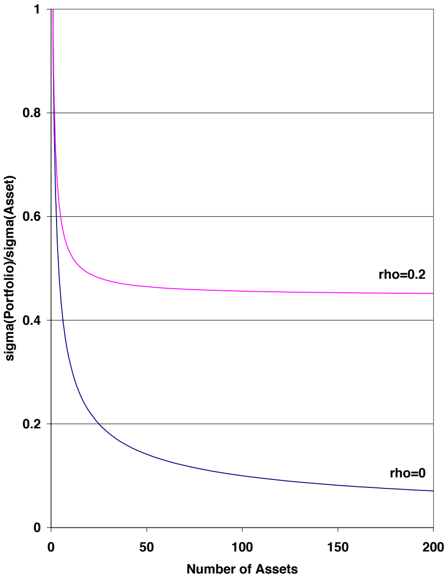


Limits of Diversification: Systematic vs. Idiosyncratic Risk

- Adding assets reduces the variance that can be achieved for a given expected return.
- This is simply another way to say that diversification reduces risk.
- **Question:** How much can risk be reduced? Can it be reduced to zero, by adding very many assets?
- Assume that all stocks are the same:
  - $E(R_i) = e$
  - $\sigma(R_i) = s$
  - $\rho(R_i, R_j) = \rho$
- Consider an equally-weighted portfolio: $E(R) = e$,

$$\sigma(R)^2 = s^2 \left(\frac{1}{N} + \left(1 - \frac{1}{N} \right) \rho \right) \to s^2 \rho$$

Theory ..



sigma(Portfolio)/sigma(Asset)
Number of Assets
rho=0.2
rho=0
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200

.. and some Evidence

Consider an equally weighted portfolio of randomly selected NYSE stocks.

Number of Stocks in Portfolio	Standard Deviation of Portfolio	Ratio of Portfolio Std. Dev. to Std. Dev. of a Single Stock
1	49.2%	1.00
2	37.4	0.76
4	29.7	0.60
8	25.0	0.51
20	21.7	0.44
50	20.2	0.41
200	19.4	0.39
500	19.2	0.39
1000	19.2	0.39

This is consistent with the theory, assuming that the average correlation is around 0.2.

Systematic vs. Idiosyncratic Risk

Consider a group of assets.

- **Systematic Risk:** Risk which affects all assets.
  - If, for instance, the assets are US stocks, systematic risk corresponds to events affecting the US economy.
- **Idiosyncratic Risk:** Risk which affects only one asset.
  - For US stocks, idiosyncratic risk corresponds to events affecting only the particular company or industry.
- Diversification within the group of assets reduces, and eventually eliminates, idiosyncratic risk.
- However, it cannot reduce systematic risk.
- Diversification outside the group of assets (if it is possible) is more effective in reducing risk.

6. Selecting a risky portfolio

Selecting a risky portfolio

- We know how to construct the PF now

- Which portfolio should we pick?
  - Smallest risk?
  - Maximum utility?
  - Highest Sharpe ratio?

Minimum Variance Portfolio

The Optimization Problem

- Among all portfolios, which is the portfolio with the minimum variance?
- Choose portfolio weights w to minimize

$$V(R) = w^\top V(R_r) w$$

subject to

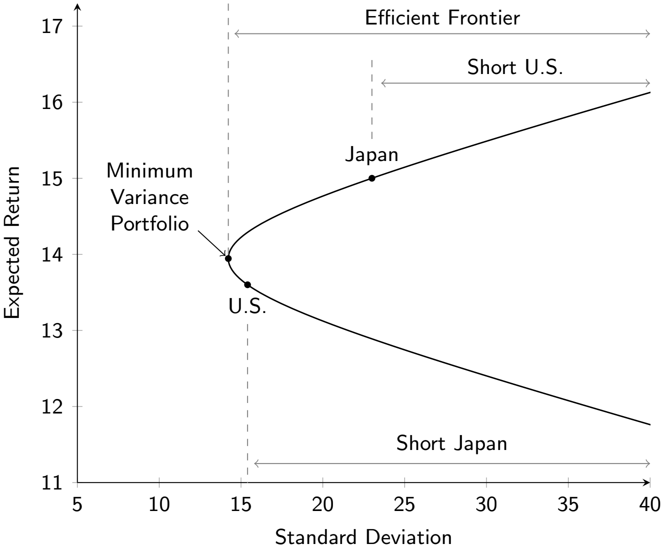
$$w^\top 1_N = 1.$$

- In Python:

```
arg1 = (2*vrr).astype("double")
arg2 = np.repeat(0,n).astype("double")
A = np.repeat(1,n).reshape(n,1)
arg3 = A.astype("double")
mrf2 = QD.solve_qp(arg1,arg2, arg3, np.array([1]).astype("double"), meq=1)

mrf2[0]
```

```
array([ 0.05652699, -0.03198421,  0.05996218, -0.00178172,  0.22227172,
        0.02844034,  0.66656469])
```

Minimum Variance Portfolio



Other options

- Maximize utility:

$$
\begin{aligned}
w^* &= argmax_w U(R, w) \\
U(R, w) &= w(E(R) - R_f) + R_f - \frac{A}{2} \cdot w^2 V(R)
\end{aligned}
$$

- Need riskless asset

- The same as maximizing the Sharpe ratio

7. Portfolio Frontier with a Riskless Asset

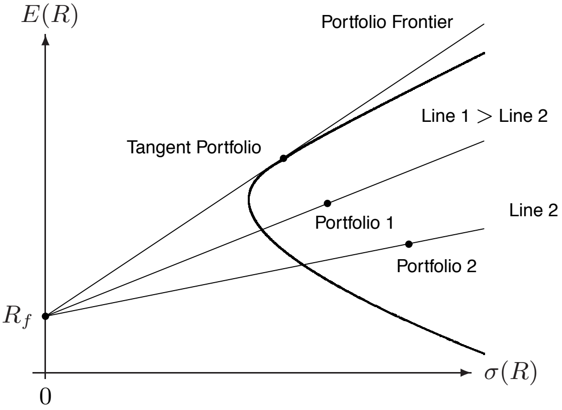
Portfolio Frontier with a Riskless Asset

The Optimization Problem

- We only need to choose the weights of the risky assets, w, $n = 1, .., N$. The weight of the riskless asset is
$$1 - w^\top 1_N.$$
- The variance of the riskless asset is 0, as is its covariance with all risky assets.
- Optimization problem: Choose weights w to minimize
$$V(R) = w^\top V(R_r) w$$
subject to
$$E(R) = w^\top E(R_r) + (1 - w^\top 1_N) R_f = E.$$

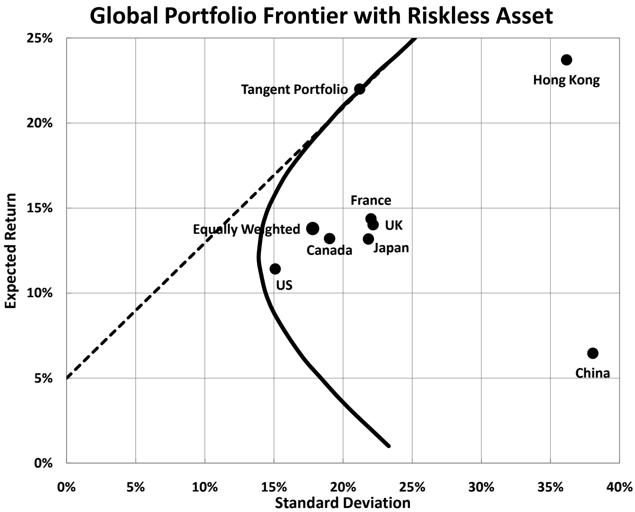
Portfolio Frontier with a Riskless Asset

Consider the lines linking the riskless asset with the points on and inside the hyperbola. The portfolio frontier is the line with the steepest slope.



Properties of the Portfolio Frontier

- The PF is the line linking the riskless asset with the tangent portfolio (TP).
- All frontier portfolios are combinations of the riskless asset and the TP.
- The portfolios below the TP involve a positive weight on the riskless asset (lending).
- The portfolios above the TP involve a negative weight (borrowing).
- To construct the PF, we only need to
  - determine one frontier portfolio, i.e., solve the optimization problem once
  - draw the line linking that portfolio to the riskless asset.



We use the US T-bills as the riskless asset, and assume a return of 5%.

The Sharpe Ratio

- We have selected a portfolio on PF that has the largest slope on Capital Allocation Line?

- What was the equation for CAL?

$$E(R) = R_f + \frac{E(R_r) - R_f}{\sigma(R_r)} \cdot \sigma_t(R)$$

- Slope is the Sharpe Ratio

- So, we are maximizing the Sharpe Ratio

Wrap Up

- Given a set of N risky assets, construct the tangency portfolio, w_T
    - Construct a portfolio frontier
    - Select a portfolio with the highest Sharpe ratio

- Combine the tangency portfolio with the risky free asset, w^*
    - Maximize your utility

- Risk-free asset gets a fraction of $1 - w^*$ of your capital, risk assets get $w^* w_T$