Lecture 12
Factor Models for Expected Returns: Estimation and Tests

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Overview of Lecture 12

Equilibrium factor models

Time-series regression tests of factor models

1. Alpha and mispricing
2. The 25 Fama-French portfolios as test assets
   1. Testing the CAPM
   2. Testing the Fama-French 3-factor model
3. Alpha and mean-variance efficiency (max Sharpe ratios)
4. Appendix: The Arbitrage Pricing Theory (APT)

Risk comes in many flavors

we used to think that risk comes in a single flavor

- a stock's β_i with the market factor
- β_i measures the 'quantity of risk' or amount of exposure of asset i to the market.

early empirical evidence originally supported the CAPM...now it is resoundingly rejected

now, we know that risk comes in many flavors

- that complicates portfolio advice
- it makes performance analysis more challenging!

Why should we expect multiple factors?

the average investor has a job

compare two stocks with the same market beta

- stock A does well in a recession
- stock B does poorly in a recession

CAPM says we are indifferent between the two stocks

are we?

some evidence for a missing recession risk factor!

Why should we expect multiple factors?

- suppose some less sophisticated investors have portfolios that are biased towards large, growth firms (because these are more glamorous)
- then, the sophisticated investors have to overweight the small, value firms in their portfolio of risky assets
- these sophisticated investors do not and cannot hold the market portfolio
- in fact, the market portfolio is no longer efficient
- this gives rise to new factors (like value and size)

Why estimate factor models?

test your model

- can it account for interesting cross-sectional variation in average returns on assets in a particular asset class?

once you have a ‘good model’

- estimate cost of capital for company
- estimate risk-adjusted returns on trading strategy
- do performance attribution, style analysis
- look for skill in risk-adjusted returns of pension funds and hedge funds

An expected return-beta pricing model

The CAPM is the example you already know:

$$E\left[R_{i,t} - R_{f,t}\right] = \beta_{M,i} E\left[R_{M,t} - R_{f,t}\right]$$

What is the property of the market portfolio?

- It is mean-variance efficient – it has the maximal Sharpe ratio

In fact, mathematically it is true that:

$$E\left[R_{i,t} - R_{f,t}\right] = \beta_{MVE,i} E\left[R_{MVE,t} - R_{f,t}\right].$$

where R_{MVE} is the return to the mean-variance efficient portfolio!

The unicorn: The Mean-Variance Efficient Portfolio

'All' we need is to find the mean-variance efficient portfolio.

- This is what multifactor expected return models are all about
- Find, say, three excess return factors that span the excess return mean-variance efficient portfolio (the 'e' superscript is for excess returns):

$$R^e_{MVE,t} = a_1 R^e_{F1,t} + a_2 R^e_{F2,t} + a_3 R^e_{F3,t}.$$

Then:

$$\begin{aligned}
E\left[R^e_{i,t}\right] &= \beta_{MVE,i} E\left[R^e_{MVE,t}\right] \\
&= \beta_{MVE,i} E\left[a_1 R^e_{F1,t} + a_2 R^e_{F2,t} + a_3 R^e_{F3,t}\right] \\
&= \beta_{MVE,i} a_1 E\left[R^e_{F1,t}\right] + \beta_{MVE,i} a_2 E\left[R^e_{F2,t}\right] + \beta_{MVE,i} a_3 E\left[R^e_{F3,t}\right] \\
&= \beta_1 E\left[R^e_{F1,t}\right] + \beta_2 E\left[R^e_{F2,t}\right] + \beta_3 E\left[R^e_{F3,t}\right]
\end{aligned}$$

How do you estimate the betas?

- Linear regression, as usual

Different Estimation Methods for Linear Factor Model

1. Time series regressions (this lecture)
   - OLS

2. Cross-sectional regressions (next lecture)
   - OLS
   - Fama and MacBeth (1973)

Single Factor Equilibrium Model: APT version

Assume there is only one factor that explains covariation in stock returns:

$$R_{it}^e = \alpha_i + \beta_i f_t + \varepsilon_{it},$$

where ε_{it} are uncorrelated across stocks and time, and f_t is a traded factor (e.g., excess market returns)

Since ε-risk can be diversified away in large portfolios, it cannot will not earn a risk premium if markets are perfectly competitive

- Two sources of variation in returns: $\beta_i f_t$ and ε_{it}
  - Risk premium on $\beta_i f_t$ is $\beta_i E[f_t]$
  - Risk premium on ε_{it} is zero
  - Thus, risk premium on stock should be $E[R_{it}^e] = \beta_i E[f_t]$ and so ... $\alpha_i = 0$!

Let's test this model using the **25 Fama-French portfolios**!

- Let the factor be the market factor

The 25 Fama-French portfolios

Fama and French (1993) sort the data into 25 portfolios.

they construct portfolios by sorting them along two dimensions:

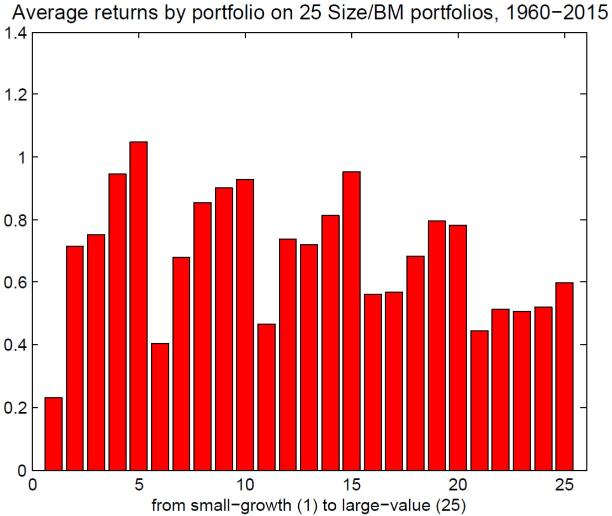
1. 5 bins sorted on size
2. 5 bins sorted on book-to-market

25 Portfolios: large spread in average excess returns

the goal is to maximize the variation in the expected excess returns $E\left[R_{it}^{e}\right]$ vs. $E\left[R_{jt}^{e}\right]$, i.e. the left hand side variables in the regression.

Then, we want to see if the factors $\mathbf{f}_t$ can capture this variation.

Fama-French 25 b/m and size sorted portfolios



25 Fama-French portfolios. Monthly data. 1960-2015.

Testing the CAPM

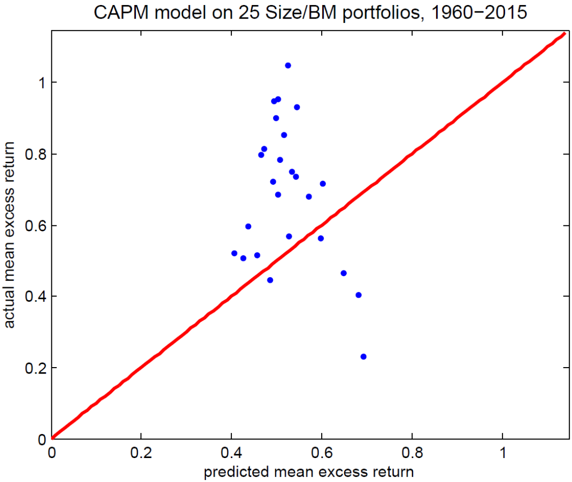
the CAPM predicts that $\alpha_i = 0$ for all assets and that variation in the cross-section of expected returns can be explained by variation in the market times an assets β_i

one-step procedure: run a time series regression of excess returns on the factor to estimate the β_i's

$$R^e_{it} = \alpha_i + \beta_i R^e_{mt} + \varepsilon_{it}$$

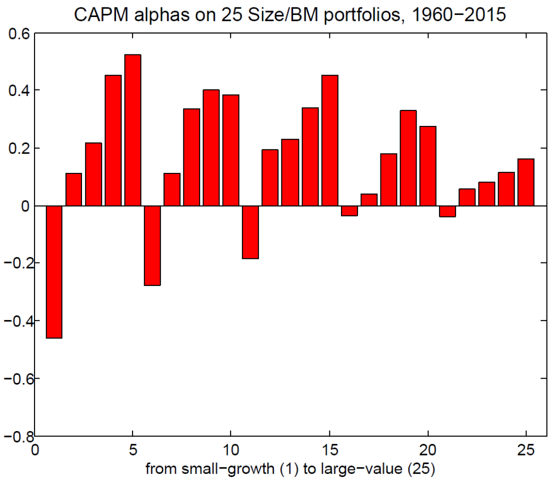
let's plot the predicted excess returns $\hat{R}^e_i = \hat{\beta}_i \times \frac{1}{T}\sum_{t=1}^{T} R^e_{mt}$ against the realized excess returns $\bar{R}^e_i = \frac{1}{T}\sum_{t=1}^{T} R^e_{i,t}$

Failure of the CAPM



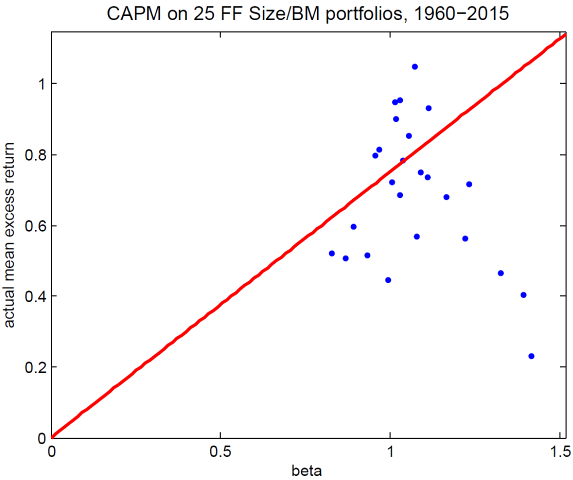
25 Fama-French portfolios. Monthly data. 1960-2015. Plot of the predicted excess returns $\hat{\beta}_i \times \frac{1}{T}\Sigma_{t=1}^{T} R^e_{mt}$ against the realized average excess returns $\frac{1}{T}\Sigma_{t=1}^{T} R^e_{it}$

CAPM alphas



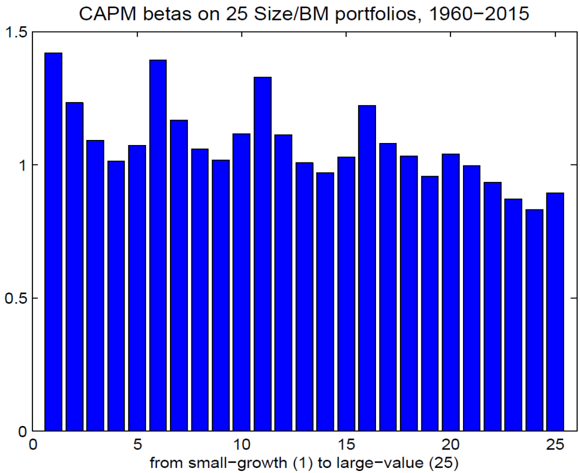
CAPM α's for the 25 Fama-French portfolios from a time series regression of returns on the factors. Monthly data. 1960-2015.

CAPM betas



CAPM β's for the 25 Fama-French portfolios from a time series regression of returns on the factors. Monthly data. 1960-2015.

CAPM betas (bar chart)



CAPM β's for the 25 Fama-French portfolios from a time series regression of returns on the factors. Monthly data. 1960-2015.

Multifactor model

the CAPM fails to capture the cross-section of expected returns built from the FF 25 portfolios sorted on size and book-to-market.

we need more factors $\mathbf{f}_t$ to explain variation in returns.

Fama and French: construct factors $\mathbf{f}_t$ from returns

Fama and French (1993) 3-factor model

The Fama/French factors are constructed using the 6 value-weight portfolios formed on size and book-to-market.
SMB (Small Minus Big) :

$$\begin{aligned} SMB &= 1/3(SmallValue + SmallNeutral + SmallGrowth) \\ &\quad -1/3(BigValue + BigNeutral + BigGrowth). \end{aligned}$$

HML (High Minus Low) :

$$HML = 1/2(SmallValue + BigValue) - 1/2(SmallGrowth + BigGrowth).$$

$R_t^m \equiv R_{mt} - R_{ft}$, the excess return on the market, is the value-weight return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates).

Time Series Regression

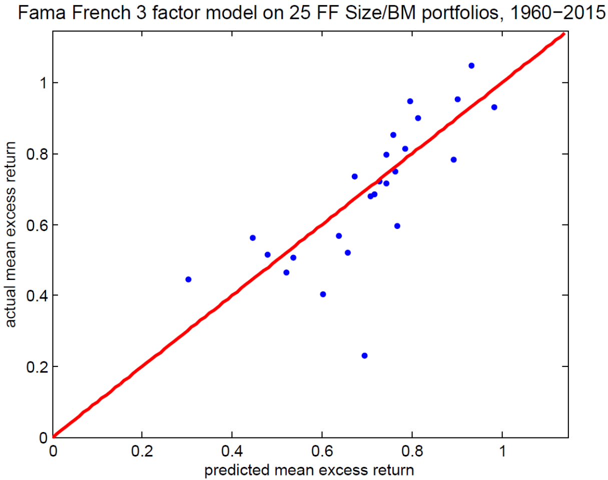
one-step procedure: run a time series regression of excess returns on the factor to estimate the β_i's

$$R_{it}^e = \alpha_i + \beta_i^m R_t^m + \beta_i^{smb} R_t^{smb} + \beta_i^{hml} R_t^{hml} + \varepsilon_{it}$$

Define the factor "risk premium" or "risk price" as $\hat{\boldsymbol{\lambda}} = \left(\lambda^m, \lambda^{smb}, \lambda^{hml}\right)$ where $\hat{\lambda}^j = \frac{1}{T}\sum_{t=1}^T R_t^j$ for $j = m, smb, hml$

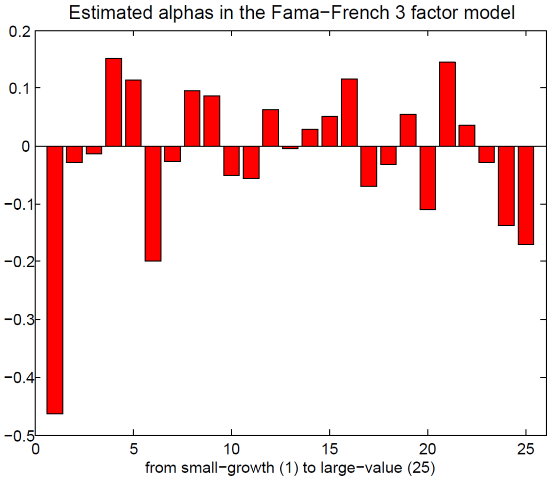
let's plot the predicted excess returns $\hat{\boldsymbol{\beta}}_i'\hat{\boldsymbol{\lambda}}$ against the realized excess returns $\bar{R}_i^e = \frac{1}{T}\sum_{t=1}^T R_{it}^e$

FF 3-factor model



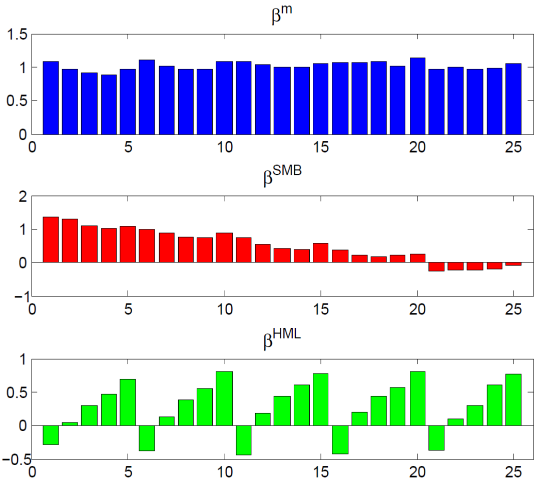
25 Fama-French portfolios. Monthly data. 1960-2015. Plot of the predicted excess returns $\hat{\beta}_i' \hat{\lambda}$ against the realized average excess returns $\frac{1}{T}\Sigma_{t=1}^{T} R_{it}^e$. The risk prices, $\hat{\lambda}$, are the means of the factors.

FF 3-factor alphas



Fama-French 3-factors $\hat{\alpha}$'s for the 25 Fama-French portfolios from a time series regression on the factors. Monthly data. 1960-2015.

FF 3-factor betas



Fama-French 3-factors $\hat{\beta}$'s for the 25 Fama-French portfolios from a time series regression on the factors. Monthly data. 1960-2015.

Testing the model: Single factor

We can test the model

$$E\left[R_{it}^{e}\right] = \beta_i E\left[f_t\right]$$

by running time series regressions:

$$R_{it}^{e} = \alpha_i + \beta_i f_t + \varepsilon_{it}, \qquad t = 1, \ldots, T$$

With i.i.d errors, the asymptotic test statistic for the pricing errors is:

$$T\left[1 + \frac{\bar{f}^2}{\hat{\sigma}_f^2}\right]^{-1} \hat{\boldsymbol{\alpha}}' \hat{\Sigma}^{-1} \hat{\boldsymbol{\alpha}} \sim \chi_N^2.$$

where $\hat{\Sigma}$ denotes the covariance matrix of $\boldsymbol{\varepsilon}_{\mathbf{t}}$.

- The CAPM is resoundingly rejected by Fama-French (1993)
- This is the 'GRS test' (from Gibbons, Ross, and Shanken (1987))

Testing the model: Multiple factors

We can test the model

$$E\left[R_{it}^{e}\right] = \boldsymbol{\beta}_i' E\left[\mathbf{f}_t\right]$$

by running time series regressions:

$$R_{it}^{e} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}, \qquad t = 1, \ldots, T$$

With i.i.d Normally distributed errors, the exact small-sample test statistic for the pricing errors is:

$$(T - N - K)/N \left[1 + \bar{\mathbf{f}}' \hat{\boldsymbol{\Sigma}}_f^{-1} \bar{\mathbf{f}}\right]^{-1} \hat{\boldsymbol{\alpha}}' \hat{\Sigma}_{\varepsilon}^{-1} \hat{\boldsymbol{\alpha}} \sim F_{N, T-N-K}$$

where $\hat{\Sigma}_{\varepsilon}$ denotes the covariance matrix of $\boldsymbol{\varepsilon}_{\mathbf{t}}$, $\hat{\boldsymbol{\Sigma}}_f$ denotes the covariance matrix of the factors f_t, $\bar{\mathbf{f}}$ is the average factor, and $\hat{\boldsymbol{\alpha}}$ are the OLS estimates of α.

- The Fama-French 3-factor model is also rejected...! (though not nearly at the same significance level as the CAPM)

Factor Models and Mean-Variance Efficiency: The Data Mining Concern

Properties of the in-sample MVE

We can, given a set of assets, easily compute the *in-sample* MVE portfolio:

$$\min_{\mathbf{w}} \mathbf{w}'\hat{\Omega}\mathbf{w} \text{ such that } \mathbf{w}'\bar{R}^e = m$$

where $\mathbf{w} = [w_1 \;\; w_2 \;\; ... \; w_N]'$, $\hat{\Omega}$ is the sample variance-covariance matrix of excess returns, and $\bar{R}^e_t = \left[\bar{R}^e_{1t} \;\; \bar{R}^e_{2t} \;\; ... \;\; \bar{R}^e_{Nt}\right]'$ is the vector of sample average excess returns for each asset.

The optimal portfolio weights are (up to a constant of proportionality):

$$\mathbf{w}^{MVE} \propto \hat{\Omega}^{-1}\bar{R}^e$$

The in-sample mean variance efficient portfolio therefore has *squared* Sharpe ratio:

$$SR^2_{MVE} = \bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e$$

- Next slide has the derivations

MVE derivations

Objective function in Lagrangian form:

$$\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\hat{\Omega}\mathbf{w} - k\left(\mathbf{w}'\bar{R}^e - m\right)$$

First order condition wrt $\mathbf{w}$ (an $N \times 1$ vector):

$$\hat{\Omega}\mathbf{w} - k\bar{R}^e = \mathbf{0}$$

Thus:

$$\mathbf{w}^{MVE} = k\hat{\Omega}^{-1}\bar{R}^e.$$

MVE derivations (cont'd)

Recall, $\mathbf{w}^{MVE} = k\hat{\Omega}^{-1}\bar{R}^e$.

Average excess return on MVE portfolio

$$\bar{R}^e_{MVE} = \left(\mathbf{w}^{MVE}\right)' \bar{R}^e = k\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e$$

Variance of excess return on MVE portfolio:

$$\begin{aligned} var\left(R^e_{MVE}\right) &= \left(\mathbf{w}^{MVE}\right)' \hat{\Omega}\mathbf{w}^{MVE} = k^2\bar{R}^{e\prime}\hat{\Omega}^{-1}\hat{\Omega}\hat{\Omega}^{-1}\bar{R}^e \\ &= k^2\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e \end{aligned}$$

Squared Sharpe ratio of MVE:

$$SR^2\left(R^e_{MVE}\right) = \frac{\left(k\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e\right)^2}{k^2\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e} = \bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^e$$

The GRS statistic revisited

Consider an asymptotic version (with i.i.d. residuals) of the Gibbons-Ross-Shanken (GRS) statistic for testing whether a factor model is rejected or not:

$$T\frac{\hat{\alpha}'\hat{\Sigma}_{\varepsilon}^{-1}\hat{\alpha}}{1+\bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e}} \sim \chi^2\left(N\right)$$

Note that $\bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e}$ is the in-sample maximum Sharpe ratio squared obtained using the factor portfolios only.

- Σ_F is the $K \times K$ variance-covariance matrix of the factors

Note that $\hat{\alpha}'\hat{\Sigma}_{\varepsilon}^{-1}\hat{\alpha}$ is the maximum Sharpe ratio squared of hedged stock returns. Recall:

$$\begin{aligned} E\left[\hat{\alpha}_i+\hat{\varepsilon}_{it}\right] &= E\left[R_{it}^e-\hat{\boldsymbol{\beta}}_i' R_{Ft}^e\right]=\hat{\alpha}_i \text{ for all } i \text{ and} \\ var\left(\hat{\alpha}+\hat{\varepsilon}_t\right) &= \hat{\Sigma}_{\varepsilon} \quad \text{(an } N\times N \text{ matrix)} \end{aligned}$$

A Mean-Variance Decomposition

Since hedged stock returns are uncorrelated with the factor returns (by construction), we have that

$$\bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^{e} = \hat{\alpha}'\hat{\Sigma}_{\varepsilon}^{-1}\hat{\alpha} + \bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e}$$

- We show this is true in a couple of slides

Then, we have that

$$\begin{aligned}
\frac{\hat{\alpha}'\hat{\Sigma}_{\varepsilon}^{-1}\hat{\alpha}}{1 + \bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e}} &= \frac{1 + \bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e} + \hat{\alpha}'\hat{\Sigma}_{\varepsilon}^{-1}\hat{\alpha}}{1 + \bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e}} - 1 \\
&= \frac{1 + \bar{R}^{e\prime}\hat{\Omega}^{-1}\bar{R}^{e}}{1 + \bar{R}_F^{e\prime}\hat{\Sigma}_F^{-1}\bar{R}_F^{e}} - 1
\end{aligned}$$

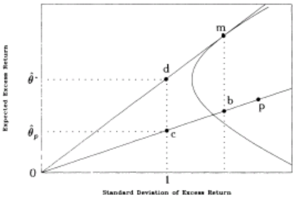
Thus, all alphas are zero if the maximum in-sample Sharpe ratio obtained using the factors equals the maximum in-sample Sharpe ratio of the test assets!

- In other words, if a linear combination of the factors is the in-sample mean-variance efficient portfolio, the factor model prices all assets perfectly in the sense that all alphas equal zero

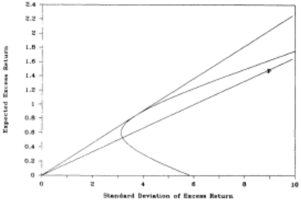
The failure of the CAPM revisited

The fact that the CAPM does not work means the Sharpe ratio of the market portfolio is far from the maximum Sharpe ratio obtainable using the test assets

PORTFOLIO EFFICIENCY 1127



1a.) Geometric intuition for W. Note the distance Oc is $\sqrt{1+\hat{\theta}_p^2}$, and the distance Od is $\sqrt{1+\hat{\theta}^{*2}}$.



1b.) *Ex post* efficient frontier based on 10 beta-sorted portfolios and the CRSP Equal-Weighted Index using monthly data, 1931-1965. Point p represents the CRSP Equal-Weighted Index.

A corollary on the implication of alpha

If a strategy has 'alpha' different from zero, it means it can be combined with the factor portfolios to obtain a higher Sharpe ratio than what one could get using the factor portfolios alone

Implications of alpha: Implementation of Max SR Portfolio

The investable set of assets are N stocks, labelled $i = 1, ..., N$ and K factor portfolios, labelled $j = 1, ..., K$.

Define the factor-neutral assets as:

$$R_{it}^{\alpha} \equiv R_{it}^{e} - \hat{\beta}' R_{Ft}^{e} = \hat{\alpha}_i + \hat{\varepsilon}_{it}, \quad \text{for all } i$$

Put all of the investable asset returns in an $(N+K) \times 1$ vector $\mathbf{R}_t = \left[R_{F_1 t}^{e} \; R_{F_2 t}^{e} \; ... \; R_{F_K t}^{e} \; R_{1t}^{\alpha} \; R_{2t}^{\alpha} \; ... \; R_{Nt}^{\alpha} \right]'$. Note that we only use the factor-neutral assets and the factors. Otherwise, there would be collinearity between R_i^e, R_i^{α}, and R_F^e.

Let $\hat{\Omega}$ denote the sample variance covariance matrix of $\mathbf{R}_t$. Note that it is block diagonal:

$$\hat{\Omega} = \begin{bmatrix} \hat{\Sigma}_F & 0 \\ 0 & \hat{\Sigma}_{\varepsilon} \end{bmatrix}$$

Implementation of Max SR Portfolio (cont'd)

Now, let's find the portfolio weights that makes the maximum Sharpe ratio:

$$\mathbf{w}^{MVE} \propto \hat{\Omega}^{-1}\overline{\mathbf{R}_t}$$

Note that $\overline{\mathbf{R}_t} = \left[\bar{R}_F' \ \hat{\alpha}'\right]'$, where $\bar{R}_F$ is the $K \times 1$ vector of sample factor means and $\hat{\alpha}$ is the $N \times 1$ vector of estimated alpha for the individual assets. Thus:

$$\mathbf{w}^{MVE} \propto \begin{bmatrix} \hat{\Sigma}_F^{-1} & 0 \\ 0 & \hat{\Sigma}_\varepsilon^{-1} \end{bmatrix} \overline{\mathbf{R}_t} = \begin{bmatrix} \hat{\Sigma}_F^{-1}\bar{R}_F \\ \hat{\Sigma}_\varepsilon^{-1}\hat{\alpha} \end{bmatrix}$$

From our earlier results, the max Sharpe ratio squared is:

$$\begin{aligned} SR^2_{MVE} &= \overline{\mathbf{R}_t}'\hat{\Omega}^{-1}\overline{\mathbf{R}_t} \\ &= \left[\bar{R}_F' \ \hat{\alpha}'\right] \begin{bmatrix} \hat{\Sigma}_F^{-1} & 0 \\ 0 & \hat{\Sigma}_\varepsilon^{-1} \end{bmatrix} \begin{bmatrix} \bar{R}_F \\ \hat{\alpha} \end{bmatrix} = \bar{R}_F'\hat{\Sigma}_F^{-1}\bar{R}_F + \hat{\alpha}'\hat{\Sigma}_\varepsilon^{-1}\hat{\alpha} \end{aligned}$$

We're always only one factor away from the MVE portfolio

Consider the misspecified factor model

$$R_{it}^e = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}.$$

where $\alpha_i \neq 0$ for all i.

Which factor is missing?

- The factor with portfolio weights

  $$w_\alpha \propto \Sigma_\varepsilon^{-1} \alpha$$

  Adding this factor mechanically ensures, in-sample, that the factors span the in-sample mean-variance efficient portfolio

The Missing Factor and Data Mining

Let's think more about the portfolio weights of the 'final' factor

$$w_\alpha \propto \Sigma_\varepsilon^{-1}\alpha$$

Let's assume, for discussion purposes, that Σ_ε is diagonal.

- The 'final' factor goes long positive alpha assets and short negative alpha assets
- Seems a lot like how we do our characteristics-based factor. E.g., value and momentum
  - Go long value stocks, short growth stocks
  - Go long winners, short losers, etc.

Data Mining and Economic Rationale

The fact that we know how to make our model work in-sample, for a given set of test assets, makes introducing this factor vacuous in itself

We need:

- An economic story of what risk or behavioral phenomenon the factor represents
- Out-of-sample, cross-country and/or cross-asset corroborating evidence

Thus, given how easy it is to data-mine, we are looking for economics that explain why we would think the phenomenon persists out-of-sample

Additional reading

I have written a note on factor model testing—time-series and cross-sectional regressions, as well as the mean-variance math laid out in the previous slides

It's posted on BruinLearn under 'Week 9' and may serve as a useful background reading in addition to the slides

I've also posted the Fama-French (1993) paper, as well as the Gibbons, Ross, Shanken (1987) paper.

APPENDIX

Arbitrage Pricing Theory (APT) of Ross (1976)

assume the data are generated by a multi-factor model

$$\begin{aligned} R^e_{it} &= \alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + \ldots + \beta_{iK} f_{Kt} + \varepsilon_{it}, \qquad i = 1, \ldots, N \\ &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \end{aligned}$$

- the APT does not identify what the factors are
- the factors $\mathbf{f}_t$ could be traded assets, macro variables, or latent factors

assume the errors are uncorrelated with each factor:

$$E\left[\varepsilon_{it}(f_{jt} - \overline{f}_{jt})\right] = 0 \qquad \forall\ i, j$$

some textbooks/authors (implicitly) assume the factors are uncorrelated with one another...an orthogonal factor model.

Arbitrage Pricing Theory of Ross (1976)

assumptions:

1. disturbances are independent of the factors:

$$\text{Cov}[\varepsilon_{it}, f_{jt}] = 0$$

2. ε_{it} is independent of ε_{jt}:

$$\text{Cov}[\varepsilon_{it}, \varepsilon_{jt}] = 0$$

   This implies $\mathbf{\Sigma}_{\varepsilon}$ is diagonal.

this is 'like' the multiple factor model

contribution of Ross (1976) and **APT**: derive equilibrium implications

see also Chamberlain and Rothschild (1983)

Arbitrage Pricing Theory of Ross (1976): Equilibrium

Suppose that excess returns are generated by a linear factor model:

$$R_{it}^e = a_i + b_{i1} f_{1t} + b_{i2} f_{2t} + \ldots + b_{iK} f_{Kt} + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

and assume no risk-free arbitrage opportunities exist.

Then, there exist risk prices λ_j for each factor such that the expected return on any security j can be stated as:

$$E\left[R_{it}^e\right] = \lambda_0 + b_{i1}\lambda_1 + b_{i2}\lambda_2 + \ldots + b_{iK}\lambda_K, \qquad i = 1, \ldots, N$$

The theory puts no restrictions on these *risk prices*, except when the factors are traded assets.

(for details; see Chapter 9 of Cochrane (2005) and Chapter 6 of Campbell, Lo, and MacKinley (1997))

Example with two factors

assume now the data are generated by a two-factor model:

$$R^e_{it} = \alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

suppose we build an equally weighted $w_i = \frac{1}{N_P}$ portfolio using N_P assets

$$\begin{aligned} R^e_{pt} &= \frac{1}{N_P} \sum_{i=1}^{N_P} \alpha_i + \frac{1}{N_P} \sum_{i=1}^{N_P} \beta_{i1} f_{1t} + \frac{1}{N_P} \sum_{i=1}^{N_P} \beta_{i2} f_{2t} + \frac{1}{N_P} \sum_{i=1}^{N_P} \varepsilon_{it} \\ &= \alpha_p + \beta_{p1} f_{1t} + \beta_{p2} f_{2t} + \frac{1}{N_P} \sum_{i=1}^{N_P} \varepsilon_{it} \end{aligned}$$

in a 'well-diversified' portfolio, **the residual risk disappears** because of the Law of Large Numbers

only systematic risk is left: $R^e_{pt} \approx \alpha_p + \beta_{p1} f_{1t} + \beta_{p2} f_{2t}$

well-diversified $\Rightarrow$ weights w_i cannot be too extreme

Example with two factors

in a well-diversified portfolio, **the residual risk disappears**
only systematic risk is left:

$$R_{it}^{e} \approx \alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t}, \qquad i = 1, \ldots, P$$

1. zero aggregate risk portfolio
   1. construct a portfolio with $\beta_1 = 0$ and $\beta_2 = 0$
   2. the expected excess return on this portfolio is 0
2. factor-1-mimicking portfolio
   1. construct a zero-investment portfolio with $\beta_1 = 1$ and $\beta_2 = 0$
   2. the expected (excess) return on this portfolio is: $E\left[R_{1t}^{e}\right]$
3. factor-2-mimicking portfolio
   1. construct a zero-investment portfolio with $\beta_1 = 0$ and $\beta_2 = 1$
   2. the expected return on this portfolio is: $E\left[R_{2t}^{e}\right]$

Recovering Risk Prices

The expected return on asset i is:

$$E\left[R_{it}^{e}\right] \approx \lambda_0 + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2,$$

1. zero aggregate risk portfolio
   1. construct a portfolio with $\beta_1 = 0$ and $\beta_2 = 0$
   2. the expected excess return on this portfolio is $\underline{\lambda_0 = 0}$
2. factor-1-mimicking portfolio
   1. construct a zero-investment portfolio with $\beta_1 = 1$ and $\beta_2 = 0$
   2. the expected (excess) return on this portfolio is the **risk price** $\underline{\lambda_1 = E\left[R_{1t}^{e}\right]}$
3. factor-2-mimicking portfolio
   1. construct a zero-investment portfolio with $\beta_1 = 0$ and $\beta_2 = 1$
   2. the expected return on this portfolio is the **risk price** $\underline{\lambda_2 = E\left[R_{2t}^{e}\right]}$

APT Pricing for Asset B

Consider another asset B with factor loadings given below:

$$R^e_{Bt} \approx constant + .5 \times f_{1t} + .5 \times f_{2t}$$

construct a portfolio A using the portfolios that 'mimick' the factors:

1. invest .5 in factor-1-mimicking portfolio
2. invest .5 in factor-2-mimicking portfolio
3. invest $1 - .5 - .5$ in risk-free asset

then the expected return on this portfolio A is:

$$E\left[R_{At}\right] \approx R_{ft} + .5\lambda_1 + .5\lambda_2$$

hence the expected return on asset B should be equal to:

$$E\left[R_{Bt}\right] \approx R_{ft} + .5\lambda_1 + .5\lambda_2$$

APT Pricing for Asset D

Consider another asset D. How do we price this asset?

$$R^e_{Dt} \approx constant + .75 \times f_{1t} + .75 \times f_{2t}$$

construct a portfolio C:

1. invest .75 in factor-1-mimicking portfolio
2. invest .75 in factor-2-mimicking portfolio
3. invest $1 - .75 - .75$ in risk-free

then the expected return on this portfolio C is:

$$E\left[R_{Ct}\right] \approx R_{ft} + .75\lambda_1 + .75\lambda_2$$

hence the expected return on asset D should be equal to:

$$E\left[R_{Dt}\right] \approx R_{ft} + .75\lambda_1 + .75\lambda_2$$

APT pricing with two factors

we assume the data are generated by a two-factor model:

$$R_{it}^{e} = \alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

- in a well-diversified portfolio, **the residual risk disappears**
- only systematic risk is left

the **quantity of risk** is determined by the loadings β_{i1} and β_{i2}

all investments must be on a plane in the space of $\left(E\left[R_{it}\right], \beta_{i1}, \beta_{i2}\right)$

hence, we can find **the price of risk** for each factor (λ_1, λ_2) such that:

$$E\left[R_{it}^{e}\right] = \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2, \qquad i = 1, \ldots, N$$

APT in Equilibrium

Pricing in equilibrium: *There exist risk prices λ_j for each factor such that the expected return on any security i can be stated as:*

$$E\left[R_{it}^e\right] = \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + \ldots + \beta_{iK}\lambda_K, \qquad i = 1, \ldots, N$$

- very general
- relative pricing: price one asset relative to others
- no need to measure the return on the total wealth portfolio (or 'the market')
- the theory does not actually tell you which factors to use!
  - for a larger set of securities, you probably need more factors

Testing the model: Robust test (Optional Material)

White covariance matrix of residuals allows for non-normal errors terms with time-varying volatility

- Asymptotic test

A little cumbersome in terms of notation

- Have to set up the system as a big panel regression
- Here goes...

Testing the model: Robust test

$$\underset{N\times 1}{r_t} = \begin{bmatrix} r_{1t} & r_{2t} & \dots & r_{Nt} \end{bmatrix}', \quad \underset{N\times 1}{\varepsilon_t} = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \dots & \varepsilon_{Nt} \end{bmatrix}'$$

$$\underset{(N\times K)\times N}{x_t} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ R^e_{m,t} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & R^e_{m,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & R^e_{m,t} \end{bmatrix},$$

$$\boldsymbol{\beta} = \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \cdots & \alpha_N & \beta_N \end{bmatrix}',$$

where $K = 2$ and the $\boldsymbol{\beta}$ vector is obtained by running the N univariate regressions.

Then, $f(\boldsymbol{\beta}) = x_t \varepsilon_t$ is a $(N \times K) \times 1$ vector.

Robust test (cont'd)

Referring back to asymptotics slides in Lecture 2 and the Asymptotics Note on CCLE, let the sample mean of the moment condition be:

$$g_T\left(\hat{\beta}\right) = \frac{1}{T}\sum_{t=1}^{T} f_t\left(\hat{\beta}\right) = 0$$

From the Central Limit Theorem:

$$\sqrt{T} g_T\left(\hat{\beta}\right) \sim N\left(0, S_T\right)$$

where

$$S_T = \frac{1}{T}\sum_{t=1}^{T} f_t\left(\hat{\beta}\right) f_t\left(\hat{\beta}\right)' = \frac{1}{T}\sum_{t=1}^{T} x_t \hat{\varepsilon}_t \hat{\varepsilon}_t' x_t'$$

is an $(N \times K) \times (N \times K)$ matrix.

Robust test (cont'd)

Then, using the Law of Large numbers and Central Limit Theorem, we have

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \sim N\left(0, \Sigma\left(\boldsymbol{\beta}\right)\right)$$

where $\Sigma\left(\boldsymbol{\beta}\right) = \frac{1}{T} E_T\left[x_t x_t'\right]^{-1} S_T E_T\left[x_t x_t'\right]^{-1}$ is the $(N \times K) \times (N \times K)$ covariance matrix of $\boldsymbol{\beta}$.

Remember now that $\hat{\boldsymbol{\beta}}$ is is an $(N \times K) \times 1$ vector. To test the α's, we are only interested in the odd rows and columns of the full variance-covariance matrix, $\Sigma\left(\boldsymbol{\beta}\right)$.

Call the $N \times N$ covariance matrix of the α's that result from extracting the odd rows and columns from $\Sigma\left(\boldsymbol{\beta}\right)$ for $\Sigma_{\alpha, White}$. Then the asymptotic test is

$$\hat{\alpha}' \Sigma_{\alpha, White}^{-1} \hat{\alpha} \sim \chi^2\left(N\right)$$