Lecture Note 10
The Cross-Section of Stock Returns: Factor Models

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Overview of Lecture Note 10: The Cross-Section

1. The single-factor model
   - Portfolio choice and the curse of dimensionality
   - The simplest covariance matrix: A single factor
   - Single-factor model math
   - Portfolios or single stocks: estimation error and beta stability
2. Multi-factor models
   - Math
   - Properties
   - Estimation
3. Applications
   - Firm characteristics-based factors (the Fama-French model)

Portfolio Problem

Portfolio of N securities with $N \times N$ variance-covariance matrix $\boldsymbol{\Sigma}$

Vector of $N \times 1$ portfolio weights: $\mathbf{w}$

Variance of portfolio return is given by:

$$\begin{aligned}
\sigma_P^2 &= \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\
&= \sum_{i,j=1}^{N} w_i w_j \sigma_{ij} \\
&= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}
\end{aligned}$$

Markowitz (1952): choose $\mathbf{w}$ to minimize the variance of the portfolio for a given mean $\boldsymbol{\mu}_P$.

Inputs to Portfolio Analyssis

Standard deviation of portfolio return:

$$\sigma_P = \left[\sum_{i,j=1}^{N} w_i w_j \sigma_{ij}\right]^{1/2}$$

$$= \left[\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij}\right]^{1/2}$$

$$= \left[\mathbf{w}'\mathbf{\Sigma}\mathbf{w}\right]^{1/2}$$

Mean of portfolio return:

$$E\left[R_P\right] = \sum_{i=1}^{N} w_i E\left[R_i\right] = \sum_{i=1}^{N} w_i \mu_i = \boldsymbol{\mu}_P$$

Inputs to Portfolio Analysis

1. Expected returns $E\left[R_i\right] = \mu_i$ and risk $\mathbf{\Sigma}$:
    - we could use historical data to estimate expected returns
    - we could use equilibrium models to estimate expected returns (see Black and Litterman (1992))
2. Pairwise correlations:
    - hard to estimate
    - unstable through time

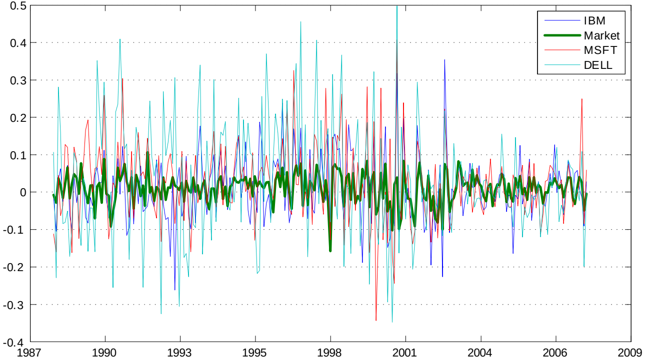
Curse of Dimensionality

- number of parameters to estimate necessary to carry out portfolio analysis is staggering.
- suppose you follow between 150 and 250 stocks
- we need all correlation coefficients ρ_{ij} for stocks i and j
- total number of free correlations in an $N \times N$ covariance matrix $\mathbf{\Sigma}$ is $\frac{N(N+1)}{2}$ (11,325 in the case of $N =$150 stocks)
- in the case of a large number of assets: traditional mean-variance analysis tends to perform very poorly out of sample; see, e.g. DeMiguel, Garlappi, and Uppal (2009)
- For more on the impact of the curse of dimensionality in statistics, see Chapter 2 of Hastie, Tibshirani, and Friedman (2009)

Single Factor Model

- single factor models: co-movement is due to a single factor.
- **economic factor models**: use economic theory or economic intuition to determine the factor.
- **statistical factor models**: use the data to determine the factor.
- some textbooks call this a **single index model**. The factor is often a stock market index.
- Capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965), and Mossin (1966) is the most famous single-factor model.
  - The 'market return' is the factor

Stock Returns



Monthly data from CRSP: The Market Return is the CRSP Value-Weighted Return on NYSE-AMEX-NASDAQ. Sample 1987:1-2007.12

Single Factor Model: The Market Model

- stocks move in tandem due to a common factor f_t
- the return on a stock i can be written as:

$$R_{it} = \alpha_i + \beta_i f_t + \varepsilon_{it}$$

- let R_{mt} denote the market return.
- suppose the common factor is the market. Then, we have

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$$

  - ε_{it} is the component of the return that is independent of the market return R_{mt}.
  - assume $E\left[\varepsilon_{it}\right] = 0$.
  - β_i is a constant.

Single Factor Model

Two key (simplifying) assumptions:

1. disturbances are independent of the market return:

$$\text{Cov}[\varepsilon_{it}, R_{mt}] = 0$$

2. ε_{it} is independent of ε_{jt}:

$$\text{Cov}[\varepsilon_{it}, \varepsilon_{jt}] = 0$$

$\Rightarrow$ no effects beyond the market that account for covariation (e.g. industry effects)

Moments of the Single Factor Model

- This model then implies:

$$E\left[R_{it}\right] = \alpha_i + \beta_i E\left[R_{mt}\right], \qquad i = 1, \ldots, N$$
$$\mu_i = \alpha_i + \beta_i \mu_m, \qquad i = 1, \ldots, N$$

$$V\left[R_{it}\right] = \beta_i^2 V\left[R_{mt}\right] + V\left[\varepsilon_{it}\right], \qquad i = 1, \ldots, N$$
$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2, \qquad i = 1, \ldots, N$$

$$Cov\left[R_{it}, R_{jt}\right] = \beta_i \beta_j \sigma_m^2, \qquad i = 1, \ldots, N$$

Single Factor Model Portfolio Math

The expected return on a portfolio is:

$$\begin{aligned} \mu_p &= \sum_{i=1}^{N} w_i \mu_i \\ &= \sum_{i=1}^{N} w_i \alpha_i + \mu_m \sum_{i=1}^{N} w_i \beta_i \end{aligned}$$

Recall that the variance of the portfolio return is:

$$V[R_P] = \left[\sum_{i=1}^{N} w_i^2 (\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2) + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \beta_i \beta_j \sigma_m^2 \right]$$

Note the dramatic decrease in number of parameters to be estimated

- $3N + 2$ estimates: $\alpha_i, \beta_i, \sigma_{e_i}^2, i = 1, \ldots, N$ and $\sigma_m^2, E[R_m]$

Single Factor Model Portfolio Math (cont'd)

- The beta of a portfolio β_p is:

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i$$

where w_i are the portfolio weights

- The alpha of a portfolio α_p is:

$$\alpha_p = \sum_{i=1}^{N} w_i \alpha_i$$

where w_i are the portfolio weights

- then, given the factor model, the expected return on the portfolio is:

$$\mu_p = \alpha_p + \beta_p \mu_m$$

Singel Factor Model: Market Model

- if the factor f_t is the market return R_{mt}, then the factor is a traded asset.
- the model should apply to the market return as well.
- then the expected return on the market portfolio is:

$$\mu_m = \alpha_m + \beta_m \mu_m$$

- this immediately implies that:
  - $\alpha_m = 0$
  - $\beta_m = 1$
- the market portfolio has a beta of one and an alpha of zero.

Single Factor Model Parameters

- to use the single-factor model, we need *estimates* of α_i and β_i

- what you need, is the betas going forward (i.e. the beta for a security that applies over the next say 5 years)

- we typically use historical betas as estimates of future betas
  - these are **backward** looking
  - but they are informative about **future** betas

Single Factor Model Estimation

- define

$$\mathbf{X} = \begin{bmatrix} 1 & R_{m,1} \\ 1 & R_{m,2} \\ \vdots & \vdots \\ 1 & R_{m,T} \end{bmatrix}, \mathbf{y}_i = \begin{bmatrix} R_{i,1} \\ R_{i,2} \\ \vdots \\ R_{i,T} \end{bmatrix}$$

- write the single factor model as:

$$\mathbf{y}_i = \mathbf{X}\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i$$

with $\boldsymbol{\beta}_i = \begin{bmatrix} \alpha_i & \beta_i \end{bmatrix}'$

- Use least squares to estimate $\boldsymbol{\beta}_i$ for each asset i

Sampling Error

- consider the case of individual securities
- the estimator of the slope coefficient is :

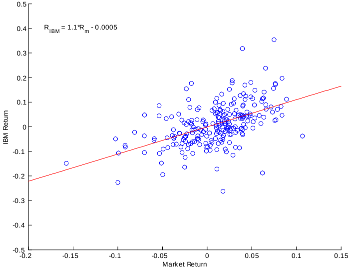
$$\widehat{\beta}_i = \frac{\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)(R_{it} - \overline{R}_i)}{\sum_{t=1}^{T}\left(R_{mt} - \overline{R}_m\right)^2}$$

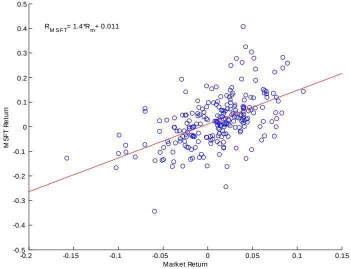
- the variance of the estimator of the slope coefficient is :

$$V\left(\widehat{\beta}_i - \beta_i\right) = \frac{1}{T}\frac{var\left(\hat{\varepsilon}_{it}\right)}{var\left(R_{mt}\right)}$$

- measure of sampling error
- sampling error is influenced by $\sigma^2_{\varepsilon_i}$, which can be large for individual stocks

IBM and MSFT



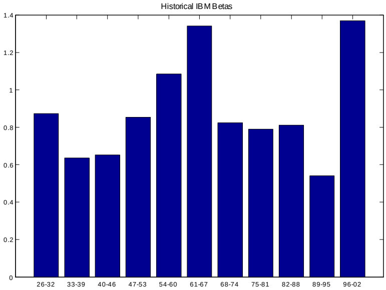


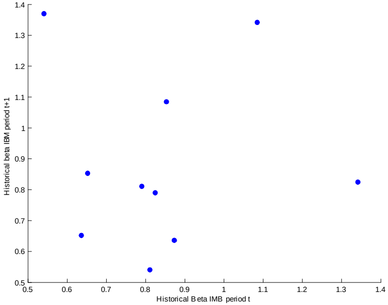
Monthly data from CRSP. Sample 1987.1-2007.12. IBM is left and MSFT is right.

Time-Variation in Betas

- empirical evidence suggests there is quite a lot of time variation in historical estimates of β's over different samples
  - could be because true betas are varying over time
  - could be sampling error
- for individual securities, little relation between estimated betas in subsequent time samples
- for portfolios of securities sorted on certain characteristics $\Rightarrow$ more stability
- for portfolios of securities $\Rightarrow$ less sampling error

Instability in IBM Betas





Monthly data from CRSP. Sample 1926.1-2002.12. Estimates over 7-year non-overlapping intervals.

Portfolio sorts

- the academic finance literature sorts stocks into portfolios
- statistically, this reduces idiosyncratic volatility of the 'asset'
  - more noisy measurement of the true beta at the individual security level (larger 'sampling error')
  - less at the portfolio level
- economically, risk characteristics of individual securities might change over time
  - key idea: sort individual securities into portfolios based on characteristics
  - examples:
    1. size (market cap)
    2. book-to-market
    3. industries

Portfolio Sampling Error

- consider the case of a portfolio with N assets.
- the estimator of the slope coefficient is :

$$\widehat{\beta}_p = \frac{\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)(R_{pt} - \overline{R}_p)}{\sum_{t=1}^{T}\left(R_{mt} - \overline{R}_m\right)^2}$$

- the variance of the estimator of the slope coefficient is :

$$V\left(\widehat{\beta}_p - \beta_p\right) = \frac{1}{T}\frac{var\left(\hat{\varepsilon}_{pt}\right)}{var\left(R_{mt}\right)}$$

where

$$var\left(\hat{\varepsilon}_{pt}\right) = \sum_{i=1}^{N} w_i^2 \, var\left(\hat{\varepsilon}_{it}\right)$$

Sampling Error (cont'd)

- originally, we have N total firms.
- separate N firms into a smaller set of $j = 1, \ldots, J$ portfolios. Each portfolio has $N^{(j)}$ assets in it. Assume no firm shows up in more than 1 portfolio at a given time.
- consider an equally-weighted portfolio with $w_i = \frac{1}{N^{(j)}}$, $i = 1, \ldots, N^{(j)}$
- by building portfolios we reduce sampling error, by pushing this term

$$\sigma^2_{\varepsilon_p} = \frac{1}{\left(N^{(j)}\right)^2} \sum_{i=1}^{N^{(j)}} \sigma^2_{\varepsilon_i}$$

  to zero as $N^{(j)}$ increases.
- moreover, we might create more stable betas

Size and Book-to-Market

Fama and French (1992, 1993) argue that *size* and *book/market* are key risk characteristics

They construct portfolios by sorting stocks on these characteristics

- The portfolios, which are constructed at the end of each June, are the intersections of 2 portfolios formed on size (market equity, ME) and 3 portfolios formed on the ratio of book equity to market equity (BE/ME).
  - The size breakpoint for year t is the median NYSE market equity at the end of June of year t.
  - BE/ME for June of year t is the book equity for the last fiscal year end in t-1 divided by ME for December of t-1.
  - The BE/ME breakpoints are the 30th and 70th NYSE percentiles.

Size and Book-to-Market (cont'd)

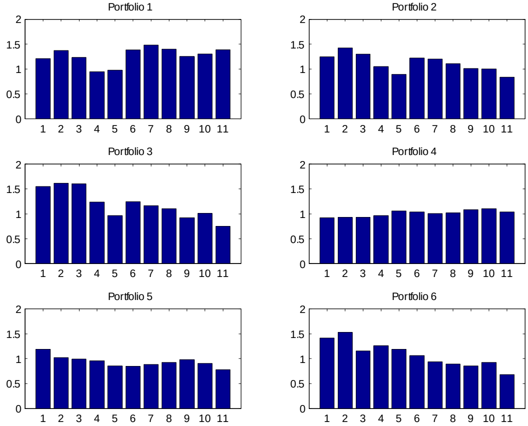
The six portfolios are

size vs. B/M	low	medium	high
small	1	2	3
big	4	5	6

We can give them nice labels

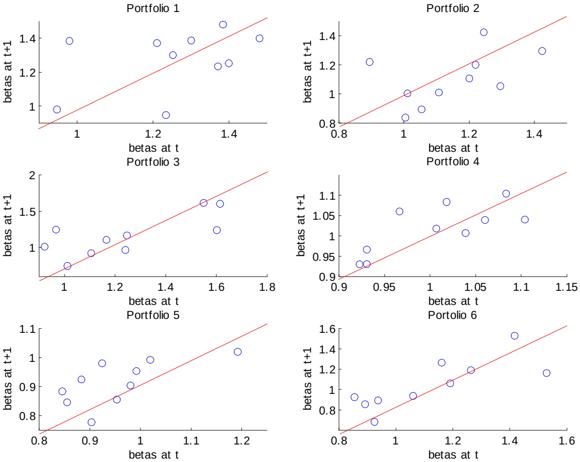
size vs. B/M	low	medium	high
small	Small Growth	Small Neutral	Small Value
big	Big Growth	Big Neutral	Big Value

Portfolio Betas



6 Fama-French portfolios sorted on size and book-to-market. Monthly data from CRSP. Sample 1926.1-2002.12. Estimates over 11 non-overlapping 7-year intervals.

Stability in Portfolio Betas



6 Fama-French portfolios sorted on size and book-to-market. Monthly data from CRSP. Sample 1926.1-2002.12. Estimates over 11 non-overlapping 7-year intervals.

Accuracy in Historical Betas

- empirical evidence suggests historical portfolio betas are better predictors of future portfolio betas
- substantial gain in stability
- sorting stocks into portfolios is one way to produce stability.
- there are other ways to produce stability. In statistics, these are called 'shrinkage estimators' or Bayesian estimators
  - the basic idea is to push our parameter estimates to a prior value that is economically plausible.
  - Vasicek (1976) and Blume (1975) are early applications of this idea

Multi-factor Models

- so far, we have assumed that all of the covariation among securities is due to a single common factor f_t
- the common factor f_t is typically taken to be the market return R_{mt}
- however, there might be other reasons security prices co-move $\Rightarrow$ multiple factors
- what should the extra factors be?
- some examples:
  1. industry factors
  2. macro-economic factors
  3. factors built from accounting variables
  4. statistical (& latent) factors

Multi-factor Models (cont'd)

- stocks move in tandem
- the return on a stock i can be written as:

$$R_{it} = \alpha_i + \beta_{i1} f_{1t} + \beta_{i2} f_{2t} + \ldots + \beta_{iK} f_{Kt} + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

- f_{it} are correlated with each other.
- for example:
  - f_{1t} is a market index
  - f_{2t} is an interest rate index

Multi-factor Models in Matrix Notation

- let $\mathbf{f}_t$ denote a $K \times 1$ vector of factors
- let $\boldsymbol{\beta}_i$ denote a $K \times 1$ vector of factor loadings
- we can write the model in matrix notation as

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

- we can then stack all N assets together

$$\mathbf{R}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{f}_t + \boldsymbol{\varepsilon}_t$$

- $\mathbf{R}_t$ is a $N \times 1$ vector of returns, $\boldsymbol{\beta}$ is a $N \times K$ matrix of factor loadings
- let $\boldsymbol{\Sigma}_f$ denote the $K \times K$ covariance matrix of the factors.
- let $\boldsymbol{\Sigma}_\varepsilon$ denote the $N \times N$ covariance matrix of the idiosyncratic errors

Multi-factor Model: Assumptions

1. disturbances are independent of the factor:

$$\mathrm{Cov}[\varepsilon_{it}, f_{jt}] = 0$$

2. ε_{it} is independent of ε_{jt}:

$$\mathrm{Cov}[\varepsilon_{it}, \varepsilon_{jt}] = 0$$

no effects beyond the factors that we have included that account for covariation. This means that $\mathbf{\Sigma}_{\varepsilon}$ is diagonal.

Multi-factor Model: Moments

Mean:

$$\begin{aligned} E\left[\mathbf{R}_t\right] &= \boldsymbol{\alpha} + \boldsymbol{\beta} E\left[\mathbf{f}_t\right] \\ &= \boldsymbol{\alpha} + \boldsymbol{\beta}\boldsymbol{\mu}_f \end{aligned}$$

Covariance matrix:

$$\begin{aligned} Var\left[\mathbf{R}_t\right] &= \boldsymbol{\beta} Var\left[\mathbf{f}_t\right] \boldsymbol{\beta}' + Var\left[\boldsymbol{\varepsilon}_t\right] \\ &= \boldsymbol{\beta}\Sigma_{\mathbf{f}}\boldsymbol{\beta}' + \Sigma_{\boldsymbol{\varepsilon}} \end{aligned}$$

Number of Parameters to be Estimated

- suppose the factors f_t are observable
- total number of parameters to be estimated:

$$K + 2N + NK + \frac{K(K+1)}{2}$$

  - we need estimates of the $K \times 1$ vector $\boldsymbol{\mu}_f$
  - we need estimates of $\boldsymbol{\alpha}$ and the diagonal elements of $\boldsymbol{\Sigma}_{\varepsilon}$
  - we need estimates of the $N \times K$ matrix of factor loadings $\boldsymbol{\beta}$
  - we need estimates of the covariance matrix $\boldsymbol{\Sigma}_{\mathbf{f}}$. It has $\frac{K(K+1)}{2}$ free parameters.

Industry Factor Model

- common co-movement between stocks not driven by the market
- some of this was related to industry effects
  - 30 to 50 % of variation in stock returns is due to the market
  - 10 % of variation in stock returns is due to industry effects

Industry Factor Model: General

Seems most appropriate for firms that operate in multiple industries

The return on a stock i can be written as:

$$R_{it} = \alpha_i + \beta_{im} R_{mt} + \beta_{i1} R_{1t} + \ldots + \beta_{iK} R_{Kt} + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

- the $R_{jt}, j = 1, \ldots, K$ are returns on industries
- R_{mt} is the market return

Industry Factor Model: Simple

Seems more appropriate for firms that operate in a single industry

The return on a stock i can be written as:

$$R_{it} = \alpha_i + \beta_{im} R_{mt} + \beta_{ij} R_{jt} + \varepsilon_{it}, \qquad i = 1, \ldots, N$$

- the R_{jt} is the return on an industry based index in which firm i operates
- R_{mt} is the market index

Fama and French (1993) Model

The Fama/French factors are constructed using the 6 value-weighted portfolios formed on size and book-to-market.

- SMB (Small Minus Big)
  - SMB = 1/3 (Small Value + Small Neutral + Small Growth) - 1/3 (Big Value + Big Neutral + Big Growth).
- HML (High Minus Low)
  - HML = 1/2 (Small Value + Big Value) - 1/2 (Small Growth + Big Growth).
- $R_{mt} - R_{ft}$, the excess return on the market, is the value-weight return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates).

Fama and French (1993) Model (cont'd)

- multi-factor model for monthly returns:

$$R_{it} = \alpha_i + \beta_{i,M} R_t^m + \beta_{i,SMB} R_t^{SMB} + \beta_{i,HML} R_t^{HML} + \varepsilon_{it}$$

- we will use $N = 10$ test assets formed on industries

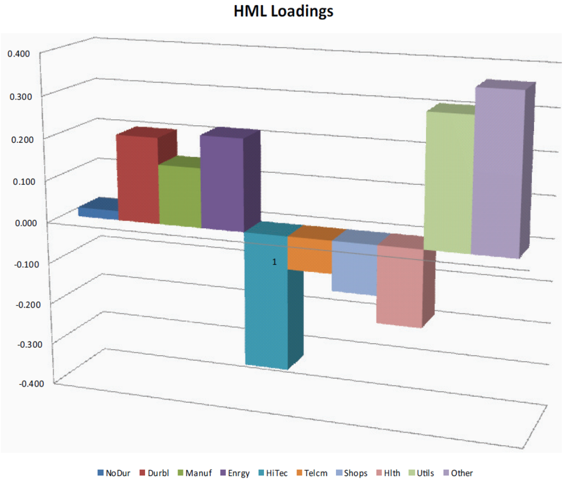
Fama and French (1993): Industry Loadings

Column1	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
HML	0.022	0.205	0.142	0.219	-0.321	-0.078	-0.118	-0.182	0.308	0.368
se	0.021	0.034	0.016	0.034	0.027	0.027	0.025	0.031	0.032	0.018
SMB	-0.031	0.007	-0.013	-0.219	0.080	-0.133	0.078	-0.093	-0.160	0.064
se	0.023	0.038	0.017	0.037	0.029	0.030	0.028	0.034	0.035	0.020
Market	0.760	1.192	1.105	0.862	1.274	0.683	0.954	0.896	0.785	1.049
se	0.014	0.023	0.011	0.023	0.018	0.019	0.017	0.021	0.022	0.012
R^2	0.769	0.764	0.928	0.621	0.848	0.587	0.780	0.660	0.622	0.909

Sample: 1926.7 – 2008.12

HML Loadings

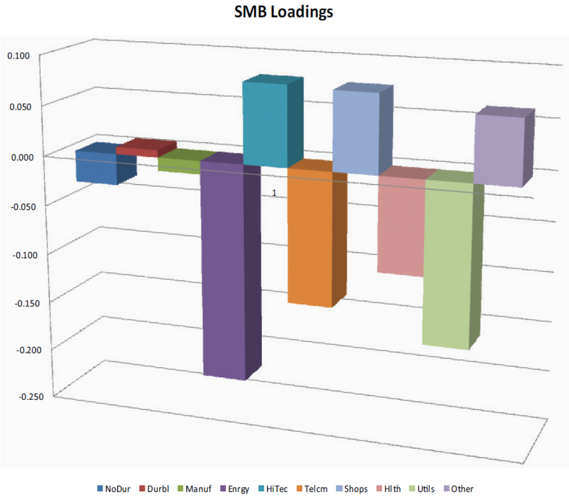
- b_{HML} for each industry:



Sample: 1926.7 – 2008.12

SMB Loadings

- b_{SMB} for each industry:



Sample: 1926.7 – 2008.12