MFE 409 Lecture 4
Measuring Value-at-Risk: Model-Building Approach

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Coin → Heads 51% +\$1
→ Tails 49% −\$1

How many draws do you need to do to gain \$10 or more with probability ≥ 99% ?

$P(\text{gain } k \text{ heads with } n \text{ draws}) \approx \text{binomial}(n, k, 51\%)$

$n \approx 14{,}000$

We need $k > \frac{n}{2} + 5 \iff$ 10 more H than T

$P(k > \frac{n}{2} + 5 \text{ with } n \text{ draws}) = 1 - \text{binomialCDF}(n, \frac{n}{2}+5, 51\%)$

$E(\text{Gain}) = 51\% - 49\% = \$0.02$ Variance(Gain) $= 1 - 0.0004 = 0.9996$

99% worst case after n draws: $0.02\,n - 2.33\sqrt{0.9996\,n} \geq 10$

Lecture Objectives

How to measure risk using data on the performance of a strategy?

Last week:

- How to judge validity of a VaR estimate?
- Historical approach

Today:

- Model-building approach
- How to get a measure for a given approach but also how to choose an appropriate approach

Outline

Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the returns on the market variables

- Sometimes called the variance-covariance approach

Normal Model

- Simplest and often-used assumption: normal distribution

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- VaR has a simple expression:

$$\text{VaR} = -\mu - \sigma \times z(c)$$

$$99\%\ \text{VaR} = -\mu + 2.32\,\sigma$$

- Portfolios of normal returns are also normally distributed

- Estimation of normal distributions very developed

Portfolios

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
  - Each asset return R_i is normally distributed with mean 0 and variance σ_i^2
  - Pairwise correlations: ρ_{ij}
  - investment in each asset α_i (in dollars)

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \sum_i \alpha_i R_i \sim \mathcal{N}(0, \sigma_P^2)$$

$$\sigma_P^2 = \sum_i \sum_j \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$$

$$= \sum_i \alpha_i^2 \sigma_i^2 + 2 \sum_i \sum_{j<i} \alpha_i \alpha_j \underbrace{\mathrm{cov}(R_i, R_j)}_{\rho_{ij}\sigma_i\sigma_j}$$

Portfolios

$$R = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}$$

- With multivariate normal returns, portfolio returns are normally distributed
- Assume:
  - Asset return $\mathbf{R}$ normally distributed with mean 0 and variance Σ $= \begin{array}{c} \\ \\ i \end{array} \begin{pmatrix} \sigma_1^2 & & j & \\ & \ddots & \vdots & \\ - & - & \rho_{ij}\sigma_i\sigma_j & \\ & & & \sigma_n^2 \end{pmatrix}$
  - Investment vector α

$$\underbrace{\Delta P}_{\text{portfolio gain}} = \alpha'\mathbf{R} \sim \mathcal{N}(0, \sigma_P^2)$$

$$\sigma_P^2 = \alpha'\Sigma\alpha$$

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$$\sigma_P^2 = \alpha' \Sigma \alpha$$

- Multiply by $-z(c)$ to obtain VaR

Example: Imperfect Hedge

- Previous example:
  - Long position EUR 10m, $M_t =$ USD/EUR $= \$1.436$, $\sigma_M = 0.65\%$
  - Dollar position \$14.36m
  - 99% VaR$= \$217,204$
- Suppose you want to hedge with Japanese Yens: $\sigma_J = 0.69\%$, $\rho_{MJ} = 0.2775$
  - What Yen position do you choose to hedge as well as possible?
  - What is your hedged VaR?

$R_{portfolio} = 14.36m \, R_M + x \, R_J$

$\hookrightarrow$ choose x to make portfolio VaR small

$$\sigma^2_{portfolio} = \alpha' \Sigma \alpha = 14.36^2 \sigma_M^2 + x^2 \sigma_J^2 + 2\rho \, \sigma_M \sigma_J \, 14.36 \times x$$

$$\alpha = \begin{pmatrix} 14.36 \\ x \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_M^2 & \rho \sigma_M \sigma_J \\ * & \sigma_J^2 \end{pmatrix}$$

Minimize with respect to x $\Leftrightarrow$ set derivative to 0

$$0 = 2x \, \sigma_J^2 + 2\rho \, \sigma_M \sigma_J \, 14.36$$

$$x = -14.36 \frac{\sigma_M}{\sigma_J} \rho = -\$2.587m$$

$$\text{VaR} = 2.33 \sqrt{\sigma^2_{portfolio}}$$

Imperfect Hedge

- Want to hedge a position R_p using a hedging instrument R_h
- Optimal hedging position:

$$\alpha_{\text{hedge}} = -\rho \frac{\sigma_p}{\sigma_h} = - \frac{\rho\, \sigma_p\, \sigma_h}{\sigma_h^2} = -\beta_{p,h}$$

$$= -\frac{\mathrm{Cov}(R_p R_h)}{\mathrm{Var}(R_h)}$$

- Variance of the hedged portfolio:

$$\text{Minimum variance} = \sigma_p^2(1 - \rho^2)$$

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$$\text{Minimum variance} = \sigma_p^2(1 - \rho^2)$$

- VaR of the hedged portfolio:

$$\text{Minimum VaR} = \text{VaR}_p \sqrt{1 - \rho^2}$$

  - Only depends on correlation ρ

Outline

1. The Model-Building Approach
2. Using Return Data
3. Using Options
4. Choosing an Approach

Volatility

- Often, volatility is defined as standard deviation of *log return*

$$\log\left(\frac{P_{t+1}}{P_t}\right)$$

- In risk management, typically the standard deviation of *simple return*

$$\frac{P_{t+1}}{P_t} - 1$$

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- Some conventions:
  - Only count trading days: $\sigma_{\mathsf{yr}} = \sigma_{\mathsf{day}} \times \sqrt{252}$
  - Variance rate: σ^2

Estimating Volatility

- Assume today is date t and we have data for n past dates
- Unbiased estimates
  - Mean

$$\bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_{t-i}$$

  - Volatility

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(R_{t-i} - \bar{R}\right)^2$$

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- Risk management practice:
  - Assume $\bar{R} = 0$: mean small relative to standard deviation for one day
  - Replace $n-1$ by n

Estimating Volatility: Maximum Likelihood

$$R \sim \mathcal{N}(0, \sigma^2)$$

$$R_1, \ldots, R_N$$

$$\mathcal{L}(R_1, \ldots, R_N \mid \sigma^2) = \mathcal{L}(R_1 \mid \sigma^2) \times \mathcal{L}(R_2 \mid \sigma^2) \times \ldots \times \mathcal{L}(R_N \mid \sigma^2)$$

$$\downarrow$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{R_i^2}{2\sigma^2}\right)$$

$$\log(\mathcal{L}(R_1, \ldots R_N \mid \sigma^2)) = -\frac{N}{2} \log(\sigma^2) - \sum_{i=1}^{N} \frac{R_i^2}{2\sigma^2} + \ldots$$

$$0 = -N \frac{1}{\sigma^2} + \frac{1}{\sigma^4} \sum_{i=1}^{N} R_i^2$$

$$\sigma^2_{MLE} = \frac{\sum_{i=1}^{N} R_i^2}{N}$$

Estimating Volatility: Maximum Likelihood

- Likelihood for one observation

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-R_i^2}{2\sigma^2}\right)$$

- Log likelihood

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \left[-\log(\sigma^2) - \frac{R_{t-i}^2}{\sigma^2} \right]$$

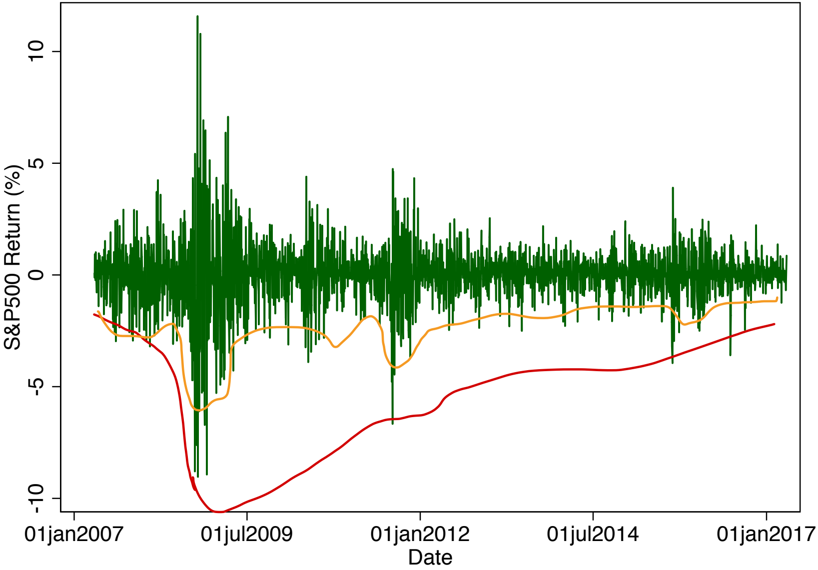
- First-order condition w.r.t. σ^2

$$0 = -\frac{n}{\sigma^2} + \frac{1}{\sigma^4} \sum_{i=1}^{n} R_{t-i}^2$$

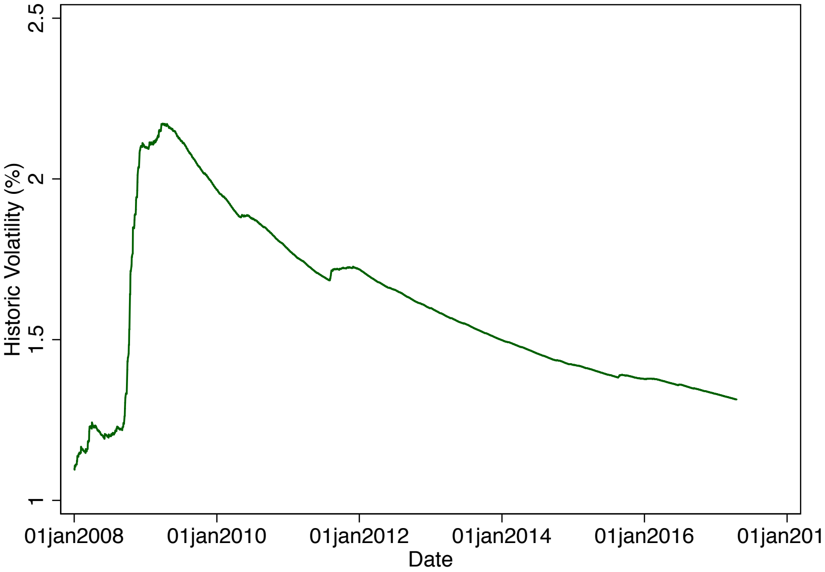
- Estimator

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} R_{t-i}^2$$

S&P500: Returns



S&P500: Historic Volatility



Weighting Schemes

- Volatility moves over time: recent information more relevant than older one

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- Weighting scheme:

$$\sigma_t^2 = \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \sum_{i=1}^{n} \alpha_i$$

Heston

$\frac{dS}{S} = \mu\, dt + \sqrt{V_t}\, dW_t$

$dV_t = \dots dt + \dots dW_t$

Weighting Schemes

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- Weighting scheme:

$$\sigma_t^2 = \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

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- Weighting scheme + long-run variance

$$\sigma_t^2 = \gamma V_L + \sum_{i=1}^{n} \alpha_i R_{t-i}^2$$

$$\text{with } 1 = \gamma + \sum_{i=1}^{n} \alpha_i$$

ARCH

- ARCH(m), autoregressive conditional heteroskedasticity: $R_t \sim \mathcal{N}(0, \sigma_t^2)$ with:

$$\sigma_t^2 = \omega + \sum_{i=1}^{m} \alpha_i R_{t-i}^2$$

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  - If $\alpha_i = 1/m$ and $\omega = 0$, rolling window estimate

EWMA

- EWMA, exponentially weighted moving average

$$\alpha_i = \frac{1-\lambda}{\lambda}\lambda^i$$

  - Simple volatility updating:

$$\sigma_t^2 = \frac{1-\lambda}{\lambda}\left[\lambda R_{t-1}^2 + \lambda^2 R_{t-2}^2 + \dots + \dots\right]$$

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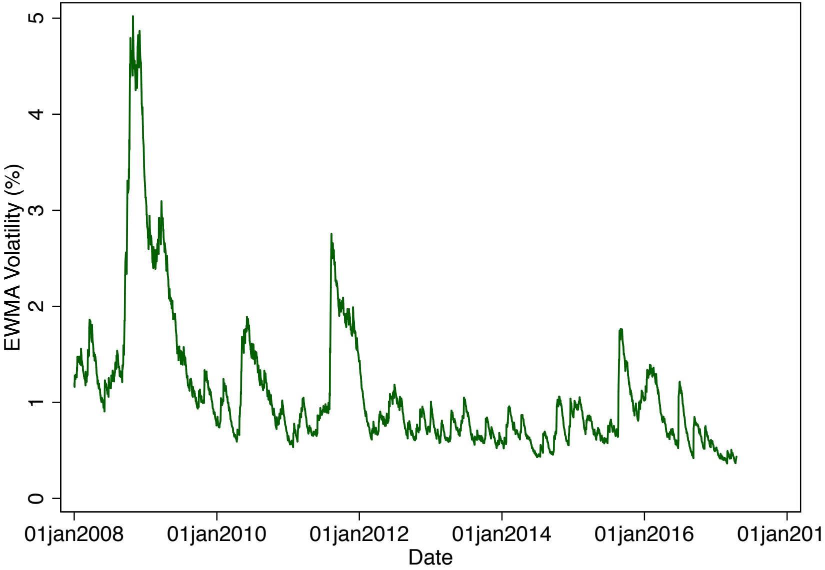
$$\sigma_t^2 = (1-\lambda)R_{t-1}^2 + \lambda(1-\lambda)R_{t-2}^2 + \lambda^2(1-\lambda)R_{t-3}^2 + \cdots$$

$$= (1-\lambda)R_{t-1}^2 + \lambda\underbrace{\left[(1-\lambda)R_{t-2}^2 + \lambda(1-\lambda)R_{t-3}^2 + \cdots\right]}_{\sigma_{t-1}^2}$$

  - RiskMetrics reported with $\lambda = 0.94$ until 2006

historical : $\lambda = 0.995$

S&P500: EWMA Volatility



GARCH(1,1)

- GARCH(1,1), generalized autoregressive conditional heteroskedasticity

$$\sigma_t^2 = \underbrace{\gamma V_L}_{\omega} + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2$$

  - EWMA + long-run average
  - If $\gamma = 0$, EWMA
  - For stability, $\alpha + \beta < 1$

$$R_t \sim \mathcal{N}(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_1 R_{t-1}^2 + \alpha_2 R_{t-2}^2$$

$$\mathcal{L}(R_1, R_2, \ldots R_N \mid \alpha_1, \alpha_2) = \mathcal{L}(R_1 \mid \alpha_1, \alpha_2)$$

$$\times \mathcal{L}(R_2 \mid R_1, \alpha_1, \alpha_2)$$

$$\times \mathcal{L}(R_3 \mid R_1, R_2, \alpha_1, \alpha_2)$$

$$\times \mathcal{L}(R_4 \mid \cancel{R_1}, R_2, R_3, \alpha_1, \alpha_2)$$

$$\times$$

$$\vdots$$

$$\times \mathcal{L}(R_n \mid \cancel{R_1 \ldots} R_{n-2} R_{n-1}, \alpha_1, \alpha_2)$$

$$\mathcal{L}(R_n \mid R_{n-1}, R_{n-2}, \alpha_1, \alpha_2)$$

$$= \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(- \frac{R_n}{\sigma_n^2}\right)$$

$$\alpha_1 R_{n-1}^2 + \alpha_2 R_{n-2}^2$$

MLE Estimation of GARCH(1,1)

- Parameters: ω, α, β
- Log-likelihood

$$\sum_{i=1}^{n}\left[-\log(\sigma_{t-i}^2)-\frac{R_{t-i}^2}{\sigma_{t-i}^2}\right]$$

- Compute σ_{t-i}^2:
  - Initialize at $\sigma_0=\sqrt{V_L}=\sqrt{\omega/(1-\alpha-\beta)}$
  - Use formula to iterate

$$\sigma_t^2=\omega+\alpha R_{t-1}^2+\beta\sigma_{t-1}^2$$

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     - ★ Compare autocorrelations $c_k = \text{cor}(R_t^2/\sigma_t^2, R_{t-k}^2/\sigma_{t-k}^2)$

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- Ljung-Box Statistic

$$n \sum_{k=1}^{K} w_k c_k^2$$

$$w_k = \frac{n+2}{n-k}$$

  - For $K = 15$, 95% threshold is 25

Volatility Forecasts

- If we want to forecast k days in the future:

$$\mathbb{E}_t\left[\sigma_{t+k}^2\right] = V_L + (\alpha+\beta)^k(\sigma_t^2 - V_L)$$

  - Exponential mean reversion

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- *Remark:* If we want to hedge volatility risk, need to consider how shocks today will affect volatility during the lifetime of the option

Outline

Implied volatility

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- Can use market prices to obtain expectation of future volatility

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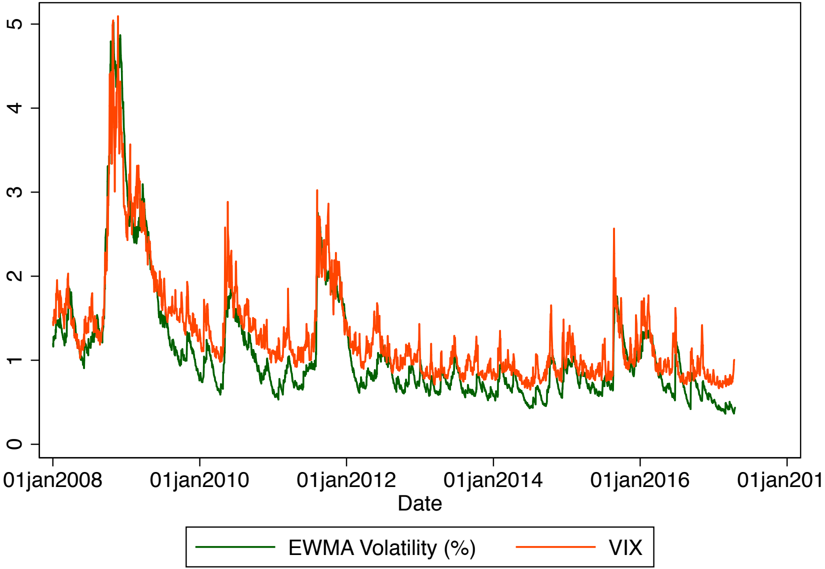
- Can use market prices to obtain expectation of future volatility
- Derivative contracts on volatility: VAR swaps, ...
- *Implied volatility* from calls and puts
  - Volatility so that Black-Scholes formula matches price

VIX


0
1
2
3
4
5
01jan2008
01jan2010
01jan2012
01jan2014
01jan2016
01jan201
Date
EWMA Volatility (%)
VIX

VIX

- The VIX index is published by CBOE
- It is no longer the Black and Scholes implied volatility
- But it is computed from a portfolio of options on the S&P500 index
  - It is deemed to better capture market “expected” volatility over the next 30 days without relying on any model

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- But it is computed from a portfolio of options on the S&P500 index
  - It is deemed to better capture market "expected" volatility over the next 30 days without relying on any model
- Why is VIX systematically higher than realized volatility?
  - Risk adjustment implicit in options, that make VIX higher than future realized volatility
  - It does not mean that market expectations are systematically too high

Non-Normal Assumptions

- Market information can be useful beyond the normal distribution

- Directly obtain measures of downside risks from options

Using Options to Infer Downside Risk

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- Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option $\rightarrow$ market expects negative skewness

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- Because higher volatility implies a higher price, if OTM options have higher implied volatility than ATM option $\rightarrow$ market expects negative skewness
  - Since the crash of October 1987, OTM put options have a higher implied volatility than ATM put options.
  - Moreover, the difference in implied volatilities is time varying.

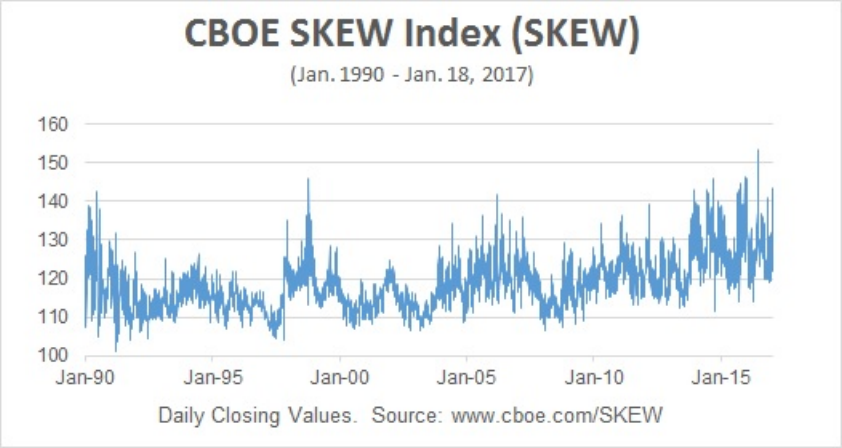
Skewness Index

- CBOE publishes a Skew Index
- Implied expected skewness is computed from option prices with a more elaborate methodology than the Black and Scholes implied volatilities, but the logic is similar.
- Recall that high *negative* skewness imply high downside risk.
  - CBOE define the Skew Index as

$$\text{Skew Index} = 100 - 10 \times \text{Implied Expected Skewness}$$

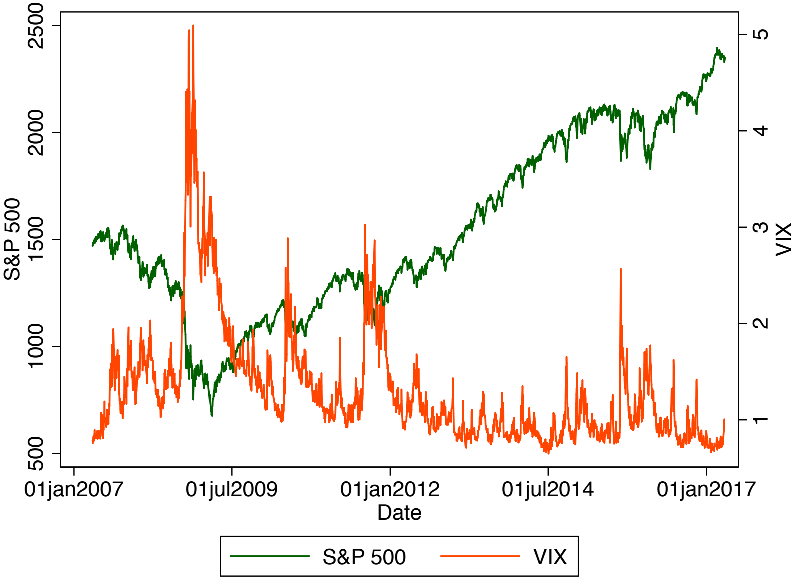
  - Higher positive index $\rightarrow$ higher downside risk

Skew Index


CBOE SKEW Index (SKEW)
(Jan. 1990 - Jan. 18, 2017)
160
150
140
130
120
110
100
Jan-90
Jan-95
Jan-00
Jan-05
Jan-10
Jan-15
Daily Closing Values. Source: www.cboe.com/SKEW

Outline

S&P500 and VIX



Model-Building

- To accurately model risk, necessary to understand interactions between different risks
- Lots of models, for each asset class
- Key question: what can go wrong?
- If model is too complex to compute VaR explicitly: Monte-Carlo simulations

Model-Building vs. Historical Simulation

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  - Large portfolios
  - Limited data
  - Taking account of nonlinearities
- Historical simulations useful for:
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  - Unknown structure of investment performance

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- Historical simulations useful for:
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  - Unknown structure of investment performance
- Key trade-off: making more assumptions vs. using a small part of the data
  - Always the same in statistics ... and finance!