Chapter 8. Interest Rate Models. Pricing Fixed Income Securities (v.A.5)

One-factor models

Assume the instantaneous spot interest rate follows a one-dimensional stochastic process, or a one-factor model. Assuming that the risk-neutral measure exists, the price of any security will be the present value of the expected future payoff, under that measure:

$$P_t = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds} V_T\right)$$

where P_t is the price of the security at time $t, \{r_s\}_{s\geq 0}$ is the stochastic process of short-term rates, and V_T is the payoff of the security at time $T > t$. The $(*)$ indicates that the expectation is computed under the risk-neutral measure.

For example, the price of a pure discount bond (zero-coupon bond) at time *t* maturing at time *T* (and having a face value of $1) will be given by

$$\boldsymbol{P}(t, T, r_t) = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds}\right).$$

Assume the dynamics of a short-term rate are given by the following SDE:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

Then, there exists a measure $*$, the risk-neutral measure, under which

$$dr_t = \big(\mu(t, r_t) - \lambda_t \cdot \sigma(t, r_t)\big)dt + \sigma(t, r_t)dW_t^*.$$

Here, $W_t^* = W_t + \int_0^t \lambda_s ds$ is a Brownian Motion process under the $* -measure$; and λ is the market price of risk. Under this measure, pricing securities is simple: just find the present value of expected future payoff.

We will consider a few important one-factor models.

(1) Vasicek Model.

Assume the dynamics of a short-term rate are given by the following SDE:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$$

This model possesses the following important properties:

- The distribution of r is Gaussian,
- The model can be solved explicitly,
- Rates, however, can be negative with positive probability.

(2) CIR Model

Assume the dynamics of a short-term rate are given by the following SDE:

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

This model possesses the following important properties:

- The rate is always positive,
- The rate has a non-central χ^2 distribution,
- The model is analytically tractable,
- The model is less tractable than the Vasicek model.

Note: Imposing the Feller condition, $2\kappa\bar{r} > \sigma^2$, guarantees that $P(r_t > 0) = 1$.

(3) Dothan Model

Assume the dynamics of a short-term rate are given by the following SDE:

$$dr_t = ar_t dt + \sigma r_t dW_t$$

This model possesses the following important properties:

- The rate is always positive,
- The distribution of r is log-normal.

(4) Hull-White Model

Assume the dynamics of a short-term rate are given by the following SDE:

$$dr_t = k[\theta_t - r_t]dt + \sigma dW_t$$

This model possesses the following important properties:

- The rate can be negative with positive probability,
- The distribution of r is normal,
- The model is tractable - it yields closed –form formulas for bond and option prices.

(5) Black-Karasinski Model

Assume the dynamics of a short-term rate are given by the following SDE:

$$dr_t = r_t[\eta_t - a \ln r_t]dt + \sigma r_t dW_t$$

This model possesses the following important properties:

- The rate is always positive,
- The distribution of r is log-normal;
- The model is not very tractable - it does not yield closed–form formulas for bond and option prices.

The following table summarizes the properties of some popular short-term interest-rate models, including the models listed above.

Table 1: Overview of Strength and Weaknesses of Basic Short Rate Models.
The following table contains an overview of short rate models. Here we will have the following notations: AB≈Analytical Bond price, AO ≈Analytical Option price, $\mathcal{N}$ ≈ normal distribution, $L\mathcal{N}$ ≈ lognormal distribution, $NC\chi^2$ ≈Non-Central Chi-Square distribution, $\mathrm{SNC}\chi^2$ ≈ Shifted Non-Central χ^2, $\mathrm{SL}\mathcal{N}$ ≈ Shifted LogNormal, MM≈Market Model, EEV≈ Extended Exponential Vasicek model. The Y* indicates the rates are positive under suitable conditions for a deterministic function φ_t.

Model	**Dynamics**	**Is $r > 0$**	**Distribution of r**	**AB**	**AO**
Vasicek	$dr_t = k[\theta - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
CIR	$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t$	Y	$\mathrm{NC}\chi^2$	Y	Y
D	$dr_t = ar_t dt + \sigma r_t dW_t$	Y	$\mathrm{L}\mathcal{N}$	Y	N
EV	$dr_t = r_t[\eta - a\ln r_t]dt + \sigma r_t dW_t$	Y	$\mathrm{L}\mathcal{N}$	N	N
HW	$dr_t = k[\theta_t - r_t]dt + \sigma dW_t$	N	$\mathcal{N}$	Y	Y
BK	$dr_t = r_t[\eta_t - a\ln r_t]dt + \sigma r_t dW_t$	Y	$\mathrm{L}\mathcal{N}$	N	N
MM	$dr_t = r_t\left[\eta_t - \left(\lambda - \frac{\gamma}{1+\gamma t}\right)\ln r_t\right]dt + \sigma r_t dW_t$	Y	$\mathrm{L}\mathcal{N}$	N	N
CIR++	$r_t = x_t + \varphi_t, \quad dx_t = k[\theta - x_t]dt + \sigma\sqrt{x_t}dW_t$	Y*	$\mathrm{SNC}\chi^2$	Y	Y
EEV	$r_t = x_t + \varphi_t,$ $dx_t = x_t[\eta - a\ln x_t]dt + \sigma x_t dW_t$	Y*	$\mathrm{SL}\mathcal{N}$	N	N

Source: *Brigo and Mercurio, Interest Rate Models: Theory and Practice. Springer, 2001.*

The table compares various models with each other in terms of model attractiveness, tractability to price bonds or options on bonds. Based on the subjective assessments, none of the models performs well when pricing interest rate related products. The main advantage of these models is their relative tractability (one factor) compared with other, multi-factor models.

Comparison of Models

In this section we will attempt to compare various models of short-term rates. We compare models according to two different sets of criteria. The first comparison is on the strengths and weaknesses of the models and the second comparison is based on explaining of historical fixed income prices.

Strengths and Weaknesses of the Models

This comparison extends the earlier comparison to include other models of interest rates. Table 2 compares the major classes of models. In the table we use the following abbreviations: DM ≈ Dynamic Mean models, GDM ≈ Generalized Dynamic Mean models, AY ≈Affine Yield models, HJM ≈ Heath-Jarrow-Morton framework, and MM≈ Market Models.

Table 2: *Comparison of various Models of Interest Rates.*

Models:	DM	GDM	AY	HJM	MM
Features:					
Statics					
Bond prices	Exact	OK	Good	Exact	Exact
Caplet/Bond options	~	~	~	Exact	Exact
Volatility structure	NO	~	OK	Excellent	Excellent
Dynamics					
Short rate	Good	Good	OK	~	~
Yield curve	NO	~	Good	Excellent	Excellent
Tractable					
Simple	Good	Good	OK	~	Good
Complex	OK	OK	OK	~	OK

Source: *James and Weber, Interest Rate Modelling, Wiley and Sons, 2000*

The HJM and Market Models give the best results overall, but for specific applications. Affine Yield or Dynamic Mean Models may be more appropriate for bond pricing.

The choice of a model for short term rate is not straightforward. It depends on both the modeling objectives and on the nature of the risk to be managed. For example, a fund that has large exotics exposure may be more likely to implement whole yield curve models. However, simple bond or bond option trading do not require anything sophisticated.

Comparison Based on Historical Criteria

We have identified some good properties of models to incorporate into interest rate models. There are four properties that we will emphasize here. These features are:

1. *The behavior of the short rate:* Does the model have a time-varying reversion level for the short rate?
2. *The behavior of the long rate:* Does a long rate (for example a 10-year rate) have a sufficient range of movement?
3. *Term structure tilts:* Can the long end and the short end move simultaneously and roughly equally in opposite directions?
4. *Hyperinflations:* Is there a non-vanishing chance that the short rate becomes unbounded in finite time?

Table 3 compares a number of models, grouped into categories, by each of these four features. All of the models considered in the table assume that the term structure is default-free. Duffie and Singleton (1997) have indicated that default may be accommodated by suitably adjusting the volatility structure. Table entries are of course a matter of opinion, and a judgment must be made as to what each feature means in the context of each model. For instance, whole yield curve models contain all the other models, but the table describes their features as they are usually implemented. Dynamic Mean models can be forced to hyperinflate by making their mean reversion levels explode, but this would then destroy their ability to capture any of the other features.

Table 3: *Comparison of Interest Rate Models with Respect to Short-Rate, Long-Rate Behavior, Term Structure Tilts, and Hyperinflation*

Features:	S	L	S/L	H
Model:				
Whole yield curve				
HJM (92), BGM (97)	×	Yes	Yes	×
Sommer (96)	~	Yes	Yes	~
Affine				
Duffie and Kan (96)	Yes	Yes	Yes	×
Babbs and Nowman (97)	Yes	Yes	Yes	×
Longstaff and Schwartz (92)	Yes	×	~	×
CIR (85), 'standard' model	×	×	×	×
CIR (85), 'inflation' model	~	~	~	×
Dynamic mean				
Tice and Webber (97)	Yes	×	~	×
Hull and White (90)	Yes	×	×	~
Fong and Vasicek (91)	Yes	×	~	×
Chen (96)	Yes	×	~	×
Ho and Lee (86)	Yes	~	×	*Yes*
Price kernel				
Bakshi and Chen (96)	×	×	×	×
Constantinides (92)	×	×	×	×
Jump models				
Babbs and Webber (94)	Yes	Yes	×	×

In this Table, the following notations were used: S: short rate behavior, L: Long rate behavior, H: hyperinflation, ~: the model has the feature, x: the model does not have this feature.

Source: *James and Weber, Interest Rate Modelling, Wiley and Sons, 2000.*

Below, we will study a few selected models in more details and provide some of their important properties.

Vasicek Model

Assume r_t is governed by:

$$dr_t = \kappa(\mu - r_t)dt + \sigma dW_t$$

Then, under the risk-neutral measure,

$$dr_t = (\kappa(\mu - r_t) - \lambda\sigma)dt + \sigma dW_t^* = \kappa(\tilde{\mu} - r_t)dt + \sigma dW_t^* \qquad \left(\tilde{\mu} = \mu - \frac{\lambda \cdot \sigma}{\kappa}\right)$$

Denoting $\bar{r} = \tilde{\mu}$, we have $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ under the risk-neutral measure. From here on, we will use the dynamics of the short-term rate models under the risk-neutral measure.

That is, assume $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ and that r_0 is given.

Lemma 1. Let $\boldsymbol{P}(t, T, r_t)$ be the price of a pure discount bond at time t, maturing at time T, with a par value of \$1. Then,

$$\boldsymbol{P}(t, T, r_t) = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds}\right) = A(t, T)e^{-B(t,T)\cdot r_t}$$

where

$$A(t, T) = exp\left\{\left(\bar{r} - \frac{\sigma^2}{2\kappa^2}\right)[B(t, T) - (T - t)] - \frac{\sigma^2}{4\kappa}B^2(t, T)\right\}, \qquad B(t, T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

Note: The formula for $\boldsymbol{P}(0, T, r_0)$ depends on the parameters of the Vasicek model and r_0.

Lemma 2. The conditional distribution of $r_t|\mathcal{F}_s$ is Gaussian with:

$$\mathbb{E}(r_t|\mathcal{F}_s) = \bar{r} + (r_s - \bar{r})e^{-\kappa(t-s)} \text{ and } Var(r_t|\mathcal{F}_s) = \sigma^2 \frac{(1 - e^{-2\kappa(t-s)})}{2\kappa}$$

Lemma 3. (a) The formula for the instantaneous interest rate r_t is given by

$$r_t = r_s e^{-\kappa(t-s)} + \bar{r}\left(1 - e^{-\kappa(t-s)}\right) + \sigma \int_s^t e^{-\kappa(t-u)} dW_u$$

(b) The spot rate $r_t(\tau)$ at time t for $\tau = T - t$ is given by:

$$r_t(\tau) = r_\infty + (r_t - r_\infty)\left(\frac{1 - e^{-\kappa\cdot\tau}}{\kappa\tau}\right) + \left(\frac{\sigma^2\tau}{4\kappa}\right)\left(\frac{1 - e^{-\kappa\cdot\tau}}{\kappa\tau}\right)^2$$

where $r_\infty = \bar{r} - \frac{\sigma^2}{2\kappa^2}, \quad \tau = T - t.$

Note: We can solve for r_t if we apply the Ito's Lemma on $(e^{kt} r_t)$, and integrate.

Lemma 4. The price of the Pure Discount Bond, $\boldsymbol{P}(t, T, r_t)$, satisfies the following PDE:

$$\frac{\partial P}{\partial t} + \kappa(\bar{r} - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} - rP = 0$$

with the terminal condition

$$\boldsymbol{P}(T, T, r_T) = 1$$

Note: In general, if the interest rate process is given by the SDE

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

then, the PDE for the price Pure Discount Bond, $\boldsymbol{P}(t, T, r_t)$, would solve the PDE:

$$\frac{\partial P}{\partial t} + \mu\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} - rP = 0$$

Now, we will consider European options on Pure Discount Bonds (zero-coupon bonds).

The payoff of a European Call option with expiration at time T, and strike price $\boldsymbol{K}$, on a pure discount bond that matures at time S (where $S > T$), and has par value of \$1, is given by: $(\boldsymbol{P}(T,S,r_T) - \boldsymbol{K})^+$.

Lemma 5. The price of a European Call option $c(t,T,S)$ at time t, with expiration at time T and strike price $\boldsymbol{K}$, on a Pure Discount Bond that matures at time S (where $S > T$), is given by

$$c(t,T,S) = \boldsymbol{P}(t,S,r_t)\, N(d_1) - \boldsymbol{K}\, \boldsymbol{P}(t,T,r_t) N(d_2)$$

The price of a European Put option is given by the following formula:

$$p(t,T,S) = \boldsymbol{K}\, \boldsymbol{P}(t,T,r_t)\, N(-d_2) - \boldsymbol{P}(t,S,r_t)\, N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{\boldsymbol{P}(t,S,r_t)}{\boldsymbol{K}\cdot\boldsymbol{P}(t,T,r_t)}\right)}{\sigma_p} + \frac{\sigma_p}{2}; \qquad d_2 = d_1 - \sigma_p; \qquad \sigma_p = \sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}} \cdot \left(\frac{1-e^{-\kappa(S-T)}}{\kappa}\right) \cdot \sigma.$$

Comment: If the par value of the bond is L (and not \$1) then, the formulae for the prices of European Call and European Put options, with Strike prices of $\boldsymbol{K}$ are given as follows:

The price of a European Call option:

$$c(t,T,S) = L \cdot \boldsymbol{P}(t,S,r_t)\, N(d_1) - \boldsymbol{K} \cdot \boldsymbol{P}(t,T,r_t) N(d_2)$$

The price of a European Put option:

$$p(t,T,S) = \boldsymbol{K} \cdot \boldsymbol{P}(t,T,r_t)\, N(-d_2) - L \cdot \boldsymbol{P}(t,S,r_t)\, N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{L \cdot \boldsymbol{P}(t,S,r_t)}{K \cdot \boldsymbol{P}(t,T,r_t)}\right)}{\sigma_p} + \frac{\sigma_p}{2}; \quad d_2 = d_1 - \sigma_p, \quad \sigma_p = \sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}} \cdot \left(\frac{1-e^{-\kappa(S-T)}}{\kappa}\right) \cdot \sigma$$

and, $\boldsymbol{P}(t,T,r_t)$ is the price of the Pure Discount Bond at time *t* that matures at time *T* and pays $1.

Fitting the Vasicek Model

Sophisticated estimation/calibration techniques are necessary for calibration of the parameters of interest rate models. In this section we will consider the calibration of one of the most popular one-factor models – the Vasicek model.

Suppose we have a time series $\{r_t\}_{t=1,\dots,n}$, of short-term rate data. Discretize the Vasicek's SDE,

$$dr_t = \alpha(\mu - r_t)dt + \sigma dz_t$$

using the Euler discretization, to get the following scheme:

$$r_{t+\Delta t} = (1 - \alpha\Delta t)r_t + \alpha\mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t.$$

One may attempt to calibrate the model by performing the following linear regression:

$$r_{t+\Delta t} = a + br_t + \eta_t$$

where η_t is a normal noise. The regression coefficients, $a, b, Var(\eta_t)$, will help estimate α, μ, σ as follows. The estimates $\hat{\alpha}$ and $\hat{\mu}$ (of α and μ of the Vasicek model) are given by:

$$\hat{\alpha} = \frac{1-b}{\Delta t}, \qquad \hat{\mu} = \frac{a}{1-b}, \quad \text{and } \sigma^2\Delta t = var(\eta_t).$$

Of course, these estimates are only sensible if the model is not mis-specified. For example, if the residuals $\varepsilon_t = \eta_t / \sigma\sqrt{\Delta t}$ are not normality distributed, or there can be heteroscedasticity or serial correlation in the model, then, the model does not fit the data well and its findings can't be used in interpreting any inferences from the model. There are a few other potential problems with the above-mentioned estimation technique, stated below, that could prevent proper calibration of the model and interpretation of its findings.

1. Unless Δt is very small the estimates of $\hat{\mu}$ and in particular $\hat{\alpha}$ are significantly biased. This is because the process above is close to having a unit root.
2. The OLS regression is equivalent to minimizing σ, and is not equivalent to ensuring that ε_t are *i.i.d.* standard normal.

Numerical Implementation

How to implement the Vasicek Model to price Pure Discount Bonds and Options on Pure Discount Bonds? Below we describe the numerical method of implementation of the Vasicek Model.

1. Pure Discount Bonds.

$\boldsymbol{P}(0,T) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\right)$ is the formula for the price of a Pure Discount Bond, and we will show steps to use Monte Carlo simulation to estimate that price.

Define $R = -\int_0^T r_s ds$. Note that, R is a random variable. Then,

$\boldsymbol{P}(0,T,r_0) = \mathbb{E}(e^R) \approx \frac{1}{N}\sum_{i=1}^{N} e^{R_i}$, where R_i is a simulation of $R = -\int_0^T r_s ds$.

Using Euler's method of integral estimation, we will write $R = -\int_0^T r_s ds = \Delta(\sum_{j=1}^{n} r_{t_j})$, where $\Delta = T/n$, $t_j = j\Delta$.

Then, we will simulate N paths of the process r_s, from 0 to T, and define the $i-th\ R$ as follows: $R^i = \Delta(\sum_{j=1}^{n} r^i{}_{t_j})$, for $i = 1,2, \dots, N$.

We can estimate the price of the Pure Discount Bond as: $\boldsymbol{P}(0,T,r_0) \approx \frac{1}{N}\sum_{i=1}^{N} e^{R_i}$.

2. Options on Pure Discount Bonds.

   $c(0,T,S) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\ (\boldsymbol{P}(T,S,r_T) - \boldsymbol{K})^+\ \right)$ is the formula for the price of a European Call option on a Pure Discount Bond.

We have:

$$c(0,T,S) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\ (\boldsymbol{P}(T,S,r_T) - \boldsymbol{K})^+\ \right) \approx \frac{1}{N}\sum_{i=1}^{N} e^{R_i}\,(\boldsymbol{P^i}(T,S,r_T) - \boldsymbol{K})^+.$$

The implementation here is very similar to the previous case, except for the price of the bond. To Estimate the price of the Bond, we will consider two sub-cases.

(a) Use the explicit formula for the Price of the Pure Discount Bond.

Then, $\boldsymbol{P^i}(T,S,r_T) = (\boldsymbol{P}(T,S,r_T) - \boldsymbol{K})^+$ for every $i = 1,2,\dots,N,$ and $\boldsymbol{P}(T,S,r_T)$ is given by the closed-form solution.

(b) Use Monte Carlo simulations to estimate the Price of the Pure Discount Bond - $\boldsymbol{P^i}(T,S,r_T)$.

For every path of the r-process, we start at 0, and simulate the path of the process until time T, and we have ${r^i}_{t_N} = {r^i}_T$, which is the value of r at time T.

We will use that value as a starting value for r, and simulate M paths of r from time T to time S, to price the Pure Discount Bond $\boldsymbol{P^i}(T,S,r_T)$, as was described earlier. Then,

$\boldsymbol{P^i}(T,S,r_T) \approx \frac{1}{M}\sum_{l=1}^{M} e^{R_l}$.

The CIR Model

Assume the dynamics of r_t, under the risk-neutral measure, follow the SDE

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

Assume that $2\kappa \cdot \bar{r} > \sigma^2$ so that the origin is inaccessible.

Lemma 6. The price $\boldsymbol{P}(t,T,r_t)$ of a Pure Discount Bond at time t, maturing at time T, is given by

$$\boldsymbol{P}(t,T,r_t) = A(t,T) \cdot e^{-B(t,T) \cdot r_t}$$

where,

$A(t,T) = \left(\frac{h_1 \cdot e^{h_2(T-t)}}{h_2 \cdot \left(e^{h_1(T-t)}-1\right)+h_1}\right)^{h_3}$, $B(t,T) = \frac{e^{h_1(T-t)}-1}{h_2 \cdot \left(e^{h_1(T-t)}-1\right)+h_1}$, $h_1 = \sqrt{\kappa^2 + 2\sigma^2}$, $h_2 = \frac{\kappa + h_1}{2}$, $h_3 = \frac{2 \cdot \kappa \cdot \bar{r}}{\sigma^2}$.

Lemma 7. The conditional distribution of $r_t|\mathcal{F}_s$ is non-central χ^2 with:

$$\mathbb{E}(r_t|\mathcal{F}_s) = r_s e^{-\kappa(t-s)} + \bar{r}\left(1 - e^{-\kappa(t-s)}\right)$$

$$Var(r_t|\mathcal{F}_s) = r_s \frac{\sigma^2}{\kappa}\left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) + \frac{\bar{r}\sigma^2}{2\kappa}\left(1 - e^{-\kappa(t-s)}\right)^2$$

Lemma 8. The SDE for the price-process of Pure Discount Bond, $\boldsymbol{P}(t,T,r_t)$, of a at time t, maturing at time T, is given by

$$d\boldsymbol{P}(t,T,r_t) = r_t \boldsymbol{P}(t,T,r_t)dt - \sigma B(t,T)\, \boldsymbol{P}(t,T,r_t)\, \sqrt{r_t} dW_t$$

Lemma 9. The spot rate $r_t(\tau)$ at time t, for $\tau = T - t$ is given by:

$$r_t(\tau) = -\frac{2\kappa\bar{r}}{\sigma^2 T} \ln(A(T) + \frac{r_t}{T} D(T)$$

where $A(T)$and $D(T)$are provided on page 390 in Martellini et al.

Lemma 10. The price of a European Call Option is given by:

$$c(t,T,S) = \boldsymbol{P}(t,S,r_t)\cdot\chi^2\left(2r^*(\phi+\psi+B(T,S);\frac{4\kappa\cdot\bar{r}}{\sigma^2},\frac{2\phi^2\cdot r_t\cdot e^{\theta(T-t)}}{\phi+\psi+B(T,S)}\right)-\boldsymbol{K}\cdot\boldsymbol{P}(t,T,r_t)$$

$$\cdot\chi^2\left(2r^*(\phi+\psi);\frac{4\kappa\cdot\bar{r}}{\sigma^2},\frac{2\phi^2\cdot r_t\cdot e^{\theta(T-t)}}{\phi+\psi}\right)$$

$$\theta=\sqrt{\kappa^2+2\sigma^2},\phi=\frac{2\theta}{\sigma^2(e^{\theta(T-t)}-1)},\psi=\frac{\kappa+\theta}{\sigma^2},r^*=\ln\left(\frac{A(T,S)}{\boldsymbol{K}}\right)/B(T,S)$$

Here, $\chi^2(x,p,q)$ is the value at x of the distribution function of Non-Central χ^2 with p-degrees of freedom and non-centrality parameter q. The density function of such distribution is given by

$$f(x)_{\chi^2(p,q)}=\sum_{i=0}^{\infty}\frac{e^{-\frac{q}{2}}\cdot\left(\frac{q}{2}\right)^i}{i!}\cdot\frac{\left(\frac{1}{2}\right)^{i+\frac{p}{2}}}{r(i+\frac{p}{2})}\cdot x^{i-1+\frac{p}{2}}\cdot e^{-\frac{x}{2}}$$

Comment: If the par value of the bond is L (and not \$1) then, in the formulae for the prices of European Call and Put options we will replace $\boldsymbol{P}(t,S,r_t)$ by $L\cdot\boldsymbol{P}(t,S,r_t)$, where $\boldsymbol{P}(t,S,r_t)$ is the price of the Pure Discount Bond at time *t*, that matures at time *S* and pays \$1.

Affine Term-Structure Models

Definition: If the Pure Discount Bond prices (also called zero-coupon bonds) are given by $\boldsymbol{P}(t,T,r_t)=A(t,T)\cdot e^{-B(t,T)\cdot r_t}$ for all $0\le t\le T$, where $A(t,T)$and $B(t,T)$ are deterministic functions, then, we say that the model possesses an affine term structure.

Lemma 11. In an Affine Term-Structure Model, in which the short-term interest rate follows the SDE $dr_t=(a_t-b_tr_t)dt+\sigma\sqrt{(c_t+d_tr_t)}\,dW_t$, the price of a Pure Discount Bond (zero-coupon bond) $\boldsymbol{P}(t,T,r_t)$, is given by: $\boldsymbol{P}(t,T,r_t)=A(t,T)\cdot e^{-B(t,T)\cdot r_t}$.

Here the $A(t,T)$and $B(t,T)$ functions satisfy the following differential equations:

$$\frac{dA(t,T)}{dt} = A(t,T)B(t,T)\left(a_t - \frac{c_t B(t,T)}{2}\right), \text{ and } A(T,T) = 1,$$

$$\frac{dB(t,T)}{dt} = b_t B(t,T) + \left(\frac{d_t B^2(t,T)}{2} - 1\right), \text{ and } B(T,T) = 0.$$

Note: In Affine Term Structure Models, we will denote by $\boldsymbol{P}(t,T,r_t)$ the price of the Pure Discount Bond at time t, that matures at time T, and the rate at time t is r_t.

Coupon-Paying Bonds and Options on Coupon-Paying Bonds

Jamshidian (1989) derived a valuation method for pricing coupon-paying bonds and options on such bonds. The main idea of the method is to view the coupon-paying bond as a portfolio of Pure Discount Bonds.

Define $C(t,T,\{c_i\}_{i=1}^n,\{T_i\}_{i=1}^n,K)$ to be the price at time t of a European Call option with strike price K, maturity T, on a coupon-paying bond that pays coupons c_i (this is the coupon amount and not a percentage) at times T_i, $T_i \geq T$. Thus, T is the expiration of the option.

Then, based on Jamshidian (1989), one can write

$$C(t,T,\{c_i\}_{i=1}^n,\{T_i\}_{i=1}^n,K) = \sum_{i=1}^{n} c_i \cdot C(t,T,T_i,K_i)$$

where

- n is the number of coupons payable after the expiration T of the option;
- $C(t,T,T_i,K_i)$is the value at time t of the European option that expires at time T, has a strike price K_i, on a zero-coupon bond that matures at time T_i.

- K_i is the exercise price of i^{th} option determined as follows:

  $K_i = \boldsymbol{P}(T, T_i, r^*)$ which is the price (at time T) of Pure Discount Bond with maturity at time T_i, with r^* as a short rate, and r^* is chosen so that

$$\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) \ = K$$

That is, the price of the bond at time T, using the rate r^* is K.

Note: r^* is not known and needs to be estimated numerically.

Below are the details of the method for pricing options on coupon-paying bonds.

Consider a coupon-paying bond described as follows:

The bond will pay n-coupons c_i at times T_i, where all $T_i > T, i = 1,2, \dots, n$.

Define $\mathfrak{F} = \{T_i, \dots, T_n\}$, $\quad C = \{c_i, \dots, c_n\}$.

Define r^* to be the constant spot rate at time T, for which the bond price at time T (that pays the coupons c_i at times T_i, $T_i \geq T$) is equal to the strike price K of the option we are trying to price:

$$\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) \ = K$$

That is, r^* solves $\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) \ = K$.

Define $K_i$to be time-T value of a Pure Discount Bond (that pays \$1 at maturity) with maturity at T_i when the spot rate is r^*: $\quad K_i = \boldsymbol{P}(T, T_i, r^*)$. The price of an option on a coupon paying bond at time t **(C)** is given by: $\quad \mathbf{C}(t, T, \mathfrak{F}, C, K) = \sum_{i=1}^{n} c_i \cdot \boldsymbol{c}(t, T, T_i, K_i)$

Here, $\boldsymbol{c}(t, T, T_i, K_i)$ is the price of an option at time t with expiration at time T and a strike price of K_i on a pure discount bond that matures at time T_i.

The price of the Coupon-Paying Bond **(CPB)** at time T is given by:

$$\mathbf{CPB}(T, \mathfrak{F}, C) = \sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r)$$

The details of the Jamshidian (1989) method are as follows:

When the model is such that the price of the Pure Discount Bond at time T, maturing at time T_i and having \$1 par value is given by a formula in which there is an explicit dependence on r_T (such as the Vasicek, CIR models), then, we will denote the price of that bond by $\boldsymbol{P}(T, T_i, r_T)$.

Then, the European Call option payoff at maturity is

$$(\boldsymbol{CBP}(T, \mathfrak{F}, C) - K)^+ = \left(\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r_T) - K\right)^+$$

The main idea is to convert this positive part of the sum (the right side) into a sum of positive parts.

First, find r^* using, for example, the Newton-Raphson method, so that

$$\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T, T_i, r^*) = K$$

Note: $\sum_{i=1}^{n} c_i \cdot K_i = K$.

Assume $\frac{\partial \mathbf{P}(t,s,r)}{\partial r} < 0$ for any $0 < t < s$. This property is satisfied for the Vasicek, CIR and Hull-White models. Then, the payoff of the call option can be written as follows:

$$\left(\sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T,T_i,r_T) - \sum_{i=1}^{n} c_i \cdot \boldsymbol{P}(T,T_i,r^*)\right)^+ = \sum_{i=1}^{n} c_i \cdot \left(\boldsymbol{P}(T,T_i,r_T) - \boldsymbol{P}(T,T_i,r^*)\right)^+,$$

which is a sum of payoffs of n options on Pure Discount Bonds.

The price of the coupon-paying bond option, with strike K and maturity T, is given by:

$$\boldsymbol{C}all(t,T,\mathfrak{F},C,K) = \sum_{i=1}^{n} c_i \cdot \boldsymbol{c}\left(t,T,T_{i,}\boldsymbol{P}(T,T_i,r^*)\right)$$

$$\boldsymbol{C}all = \sum_{i=1}^{n} c_i \cdot \left\{\boldsymbol{P}(t,T_i) \cdot N\left(d_{i,+}\right) - K_i \boldsymbol{P}(t,T) N(d_{i,-})\right\}$$

$$d_{i,\pm} = \frac{1}{\sigma_p(t,T,T_i)} \cdot \ln\left(\frac{\boldsymbol{P}(t,T_i,r_t)}{K_i \cdot \boldsymbol{P}(t,T,r_t)}\right) \pm \frac{\sigma_p(t,T,T_i)}{2}$$

$$\sigma_p(t,T,T_i) = \frac{\sigma}{\kappa}\left(1 - e^{-\kappa(T_i - T)}\right)\sqrt{\frac{1}{2\kappa}(1 - e^{-2\kappa(T-t)})}$$

Comment: This method works in cases when the price of a bond $\boldsymbol{P}(t,T,r_t)$ is a known function of short-term rate r_t (such as in the Vasicek or CIR models).

Demonstration for the Vasicek model

Example: Consider a coupon paying bond, that pays semiannual coupons of $c = \$2$; has a Face Value of $FV = \$100$, and matures in 4 years. We would like to find the Price a European Call option that has maturity of 4 months and strike price of 98% of Par Value of the bond.

Here, $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c = \$2$

$c_8 = FV + c = \$(100+2) = \$102,\ K = \$98,\ t = 0,\ T = \frac{4}{12}, T_i = i * \frac{6}{12},$ for $i = 1,2,\dots,8$.

$\mathbf{P}(T, T_i, r^*)$ is the price at time T of a Pure Discount Bond maturity at time T_i with $r_T = r^*$.

That is,

$$\mathbf{P}(T, T_i, r^*) = A(T, T_i) \cdot e^{-B(T,T_i)\cdot r^*} \text{ with}$$

$$A(T, T_i) = \exp\left(\left(\bar{r} - \frac{\sigma^2}{2\kappa^2}\right)\left(B(T, T_i) - (T_i - T)\right) - \frac{\sigma^2}{4\kappa} \cdot B^2(T, T_i)\right), \text{ and}$$

$$B(T, T_i) = \frac{1}{\kappa} \cdot \left(1 - e^{-\kappa(T_i - T)}\right).$$

Note: $T_1 - T = \frac{2}{12}, T_2 - T = \frac{8}{12}, T_3 - T = \frac{14}{12}, T_4 - T = \frac{20}{12}, T_5 - T = \frac{26}{12}, T_6 - T = \frac{32}{12},$

$T_7 - T = \frac{38}{12}, T_8 - T = \frac{44}{12}.$

Step 1: Find $\mathbf{P}(T, T_i, r^*)$ for $i = 1, \dots, 8$ as a function of r^*.

Step 2: Solve for r^* so that $\sum_{i=1}^{8} c_i \cdot \mathbf{P}(T, T_i, r^*) = K$.

Step 3: Set $K_i = \mathbf{P}(T, T_i, r^*)$ for $i = 1, \dots, 8$,

Step 4: Find the price of a European Call option with strike price K_i, maturity T, on a Pure Discount bond maturing at time T_i, paying \$1 as follows:

$$\boldsymbol{c}(t, T, T_i, K_i) = \boldsymbol{P}(t, T_i, r_t) \cdot N(d_1) - K_i \cdot \boldsymbol{P}(t, T, r_t) N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{P(t,T_i,r_t)}{K_i \cdot P(t,T,r_t)}\right)}{\sigma_p} + \frac{\sigma_p}{2}, d_2 = d_1 - \sigma_p, \sigma_p = \sigma \cdot \frac{1-e^{-\kappa(T_i-T)}}{\kappa} \cdot \sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}}, T-t = \frac{4}{12}$$

Step 5: The price of the European Call option on the coupon-paying bond is given by:

$$C = \sum_{i=1}^{8} c_i \cdot \boldsymbol{c}(t,T,T_i,K_i)$$

Two-Factor Models of Short-Rate

We use short rates – regardless of the number of factors - to characterize the entire yield curve. Having the dynamics of the short rate, Pure Discount Bond prices can be computed as

$$\boldsymbol{P}(t,T,r_t) = \mathbb{E}_t\left[\exp\left\{-\int_t^T r_s ds\right\}\right]$$

From the bond prices, it is possible to construct the zero-interest rate curve. Thus, the dynamics of the zero-coupon yield curve is characterized by the dynamics of short-term rates.

In some cases, this approach may result in a poor model of the yield curve. When the security (to be priced) depends on correlations of rates of different maturities (say 1 year and 10 years), then this approach is not reliable. In such cases, a more realistic correlation structure is needed, and thus, multi-factor models come into play. In what follows, we will consider simple multi-factor models of short-term rates.

Gaussian-Vasicek Two-Factor Model

The Gaussian-Vasicek two-factor model is used to model the movements in short-term rate and is driven by two sources of uncertainties. The model can be written as follows:

$$\begin{cases} dx_t = \kappa_x(\bar{x} - x_t)dt + \sigma_x dW_t^1 \\ dy_t = \kappa_y(\bar{y} - y_t)dt + \sigma_y dW_t^2 \\ r_t = x_t + y_t \end{cases}$$

and $dW_t^1 \cdot dW_t^2 = \rho dt$

Here the correlation between zero-rates of maturities T_1, and T_2 is

$$Corr(R(t,T_1), R(t,T_2))$$
$$= Corr(b^x(t,T_1) \cdot x_t + b^y(t,T_1) \cdot y_t, b^x(t,T_2) \cdot x_t + b^y(t,T_2) \cdot y_t)$$

which depends on the correlation between x_t and y_t, which, in its turn, depends on ρ.

As can be seen, the rate is broken down to two correlated mean-reverting processes, with potentially different speeds of reversion and different long-term means and volatilities. This model is richer than its one-dimensional counterpart as it gives more flexibility and more freedom for calibration and fitting.

One may define multifactor models of interest rate with 3, 4, or more factors. So, a natural question to ask is: how many factors are needed and why so many?

The answer depends on the compromise between numerical implementation/tractability and capability of the model to capture realistic correlation pattern of fixed-income securities.

Using the Principal Components Analysis approach, one can easily demonstrate that two to three components (factors) can represent more than 90% of the variation in the yield curve. This suggests that 2 or 3 factor models may be adequate for capturing important features of the yield curve.

The next model is another two-factor model of short-term rates.

The G2++ Model

Assume the short-term rate is a sum of two mean-reverting stochastic processes as follows:

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t^1 \quad x_0 = 0 \\ dy_t = -by_t dt + \eta dW_t^2 \quad y_0 = 0 \\ r_t = x_t + y_t + \varphi_t \\ \varphi_0, r_0 \text{ are given} \\ dW_t^1 \cdot dW_t^2 = \rho dt, \quad -1 \le \rho \le 1 \end{cases}$$

Here φ_t is a deterministic shift function (to help better fit the zero-coupon curve).

Note: Here, W_t^1 and W_t^2 are NOT independent as in the Longstaff-Schwartz model that will be introduced below.

Lemma 12. Consider the G2++ Model. Assume $r_0, a, b, \sigma, \eta > 0$. We have

$$\mathbb{E}(r_t|\mathcal{F}_s) = x_s \cdot e^{-a(t-s)} + y_s \cdot e^{-b(t-s)} + \varphi_t$$

$$Var(r_t|\mathcal{F}_s) = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right) + \frac{\eta^2}{2b}\left(1 - e^{-2b(t-s)}\right) + 2\rho\frac{\sigma\eta}{a+b}\left(1 - e^{-(a+b)(t-s)}\right)$$

Lemma 13. Consider the G2++ Model. The price of a zero-coupon bond is given by

$$\boldsymbol{P}(t,T,r_t) = \mathbb{E}_t\left(e^{-\int_t^T r_s ds}\right)$$

$$= \exp\left\{-\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}x_t - \frac{1-e^{-b(T-t)}}{b}y_t + \frac{1}{2}V(t,T)\right\}$$

where

$$V(t,T) = \frac{\sigma^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

$$+\frac{\eta^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right]$$

$$+2\rho\frac{\sigma\eta}{ab}\left[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b)(T-t)}-1}{a+b}\right]$$

<u>European Call Option (G2++ Model)</u>

Let $\boldsymbol{Call}(t,T,S,K)$ be price of a European Call Option at time t, on a zero-coupon bond that matures at time S, with option strike K and option expiration at time T.

$$\boldsymbol{C}(t,T,S,K) = \mathbb{E}_t^*\left(e^{-\int_t^T r_s ds}\cdot(\boldsymbol{P}(T,S,r_T)-K)^+\right)$$

Then,

$$\boldsymbol{C}(t,T,S,K) = P(t,S)\cdot\boldsymbol{N}\left(\frac{\ln\left(\frac{\boldsymbol{P}(t,S,r_t)}{K\cdot\boldsymbol{P}(t,T,r_t)}\right)}{\Sigma}+\frac{1}{2}\Sigma\right) - P(t,T)\cdot K$$

$$\cdot\boldsymbol{N}\left(\frac{\ln\left(\frac{\boldsymbol{P}(t,S,r_t)}{K\cdot\boldsymbol{P}(t,T,r_t)}\right)}{\Sigma}-\frac{1}{2}\Sigma\right)$$

where $\boldsymbol{N}(\cdot)$ is the Standard Normal CDF, and

$$\Sigma^2 = \frac{\sigma^2}{2a^3}\left[1 - e^{-a(S-T)}\right]^2 \cdot \left(1 - e^{-2a(T-t)}\right) + \frac{\eta^2}{2b^3}\left[1 - e^{-b(S-T)}\right]^2 \cdot \left(1 - e^{-2b(T-t)}\right) + 2\rho\frac{\sigma\eta}{ab(a+b)} \cdot \left(1 - e^{-a(S-T)}\right)\left(1 - e^{-b(S-T)}\right)\left(1 - e^{-(a+b)(T-t)}\right)$$

<u>European Put option (G2++ Model)</u>

$$\boldsymbol{Put}(t,T,S,K) = -\boldsymbol{P}(t,S,r_t) \cdot \boldsymbol{N}\left(\frac{\ln\left(\frac{K \cdot \boldsymbol{P}(t,T,r_t)}{\boldsymbol{P}(t,S,r_t)}\right)}{\Sigma} - \frac{1}{2}\Sigma\right) + P(t,T,r_t) \cdot K \cdot \boldsymbol{N}\left(\frac{\ln\left(\frac{K \cdot \boldsymbol{P}(t,T,r_t)}{\boldsymbol{P}(t,S,r_t)}\right)}{\Sigma} + \frac{1}{2}\Sigma\right)$$

with the same notations as above for the call option price formula.

Comment: If the par value of the bond is L (and not \$1) then in the formulae for the prices of European Call and Put options we will replace $\boldsymbol{P}(t,S,r_t)$ by $L \cdot \boldsymbol{P}(t,S,r_t)$, where $\boldsymbol{P}(t,S,r_t)$ is the price of the Pure Discount Bond at time *t*, that matures at time *S* and pays \$1,

Longstaff-Schwartz-1992 Model

Assume the short-term rate is a weighted sum of two mean-reverting stochastic processes as follows: r_t **is defined as** $r_t = \alpha \cdot x_t + \beta \cdot y_t \; (\alpha \neq \beta)$**, where**

$$\begin{cases} dx_t = (\gamma - \delta x_t)dt + \sqrt{x_t}dW_t^1 \\ dy_t = (\eta - \theta y_t)dt + \sqrt{y_t}dW_t^2 \end{cases}$$

In this model, W_t^1 and W_t^2 are independent. In this case we have the following closed-form formulae for pricing bonds and options on bonds.

Discount Bond price:

$$\boldsymbol{P}(t,s,r_t) = \exp\{G(t,s) + C(t,s)r_t + D(t,s)v_t\}$$

where $v_t = \alpha^2 x_t + \beta^2 y_t$ is the volatility of r_t,

$$G(t,s) = \kappa\tau + 2\gamma \ln A(t,s) + 2\eta \ln B(t,s)$$

$$A(t,s) = \frac{2\phi}{(\delta+\phi)(e^{\theta\tau}-1)+2\phi}$$

$$B(t,s) = \frac{2\Psi}{(\theta+\Psi)(e^{\Psi\tau}-1)+2\Psi}$$

$$C(t,s) = \frac{\alpha\phi \cdot (e^{\Psi\tau}-1)\cdot B(t,s) \; - \; \beta\cdot\Psi\cdot\left(e^{\phi\tau}-1\right)A(t,s)}{\phi\cdot\Psi\cdot(\beta-\alpha)}$$

$$D(t,s) = \frac{\Psi\cdot\left(e^{\phi\tau}-1\right)\cdot A(t,s) - \phi\cdot(e^{\Psi\tau}-1)\cdot B(t,s)}{\phi\cdot\Psi\cdot(\beta-\alpha)}$$

$$\tau = s-t, \phi = \sqrt{2\alpha+\delta^2}, \psi = \sqrt{2\beta+e^2}, \kappa = \gamma(\delta+\phi)+\eta(\theta+\Psi)$$

European Call option price

$$C(t,T,S) = \boldsymbol{P}(t,S,r_t)\cdot\boldsymbol{\Psi}(\theta_1,\theta_2,\cdot 4\gamma,4\eta,w_1,w_2) - K\cdot\boldsymbol{P}(t,T,r_t)\cdot\boldsymbol{\Psi}(\theta_3,\theta_4,\cdot 4\gamma,4\eta,w_3,w_4)$$

$\theta_1 = \frac{4\cdot\zeta\cdot\phi^2}{\alpha\cdot\left(e^{\Phi(\mathrm{T-t})}-1\right)^2\cdot A(t,S)}$, $\theta_2 = \frac{4\cdot\zeta\cdot\Psi^2}{\beta\cdot\left(e^{\Psi(\mathrm{T-t})}-1\right)^2\cdot B(t,S)}$, $\theta_3 = \frac{4\cdot\zeta\cdot\phi^2}{\alpha\cdot\left(e^{\Phi(\mathrm{T-t})}-1\right)^2\cdot A(t,T)}$, $\theta_4 = \frac{4\cdot\zeta\cdot\Psi^2}{\beta\cdot\left(e^{\Psi(\mathrm{T-t})}-1\right)^2\cdot B(t,T)}$, $w_1 = \frac{4\phi\cdot e^{\Phi(\mathrm{T-t})}\cdot A(t,S)\cdot(Br_t - v_t)}{\alpha(\beta-\alpha)\left(e^{\Phi(\mathrm{T-t})}-1\right)A(t,T-S)}$, $w_2 = \frac{4\psi\cdot e^{\Psi(\mathrm{T-t})}\cdot B(t,S)\cdot(v_t-\alpha\cdot r_t)}{\beta(\beta-\alpha)\left(e^{\Psi(\mathrm{T-t})}-1\right)B(t,T-S)}$, $w_3 = \frac{4\phi\cdot e^{\Phi(\mathrm{T-t})}\cdot A(t,T)\cdot(\beta r_t - v_t)}{\alpha(\beta-\alpha)\left(e^{\Phi(\mathrm{T-t})}-1\right)}$, $w_4 = \frac{4\psi\cdot e^{\Psi(\mathrm{T-t})}B(t,T)\cdot(v_t-\alpha\cdot r_t)}{\beta(\beta-\alpha)\left(e^{\Psi(\mathrm{T-t})}-1\right)}$,

$\zeta = \kappa(S-T) + 2\gamma \ln A(t, S-T) + 2\eta \ln B(t, S-T) - \ln K$,

and $\mathbf{\Psi}$~distribution function of bivariate, non-central χ^2.

Since the pricing of bonds or options on bonds can be done by using the PDE approach and by using the probabilistic approach, here we will provide a result – the well-known Feyman-Kac Theorem – that relates the two approached to each other.

Theorem (Feyman-Kac).

(a) Assume under the risk-neutral measure the price-process is given by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and the price of a contingent claim at time T is given by

$$V(t, S_t) = \mathbb{E}_t^*\left(e^{-\int_t^T r(u,S_u)du} \cdot H(T, S_T)\right)$$

Then, V solves the following PDE:

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

with $V(T, S_T) = H(T, S_T)$

(b) In a multidimensional case, when $dS_t^i = \mu_i\big(t, S_t^i\big)dt + \sigma_i\big(t, S_t^i\big)dW_t^i$ for $i = 1, \dots, d$, the corresponding PDE is given by:

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{d} \mu_i \frac{\partial V}{\partial S_i} + \frac{1}{2}\sum_{i,j=1}^{d} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} - rV = 0.$$

(c) Assume under the risk-neutral measure the price-process is given by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and the price of a contingent claim at time T is given by

$$V(t, S_t) = \mathbb{E}_t^*\left(\int_t^T h(u, S_u)\, e^{-\int_t^u r(y, S_y)dy} du + e^{-\int_t^T r(y, S_y)dy} \cdot g(T, S_T)\right)$$

Then, V solves the following PDE:

$$\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - rV + h = 0$$

with $V(T, S_T) = g(T, S_T)$.

Example. Implementation of the IFD method in pricing a European Call Option on a Pure Discount Bond in the CIR framework.

Consider the CIR model of short-term rate: $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$.

The goal is to compute the price $C(0, T, S)$ of the European Call option on a pure discount bond (zero-coupon bond), which matures at time *S* and pays *FV* at maturity. The option's expiration is at time T (where $S > T$), and the strike price of the option is *K*.

Under the risk-neutral measure, the price of the call option, at time 0, is given by:

$$C(0,T,S) = \mathbb{E}\left(e^{-\int_0^T r_s ds}\ (\boldsymbol{P}(T,S,r_T)\ - K)^+\ \right)$$

where $\boldsymbol{P}(T,S,r_T)$ is the price of the pure discount bond at time T, maturing at time S and paying FV at maturity.

Using the Feyman-Kac Theorem, we can transform the problem into another one in which we solve the following PDE for C

$$\boldsymbol{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 C}{\partial r^2} + \kappa(\bar{r} - r)\ \frac{\partial C}{\partial r} - rC = 0}$$

With the terminal condition, $C(T,T,S) = \max(\boldsymbol{P}(T,S,r_T)\ - K,\ 0)$, where $\boldsymbol{P}(T,S,r_T)$ is the price of the pure discount bond at time T, maturing at time S and paying FV at maturity.

While there is a closed-form solution to the above-posed problem, below we provide the details of an algorithm for numerically solving the above PDE, by using the Implicit Finite Difference (IFD) method.

Consider a grid of time (t) and rate (r) as follows:

Take a uniform partition of the time interval [0,T] by dividing the time-interval [0,T] into M equal parts: $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$, where $t_k = \frac{T}{M}k = \Delta k$, where $\Delta = \frac{T}{M}$.

Take a truncated range of r as follows: $[0, r_{max}]$, where $r_{max} = N\Delta r,\ r$ takes on the following values $\{0, \Delta r, 2\Delta r, \dots, N\Delta r\}$. That is, $r_{i,j} = j\Delta r$, for any $i = 0, \dots, M\ j = 0, \dots, N$.

<u>The boundary conditions/values of option prices can be used as follows:</u>

(1) $C_{M,j} = (P_{M,j} - K)^+$ for $j = 0, \dots, N$ (this is the payoff condition at option's expiration)

(2) $C_{i,N} = 0$, for $i = 0, \dots, M$ (since for very large values of r, the call option is deep out of the money, and the call value is worthless)

(3) $C_{i,0} = PV - K$, for $i = 0, \dots, M$ $(\text{ since for small r, r} = 0, \text{the option is deep in} - \text{the} - \text{money and its value is } PV - K.$

The discretized version of this PDE (using the Implicit FD method) can be written as follows:

$$\frac{C_{i+1,j} - C_{i,j}}{\Delta t} + \frac{1}{2}\sigma^2 (j\Delta r)\frac{C_{i,j+1} - 2C_{i,j} + C_{i,j-1}}{(\Delta r)^2} + \kappa(\bar{r} - j\Delta r)\frac{C_{i,j+1} - C_{i,j-1}}{2\Delta r} - rC_{i,j} = 0$$

By combining the similar terms, we can rewrite the above scheme as:

$$C_{i+1,j} = P_u^j \cdot C_{i,j+1} + P_m^j \cdot C_{i,j} + P_d^j \cdot C_{i,j-1}$$

where

$$\begin{cases} P_u^j = \Delta_t \left(-\dfrac{\sigma^2 j}{2\Delta r} - \dfrac{\kappa \bar{r}}{2\Delta r} + \dfrac{\kappa j}{2} \right) \\ P_m^j = \Delta_t \left(\dfrac{1}{\Delta_t} + \dfrac{\sigma^2 j}{\Delta r} + r \right) \\ P_d^j = \Delta_t \left(-\dfrac{\sigma^2 j}{2\Delta r} + \dfrac{\kappa \bar{r}}{2\Delta r} - \dfrac{\kappa j}{2} \right) \end{cases}$$

for $j = N - 1, \dots, 1$.

Also, we have the boundary conditions for C: $C_{i,0} = PV - K$ and $C_{i,N} = 0$, for $i = 0, \dots, M$.

Putting all equations together, we will have the following system of equations in a matrix form:

$$C_{i+1} = AC_i + B_i$$

where

$$C_i = \begin{pmatrix} C_{i,N-1} \\ C_{i,N-2} \\ C_{i,N-3} \\ \cdot \\ \cdot \\ C_{i,1} \end{pmatrix}; \quad A = \begin{pmatrix} P_m^{N-1} & P_d^{N-1} & 0 & \vdots & \vdots & \vdots & \vdots & 0 \\ P_u^{N-2} & P_m^{N-2} & P_d^{N-2} & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & P_u^{N-3} & P_m^{N-3} & P_d^{N-3} & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & P_u^{N-4} & P_m^{N-4} & P_d^{N-4} & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & 0 & P_u^2 & P_m^2 & P_d^2 \\ 0 & 0 & \vdots & \vdots & 0 & 0 & P_u^1 & P_m^1 \end{pmatrix}; \quad B_i = \begin{pmatrix} P_u^N C_{i,N} \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ P_d^1 C_{i,0} \end{pmatrix}$$

The goal is to find C_0.

The matrix equations $C_{i+1} = AC_i + B_i$ can be solved, starting at $i = N - 1$ and moving backwards in time to $i = 0$.

Remarks:

1. We did not include the extreme values of *C* in the matrix equations above. That is, the values of $C_{i,N}$ and $C_{i,0}$, which correspond to option values for “very large” and “very small” bond prices, respectively, were not included in the matrix equation above.
2. One obtains a “very large” (or small) bond price when the rate is 0 (or r_{max}).
3. For call options, and for $i = 0, \dots, M$ we can use $C_{i,0} = PV - K$ as this is the case when *r=0* and the call option is deep in-the-money. Also, $C_{i,N} = 0$, for $i = 0, \dots, M$. This corresponds to the case when $r = r_{max}$, when the underlying bond price is very low, so the option is close-to-being worthless.

Exercises:

1. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**Vasicek Model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$$

with $r_0 = 5\%,\ \sigma = 12\%,\ \kappa = 0.82,\ \bar{r} = 5\%$.

(a) Use Monte Carlo Simulation (assume each time step is a day) to find the price of a pure discount bond, with Face Value of \$1,000, maturing in $T = 0.5$ years (at time $t = 0$):

$$P(t,T) = \mathbb{E}_t^*\left[\$1{,}000\ exp\left(-\int_t^T r(s)ds\right)\right]$$

(b) Use Monte Carlo Simulation to find the price of a coupon paying bond, with Face Value of \$1,000, paying semiannual coupons of \$30, maturing in $T = 4$ years:

$$P(0,C,T) = \mathbb{E}_0^*\left[\sum_{i=1}^{8} C_i\ exp\left(-\int_0^{T_i} r(s)ds\right)\right]$$

where $C = \{C_i = \$30 \text{ for } i = 1,2,\ldots,7; \text{ and } C_8 = \$1{,}030\}$,
$T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \{0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4\}$.

(c) Use Monte Carlo Simulation to find the price of a European Call option on the pure discount bond in part (a). The option matures in 3 months and has a strike price of $K = \$980$. Use the explicit formula for the underlying bond price (only for the bond price).

(d) Use Monte Carlo Simulation to find the price of a European Call option on the coupon paying bond in part (b). The option matures in 3 months and has a strike price of $K = \$980$. Use Monte Carlo simulation for pricing the underlying bond.

(e) Find the price of a European Call option of part (d) by using the explicit formula for the underlying bond price, and reconcile the findings with the ones of part (d).

2. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following SDE (**CIR model**):

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$$

with $r_0 = 5\%, \sigma = 12\%,\ \kappa = 0.92,\ \bar{r} = 5.5\%$.

(a) Use Monte Carlo Simulation to find at time $t = 0$ the price $c(t,T,S)$ of a European Call option, with strike price of $K = \$980$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1$ year:

$$c(t,T,S) = \mathbb{E}_t^*\left[exp\left(-\int_t^T r(u)du\right) * \max(P(T,S) - K, 0)\right]$$

(b) Use the *Implicit Finite-Difference Method* to find at time $t = 0$ the price $c(t,T,S)$ of a European Call option, with strike price of $K = \$980$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1$ year. The PDE for c is given as

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 r \frac{\partial^2 c}{\partial r^2} + \kappa(\bar{r} - r)\ \frac{\partial c}{\partial r} - rc = 0$$

with $c(T,T,S) = \max(P(T,S) - K, 0)$, and $P(T,S)$ is computed explicitly.

(c) Compute the price $c(t,T,S)$ of the European Call option above using the explicit formula, and compare it to your findings in parts (a) and (b) and comment on your findings.

3. Assume the dynamics of the short-term interest rate, under the risk-neutral measure, are given by the following system of SDE (**G2++ model**):

$$\begin{cases} dx_t = -ax_t dt + \sigma dW_t^1 \\ dy_t = -by_t dt + \eta dW_t^2 \\ r_t = x_t + y_t + \phi_t \end{cases}$$

$x_0 = y_0 = 0$, $\phi_0 = r_0 = 3\%$, $dW_t^1 dW_t^2 = \rho dt$, $\rho = 0.7$, $a = 0.1$, $b = 0.3$, $\sigma = 3\%$, $\eta = 8\%$. Assume $\phi_t = const = 3\%$ for any $t \geq 0$. Use Monte Carlo Simulation to find at time $t = 0$ the price $p(t,T,S)$ of a European Put option, with strike price of $K = \$950$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face value of \$1,000, that matures in $S = 1$ year. Compare it with the price found by the explicit formula and comment on the estimation.

4. Consider a European Put option, with strike price of $K = \$970$, maturity of $T = 0.5$ years on a Pure Discount Bond with Face Value of \$1,000, that matures in $S = 1.5$ years. Which of the two models below would result in a more expensive price for the option?
   (a) The Vasicek Model $dr_t = \kappa(\bar{r} - r_t)dt + \sigma dW_t$ with $r_0 = 5\%$, $\sigma = 12\%$, $\kappa = 0.82$, $\bar{r} = 5\%$.
   (b) The CIR Model $dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_t$ with $r_0 = 5\%$, $\sigma = 54\%$, $\kappa = 0.82$, $\bar{r} = 5\%$.
   Answer by using explicit formulas or by Monte Carlo simulations.
   Is the answer consistent with your intuition?

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