Chapter 4. Binomial Tree Method of Option Pricing (v.A.3)

The major task of this chapter is to develop discrete option pricing formulas and their numerical implementation algorithms under reasonable and easily-implementable models of stock prices. The simple but yet powerful binomial option pricing model is the focus of this chapter, and the Black-Scholes formula is derived as a limiting case of it.

Introduction

In option pricing, the probability distribution of the underlying security's price and the risk-free interest rate are used to determine present values of options' payoffs, and to price options. In 1973 Black, Scholes, and Merton were able to explicitly derive option pricing formulas. The model now is known as the Black-Scholes model or, sometimes it is referred to as the Black-Scholes-Merton model.

The Black-Scholes model is based on a few important assumptions, one of which is that underlying security prices move continuously according to a Geometric Brownian Motion processes. An alternative to the continuous-time model is a discrete model of the underlying price, referred to as the **Binomial model,** which assumes only two possible price movement in every (small) time-period. Dividing the options' maturity into enough number of time-periods for the underlying security price movements makes the price of the option (in this discrete case) converge to the price of the option implied from the Black-Scholes model (in the continuous-time case). In fact, we provide a result to show that options' prices in the Binomial model converge to the ones of the Black–Scholes model, as the number of time-periods increases.

Throughout this chapter, C or c denote the American or European call option prices respectively, and P or p denote the American or European put option prices respectively, X is the strike price, S is the stock price, q is the dividend yield, and D *is* the (discrete) dividend amount.

The Binomial Option Pricing Model

In the Binomial model, the option's expiration is fixed and is measured in periods, the length of each period being δ. The model assumes that if the current stock price is S_0, then in the next period it can go up to $S_u = S_0 u$ with some probability q, or go down to $S_d = S_0 d$ with probability $1 - q$, where $0 < q < 1$ and $\boldsymbol{d < e^{r\delta} < u}$, where r is the risk-free interest rate (annualized). The latter condition is the no-arbitrage condition. What if this condition is violated? If $\boldsymbol{d < e^{r\delta} < u}$ condition is violated then, we can't apply the Binomial method to price options.

In particular,

(a) if $d \geq e^{r\delta}$ then one would borrow money and buy the stock. This strategy will lead to arbitrage profits (the expected payoff being positive and the payoff being non-negative in every state).

(b) On the other hand, if $e^{r\delta} \geq u$, then one would short-sell the stock and invest the proceeds at the risk-free rate. This strategy will lead to arbitrage profits.

Options on a Non-Dividend-Paying Stock: 1-period case

Suppose that the expiration date is only one period. Let c_u be the value of the option at option's expiration (at time δ) if the stock price moves up to S_u and c_d be the value of the option at option's expiration (at time δ) if the stock price moves down to S_d.

We have,

$$c_u = \max(S_u - K, 0) = (S_u - K)^+, c_d = \max(S_d - K, 0) = (S_d - K)^+$$

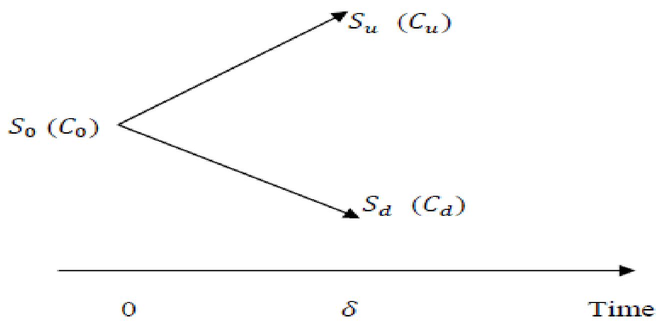

Figure 1: One Step Binomial Tree.

A Replicating Portfolio

Set up a portfolio π of Δ shares of the underlying stock and B dollars invested in the risk-free security (which pays at the rate r). The value of this portfolio at the time of creation (time 0) is $\pi_0 = \Delta S_0 + B$.

The value of this portfolio at time δ is either $\pi_u = \Delta S_0 u + Be^{r\delta}$ or $\pi_d = \Delta S_0 d + Be^{r\delta}$, depending on the movement of the stock price.

The idea of the portfolio replication is to choose Δ and B in such a way that the portfolio replicates the payoff of the call option in both up and down states for the stock price at time δ:

$$\begin{cases} \pi_u = \Delta S_0 u + Be^{r\delta} = c_u \\ \pi_d = \Delta S_0 d + Be^{r\delta} = c_d \end{cases}$$

Solving this system for Δ and B will yield the following:

$$\Delta = \frac{C_u - C_d}{S_0(u-d)}, \qquad B = e^{-r\delta}\frac{uC_d - dC_u}{(u-d)}$$

Thus, we have created a portfolio π that replicates the payoff of the call option at time δ.

By the no-arbitrage principle (or the Law of One Price), the call option should have the same price at time 0 as the cost of the portfolio π:

$$c_0 = \pi_0 = \Delta S_0 + B$$

By using the expressions for B and Δ in $c_0 = \Delta S_0 + B$, we get the following expression for the price of the call option:

$$c_0 = S_0 \frac{c_u - c_d}{S_0(u-d)} + e^{-r\delta} \frac{uc_d - dc_u}{(u-d)} = e^{-r\delta}(pc_u + (1-p)c_d) = e^{-r\delta}\, \mathbb{E}_p(c_\delta)$$

where

$$p = \frac{e^{r\delta} - d}{u-d}$$

That is, we can **think of c_0 as the discounted** (by the risk–free rate) **expected payoff of the option, under a certain measure-p**. This measure is the so-called **Risk-Neutral measure.**

Surprisingly, the option value is independent of q, the probability of an upward movement in price, and hence the expected return of the stock $qS_u + (1-q)\,S_d$. It therefore does not directly depend on investors' **risk preferences** and will have the same price regardless of investors' risk aversions.

That is, under the newly constructed measure p, which we called the Risk-Neutral measure, the price of the option is the present value of the expected payoff at its maturity. This is the significance of the Risk-Neutral measure: it allows us to take the risk into account and just find the expected payoff of the option and discount it at the risk-free rate to find its price. We could not accomplish the same task under the measure q (called physical measure).

Options on a Non-Dividend-Paying Stock: 2-period case

Now we will consider a case in which the option's time to expiration is divided into two periods. Thus, we will be dealing with a two-step binomial model of security prices.

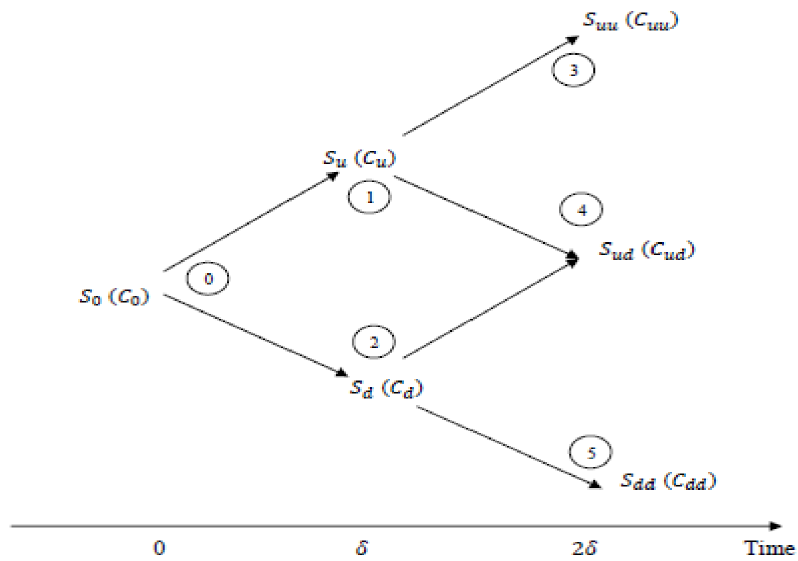

Figure 2: Two Step Binomial Tree.

To find the value of the option at time 0, we start at the expiration date, which is 2δ, and find the payoff of the option in all nodes (nodes 3, 4, and 5). Then, using the 1-period model, we can price the option at the nodes 1 and 2. For the node 1: we can apply the 1-period model (with nodes 3 and 4) and find the option's price at the node 1. For the node 2: we can apply the 1-period model (with nodes 4 and 5) and find the option's price at the node 2.

Using the same logic, and prices found at nodes 1 and 2, we find the price of the option at node 0, using the 1-period model again. The calculations are shown below.

In nodes 3, 4 and 5 the payoff/value of the call option is just its exercise value: $(S_{2\delta} - X)^+$

In node 1: $c_u = e^{-r\delta}(pc_{uu} + (1-p)c_{ud})$ (using the formula for 1-period)

In node 2: $c_d = e^{-r\delta}(pc_{ud} + (1-p)c_{dd})$

In node 0: $c_0 = e^{-r\delta}(pc_u + (1-p)c_d) = e^{-r2\delta}(p^2c_{uu} + 2p(1-p)c_{ud} + (1-p)^2c_{dd})$

$$= e^{-r2\delta}\,\mathbb{E}_p(c_{2\delta})$$

Thus, the price of the call option at time 0 is the **present value of the expected payoff at maturity under the Risk-Neutral Measure p** (defined earlier).

Options on a Non-Dividend-Paying Stock: n −period case

Now we will consider a case in which the option's time to expiration is divided into *n* equal sub-periods. Thus, we will be dealing with a n-period binomial model of security prices.

Under the binomial model, the stock can take on the following $n+1$ possible values at maturity: $S_0 u^k d^{n-k}$ for any $k = 0,1,\dots,n$.

Using the same argument of the 2-period case, we can write the value of the call option at time 0 to be:

$$\begin{aligned} c_0 &= e^{-rn\delta}\left(\binom{n}{n} p^n (1-p)^0 c_{u^n} + \binom{n}{n-1} p^{n-1}(1-p)^1 c_{u^{n-1}d} + \cdots \right. \\ &\qquad \left. + \binom{n}{k} p^k (1-p)^{n-k} c_{u^k d^{n-k}} + \cdots + \binom{n}{0} p^0 (1-p)^n c_{d^n}\right) \\ &= e^{-rn\delta} \sum_{k=0}^{n} \binom{n}{k} p^n (1-p)^{n-k} (S_0 u^k d^{n-k} - X)^+ = e^{-rn\delta}\, \mathbb{E}_p(c_{n\delta}) \end{aligned}$$

Similarly, the value of a European put option is

$$p_0 = e^{-rn\delta} \sum_{k=0}^{n} \binom{n}{k} p^n (1-p)^{n-k} (X - S_0 u^k d^{n-k})^+ = e^{-rn\delta}\, \mathbb{E}_p(P_{n\delta})$$

Notice that the above formulas are closed-form representations of the call and put option prices respectively. But, when implementing the binomial pricing approach, it is computationally more efficient to start at the maturity and go backwards step-by-step, one period at a time, to find the prices at time 0. (Why would that be more efficient?)

One of the big advantages of the binomial pricing model, compared with the Black-Scholes model, is that it can be used to price not only European but also American options.

The process of pricing American-type options is as follows.

Starting at the option's expiration and estimate the option's payoffs in all nodes.

Then, move backwards by one period and for every node, estimate the option's payoff by using the 1-period model (this is if the option is kept alive and not exercised). Then, estimate the exercise-value of the option on the same node.

Compare the exercise value to the continuation value of the option and keep the larger of the two. This is the option's value at that node.

Continue this process until time 0. There is only one node at time 0 and the estimated price at that node is the option price.

A demonstration of pricing of an American option is below.

Exercises.

1. Show that the price of the estimated call option in the n-period binomial model will converge to the Black-Scholes value of the option, as $n \to \infty$.
2. What will be the convergence rate of the estimated call option price in the Binomial Model to the one of the Black-Scholes Model?
3. Assume the current stock price is $S_0 = \$10$ and with 65% probability it may go up to \$12, and with probability of 35% will stay at \$10 level in a month. Assume the risk-free rate is 0. What's the price of the European call option that expires in a month and has a strike price of \$10? What is the put price with strike price \$10?

Selection of Parameters

How to select the model parameters to price options in the Binomial Model?

We will use the continuous-time Black-Scholes model of stock prices to derive the parameters of the Binomial model in such a way that they are consistent with the Black-Scholes model.

Assume the stock price, that pays no dividends, follows a Geometric Brownian Motion (GBM) model under the risk-neutral measure:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and S_0 is given.

The value of the stock S_δ at time δ can be written as

$$S_\delta = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)\delta + \sigma W_\delta} = S_0 e^{r\delta}\, e^{-\frac{\sigma^2}{2}\delta + \sigma W_\delta}$$

The $k - th$ moment of S_t can be calculated to be:

$$ES_t^k = S_0^k e^{rtk}\, e^{\frac{k\sigma^2 t}{2}(k-1)}$$

This implies that the first two moments of S_δ are given by

$$ES_\delta = S_0 e^{r\delta}, \qquad ES_\delta^2 = S_0^2 e^{(2r+\sigma^2)\delta}$$

Now, consider the Binomial Model, in which over a small time interval δ, the stock price can go up to $S_\delta = S_0 u$ with probability p, or go down to $S_\delta = S_0 d$ with probability $1 - p$. The relationship between the parameters of the continuous-time process and the Binomial process is obtained by **equating the first and second moments of the processes** (continuous-time and discrete-time) over the time interval δ:

$$E[S_\delta] = pS_0 u + (1-p)S_0 d = S_0 e^{r\delta}$$

$$E[{S_\delta}^2] = pS_0^2 u^2 + (1-p)S_0^2 d^2 = S_0^2 e^{(2r+\sigma^2)\delta}$$

We have two equations (above) and three unknowns: $\boldsymbol{p, u, d}$. We will consider various cases of parameter choices for the binomial model.

Notice that, the normal or lognormal distributions are determined by their first two moments. Therefore, it suffices to equate the first two moments of the two cases (continuous and discrete) to select the parameters that would make the two cases consistent with each other.

Case (a): Assume a symmetric and recombining tree: $u = 1/d$[1]. We need to solve for p and d. Solving for p in both equations yields:

$$p = \frac{e^{r\delta} - d}{u - d} \text{ and } p = \frac{e^{(2r+\sigma^2)\delta} - d^2}{u^2 - d^2}$$

Setting the above two equal to each other, using $u = 1/d$, and getting an equation for d, and solving it gives us the following result:

$$\begin{cases} d = g - \sqrt{g^2 - 1} \\ u = g + \sqrt{g^2 - 1} \\ g = \frac{1}{2}\left(e^{-r\delta} + e^{(r+\sigma^2)\delta}\right) \\ p = \frac{e^{r\delta} - d}{u - d} \end{cases}$$

Case (b): Take $p = 1/2$. Solve for u and d.

Now we have

$$u + d = 2e^{r\delta}$$
$$u^2 + d^2 = 2e^{(2r+\sigma^2)\delta}$$

Solving for u and d gives

$$\begin{cases} d = e^{r\delta}\left(1 - \sqrt{e^{\sigma^2\delta} - 1}\right) \\ u = e^{r\delta}\left(1 + \sqrt{e^{\sigma^2\delta} - 1}\right) \\ p = 1/2 \end{cases}$$

Two particular cases of the above cases that are popular in the literature are the CRR (Cox-Ross-Rubinstein) model and the JR (Jarrow-Rudd) model.

[1] The stop price moving up then down is the same as moving down then up, and, it arrives at the same spot.

The parameter choices for both models can be obtained as particular cases when taking a few terms of the Taylor expansion in the general formula in case (a).

CRR (Cox-Ross-Rubinstein) Model:

$$\begin{cases} d = e^{-\sigma\sqrt{\delta}} \\ u = e^{\sigma\sqrt{\delta}} \\ p = \dfrac{1}{2}\left(1 + \dfrac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\delta}}{\sigma}\right) \end{cases}$$

The parameters in this case can be derived from case (a) by taking the first order approximation to the Taylor series expansion of e^x. See the original paper by Cox, Ross Rubinstein (1979) for an alternative derivation. Notice that, under the CRR parameterization the tree is perfectly balanced in space and in time.

JR (Jarrow-Rudd) Model:

$$\begin{cases} d = e^{\left(r-\frac{\sigma^2}{2}\right)\delta - \sigma\sqrt{\delta}} \\ u = e^{\left(r-\frac{\sigma^2}{2}\right)\delta + \sigma\sqrt{\delta}} \\ p = \dfrac{1}{2} \end{cases}$$

Notice that, we also obtain this model if we assume that the stock price follows a Geometric Brownian Motion process. Thus, in a time step δ the stock price moves from S_0 to either $S_0 e^{\left(r-\frac{\sigma^2}{2}\right)\delta+\sigma\sqrt{\delta}}$ or to $S_0 e^{\left(r-\frac{\sigma^2}{2}\right)\delta-\sigma\sqrt{\delta}}$.

The JR binomial model is not balanced in the stock price space since it grows at the rate of $e^{\left(r-\frac{\sigma^2}{2}\right)\delta}$. This qualitative behavior is shown in Figure 3. It does not grow exactly at the forward risk-free interest rate curve, but it is possible to construct binomial trees with this property. See “Growing a Smiling Tree” by Barle, S. and N. Cakici (1995) for more details.

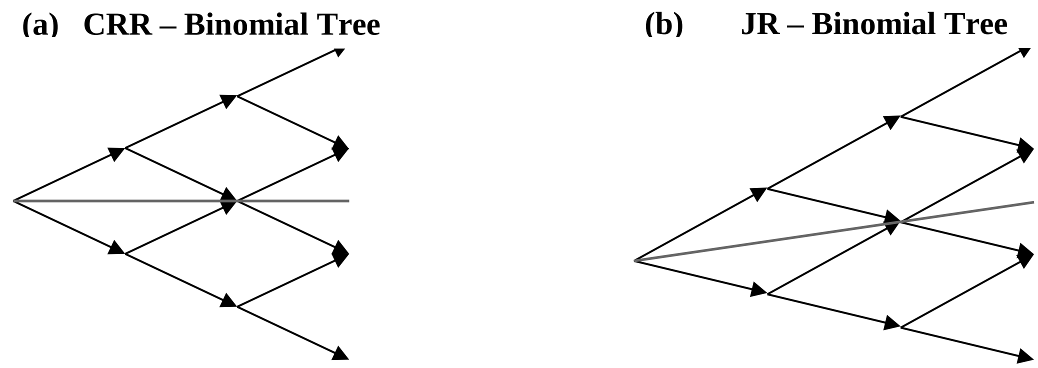

Figure 3: *Comparison Between a CRR– and JR-Binomial Tree.*

More generally, in a recombining constant volatility tree u and d have the general form: $e^{\pi\delta + \sigma\sqrt{\delta}}$ and $e^{\pi\delta - \sigma\sqrt{\delta}}$ for any reasonable number π.

Remark:

Both, the CRR and JR binomial models converge to the Black-Scholes model in the continuous time limit.

Another way to parameterize the binomial tree is to use the log-price processes (or returns), instead of stock-price processes. We demonstrate this implementation below.

Using the Log-price Process

Assume the price of a stock that pays no dividends follows a Geometric Brownian Motion (GBM) process, which under the risk-neutral measure is given by:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and S_0 is given. Using the $X_t = \ln S_t$ transformation, we can write the SDE for X_t as:

$$dX_t = (r - \frac{\sigma^2}{2})dt + \sigma dW_t$$

and $X_0 = \ln S_0$. Denote by $\gamma = r - \frac{\sigma^2}{2}$.

Consider the Binomial model, in which over a small time-interval δ, the stock price can go up by ΔX_u to $X_\delta = X_0 + \Delta X_u$ with probability p, or go down by ΔX_d to $X_\delta = X_0 + \Delta X_d$ with probability $1 - p$.

The relationship between the parameters of the continuous time process and the binomial process is obtained by equating the first and second moments of the processes over the time interval δ:

$$E(\Delta \mathrm{X}) = E(X_\delta - X_0) = p\Delta X_u + (1-p)\Delta X_d = \gamma\delta$$

$$E(\Delta \mathrm{X})^2 = E(X_\delta - X_0)^2 = p\,(\Delta X_u)^2 + (1-p)(\Delta X_d)^2 = \gamma^2\delta^2 + \sigma^2\delta$$

We have two equations (above) and three unknowns: $\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{d}$. We will consider various cases of parameter choices for this model below.

Case (a): $\Delta X = \Delta X_u = -\Delta X_d$.

Solving the above equations will yield

$$\begin{cases} \Delta X = \sqrt{\sigma^2\delta + \gamma^2\delta^2} \\ p = \dfrac{1}{2}\left(1 + \dfrac{\gamma\delta}{\Delta X}\right) \end{cases}$$

As in the case of the CRR Binomial Model, this parameterization results in a symmetric and recombining tree. That is, up-down move in the stock price ends up in the same spot as the down-up move.

Case (b): $p = 1/2$.

Solving the above equations for ΔX_u and ΔX_d we get

$$\begin{cases} \Delta X_u = \dfrac{1}{2}\gamma\delta + \dfrac{1}{2}\sqrt{4\sigma^2\delta - 3\gamma^2\delta^2} \\ \Delta X_d = \dfrac{3}{2}\gamma\delta - \dfrac{1}{2}\sqrt{4\sigma^2\delta - 3\gamma^2\delta^2} \\ p = \dfrac{1}{2} \end{cases}$$

where $\gamma = r - \frac{\sigma^2}{2}$.

There are many other choices of parameters $p, \Delta X_u,$ and ΔX_d that would result in consistent results for the discrete vs continuous-time cases. We can derive various models as long as they satisfy the following two equations:

$$p\Delta X_u + (1-p)\Delta X_d = \gamma\delta$$

$$p\,(\Delta X_u)^2 + (1-p)(\Delta X_d)^2 = \gamma^2\delta^2 + \sigma^2\delta$$

Example: Pricing an American Put Option

Binomial trees can be used to price American options which we will demonstrate in this section. An American option allows the holder to exercise the option any time before its expiration. Since an American option carries all the rights of a European option with the same strike price and written on the same underlying, it is clear that an American option should be at least as valuable as its European counterpart should. It can be shown that the price of an American call option written on a non-dividend paying stock is *the same* as the price of the equivalent European option.

To price an American option in the binomial tree framework, it is necessary to compare the continuation value and the exercise value of the option at each node of the tree and take the larger of the two as the value of the option on that node. If the exercise value is larger than the continuation value, then the option is exercised. The following example illustrates the pricing of an American put option.

The model parameters are as follows: $S_0 = 65, X = 60, T = 3, u = 1.25, d = 0.8, r = 9.531\%$.

Then, $p = \frac{e^r - d}{u - d} = 0.667$. Also, $e^{-0.09531} = \frac{1}{1.1}$ (this is only for demonstration purposes).

Here the nodes are numbered and in orange blocks; the stock prices in each node are the first numbers at that node, and the value of the option is in parenthesis, under the stock prices (in blue colors). **EV** is the Exercise Value, and the **CV** is the continuation value of the option.

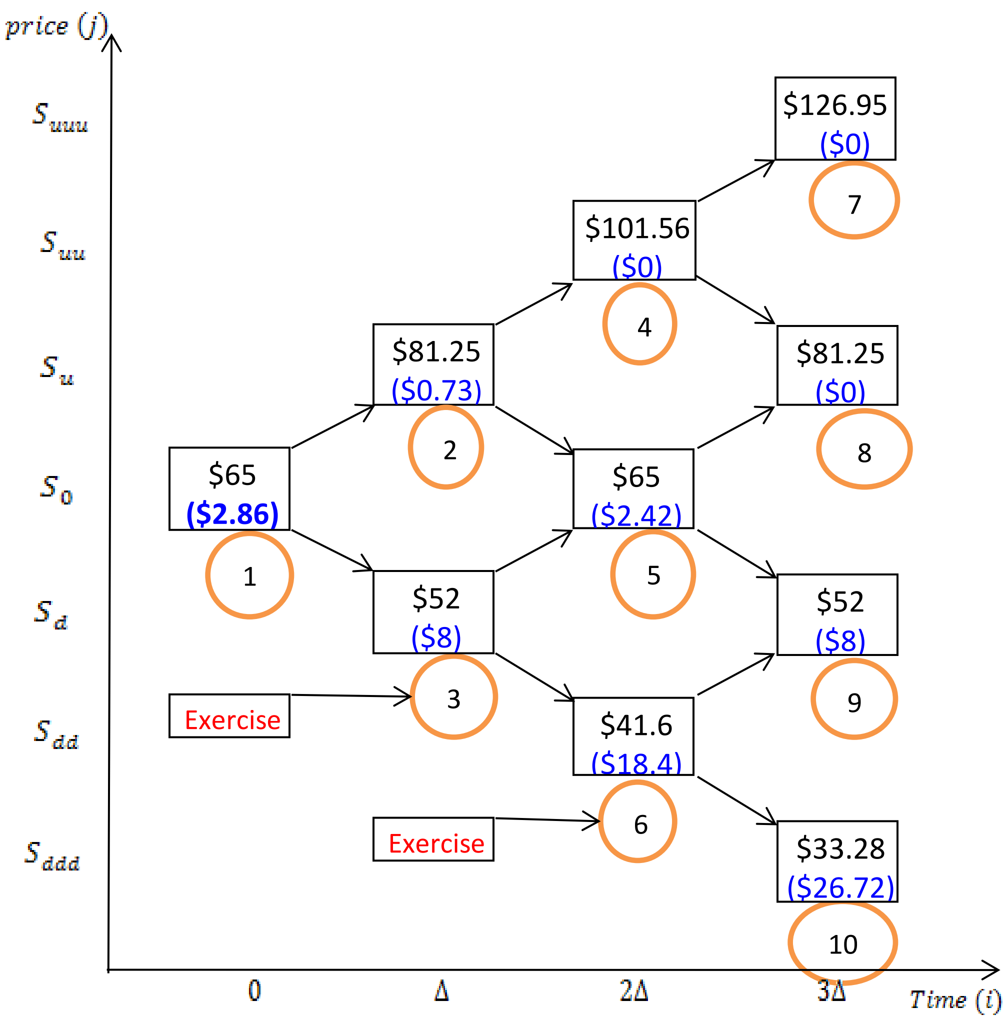

Figure 4: Evaluating an American Put Option with a Binomial Tree.

In every node, we will compute the exercise value, EV, and the continuation value, CV (which will be computed using a simple 1-period Binomial Model); compare these two, and take the larger of the two, and assign that to the option's value at that node.

At node 10: The payoff of the put option is $(X - S_T)^+ = (\$60 - \$33.28)^+ = \$26.72$. Therefore, the EV=\$26.72. The CV=\$0. Thus, the value of the option is **\$26.72.**

At node 9: The payoff of the put option is $(X - S_T)^+ = (\$60 - \$52)^+ = \$8$. Therefore, the EV=\$8. The CV=\$0. Thus, the value of the option is **\$8.**

At node 8: The payoff of the put is $(X - S_T)^+ = (\$60 - \$81.25)^+ = \$0$. Therefore, the EV=\$0. The CV=\$0. Thus, the value of the option is **\$0.**

At node 7: The payoff of the put option is $(X - S_T)^+ = (\$60 - \$126.95)^+ = \$0$. Therefore, the EV=\$0. The CV=\$0. Thus, the value of the option is **\$0.**

At node 6: $EV = (X - S_t)^+ = (\$60 - \$41.6)^+ = \$18.4,$

$$CV = e^{-r}(pc_u + (1-p)c_d) = e^{-0.09531}(0.667 * \$8 + 0.333 * \$26.72) = \$12.94$$

Thus, it is **optimal to exercise**, and the value of the option is **\$18.4.**

At node 5: $EV = (X - S_t)^+ = (\$60 - \$65)^+ = \$0.$

$$CV = e^{-r}(pc_u + (1-p)c_d) = e^{-0.09531}(0.667 * \$0 + 0.333 * \$8) = \$2.42$$

Thus, the value of the option is **\$2.42.**

At node 4: $EV = (X - S_t)^+ = (\$60 - \$101.56)^+ = \$0.$

$$CV = e^{-r}(pc_u + (1-p)c_d) = e^{-0.09531}(0.667 * \$0 + 0.333 * \$0) = \$0$$

Thus, the value of the option is **\$0.**

At node 3: $EV = (X - S_t)^+ = (\$60 - \$52)^+ = \$8.$

$$CV = e^{-r}(pc_u + (1-p)c_d) = e^{-0.09531}(0.667 * \$2.42 + 0.333 * \$18.4) = \$7$$

Thus, it is **optimal to exercise**, and the value of the option is **\$8.**

At node 2: $EV = (X - S_t)^+ = (\$60 - \$81.25)^+ = \$0.$

$$CV = e^{-r}(pc_u + (1-p)c_d) = e^{-0.09531}(0.667 * \$0 + 0.333 * \$2.42) = \$0.73$$

Thus, the value of the option is **\$0.73.**

At node 1: $EV = (X - S_t)^+ = (\$60 - \$65)^+ = \$0.$

$$CV = e^{-r}(pc_u + (1-p)c_d) = e^{-0.09531}(0.667 * \$0.73 + 0.333 * \$8) = \$2.86$$

Thus, the value of the option is **\$2.86.**

Exercises

Why is it not optimal to exercise an American call option on a non-dividend paying stock before maturity?

1. Can it be optimal to exercise an American call option before maturity on any stock?
2. Why can it be optimal to exercise an American put option early?

Trinomial Trees

In the Trinomial model, we assume that in every time period the stock price can move to 3 possible states.

The obvious advantage of the trinomial tree model of stock prices, compared to the binomial tree model, is that it provides more flexibility for the stock price movements. For example, we can choose the model parameters in such a way that the stock price not only can increase and decrease, but can also stay the same level in every period. This seems to be a more realistic model for the evolution of the stock prices, than the binomial model.

In order to calibrate the parameters of a trinomial tree (and be consistent with the Black-Scholes model) we need to choose five parameters: three stock price changes and two probabilities of moving in the 3 states in the next period. There are many ways to choose these parameters. We will discuss two approaches in this section.

Assume the price of a non-dividend paying stock follows a Geometric Brownian Motion (GBM) process under the risk-neutral measure:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and S_0 is given. Using the $X_t = \ln S_t$ transformation, and denoting $\gamma = r - \frac{\sigma^2}{2}$, we can write

$$dX_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t = \gamma dt + \sigma dW_t$$

and $X_0 = \ln S_0$.

Consider the Trinomial model, in which over a small time-interval δ, the log-stock price can go up by ΔX with probability p_u; stay at the same level X with probability p_m; or go down by ΔX with probability p_d.

The relationship between the parameters of the continuous time process and the trinomial process is obtained by equating the first and second moments of the processes over the time interval δ:

Case 1: Take $\Delta X = \sigma\sqrt{3\delta}$.

By taking $\Delta X = \sigma\sqrt{3\delta}$ (this choice will be explained later when discussing the convergence of stability of the method), and equating the first and second moments of the processes over the time interval Δ, we can solve and get p_u, p_m, p_d:

$$\begin{cases} E[\Delta X] = p_u(\Delta X) + p_m(0) + p_d(-\Delta X) = \gamma\delta \\ E[\Delta X^2] = p_u(\Delta X^2) + p_m(0) + p_d(-\Delta X^2) = \sigma^2\delta + \gamma^2\delta^2 \\ p_u + p_m + p_d = 1 \end{cases}$$

Solving the above equations gives

$$\begin{cases} p_u = \frac{1}{2}\left(\frac{\sigma^2\delta + \gamma^2\delta^2}{\Delta X^2} + \frac{\gamma\delta}{\Delta X}\right) \\ p_d = \frac{1}{2}\left(\frac{\sigma^2\delta + \gamma^2\delta^2}{\Delta X^2} - \frac{\gamma\delta}{\Delta X}\right) \\ p_m = 1 - p_d - p_u \end{cases}$$

In a 1-period case the price of a call option will be given by

$$C = e^{-r\delta}(p_u C_u + p_m C_m + p_d C_d)$$

In a 2-period case the price of a call option will be given by

$$C = e^{-r\delta}(p_u C_u + p_m C_m + p_d C_d)$$

where

$$C_u = e^{-r\delta}(p_u C_{uu} + p_m C_{um} + p_d C_{mm})$$

$$C_m = e^{-r\delta}(p_u C_{um} + p_m C_{mm} + p_d C_{md})$$

$$C_d = e^{-r\delta}(p_u C_{du} + p_m C_{dm} + p_d C_{dd})$$

Using the above three expressions in the pricing formula, we get

$$C = e^{-r2\delta}({p_u}^2 C_{uu} + 2p_u p_m C_{um} + (p_u p_d + {p_m}^2 + p_d p_u)C_{mm}) + 2p_m p_d C_{md} + {p_d}^2 C_{dd})$$

Generally, in an n −period model, the algorithm for pricing options is similar to the one of the binomial model: start from the last (terminal) time-point, find the option's payoffs and move backward in time, one period at a time, by computing the option values in every node as discounted expected payoffs. Denoting by i the time and by j the price axis, in node (i, j) we will have:

$$C_{i,j} = e^{-r\delta}(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1})$$

The price of the option will be $C_{0,0}$.

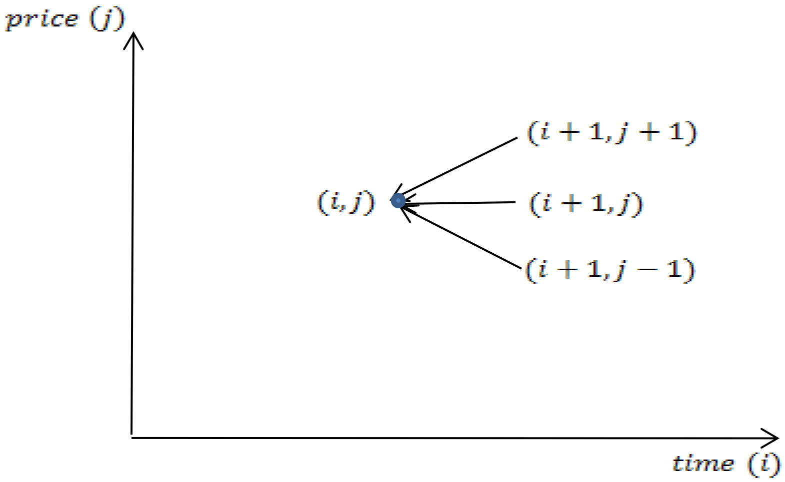

Figure 5: Trinomial Tree.

Now we'll use the Geometric Brownian Motion (GBM) price process to solve for the parameters of the trinomial tree.

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and S_0 is given.

Case 2:

Consider the Trinomial model, in which over a small time-interval δ, the asset price can go up from S_0 to $S_0 u$ with probability p_u, stay at the same level S_0 with probability p_m, or go down to $S_0 d$ with probability p_d.

By equating the first and second moments of the processes over the time interval δ, we get:

$$\begin{cases} E[S_\delta] = p_u(S_0 u) + p_m(S_0) + p_d(S_0 d) = S_0 e^{r\delta} \\ E[S_\delta^2] = p_u(S_0 u)^2 + p_m(S_0)^2 + p_d(S_0 d)^2 = {S_0}^2 e^{(2r+\sigma^2)\delta} \\ p_u + p_m + p_d = 1 \end{cases}$$

Solving the above equations, we get:

$$\begin{cases} p_u = \dfrac{e^{(2r+\sigma^2)\delta} - (d+1)e^{r\delta} + d}{(u-1)(u-d)} \\ p_d = \dfrac{e^{(2r+\sigma^2)\delta} - (u+1)e^{r\delta} + u}{(1-d)(u-d)} \\ p_m = 1 - p_d - p_u \end{cases}$$

We can take $u = e^{\sigma\sqrt{3\delta}}$ and $d = \frac{1}{u}$.

This choice of parameters will be explained later, when discussing the convergence and the stability of the method. By using a simpler version of the above equations, and equating the first and the second moments of the processes over the time interval δ, and by using the Taylor approximation of the exp(.) function of the right-hand sides gives:

$$\begin{cases} E[S_\delta] = p_u(S_0 u) + p_m(S_0) + p_d(S_0 d) = S_0(1 + r\delta) \\ E[S_\delta^2] = p_u(S_0 u)^2 + p_m(S_0)^2 + p_d(S_0 d)^2 = {S_0}^2(1 + r\delta)^2 + {S_0}^2\sigma^2\delta \\ p_u + p_m + p_d = 1 \end{cases}$$

Solving the above equations results in the following probabilities:

$$\begin{cases} p_u = \dfrac{\sigma^2\delta + r^2\delta^2 - (d-1)r\delta}{(u-1)(u-d)} \\ p_d = \dfrac{\sigma^2\delta + r^2\delta^2 - (u-1)r\delta}{(1-d)(u-d)} \\ p_m = 1 - p_d - p_u \end{cases}$$

There are many other possibilities to choose the parameters for a trinomial tree.

In order to reduce the variance of the option price estimates it is possible to use variance reduction techniques from the previous chapter. For example, if we want to estimate the price of an American Put on a stock without dividends, we could use the price of an American Call as the control variate, because we have an explicit formula for the latter - the price of the American Call in this case is equal to the Black Scholes price of the European Call option.

It is also an interesting question to compare the performance of binomial and trinomial models. For a comparison for American puts, see for example “Trinomial or Binomial: Accelerating American Put Option Price on Trees” by Chan, Joshi, Tang, and Yang (2008).

So far, we have only considered binomial and trinomial trees with constant volatility. One major advantage of tree models is that it is possible to construct binomial and trinomial trees with changing volatility, so-called implied binomial and trinomial trees. These trees make it possible to model volatility smiles or skews. Derman, Kani, and Chris discuss the construction of implied trinomial tree in "Implied Trinomial Trees of the Volatility Smile" by Derman, Kani and Chriss (1996).

Exercises

1. Use the Binomial Method to price a 6-month European Call option with the following information: the risk-free interest rate is 5% per annum and the volatility is 24%/annum, the current stock price is \$32 and the strike price is \$30. Divide the time interval into n parts to estimate the price of this option. Use $n = 10, 15, 20, 40, 70, 80, 100, 200$ and 500 to compute the approximate price and draw them in one graph, where the horizontal axis measures n, and the vertical one– the price of the option. Compare the convergence rates of the four methods below:

   (a) Use the binomial method in which

   $$u = \frac{1}{d}, d = c - \sqrt{c^2 - 1}, \qquad c = \frac{1}{2}\left(e^{-r\Delta} + e^{(r+\sigma^2)\Delta}\right), \qquad p = \frac{e^{r\Delta} - d}{u - d}$$

   (b) Use the binomial method in which

   $$u = e^{r\Delta}\left(1 + \sqrt{e^{\sigma^2\Delta} - 1}\right), \qquad d = e^{r\Delta}\left(1 - \sqrt{e^{\sigma^2\Delta} - 1}\right), \qquad p = 1/2$$

   (c) Use the binomial method in which

   $$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}}, \qquad d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta - \sigma\sqrt{\Delta}}, \qquad p = 1/2$$

   (d) Use the binomial method in which

   $$u = e^{\sigma\sqrt{\Delta}}, \qquad d = e^{-\sigma\sqrt{\Delta}}, \qquad p = \frac{1}{2} + \frac{1}{2}\left(\frac{\left(r - \frac{\sigma^2}{2}\right)\sqrt{\Delta}}{\sigma}\right)$$

2. Take the current price of AMZN. Use risk-free rate of 2% per annum, and strike price that is the closest integer (divisible by 10) to 110% of the current price. Estimate the price of the call option that expires on January of next year, using the binomial model. AMZN does not pay dividends. To estimate the historical volatility, use 60 months of historical stock price data on the company. You may use *Bloomberg* or *finance.yahoo.com* to obtain historical prices and the current price of AMZN. Compare your price with the one you can get from *Bloomberg* or *finance.yahoo.com.* If the two are different, how would you change the volatility in your code to get the market price?

3. Consider the following information on the stock of a company and options on it: $S_0 = \$49, K = \$50, r = 0.03, \sigma = 0.2, T = 0.3846$ (20 weeks), $\mu = 0.14$. Using the binomial method (any one of them) estimate the following and draw the graphs:

(i) Delta of the call option as a function of S_0, for S_0 ranging from \$10 to \$80, in increments of \$2.

(ii) Delta of the call option, as a function of T (time to expiration), from 0 to 0.3846 in increments of 0.01.

(iii) Theta of the call option, as a function of S_0, for S_0 ranging from \$10 to \$80 in increments of \$2.

(iv) Gamma of the call option, as a function of S_0, for S_0 ranging from \$10 to \$80 in increments of \$2.

(v) Vega of the call option, as a function of S_0, for S_0 ranging from \$10 to \$80 in increments of \$2.

(vi) Rho of the call option, as a function of S_0, for S_0 ranging from \$10 to \$80 in increments of \$2.

4. Consider 12-month put options on a stock of company XYZ. Assume the risk-free rate is 5%/annum and the volatility of the stock price is 30 % /annum and the strike price of the option is \$100. Use binomial method to estimate the prices of European and American Put options with current stock prices varying from \$80 to \$120 in increments of \$4. Draw them all in one graph and compare.

5. Use the Trinomial Method to price a 6-month European Call option with the following information: the risk-free interest rate is 5% per annum and the volatility is 24%/annum, the current stock price is \$32 and the strike price is \$30. Divide the time interval into n parts to estimate the price of this option. Use $n = 10, 15, 20, 40, 70, 80, 100, 200$ and 500 to compute the approximate price and draw them in one graph, where the horizontal axis measures n, and the vertical one– the price of the option. Compare the convergence rates of the two methods below:

   (a) Use the trinomial method applied to the stock price-process (S_t) in which

$$u = \frac{1}{d}, \qquad d = e^{-\sigma\sqrt{3\Delta}},$$

$$p_d = \frac{r\Delta(1-u)+(r\Delta)^2+\sigma^2\Delta}{(u-d)(1-d)},\ p_u = \frac{r\Delta(1-d)+(r\Delta)^2+\sigma^2\Delta}{(u-d)(u-1)},\ p_m = 1 - p_u - p_d$$

   (b) Use the trinomial method applied to the Log-stock price-process (X_t) in which in which

$$\Delta X_u = \sigma\sqrt{3\Delta},\ \ \Delta X_d = -\sigma\sqrt{3\Delta}$$

$$p_d = \frac{1}{2}\left(\frac{\sigma^2\Delta+\left(r-\frac{\sigma^2}{2}\right)^2\Delta^2}{\Delta {X_u}^2} - \frac{\left(r-\frac{\sigma^2}{2}\right)\Delta}{\Delta X_u}\right),\ p_u = \frac{1}{2}\left(\frac{\sigma^2\Delta+\left(r-\frac{\sigma^2}{2}\right)^2\Delta^2}{\Delta {X_u}^2} + \frac{\left(r-\frac{\sigma^2}{2}\right)\Delta}{\Delta X_u}\right),$$

$$p_m = 1 - p_u - p_d$$

6. Use Halton's Low-Discrepancy Sequences to price European call options. The code should be generic: it will ask for the user inputs for S_0, K, T, r, σ, N (number of points) and b_1 (base 1) and b_2 (base 2). Use the Box-Muller method to generate Standard Normals as follows:

$$\begin{cases} Z_1 = \sqrt{-2Ln(H_1)}\cos(2\pi H_2) \\ Z_2 = \sqrt{-2Ln(H_1)}\, sin(2\pi H_2) \end{cases}.$$

   where H_1 and H_2 will be the Halton's numbers with base b_1 and base b_2 accordingly.

   For the price of the call option you may use the following formula:

$$c = e^{-(rT)}\boldsymbol{E}^*(S_T - K)^+ = e^{-(rT)}\boldsymbol{E}^*\left(S_0\ e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T} - K\right)^+ = \boldsymbol{E}^* f(W_T)$$

7. Assume the current stock price is $S_0 = \$30$ and with 70% probability it may go up to $31 and with probability of 30% will go down to $28 in a year. Assume the risk-free rate is 5% a year. What's the price of the European call option that expires in a year and has a strike price of $30? What is the European put option price with strike price $27.50?

8. How much are you willing to pay to play this game: You toss a fair coin. If the outcome is a Tail then you get $7 in 18 months. If it is a Head then you lose $2 immediately. The one- and two-year zero-coupon rates are 4% and 6% respectively. Would the amount you are willing to pay to play this game increase or decrease if the payoff in case of Tails is in 36-months instead of 18 months?

9. Value an American Put option that has no maturity (perpetual option). What's the delta of the option if it is at-the-money? Try to get an explicit formula for the price and estimate it by Monte Carlo simulation.