3. Non-linear Payoffs: Options Markets

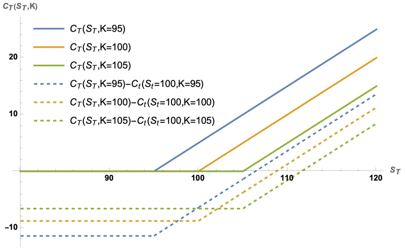
Prologue

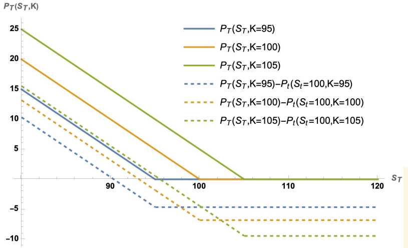
- In this section, we review the payoffs (both outright and P&L net of initial premium) of calls, puts, and portfolios/strategies built by combining them
- This seems as good a place as any to say a bit about the valuation model inputs we'll be using during the remainder of the course: both for in-class examples and for assignments (which serve slightly different purposes)
  - In-class examples: our objective is to use consistent, simplified inputs to "get a feel" for the relative values of different structures
    - It makes the most sense to use rounded, generic values that are easy to compare (for prices) and of typical magnitudes (for rates, yields, and vols)
    - Hence, we will base prices around $S = K = 100$ and take $r = 4\%, y = 2\%, \sigma = 20\%$
  - Homework: we'll use figures as close as possible to real market values, e.g., SPX $\simeq$ 5827 (as of Friday, 10-Jan-2025) and as consistently as possible thereafter – unless market conditions change significantly [$\rightarrow$ 5996.66 $\simeq$ 6000 close as of F 17 Jan].
- In discussing portfolios/strategies in this section of the notes, we can only hope to cover those that are most relevant to the narrative later in the course.
  - Even though it's more than 20 years old, the LIFFE (now ICE Futures Europe) document is still a great reference for the menagerie of vanilla option strategies: www.globalriskguard.com/resources/deriv/liffe_options.pdf

3. Non-linear Payoffs: Options Markets (2)

Calls and Puts

- A call option represents the right (but not the obligation) to purchase an asset S on [or before] the maturity date T for a given (*strike*) price K.
  - Payoff at time T : $C_T = \max[S_T - K, 0] = [S_T - K]^+$
  - Think of it as the upside of a forward contract – hence the famous "hockey stick" payoff.
  - At-the-money (ATM): $K = S_t$; In-the-money (ITM): $K < S_t$; Out-of-the-money (OTM): $K > S_t$





- A put option represents the right (but not the obligation) to sell an asset S on [or before] the maturity date T for the strike price K.
  - Payoff at time T : $P_T = \max[K - S_T, 0] = [K - S_T]^+$
  - Think of it as the (negative) downside of a forward contract.
  - At-the-money (ATM): $K = S_t$; In-the-money (ITM): $K > S_t$; Out-of-the-money (OTM): $K < S_t$

3. Non-linear Payoffs: Options Markets (3)

S&P 500® Index Options

Symbol
SPX/SPXW

Underlying
The Standard & Poor's 500 Index is a capitalization-weighted index of 500 stocks from a broad range of industries. The component stocks are weighted according to the total market value of their outstanding shares. The impact of a component's price change is proportional to the issue's total market value, which is the share price times the number of shares outstanding. These are summed for all 500 stocks and divided by a predetermined base value. The base value for the S&P 500 Index is adjusted to reflect changes in capitalization resulting from mergers, acquisitions, stock rights, substitutions, etc.

Multiplier
$100.

Strike Price Intervals
Generally, $5, $10, $25, $50, $100, and $200 strike prices are available. Newly added and longer-term expirations have less granularity. Granularity is added over the life of the expiration with nearer term options having the most granularity. Standard 3rd Friday AM and PM options always have the most granularity and widest ranges.

Strike (Exercise) Prices
In-, at- and out-of-the-money strike prices are initially listed. New strikes can be added as the index moves up or down.

Premium Quote
Stated in decimals. One-point equals $100. Minimum tick for options trading below 3.00 is 0.05 ($5.00) and for all other series, 0.10 ($10.00). For complex orders legs may trade in .01 ($1.00) increments, but net package price must be in .05 ($5.00) increments except for Boxes and Box Swaps which are eligible to trade in .01 ($1.00) increments.

Exercise Style
European - SPX/SPXW options generally may be exercised only on the expiration date.

Last Trading Day
Trading in SPX options will ordinarily cease on the business day (usually a Thursday) preceding the day on which the exercise-settlement value (i.e., the expiration date) is calculated, 3:15pm CT.
Trading in SPXW options will ordinarily cease on the day of expiration, 3:00 pm CT.

Expiration Date for Standard, Weekly and EOM Option
SPX AM options expire on the third Friday of the expiration month or the immediately preceding business day if the Exchange is not open on that Friday.
SPXW PM expiring options including Monday thru Friday, EOM and EOQ settle on their expiration date. If the Exchange is not open on that Friday. SPXW options expire on a Monday, Tuesday, Wednesday, Thursday, or a Friday. If the Exchange is not open on a Tuesday, Wednesday, Thursday, or Friday, the normally expiring Tuesday, Wednesday, Thursday, or Friday Weekly will expire on the immediately preceding business day. If the Exchange is not open on a Monday, the normally expiring Monday XSP Weekly will expire on the first business day immediately following that Monday. EOM XSP options expire on the last business day of the expiration month.

- Source: CBOE (www.cboe.com/tradable_products/sp_500/spx_options/specifications/)

3. Non-linear Payoffs: Options Markets (4)

Expiration Months

Cboe may list up to twelve standard monthly expirations. Cboe may list up to ten (10) SPX LEAPS® monthly expirations at one time that expire from 12 to 180 months from the date of issuance.

For SPXW, the exchange lists 5 weeks of daily expirations for each Monday-Thursday expirations, if an EOM or EOQ falls on a Monday-Thursday it is counted towards that total. The Exchange lists 5 weeks of EOW expiring options, if a 3rd Friday, EOM or EOQ expiration falls on the EOW it is not counted towards the total. The Exchange may list series that expire at the end of the next consecutive four calendar quarters (EOQ), as well as the fourth quarter of the next calendar year. The Exchange may list 12 expirations for Monthly Options Series (EOM). Monthly Option Series expirations need not be for consecutive months; however, the expiration date of a nonconsecutive expiration may not be beyond what would be considered the last expiration date if the maximum number of expirations were listed consecutively.

Settlement of Options Exercise

Exercise will result in delivery of cash on the business day following expiration. The exercise-settlement value, SET, is calculated using the opening sales price in the primary market of each component security on the expiration date. The exercise-settlement amount is equal to the difference between the exercise-settlement value and the exercise price of the option, multiplied by $100.

SPXW exercise will result in delivery of cash on the business day following expiration. The exercise-settlement value is calculated using the closing sales price in the primary market of each component security on the expiration date. The exercise-settlement amount is equal to the difference between the exercise-settlement value and the exercise price of the option, multiplied by $100.

Position and Exercise Limits

No position and exercise limits are in effect.

Margin

Purchases of puts or calls with 9 months or less until expiration must be paid for in full. Writers of uncovered puts or calls must deposit / maintain 100% of the option proceeds* plus 15% of the aggregate contract value (current index level x $100) minus the amount by which the option is out-of-the-money, if any, subject to a minimum for calls of option proceeds* plus 10% of the aggregate contract value and a minimum for puts of option proceeds* plus 10% of the aggregate exercise price amount. (*For calculating maintenance margin, use option current market value instead of option proceeds.) Additional margin may be required pursuant to Exchange Rule 12.10.

Cusip Number

648815

Trading Hours

Regular Hours

8:30 a.m. to 3:15 p.m. (CT)

Note SPXW expiring options trade until 3:00 p.m. (CT) on their expirations.

Curb

3:15 p.m. to 4:00 p.m. (CT)

Global Trading Hours

7:15 p.m. to 8:25 a.m. (CT)

*Position and Exercise limits are subject to change.

- Source: CBOE (www.cboe.com/tradable_products/sp_500/spx_options/specifications/)

3. Non-linear Payoffs: Options Markets (5)

SP 500 INDEX (^spx)

2025-01-15 16:14:59 ET (Delayed)

Bid: **5880.9199** Ask: **6003.5601** Vol: **0** Last: **5,949.91** Change: **+107.0002 (+1.8313%)**

Filters By:

Volume: All | Expiration Type: All | Options Range: Near the Money | Size: 3 | Expiration: 2025 January | View Chain

Options Chain

Total Records: 72

Wed Jan 15 2025

Calls Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Puts Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
14.7	+12.2	14.5	15.7	25,760	0	0	0	1872	SPXW 5935.000	0.05	-113.25	0	0.05	46,250	0	0	0	285
9.5	+7.4	9.1	11.1	43,081	0	0	0	2481	SPXW 5940.000	0.05	-117.7	0	0.05	46,473	0	0	0	238
4.8	+3	4.1	5.6	38,426	0	0	0	2109	SPXW 5945.000	0.05	-122.4	0	0.05	45,159	0	0	0	736
0.35	-1.125	0.2	0.75	97,344	0	0	0	5081	SPXW 5950.000	0.65	-126.5	0.55	0.7	61,433	0	0	0	2273
0.05	-1.2	0	0.05	64,869	0	0	0	1071	SPXW 5955.000	5	-126.9	4.5	5.7	27,889	0	0	0	78
0.05	-1	0	0.05	100,117	0	0	0	2419	SPXW 5960.000	9.93	-126.72	9.5	11.3	11,436	0	0	0	112

Fri Jan 17 2025

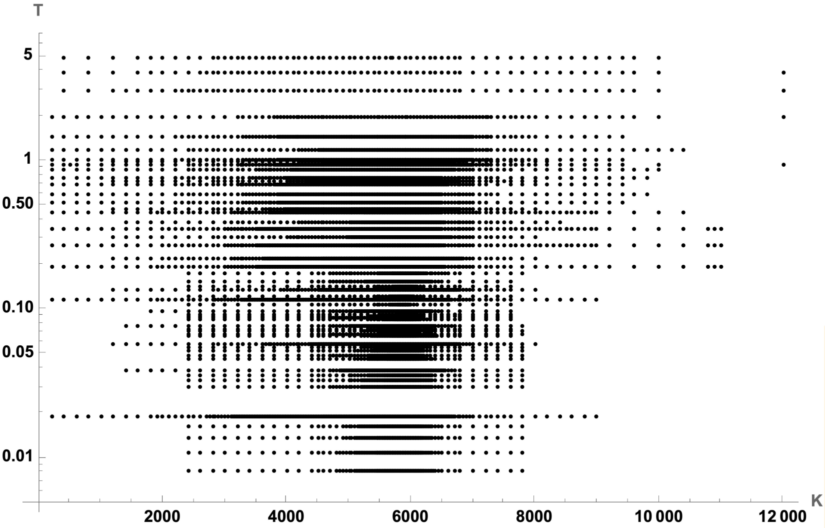
Calls Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Puts Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
35.1	+27.3	31.9	32.3	951	0.15	0.5675	0.0058	1082	SPXW 5940.000	21.99	-99.35	22.2	22.6	941	0.16	-0.4323	0.0058	573
26.38	+21.03	25.3	25.9	389	0.14	0.546	0.0068	1633	SPX 5945.000	20.89	-102.45	20.7	21.3	299	0.14	-0.4539	0.0068	497
29.8	+22.8	29	29.4	1,089	0.15	0.5387	0.0059	2646	SPXW 5945.000	24.4	-99.85	24.3	24.7	533	0.15	-0.4611	0.0059	445
23.52	+18.64	22.6	23.1	28,446	0.14	0.5118	0.0069	26624	SPX 5950.000	22.9	-105.29	22.9	23.5	28,094	0.14	-0.488	0.0069	32348
26.78	+21.05	26.3	26.7	5,467	0.15	0.5092	0.006	2975	SPXW 5950.000	26.52	-103.4	26.6	26.9	3,734	0.15	-0.4906	0.006	2571
21.4	+15.9	20	20.6	445	0.14	0.477	0.007	447	SPX 5955.000	25.3	-106.8	25.3	26	170	0.14	-0.5228	0.007	423

Jan 2025 | Feb 2025 | Mar 2025 | Apr 2025 | May 2025 | Jun 2025 | Jul 2025 | Aug 2025
Sep 2025 | Oct 2025 | Nov 2025 | Dec 2025 | Jan 2026 | Mar 2026 | Jun 2026 | Dec 2026
Dec 2027 | Dec 2028 | Dec 2029

- Source: CBOE, 2025-01-15 (www.cboe.com/delayed_quotes/spx/quote_table)

3. Non-linear Payoffs: Options Markets (6)

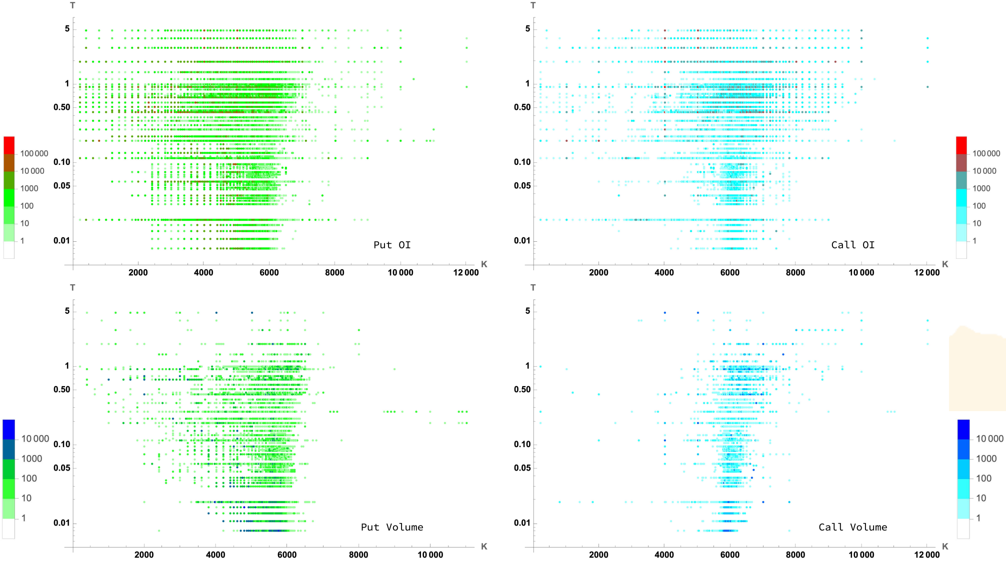
Listed SPX Options by Strike (K) & Maturity (T, years)



- Source: CBOE, 2025-01-10 (www.cboe.com/delayed_quotes/spx/quote_table)

3. Non-linear Payoffs: Options Markets (7)

Listed SPX Options Open Interest & Volume by Strike (K) & Maturity (T, years)



- Source: CBOE, 2025-01-10 (www.cboe.com/delayed_quotes/spx/quote_table)

3. Non-linear Payoffs: Options Markets (8)

Equity Options Product Specifications

Symbol

The option symbols are the same as for the underlying equity security. Visit the Cboe Symbol Directory for specific symbols.

Underlying

Generally, 100 shares of the underlying equity security.

Strike Price Intervals

Generally, 2 1/2 points when the strike price is between $5 and $25, 5 points when the strike price is between $25 and $200, and 10 points when the strike price is over $200. Strikes are adjusted for splits, re-capitalizations, etc.

Strike (Exercise) Prices

In-, at- and out-of-the-money strike prices are initially listed. New series are generally added when the underlying trades through the highest or lowest strike price available.

Premium Quotation

Stated in decimals. One point equals $100. Generally, minimum tick for options trading below $3 is $0.05 and for all other series, $0.10. For classes participating in the *Penny Pilot Program*, the minimum tick for options trading below $3 is $0.01 and $0.05 for options trading at $3 or above.

Expiration Date

The third Friday of the expiration month.

Expiration Months

Two near-term months plus two additional months from the January, February or March quarterly cycles.

Exercise Style

American - Equity options generally may be exercised on any business day up to and including on the expiration date.

Settlement of Option Exercise

Exercise notices properly tendered on any business day will result in delivery of the underlying stock on the second business day following exercise.

- Source: CBOE (www.cboe.com/exchange_traded_stock/equity_options_spec/)

3. Non-linear Payoffs: Options Markets (9)

Apple Inc. (aapl) Bid: **238.27** Ask: **238.48** Vol: **39,761,757** Last: **238.45** Change: **+4.59 (+1.9676%)**

2025-01-15 16:00:00 ET (Delayed)

Filters By:

Volume: All | Expiration Type: All | Options Range: Near the Money | Size: 3 | Expiration: 2025 January | View Chain

Options Chain

Total Records: 18

Calls — Fri Jan 17 2025 — Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
5.67	+2.875	5.6	5.85	2,366	0.26	0.8792	0.0423	4933	AAPL 232.500	0.27	-1.875	0.27	0.28	10,941	0.27	-0.1214	0.0426	5983
3.6	+2.025	3.55	3.7	23,006	0.25	0.7521	0.0715	44493	AAPL 235.000	0.64	-2.81	0.63	0.66	37,360	0.25	-0.2502	0.0727	27439
1.93	+1.155	1.88	1.94	34,249	0.24	0.5474	0.0946	10238	AAPL 237.500	1.47	-3.68	1.41	1.45	19,148	0.24	-0.4599	0.0973	5108
0.78	+0.445	0.78	0.82	63,913	0.23	0.3126	0.087	58542	AAPL 240.000	2.87	-4.43	2.78	2.89	5,816	0.23	-0.7017	0.0893	33028
0.28	+0.145	0.26	0.28	21,967	0.23	0.1343	0.0537	11625	AAPL 242.500	4.98	-4.645	4.7	5	862	0.22	-0.884	0.0547	7884
0.07	+0.005	0.08	0.09	15,685	0.23	0.0457	0.0235	49846	AAPL 245.000	7.4	-4.675	6.95	7.4	1,841	0	-0.9721	0.0216	14870

Calls — Fri Jan 24 2025 — Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
6.6	+2.55	6.6	6.85	652	0.21	0.7736	0.0388	1494	AAPL 232.500	1.05	-2.15	1.02	1.05	6,088	0.21	-0.2295	0.0394	3270
4.75	+1.99	4.75	4.9	4,342	0.2	0.6698	0.0485	5339	AAPL 235.000	1.65	-2.8	1.62	1.68	7,339	0.2	-0.3351	0.0493	7214
3.2	+1.41	3.2	3.3	6,654	0.19	0.5419	0.0547	3470	AAPL 237.500	2.63	-3.37	2.54	2.6	2,406	0.19	-0.4654	0.0556	1270
1.98	+0.88	1.98	2.03	11,152	0.19	0.404	0.0546	5185	AAPL 240.000	3.95	-3.875	3.75	3.95	3,082	0.19	-0.6059	0.0555	3272
1.15	+0.5	1.13	1.18	12,620	0.19	0.2741	0.0478	3049	AAPL 242.500	5.5	-4.45	5.35	5.65	365	0.19	-0.7386	0.049	474
0.63	+0.245	0.62	0.64	11,796	0.19	0.1697	0.0364	8204	AAPL 245.000	7.5	-4.75	7.25	7.8	607	0.19	-0.8478	0.0391	2149

Jan 2025 | Feb 2025 | Mar 2025 | Apr 2025 | May 2025 | Jun 2025 | Jul 2025 | Aug 2025
Sep 2025 | Oct 2025 | Dec 2025 | Jan 2026 | Jun 2026 | Dec 2026 | Jan 2027

- Source: CBOE, 2025-01-15 (www.cboe.com/delayed_quotes/aapl/quote_table)

3. Non-linear Payoffs: Options Markets (10)

Calls and Puts (continued)

- Equity index options can usually only be exercised at maturity T (*European Style*).
- Single stock options can usually be exercised at any time up to T (*American Style*).
- For FX and commodities, exercise style varies:
  - For FX and commodities with a robust spot market, exercise is American
  - Otherwise, where primary underlying asset is a (near) futures contract, exercise is typically European (and occasionally American).
- In some OTC markets or for some corporate securities, exercise is at a discrete set of dates up to T (*Bermud[i]an Style*).

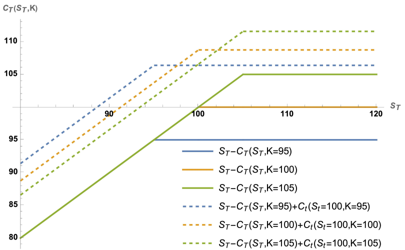
- Of interest in equity markets: Triple witching expirations
  - Days (3rd Fridays) on which single-stock options, index futures, and index options all expire.
  - Four times a year (quarterly expiries): third Friday of March, June, September, and December.

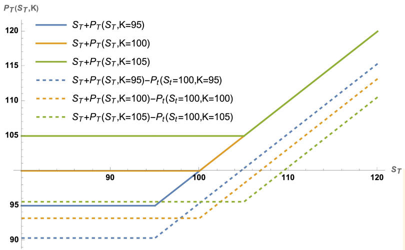
- ATM conventions
  - In equity markets, ATM refers to at-the-money relative to Spot S_t (ATMS).
  - In currency markets, ATM refers to at-the-money relative to Forward $F_{t,T}$ (ATMF).
  - ITM, OTM conventions follow those for ATM.
  - Intrinsic value at t: $[S_t - K]^+$ for calls; $[K - S_t]^+$ for puts; map $S_t \mapsto F_{t,T}$ for FX

3.1. Option Strategies

Stock ± Option Strategies: Covered Calls and Protective Puts

- Covered call strategy: long stock, short a (usually OTM) call.
  - Payoff at time $T : S_T - \max[S_T - K, 0] = S_T - [S_T - K]^+$ (plus any dividends)
  - Call over-writing strategies are very common buy-side "yield enhancement" techniques
  - Buy-side is generally short OTM calls:
    Implied volatilities for $K > S_t$ (and hence OTM call prices) are reduced relative to ATM



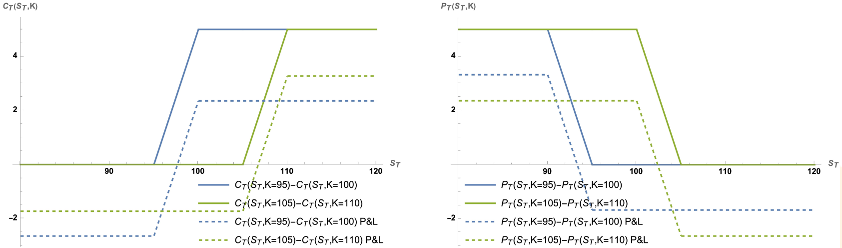


- Protective put strategy: long stock, long a (usually OTM) put.
  - Payoff at time $T : S_T + \max[K - S_T, 0] = S_T + [K - S_T]^+$ (plus any dividends)
  - Simplest form of *portfolio insurance*
  - Buy-side is generally long OTM puts:
    Implied volatilities for $K < S_t$ (and hence OTM put prices) are increased relative to ATM

3.1. Option Strategies (2)

Long/Short Option Strategies: Call and Put Spreads

- Bull spread, vertical call spread, or capped call:
  Long call with K_1 (usually near ATM), short call with $K_2 > K_1$, both with same maturity T
  - Payoff at time T : $\max[S_T - K_1, 0] - \max[S_T - K_2, 0] = [S_T - K_1]^+ - [S_T - K_2]^+$
  - Maximum payoff = $K_2 - K_1$: capture market gains over a range, at reduced cost relative to a single option
  - Valuation bounds: payoff is non-negative everywhere, so call option values must be *decreasing* in strike.

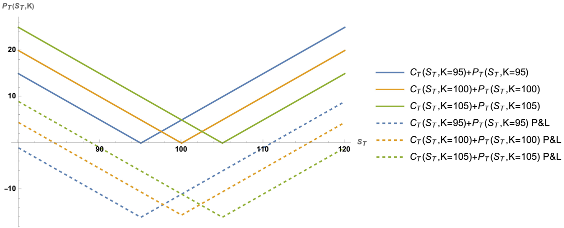


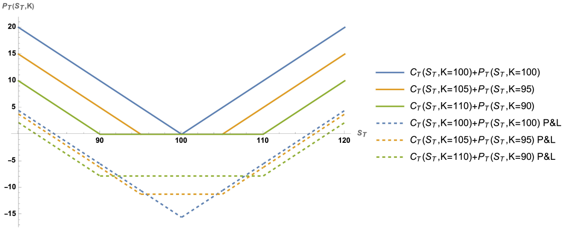
- Bear spread, vertical put spread, or capped put:
  Long put with K_1 (usually near ATM or OTM), short put with $K_2 < K_1$, both with same maturity T
  - Payoff at time T : $\max[K_1 - S_T, 0] - \max[K_2 - S_T, 0] = [K_1 - S_T]^+ - [K_2 - S_T]^+$
  - Maximum payoff = $K_1 - K_2$: capture market losses over a range, at reduced cost relative to a single option
  - Valuation bounds: payoff is non-negative everywhere, so put option values must be *increasing* in strike.

3.1. Option Strategies (3)

Both Long (or Short) Call + Put Strategies: Straddles and Strangles

- Straddle: long a call & a put, with identical strike K & maturity T, usually at (or near) the money
  - Payoff at time T : $\max[S_T - K, 0] + \max[K - S_T, 0] = [S_T - K]^+ + [K - S_T]^+$
  - Generally not a directional or "first-moment" play
    - Instead, used to express views on volatility, second-moment, or large vs. short moves. E.g., long straddle: buy vol cheaply, wait until implied volatility rises or market moves away from ATM, then close out position



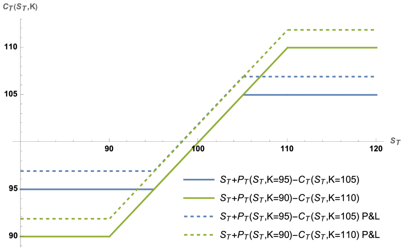


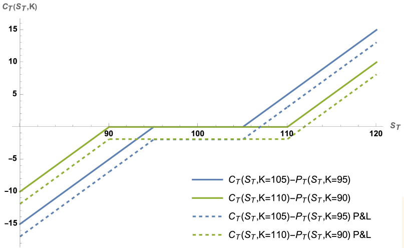
- Strangle: long call with strike K_c & put with strike $K_p < K_c$, but same maturity T, again, usually at (or near) the money
  - Payoff at time T : $\max[S_T - K_c, 0] + \max[K_p - S_T, 0] = [S_T - K_c]^+ + [K_p - S_T]^+$
  - Conceptually similar to a straddle, but requiring less premium up-front... and larger moves away from ATM to lock in profit

3.1. Option Strategies (4)

Long/Short Call/Put (Risk Reversal) Strategies: Collars and Reverse Collars

- Collar or range forward strategy: long stock, long a (usually OTM) put, short a (usually OTM) call.
  - Payoff at time T : $S_T + [K_p - S_T]^+ - [S_T - K_c]^+$ (plus any dividends)
  - Combination of protective put and covered call strategies: allow gains or losses within a limited range
  - Call and put strikes are often selected so that initial value of the collar overlay is zero: *zero-cost collar*



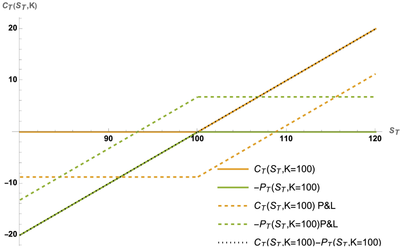


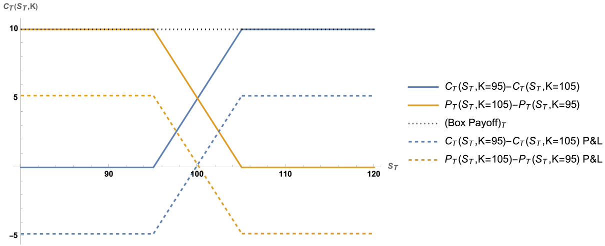
- Reverse collar or participating forward: short a (usually OTM) put, long a (usually OTM) call
  - Payoff at time T : $[S_T - K_c]^+ - [K_p - S_T]^+$
  - Lock-in a future price for small moves; participate in gains or losses for larger moves.
  - Like zero-cost collar, call and put strikes are often selected so that initial value of the position is zero.
  - Like many other strategies, can combine with a zero-coupon bond (ZCB) to create a structured note (in this case, *not* principal-protected)

3.1. Option Strategies (5)

Long Call – Short Put (Conversion) Strategies: Synthetic Forwards and Boxes

- Conversion or synthetic forward: long a call, short a put, both with the same strike K & maturity T
  - Payoff at time T : $C_T - P_T = \max[S_T - K, 0] - \max[K - S_T, 0] = S_T - K$
  - Hence (long) a European Call & (short) a European Put is a *synthetic forward* contract with forward price K.
  - Valuation by the same replication argument as for forwards: $C_t - P_t = S_t - e^{-r(T-t)}K$ (if no divs)
  - With a continuous yield y: $C_t - P_t = e^{-y(T-t)}S_t - e^{-r(T-t)}K$ **(European) Put-Call parity relationship**
  - Does Put-Call parity hold in reality?
    We'll look at some data when we study the option price (implied volatility) surface in detail.



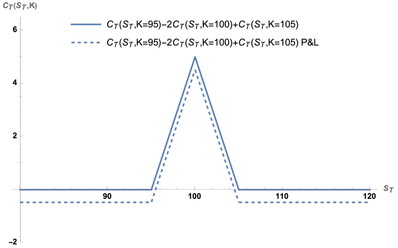


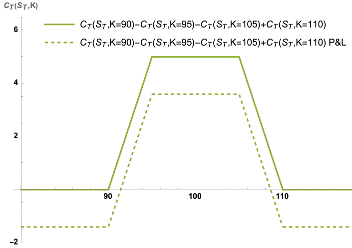
- Box: long synthetic forward of strike K_1, short synthetic forward of strike K_2 (same maturity T)
  - Equivalent to long $\{K_1, K_2\}$ call spread and long $\{K_2, K_1\}$ put spread
  - Payoff at time T : $(S_T - K_1) - (S_T - K_2) = K_2 - K_1$
  - Objective: lock-in differential mispricings of call & put spreads (or conversions)

3.1. Option Strategies (6)

Butterflies and Condors

- Butterfly Call Spread: long 1 call struck at $K-\Delta K$, short 2 calls at K, long 1 call at $K+\Delta K$.
  - Butterfly Put Spread: long 1 put struck at $K+\Delta K$, short 2 puts at K, long 1 put at $K-\Delta K$.
  - Also equivalent to long the $(K-\Delta K, K)$ call spread, short the $(K, K+\Delta K)$ call spread
  - Payoff at time T : $[S_T-(K-\Delta K)]^+ - 2[S_T-K]^+ + [S_T-(K+\Delta K)]^+$
  - Similar to argument for bull and bear spreads, payoff is everywhere non-negative $\Rightarrow$ present value of a butterfly is also non-negative ($\Rightarrow$ option prices are *convex* in strike)





- Condor: similar to Butterfly, but short 2 calls at $K \to$ short 1 at K, short another at $K' > K$
  - Final long call strike is also shifted: $K+\Delta K \mapsto K'+\Delta K$
  - Equivalent put portfolio can again be constructed by reversing the order of the strikes
  - Also equivalent to long the $(K-\Delta K, K)$ call spread, short the $(K', K'+\Delta K)$ call spread
  - Same arguments hold as for butterflies, but resulting constraints (and P&L) are not as tight.

3.1. Option Strategies (7)

Calendar (Horizontal) Spreads

- Long a call (or put) with strike K and maturity T_2, short the corresponding call (or put) (usually) with strike K and maturity $T_1 < T_2$
- Not usually a directional position, except in a divergence or mean-reversion sense.
- Generally used to gain exposure to volatility between T_1 and T_2, though more modern instruments (e.g., variance/volatility swaps) allow this to be done in a more precise way
- Under certain conditions, it's possible to say that the T_2-maturity option must have greater value than the T_1-maturity option $\Rightarrow$ potential arbitrage strategies.
- Also *diagonal spreads*: going long vs. short options of differing strikes and maturities.

3.2. Option Valuation: Bounds & Constraints

- Easy enough to say that calls and puts, as rights but not obligations to buy (or sell) the underlying asset for a given strike price K, must have non-negative values.
- Similar conclusions for values of vertical spreads and butterflies:
  - Bull/bear spreads must have non-negative value
    $\Rightarrow$ calls must be decreasing in K and puts must be increasing in K
  - Butterfly spreads must have non-negative value
    $\Rightarrow$ call and put values must be convex in K
- Another set of fairly simple relationships:
  (American option value) $\geq$ (Bermud[i]an option value) $\geq$ (European option value)
  - What arbitrage trade could we put on if any of these inequalities were violated?
- Can we say a bit more, particularly about option values *per se*?

3.2. Option Valuation: Bounds & Constraints (2)

Upper Bounds

- Calls:
  - American: $C_t(S, K, T) < S_t$ (why?)
  - European: $C_t(S, K, T) < e^{-y(T-t)} S_t$ (why?)
- Puts:
  - American: $P_t(S, K, T) < K$ (why?)
  - European: $C_t(S, K, T) < e^{-r(T-t)} K$ (why?)

Lower Bounds

- Calls:
  - American: $C_t(S, K, T) \geq \max[S_t - K, 0]$ (why?)
    - Corollary: American calls are increasing in T
  - European: $C_t(S, K, T) \geq \max[e^{-y(T-t)} S_t - e^{-r(T-t)} K, 0]$ (why?)
    - Corollary: American calls on a non-dividend paying stock should never be exercised early.
    - Corollary: European calls on a non-dividend paying stock are increasing in T
- Puts:
  - American: $P_t(S, K, T) \geq \max[K - S_t, 0]$ (why?)
    - Corollary: American puts are increasing in T
  - European: $P_t(S, K, T) \geq \max[e^{-r(T-t)} K - e^{-y(T-t)} S_t, 0]$ (why?)
    - No analogous corollaries can be deduced for put options (unless $r = 0$).

3.3. Option Valuation: Binomial Model

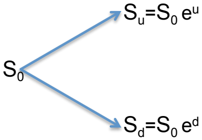
Binomial Model Overview

- Binomial Model starts from viewpoint of discrete approximation of the asset price process S_t (or, equivalently, the underlying Brownian motion). Over each (time-)step Δt:
  - Define the set of two nodes $\{S_{t+\Delta t}\} = \{S_{t+\Delta t,u}, S_{t+\Delta t,d}\}$ reachable from S_t over Δt
  - Apply replication argument to determine risk-neutral probabilities of reaching the two nodes
  - Calculate expected payoff $\mathbb{E}[\{C(S_{t+\Delta t}, t+\Delta t)\}|S_t]$ using the risk-neutral probabilities
  - Discount at risk-neutral rate over Δt to obtain $C(S_t, t)$
  - Linkage to hedging strategy clear:
    Risk-neutral probabilities of reaching $\{S_{t+\Delta t}\}$ from S_t are induced by no-arbitrage condition
- Historical background
  - Seminar on Financial Decision-making under Uncertainty, sponsored by The Israel National Council for Research and Development, Ein Bokek (Dead Sea beach resort), March 1975.
  - Two years after Black-Scholes (& Merton), discussion emphasized how difficult it was to build intuition around the continuous time theory and results.
  - Comment attributed to Bill Sharpe: essentially, “What if there are effectively just two states of nature spanned by the stock and bond?”
  - Resulting paper: John Cox, Stephen Ross, and Mark Rubinstein, “Option pricing: A simplified approach,” *Journal of Financial Economics* **7**(3): 229-63, (September 1979). doi.org/10.1016/0304-405X(79)90015-1.

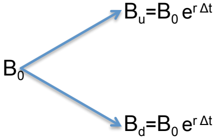
3.3. Option Valuation: Binomial Model (2)

Binomial Model Framework

- $1+1$ element (vector) stochastic process framework, with assets $\{S, B\}$
- Two possible outcomes for the risky asset price S over time Δt:



- Assume probabilities of p_u for the up state and $p_d = 1 - p_u$ for the down state
  - We shall see that we don't care about and don't even need to know $\{p_u, p_d\}$.
- Assume the riskless bond or money-market account B grows at the continuously-compounded rate r over Δt. Then for the two states of nature $\{u, d\}$:



3.3. Option Valuation: Binomial Model (3)

Binomial Model Framework, continued

- Assume a derivative C that pays off C_u in the up state and C_d in the down state. What is the value C_0 of C today?

$$
\begin{array}{ccl}
 & \nearrow & C_u = \Delta_S S_u + \Delta_B B_u \\
C_0(?) & & \\
 & \searrow & C_d = \Delta_S S_d + \Delta_B B_d
\end{array}
$$

- Basic idea of Cox-Ross-Rubinstein (CRR) model:
  - The two assets $\{S, B\}$ suffice to span all states of nature (market is complete), so the payoffs of C can be replicated by a portfolio of $\{\Delta_S, \Delta_B\}$ units of the two, respectively $\Rightarrow$ C is *redundant*.
  - By the Law Of One Price (LOOP: absence of arbitrage), the value C_0 of C today must be equal to the value of the replicating portfolio today: $$C_0 = \Delta_S S_0 + \Delta_B B_0$$
- What are $\{\Delta_S, \Delta_B\}$? Solve the replicating equations:

$$
\begin{pmatrix} S_u & B_u \\ S_d & B_d \end{pmatrix} \cdot \begin{pmatrix} \Delta_S \\ \Delta_B \end{pmatrix} = \begin{pmatrix} C_u \\ C_d \end{pmatrix}
$$

$$
\Longrightarrow \quad \begin{pmatrix} \Delta_S \\ \Delta_B \end{pmatrix} = \frac{1}{S_u B_d - S_d B_u} \begin{pmatrix} B_d & -B_u \\ -S_d & S_u \end{pmatrix} \cdot \begin{pmatrix} C_u \\ C_d \end{pmatrix} = \frac{1}{S_u B_d - S_d B_u} \begin{pmatrix} B_d C_u - B_u C_d \\ S_u C_d - S_d C_u \end{pmatrix}
$$

$$
= \frac{1}{S_0 (e^u - e^d)\, B_0 e^{r \Delta t}} \begin{pmatrix} B_0 e^{r \Delta t} (C_u - C_d) \\ S_0 e^u C_d - S_0 e^d C_u \end{pmatrix}
$$

3.3. Option Valuation: Binomial Model (4)

Binomial Model Framework, continued

- Substitute to obtain C_0:

$$C_0 = \frac{C_u - C_d}{e^u - e^d} - e^{-r\Delta t}\frac{e^d C_u - e^u C_d}{e^u - e^d} = \frac{1 - e^{d-r\Delta t}}{e^u - e^d}C_u + \frac{e^{u-r\Delta t} - 1}{e^u - e^d}C_d$$

$$= e^{-r\Delta t}\left(\frac{e^{r\Delta t} - e^d}{e^u - e^d}C_u + \frac{e^u - e^{r\Delta t}}{e^u - e^d}C_d\right) = e^{-r\Delta t}(q_u C_u + q_d C_d) \text{ with } q_u + q_d = 1$$

- Comment 1: Interpretation of $\{q_u, q_d\}$ as probabilities (existence of equivalent martingale measure or EMM) holds as long as $u > r\Delta t > d$, i.e. neither asset dominates the other. (First Law: absence of arbitrage)
- Comment 2: Uniqueness of q-measure is independent of and indeed requires no knowledge (or uniqueness) of p-measure. What is required is the (unique) hedging strategy that allows us to span the set of possible payoffs. (Second Law: complete markets)
- Comment 3: To modify for a (re-invested) dividend yield y, we must recognize that Δ_S shares at time Δt are equal to $e^{-y\Delta t}\Delta_S$ shares today. Then:

$$C_0 = e^{-y\Delta t}\frac{C_u - C_d}{e^u - e^d} - e^{-r\Delta t}\frac{e^d C_u - e^u C_d}{e^u - e^d} = \frac{e^{-y\Delta t} - e^{d-r\Delta t}}{e^u - e^d}C_u + \frac{e^{u-r\Delta t} - e^{-y\Delta t}}{e^u - e^d}C_d$$

$$= e^{-r\Delta t}\left(\frac{e^{(r-y)\Delta t} - e^d}{e^u - e^d}C_u + \frac{e^u - e^{(r-y)\Delta t}}{e^u - e^d}C_d\right) = e^{-r\Delta t}(q_u C_u + q_d C_d) \text{ with } q_u + q_d = 1$$

3.3. Option Valuation: Binomial Model (5)

Binomial Model Framework, continued

- Comment 4: Existence & uniqueness hold even if B has stochastic returns as long as returns of B & S are not collinear $(S_uB_d \neq S_dB_u)$ and neither asset dominates the other. Then:

$$\begin{aligned} C_0 = S_0\Delta_S + B_0\Delta_B &= S_0\frac{B_uC_d - B_dC_u}{B_uS_d - B_dS_u} + B_0\frac{S_uC_d - S_dC_u}{S_uB_d - S_dB_u} \\ &= \frac{B_0S_d - B_dS_0}{B_uS_d - B_dS_u}C_u + \frac{B_uS_0 - B_0S_u}{B_uS_d - B_dS_u}C_d \\ &= \frac{\frac{B_u - B_d}{B_0} - \frac{S_u - S_d}{S_0}}{\frac{B_u}{B_0}\frac{S_d}{S_0} - \frac{B_d}{B_0}\frac{S_u}{S_0}}\left(\frac{\frac{S_d}{S_0} - \frac{B_d}{B_0}}{\frac{B_u - B_d}{B_0} - \frac{S_u - S_d}{S_0}}C_u + \frac{\frac{B_u}{B_0} - \frac{S_u}{S_0}}{\frac{B_u - B_d}{B_0} - \frac{S_u - S_d}{S_0}}C_d\right), \text{ etc.} \end{aligned}$$

  - Interpretation of probabilities is reasonably straightforward; interpretation of discount factor not so easy (it's the present value of the synthetic "riskless" security paying off $\{\$1, \$1\}$)
  - We don't always need to have a riskless numeraire asset to be able to derive option prices (and hedges). E.g.: outperformance option.

3.3. Option Valuation: Binomial Model (6)

Binomial Model $\Delta t \searrow 0$ Limit

- How do we adapt this model for use as a numerical method that represents the process $dS/S = (r-y)dt + \sigma dW$ and in particular converges to the distribution of S_T as $\Delta t \searrow 0$?
- Use idea that we can represent a Brownian motion as the limit of a Bernoulli (binomial) up/down process, with increments ΔW over Δt:
  - $\mathbb{E}[\Delta W] = 0$
  - $\mathbb{E}[(\Delta W)^2] = \text{var}[\Delta W] = \Delta t \Rightarrow \text{std}[\Delta W] = \sqrt{\Delta t}$
- These conditions imply that $\Delta W = \pm\sqrt{\Delta t}$ + higher order terms in $\sqrt{\Delta t}$, i.e.: $\Delta W = \pm\sqrt{\Delta t} + \alpha \cdot \Delta t + \ldots$
- Assuming equal probability weights (1/2) for up and down moves:
  1. Divide T into m increments of length $\Delta t = T/m$;
  2. Take $\Delta W_j = \pm\sqrt{\Delta t}$, each with probability $1/2, \forall j = 1, \ldots, m$;
  3. Let $W_m = \sum_{j=1}^{m} \Delta W_j$;
  4. Take the limit $m \to \infty$.

3.3. Option Valuation: Binomial Model (7)

Binomial Model $\Delta t \searrow 0$ Limit, continued

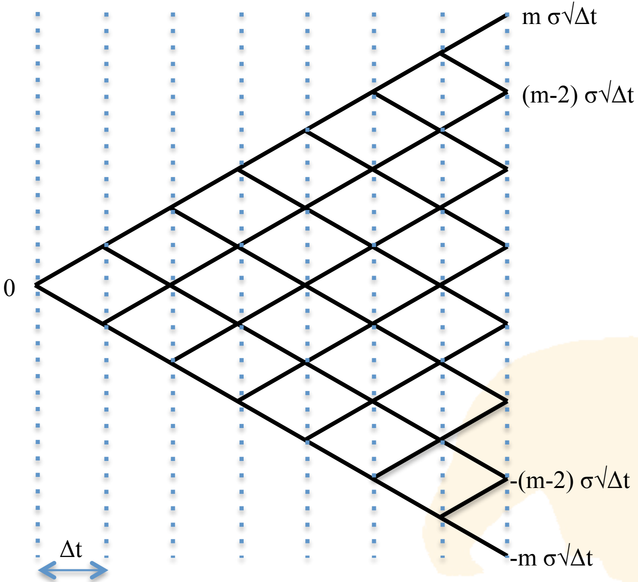
- Just as an Itô process for an asset price transforms a Brownian increment into an increment of the asset price, we (may) need to transform our Bernoulli increments to obtain a discrete approximation that converges to the desired price process as $\Delta t \searrow 0$.
- Log-normal model: asset prices look like $S_0 e^{\sigma W_t}$, which we approximate as $\{S_0 e^u, S_0 e^d\}$
- Hence, $u \sim \sigma\sqrt{\Delta t}$, $d \sim -\sigma\sqrt{\Delta t}$. Choosing these specific values leads to the simplest (and probably most widely used) version of the binomial model, with:

$$q_u = \frac{e^{(r-y)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \sim \frac{1}{2}\left[1 + \frac{(r-y-\sigma^2/2)\Delta t}{\sigma\sqrt{\Delta t}} + \mathcal{O}(\Delta t^{3/2}) + \dots\right] = \frac{1}{2} + \mathcal{O}(\Delta t^{1/2})$$

  - At the m^{th} time step, we will have $m+1$ nodes, with the maximum and minimum nodes at $S_0 e^{\pm m\sigma\sqrt{\Delta t}} = S_0 e^{\pm\sqrt{m}\sigma\sqrt{T}}$: potential to “prune” tree beyond some large multiple of $\sigma\sqrt{T}$
  - Advantage of this method: lattice/grid points always at integer multiples of $\sigma\sqrt{\Delta t}$, so path-dependence (tracking of barriers, extrema, averages...) can be handled cleanly

3.3. Option Valuation: Binomial Model (8)

Binomial Grid in Log-Price Space



3.3. Option Valuation: Binomial Model (9)

Binomial Model $\Delta t \searrow 0$ Limit, continued

- This solution is not unique. Indeed, almost any (reasonably bounded) choice of α in: $u = \sigma\sqrt{\Delta t} + \alpha\Delta t$, $d = -\sigma\sqrt{\Delta t} + \alpha\Delta t$ (drifting lattice) will work.
  Then: $q_u = \frac{e^{(r-y)\Delta t} - e^{-\sigma\sqrt{\Delta t}+\alpha\Delta t}}{e^{\alpha\Delta t}(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}})} \sim \frac{1}{2}\left[1 + \frac{(r-y-\alpha-\sigma^2/2)\Delta t}{\sigma\sqrt{\Delta t}} + \mathcal{O}(\Delta t^{3/2}) + \ldots\right]$
- Two obvious choices of drifty lattice:
  - $\alpha = r-y$ (drifting with the forward): $q_u = \frac{1}{2} - \frac{\sigma\sqrt{\Delta t}}{4} + \frac{\sigma^3\Delta t^{3/2}}{48} + \mathcal{O}(t^{5/2}) + \ldots$
    - Natural interpretation of lattice evolution
  - $\alpha = r-y-\frac{\sigma^2}{2}$ (drifting with the log-forward): $q_u = \frac{1}{2} + \frac{\sigma^3\Delta t^{3/2}}{24} + \mathcal{O}(t^{5/2}) + \ldots$
    - Only slightly less natural interpretation of lattice evolution.
    - Superior theoretical convergence for smooth payoffs... but that may not be the limiting factor for convergence...

3.3. Option Valuation: Binomial Model (10)

Binomial Model $\Delta t \searrow 0$ Limit, continued

- General valuation approach: backward induction of values at $j\Delta t$ from those at $(j+1)\Delta t$
  - $j\Delta t$ values determined by probability-weighting those at $(j+1)\Delta t$, then discounting by $e^{-r\Delta t}$
- Generic algorithm:
  for $(k=-m, k\leq m, k{+}{=}\ 2)$ $C[m\Delta t, S_k = S_0 e^{k\sigma\sqrt{\Delta t}}]$ = payoff condition $C_T(S_k)$
  for $(j=m-1, j\geq 0, j{-}{-})$
  for $(k=-j, k\leq j, k{+}{=}\ 2)$ $C[j\Delta t, S_k] = e^{-r\Delta t}\left(q_u C[(j+1)\Delta t, S_{k+1}] + q_d C[(j+1)\Delta t, S_{k-1}]\right)$
- With some careful bookkeeping and a bit of clever indexing to take advantage of vectorized functionality, this can be made pretty efficient
  - Nevertheless, the required computational effort/time is of order $\mathcal{O}(m^2)$
- For European payoffs, there may not be a reason to do full time-stepping.

  Rather, just recognize that at maturity (time step m), the (risk-neutral) probability of state:
  $ku+(m-k)d = (2k-m)\sigma\sqrt{\Delta t} + m\alpha\Delta t$, $0\leq k\leq m$, is:

$$\frac{m!}{k!(m-k)!} q_u^k q_d^{m-k} = \binom{m}{k} q_u^k q_d^{m-k} = q_d^m \binom{m}{k}\left(\frac{q_u}{q_d}\right)^k$$

- Hence, European option valuation can be written as:

$$C(0\Delta t, S_0) = e^{-rT} q_d^m \sum_{k=0}^{m} \binom{m}{k}\left(\frac{q_u}{q_d}\right)^k C_T\left(S_k = S_0 e^{(2k-m)\sigma\sqrt{\Delta t}+m\alpha\Delta t}\right),$$

  requiring $\mathcal{O}(m)$ computational effort.

3.3. Option Valuation: Binomial Model (11)

American/Bermudian option valuation

- Initialize lattice with terminal payoff values $C_T(S_k)$ as before
- Proceed with backward induction, calculating *continuation* values $C(j\Delta t, S_k)$ at time $j\Delta t$ by probability-weighting those at $(j+1)\Delta t$, then discounting by $e^{-r\Delta t}$
- Additional step(s) at a time $j\Delta t$ at which exercise is permitted:
  - Identify range of nodes S_k for which exercise value (e.g., $[S_k - K]^+$) $>$ continuation value $C(j\Delta t, S_k)$
  - At each such node, replace continuation value by exercise value
- Continue iterating backward in time.
- Again, there are tweaks to make this algorithm reasonably efficient but it almost inherently requires computational effort/time of order $\mathcal{O}(m^2)$

3.3. Option Valuation: Binomial Model (12)

American Put Illustration

- $S_0 = K = 100, T = 1.0, r = 0.04, y = 0.02, \sigma = 0.2, m = 100$