Week 3: Stochastic Differential Equations

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I. Stochastic Differential equations (SDE)

- Is there a stochastic process X_t that satisfies the following stochastic differential equation

$$\begin{aligned} dX_t &= \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \\ X_0 &= x_0 \end{aligned}$$

- To be more precise, the above notation is shorthand for the following:

$$X_t = x_0 + \int_0^t \mu(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s \text{ for all } t \geq 0$$

- Result: Such a solution exists as long as there exists a constant K such that

$$\begin{aligned} |\mu(t,x) - \mu(t,y)| &\leq K|x-y| \\ |\sigma(t,x) - \sigma(t,y)| &\leq K|x-y| \\ |\mu(t,x)| + |\sigma(t,x)| &\leq K(1+|x|) \end{aligned}$$

- The symbol $|\cdot|$ is the Euclidean norm.

- Assuming these conditions are satisfied, X_t exists and is unique and satisfies
  1. X is $\mathcal{F}_t^W$ adapted,
  2. It has continuous trajectories
  3. It is a Markov process (the distribution of X_{t+s} depends at most on X_t for all $s \geq 0$)
  4. There exists a constant C such that

$$E\left[|X_t|^2\right] \leq Ce^{Ct}\left(1+|x_0|^2\right)$$

- Except for very special cases, a SDE cannot be solved analytically.

- There are a few cases, however, where this is possible

- We saw already the example of Geometric Brownian motion

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

  whose transition densities are given by

$$X_t = X_0 \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}.$$

- Another very popular class is the linear SDE.

$$dX_t = \alpha X_t dt + \sigma dW_t$$

- To solve the linear SDE, we proceed as with regular linear ODEs. Specifically note that Ito's Lemma implies that

$$
\begin{aligned}
d\left(e^{-\alpha t}X_t\right) &= -\alpha e^{-\alpha t}X_t dt + e^{-\alpha t}dX_t = \\
&= e^{-\alpha t}\sigma dW_t
\end{aligned}
$$

Accordingly

$$e^{-\alpha t}X_t - X_0 = \sigma \int_0^t e^{-\alpha s} dW_s$$

or

$$X_t = X_0 e^{\alpha t} + \sigma \int_0^t e^{\alpha(t-s)} dW_s$$

- What to do when an SDE does not have an analytical solution?
- Numerical solutions
- Euler Scheme is the simplest (albeit somewhat numerically ineffecient. Split $0..T$ into N equidistant intervals. Let $\Delta = \frac{T}{N}$. Draw N i.i.d. random normal variables ε_i and simulate paths according to

$$X_{t+\Delta} = X_t + \mu\left(t, X_t\right)\Delta + \sigma\left(t, X_t\right)\sqrt{\Delta}\varepsilon_{t+\Delta}$$

II. The Infinitesimal Operator and Partial Differential Equations

- Let X_t be the solution to the SDE $dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$
- Let the infinitesimal operator for a C^2-function h be defined as

$$\mathcal{A}h(t,x) = \sum_{i=1}^{n} \mu_i(t,x) \frac{\partial h}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} C_{i,j}(t,x) \frac{\partial^2 h}{\partial x_i \partial x_j}(x)$$

where $C(t,x) = \sigma(t,x)\, \sigma^*(t,x)$

- Note that using the infinitesimal operator, Ito's Lemma can be written compactly as

$$df(t, X_t) = \left\{ \frac{\partial f}{\partial t} + \mathcal{A}f \right\} dt + [\nabla_x f]\, \sigma dW_t$$

where the gradient ∇_x is short-hand notation for

$$\nabla_x f = \left[\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right]$$

- There are important connections between boundary value problems for Partial differential equations (PDE) and the theory of Stochastic Differential Equations.

- Consider the so-called "Cauchy" problem to find a solution to the following boundary value problem

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x) \frac{\partial F}{\partial x} + \frac{\sigma^2(t,x)}{2} \frac{\partial^2 F}{\partial x^2} &= 0 \\ F(T,x) &= \Phi(x) \end{aligned}$$

where $T > t$ and $\Phi(x)$ is a pre-determined function of x.

- We will next give an expression for the solution of the above PDE in terms of an appropriately defined conditional expectation

- Step 1: Define the stochastic process X on the interval $[t, T]$ as the solution to the SDE

$$dX_s = \mu(s, X_s)\, ds + \sigma(s, X_s)\, dW_s \tag{1}$$

- Using the notion of the infinitesimal operator, we may therefore write the Cauchy problem as

$$\begin{aligned} \frac{\partial F}{\partial t} + \mathcal{A}F(t, x) &= 0 \\ F(T, x) &= \Phi(x) \end{aligned}$$

- Step 2: Applying Ito's Formula implies that

$$F(T, X_T) = F(t, X_t) + \int_t^T \left\{ \frac{\partial F}{\partial t}(s, X_s) + \mathcal{A}F(s, X_s) \right\} ds + \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)\, dW_s$$

- Step 3: Take expectations on both sides

$$\begin{aligned} E_t[F(T, X_T)] &= F(t, X_t) + E_t \int_t^T \left\{ \frac{\partial F}{\partial t}(s, X_s) + \mathcal{A}F(s, X_s) \right\} ds + \\ &\quad E_t \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)\, dW_s \end{aligned}$$

A solution to the Cauchy problem neede s to satisfy $\frac{\partial F}{\partial t}(s, X_s) + \mathcal{A}F(s, X_s) = 0$ and hence the first integral is zero. The second integral is a stochastic integral. Assuming $\sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)$ is square integrable, its expectation is zero.

- Step 4: Impose the boundary condition $F(T, X_T) = \Phi(X_T)$ to arrive at

$$F(t,x) = E_t\left[\Phi(X_T)\,|X_t = x\right]$$

- This is the statement of the Feynman Kac Theorem: A solution to the Cauchy problem can be represented as a conditional expectation of $\Phi(X_T)$, where X_t follows the SDE (1)

- An example: Solve the Partial differential equation

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2} &= 0 \\ F(T,x) &= x^2 \end{aligned}$$

- We know from the Feynman Kac Theorem that

$$F(t,x) = E_t\left[X_T^2\right]$$

where

$$dX_s = 0ds + \sigma dW_s$$

- Therefore

$$X_T = x + \sigma(W_T - W_t)$$

- Thus

$$\begin{aligned} F(t,x) &= E_t\left(X_T^2\right) = Var(X_T) + \{E[X_T]\}^2 \\ &= \sigma^2(T-t) + x^2 \end{aligned}$$

- An important generalization of the Feynman Kac theorem. Consider the PDE

$$\begin{aligned} \frac{\partial F}{\partial t} + \mathcal{A}F(t,x) - r(x)F(t,x) &= -h(x) \\ F(T,x) &= \Phi(x) \end{aligned}$$

- Letting, as before,

$$dX_s = \mu(s, X_s)\,ds + \sigma(s, X_s)\,dW_s$$

the solution for $F(t,x)$ can be expressed as

$$F(t,x) = E_t\left[\int_t^T e^{-\int_t^s r(X_u)du} h(X_s)\,ds + e^{-\int_t^T r(X_u)du}\Phi(X_T)\right]$$

- The proof is in your problem set

III. Moment generating functions of Diffusion Processes

- As we discussed earlier, "solving" SDEs can be quite tricky.
- Applying the Feynman Kac theorem "in reverse" is useful for deriving the moment generating function of diffusion processes
- Here is the approach. Consider an SDE with some dynamics

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

- Suppose that we wish to determine the characteristic function of the distribution of X_T at some time $T > t$. Consider the function

$$\Phi(X_T) = e^{\lambda X_T}$$

and its conditional expectation

$$f(t,x) = E\left[\Phi(X_T)\,|X_t = x\right]$$

- Let's proceed slightly heuristically. Because $f(t,x)$ is a conditional expectation, it follows that

$$f(t,X_t) = E_t\left[f\left(t+dt, X_{t+dt}\right)\right]$$

or

$$E_t[df_t] = 0$$

- Assuming that $f(t,x)$ is twice continuously differentiable, we can apply Ito's Lemma to obtain

$$df_t = \left(\frac{\partial f}{\partial t}(t,X_t) + \mathcal{A}f(t,X_t)\right)dt + f_x dW_t$$

- Taking expectations on both sides shows that $E_t[df_t] = 0$ when and only when

$$\frac{\partial f}{\partial t}(t,X_t) + \mathcal{A}f(t,X_t) = 0$$

subject to the boundary condition $f(T,X_T) = \Phi(X_T)$.

- In some ways this is the Feynman Kac theorem in reverse.

- An example. Find the moment generating function of

$$dX_t = -\eta \left(X_t - \overline{X} \right) dt + \sigma dW_t$$

- Compute the moment generating function

$$f(t,x) = E\left(e^{\lambda X_T} | X_t = x \right).$$

- f needs to satisfy the PDE

$$f_t - \eta f_x \left(x - \overline{X} \right) + \frac{\sigma^2}{2} f_{xx} = 0, f(T,x) = e^{\lambda x}$$

- We can solve this PDE explicity. Guess that $f(t,x)$ takes the form $f(t,x) = \exp[\alpha_0(t) + \alpha_1(t)x]$. Substituting this guess into the above PDE we obtain

$$\begin{aligned} \dot{\alpha}_0 &= -\eta \overline{X} \alpha_1(t) - \frac{\sigma^2}{2} \alpha_1^2(t), \quad \alpha_0(T) = 0 \\ \dot{\alpha}_1 &= \eta \alpha_1(t), \quad \alpha_1(T) = \lambda \end{aligned}$$

- Integrating the above system gives

$$\begin{aligned} \alpha_1(0) &= \lambda e^{-\eta T} \\ \alpha_0(0) &= \lambda \overline{X} \left(1 - e^{-\eta T} \right) + \lambda^2 \frac{\sigma^2}{4\eta} \left(1 - e^{-2\eta T} \right) \end{aligned}$$

- Therefore

$$f(0,x) = \exp\left[\lambda \left(\overline{X} + \left(x - \overline{X} \right) e^{-\eta T} \right) + \frac{1}{2} \lambda^2 \left(\frac{\sigma^2}{2\eta} \left(1 - e^{-2\eta T} \right) \right) \right]$$

IV. Appendix: The Dynkin formula

- Let $f \in C_0^2(R^n)$. Suppose τ is a stopping time with $E^x[\tau] < \infty$. Then

$$E(f(X_\tau)|X_t = x) = f(x) + E\left[\int_0^\tau \mathcal{A}f(X_s)\,ds\right]$$

- Example: Consider $n-$dimensional Brownian motion $W = (W_1...W_n)$ starting at $a = (a_1, .., a_n)$ and assume that $|a| < R$. What is the expected value of the first exit time τ_K of W from the ball

$$K = K_R = \{x \in R^n; |x| < R\}$$

- Choose an integer k and apply Dynkin's fomula with $X = W$, $f = |x|^2$ and $\tau = \min(k, \tau_K)$. Then

$$E[f(W_\tau)] = |a|^2 + nE\left[\int_0^\tau ds\right] = |a|^2 + nE(\tau)$$

- Hence

$$E(\tau) \leq \frac{1}{n}\left(R^2 - |a|^2\right) \text{ for all } k.$$

- Letting $k \to \infty$ we conclude that $\tau_k < \infty$ $a.s.$ and

$$E(\tau_K) = \frac{1}{n}\left(R^2 - |a|^2\right)$$

- Next suppose that $n \geq 2$ and assume that $|b| > R$. What is the probability that B starting at b will ever hit K?
- Let a_k be the first exit time from the annulus

$$A_k = \left\{x; R < |x| < 2^k R\right\}, \ k = 1, 2, ...$$

- Let $f = f_{n,k}$ be a C^2 function with compact support such that if $R \leq x \leq 2^k R$,

$$f(x) = \begin{cases} -\log|x| & \text{when } n = 2 \\ |x|^{2-n} & \text{when } n > 2 \end{cases}$$

- By construction of $f(x)$, $\mathcal{A}f = 0$ in A_k. and therefore

$$E\left(f\left(W_{a_k}\right)\right) = f(b) \ \text{ for all } k \tag{2}$$

- Put $p_k = P\left[\left|W_{a_k}\right| = R\right], q_k = P\left[\left|W_{a_k}\right| = 2^k R\right]$

- When $n = 2$, equation (2) implies

$$-\log R \times p_k - (\log R + k \log 2) \times q_k = -\log|b|$$

- This implies that $q_k \to 0$ as $k \to \infty$. so that

$$P(\tau_K < \infty) = 1$$

- Hence Brownian motion is recurrent in R^2.

- When $n > 2$, equation (2) implies

$$R^{2-n} \times p_k + \left(2^k R\right)^{2-n} \times q_k = |b|^{2-n}$$

- Letting $k \to \infty$, this equation implies

$$\lim_{k\to\infty} p_k = P\left(\tau_K < \infty\right) = \left(\frac{|b|}{R}\right)^{2-n}$$

- Hence Brownian motion is transient when $n > 2$.