Investments

Topic 5: Equities

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by Mikhail Chernov

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Overview of Topic 5

1. What Stocks Are.
2. Characterizing Stock Returns.
3. Return on a Portfolio.
4. Investing for a "typical" investor.
5. Evidence.
6. Expected Return and Variance of a Portfolio.
7. Risk-Return Relationship.
8. Utility.

1. What Stocks Are

What Stocks Are

- Different types of stocks: common stock, non-voting stock, preferred stock.
- Main features of common stock:
  - Voting rights
  - Rights to dividends
  - Limited liability
- Dividends are the cash flows of common stock. They are:
  - discretionary. By contrast, debt payments are mandatory.
  - not tax-deductible (because they are not business expenses). By contrast, debt payments are tax-deductible.
  - generally paid quarterly.

Market Cap, or Size

- Market Capitalization: #shares $\times$ price
  - Microsoft: \$3.18 Trillion
  - 7.4B shares outstanding @428/share
- Vanguard Fund Classification
  - Small Cap: $< \$2.1B$
  - Mid Cap: $\$2.1 - 15.6B$
  - Large Cap: $> \$15.6B$
- Managers often focus on such characteristics as size
  - Also, value/growth stocks
- Mutual funds mimic that cheaply

20 Largest Companies Worldwide

Company Name	Nationality	Industry	Market Cap ($bn)
Apple Inc	United States	Technology	3549
NVIDIA Corp	United States	Technology	3447
Microsoft Corp	United States	Technology	3171
Alphabet (Google)	United States	Technology	2030
Amazon.com Inc	United States	Consumer Discretionary	1977
Saudi Aramco	Saudi Arabia	Oil	1740
Meta Platforms (Facebook)	United States	Technology	1438
TSMC	Taiwan	Semiconductors	1010
Berkshire Hathaway	United States	Financials	1003
Tesla Inc	United States	Consumer Discretionary	843
Eli Lilly	United States	Pharmaceuticals	813
Broadcom	United States	Semiconductors	806
Walmart Inc	United States	Consumer Discretionary	669
JPMorgan Chase	United States	Financials	635
Visa Inc	United States	Financials	552
Exxon Mobil	United States	Oil	534
UnitedHealth Group	United States	Health Care	522
Novo Nordisk	Denmark	Pharmaceuticals	513
Tencent	China	Technology	505
Oracle	United States	Technology	478

Note: Market Cap as of October 28, 2024.

20 Largest Companies in the United States

Company Name	Nationality	Industry	Market Cap ($bn)
Apple Inc	United States	Technology	3549
NVIDIA Corp	United States	Technology	3447
Microsoft Corp	United States	Technology	3171
Alphabet (Google)	United States	Technology	2030
Amazon.com Inc	United States	Consumer Discretionary	1977
Meta Platforms (Facebook)	United States	Technology	1438
Berkshire Hathaway	United States	Financials	1003
Tesla Inc	United States	Consumer Discretionary	843
Eli Lilly	United States	Pharmaceuticals	816
Broadcom	United States	Semiconductors	806
Walmart Inc	United States	Consumer Discretionary	671
JPMorgan Chase	United States	Financials	638
Visa Inc	United States	Financials	557
Exxon Mobil	United States	Oil	532
UnitedHealth Group	United States	Health Care	522
Oracle	United States	Technology	478
Mastercard	United States	Financials	475
Procter Gamble	United States	Consumer Staples	402
Johnson Johnson	United States	Health Care	401
Home Depot Inc	United States	Consumer Discretionary	401

Note: Market Cap as of October 28, 2024.

Growth and Value

Vanguard's classification

- **Growth stocks**: Stocks of companies with above-average prospects for long-term growth based on such measures as revenue, earnings, and book value. The stocks generally produce little dividend income, because the companies prefer to reinvest earnings in research and development.
- **Value stocks**: Stocks that are out of favor with the investment community, that is, they sell at relatively low prices in relation to their earnings or book value. These stocks typically produce above-average dividend income.
- A typical measure is "Book-to-Market": accounting, or book, value of assets divided by the market cap of the firm
  - Small BM indicates a growth company

2. Characterizing Stock Returns

Computing Stock Returns

Return between two dates, 0 and 1:

- Invest X dollars in the stock at date 0.
- If there are any dividend payments between dates 0 and 1, reinvest them in the stock.
- Suppose that value of the stock investment at date 1 is P dollars.
- Return on the stock, R, is defined as

$$X(1 + R) = P.$$

A Simple Formula

We will frequently assume that the only dividend payment between dates 0 and 1 is at date 1.

- With X dollars, buy $x = X/P_0$ shares of the stock at date 0, where P_0 is the date 0 price.

- At date 1, these x shares are worth xP_1, where P_1 is the date 1 price. Moreover, they pay a dividend xD_1, where D_1 is the date 1 dividend.

- Value of the stock investment in period 1 is $x(P_1 + D_1)$.

- Return on the stock is

$$R = \frac{x(P_1 + D_1)}{X} - 1 = \frac{D_1 + (P_1 - P_0)}{P_0}.$$

- Return is due to dividend payments and capital gains.

Definitions

- We can define expectation, variance, and st. dev. in a **statistics** sense, for return series R_t.
- A measure of the average value of the series is the **sample average**

$$\overline{R} = \frac{R_1 + \cdots + R_T}{T}.$$

- Two measures of the dispersion of the series around the average value are the **sample variance**

$$s(R)^2 = \frac{(R_1 - \overline{R})^2 + \cdots + (R_T - \overline{R})^2}{T - 1}$$

and the **sample standard deviation**

$$s(R) = \sqrt{s(R)^2}.$$

- The sample standard deviation is in the same units as the data series. It measures the "typical" distance of the series elements from the average value.

Expectation, Variance, and Standard Deviation

Using historical returns[1] 1926-2024, we can compute:

	Sample Average	Sample St. Dev.
Small Stocks	16.0%	31.1%
Large Stocks	12.2%	19.8%
Long-Term Govt Bonds	5.6%	10.2%
T-Bills	3.3%	3.1%
Inflation	3.0%	4.0%

First basic fact:

$\boxed{\text{Riskier investments have higher returns on average}}$

[1] Source: CFA Institute SBBI Data

Covariance and Correlation

- We want to measure the association between two data series R_{1t} and R_{2t}.
- Note: R_{2t} could serve as R_{1t-1} to measure dependence across time
- Two related measures of association are the **sample covariance**

$$Cov(R_1, R_2) = \frac{\Sigma_{t=1}^{T}(R_{1t} - \overline{R}_1)(R_{2t} - \overline{R}_2)}{T - 1}$$

and the **sample correlation**

$$\rho(R_1, R_2) = \frac{Cov(R_1, R_2)}{s(R_1)s(R_2)}.$$

Probability and Random Variables

- We can also define expectation, variance, and standard deviation in a **probability** sense, for a **random variable** Z.
- Suppose that Z takes the values Z_1, ..., Z_K, with probabilities p_1, ..., p_K.
- The **expectation** of Z is

$$E(Z) = p_1 Z_1 + \cdots + p_K Z_K.$$

- The **variance** of Z is

$$V(Z) = p_1 (Z_1 - E(Z))^2 + \cdots + p_K (Z_K - E(Z))^2,$$

  and the **standard deviation** of Z is

$$\sigma(Z) = \sqrt{V(Z)}.$$

- Covariance and correlation can also be defined for random variables.

Example

- Consider the random variable Z describing the outcomes of rolling a dice. Z can take the values 1, 2, 3, 4, 5, 6, each with probability $1/6$.
- The expectation of Z is

$$E(Z) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5.$$

- The variance of Z is

$$V(Z) = \frac{1}{6} \times (1 - 3.5)^2 + \frac{1}{6} \times (2 - 3.5)^2 + \frac{1}{6} \times (3 - 3.5)^2$$

$$+ \frac{1}{6} \times (4 - 3.5)^2 + \frac{1}{6} \times (5 - 3.5)^2 + \frac{1}{6} \times (6 - 3.5)^2 = 2.9167.$$

- The standard deviation of Z is

$$\sigma(Z) = \sqrt{2.9167} = 1.7078.$$

Statistics vs. Probability Definitions

- The sample statistics and probability definitions of expectation, variance, and standard deviation, are related.
- Suppose that we roll a dice a number of times, and record the data series formed by the outcomes.
- For that data series, we can compute a sample average, sample variance, and sample standard deviation.
- These will generally be different than the expectation (3.5), variance (2.9167), and standard deviation (1.7078), of Z.
- However, if we roll the dice very many times, they will get very close. If we roll the dice an "infinite" number of times, they will become equal.
- **General result (Law of Large Numbers):** Sample statistics converge to their probability counterparts as the sample grows.

Geometric vs arithmetic averages

- Arithmetic average returns as described above are the most commonly reported measure of performance
- They correspond to a trading strategy where you always keep \$1 invested in the strategy
  - No reinvestment of profits
  - Top up in case of losses
- Geometric average corresponds to the full re-investment strategy $(1 + R_1)(1 + R_2)\cdots(1 + R_T)$

$$\overline{R}^G = [(1 + R_1)(1 + R_2)\cdots(1 + R_T)]^{1/T} - 1.$$

- Log of (gross) geometric average returns is arithmetic average of log (gross) returns
- Example: the strategy earned +200% followed by -100%

$$\begin{aligned} \overline{R} &= 50\% \\ \overline{R}^G &= -100\% \end{aligned}$$

Decimalization and annualization

- It is a good practice to express returns in decimal form
  - It is very easy to get confused when computing variance of returns expressed in %
  - You may always express the relevant quantities in % as a last step of your analysis

- It is common to annualize *arithmetic* returns
  - For example, use weekly returns to compute the arithmetic average and standard deviation
  - Annualize them by multiplying by 52 and $\sqrt{52}$, respectively

- A note of caution: if your weekly return is 0.23% then your annual return is $0.23\% \times 52 = 11.96\%$ only if returns are iid (not predictable)

Details of annualization

- Consider two periods 1 and 2
- The cumulative return is (approximately) $R_1 + R_2$

$$(1 + R_1)(1 + R_2) = 1 + R_1 + R_2 + R_1 R_2$$

  Last term is very small if returns are relatively small
- Suppose each return has the same mean $\overline{R}$, then the two-period return has a mean of $2\overline{R}$
- Suppose each return has the same variance $V(R)$, then the two-period variance is

$$V(R_1 + R_2) = 2V(R) + 2cov(R_1, R_2) = 2V(R)$$

  The last expression is correct if returns are iid
- An alternative: log-returns
  - $\log(1 + x) = x - x^2/2 + \dots$
  - $\log[(1 + R_1)(1 + R_2)] \approx R_1 + R_2$
  - If returns are not small (high volatility), have to use second order

3. Return on a Portfolio

Return on a Portfolio

Determine the return on an asset portfolio given the returns on the individual assets.

- Consider a portfolio consisting of X_1 dollars in S&P 500 and X_2 dollars in UST bonds.
- The value of the portfolio at date 0 is $X = X_1 + X_2$.
- The value of the portfolio at date 1 is

$$X_1(1 + R_1) + X_2(1 + R_2),$$

  where R_1 is the return on S&P 500 and $R_2 = R_f$ the return on a bond between dates 0 and 1.
- The return on the portfolio is

$$\begin{aligned} R &= \frac{X_1(1 + R_1) + X_2(1 + R_2) - X}{X} \\ &= \frac{X_1}{X} R_1 + \frac{X_2}{X} R_2. \end{aligned}$$

Portfolio Weights

- The **portfolio weights** are

$$w_1 = \frac{X_1}{X} \qquad \text{and} \qquad w_2 = \frac{X_2}{X}.$$

  They represent the fraction of portfolio value invested in each stock. They sum to 1.

- The return on the portfolio is

$$R = w_1 R_1 + w_2 R_2 \equiv w R_r + (1 - w) R_f.$$

  It is a weighted average of the returns on the individual stocks.

- Returns on the two asset could be time-varying
  - The key difference is that R_f is known today, but not R_r
- Portfolio weight w could be time-varying as well
  - It is known today

Example 1

Compute the return on a portfolio consisting of \$300 in AMZN and \$100 in UST.

- We have

$$w_1 = \frac{300}{400} = 0.75$$

  and

$$w_2 = \frac{100}{400} = 0.25.$$

- The return on the portfolio is

$$R = 0.75 \times R_1 + 0.25 \times R_2.$$

Example 2

Start with the same amount of initial capital \$400. Instead of buying \$100 of UST, we borrow, aka sell short \$100 of UST. Compute the return on the portfolio.

- Date 0: Get \$100 from selling UST. Invest $\$100 + \$400 = \$500$ in AMZN.
- Date 1:
  - AMZN is worth $\$500(1 + R_1)$.
  - Pay $\$100(1 + R_2)$ to buy UST.
  - Portfolio value is

$$500(1 + R_1) - 100(1 + R_2)$$

- Return on the portfolio is

$$\begin{aligned} R &= \frac{500(1 + R_1) - 100(1 + R_2) - 400}{400} = \frac{500}{400} R_1 - \frac{100}{400} R_2 \\ &= 1.25 \times R_1 - 0.25 \times R_2. \end{aligned}$$

- **Conclusion:** Portfolio return is still weighted average of returns on the individual stocks. However, weights of stocks that are sold short are negative. The weights still sum to 1.

Stock Indexes are portfolios designed to track the market

- Value-weighted index, e.g., S&P 500, Wilshire 5000

$$I_t^V = \sum_{i=1}^{N} w_t^i \cdot P_t^i, \quad w_t^i = \text{market cap}^i / \sum_{i=1}^{N} \text{market cap}^i$$

  More weight on larger companies; no need to rebalance

- Equally-weighted, e.g., DJIA 30, S&P 500 EWI

$$I_t^E = N^{-1} \sum_{i=1}^{N} P_t^i$$

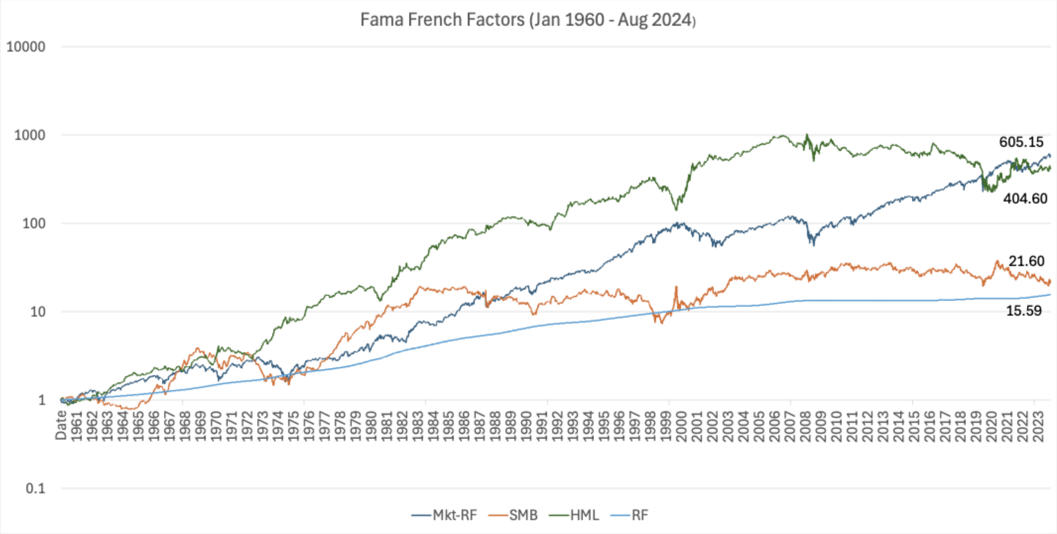
  - buy more shares of a company with a declining share price and sell off shares when a company sees its share price rise.

4. Evidence

Using Statistics

- The return next year can be viewed as a random variable. We can estimate its
  - expectation (the expected return) by the sample average
  - variance by the sample variance
  - standard deviation by the sample standard deviation.
- For variance and standard deviation, we can obtain quite precise estimates using a sample of even one year (the most recent), provided that we use daily or weekly returns.
- For expected return, we should use as large a sample as possible. But even for a 75-year sample the estimates are quite imprecise. (Standard error is around 2.5%.)

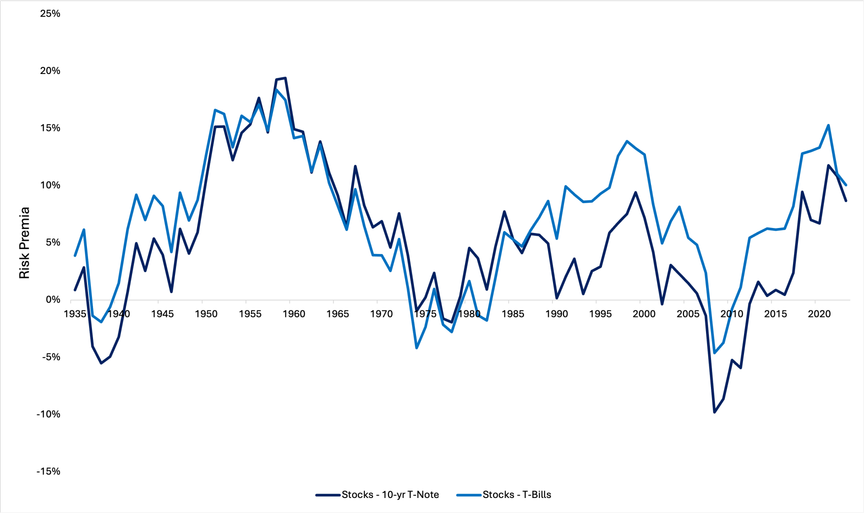
Perspective on the U.S. market performance



An Example: The Market Risk Premium

- We want to know the expected return of large stocks relative to T-bills. We refer to this as the **market risk premium** (MRP).
- We can obtain an estimate of the MRP using sample averages.
- The 1926-2024 sample averages for large stocks and T-bills are 12.2% and 3.3%, respectively. Large stocks outperformed by 8.9
- An estimate of the MRP is 8.9%.
- We will revisit the MRP .

The Equity Premium



An Aside: Realized vs. Expected Returns

- The 8.9% is (an estimate of) the **expected return** of large stocks relative to T-bills.
- We expect that large stocks will, on average, outperform T-bills by 8.9%.
- This does not mean that in 2024 large stocks will outperform T-bills by 8.9% for sure.
- The return of large stocks relative to T-bills in 2024, will be the **realized return** in 2024.

Individual Stocks vs. Indices

Using annual returns 1981-2024[2], we can compute:

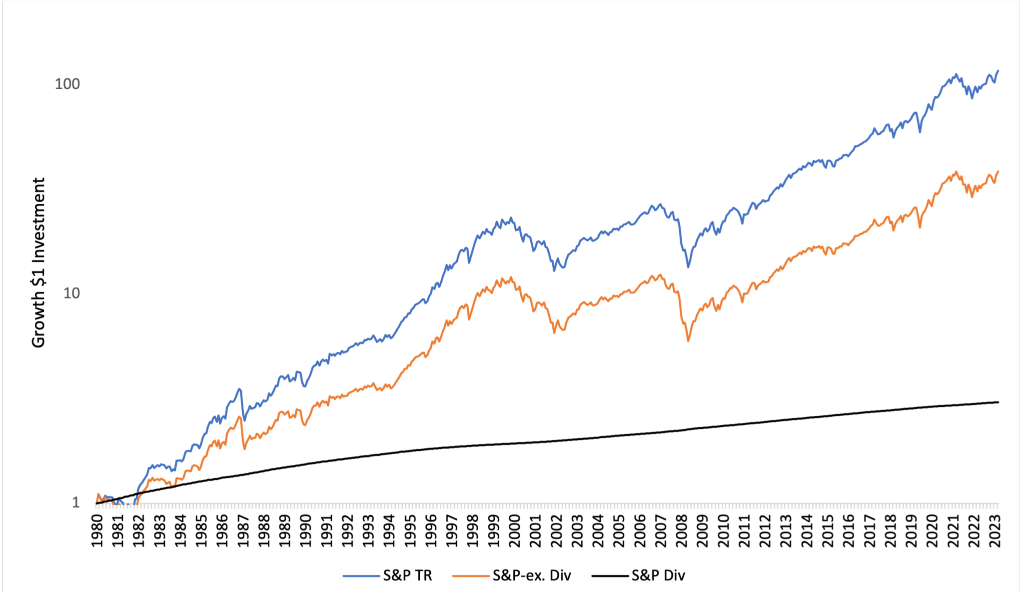
	Sample Average	Sample St. Dev.
Coca Cola (KO)	24.2	35.6
Apple (AAPL)	34.0	65.7
Disney (DIS)	14.7	28.7
IBM (IBM)	9.2	27.5
Xerox (XRX)	9.7	45.4
S&P 500	12.7	16.6

[2]Source: Yahoo Finance, CRSP

Sources of Income

- Expected dividends and prices
  - The longer one holds the stock, the greater the role played by dividends (coupons) and the lesser the role of price appreciation.
- Classic definitions
  - Investor: one who buys/sells based on future expected dividend income
  - Speculator: one who buys/sells based on future price appreciation
- Typical investor: a hybrid who is interested in both components.

How much income from each source?



Stock Valuation

- Taking expected values in the return definition and solving for price:

$$
\begin{aligned}
P_t &= \frac{E_t P_{t+1} + E_t D_{t+1}}{1 + E_t R_{t+1}} = \frac{E_t D_{t+1}}{1 + E_t R_{t+1}} \\
&+ \frac{E_t P_{t+2} + E_t D_{t+2}}{(1 + E_t R_{t+1})(1 + E_t R_{t+2})} \\
&= \sum_{j=1}^{\infty} \frac{E_t D_{t+j}}{\prod_{k=1}^{j}(1 + E_t R_{t+k})}
\end{aligned}
$$

- Interesting special case:
  - constant expected returns, $E_t R_{t+k} = R$
  - constant dividend growth, G
  - perpetuity! (a.k.a. Gordon growth model)

$$
P_t = \sum_{j=1}^{\infty} \frac{D_t(1+G)^j}{(1+R)^j} = \frac{D_t}{R-G}
$$

Conclusions

- Note that stocks with higher sample standard deviations tend to have higher sample average returns.
- We will study the relation between a stock's expected return and risk.
- Note also that the sample standard deviation of the S&P500 is much smaller than those of the individual stocks. By contrast, the sample average returns are comparable.
- Second basic fact:

> There are benefits to diversification.
> By holding a diversified portfolio,
> we can reduce risk
> without sacrificing expected return.

5. Investing for a typical investor

Investing for a typical investor

- A typical investor holds most of her wealth in the market, say S&P 500
  - Why?
- The rest of the wealth is invested in bonds/money market
  - What's the optimal allocation in each?
  - Should the allocation be constant over time?
- Determinants of the portfolio rule
  - Expected return for market
    - Equity Premium: the additional return of the market over the risk free rate.
  - Risk-free rate
    - The proxy depends on the investor's horizon
  - Risk: How to measure risk?
    - Typically measured by standard deviation ("volatility") of historical returns.
  - Preferences of the investor
    - Is the investor risk averse? How much does the investor like or dislike risk?

6. Expected Return and Variance of a Portfolio

Expected Return of a Portfolio

- The expected return of the market based on the information at time t, I_t, is

$$E(R_{r,t+1}|I_t) = E_t(R_{r,t+1}).$$

- The expected return on the risk-free bond is $E_t(R_{f,t}) = R_{f,t}$
- The expected return on the portfolio is

$$E_t(R_{t+1}) = w_t E_t(R_{r,t+1}) + (1 - w_t) R_{f,t}$$

- Convenient to re-write as

$$E_t(R_{t+1}) = w_t \underbrace{\left(E_t(R_{r,t+1}) - R_{f,t}\right)}_{\text{Risk premium}} + R_{f,t}$$

- In practice, we do not know the expected returns of the risky securities, but can estimate them using sample averages.

Expected Return: Example

Compute (an estimate of) the expected return of a portfolio consisting of \$300 in S&P and \$100 in UST.

- We have
$$w_1 = \frac{300}{400} = 0.75$$
and
$$w_2 = \frac{100}{400} = 0.25.$$
- Estimating the expected returns $E(R_1)$ and $E(R_2)$ from the 1926-2017 data, we have
$$\overline{R}_1 = 10.8\%$$
and
$$\overline{R}_2 = 7.1\%.$$
- The estimate of the expected return of the portfolio $E(R)$ is
$$\overline{R} = 0.75 \times \overline{R}_1 + 0.25 \times \overline{R}_2 = 9.875\%.$$

Variance of the Portfolio (Risk)

- Suppose the risky's asset variance is

$$V_t(R_{r,t+1}) = E_t\left(R_{r,t+1} - E_t(R_{r,t+1})\right)^2$$

- Volatility is $\sigma_t(R_{r,t+1}) = V_t^{1/2}(R_{r,t+1})$

- Then, the volatility of the portfolio is $\sigma_t(R_{t+1}) = |w_t|\sigma_t(R_{r,t+1})$
  - Why?

- Issues:
  - Is the risk-free really risk free?
  - Is variance a good measure of risk?

Example: changing allocation to S&P

- Assume expected return and variance are constant
- S&P from 1956 to 2015: $E(R_r) = 0.63\%$, $\sigma(R_r) = 4.21\%$ (monthly)
- 1-month UST from 1956 to 2015: $R_f = 0.37\%$
- How to choose w?

w	$E(R)$	$\sigma(R)$
5.76	1.84%	24.25%
2.88	1.11%	12.13%
1.44	0.74%	6.06%
0.72	0.55%	3.03%
0.36	0.46%	1.52%

7. Risk-Return Relationship

Risk-Return Relationship

- Portfolio's risk implies $|w| = \sigma(R)/\sigma(R_r)$
- When $w > 0$, substitute into portfolio's expected return

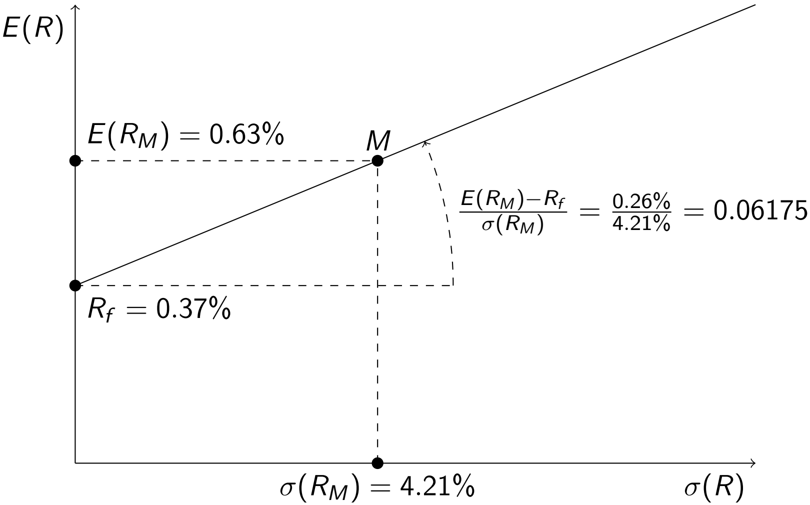
$$E_t(R_{t+1}) = R_{f,t} + \underbrace{\frac{E_t(R_{r,t+1}) - R_{f,t}}{\sigma_t(R_{r,t+1})}}_{\text{Reward-to-risk ratio}} \cdot \sigma_t(R_{t+1})$$

- The reward-to-risk ratio is the return premium per unit of risk, a.k.a. the Sharpe ratio
- For the US, we have (monthly)

$$E_t(R_{t+1}) = 0.0037 + \frac{0.0026}{0.0421} \cdot \sigma_t(R_{t+1})$$

- Trade-off is just a line as a function of portfolio risk: capital market line
- (capital allocation line refers to the same object when the risky asset is a stock)

The Capital Market Line



Risk and return in reality

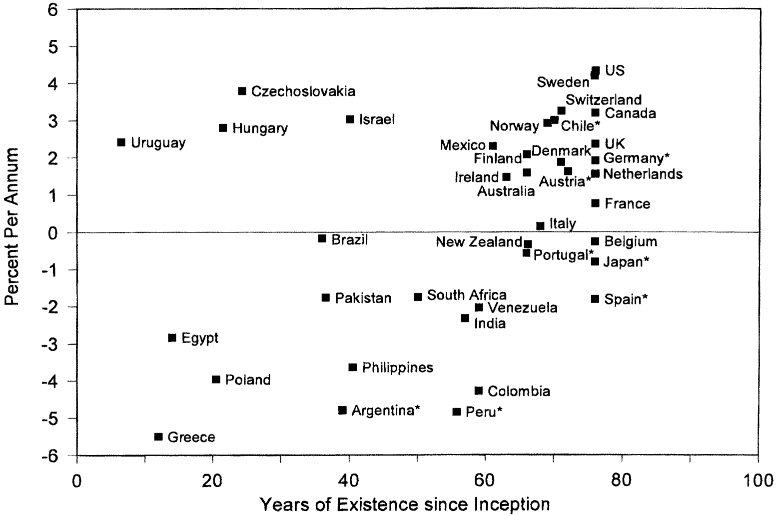
- Since 1926, the market in U.S. delivers
  - Equity premium of about 5% per year
  - Annual volatility of 15%

- Is the equity premium really 5% on average?
  - Survivorship bias: The fact that the US market has continuously operated implies that it is biased upwards.

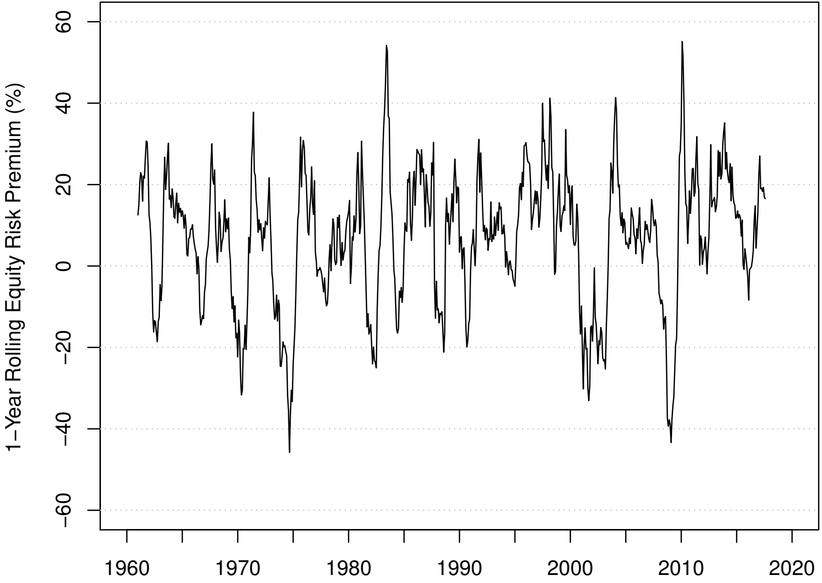
- Is the equity premium really constant?
  - If it is, how can the market go down for 3 years in a row?
  - If not, there are opportunities for market timing.

- Is volatility really constant?
  - No $\Rightarrow$ Volatility timing

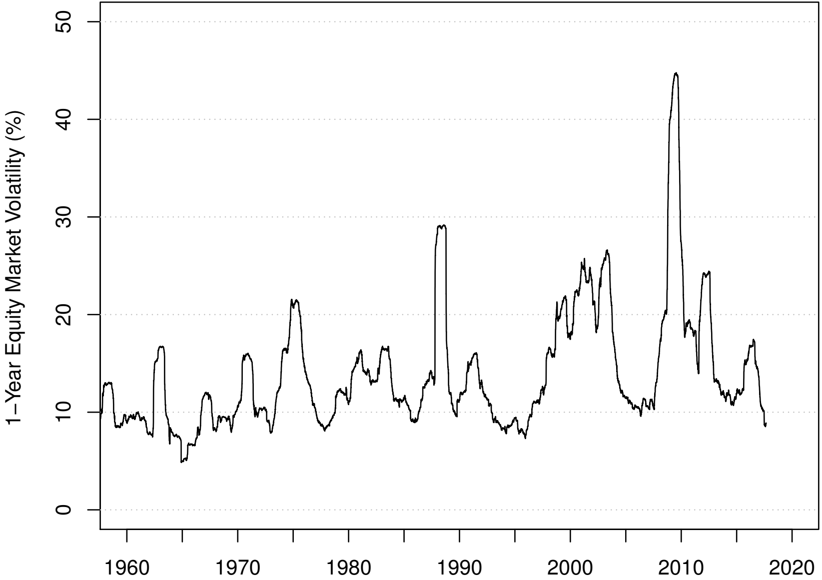
Survivorship bias



Time Variation in the Equity Premium



Time Variation in the Market Volatility



Changing Equity Premium and Volatility

- What are the implications?
  - Market timing: Increase allocations when expected returns are high
  - Volatility timing: Decrease allocations when volatility is high.
- How to predict returns?
  - Strategies based on historical data.
  - Strategies based on economic variables.
- Given the portfolio risk and expected return, we would like to find the "optimal" w_t
- What do we mean by optimal?
  - The best
  - What is the metric to judge performance?
- Use a model of risk preferences
  - Risk and reward
  - Higher risk offers higher returns (on average!), so it may pay off well to take on some risk

8. Utility

Utility

- Utility theory quantifies the subjective desirability of something
  - Use an investor's utility to compare different investors
  - The investor will choose the portfolio that delivers the highest utility.
- What does your utility depend on?
- Intuition:
  - We want high expected returns
  - We want low risk
- How much of each?
  - We have to come up with a penalty for risk for a given return
- Risk aversion: quantifies the risk/return trade-off.

Mean-Variance Utility

- Assume

$$U(R_{t+1}) = E_t(R_{t+1}) - \frac{A}{2} \cdot V_t(R_{t+1})$$

- A measures an investor's level of risk aversion:
  - $A > 0$: investor is risk-averse
  - $A = 0$: investor is risk-neutral
  - $A < 0$: investor is risk-loving

- S&P from 1956 to 2015:
  $E(R_r) = 0.63\%$,
  $\sigma(R_r) = 4.21\%$ (monthly)
- What does level of utility mean?
- As A increases, you dislike risk more and more

A	$U \times 10^2$
0.25	0.60
0.5	0.58
1	0.54
2	0.45
4	0.27

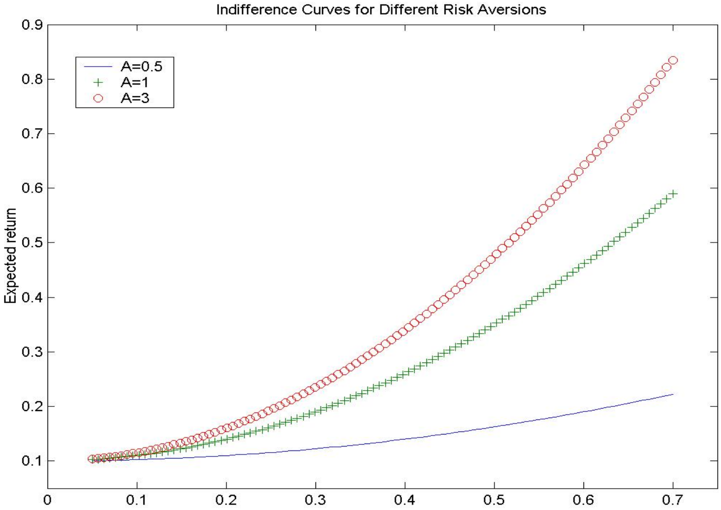
Indifference curves

- For a given level of risk aversion, A, there are multiple combinations of $E(R)$ and $\sigma(R)$ that give the same value of utility
- For example: for $A = 1$, the following combinations give the same level of utility, $U = 0.15$

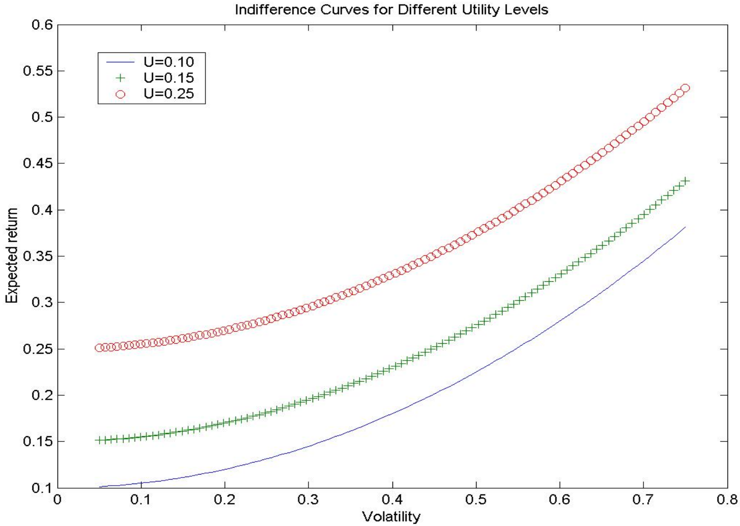
$E(R)$	0.15500	0.16125	0.17000	0.18125	0.19500
$\sigma(R)$	0.10	0.15	0.20	0.25	0.30

- This also known as *certainty equivalent* – return that you receive for sure
- Because these combinations yield the same level of utility, you must be indifferent among the choices
- Indifferent curves are all possible combination of $E(R)$ and $\sigma(R)$ that yield the same U
  - as you move along the curve, expected return and risk are changing, but in combination they give the same utility level

Indifference Curves and Risk Aversion



Indifference Curves and Utility Levels



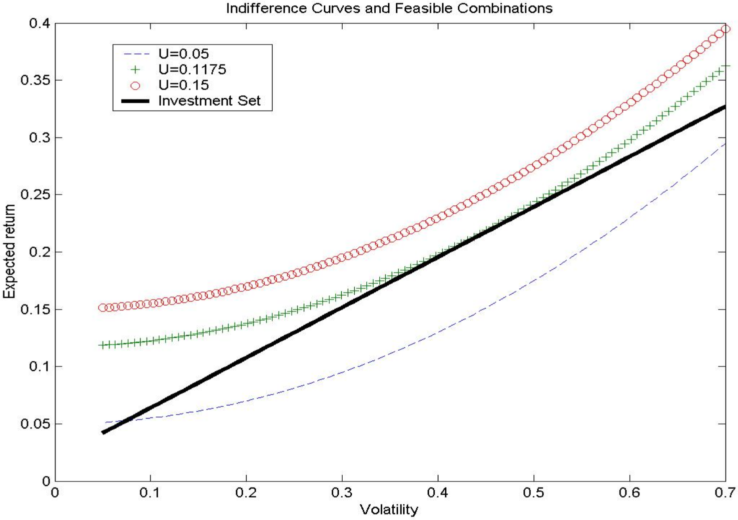
Last piece

- Higher risk aversion means that investors require higher returns for the same level of risk in order to preserve the same utility level
- Investors would prefer a higher curve to ones below.
- Is any allocation feasible? Can you invest on any part of the line?
  - Depends on securities available to investor
- Which means and variance are feasible?
- The ones generated by the given expected returns, volatility and portfolio weights.

$$\begin{aligned} E_t(R_{t+1}) &= w_t(E_t(R_{r,t+1}) - R_{f,t}) + R_{f,t} \\ \sigma_t(R_{t+1}) &= |w_t|\sigma_t(R_{r,t+1}) \end{aligned}$$

- What does the investment opportunity set look like?
  - It is the capital market line: a line on the expected return/volatility graph.

Putting the Pieces Together



9. Optimal Portfolios

Optimal Portfolios

- CML gives the possible risk-return trade-offs

- Find the portfolio weight that maximizes utility:

$$
\begin{aligned}
w_t^* &= argmax_w U(R_{t+1}, w_t) \\
U(R_{t+1}, w_t) &= w_t(E_t(R_{r,t+1}) - R_{f,t}) + R_{f,t} - \frac{A}{2} \cdot w_t^2 V_t(R_{r,t+1})
\end{aligned}
$$

- To find the maximum, take derivative and set equal to zero to get the optimal portfolio:

$$
w_t^* = \frac{E_t(R_{r,t+1}) - R_{f,t}}{A \cdot V_t(R_{r,t+1})}
$$

- The risky asset holdings are larger when
  - Risk premium is larger
  - Investor is less risk averse (A is smaller)
  - Volatility is lower

Example: S&P

- Assume means and variances are constant over time
- S&P from 1956 to 2015: $E(R_r) = 0.63\%$, $\sigma(R_r) = 4.21\%$ (monthly)
- 1-month UST from 1956 to 2015: $R_f = 0.37\%$

$$w_t^* = \frac{E_t(R_{r,t+1}) - R_{f,t}}{A \cdot V_t(R_{r,t+1})}$$

A	w^*	$E(R)$	$\sigma(R)$
0.25	5.76	1.84%	24.25%
0.50	2.88	1.11%	12.13%
1.00	1.44	0.74%	6.06%
2.00	0.72	0.55%	3.03%
4.00	0.36	0.46%	1.52%

A historical perspective

- Since 1926, the “market” in U.S. delivers, per annum
  - Equity premium of about 6%
  - Volatility of 15%

- This implies a long horizon investor should allocate

$$w_t^* = \frac{E_t(R_{r,t+1}) - R_{f,t}}{A \cdot V_t(R_{r,t+1})} = \frac{0.06}{A \cdot 0.15^2} = \frac{2.67}{A} = \begin{cases} 1.33 & A = 2 \\ 0.67 & A = 4 \\ 0.33 & A = 8 \end{cases}$$