MGMT MFE 406 – Derivative Markets (4 units)

Part 3: Analytical Methods for Single-Period Payoffs

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Outline

1. Arbitrage Conditions

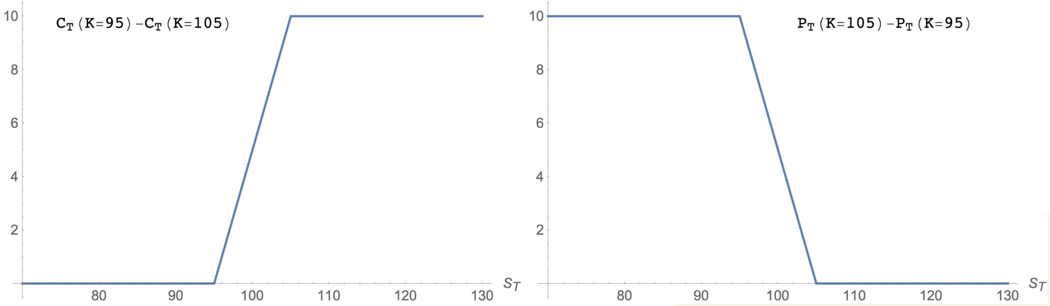
- Central notion in option pricing theory is *absence of arbitrage*
- Two types of arbitrage-related conditions between/amongst European option prices:
  - Those as a function of strike (for fixed maturity)
  - Those as a function of maturity (for more-or-less fixed strike)

    We considered some of these in §3.2 of part 1 of the notes and in problem 1 of PS 3.
- Strike-related conditions are relatively easy to understand and are based on the idea that certain simple portfolios of options with the same maturity and non-negative payoffs must have non-negative value:
  - First-order (call spread and put spread) conditions
  - Second-order (butterfly spread) conditions

1.1. First-Order Strike Conditions

- A bull spread of calls with maturity T has payoff ≥ 0 and must (for $t < T$) have positive value
  - Present value $C_t(S_t, K)$ of a call option with maturity T must be decreasing in strike K:

$$C_t(K,T) - C_t(K',T) \geq 0 \ \ \forall K' > K$$



- Correspondingly, a bearish put spread must also have positive value
  - Present value $P_t(S_t, K)$ of a put option must be increasing in strike K:

$$P_t(K,T) - P_t(K',T) \geq 0 \ \ \forall K' < K$$

1.1. First-Order Strike Conditions (2)

- Express option values as discounted expectations with respect to the risk-neutral (**Q**-measure) probability density q of the terminal asset price S_T:

$$C_t(S_t, K, T) = \mathcal{R}(t,T) \int_K^{\infty} dS_T \, q(S_T|S_t) \, (S_T - K)$$
$$P_t(S_t, K, T) = \mathcal{R}(t,T) \int_0^{K} dS_T \, q(S_T|S_t) \, (K - S_T)$$

with the discount factor: $\mathcal{R}(t,T) \doteq \exp[-R(t,T)] \doteq \exp\left[-\int_t^T dt' \, r(t')\right]$ and $r(t')$ equal to the (deterministic) interest rate at time t'

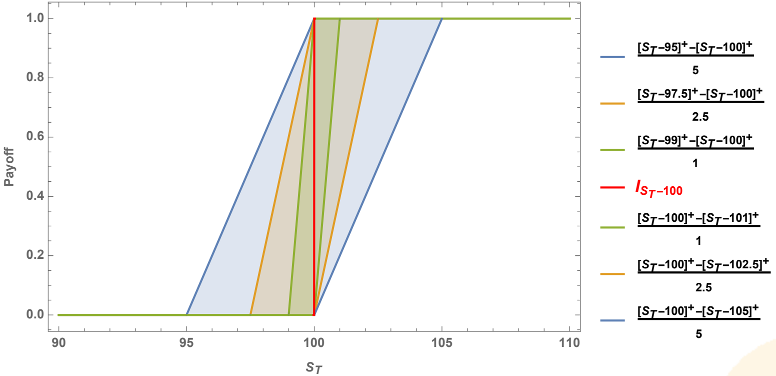
- Dividing by $|K' - K|$ and taking limits, the spread conditions become:

$$\lim_{K' \searrow K} \frac{C_t(K,T) - C_t(K',T)}{K' - K} = -\frac{dC_t(K,T)}{dK} = \mathcal{R}(t,T) \int_K^{\infty} dS_T \, q(S_T) \geq 0 \;\; (K' > K)$$
$$\lim_{K' \nearrow K} \frac{P_t(K,T) - P_t(K',T)}{K - K'} = \frac{dP_t(K,T)}{dK} = \mathcal{R}(t,T) \int_0^{K} dS_T \, q(S_T) \geq 0 \;\; (K' < K)$$

- Since the discount factor $\mathcal{R}(t,T) \geq 0$, these conditions simply require non-negativity of the cdf of the terminal asset price (more precisely, $0 \leq \text{cdf} \leq 1$)
- They define binary (cash-or-nothing) option payoffs: $B_T(S_T, K) = BC_T(S_T, K) = \mathbf{1}_{S_T - K}$ (binary call) and $BP_T(S_T, K) = \mathbf{1}_{K - S_T}$ (binary put)

1.1. First-Order Strike Conditions (3)

Convergence of Call Spreads to Binary Option

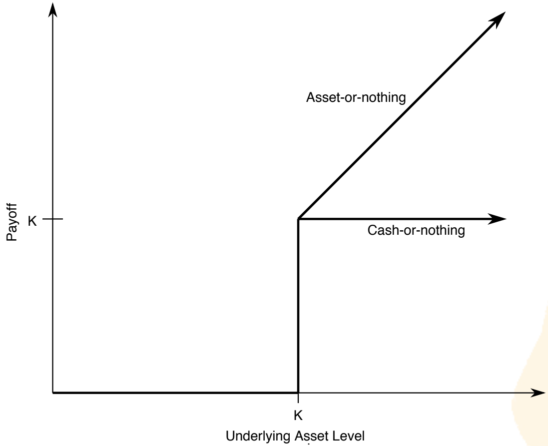


- We can super-replicate the binary payoff using vanilla strikes $\leq K \rightarrow$ upper bound
- We can also sub-replicate the payoff using strikes $\geq K \rightarrow$ lower bound
- We can seek a best-fit (in some measure & metric) portfolio of vanilla options
- Tightness of bounds / goodness of fit depends on what strike prices are available $\rightarrow$ convergence to theoretical value in $\Delta K \searrow 0$ limit

1.1. First-Order Strike Conditions (4)

Diversion on Binary Options

- Two basic kinds of binary (also known as digital or bet) payoff:
  - Cash-or-nothing (call): $B_T(S_T, K) = BC_T(S_T, K) = CONC_T(S_T, K) \doteq (K)\mathbf{1}_{S_T - K}$
  - Asset-or-nothing (call): $AONC_T(S_T, K) \doteq S_T\,\mathbf{1}_{S_T - K}$



1.1. First-Order Strike Conditions (5)

Diversion on Binary Options (continued)

- Relating valuation of binary (call) options to the risk-neutral density q:
  - Cash-or-nothing call:
$$BC_T = CONC_T = \mathbf{1}_{S_T - K} \;\Rightarrow\; CONC_t = \mathcal{R}(t,T)\int_0^\infty dS_T\, q(S_T)\, \mathbf{1}_{S_T - K} = \mathcal{R}(t,T)\int_K^\infty dS_T\, q(S_T)$$
  - Asset-or-nothing call:
$$AONC_T = S_T\, \mathbf{1}_{S_T - K} \;\Rightarrow\; AONC_t = \mathcal{R}(t,T)\int_0^\infty dS_T\, q(S_T)\, S_T\, \mathbf{1}_{S_T - K} = \mathcal{R}(t,T)\int_K^\infty dS_T\, S_T\, q(S_T)$$
- Analogously for binary (put) options:
  - Cash-or-nothing put:
$$BP_T = CONP_T = \mathbf{1}_{K - S_T} \;\Rightarrow\; CONP_t = \mathcal{R}(t,T)\int_0^\infty dS_T\, q(S_T)\, \mathbf{1}_{K - S_T} = \mathcal{R}(t,T)\int_0^K dS_T\, q(S_T)$$
  - Asset-or-nothing put:
$$AONP_T = S_T\, \mathbf{1}_{K - S_T} \;\Rightarrow\; AONP_t = \mathcal{R}(t,T)\int_0^\infty dS_T\, q(S_T)\, S_T\, \mathbf{1}_{K - S_T} = \mathcal{R}(t,T)\int_0^K dS_T\, S_T\, q(S_T)$$

1.1. First-Order Strike Conditions (6)

Diversion on Binary Options (continued)

- Valuation in Black-Scholes-Merton world is easy enough by integration method, but using a trick gives us some insight:
  - Rewrite standard call option payoff in terms of stock-or-nothing and cash-or-nothing (call) payoffs:

$$C_T(S_T, K) = \max[S_T - K, 0] = S_T \mathbf{1}_{S_T - K} - K\,\mathbf{1}_{S_T - K}$$

  - Now, remember that terms in the integral don't mix:

$$C_t(S_t, K) = S_t\, e^{-y(T-t)}\,\mathcal{N}\left[z_+\right] - K\, e^{-r(T-t)}\,\mathcal{N}\left[z_-\right]$$

$$\text{with } z_\pm = \frac{\ln\left(S_t/K\right) + (r-y)(T-t)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

- We can interpret the first term in the Black-Scholes formula as the value of an asset-or-nothing call struck at K and the second term as $(-K)$ cash-or-nothing calls also struck at K:

$$AONC_t(S_t, K) = S_t\, e^{-y(T-t)}\,\mathcal{N}\left[z_+\right]$$

$$B_t(S_t, K) = BC_t(S_t, K) = CONC_t(S_t, K) = e^{-r(T-t)}\,\mathcal{N}\left[z_-\right]$$

1.1. First-Order Strike Conditions (7)

- Valuation of "put" versions of binary options follows analogously (e.g. by decomposing the Black-Scholes put option formula, or by applying put-call parity)
  - Since the sum of the payoffs of a binary call and a binary put equals unconditional delivery of cash (for cash-or-nothing binaries) or stock (for asset-or-nothings), their present value must sum to the present value of future delivery of cash or stock, respectively:

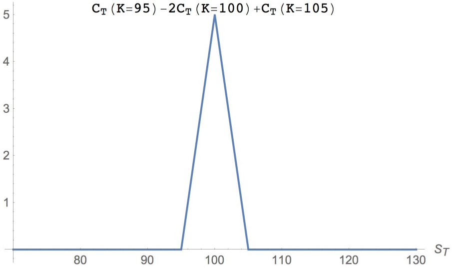
$$
\begin{aligned}
& CONC_T(S_T, K) + CONP_T(S_T, K) = 1 \\
\Longrightarrow\ & CONC_t(S_t, K) + CONP_t(S_t, K) = e^{-r(T-t)} \\
\Longrightarrow\ & CONP_t(S_t, K) = e^{-r(T-t)}(1 - \mathcal{N}[z_-]) = e^{-r(T-t)}\mathcal{N}[-z_-]
\end{aligned}
$$

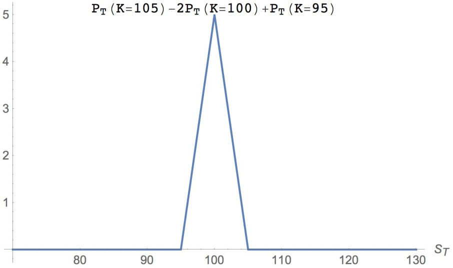
$$
\begin{aligned}
& AONC_T(S_T, K) + AONP_T(S_T, K) = S_T \\
\Longrightarrow\ & AONC_t(S_t, K) + AONP_t(S_t, K) = S_t\, e^{-y(T-t)} \\
\Longrightarrow\ & AONP_t(S_t, K) = S_t\, e^{-y(T-t)}(1 - \mathcal{N}[z_+]) = S_t\, e^{-y(T-t)}\mathcal{N}[-z_+]
\end{aligned}
$$

- Basic reference: Rubinstein and Reiner, "Unscrambling the Binary Code," *Risk Magazine* **4**(9) (Oct 1991), 75-83.

1.2. Second Order Strike Conditions

- Second-order conditions require that butterfly spreads have non-negative values since they correspond to payoffs that are everywhere non-negative





- We can think of the butterfly spread position as long a bullish call spread and short another whose lower strike is equal to the upper strike of the first spread, or the analogous position in bearish put spreads (with the order of strikes reversed):

$$[C_T(K-\Delta K,T) - C_T(K,T)] - [C_T(K,T) - C_T(K+\Delta K,T)] = [P_T(K+\Delta K,T) - P_T(K,T)] - [P_T(K,T) - P_T(K-\Delta K,T)]$$

- This relation holds at any time $t \leq T$:

$$[C_t(K-\Delta K,T) - C_t(K,T)] - [C_t(K,T) - C_t(K+\Delta K,T)] = [P_t(K+\Delta K,T) - P_t(K,T)] - [P_t(K,T) - P_t(K-\Delta K,T)]$$

1.2. Second Order Strike Conditions (2)

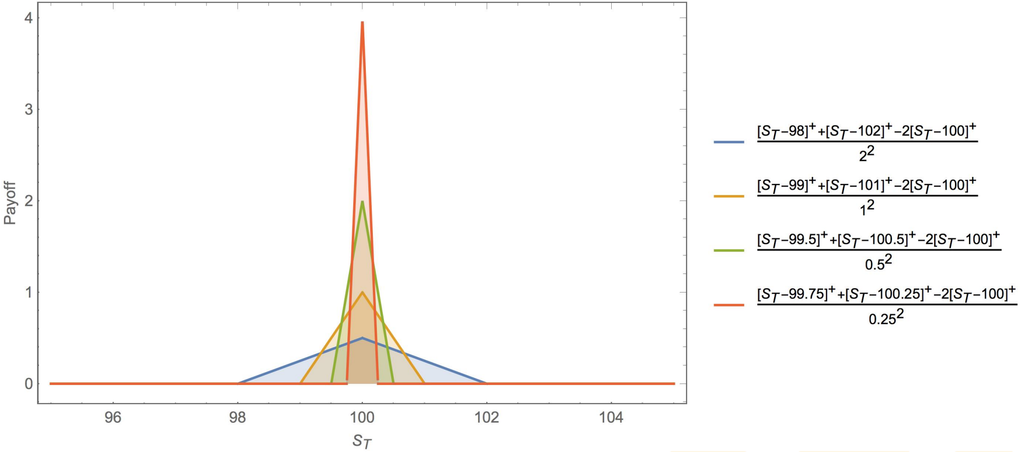
- Taking the limit as $\Delta K \searrow 0$ and applying the representation in terms of q, we find:

$$
\begin{aligned}
&\lim_{\Delta K \searrow 0} \frac{1}{\Delta K}\left(\frac{[C_t(K-\Delta K,T) - C_t(K,T)]}{\Delta K} - \frac{[C_t(K,T) - C_t(K+\Delta K,T)]}{\Delta K}\right) = \\
&\lim_{\Delta K \searrow 0} \frac{1}{\Delta K}\left(\frac{[P_t(K+\Delta K,T) - P_t(K,T)]}{\Delta K} - \frac{[P_t(K,T) - P_t(K-\Delta K,T)]}{\Delta K}\right) \\
&\qquad = -\frac{d}{dK}\left(-\frac{dC_t(K,T)}{dK}\right) = \frac{d}{dK}\left(\frac{dP_t(K,T)}{dK}\right) \\
&\qquad = \frac{d^2C_t(K,T)}{dK^2} = \frac{d^2P_t(K,T)}{dK^2} \\
&\qquad = \mathcal{R}(t,T)\, q(S_T)|_{S_T=K} \\
&\qquad \geq 0
\end{aligned}
$$

- This condition is equivalent to requiring non-negativity of the risk-neutral density for any terminal asset price $\implies$ we can infer the state price density from option prices.
- Note that both first and second-order conditions are independent of the asset price process that generates returns.
- Breeden, D. and Litzenberger, R., "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business* **51**(4) (Oct 1978), 621-651.

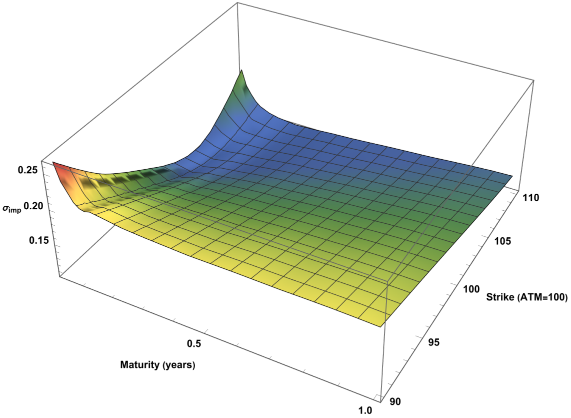
1.2. Second Order Strike Conditions (3)

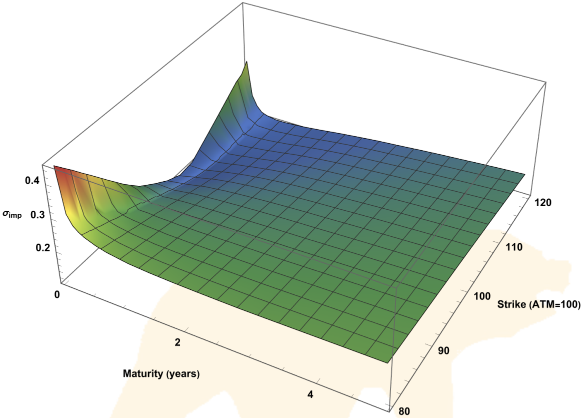
Convergence of Butterfly Spreads to Delta Function



1.3. Volatility Skew and Smile

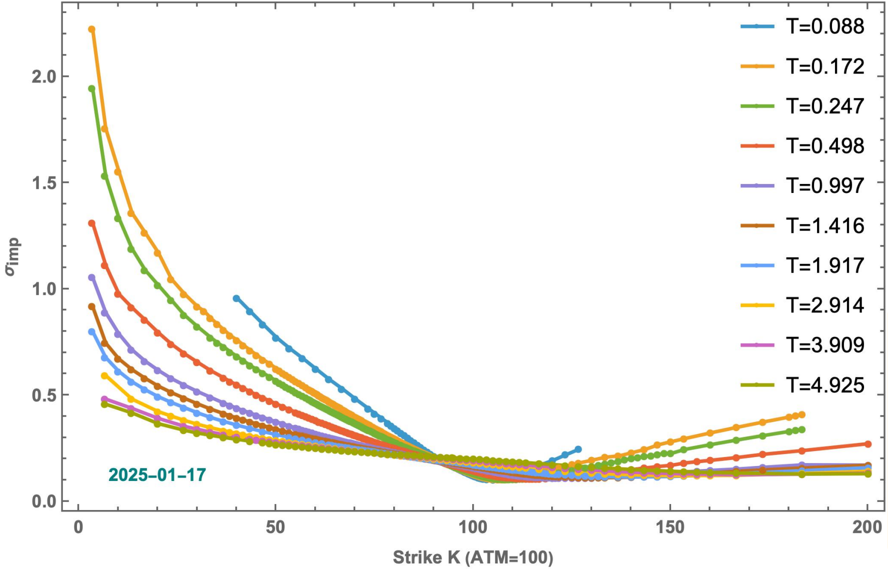
- Why do we care about this?
- Volatility surface (equity index example: SPX CBOE options @ close 2025-01-17):





1.3. Volatility Skew and Smile (2)

- 17 Jan 2025 closing data for SPX ($S = 5996.66 \simeq 6000$, source: CBOE):



1.3. Volatility Skew and Smile (3)

- For equity (indices), typical convention for implied vols for strikes not too far from at-the-money:
$$\sigma_{imp}(K,T;S) = \sigma_{atm}(T) + \xi(T)\ln(S/K) + \kappa(T)[\ln(S/K)]^2 + \ldots$$

  - Smile $\rightarrow$ skew, smirk, sneer or leer
  - Some attempts have been made to posit general forms for $\xi(T), \kappa(T)$, e.g. $\xi(T) \sim \xi_0 T^{-1/2}$ (traders' rules)
  - Slightly different (theoretically preferable) convention for currency options:
$$\sigma_{imp}(K,T;F) = \sigma_{atmf}(T) + \xi(T)\ln(F/K) + \kappa(T)[\ln(F/K)]^2 + \ldots$$
- Implications:
  - $\dfrac{d}{dK} \neq \dfrac{\partial}{\partial K}$, but rather $\dfrac{d}{dK} = \dfrac{\partial}{\partial K} + \dfrac{\partial \sigma}{\partial K}\dfrac{\partial}{\partial \sigma}$
  - $\dfrac{d^2}{dK^2} \neq \dfrac{\partial^2}{\partial K^2}$, but rather $\dfrac{d^2}{dK^2} = \left(\dfrac{\partial}{\partial K} + \dfrac{\partial \sigma}{\partial K}\dfrac{\partial}{\partial \sigma}\right)^2$
  - Entirely non-classical effects!

1.3. Volatility Skew and Smile (4)

Where do the volatility skew and smile come from? Some explanations & stylized models

- Skew
  - Leverage (capital structure) effects
    - Merton's (1974) capital structure model: equity as call option on firm assets (struck at value of debt)
      Underpinning all modern structural credit models, e.g., Moody's-KMV, Leland-Toft, ...
    - Some simpler modelling alternatives: displaced diffusion; CEV (constant elasticity of variance):
      $$dS_t = (r-y)S_t\,dt + \hat{\sigma}\,S_t^{1-\alpha}\,dW_t^Q = (r-y)S_t\,dt + \sigma_0(S_0/S_t)^{\alpha}S_t\,dW_t^Q$$
    - Difficult to reconcile with observed steepness of index skew, skew presence for stocks with little to no leverage
    - Generalization of this theme – "stochastic" vol as a deterministic function $\sigma(S_t,t)$: Dupire's (1994) local vol (or ***implied diffusion***) model, with $\sigma(S_t,t)$ determined to fit entire volatility surface (term structure + skew + smile)
  - Jump (crash) risk
    - E.g., Merton's (1976) jump diffusion model:
      $$\frac{dS_t}{S_t} = \left[(r-y)\,dt + \sigma\,dW_t^{\mathbf{Q}}\right] + \left[-\lambda_{J,t}\,dt + dJ_t\right],$$
      where J_t is a ***compound Poisson*** process (independent jumps with some size distribution) and $\lambda_{J,t}$ is a ***compensator*** that ensures the martingale property.
    - Solvable in closed form for simple (e.g., log-normal) assumptions about jump distribution
    - Inspiration behind a plethora of more complex models, e.g., variance gamma (Madan), pure jump (Lèvy) processes
      Some of these models can be made to align with phenomenological traders' rules, stylized "universality" patterns
    - **P**-measure jump risk (incompleteness) amplified via market participants' risk aversion, consumption preferences, appetite for insurance, transaction costs... $\Longrightarrow$ **Q**-measure vol skew

1.3. Volatility Skew and Smile (5)

Where do the volatility skew and smile come from? Some explanations & stylized models (cont.)

- Smile
  - Stochastic volatility: variance of variance
    $\Longrightarrow$ leptokurtic (fat tailed) returns distribution and increasing σ_{imp} in the tails.
  - One popular model, arguably the benchmark: Heston (1993):

$$dS_t = (r-y)S_t\,dt + \sqrt{v_t}\,S_t\,dW^Q_{S,t}$$

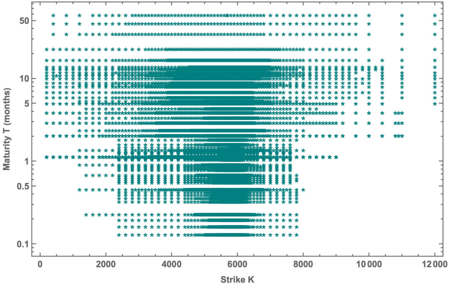
$$dv_t = \kappa(v_\infty - v_t) + \sigma_v\sqrt{v_t}\,dW^Q_{v,t} \quad \text{with: } (dW^Q_{v,t}\,dW^Q_{S,t}) = \rho_{v,S}\,dt$$

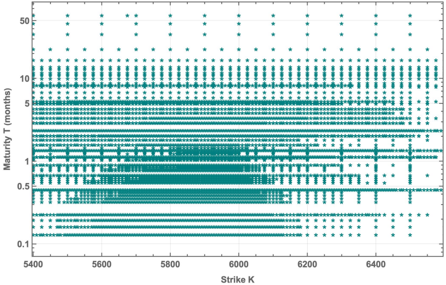
  - Note that $\rho_{v,S}$ can make the smile lean (in either direction):
    - alternative explanation of vol skew consistent with stylized facts like “volatility increases in down markets”
    - useful in currency and commodity settings with directional pressure/jump risk
  - The characteristic function (i.e., Fourier transform of the density of) $\ln(S_t)$ can be written in closed form.
  - Contemporary extensions include “rough” Heston models (Gatheral *et al.*) in which $W_{v,t}$ is taken to be a ***fractional*** Brownian motion.
  - There is a good bit of anecdotal evidence that “the market” prices (or at least fits slices of) the volatility surface using Heston or a simplified version thereof, e.g., SVI (*Stochastic Volatility Inspired*):

$$\sigma^2_{imp}(K) \sim a + b\left(\rho\ln(K/K^*) + \sqrt{\ln^2(K/K^*) + \sigma_0^2}\right)$$

1.3. Volatility Skew and Smile (6)

- Example: CBOE 17 Jan 2025 closing data for SPX+SPXW European Options ($S = 5996.66$)
  - 11987 lines of bid-ask data for calls and puts
    - lines deleted: 433 that day's expiry ⇒ 11554 valid lines (over 23k options/46k prices)
    - 59 distinct maturities from Tue 21 Jan 2025 to Fri 21 Dec 2029; strike prices K from 200 to 12000



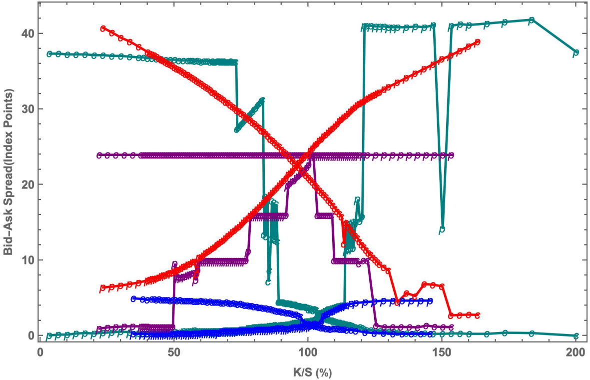


  - Downsample to monthly options (19 maturities, same strike range: 3657 lines)
- We'll also consider data from three other dates, with very similar overall statistics, exemplifying 3 different market regimes:
  - 11 Jun 2019 (calm, pre-COVID), $S = 2885.72$
  - 05 Jun 2020 (excited, a few months into COVID), $S = 3193.93$
  - 11 Jun 2020 (crisis: SPX down nearly 6% on the day), $S = 3002.10$
  - 17 Jan 2025 (current: SPX up 1% on the day), $S = 5996.66$

1.3. Volatility Skew and Smile (7)

Market Liquidity: Bid-Ask Spread Data

- Focus on $T \approx 1$ year tenors (e.g., next year's January expiry)

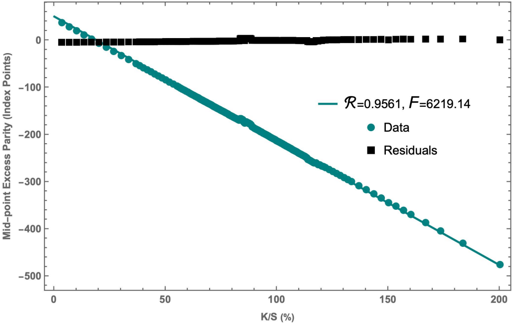


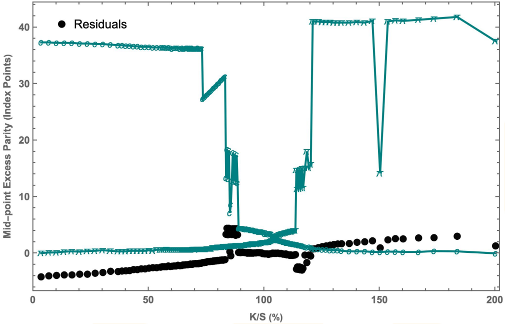
- Also examine historical evolution
  - Spreads progressively widen as conditions go from calm to excited to crisis, then back

1.3. Volatility Skew and Smile (8)

Put-Call Parity

- Fit $C(K,T) - P(K,T) = \mathcal{R}(T)(F(T) - K) = S\,e^{-y_{imp,T}T} - K\,e^{-r_{imp,T}T}$ for $\{\mathcal{R}(T), F(T)\} \leftrightarrow \{r_{imp,T}, y_{imp,T}\}$ to midpoint data across strikes for each tenor
- Data model: $C \sim N\left[C_{mid}, (C_{ask} - C_{bid})^2\right], P \sim N\left[P_{mid}, (P_{ask} - P_{bid})^2\right]$: price uncertainty proportional to (bid-ask spread width), so points weighted inversely to (spread width)^2
- We actually fit the *excess* parity $(S - K) - (C - P) = (S - \mathcal{R}F) - (1 - \mathcal{R})K$: removing the major contributions to slope and intercept produces a **much** cleaner test of Put-Call Parity

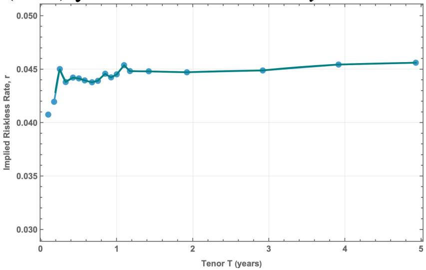


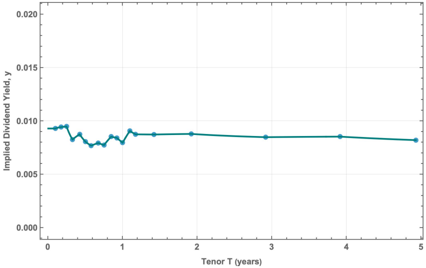


- Adj. R^2 for 1 yr tenor $= 0.999991$. All errors/residuals well inside bid-ask spreads.

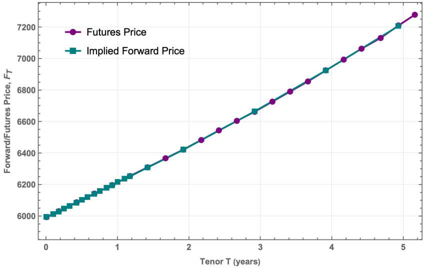
1.3. Volatility Skew and Smile (9)

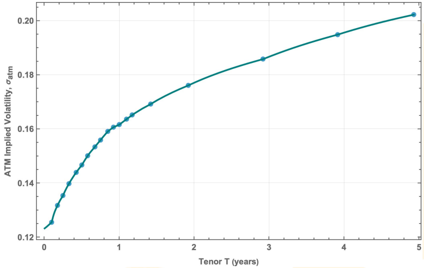
- Implied (zero) yield curves for r and y:





- Implied Forward and ATM Vol curves:





1.3. Volatility Skew and Smile (10)

- CBOE 17 Jan 2025 closing data for SPX ($S = 5996.66$), 16 Jan 2026 expiry ($T \approx 0.997$ yrs)

0.996518 1.00822

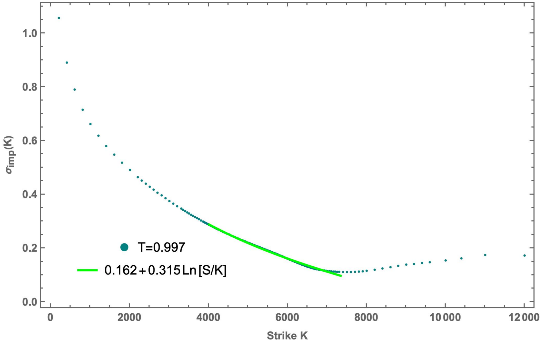
	Estimate	Standard Error	t-Statistic	P-Value
DF	0.956073	0.0000793318	12 051.6	0.
F	6219.14	0.0523081	118 894.	0.

0.999991

	Estimate	Standard Error	t-Statistic	P-Value
r	0.0445549	0.0000823002	541.37	8.26791×10^{-294}
y	0.00799934	0.000080468	99.4101	1.54214×10^{-160}

0.999991

  - Implied $r \approx 4.45\%$ (in neighborhood of 1-year USD term rates)
  - Implied $y \approx 0.80\%$ (a little below current backward-looking 1-year SPX div yield of $\sim 1.25\%$)
  - Enforce parity $\implies$ same σ_{imp} for both calls & puts, with (bid-ask)-adjusted mid-prices.

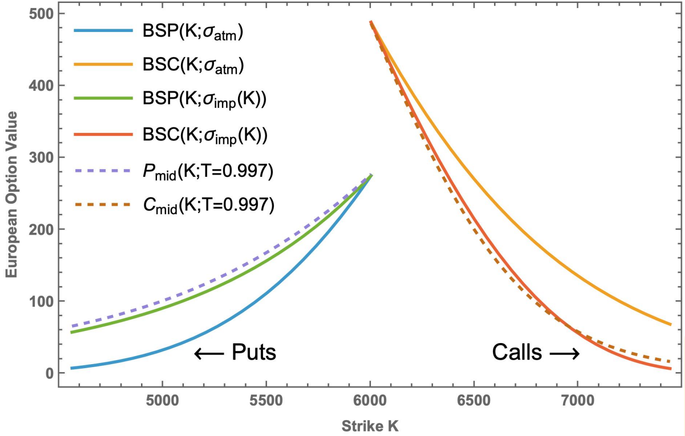


- ATM $\{-2.5, +1.25\}$ s.d.: $\sigma_{imp}(K, T \approx 0.997; S \approx 6000) = 16.2\% + 31.5\% \ln(S/K)$ (adj. $R^2 = 0.9971$)

1.3. Volatility Skew and Smile (11)

Impact of implied vol on vanilla European option values

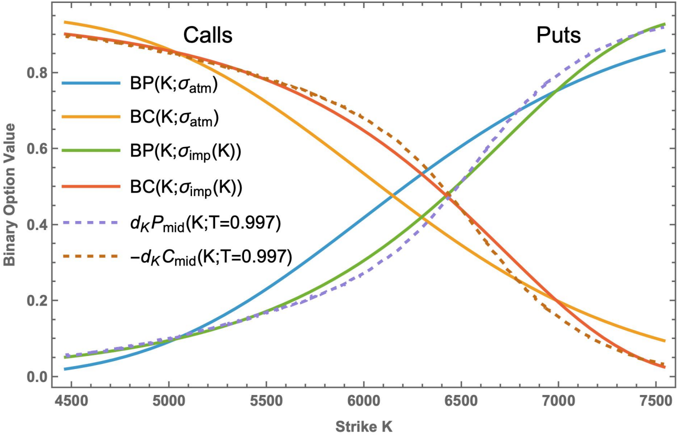
- "Problem set parameters": $S = 6000$, $T = 1.0$ yr, $r = 445$ bp, $y = 0.80\%$, $\sigma_{atm} = 16\%$, $\xi = 30\%$, $\kappa = 0$



- Volatility "skew" makes downside (puts) more expensive and upside (calls) cheaper.

1.3. Volatility Skew and Smile (12)

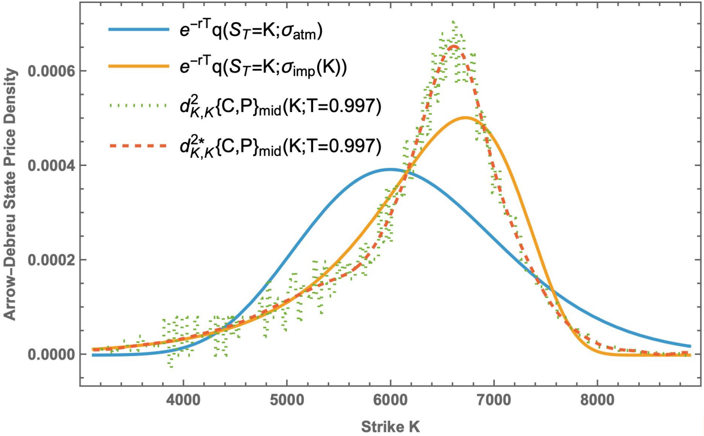
Impact of implied vol on binary option values



- Binary put-call parity is apparent in these results.
- Fatter tail on the (far) downside and thinner tail on the (far) upside are as expected, but why the downward bulge in the cdf ($\sim$ binary put values) at intermediate strikes?

1.3. Volatility Skew and Smile (13)

Impact of implied vol on Arrow-Debreu state price density



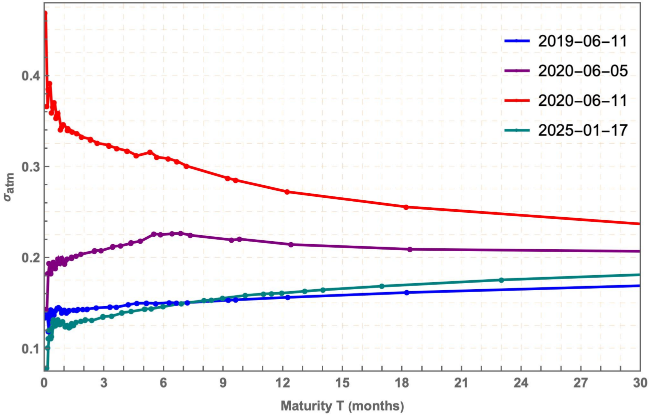
- Note higher moment effects that are unexpected *ex ante*
- Why do these arise? (hint: $S_0 e^{(r-y)T}$)
- Multiple differentiation of even slightly noisy input data is perilous

  $\Longrightarrow$ avoid working directly with empirical $\mathbf{Q}$ density if at all possible.

1.3. Volatility Skew and Smile (14)

Cross-sectional behavior of the implied volatility surface vs. t and T (1)

- Observations of the ATM volatility term structure:

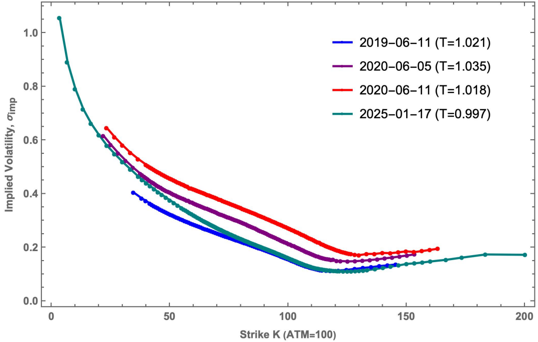


- These results suggest that the most persistent characteristics are:
  1. some form of mean-reversion, especially out toward long-term tenors and
  2. (perhaps) persistence in perceptions of elevated volatility around specific future events.

1.3. Volatility Skew and Smile (15)

Cross-sectional behavior of the implied volatility surface vs. t and T (2)

- But is the overall shape of the volatility surface persistent?
  Our "three-regime" results vs. strike K for the $\sim$1 year tenors are suggestive:

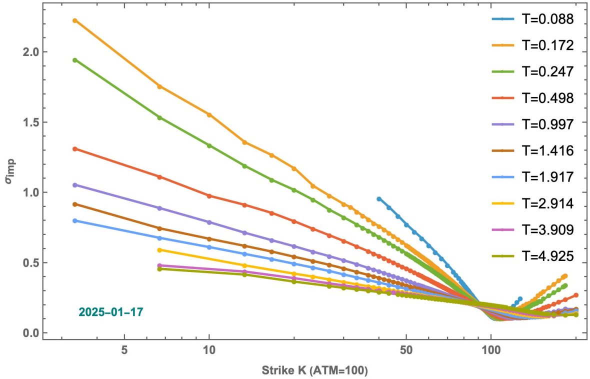


- Overall shape is remarkably consistent, although the curve shifts up-and-down and, to a more limited extent, left-right.
- Fitted implied vol "skewness" coefficient ξ – multiplying $\ln(S/K)$ – only rises from 32.4% (calm) to 37.3% (elevated) to 38.4% (crisis), then back to 31.5% over the $\sim$4.5 years.

1.3. Volatility Skew and Smile (16)

Cross-sectional behavior of the implied volatility surface vs. t and T (3)

- Is there any consistency/universality across maturity tenors T at a given time t?
- Our cross-sectional slices for various T (σ_{imp} vs. K) appear to reveal only a general, overall similarity in shape:

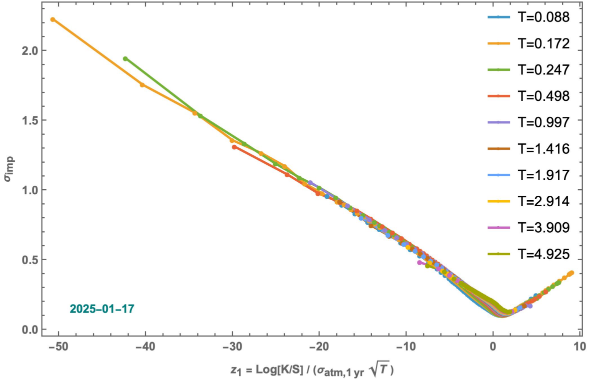


- But a change to log K/S co-ordinates hints at more...

1.3. Volatility Skew and Smile (17)

Cross-sectional behavior of the implied volatility surface vs. t and T (4)

- What happens if we take inspiration from the "old (equity) traders' tale"?
  - Volatility skew (i.e. ξ) should scale as $1/\sqrt{T}$, so define a normalized co-ordinate: $z_1 \doteq \ln(K/S)/(\sigma_1\sqrt{T})$, always using the 1-year tenor ATM vol σ_1 to non-dimensionalize.

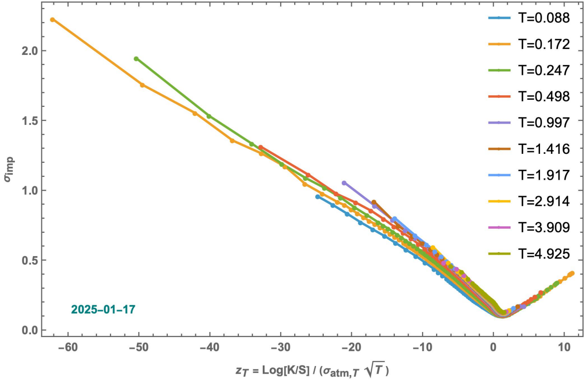


  - Remarkably good fit, even for extreme strikes at the smallest (1 month) tenor
  - Generally consistent during calm, excited, and crisis periods
  - Impact of ATM vol term structure in the tails isn't completely clear.

1.3. Volatility Skew and Smile (18)

Cross-sectional behavior of the implied volatility surface vs. t and T (5)

- One last experiment, testing common FX market convention:
  - $\sigma_{imp}(\Delta = \mathcal{N}(z_+)) \leftrightarrow \sigma_{imp}\left(z_+ = \frac{\ln(F/K)}{\sigma_{atm,T}\sqrt{T}} + \frac{\sigma_{atm,T}\sqrt{T}}{2}\right)$. At these scales, $\frac{\ln(F/S)}{\sigma_{atm}\sqrt{T}}$ & $\frac{\sigma_{atm}\sqrt{T}}{2}$ offsets are negligible, so define normalized co-ordinate: $z_T \doteq \ln(K/S)/(\sigma_{atm,T}\sqrt{T})$:



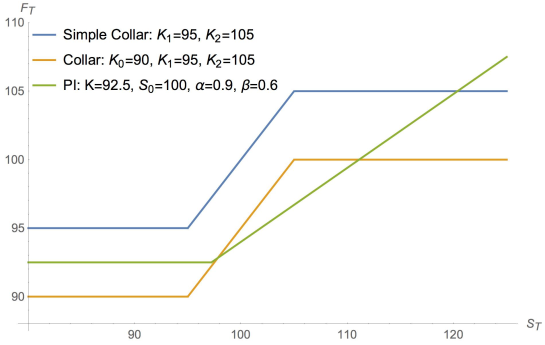
  - Fits are no better or somewhat worse, including those for calm, excited, and crisis periods.
  - For SPX, it appears the old equity traders' rule works better than FX convention.

2. Pricing via (Static) Replication

- Turn Breeden & Litzenberger's result on its head
- First, consider piecewise linear payoffs
- Then generalize to arbitrary (convex/concave) payoffs

2.1. Piecewise Linear Payoffs (Packages)

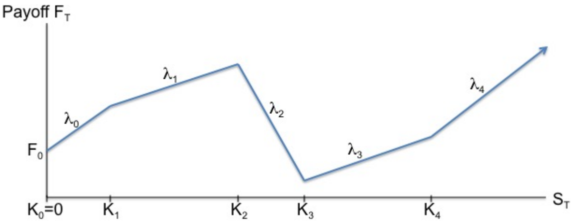
- Many strategies involving options of a single maturity can be written as piecewise linear payoffs:
  - Bull (call) and bear (put) spreads
  - Simple collars: $F_T = \min[\max(S_T, K_1), K_2] = K_1 + \max[0, S_T - K_1] - \max[0, S_T - K_2]$ (where $0 < K_1 < K_2$)
    - Generalized collars, including range forwards (zero-cost collars): $F_T = K_0 + \max[0, S_T - K_1] - \max[0, S_T - K_2]$
  - Portfolio insurance-like strategies: $F_T = \max[K, S_0 + \beta(\alpha S_T - S_0)] = K + \alpha\beta \max[0, S_T - K^*]$, where: $0 < \alpha < 1$, $\beta > 0$, and $K^* \doteq [K - S_0(1-\beta)]/\alpha\beta$
  - Straddles, butterflies, etc.



  - Source: Rubinstein, M., "Packages," (9 Dec 1991), 1-6.

2.1. Piecewise Linear Payoffs (Packages) (2)

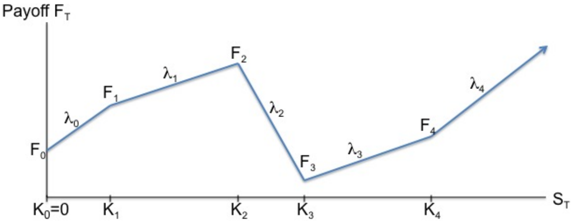
- Generic representation of piecewise linear payoff:



  - Kink points and slopes: K_i, λ_i
  - Value and slope at $S_T = 0$: F_0, λ_0
  - Note that if $F_0 = 0$, $\lambda_0 = 0$, payoff effectively begins at K_1
- Valuation: F_T decomposes into: $F_T = F_0 + \lambda_0 S_T + \sum_i (\lambda_i - \lambda_{i-1})[S_T - K_i]^+$
- Hence: $F_t = e^{-r(T-t)} F_0 + \lambda_0 e^{-y(T-t)} S_t + \sum_i (\lambda_i - \lambda_{i-1}) C_t[S_t; K_i, T-t]$
  - Process-independent results
  - Can also begin decomposition at any particular K_i, with calls on the upside and puts on the downside

2.1. Piecewise Linear Payoffs (Packages) (3)

- What if we wish to represent payoff in terms of values F_i at kink points, and not slopes λ_i in between?



$\Longrightarrow$ Identify $\lambda_i = \frac{F_{i+1}-F_i}{K_{i+1}-K_i}$

- Rewrite F_T decomposition:

$$F_T = F_0 + \frac{F_1-F_0}{K_1-0} S_T + \sum_i \left(\frac{F_{i+1}-F_i}{K_{i+1}-K_i} - \frac{F_i-F_{i-1}}{K_i-K_{i-1}}\right)[S_T - K_i]^+$$
$$= F_0 + \frac{\Delta F_0}{\Delta K_0} S_T + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right)[S_T - K_i]^+$$

- Hence: $F_t = e^{-r(T-t)} F_0 + \frac{\Delta F_0}{\Delta K_0} e^{-y(T-t)} S_t + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right) C_t[S_t; K_i, T-t]$

2.2. General Payoffs

- Consider a general bounded payoff $F_T(S_T)$ with bounded first derivative at $S_T = 0$
  - We can (partly) relax the first boundedness restriction later
- Assume the existence of vanilla options of all strikes $\{C(K)\}$
- How can we price (and replicate) F using $\{C(K)\}$, stock, and bonds?
- Start with piece-wise linear payoffs and take many K_i, F_i, with $\Delta K_i, \Delta F_i \searrow 0$:

$$F_T = F_0 + \frac{\Delta F_0}{\Delta K_0} S_T + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right)[S_T - K_i]^+$$

$$\Rightarrow F_0 + F_0' S_T + \int_0^\infty dK\, F''(K)[S_T - K]^+$$

- Hence:

$$F_t = e^{-r(T-t)} F_0 + \frac{\Delta F_0}{\Delta K_0} e^{-y(T-t)} S_t + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right) C_t[S_t; K_i, T-t]$$

$$\Rightarrow e^{-r(T-t)} F_0 + F_0' e^{-y(T-t)} S_t + \int_0^\infty dK\, F''(K) C_t[S_t; K, T-t]$$

2.2. General Payoffs (2)

- Representation of general payoff in terms of bond + stock + vanilla (call) options:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = 0) + F_T'(0)\, C_t(S_t, K = 0) + \int_0^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

- Breeden-Litzenberger results are distribution- and process- independent: so are these!
  - Robust, static replicating portfolio
  - But, depends on availability of options of (all) strikes

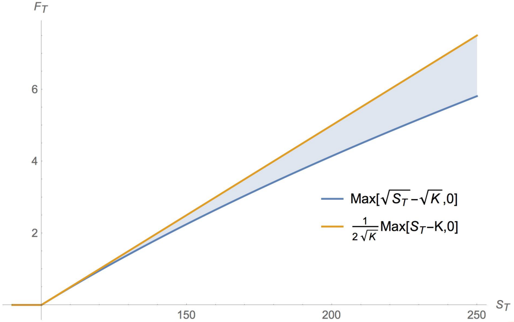
- In practice, we may not want to build the replicating portfolio out of calls starting at strike 0, but rather bifurcate the portfolio into calls with strike greater than some K^* (e.g. at-the-money) and puts with strike less than K^*:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = K^*) + F_T'(S_T = K^*) \left(S_t e^{-y(T-t)} - K^* e^{-r(T-t)} \right) + \int_0^{K^*} dK\, F_T''(K)\, P_t(S_t, K) + \int_{K^*}^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

  - Reference: Carr, P. and Madan, D., "Towards a Theory of Volatility Trading," In: *Volatility: New estimation techniques for pricing derivatives.* Risk Books (1998), 417-427.
  - Why might we want to build a replicating portfolio this way instead of with calls at all K?

2.2. General Payoffs (3)

- Example: suppose $F_T = \left[S_T^{1/2} - K^{1/2}\right]^+$ (with $K = 100$):



Then:

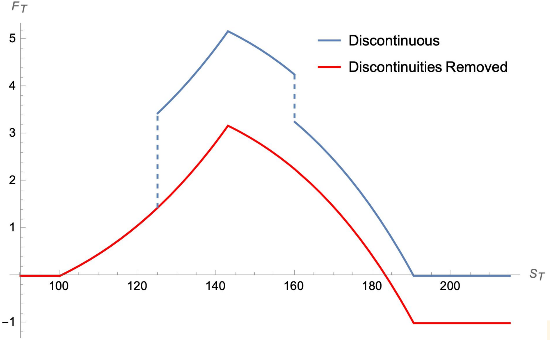
$$F_T' = \begin{Bmatrix} 0 & S_T < K \\ \frac{1}{2} S_T^{-1/2} & S_T > K \end{Bmatrix}$$

$$F_T'' = \begin{Bmatrix} 0 & S_T < K \\ \frac{1}{2} K^{-1/2} \delta(S_T - K) & S_T = K \\ -\frac{1}{4} S_T^{-3/2} & S_T > K \end{Bmatrix}$$

- Discontinuous 1^{st} derivatives lead to singular 2^{nd} derivatives and discrete distributions of options

2.2. General Payoffs (4)

- What happens when F is discontinuous so that its first derivatives are singular?



- Replicating portfolio also contains binary options:

$$F_t = e^{-r(T-t)} F_T(0) + F'_T(0)\, C_t(0) + \int_0^{\infty} dK\, F''_T(K)\, C_t(K) + \sum_i \Delta F_T(K_i)\, B_t(K_i) \; (*)$$

where $\Delta F_T\,(K_i)$ is the i^{th} jump in the payoff

(*) note that F''_T in the integral must be considered with the jumps removed

3. Introduction to Symmetry/Transformation Methods

- In the B-S-M world, we can sometimes use properties of geometric Brownian motion to simplify valuation of more complex payoffs, particularly where the payoff depends on powers of the underlying asset price.
- Example: typical power option payoff: $C_T = \max[S_T^\alpha - K, 0]$
- Again, valuation by integration is straightforward (**Q**-measure expectation discounted by $e^{-r(T-t)}$), but why not add another trick to the toolbox?
  - Our process for S is: $dS = (r-y)S\,dt + \sigma S\,dW^{\mathbf{Q}}$
  - Construct a process (using Ito's lemma) for $S' \doteq S^\alpha$ (hence $S'_t = S_t^\alpha$):

$$dS^\alpha = \alpha[(r-y) + (\alpha-1)\frac{\sigma^2}{2}]\,S^\alpha\,dt + \alpha\sigma S^\alpha\,dW^{\mathbf{Q}}$$

$$\Longrightarrow \quad dS' = \alpha[(r-y) + (\alpha-1)\frac{\sigma^2}{2}]\,S'\,dt + \alpha\sigma S'\,dW^{\mathbf{Q}}$$

  - Interpret this as a (pseudo-) price process for S' with modified volatility and payout rate:

$$dS' = (r - y_{\mathit{eff}})S'\,dt + \sigma_{\mathit{eff}} S'\,dW^{\mathbf{Q}}$$

$$\Longrightarrow \quad \sigma_{\mathit{eff}} = |\alpha|\,\sigma, \quad y_{\mathit{eff}} = r - \alpha[(r-y) + (\alpha-1)\frac{\sigma^2}{2}]$$

3. Introduction to Symmetry/Transformation Methods (2)

- So, we can write down valuation result for the power option:

$$C_t(S'_t, K) = S'_t\, e^{-y_{eff}(T-t)} \mathcal{N}(z_{eff,+}) - K\, e^{-r(T-t)} \mathcal{N}(z_{eff,-})$$
$$C_t(S^\alpha_t, K) = S^\alpha_t\, e^{-y_{eff}(T-t)} \mathcal{N}(z_{eff,+}) - K\, e^{-r(T-t)} \mathcal{N}(z_{eff,-})$$
$$\text{with } z_{eff,\pm} = \frac{\ln\left(S^\alpha_t / K\right) + (r - y_{eff})(T-t)}{\sigma_{eff}\sqrt{T-t}} \pm \frac{\sigma_{eff}\sqrt{T-t}}{2}$$

- If we have a Black-Scholes call valuation function **BSCall**$[S, K, T, \sigma, r, y]$, then we can value the power option with **BSCall**$[S^\alpha_t, K, T-t, \sigma_{eff}, r, y_{eff}]$
- Even easier in Black's model **BSCall**$[F_{eff,T}, K, T, \sigma, r]$ with:

$$F_{eff,T} \doteq \mathbb{E}^{\mathbf{Q}}[S^\alpha_T] = S^\alpha_t e^{(r-y_{eff})(T-t)} = S^\alpha_t e^{\alpha[(r-y)+(\alpha-1)\frac{\sigma^2}{2}](T-t)}$$
$$\Longrightarrow \quad C_t(F_{eff,T}, K) = e^{-r(T-t)}\left[F_{eff,T}\, \mathcal{N}(z_{eff,+}) - K\, \mathcal{N}(z_{eff,-})\right]$$
$$\text{with } z_{eff,\pm} = \frac{\ln\left(F_{eff}/K\right)}{\sigma_{eff}\sqrt{T-t}} \pm \frac{\sigma_{eff}\sqrt{T-t}}{2}$$

3. Introduction to Symmetry/Transformation Methods (3)

- Power option example 2: leveraged option payoff: $C_T = S_T^\alpha \max[S_T - K, 0]$
- Here another type of trick is useful. Write: $C(S_t, t) = S_t^\alpha D(S_t, t)$
- Substitute this into the Black-Scholes PDE: $\dfrac{\partial C}{\partial t} + (r-y)S\dfrac{\partial C}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} = rC$
- Obtain a modified PDE:

$$\frac{\partial D}{\partial t} + (r - y + \alpha\sigma^2)S\frac{\partial D}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 D}{\partial S^2} = \left[r - \alpha(r - y + \frac{\alpha-1}{2}\sigma^2)\right] D$$

- Interpret this as a B-S PDE (hence price process) with modified interest and payout rates:

$$\frac{\partial D}{\partial t} + (r_{\mathit{eff}} - y_{\mathit{eff}})S\frac{\partial D}{\partial S} + \frac{1}{2}\sigma_{\mathit{eff}}^2 S^2 \frac{\partial^2 D}{\partial S^2} = r_{\mathit{eff}}\, D$$

$$\implies \quad r_{\mathit{eff}} = r - \alpha(r - y + \frac{\alpha-1}{2}\sigma^2), \;\; y_{\mathit{eff}} = y - \alpha(r - y + \frac{\alpha+1}{2}\sigma^2), \;\; \sigma_{\mathit{eff}} = \sigma$$

- Finally, substitute for C to obtain: $C(S_t, t) = S_t^\alpha \left[S_t e^{-y_{\mathit{eff}}(T-t)} \mathcal{N}(z_{\mathit{eff},+}) - K e^{-r_{\mathit{eff}}(T-t)} \mathcal{N}(z_{\mathit{eff},-})\right]$ with $z_{\mathit{eff},\pm}$ defined in terms of effective parameters
- Again easier in Black's model with $\{F_{\mathit{eff}}, r_{\mathit{eff}}\}$, with r_{eff} here the same as y_{eff} in previous example.
  - Or even with $\{F_{\mathit{eff}} = \mathbb{E}^{\mathbf{Q}}[S_T^{\alpha+1}], K_{\mathit{eff}} = K\,\mathbb{E}^{\mathbf{Q}}[S_T^\alpha], r_{\mathit{eff}} = r\}$!

3. Introduction to Symmetry/Transformation Methods (4)

- When are these particular approaches useful?
  - When the payoff can be (re-)written in terms of the $\max$ or $\min$ of two log-normally distributed variables under the risk-neutral measure
  - E.g.: outperformance options, quantos, geometric Asian options, geometrically-weighted basket options...
- Why do we define effective parameters, in particular r and y?
  - Work from the point-of-view that we've built and debugged a standard Black-Scholes calculator, e.g.:

$$\textbf{BSCall}[S\_, K\_, T\_, \sigma\_, r\_, y\_] := \ldots$$
$$\text{or } \textbf{BlackCall}[F\_, K\_, T\_, \sigma\_, r\_] := \ldots$$

  - If we can identify the coefficients that determine the relevant/effective forward price(s), discount rates, and (log-) volatility, we can feed those (perhaps via a wrapper function) into our existing Black-Scholes calculator without having to re-program the standard option valuation formulas each time. The process of organizing the effective inputs to Black-Scholes is called "pre-washing" the inputs.
  - The process of deriving expressions for r_{eff} and y_{eff} can seem a little confusing because r plays two roles in the valuation: driving the forward price (expected value) and discounting the whole payoff. We want both roles to behave correctly, so we use $r_{eff} - y_{eff}$ as the rate at which the forward price grows and r_{eff} as the discounting rate (and so, have to solve for y_{eff}).
  - Note that to get correct sensitivities to our original input parameters, we may have to "post-wash" output sensitivities from the Black-Scholes code in our wrapper function.

4. Appendices

- Appendix 3.1: Hedge Decomposition & Conservation Principles
- Appendix 3.2: Backward and Forward Equations
- Appendix 3.3: General Payoffs Static Replication (Another Proof)

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles

- Normally, we view the option pricing problem as one of mapping a set of states (payoffs) into a value C via a probability density.
  - Feynman-Kac theorem: primary role of PDE is to generate density of states.
  - Analogous to statistical mechanics (computation of an energy measure or "fundamental equation")
- Alternative (complementary) viewpoint:
  - Consider option prices in terms of constitutive elements (hedge positions in underlying assets, incl. cash)
  - An option's value at any time t is merely the sum of the values of the components of the replicating portfolio at t
- Work with $n+1$ underlying assets (asset 0 = numéraire)
  - Slightly more general than we've done so far:

$$C = \sum_{i=0}^{n} S_i \Delta_i = \sum_{i=0}^{n} S_i \frac{\partial C}{\partial S_i} \quad \left(\frac{C}{S_0} = \sum_{i=0}^{n} \frac{S_i}{S_0} \Delta_i \right)$$

  - Technically, this requires that C be homogeneous of degree 1 in its asset price inputs.
  - This obviously is the case for vanilla options, e.g. if we consider $\{S, K\}$ as asset prices, but it's not so obvious that this is the case for binary options (for example)
  - Why does this work in all cases? Space of $n+1$ underlying assets including numéraire but only n observable underlying prices $\implies$ we can scale all $n+1$ assets by a constant leaving all relative prices invariant.

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (2)

Conservation Equations

- Consider variations in asset prices holding other parameters constant:

$$
\begin{aligned}
dC &= d\left(\sum_{i=0}^{n} S_i \frac{\partial C}{\partial S_i}\right) \\
&= \sum_{i=0}^{n} \left[\frac{\partial C}{\partial S_i} dS_i + S_i\, d\left(\frac{\partial C}{\partial S_i}\right)\right] \\
&= \sum_{i=0}^{n} \frac{\partial C}{\partial S_i} dS_i
\end{aligned}
$$

$$
\Longrightarrow \sum_{i=0}^{n} S_i\, d\left(\frac{\partial C}{\partial S_i}\right) = 0 \text{ for any set of asset price variations}
$$

- In particular, consider variations in one asset price:

$$
\begin{aligned}
\frac{\partial C}{\partial S_j} &= \Delta_j \\
&= \Delta_j + \sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial S_j}
\end{aligned}
$$

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (3)

Conservation Equations (continued)

- Consequently: $$\sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial S_j} = \sum_{i=0}^{n} S_i \Gamma_{i,j} = 0 \;\; \forall j$$ ("Gamma Sum Rule"):

  (analogous to Gibbs-Duhem equation in Thermodynamics)

- But since: $$\Gamma_{i,j} = \frac{\partial \Delta_i}{\partial S_j} = \frac{\partial^2 C}{\partial S_i \partial S_j} = \frac{\partial^2 C}{\partial S_j \partial S_i} = \frac{\partial \Delta_j}{\partial S_i} = \Gamma_{j,i}$$

  we can exchange upper and lower indices:

$$\sum_{i=0}^{n} S_i \Gamma_{i,j} = \sum_{i=0}^{n} S_i \Gamma_{j,i} = \sum_{i=0}^{n} S_i \frac{\partial \Delta_j}{\partial S_i} = 0 \;\; \forall j$$

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (4)

Example: Black-Scholes Hedge

- Hedge parameters in cash, stock:

$$\begin{aligned}\Delta_0 &= -Ke^{-r(T-t)}\,\mathcal{N}(z_-)\\ \Delta_1 &= e^{-y(T-t)}\,\mathcal{N}(z_+)\end{aligned}$$

- Use sum rule for $j = 1$:

$$1\,\frac{\partial \Delta_0}{\partial S} + S\,\frac{\partial \Delta_1}{\partial S} = 0$$

- Hence:

$$-Ke^{-r(T-t)}\,\frac{\partial \mathcal{N}(z_-)}{\partial S} + Se^{-y(T-t)}\,\frac{\partial \mathcal{N}(z_+)}{\partial S} = 0$$

- Applies to any single-asset payoff.

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (5)

Conservation Equations for Parametric Sensitivities

- Consider now sensitivity of option value with respect to any non-asset price parameter η:

$$\frac{\partial C}{\partial \eta} = \sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial \eta}$$

implies any sensitivity can be decomposed into its asset-price components (e.g., vanna)

- As an example, consider Θ:

$$\frac{\partial C}{\partial t} = \sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial t}$$

- Conservation of theta (Garman's charm)

4.2. Appendix 3.2: Backward and Forward Equations

- Given Dirichlet conditions:

$$C = C_{\partial\boldsymbol{\Omega}}(x_{\partial\boldsymbol{\Omega}}, t) \text{ on a (half) closed } \mathbb{R}^1 \text{ boundary } \partial\boldsymbol{\Omega} = x_{\partial\boldsymbol{\Omega}}(t)$$

- Feynman-Kac tells us that $C(x,t)$ satisfying the PDE:

$$\frac{\partial C}{\partial t} + \mu(x,t)\frac{\partial C}{\partial x} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2 C}{\partial x^2} = r(t)C$$

  is equal to the expectation:

$$\mathbb{E}^{\mathbf{Q}}_{x,t}\left[\exp\left(-\int_t^{t_{\partial\boldsymbol{\Omega}}} d\tau\, r(\tau)\right) C_{\partial\boldsymbol{\Omega}}(x_{\partial\boldsymbol{\Omega}}, t_{\partial\boldsymbol{\Omega}})\right]$$

  for the Ito process: $dx = \mu(x,t)\,dt + \sigma(x,t)\,dW^{\mathbf{Q}}$ starting at $x(t) = x$
  - Note also that we can introduce a "source" term $q(t,x)$
- $t \leq T$, so this implies that we are working *back* in time from the final condition $C_T(x_T, T)$ $\implies$ the Feynman-Kac PDE (of which the B-S-M PDE is an example) is a "backward" (Kolmogorov) equation.
- What equation is satisfied if we take $C(x,t)$ as given and look forward instead?

4.2. Appendix 3.2: Backward and Forward Equations (2)

- Consider Feynman-Kac expectation for $C(x_t, t)$:

$$C(x_t, t) = \mathbb{E}^{\mathbf{Q}}_{x_t, t}\left[\exp\left(-\int_t^{t_{\partial\Omega}} d\tau\, r(\tau)\right) C_{\partial\Omega}(x_{\partial\Omega}, t_{\partial\Omega})\right]$$

- Introduce Arrow-Debreu state price density $\Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})$ ("propagator"):

$$C(x_t, t) \sim \int_{\partial\Omega} d(x_{\partial\Omega}, t_{\partial\Omega})\, \Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})\, C_{\partial\Omega}(x_{\partial\Omega}, t_{\partial\Omega})$$

  - $C(x_t, t)$ is thus represented as a superposition of propagated boundary values $C_{\partial\Omega}(x_{\partial\Omega}, t_{\partial\Omega})$
  - $\Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})$ satisfies the (backward) Feynman-Kac PDE in (x_t, t)
  - What forward equation in $(x_{\partial\Omega}, t_{\partial\Omega})$ does $\Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})$ satisfy?

- Take $\partial\Omega = (x_T : -\infty < x_T < \infty)$ to make things a bit easier to interpret:

$$C(x_t, t) = \int_{-\infty}^{\infty} dx_T\, \Pi_{t, T}(x_t, t; x_T, T)\, C_T(x_T)$$

- Consider an intermediate time τ and write the iterated expectation:

$$C(x_t, t) = \int_{-\infty}^{\infty} dx_\tau\, \Pi_{t, \tau}(x_t, t; x_\tau, \tau) \int_{-\infty}^{\infty} dx_T\, \Pi_{\tau, T}(x_\tau, \tau; x_T, T)\, C_T(x_T)$$

4.2. Appendix 3.2: Backward and Forward Equations (3)

- But $C(x_t, t)$ must be independent of the choice of τ!

$$\partial_\tau C(x_t,t) \;=\; \partial_\tau\left(\int_{-\infty}^{\infty} dx_\tau\, \Pi_{t,\tau}(x_t,t;x_\tau,\tau)\int_{-\infty}^{\infty} dx_T\, \Pi_{\tau,T}(x_\tau,\tau;x_T,T)\, C_T(x_T)\right) \;=0$$

- Expand the τ derivative using the product rule (and suppress time arguments):

$$\partial_\tau C(x_t,t) = \int_{-\infty}^{\infty} dx_\tau \int_{-\infty}^{\infty} dx_T \Big[\;\Pi_{t,\tau}(x_t;x_\tau)\partial_\tau \Pi_{\tau,T}(x_\tau;x_T) + \Pi_{\tau,T}(x_\tau;x_T)\partial_\tau \Pi_{t,\tau}(x_t;x_\tau)\Big]\, C_T(x_T) = 0$$

- This must hold for arbitrary $C_T \Rightarrow$ choose C_T as a delta function at an arbitrary x_T to eliminate integral over x_T:

$$0 = \int_{-\infty}^{\infty} dx_\tau \left[\Pi_{t,\tau}(x_t;x_\tau)\partial_\tau \Pi_{\tau,T}(x_\tau;x_T) + \Pi_{\tau,T}(x_\tau;x_T)\partial_\tau \Pi_{t,\tau}(x_t;x_\tau)\right]$$

- Develop the first term: substitute F-K PDE for $\partial_\tau \Pi_{\tau,T}(x_\tau;x_T)$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx_\tau\, \Pi_{t,\tau}(x_t;x_\tau)\; \partial_\tau \Pi_{\tau,T}(x_\tau;x_T) \\
&= \int_{-\infty}^{\infty} dx_\tau\, \Pi_{t,\tau}(x_t;x_\tau)\left(r(\tau) - \mu(x_\tau,\tau)\partial_{x_\tau} - \frac{\sigma^2(x_\tau,\tau)}{2}\partial^2_{x_\tau,x_\tau}\right)\Pi_{\tau,T}(x_\tau;x_T)
\end{aligned}$$

4.2. Appendix 3.2: Backward and Forward Equations (4)

- Integrate second and third terms by parts (assuming all components go to zero sufficiently fast as $x_\tau \to \pm\infty$):

$$\int_{-\infty}^{\infty} dx_\tau\, \Pi_{t,\tau}(x_t;x_\tau)\left(-\mu(x_\tau,\tau)\partial_{x_\tau}\Pi_{\tau,T}(x_\tau;x_T)\right)$$

$$= \cancel{-[\mu(x_\tau,\tau)\Pi_{t,\tau}(x_t;x_\tau)\Pi_{\tau,T}(x_\tau;x_T)]_{x_\tau=-\infty}^{\infty}} + \int_{-\infty}^{\infty} dx_\tau\, \Pi_{\tau,T}(x_\tau;x_T)\partial_{x_\tau}[\mu(x_\tau,\tau)\Pi_{t,\tau}(x_t;x_\tau)]$$

$$\int_{-\infty}^{\infty} dx_\tau\, \Pi_{t,\tau}(x_t;x_\tau)\left(-\frac{\sigma^2(x_\tau,\tau)}{2}\,\partial^2_{x_\tau,x_\tau}\Pi_{\tau,T}(x_\tau;x_T)\right)$$

$$= \cancel{-[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)\,\partial_{x_\tau}\Pi_{\tau,T}(x_\tau;x_T)]_{x_\tau=-\infty}^{\infty}} + \int_{-\infty}^{\infty} dx_\tau\, \partial_{x_\tau}[\Pi_{\tau,T}(x_\tau;x_T)]\,\partial_{x_\tau}\left[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)\right]$$

$$= \cancel{\left[\Pi_{\tau,T}(x_\tau;x_T)\,\partial_{x_\tau}[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)]\right]_{x_\tau=-\infty}^{\infty}} - \int_{-\infty}^{\infty} dx_\tau\, \Pi_{\tau,T}(x_\tau;x_T)\,\partial^2_{x_\tau,x_\tau}\left[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)\right]$$

4.2. Appendix 3.2: Backward and Forward Equations (5)

- Putting this all together, we find:

$$0 = \int_{-\infty}^{\infty} dx_\tau \, \Pi_{\tau,T}(x_\tau; x_T) \left\{ [\partial_\tau + r(\tau)]\Pi_{t,\tau}(x_t; x_\tau) + \partial_{x_\tau}[\mu(x_\tau,\tau)\Pi_{t,\tau}(x_t; x_\tau)] - \partial^2_{x_\tau,x_\tau}\left[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t; x_\tau)\right]\right\}$$

- Since this must hold for arbitrary $\Pi_{\tau,T}(x_\tau; x_T)$, and considering in particular the limit: $\tau \nearrow T$, $\Pi_{\tau,T}(x_\tau; x_T) \to \delta(x_\tau, x_T)$, we conclude that:

$$[\partial_T + r(T)]\Pi_{t,T}(x_t; x_T) + \partial_{x_T}[\mu(x_T, T)\Pi_{t,T}(x_t; x_T)] - \partial^2_{x_T,x_T}\left[\frac{\sigma^2(x_T,T)}{2}\Pi_{t,T}(x_t; x_T)\right] = 0$$

- Standard form of (Kolmogorov) forward equation:

$$\frac{\partial \Pi_{t,T}}{\partial T} = \frac{1}{2}\frac{\partial^2[\sigma^2(x_T,T)\Pi_{t,T}]}{\partial x_T^2} - \frac{\partial[\mu(x_T,T)\Pi_{t,T}]}{\partial x_T} - r(T)\Pi_{t,T}$$

- Compare to standard form of (Kolmogorov) backward equation:

$$-\frac{\partial \Pi_{t,T}}{\partial t} = \frac{1}{2}\sigma^2(x_t,t)\frac{\partial^2 \Pi_{t,T}}{\partial x_t^2} + \mu(x_t,t)\frac{\partial \Pi_{t,T}}{\partial x_t} - r(t)\Pi_{t,T}$$

- This integration by parts of a product of Green's functions is a standard technique.
- The forward and backward PDEs related by this process are called *adjoint* equations.

4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof)

- Consider a general bounded payoff $F_T(S_T)$ with bounded first derivative at $S_T = 0$
  - We can (partly) relax the first boundedness restriction later
- Assume the existence of vanilla options of all strikes $\{C(K)\}$
- How can we price (and replicate) F using $\{C(K)\}$, stock, and bonds?

$$F_T = F_T(S_T) \implies F_t(S_t) = e^{-r(T-t)} \int_0^\infty dS_T \, q(S_T|S_t) \, F_T(S_T)$$

4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (2)

Digression on integration-by-parts

- In what follows, we will apply a variation on the integration-by-parts formula.
- You most likely remember the mnemonic $\int u\,dv = u\,v - \int v\,du$, which is shorthand for:
  $\displaystyle\int_a^b dx\,u(x)v'(x) = \big[u(x)v(x)\big]_{x=a}^{x=b} - \int_a^b dx\,u'(x)v(x)$, where: $\displaystyle v(x) = \int_c^x dy\,v'(y)$,
  x and y are both "dummy" integration variables that we can choose arbitrarily, and c is an arbitrary reference value of the dummy variables.
- If you need to convince yourself that c can be chosen arbitrarily, split up the integral:
  $$\int_a^b = \int_a^c + \int_c^b = \int_c^b - \int_c^a$$
- In particular, we can choose $c = \infty$, in which case: $\displaystyle v(x) = \int_\infty^x dy\,v'(y) = -\int_x^\infty dy\,v'(y)$
  and $\displaystyle\int_a^b dx\,u(x)v'(x) = -\left[u(x)\int_x^\infty dy\,v'(y)\right]_{x=a}^{x=b} + \int_a^b dx\,u'(x)\int_x^\infty dy\,v'(y)$

4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (3)

- Integrate by parts once with $F_T'(K) \doteq \left.\frac{\partial F_T}{\partial S_T}\right|_{S_T=K}$, using outer integration variable K and inner integration variable S_T (remember these can be chosen arbitrarily):

$$
\begin{aligned}
F_t(S_t) &= e^{-r(T-t)} \int_0^\infty dS_T \, F_T(S_T) \, q(S_T|S_t) \\
&= e^{-r(T-t)} \left\{ - \left[F_T(K) \int_K^\infty dS_T \, q(S_T|S_t) \right]_{K=0}^{K=\infty} + \int_0^\infty dK \, F_T'(K) \int_K^\infty dS_T \, q(S_T|S_t) \right\} \\
&= e^{-r(T-t)} \left\{ F_T(0) + \int_0^\infty dK \, F_T'(K) \, \mathbb{E}^q [B_T(S_T, K)] \right\} \\
&= e^{-r(T-t)} F_T(0) + \int_0^\infty dK \, F_T'(K) \, B_t(S_t, K)
\end{aligned}
$$

- Representation in terms of bond + p.v. of binary (cash-or-nothing) calls $B_t(S_t, K)$
  - Please don't confuse B_t with a money market account – using it here as abbreviation for present value of **B**inary (cash or nothing) call option struck at K

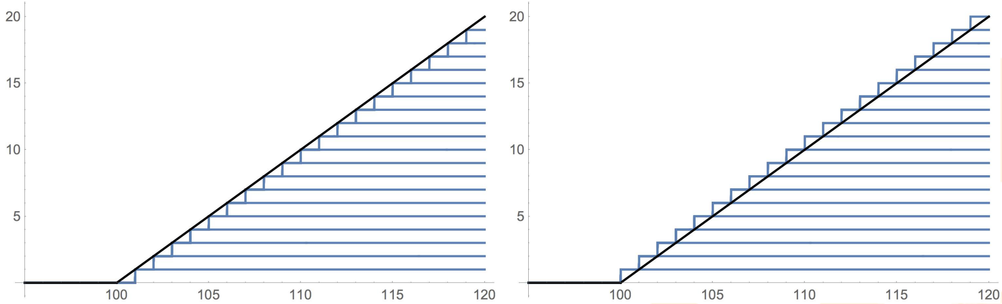
4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (4)

- Integrate by parts again with $F_T''(K) \doteq \left.\frac{\partial^2 F_T}{\partial S_T^2}\right|_{S_T=K}$, using outer integration variable K and inner integration variable K':

$$F_t(S_t) = e^{-r(T-t)} F_T(0) - \left[F_T'(K) \int_K^\infty dK'\, B_t(S_t, K') \right]_{K=0}^{K=\infty} + \int_0^\infty dK\, F_T''(K) \int_K^\infty dK'\, B_t(S_t, K')$$

- But a standard option payoff can be represented as an integral over binary options:

$$C_T(S_T, K) = \max[S_T - K, 0] = \int_K^\infty dK'\, \mathbf{1}_{S_T - K'} \implies C_t(S_t, K) = \int_K^\infty dK'\, B_t(S_t, K')$$



4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (5)

- Representation of general payoff in terms of bond + stock + vanilla (call) options:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = 0) + F_T'(0)\, C_t(S_t, K = 0) + \int_0^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

- Breeden-Litzenberger results are distribution- and process- independent: so are these!
  - Robust, static replicating portfolio
  - But, depends on availability of options of (all) strikes
- In practice, we may not want to build the replicating portfolio out of calls starting at strike 0, but rather bifurcate the portfolio into calls with strike greater than some K^* (e.g. at-the-money) and puts with strike less than K^*:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = K^*) + F_T'(S_T = K^*) \left(S_t e^{-y(T-t)} - K^* e^{-r(T-t)} \right)$$
$$+ \int_0^{K^*} dK\, F_T''(K)\, P_t(S_t, K) + \int_{K^*}^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

  - Reference: Carr, P. and Madan, D., "Towards a Theory of Volatility Trading," In: *Volatility: New estimation techniques for pricing derivatives.* Risk Books (1998), 417-427.
  - Why might we want to build a replicating portfolio this way instead of with calls at all K?