1.3. Volatility Skew and Smile (5)

Where do the volatility skew and smile come from? Some explanations & stylized models (cont.)

- Smile
  - Stochastic volatility: variance of variance
    $\Longrightarrow$ leptokurtic (fat tailed) returns distribution and increasing σ_{imp} in the tails.
  - One popular model, arguably the benchmark: Heston (1993):

$$dS_t = (r-y)S_t\,dt + \sqrt{v_t}\,S_t\,dW^Q_{S,t}$$

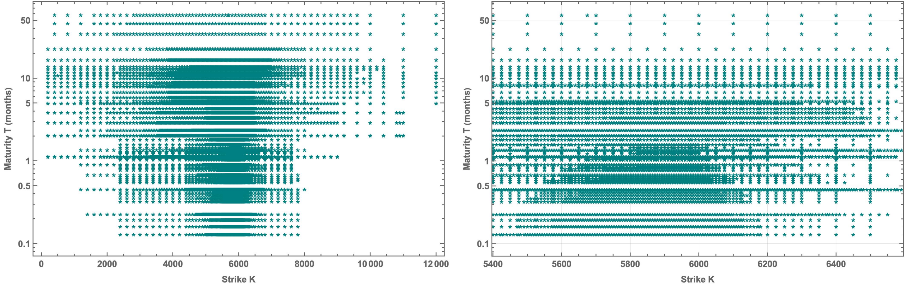
$$dv_t = \kappa(v_\infty - v_t) + \sigma_v\sqrt{v_t}\,dW^Q_{v,t} \quad \text{with: } (dW^Q_{v,t}\,dW^Q_{S,t}) = \rho_{v,S}\,dt$$

  - Note that $\rho_{v,S}$ can make the smile lean (in either direction):
    - alternative explanation of vol skew consistent with stylized facts like "volatility increases in down markets"
    - useful in currency and commodity settings with directional pressure/jump risk
  - The characteristic function (i.e., Fourier transform of the density of) $\ln(S_t)$ can be written in closed form.
  - Contemporary extensions include "rough" Heston models (Gatheral *et al.*) in which $W_{v,t}$ is taken to be a ***fractional*** Brownian motion.
  - There is a good bit of anecdotal evidence that "the market" prices (or at least fits slices of) the volatility surface using Heston or a simplified version thereof, e.g., SVI (*Stochastic Volatility Inspired*):

$$\sigma^2_{imp}(K) \sim a + b\left(\rho\ln(K/K^*) + \sqrt{\ln^2(K/K^*) + \sigma_0^2}\right)$$

1.3. Volatility Skew and Smile (6)

- Example: CBOE 17 Jan 2025 closing data for SPX+SPXW European Options ($S = 5996.66$)
  - 11987 lines of bid-ask data for calls and puts
    - lines deleted: 433 that day's expiry $\Rightarrow$ 11554 valid lines (over 23k options/46k prices)
    - 59 distinct maturities from Tue 21 Jan 2025 to Fri 21 Dec 2029; strike prices K from 200 to 12000

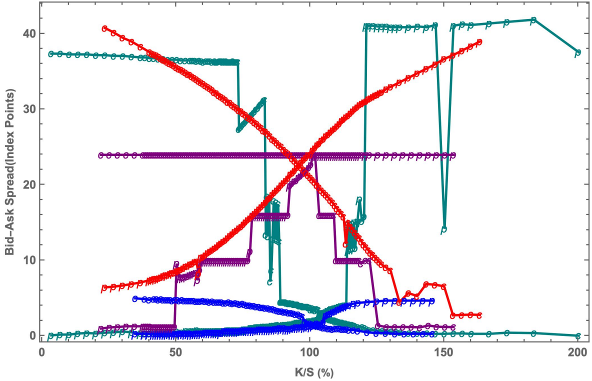


  - Downsample to monthly options (19 maturities, same strike range: 3657 lines)
- We'll also consider data from three other dates, with very similar overall statistics, exemplifying 3 different market regimes:
  - 11 Jun 2019 (calm, pre-COVID), $S = 2885.72$
  - 05 Jun 2020 (excited, a few months into COVID), $S = 3193.93$
  - 11 Jun 2020 (crisis: SPX down nearly 6% on the day), $S = 3002.10$
  - 17 Jan 2025 (current: SPX up 1% on the day), $S = 5996.66$

1.3. Volatility Skew and Smile (7)

Market Liquidity: Bid-Ask Spread Data

- Focus on $T \approx 1$ year tenors (e.g., next year's January expiry)

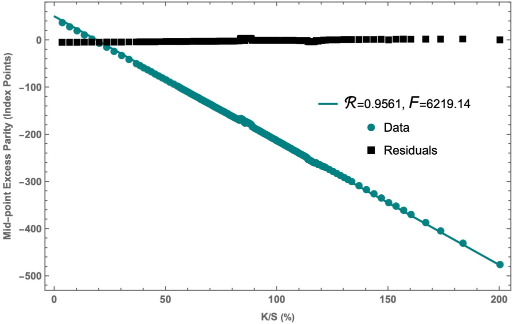


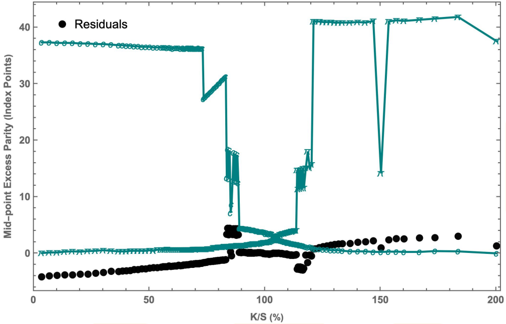
- Also examine historical evolution
  - Spreads progressively widen as conditions go from calm to excited to crisis, then back

1.3. Volatility Skew and Smile (8)

Put-Call Parity

- Fit $C(K,T) - P(K,T) = \mathcal{R}(T)(F(T) - K) = S\,e^{-y_{imp,T}T} - K\,e^{-r_{imp,T}T}$ for $\{\mathcal{R}(T), F(T)\} \leftrightarrow \{r_{imp,T}, y_{imp,T}\}$ to midpoint data across strikes for each tenor
- Data model: $C \sim N\left[C_{mid}, (C_{ask} - C_{bid})^2\right], P \sim N\left[P_{mid}, (P_{ask} - P_{bid})^2\right]$: price uncertainty proportional to (bid-ask spread width), so points weighted inversely to (spread width)^2
- We actually fit the *excess* parity $(S-K) - (C-P) = (S-\mathcal{R}F) - (1-\mathcal{R})K$: removing the major contributions to slope and intercept produces a **much** cleaner test of Put-Call Parity

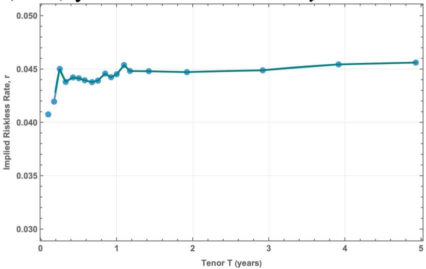


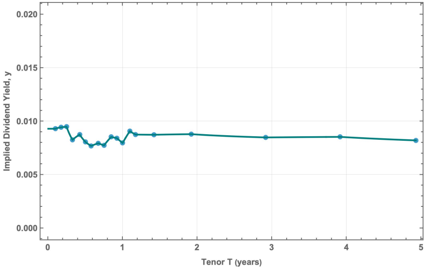


- Adj. R^2 for 1 yr tenor $= 0.999991$. All errors/residuals well inside bid-ask spreads.

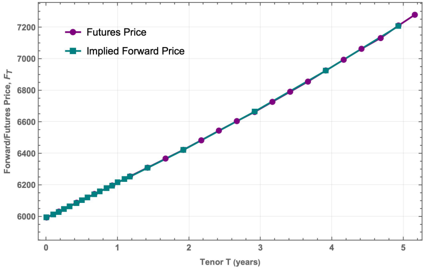
1.3. Volatility Skew and Smile (9)

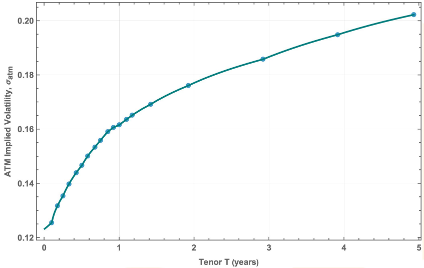
- Implied (zero) yield curves for r and y:





- Implied Forward and ATM Vol curves:





1.3. Volatility Skew and Smile (10)

- CBOE 17 Jan 2025 closing data for SPX ($S = 5996.66$), 16 Jan 2026 expiry ($T \approx 0.997$ yrs)

0.996518 1.00822

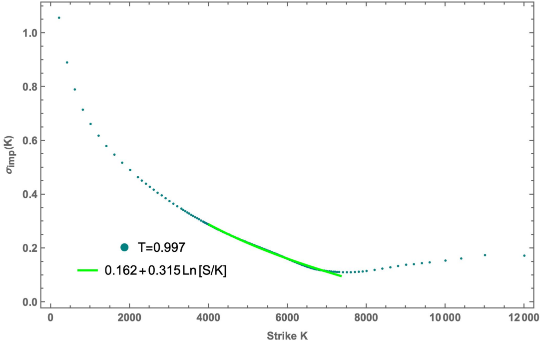
	Estimate	Standard Error	t-Statistic	P-Value
DF	0.956073	0.0000793318	12051.6	0.
F	6219.14	0.0523081	118894.	0.

0.999991

	Estimate	Standard Error	t-Statistic	P-Value
r	0.0445549	0.0000823002	541.37	8.26791×10^{-294}
y	0.00799934	0.000080468	99.4101	1.54214×10^{-160}

0.999991

  - Implied $r \approx 4.45\%$ (in neighborhood of 1-year USD term rates)
  - Implied $y \approx 0.80\%$ (a little below current backward-looking 1-year SPX div yield of $\sim 1.25\%$)
  - Enforce parity $\implies$ same σ_{imp} for both calls & puts, with (bid-ask)-adjusted mid-prices.

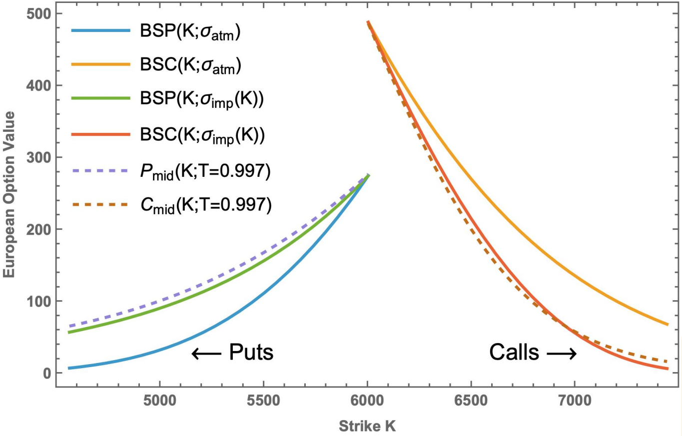


- ATM $\{-2.5, +1.25\}$ s.d.: $\sigma_{imp}(K, T \approx 0.997; S \approx 6000) = 16.2\% + 31.5\% \ln(S/K)$ (adj. $R^2 = 0.9971$)

1.3. Volatility Skew and Smile (11)

Impact of implied vol on vanilla European option values

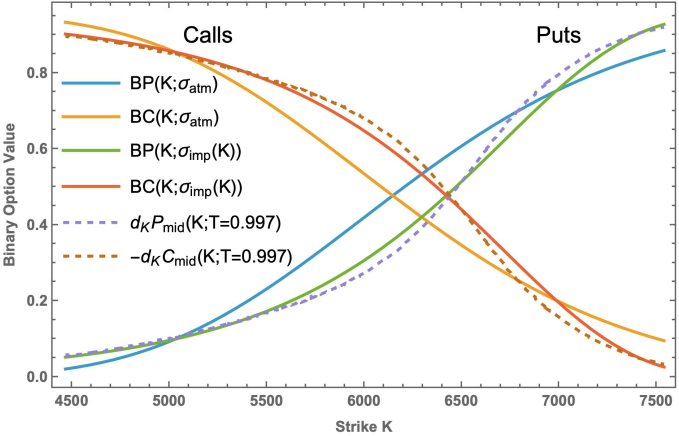
- “Problem set parameters": $S = 6000$, $T = 1.0$ yr, $r = 445$ bp, $y = 0.80\%$, $\sigma_{atm} = 16\%$, $\xi = 30\%$, $\kappa = 0$



- Volatility “skew” makes downside (puts) more expensive and upside (calls) cheaper.

1.3. Volatility Skew and Smile (12)

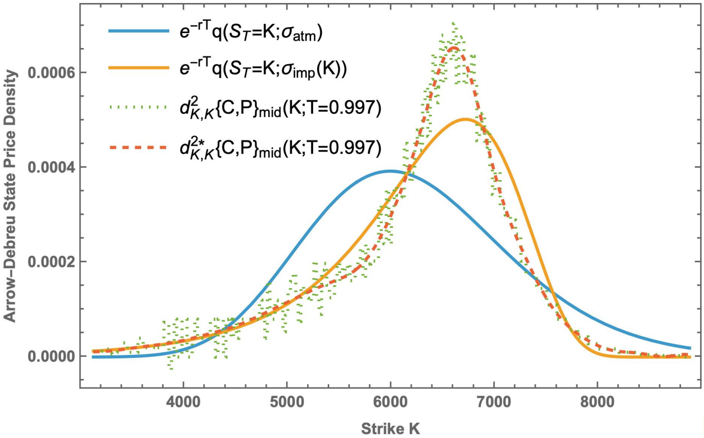
Impact of implied vol on binary option values



- Binary put-call parity is apparent in these results.
- Fatter tail on the (far) downside and thinner tail on the (far) upside are as expected, but why the downward bulge in the cdf ($\sim$ binary put values) at intermediate strikes?

1.3. Volatility Skew and Smile (13)

Impact of implied vol on Arrow-Debreu state price density



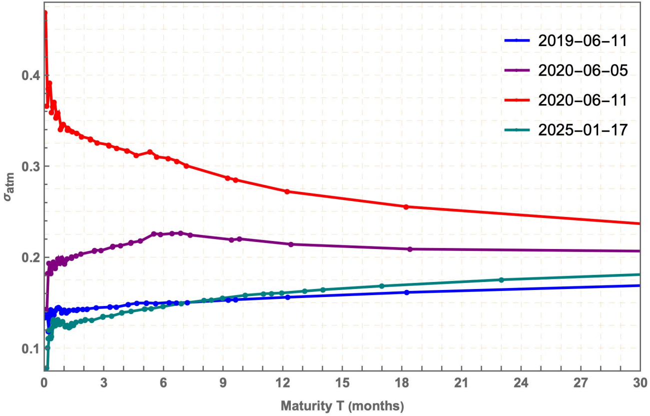
- Note higher moment effects that are unexpected *ex ante*
- Why do these arise? (hint: $S_0 e^{(r-y)T}$)
- Multiple differentiation of even slightly noisy input data is perilous

  $\Longrightarrow$ avoid working directly with empirical $\mathbf{Q}$ density if at all possible.

1.3. Volatility Skew and Smile (14)

Cross-sectional behavior of the implied volatility surface vs. t and T (1)

- Observations of the ATM volatility term structure:

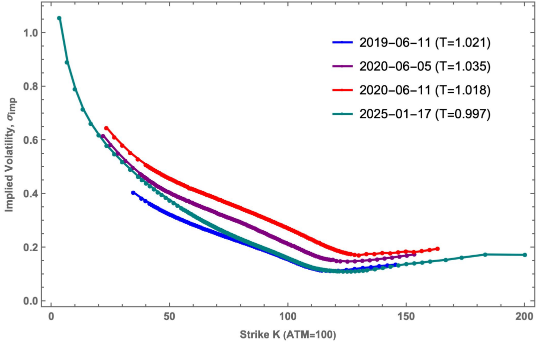


- These results suggest that the most persistent characteristics are:
  1. some form of mean-reversion, especially out toward long-term tenors and
  2. (perhaps) persistence in perceptions of elevated volatility around specific future events.

1.3. Volatility Skew and Smile (15)

Cross-sectional behavior of the implied volatility surface vs. t and T (2)

- But is the overall shape of the volatility surface persistent?
  Our "three-regime" results vs. strike K for the $\sim$1 year tenors are suggestive:

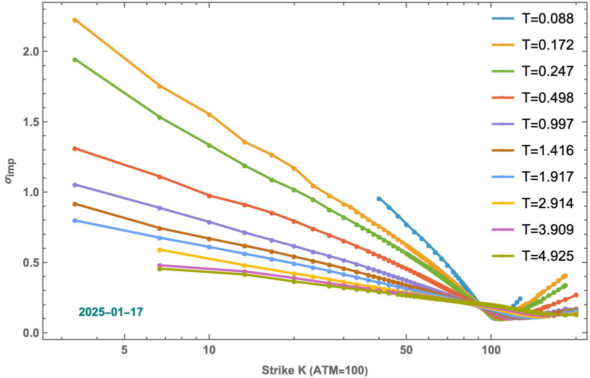


- Overall shape is remarkably consistent, although the curve shifts up-and-down and, to a more limited extent, left-right.
- Fitted implied vol "skewness" coefficient ξ – multiplying $\ln(S/K)$ – only rises from 32.4% (calm) to 37.3% (elevated) to 38.4% (crisis), then back to 31.5% over the $\sim$4.5 years.

1.3. Volatility Skew and Smile (16)

Cross-sectional behavior of the implied volatility surface vs. t and T (3)

- Is there any consistency/universality across maturity tenors T at a given time t?
- Our cross-sectional slices for various T (σ_{imp} vs. K) appear to reveal only a general, overall similarity in shape:

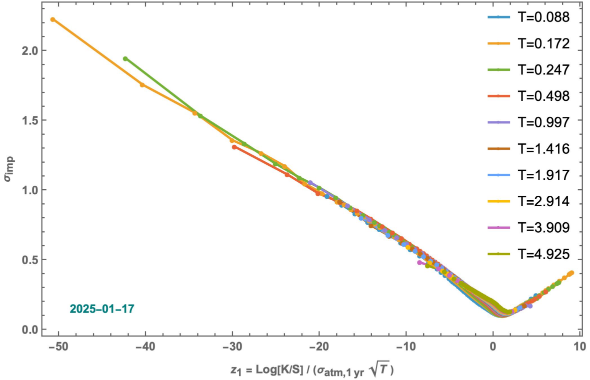


- But a change to log K/S co-ordinates hints at more...

1.3. Volatility Skew and Smile (17)

Cross-sectional behavior of the implied volatility surface vs. t and T (4)

- What happens if we take inspiration from the “old (equity) traders’ tale"?
  - Volatility skew (i.e. ξ) should scale as $1/\sqrt{T}$, so define a normalized co-ordinate: $z_1 \doteq \ln(K/S)/(\sigma_1\sqrt{T})$, always using the 1-year tenor ATM vol σ_1 to non-dimensionalize.

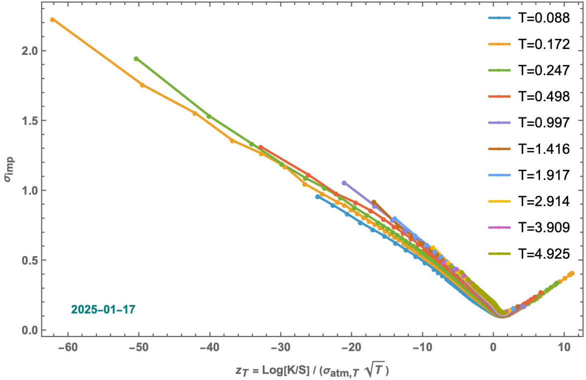


  - Remarkably good fit, even for extreme strikes at the smallest (1 month) tenor
  - Generally consistent during calm, excited, and crisis periods
  - Impact of ATM vol term structure in the tails isn’t completely clear.

1.3. Volatility Skew and Smile (18)

Cross-sectional behavior of the implied volatility surface vs. t and T (5)

- One last experiment, testing common FX market convention:
  - $\sigma_{imp}(\Delta = \mathcal{N}(z_+)) \leftrightarrow \sigma_{imp}\left(z_+ = \frac{\ln(F/K)}{\sigma_{atm,T}\sqrt{T}} + \frac{\sigma_{atm,T}\sqrt{T}}{2}\right)$. At these scales, $\frac{\ln(F/S)}{\sigma_{atm}\sqrt{T}}$ & $\frac{\sigma_{atm}\sqrt{T}}{2}$ offsets are negligible, so define normalized co-ordinate: $z_T \doteq \ln(K/S)/(\sigma_{atm,T}\sqrt{T})$:



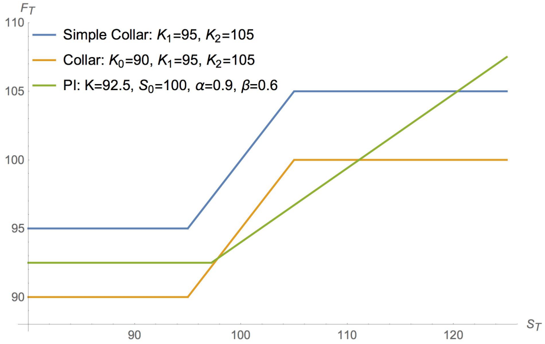
  - Fits are no better or somewhat worse, including those for calm, excited, and crisis periods.
  - For SPX, it appears the old equity traders' rule works better than FX convention.

2. Pricing via (Static) Replication

- Turn Breeden & Litzenberger's result on its head
- First, consider piecewise linear payoffs
- Then generalize to arbitrary (convex/concave) payoffs

2.1. Piecewise Linear Payoffs (Packages)

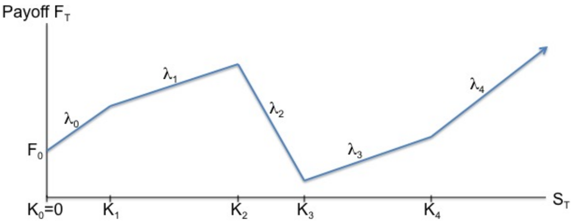
- Many strategies involving options of a single maturity can be written as piecewise linear payoffs:
  - Bull (call) and bear (put) spreads
  - Simple collars: $F_T = \min[\max(S_T, K_1), K_2] = K_1 + \max[0, S_T - K_1] - \max[0, S_T - K_2]$ (where $0 < K_1 < K_2$)
    - Generalized collars, including range forwards (zero-cost collars): $F_T = K_0 + \max[0, S_T - K_1] - \max[0, S_T - K_2]$
  - Portfolio insurance-like strategies: $F_T = \max[K, S_0 + \beta(\alpha S_T - S_0)] = K + \alpha\beta \max[0, S_T - K^*]$, where: $0 < \alpha < 1$, $\beta > 0$, and $K^* \doteq [K - S_0(1-\beta)]/\alpha\beta$
  - Straddles, butterflies, etc.



  - Source: Rubinstein, M., “Packages,” (9 Dec 1991), 1-6.

2.1. Piecewise Linear Payoffs (Packages) (2)

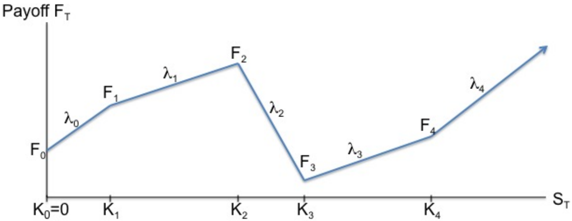
- Generic representation of piecewise linear payoff:



  - Kink points and slopes: K_i, λ_i
  - Value and slope at $S_T = 0$: F_0, λ_0
  - Note that if $F_0 = 0, \lambda_0 = 0$, payoff effectively begins at K_1
- Valuation: F_T decomposes into: $F_T = F_0 + \lambda_0 S_T + \sum_i (\lambda_i - \lambda_{i-1})[S_T - K_i]^+$
- Hence: $F_t = e^{-r(T-t)} F_0 + \lambda_0 e^{-y(T-t)} S_t + \sum_i (\lambda_i - \lambda_{i-1}) C_t[S_t; K_i, T-t]$
  - Process-independent results
  - Can also begin decomposition at any particular K_i, with calls on the upside and puts on the downside

2.1. Piecewise Linear Payoffs (Packages) (3)

- What if we wish to represent payoff in terms of values F_i at kink points, and not slopes λ_i in between?



$\Longrightarrow$ Identify $\lambda_i = \frac{F_{i+1}-F_i}{K_{i+1}-K_i}$

- Rewrite F_T decomposition:

$$
\begin{aligned}
F_T &= F_0 + \tfrac{F_1-F_0}{K_1-0} S_T + \textstyle\sum_i \left(\tfrac{F_{i+1}-F_i}{K_{i+1}-K_i} - \tfrac{F_i-F_{i-1}}{K_i-K_{i-1}}\right)[S_T - K_i]^+ \\
&= F_0 + \tfrac{\Delta F_0}{\Delta K_0} S_T + \textstyle\sum_i \left(\tfrac{\Delta F_i}{\Delta K_i} - \tfrac{\Delta F_{i-1}}{\Delta K_{i-1}}\right)[S_T - K_i]^+
\end{aligned}
$$

- Hence: $F_t = e^{-r(T-t)} F_0 + \frac{\Delta F_0}{\Delta K_0} e^{-y(T-t)} S_t + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right) C_t[S_t; K_i, T-t]$

2.2. General Payoffs

- Consider a general bounded payoff $F_T(S_T)$ with bounded first derivative at $S_T = 0$
  - We can (partly) relax the first boundedness restriction later
- Assume the existence of vanilla options of all strikes $\{C(K)\}$
- How can we price (and replicate) F using $\{C(K)\}$, stock, and bonds?
- Start with piece-wise linear payoffs and take many K_i, F_i, with $\Delta K_i, \Delta F_i \searrow 0$:

$$F_T = F_0 + \frac{\Delta F_0}{\Delta K_0} S_T + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right)[S_T - K_i]^+$$

$$\Rightarrow F_0 + F_0' S_T + \int_0^\infty dK\, F''(K)[S_T - K]^+$$

- Hence:

$$F_t = e^{-r(T-t)} F_0 + \frac{\Delta F_0}{\Delta K_0} e^{-y(T-t)} S_t + \sum_i \left(\frac{\Delta F_i}{\Delta K_i} - \frac{\Delta F_{i-1}}{\Delta K_{i-1}}\right) C_t[S_t; K_i, T-t]$$

$$\Rightarrow e^{-r(T-t)} F_0 + F_0' e^{-y(T-t)} S_t + \int_0^\infty dK\, F''(K) C_t[S_t; K, T-t]$$

2.2. General Payoffs (2)

- Representation of general payoff in terms of bond + stock + vanilla (call) options:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = 0) + F_T'(0)\, C_t(S_t, K = 0) + \int_0^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

- Breeden-Litzenberger results are distribution- and process- independent: so are these!
  - Robust, static replicating portfolio
  - But, depends on availability of options of (all) strikes

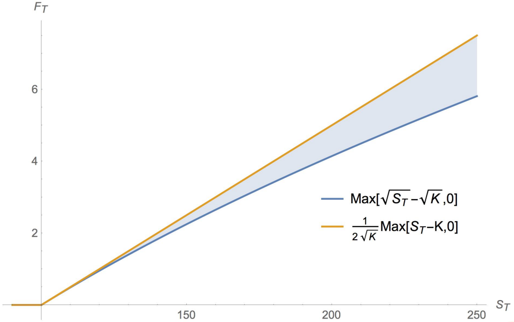
- In practice, we may not want to build the replicating portfolio out of calls starting at strike 0, but rather bifurcate the portfolio into calls with strike greater than some K^* (e.g. at-the-money) and puts with strike less than K^*:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = K^*) + F_T'(S_T = K^*) \left(S_t e^{-y(T-t)} - K^* e^{-r(T-t)} \right) + \int_0^{K^*} dK\, F_T''(K)\, P_t(S_t, K) + \int_{K^*}^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

  - Reference: Carr, P. and Madan, D., “Towards a Theory of Volatility Trading,” In: *Volatility: New estimation techniques for pricing derivatives.* Risk Books (1998), 417-427.
  - Why might we want to build a replicating portfolio this way instead of with calls at all K?

2.2. General Payoffs (3)

- Example: suppose $F_T = \left[S_T^{1/2} - K^{1/2}\right]^+$ (with $K = 100$):



Then:

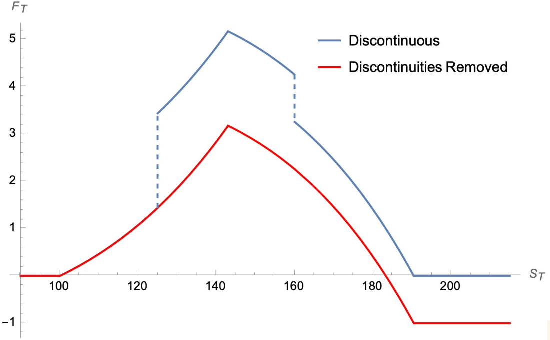
$$F'_T = \begin{Bmatrix} 0 & S_T < K \\ \frac{1}{2} S_T^{-1/2} & S_T > K \end{Bmatrix}$$

$$F''_T = \begin{Bmatrix} 0 & S_T < K \\ \frac{1}{2} K^{-1/2} \delta(S_T - K) & S_T = K \\ -\frac{1}{4} S_T^{-3/2} & S_T > K \end{Bmatrix}$$

- Discontinuous 1st derivatives lead to singular 2nd derivatives and discrete distributions of options

2.2. General Payoffs (4)

- What happens when F is discontinuous so that its first derivatives are singular?



- Replicating portfolio also contains binary options:

$$F_t = e^{-r(T-t)} F_T(0) + F_T'(0)\, C_t(0) + \int_0^\infty dK\, F_T''(K)\, C_t(K) + \sum_i \Delta F_T(K_i)\, B_t(K_i) \; (*)$$

where $\Delta F_T\,(K_i)$ is the i^{th} jump in the payoff

(*) note that F_T'' in the integral must be considered with the jumps removed

3. Introduction to Symmetry/Transformation Methods

- In the B-S-M world, we can sometimes use properties of geometric Brownian motion to simplify valuation of more complex payoffs, particularly where the payoff depends on powers of the underlying asset price.
- Example: typical power option payoff: $C_T = \max[S_T^\alpha - K, 0]$
- Again, valuation by integration is straightforward ($\mathbf{Q}$-measure expectation discounted by $e^{-r(T-t)}$), but why not add another trick to the toolbox?
  - Our process for S is: $dS = (r-y)S\,dt + \sigma S\,dW^{\mathbf{Q}}$
  - Construct a process (using Ito's lemma) for $S' \doteq S^\alpha$ (hence $S'_t = S_t^\alpha$):

$$dS^\alpha = \alpha[(r-y) + (\alpha-1)\frac{\sigma^2}{2}]\,S^\alpha\,dt + \alpha\sigma S^\alpha\,dW^{\mathbf{Q}}$$

$$\Longrightarrow\quad dS' = \alpha[(r-y) + (\alpha-1)\frac{\sigma^2}{2}]\,S'\,dt + \alpha\sigma S'\,dW^{\mathbf{Q}}$$

  - Interpret this as a (pseudo-) price process for S' with modified volatility and payout rate:

$$dS' = (r - y_{\mathit{eff}})S'\,dt + \sigma_{\mathit{eff}} S'\,dW^{\mathbf{Q}}$$

$$\Longrightarrow\quad \sigma_{\mathit{eff}} = |\alpha|\,\sigma,\ \ y_{\mathit{eff}} = r - \alpha[(r-y) + (\alpha-1)\frac{\sigma^2}{2}]$$

3. Introduction to Symmetry/Transformation Methods (2)

- So, we can write down valuation result for the power option:

$$C_t(S_t', K) = S_t'\, e^{-y_{eff}(T-t)} \mathcal{N}(z_{eff,+}) - K\, e^{-r(T-t)} \mathcal{N}(z_{eff,-})$$

$$C_t(S_t^\alpha, K) = S_t^\alpha\, e^{-y_{eff}(T-t)} \mathcal{N}(z_{eff,+}) - K\, e^{-r(T-t)} \mathcal{N}(z_{eff,-})$$

$$\text{with } z_{eff,\pm} = \frac{\ln\left(S_t^\alpha / K\right) + (r - y_{eff})(T-t)}{\sigma_{eff}\sqrt{T-t}} \pm \frac{\sigma_{eff}\sqrt{T-t}}{2}$$

- If we have a Black-Scholes call valuation function **BSCall**$[S, K, T, \sigma, r, y]$, then we can value the power option with **BSCall**$[S_t^\alpha, K, T-t, \sigma_{eff}, r, y_{eff}]$
- Even easier in Black's model **BSCall**$[F_{eff,T}, K, T, \sigma, r]$ with:

$$F_{eff,T} \doteq \mathbb{E}^{\mathbf{Q}}[S_T^\alpha] = S_t^\alpha e^{(r-y_{eff})(T-t)} = S_t^\alpha e^{\alpha[(r-y)+(\alpha-1)\frac{\sigma^2}{2}](T-t)}$$

$$\Longrightarrow \quad C_t(F_{eff,T}, K) = e^{-r(T-t)}\left[F_{eff,T}\, \mathcal{N}(z_{eff,+}) - K\, \mathcal{N}(z_{eff,-})\right]$$

$$\text{with } z_{eff,\pm} = \frac{\ln\left(F_{eff}/K\right)}{\sigma_{eff}\sqrt{T-t}} \pm \frac{\sigma_{eff}\sqrt{T-t}}{2}$$

3. Introduction to Symmetry/Transformation Methods (3)

- Power option example 2: leveraged option payoff: $C_T = S_T^\alpha \max[S_T - K, 0]$
- Here another type of trick is useful. Write: $C(S_t, t) = S_t^\alpha D(S_t, t)$
- Substitute this into the Black-Scholes PDE: $\dfrac{\partial C}{\partial t} + (r-y)S\dfrac{\partial C}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} = rC$
- Obtain a modified PDE:

$$\frac{\partial D}{\partial t} + (r - y + \alpha\sigma^2)S\frac{\partial D}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 D}{\partial S^2} = \left[r - \alpha(r - y + \frac{\alpha-1}{2}\sigma^2)\right] D$$

- Interpret this as a B-S PDE (hence price process) with modified interest and payout rates:

$$\frac{\partial D}{\partial t} + (r_{\mathit{eff}} - y_{\mathit{eff}})S\frac{\partial D}{\partial S} + \frac{1}{2}\sigma_{\mathit{eff}}^2 S^2 \frac{\partial^2 D}{\partial S^2} = r_{\mathit{eff}}\, D$$

$$\implies \quad r_{\mathit{eff}} = r - \alpha(r - y + \frac{\alpha-1}{2}\sigma^2), \;\; y_{\mathit{eff}} = y - \alpha(r - y + \frac{\alpha+1}{2}\sigma^2), \;\; \sigma_{\mathit{eff}} = \sigma$$

- Finally, substitute for C to obtain: $C(S_t, t) = S_t^\alpha \left[S_t e^{-y_{\mathit{eff}}(T-t)} \mathcal{N}(z_{\mathit{eff},+}) - K e^{-r_{\mathit{eff}}(T-t)} \mathcal{N}(z_{\mathit{eff},-})\right]$ with $z_{\mathit{eff},\pm}$ defined in terms of effective parameters
- Again easier in Black's model with $\{F_{\mathit{eff}}, r_{\mathit{eff}}\}$, with r_{eff} here the same as y_{eff} in previous example.
  - Or even with $\{F_{\mathit{eff}} = \mathbb{E}^{\mathbf{Q}}[S_T^{\alpha+1}], K_{\mathit{eff}} = K\,\mathbb{E}^{\mathbf{Q}}[S_T^\alpha], r_{\mathit{eff}} = r\}$!

3. Introduction to Symmetry/Transformation Methods (4)

- When are these particular approaches useful?
  - When the payoff can be (re-)written in terms of the $\max$ or $\min$ of two log-normally distributed variables under the risk-neutral measure
  - E.g.: outperformance options, quantos, geometric Asian options, geometrically-weighted basket options...
- Why do we define effective parameters, in particular r and y?
  - Work from the point-of-view that we've built and debugged a standard Black-Scholes calculator, e.g.:

$$\textbf{BSCall}[S\_, K\_, T\_, \sigma\_, r\_, y\_] := \ldots$$

$$\text{or}\ \ \textbf{BlackCall}[F\_, K\_, T\_, \sigma\_, r\_] := \ldots$$

  - If we can identify the coefficients that determine the relevant/effective forward price(s), discount rates, and (log-) volatility, we can feed those (perhaps via a wrapper function) into our existing Black-Scholes calculator without having to re-program the standard option valuation formulas each time. The process of organizing the effective inputs to Black-Scholes is called “pre-washing” the inputs.
  - The process of deriving expressions for r_{eff} and y_{eff} can seem a little confusing because r plays two roles in the valuation: driving the forward price (expected value) and discounting the whole payoff. We want both roles to behave correctly, so we use $r_{eff} - y_{eff}$ as the rate at which the forward price grows and r_{eff} as the discounting rate (and so, have to solve for y_{eff}).
  - Note that to get correct sensitivities to our original input parameters, we may have to “post-wash" output sensitivities from the Black-Scholes code in our wrapper function.

4. Appendices

- Appendix 3.1: Hedge Decomposition & Conservation Principles
- Appendix 3.2: Backward and Forward Equations
- Appendix 3.3: General Payoffs Static Replication (Another Proof)

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles

- Normally, we view the option pricing problem as one of mapping a set of states (payoffs) into a value C via a probability density.
  - Feynman-Kac theorem: primary role of PDE is to generate density of states.
  - Analogous to statistical mechanics (computation of an energy measure or “fundamental equation”)
- Alternative (complementary) viewpoint:
  - Consider option prices in terms of constitutive elements (hedge positions in underlying assets, incl. cash)
  - An option’s value at any time t is merely the sum of the values of the components of the replicating portfolio at t
- Work with $n+1$ underlying assets (asset 0 = numéraire)
  - Slightly more general than we’ve done so far:

$$C = \sum_{i=0}^{n} S_i \Delta_i = \sum_{i=0}^{n} S_i \frac{\partial C}{\partial S_i} \quad \left(\frac{C}{S_0} = \sum_{i=0}^{n} \frac{S_i}{S_0} \Delta_i \right)$$

  - Technically, this requires that C be homogeneous of degree 1 in its asset price inputs.
  - This obviously is the case for vanilla options, e.g. if we consider $\{S, K\}$ as asset prices, but it’s not so obvious that this is the case for binary options (for example)
  - Why does this work in all cases? Space of $n+1$ underlying assets including numéraire but only n observable underlying prices $\implies$ we can scale all $n+1$ assets by a constant leaving all relative prices invariant.

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (2)

Conservation Equations

- Consider variations in asset prices holding other parameters constant:

$$
\begin{aligned}
dC &= d\left(\sum_{i=0}^{n} S_i \frac{\partial C}{\partial S_i}\right) \\
&= \sum_{i=0}^{n} \left[\frac{\partial C}{\partial S_i} dS_i + S_i\, d\left(\frac{\partial C}{\partial S_i}\right)\right] \\
&= \sum_{i=0}^{n} \frac{\partial C}{\partial S_i} dS_i
\end{aligned}
$$

$$
\Longrightarrow \sum_{i=0}^{n} S_i\, d\left(\frac{\partial C}{\partial S_i}\right) = 0 \text{ for any set of asset price variations}
$$

- In particular, consider variations in one asset price:

$$
\begin{aligned}
\frac{\partial C}{\partial S_j} &= \Delta_j \\
&= \Delta_j + \sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial S_j}
\end{aligned}
$$

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (3)

Conservation Equations (continued)

- Consequently: $$\sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial S_j} = \sum_{i=0}^{n} S_i \, \Gamma_{i,j} = 0 \;\; \forall j$$ ("Gamma Sum Rule"):

  (analogous to Gibbs-Duhem equation in Thermodynamics)

- But since:

$$\Gamma_{i,j} = \frac{\partial \Delta_i}{\partial S_j} = \frac{\partial^2 C}{\partial S_i \partial S_j} = \frac{\partial^2 C}{\partial S_j \partial S_i} = \frac{\partial \Delta_j}{\partial S_i} = \Gamma_{j,i}$$

  we can exchange upper and lower indices:

$$\sum_{i=0}^{n} S_i \, \Gamma_{i,j} = \sum_{i=0}^{n} S_i \, \Gamma_{j,i} = \sum_{i=0}^{n} S_i \frac{\partial \Delta_j}{\partial S_i} = 0 \;\; \forall j$$

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (4)

Example: Black-Scholes Hedge

- Hedge parameters in cash, stock:

$$\begin{aligned} \Delta_0 &= -Ke^{-r(T-t)}\mathcal{N}(z_-) \\ \Delta_1 &= e^{-y(T-t)}\mathcal{N}(z_+) \end{aligned}$$

- Use sum rule for $j = 1$:

$$1\,\frac{\partial \Delta_0}{\partial S} + S\,\frac{\partial \Delta_1}{\partial S} = 0$$

- Hence:

$$-Ke^{-r(T-t)}\,\frac{\partial \mathcal{N}(z_-)}{\partial S} + Se^{-y(T-t)}\,\frac{\partial \mathcal{N}(z_+)}{\partial S} = 0$$

- Applies to any single-asset payoff.

4.1. Appendix 3.1: Hedge Decomposition & Conservation Principles (5)

Conservation Equations for Parametric Sensitivities

- Consider now sensitivity of option value with respect to any non-asset price parameter η:

$$\frac{\partial C}{\partial \eta} = \sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial \eta}$$

implies any sensitivity can be decomposed into its asset-price components (e.g., vanna)

- As an example, consider Θ:

$$\frac{\partial C}{\partial t} = \sum_{i=0}^{n} S_i \frac{\partial \Delta_i}{\partial t}$$

- Conservation of theta (Garman's charm)

4.2. Appendix 3.2: Backward and Forward Equations

- Given Dirichlet conditions:

$$C = C_{\partial\boldsymbol{\Omega}}(x_{\partial\boldsymbol{\Omega}}, t) \text{ on a (half) closed } \mathbb{R}^1 \text{ boundary } \partial\boldsymbol{\Omega} = x_{\partial\boldsymbol{\Omega}}(t)$$

- Feynman-Kac tells us that $C(x,t)$ satisfying the PDE:

$$\frac{\partial C}{\partial t} + \mu(x,t)\frac{\partial C}{\partial x} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2 C}{\partial x^2} = r(t)C$$

  is equal to the expectation:

$$\mathbb{E}^{\mathbf{Q}}_{x,t}\left[\exp\left(-\int_t^{t_{\partial\boldsymbol{\Omega}}} d\tau\, r(\tau)\right) C_{\partial\boldsymbol{\Omega}}(x_{\partial\boldsymbol{\Omega}}, t_{\partial\boldsymbol{\Omega}})\right]$$

  for the Ito process: $dx = \mu(x,t)\,dt + \sigma(x,t)\,dW^{\mathbf{Q}}$ starting at $x(t) = x$
  - Note also that we can introduce a “source” term $q(t,x)$

- $t \leq T$, so this implies that we are working *back* in time from the final condition $C_T(x_T, T)$ $\implies$ the Feynman-Kac PDE (of which the B-S-M PDE is an example) is a “backward” (Kolmogorov) equation.
- What equation is satisfied if we take $C(x,t)$ as given and look forward instead?

4.2. Appendix 3.2: Backward and Forward Equations (2)

- Consider Feynman-Kac expectation for $C(x_t, t)$:

$$C(x_t, t) = \mathbb{E}^{\mathbf{Q}}_{x_t, t}\left[\exp\left(-\int_t^{t_{\partial\Omega}} d\tau\, r(\tau)\right) C_{\partial\Omega}(x_{\partial\Omega}, t_{\partial\Omega})\right]$$

- Introduce Arrow-Debreu state price density $\Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})$ ("propagator"):

$$C(x_t, t) \sim \int_{\partial\Omega} d(x_{\partial\Omega}, t_{\partial\Omega})\, \Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})\, C_{\partial\Omega}(x_{\partial\Omega}, t_{\partial\Omega})$$

  - $C(x_t, t)$ is thus represented as a superposition of propagated boundary values $C_{\partial\Omega}(x_{\partial\Omega}, t_{\partial\Omega})$
  - $\Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})$ satisfies the (backward) Feynman-Kac PDE in (x_t, t)
  - What forward equation in $(x_{\partial\Omega}, t_{\partial\Omega})$ does $\Pi_{t, t_{\partial\Omega}}(x_t, t; x_{\partial\Omega}, t_{\partial\Omega})$ satisfy?

- Take $\partial\Omega = (x_T : -\infty < x_T < \infty)$ to make things a bit easier to interpret:

$$C(x_t, t) = \int_{-\infty}^{\infty} dx_T\, \Pi_{t,T}(x_t, t; x_T, T)\, C_T(x_T)$$

- Consider an intermediate time τ and write the iterated expectation:

$$C(x_t, t) = \int_{-\infty}^{\infty} dx_\tau\, \Pi_{t,\tau}(x_t, t; x_\tau, \tau) \int_{-\infty}^{\infty} dx_T\, \Pi_{\tau,T}(x_\tau, \tau; x_T, T)\, C_T(x_T)$$

4.2. Appendix 3.2: Backward and Forward Equations (3)

- But $C(x_t, t)$ must be independent of the choice of τ!

$$\partial_\tau C(x_t, t) = \partial_\tau \left(\int_{-\infty}^{\infty} dx_\tau \, \Pi_{t,\tau}(x_t, t; x_\tau, \tau) \int_{-\infty}^{\infty} dx_T \, \Pi_{\tau,T}(x_\tau, \tau; x_T, T) \, C_T(x_T) \right) = 0$$

- Expand the τ derivative using the product rule (and suppress time arguments):

$$\partial_\tau C(x_t, t) = \int_{-\infty}^{\infty} dx_\tau \int_{-\infty}^{\infty} dx_T \left[\; \Pi_{t,\tau}(x_t; x_\tau) \partial_\tau \Pi_{\tau,T}(x_\tau; x_T) + \Pi_{\tau,T}(x_\tau; x_T) \partial_\tau \Pi_{t,\tau}(x_t; x_\tau) \right] C_T(x_T) = 0$$

- This must hold for arbitrary $C_T \Rightarrow$ choose C_T as a delta function at an arbitrary x_T to eliminate integral over x_T:

$$0 = \int_{-\infty}^{\infty} dx_\tau \left[\Pi_{t,\tau}(x_t; x_\tau) \partial_\tau \Pi_{\tau,T}(x_\tau; x_T) + \Pi_{\tau,T}(x_\tau; x_T) \partial_\tau \Pi_{t,\tau}(x_t; x_\tau) \right]$$

- Develop the first term: substitute F-K PDE for $\partial_\tau \Pi_{\tau,T}(x_\tau; x_T)$:

$$\begin{aligned} & \int_{-\infty}^{\infty} dx_\tau \, \Pi_{t,\tau}(x_t; x_\tau) \, \partial_\tau \Pi_{\tau,T}(x_\tau; x_T) \\ = & \int_{-\infty}^{\infty} dx_\tau \, \Pi_{t,\tau}(x_t; x_\tau) \left(r(\tau) - \mu(x_\tau, \tau) \partial_{x_\tau} - \frac{\sigma^2(x_\tau, \tau)}{2} \partial^2_{x_\tau, x_\tau} \right) \Pi_{\tau,T}(x_\tau; x_T) \end{aligned}$$

4.2. Appendix 3.2: Backward and Forward Equations (4)

- Integrate second and third terms by parts (assuming all components go to zero sufficiently fast as $x_\tau \to \pm\infty$):

$$\int_{-\infty}^{\infty} dx_\tau \, \Pi_{t,\tau}(x_t;x_\tau)\left(-\mu(x_\tau,\tau)\partial_{x_\tau}\Pi_{\tau,T}(x_\tau;x_T)\right)$$

$$= \cancel{-[\mu(x_\tau,\tau)\Pi_{t,\tau}(x_t;x_\tau)\Pi_{\tau,T}(x_\tau;x_T)]_{x_\tau=-\infty}^{\infty}} + \int_{-\infty}^{\infty} dx_\tau \, \Pi_{\tau,T}(x_\tau;x_T)\partial_{x_\tau}[\mu(x_\tau,\tau)\Pi_{t,\tau}(x_t;x_\tau)]$$

$$\int_{-\infty}^{\infty} dx_\tau \, \Pi_{t,\tau}(x_t;x_\tau)\left(-\frac{\sigma^2(x_\tau,\tau)}{2}\,\partial^2_{x_\tau,x_\tau}\Pi_{\tau,T}(x_\tau;x_T)\right)$$

$$= \cancel{-[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)\,\partial_{x_\tau}\Pi_{\tau,T}(x_\tau;x_T)]_{x_\tau=-\infty}^{\infty}} + \int_{-\infty}^{\infty} dx_\tau \, \partial_{x_\tau}[\Pi_{\tau,T}(x_\tau;x_T)]\,\partial_{x_\tau}\left[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)\right]$$

$$= \cancel{\left[\Pi_{\tau,T}(x_\tau;x_T)\,\partial_{x_\tau}[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)]\right]_{x_\tau=-\infty}^{\infty}} - \int_{-\infty}^{\infty} dx_\tau \, \Pi_{\tau,T}(x_\tau;x_T)\,\partial^2_{x_\tau,x_\tau}\left[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t;x_\tau)\right]$$

4.2. Appendix 3.2: Backward and Forward Equations (5)

- Putting this all together, we find:

$$0 = \int_{-\infty}^{\infty} dx_\tau \, \Pi_{\tau,T}(x_\tau; x_T) \left\{ [\partial_\tau + r(\tau)]\Pi_{t,\tau}(x_t; x_\tau) + \partial_{x_\tau}[\mu(x_\tau,\tau)\Pi_{t,\tau}(x_t; x_\tau)] - \partial^2_{x_\tau,x_\tau}\left[\frac{\sigma^2(x_\tau,\tau)}{2}\Pi_{t,\tau}(x_t; x_\tau)\right]\right\}$$

- Since this must hold for arbitrary $\Pi_{\tau,T}(x_\tau; x_T)$, and considering in particular the limit: $\tau \nearrow T$, $\Pi_{\tau,T}(x_\tau; x_T) \to \delta(x_\tau, x_T)$, we conclude that:

$$[\partial_T + r(T)]\Pi_{t,T}(x_t; x_T) + \partial_{x_T}[\mu(x_T,T)\Pi_{t,T}(x_t; x_T)] - \partial^2_{x_T,x_T}\left[\frac{\sigma^2(x_T,T)}{2}\Pi_{t,T}(x_t; x_T)\right] = 0$$

- Standard form of (Kolmogorov) forward equation:

$$\frac{\partial \Pi_{t,T}}{\partial T} = \frac{1}{2}\frac{\partial^2[\sigma^2(x_T,T)\Pi_{t,T}]}{\partial x_T^2} - \frac{\partial[\mu(x_T,T)\Pi_{t,T}]}{\partial x_T} - r(T)\Pi_{t,T}$$

- Compare to standard form of (Kolmogorov) backward equation:

$$-\frac{\partial \Pi_{t,T}}{\partial t} = \frac{1}{2}\sigma^2(x_t,t)\frac{\partial^2 \Pi_{t,T}}{\partial x_t^2} + \mu(x_t,t)\frac{\partial \Pi_{t,T}}{\partial x_t} - r(t)\Pi_{t,T}$$

- This integration by parts of a product of Green's functions is a standard technique.
- The forward and backward PDEs related by this process are called *adjoint* equations.

4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof)

- Consider a general bounded payoff $F_T(S_T)$ with bounded first derivative at $S_T = 0$
  - We can (partly) relax the first boundedness restriction later
- Assume the existence of vanilla options of all strikes $\{C(K)\}$
- How can we price (and replicate) F using $\{C(K)\}$, stock, and bonds?

$$F_T = F_T(S_T) \implies F_t(S_t) = e^{-r(T-t)} \int_0^\infty dS_T \, q(S_T|S_t) \, F_T(S_T)$$

4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (2)

Digression on integration-by-parts

- In what follows, we will apply a variation on the integration-by-parts formula.
- You most likely remember the mnemonic $\int u\,dv = u\,v - \int v\,du$, which is shorthand for:
$$\int_a^b dx\, u(x)v'(x) = \big[u(x)v(x)\big]_{x=a}^{x=b} - \int_a^b dx\, u'(x)v(x), \text{ where: } v(x) = \int_c^x dy\, v'(y),$$
x and y are both "dummy" integration variables that we can choose arbitrarily, and c is an arbitrary reference value of the dummy variables.
- If you need to convince yourself that c can be chosen arbitrarily, split up the integral:
$$\int_a^b = \int_a^c + \int_c^b = \int_c^b - \int_c^a$$
- In particular, we can choose $c = \infty$, in which case: $v(x) = \displaystyle\int_\infty^x dy\, v'(y) = -\int_x^\infty dy\, v'(y)$

and $\displaystyle\int_a^b dx\, u(x)v'(x) = -\left[u(x)\int_x^\infty dy\, v'(y)\right]_{x=a}^{x=b} + \int_a^b dx\, u'(x)\int_x^\infty dy\, v'(y)$

4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (3)

- Integrate by parts once with $F'_T(K) \doteq \left.\frac{\partial F_T}{\partial S_T}\right|_{S_T=K}$, using outer integration variable K and inner integration variable S_T (remember these can be chosen arbitrarily):

$$
\begin{aligned}
F_t(S_t) &= e^{-r(T-t)} \int_0^\infty dS_T \, F_T(S_T)\, q(S_T|S_t) \\
&= e^{-r(T-t)} \left\{ -\left[F_T(K) \int_K^\infty dS_T \, q(S_T|S_t) \right]_{K=0}^{K=\infty} + \int_0^\infty dK \, F'_T(K) \int_K^\infty dS_T \, q(S_T|S_t) \right\} \\
&= e^{-r(T-t)} \left\{ F_T(0) + \int_0^\infty dK \, F'_T(K)\, \mathbb{E}^q[B_T(S_T, K)] \right\} \\
&= e^{-r(T-t)} F_T(0) + \int_0^\infty dK \, F'_T(K)\, B_t(S_t, K)
\end{aligned}
$$

- Representation in terms of bond + p.v. of binary (cash-or-nothing) calls $B_t(S_t, K)$
  - Please don't confuse B_t with a money market account – using it here as abbreviation for present value of **B**inary (cash or nothing) call option struck at K

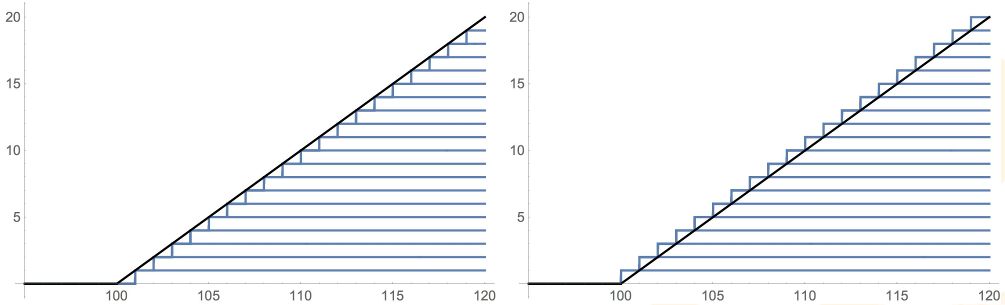
4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (4)

- Integrate by parts again with $F_T''(K) \doteq \left.\frac{\partial^2 F_T}{\partial S_T^2}\right|_{S_T=K}$, using outer integration variable K and inner integration variable K':

$$F_t(S_t) = e^{-r(T-t)} F_T(0) - \left[F_T'(K) \int_K^\infty dK'\, B_t(S_t, K') \right]_{K=0}^{K=\infty} + \int_0^\infty dK\, F_T''(K) \int_K^\infty dK'\, B_t(S_t, K')$$

- But a standard option payoff can be represented as an integral over binary options:

$$C_T(S_T, K) = \max[S_T - K, 0] = \int_K^\infty dK'\, \mathbf{1}_{S_T - K'} \implies C_t(S_t, K) = \int_K^\infty dK'\, B_t(S_t, K')$$



4.3. Appendix 3.3: General Payoffs Static Replication (Another Proof) (5)

- Representation of general payoff in terms of bond + stock + vanilla (call) options:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = 0) + F_T'(0)\, C_t(S_t, K = 0) + \int_0^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

- Breeden-Litzenberger results are distribution- and process- independent: so are these!
  - Robust, static replicating portfolio
  - But, depends on availability of options of (all) strikes
- In practice, we may not want to build the replicating portfolio out of calls starting at strike 0, but rather bifurcate the portfolio into calls with strike greater than some K^* (e.g. at-the-money) and puts with strike less than K^*:

$$F_t(S_t) = e^{-r(T-t)} F_T(S_T = K^*) \,+\, F_T'(S_T = K^*) \left(S_t e^{-y(T-t)} - K^* e^{-r(T-t)} \right)$$
$$+ \int_0^{K^*} dK\, F_T''(K)\, P_t(S_t, K) + \int_{K^*}^\infty dK\, F_T''(K)\, C_t(S_t, K)$$

  - Reference: Carr, P. and Madan, D., "Towards a Theory of Volatility Trading," In: *Volatility: New estimation techniques for pricing derivatives.* Risk Books (1998), 417-427.
  - Why might we want to build a replicating portfolio this way instead of with calls at all K?