Week 8: The Martingale Approach to Arbitrage Theory

Professor Stavros Panageas

Anderson School of Management, MGMTMFE403, Fall 2023

I. The case with zero interest rates

II. The general case

III. Completeness

IV. The Girsanov Theorem

V. The Black Scholes Model Revisited

VI. Examples

VII. Incomplete Markets and Non-traded Assets

I. The case with zero interest rate

- To set ideas, let us assume that $S_0(t) = 1$ for all t
- The dynamics of the risky assets are

$$S(t) = \begin{bmatrix} S_1(t) \\ \vdots \\ S_N(t) \end{bmatrix}$$

- For any process $h = [h_0, h_S]$, its value process $V(t;h)$ is defined by

$$V(t;h) = h_0(t) \cdot 1 + \sum_{i=1}^{N} h_i(t) S_i(t),$$

or in compact terms

$$V(t;h) = h_0(t) + h_S(t) S(t)$$

- An adapted process is called **admissible** if there exists a non-negative real number α (which may depend on the choice of h_S) such that

$$\int_0^t h_S(u)\, dS(u) \geq -\alpha \text{ for all } [t, T] \qquad (1)$$

A process $h(t)$ is called an **admissible portfolio process** if h_S is admissible

- An admissible portfolio process is said to be self-financing if

$$V(t; h) = V(0; h) + \int_0^t h_S(u)\, dS(u),$$

i.e., if

$$dV(t; h) = h_S dS(t)$$

(Note that the above equation follows from $dS_0(t) = 0$) along with

$$dV(t; h) = h_0(t)\, dS_0(t) + h_S(t)\, dS(t)$$

- A straightforward result: For any adapted process h_S satisfying the admissibility condition (1), and for any real number x, there exists a unique adapted process $h_0(t)$, such that
  - The portfolio $h(t)$ defined by $h(t) = [h_0(t), h_S(t)]$ is self-financing
  - The value process is given by

$$V(t; h) = x + \int_0^t h_S(u)\, dS(u).$$

- Proof: Define $h_0(t)$ by

$$h_0(t) = x + \int_0^t h_S(u)\, dS(u) - h_S(t)\, S(t)$$

Then, by the definition of the value process, we obviously have

$$\begin{aligned} V(t;h) &= h_0(t) + h_S(t)\, S(t) = \\ &= x + \int_0^t h_S(u)\, dS(u) \end{aligned}$$

and hence

$$dV(t;h) = h_S dS(t)$$

which shows that h is self-financing.

- Implication. The space $\mathcal{K}_0$ of portfolio values at time T, which are reachable by means of a self-financing portfolio with zero initional cost is given by

$$\mathcal{K}_0 = \left\{ \int_0^T h_S(t)\, dS(t) \right\} \text{ where } h_S \text{ is admissible}$$

- Definition: A probability measure Q on $\mathcal{F}_T$ is called an equivalent martingale measure, if
  - Q is equivalent to P on $\mathcal{F}_T$
    - This means that the same events that have positive probability under the one measure have positive probability under the other
  - All price processes $S_0, S_1, ..., S_N$ are martingales under Q on the time interval $[0, T]$

- We will next show that the **first fundamental theorem of asset pricing:**

If there exists an equivalent martingale measure, then there is no arbitrage.

- Proof: The first step. The Girsanov Theorem (we will hear more about this later) implies that if Q is equivalent to P and $S_i(t)$ are martingales under Q, then $S_i(t)$ must obey the dynamics

$$dS_i(t) = S_i(t)\,\sigma_i dW^Q(t)$$

where $dW^Q(t)$ is a multi-dimensional Brownian motion under Q.

- To prove the lack of arbitrage opportunities, we proceed as follows. First, we recall that an arbitrage is an admissible strategy h, such that

$$\begin{aligned} P(V(T;h) \geq 0) &= 1, \\ P(V(T;h) > 0) &> 0 \end{aligned}$$

- Since the probability measures Q and P are equivalent we can re-formulate these conditions as

$$\begin{aligned} P^Q(V(T;h) \geq 0) &= 1, \\ P^Q(V(T;h) > 0) &> 0 \end{aligned}$$

Since h is self-financing, we have that

$$dV(t;h) = \sum_{i=1}^{N} h_i(t)\,S_i(t)\,\sigma_i(t)\,dW^Q(t)$$

and thus

$$E^Q[V(T;h)] = V(0;h) + E^Q\left[\int_0^T dV(t;h)\right]$$

- The term $\int_0^T dV(t;h)$ is a martingale if h^S is bounded, and more generally is a supermartingale if h^S is only bounded below. Therefore

$$E^Q\left[\int_0^T dV(t;h)\right] \leq 0$$

- The conditions $P^Q(V(T;h) \geq 0) = 1$ and $P^Q(V(T;h) > 0) > 0$ implies that $E^Q[V(T;h)] > 0$ and so

$$\begin{aligned} V(0;h) &= E^Q[V(T;h)] - E^Q\left[\int_0^T dV(t;h)\right] \\ &\geq E^Q[V(T;h)] \\ &> 0 \end{aligned}$$

- What have we proved? Roughly speaking, we have proven that any value process $V(T;h)$ that is a) non-negative and positive with positive probability and b) h is admissible and self-financing implies that $V(0;h) > 0$

- Accordingly, if an equivalent martingale measure exists, there can be no arbitrage.

- The reverse implication is much harder to prove and actually is only true under some restrictive assumptions.

II. The general case

- What if the interest rate is non-zero?

- Define the normalized economy (the "Z" economy) as a price vector process

$$Z(t) = \frac{S(t)}{S_0(t)}$$

that is

$$Z(t) = \left[1, \frac{S_1(t)}{S_0(t)}, \frac{S_2(t)}{S_0(t)}, ..., \frac{S_N(t)}{S_0(t)}\right]$$

- We have two price process $(S(t), Z(t))$ and two economies (the "S-economy" and the "Z-economy") and thus two value processes

$$\begin{aligned}
V^S(t;h) &= \sum_{i=0}^{N} h_i(t) S_i(t) = \\
&= S_0(t) \times \sum_{i=0}^{N} h_i(t) \left(\frac{S_i(t)}{S_0(t)}\right) \\
&= S_0(t) \times \sum_{i=0}^{N} h_i(t) \left(\frac{S_i(t)}{S_0(t)}\right) \\
&= S_0(t) \times \sum_{i=0}^{N} h_i(t) Z_i(t) \\
&= S_0(t) \times V^Z(t;h)
\end{aligned}$$

- This relates $V^Z(t;h)$ with $V^S(t;h)$. An important property is that if h is self-financing under the price process S, it is self-financing under the process Z. To see this apply Ito's lemma to obtain

$$\begin{aligned} dV^Z(t) &= d\left(S_0^{-1}(t)\right) \times V^S(t;h) + dV^S(t;h) \times \left(S_0^{-1}(t)\right) + dV^S d\left(S_0^{-1}(t)\right) \\ &= h(t) \cdot \left(d\left(S_0^{-1}(t)\right) \times S(t) + S_0^{-1}(t) \times dS(t) + dS(t) \times d\left(S_0^{-1}(t)\right)\right) \\ &= h(t) \cdot dZ(t) \end{aligned}$$

- Repeating the same arguments as earlier, we have that if there exists a martingale measure Q, and h is an admissible, self-financing strategy, we therefore have – by repeating the previous arguments– that

$$V^Z(0;h) \geq E^Q\left(V^Z(T;h)\right)$$

and therefore

$$V^S(0;h) \geq \left(\frac{S_0(0)}{S_0(T)}\right) E^Q\left(V^S(T;h)\right)$$

Hence, as long as $\frac{S_0(t)}{S_0(T)}$ is positive, we have that if $E^Q\left(V^S(T;h)\right) > 0$, then $V^S(0;h) > 0$.

III. Completeness

- We have the following result. Consider a given claim with terminal random payoff $X(T)$ at time T. Assume that an equivalent martingale measure Q exists, and assume that the normalized claim $\frac{X}{S_0(T)}$ is integrable. If the $Q-$martingale M, defined by

$$M(t) = E^Q\left[\frac{X}{S_0(T)}|\mathcal{F}_t\right] \tag{2}$$

admits an integral representation of the form

$$M(t) = x + \sum_{i=1}^{N}\int_0^t h_i(s)\,dZ_i(s) \tag{3}$$

then X can be replicated in the "$S-$economy". Furthermore the repliacting portfolio $(h_0, h_1, ..., h_N)$ is given by (3) for $(h_1, ..., h_N)$, and h_0 is given by $M(t) - \sum_{i=1}^{N}\int_0^t h_i(s)\,dZ_i(s)$

- Proof: We are looking for $h = (h_0, ..., h_N)$ such that

$$V^Z(T; h) = \frac{X}{S_0(T)} \quad P-\text{a.s.}$$

and

$$dV^Z = \sum_{i=1}^{N} h_i dZ_i$$

where the normalized value process is given by

$$V^Z(t) = h_0(t)\cdot 1 + \sum_{i=1}^{N} h_i(t)\,Z_i(t)$$

- A reasonable guess is that $M(t) = V^Z(t)$ so we let $M(t)$ be defined by (2) and $h_1 ... h_N$ be defined by (3). Finally, we define h_0 by

$$h_0 = M(t) - \sum_{i=1}^{N} h_i(t) Z_i(t)$$

- With these definitions we have that

$$\begin{aligned} V^Z(t) &= h_0(t) \cdot 1 + \sum_{i=1}^{N} h_i(t) Z_i(t) \\ &= M(t) - \sum_{i=1}^{N} h_i(t) Z_i(t) + \sum_{i=1}^{N} h_i(t) Z_i(t) \\ &= M(t) \end{aligned}$$

- Moreover this result and (3) imply that

$$dV^Z(t) = dM(t) = \sum_{i=1}^{N} \int_0^t h_i(s)\, dZ_i(s)$$

this shows that the portfolio strategy $h_0 ... h_N$ is self-financing in the Z economy, and hence it is self-financing in the S economy. Furthermore

$$V^Z(T) = M(T) = \frac{X}{S_0(T)}$$

and hence $V^S(T) = S_0(T) V^Z(T) = X$.

- Looking one more time at the above proof, we are left with one question: Can the martingale of (2) be put into the form (3) ?

- Yes! Here is a celebrated result in mathematical finance (which we shall not prove)

- (The martingale representation theorem) Let $\mathcal{M}$ denote the (convex) set of equivalent martingale measures. Then for any fixed $Q \in \mathcal{M}$ the following statements are equivalent

  1. Every process M that is a martingale under Q can be written as
  $$dM(t) = \sum_{i=1}^{N} \int_0^t h_i(s)\, dZ_i(s) \tag{4}$$
  for an appropriate vector process $h(t)$

  2. Q is an extremal point of $\mathcal{M}$

- The second fundamental theorem of Asset Pricing

  Assume that the market is arbitrage free. Then the market is complete (all contingent claims can be replicated) if and only if the martingale measure Q is unique

- Proof: If Q is unique, then Q is the unique extremal point, and the martingale representation theorem implies that every Q martingale has a stochastic integral representation. In particular the martingale

$$M(t) = E^Q\left(\frac{X}{S_0(T)}|\mathcal{F}_t\right)$$

  has a stochastic integral representation (4) and then the previous results imply that X is reachable, that is the market is complete.

- We will skip the proof of the reverse implication

- We summarize the practical implications so far as follows

- **First Fundamental Theorem:** The following two statements are (roughly) equivalent

- There are no arbitrage opportunities

- There exists (at least one) martingale measure, i.e., it is possible to write the dynamics of all stocks as

$$dZ_i(t) = Z_i(t)\,\sigma_i dW^Q(t)$$

In particular if $S_0(t) = \exp\left\{\int_0^t r_u du\right\}$ it follows that

$$dS_i(t) = r(t)\,S(t)\,dt + \sigma_i S(t)\,dW^Q(t)$$

- **Second Fundamental Theorem:** The following two statements are equivalent
  - The martingale measure Q is unique
  - The market is complete, that is all claims are replicated.

- For a claim that can be replicated, the arbitrage-free value of the payoff X is given by

$$\Pi(t;X) = E^Q\left(\frac{S_0(t)}{S_0(T)}X|\mathcal{F}_t\right),$$

and the replicating strategy is given by

$$h_0(t) = E^Q\left(\frac{S_0(t)}{S_0(T)}X|\mathcal{F}_t\right) - \sum_{i=1}^{N}\int_0^t h_i(s)\,dZ_i(s)$$

where $h_1...h_N$ are given by

$$d\Pi(t;X) = \sum_{i=1}^{N}\int_0^t h_i(s)\,dZ_i(s)$$

IV. The Girsanov Theorem

- The Girsanov Theorem. Roughly it says the following. Fix some process ϕ_t. Then there exists a probability measure Q such that the dynamics of the $P-$Brownian motion under the probability measure Q are given by

$$dW^P = \phi_t dt + dW^Q$$

where dW^Q is a Brownian motion under Q.

V. The Black Scholes model revisited

- Start with the Black Scholes dynamics under the probability measure P :

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P$$

- Apply Girsanov's Theorem with $\phi_t = -\frac{\mu - r}{\sigma}$. Then there exist a probability measure Q such that

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^P \\ &= \mu S_t dt + \sigma S_t \left(-\frac{\mu - r}{\sigma} dt + dW^Q \right) \\ &= r S_t dt + \sigma S_t dW^Q \end{aligned}$$

- Implication: Applying Ito's Lemma to the discounted process $Z_t = e^{-rt} S_t$ we obtain that

$$dZ_t = \sigma Z_t dW^Q$$

which is a martingale.

- By the first fundamental Theorem of asset pricing, there is no arbitrage in the Black Scholes model.

- Moreover, since there is a unique ϕ_t that "makes" dZ_t a martingale, the martingale measure Q is unique and the market is complete.

- Pricing. Using the risk neutral measure Q we have that

$$\Pi(t;X) = e^{-r(T-t)} E^Q [X|\mathcal{F}_t] \tag{5}$$

where the Q dynamics of $S(t)$ are given by

$$dS_t = rS_t dt + \sigma S_t dW^Q$$

- When $X = \Phi(S_T)$, we can use the Feynman Kac theorem to express the solution of (5) as the Black Scholes PDE

$$\begin{aligned} \frac{\partial F}{\partial t} + rS_t \frac{\partial F}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} &= rF \\ F(T,s) &= \Phi(s) \end{aligned}$$

- The added benefits of the martingale approach
  - We can easily generalize the Black Sholes assumptions. μ, r, σ don't have to be constant, they can be any $\mathcal{F}_t-$adapted processes.
  - X need not be of the form $X = \Phi(S_T)$. Indeed, it can be any $\mathcal{F}_T$ measurable random variable.

VI. An example

- Suppose that $S_0 \in \left[S^l, S^h\right]$. Suppose that we wish to price a contract that pays one dollar if S_t hits S^h before it hits S^l, but becomes worthless (its value becomes zero permanently) if S_t hits S^l before S^h. Price this contract.

- Solution: Let

$$\tau^l = \inf_{t \geq 0} \left\{ S_t = S^l \right\}, \tau^h = \inf_{t \geq 0} \left\{ S_t = S^h \right\}$$

Then X can be expressed as

$$X = 1 \left\{ \tau^h < \tau^l \right\}$$

- We know that its price is given by

$$\Pi(t; X) = E^Q \left[e^{-r\tau^h} 1 \left\{ \tau^h < \tau^l \right\} | \mathcal{F}_t \right]$$

- Using the Dynkin Formula, we know that for $S_t \in \left(S^l, S^h \right)$ it must be the case that $\Pi(t; X) = F(S_t)$ where $F(S_t)$ satisfies

$$r S_t \frac{\partial F}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} - rF = 0 \tag{6}$$

- Moreover, the boundeary conditions are

$$\begin{aligned} F\left(S^h\right) &= 1 \\ F\left(S^l\right) &= 0 \end{aligned}$$

- To solve this PDE, we "guess" that it has a solution given by

$$F\left(S_t\right) = C_1 S_t^{\alpha_1} + C_2 S_t^{\alpha_2} \tag{7}$$

for four constants $\alpha_1, \alpha_2, C_1, C_2$ that need to be determined.

- To verify this conjecture, we substitute (7) into (6) to obtain

$$C_1 S_t^{\alpha_1}\left(r\alpha_1 + \frac{1}{2}\sigma^2\alpha_1\left(\alpha_1 - 1\right) - r\right) + C_2 S_t^{\alpha_2}\left(r\alpha_2 + \frac{1}{2}\sigma^2\alpha_2\left(\alpha_2 - 1\right) - r\right) = 0$$

- For this equation to be always zero for any value of S_t it must be the case that α_1 and α_2 are the two roots of the quadratic equation

$$r\alpha + \frac{1}{2}\sigma^2\alpha\left(\alpha - 1\right) - r = 0$$

- Finally to satisfy $F\left(S^h\right) = 1, F\left(S^h\right) = 0$, we need to determine C_1, C_2 so that

$$\begin{aligned} F\left(S^h\right) &= C_1\left(S^h\right)^{\alpha_1} + C_2\left(S^h\right)^{\alpha_2} = 1 \\ F\left(S^l\right) &= C_1\left(S^l\right)^{\alpha_1} + C_2\left(S^l\right)^{\alpha_2} = 0 \end{aligned}$$

- Solving this 2×2 system results in

$$C_1 = \frac{\left(S^h\right)^{-\alpha_1}}{\left[1 - \left(\frac{S^h}{S^l}\right)^{\alpha_2 - \alpha_1}\right]}$$

$$C_2 = -\frac{\left(\frac{S^h}{S^l}\right)^{-\alpha_1}\left(\frac{1}{(S^l)^{\alpha_2}}\right)}{\left[1 - \left(\frac{S^h}{S^l}\right)^{\alpha_2 - \alpha_1}\right]}$$

- Therefore

$$\begin{aligned} F\left(S_t\right) &= \frac{\left(\frac{S_t}{S^h}\right)^{\alpha_1} - \left(\frac{S^h}{S^l}\right)^{-\alpha_1}\left(\frac{S_t}{S^l}\right)^{\alpha_2}}{\left[1 - \left(\frac{S^h}{S^l}\right)^{\alpha_2 - \alpha_1}\right]} \\ &= \left(\frac{S_t}{S^h}\right)^{\alpha_1} \times \frac{1 - \left(\frac{S_t}{S^l}\right)^{\alpha_2 - \alpha_1}}{1 - \left(\frac{S^h}{S^l}\right)^{\alpha_2 - \alpha_1}} \end{aligned}$$

VII. Incomplete markets and non-traded assets

- Setup: A stochastic proxess $X(t)$ that solves the SDE

$$dX(t) = \mu(X_t)\,dt + \sigma(X_t)\,d\overline{W}(t)$$

- A constant riskless rate r

- X_t is not a traded asset. Say we are looking at a weather derivative and X_t is the temperature.

- How do we price a contingent claim of the form

$$\Phi(X_T)$$

- What is the key issue?
  - The key issue is that we can no longer build a portfolio of X_t and the underlying derivative to "synthetically" create a riskless asset
  - In the Black Sholes model, we did not have this issue because the underlying stochastic variable was traded on the market

- Key idea:
  - Assume that there are multiple derivatives that trade on the market. Suppose for instance that there are two claims with payoff functions $\Phi(X_T)$ and $\Gamma(X_T)$.
  - Suppose that the prices of the two claims are given by $F(t, X_t)$ and $G(t, X_t)$.

- Applying Ito's Lemma to the prices of the two claims leads to

$$\begin{aligned} dF &= \alpha_F F dt + \sigma_F F d\overline{W}(t) \\ dG &= \alpha_G G dt + \sigma_G G d\overline{W}(t) \end{aligned}$$

where

$$\begin{aligned} \alpha_F &= \frac{\frac{\partial F}{\partial t} + \mu(X_t)\frac{\partial F}{\partial X_t} + \frac{1}{2}\sigma^2(X_t)\frac{\partial^2 F}{\partial X_t^2}}{F} \qquad (8) \\ \sigma_F &= \sigma(X_t)\frac{\frac{\partial F}{\partial X_t}}{F} \end{aligned}$$

and similarly for α_G, σ_G.

- Now, let's try to build a riskless portfolio of the two assets. placing weights u_F and u_G in the two assets.

- The Value process for such a portfolio is

$$dV = V\left\{u_F\alpha_F + u_G\alpha_G\right\}dt + V\left\{u_F\sigma_F + u_G\sigma_G\right\}d\overline{W}(t)$$

- To create a riskless asset, and ensure that the portfolio is self-financing

$$\begin{aligned} u_F + u_G &= 1 \\ u_F\sigma_F + u_G\sigma_G &= 0 \end{aligned}$$

- The solution to this system is

$$u_F = -\frac{\sigma_G}{\sigma_F - \sigma_G}, u_G = \frac{\sigma_F}{\sigma_F - \sigma_G}$$

- Substituting u_F, u_G into the dynamics for dV leads to

$$dV = V\left\{\frac{\alpha_G\sigma_F - \alpha_F\sigma_G}{\sigma_F - \sigma_G}\right\}dt$$

- To avoid arbitrage, it must be the case

$$\frac{\alpha_G \sigma_F - \alpha_F \sigma_G}{\sigma_F - \sigma_G} = r$$

or after some re-arranging

$$\frac{\alpha_F - r}{\sigma_F} = \frac{\alpha_G - r}{\sigma_G}$$

- We thus have the following result

Assume that the market for derivatives is free of arbitrage. Then there exists a universal process $\lambda(t)$ such that

$$\frac{\alpha_F(X_t) - r}{\sigma_F(X_t)} = \lambda(t)$$

- Substituting this expression into (8) leads to

$$\begin{aligned} \frac{\partial F}{\partial t} + \left[\mu(X_t) - \lambda(X_t)\sigma(X_t)\right]\frac{\partial F}{\partial X_t} + \frac{1}{2}\sigma^2(X_t)\frac{\partial^2 F}{\partial X_t^2} &= rF(X_t) \\ F(T, X_T) &= \Phi(X_T) \end{aligned}$$

- Note that the same PDE would hold for G, the only thing that would change is the boundary condition at time T.

- Using the Feynman-Kac theorem, the solution to $F(t, X_t)$ is given by

$$F(T, X_t) = e^{-r(T-t)} E_t^Q [\Phi(X_T)]$$

and under the probability measure Q the process X_t follows the dynamics

$$dX(t) = \{\mu(X_t) - \lambda(X_t)\sigma(X_t)\} dt + \sigma(X_t) dW(t)$$

- Where does the market price of risk $\lambda(X_t)$ come from?

- We have to model the dynamics of the price of at least one derivative security on X_t.