MFE 409 Lecture 1
Value-at-Risk

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Lecture Objectives

Understanding **Value-at-Risk**

- How to compute it
- What is it useful for?
- What are its limitations?
- Some alternatives

Outline

1. What is Value-at-Risk?
2. When Value-at-Risk works well and when it does not
3. How Value-at-Risk is used

Measuring Risk

- Defining and managing risk is one of the most important issues firms are facing in their daily operations
- Especially important for financial institutions that rely on leverage.
- Find an answer to the question:

  "What is realistically the worst that could happen over one day, one week, or one year?"

Value-at-Risk

- **Value-at-risk (VaR)** is an answer to the question above where "realistically" is defined by finding an outcome that is so bad that anything worse is highly unlikely.

- Value-at-Risk is the realistically worst case outcome in the sense that anything worse only happens with probability less than some fixed level (such as 1%).

Value-at-Risk: Simple Definition

Value-at-Risk (VaR) at confidence level c is how much you loose in "c worst case":

$$\text{Prob}(W < W_0 - \text{VaR}) = 1 - c$$

Typically:

- W is the value of a portfolio at some point in the future (1 day, 1 month, 1 year)
- W_0 gives some base level: often current value of the portfolio
- Confidence level c gives concrete meaning to "worst case": 99%, 99.9%

Value-at-Risk: Formal Definition

Definition. Let W be a random variable. The *Value-at-Risk* at confidence level c relative to base level W_0 is the smallest non-negative number denoted by VaR such that

$$\text{Prob}(W < W_0 - \text{VaR}) \leq 1 - c$$

Typically:

- W is the value of a portfolio at some point in the future (1 day, 1 month, 1 year)
- W_0 gives some base level: often current value of the portfolio
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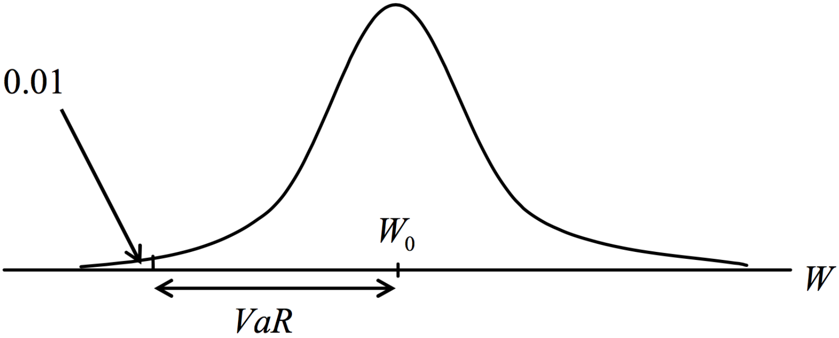
Value-at-Risk Using the PDF



- Bottom point: c-quantile, $F^{-1}(1-c)$ if F is cumulative distribution function

Value-at-Risk Using the CDF

- "Worst case": c-quantile, $F^{-1}(1-c)$ if F is cumulative distribution function

Value-at-Risk Using Simulations

- Often you do not have a formula for the distribution ... but you can simulate random draws

- Monte-Carlo estimator of the c-quantile:
  - Obtain many draws from the distribution
  - Take the $1 - c$ lowest value

Example: Uniform distribution

- All outcomes between a loss of \$50 million and a gain of \$50 million are equally likely for a one-year project

- The VaR for a one-year time horizon and a 99% confidence level is \$49 million

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Example: Normal Distribution

- January 8, 2010
  - Position: EUR 10 million
  - Exchange rate M_t= USD/EUR = \$1.436
  - Dollar position W_0 = \$14.36 million
- Assume normal distribution for FX return

$$R_{M,t+1} \sim \mathcal{N}(\mu, \sigma)$$

  - Historically (in daily units), we find: $\sigma = 0.65\%$ and $\mu \approx 0$
- We want to compute the 99% 1-day Value-at-Risk

Example: Normal Distribution

- Gain distribution: $W - W_0 = \$14.36\text{m} \times R_{M,t+1} \sim \mathcal{N}(\mu_V, \sigma_V)$
  - $\mu_V = \$14.36\text{m} \times \mu_M \approx 0$
  - $\sigma_V = \$14.36\text{m} \times \sigma_M$
- Define $z(c)$: cutoff such that there is a probability c that a standard normal is larger than $z(c)$
  - $c = 99\% \rightarrow z(c) = -2.326$
  - $c = 95\% \rightarrow z(c) = -1.645$
- We have $W - W_0 > \mu_V + z(c) \times \sigma_V$ with probability c
  - 99% 1-day VaR $= -(-2.326 \times 14.36 \times 0.0065) = \$217,204$

Role of Time

- If losses in successive days are independent, normally distributed, and have a mean of zero

$$T\text{-day VaR} = 1\text{-day VaR} \times \sqrt{T}$$

$$T\text{-day ES} = 1\text{-day ES} \times \sqrt{T}$$

- Autocorrelation ρ between losses on successive days, replace $\sqrt{T}$ by

$$\sqrt{T + 2(T-1)\rho + 2(T-2)\rho^2 + 2(T-3)\rho^3 + \ldots + 2\rho^{T-1}}$$

	T=1	T=2	T=5	T=10	T=50	T=250
ρ=0	1.0	1.41	2.24	3.16	7.07	15.81
ρ=0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ=0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ=0.2	1.0	1.55	2.62	3.79	8.62	19.35

Example: Discrete Distribution

- Project A has:
  - 98% chance of leading to a gain of \$2 million
  - 1.5% chance of a loss of \$4 million
  - 0.5% chance of a loss of \$10 million
- The VaR with a 99.9% confidence level is ...
- What if the confidence level is 99.5%?
- What if it is 99%?

Example: Discrete Distribution

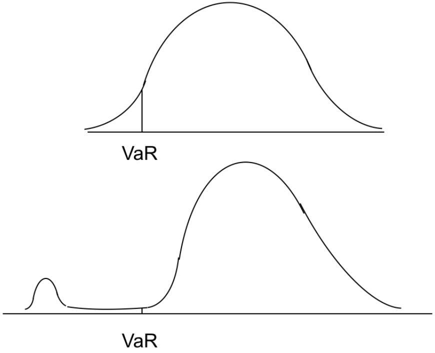
- Project A has:
  - 98% chance of leading to a gain of \$2 million
  - 1.5% chance of a loss of \$4 million
  - 0.5% chance of a loss of \$10 million
- The VaR with a 99.9% confidence level is \$10 million
- What if the confidence level is 99.5%? \$4 million
- What if it is 99%? \$4 million

Example: Discrete Distribution

- Project A - B has:
  - 98% - 98% chance of leading to a gain of \$2 million
  - 1.5% - 1.1% chance of a loss of \$4 million
  - 0.5% - 0.9% chance of a loss of \$10 million
- Which project is more risky?
- Which project has a larger 99% VaR?

Limitation of VaR

- VaR does not capture the distribution of losses below the threshold
- Formally, any 2 distributions with same $F^{-1}(1-c)$ will have the same VaR



Gaming VaR

- Banks are regulated based on VaR ...

- But would like to take more risk

- Taking extreme tail risk will not increase VaR
  - Selling puts
  - Selling disaster insurance

Expected Shortfall

- **Expected shortfall**: expected loss given loss larger than VaR

$$
\begin{aligned}
\mathsf{ES} &= W_0 - \mathbb{E}\left[W | W \leq W_0 - \mathsf{VaR}\right] \\
&= W_0 - \frac{\int_{-\infty}^{W_0 - \mathsf{VaR}} W f(W) dW}{\int_{-\infty}^{W_0 - \mathsf{VaR}} f(W) dW}
\end{aligned}
$$

- Also called C-VaR and Tail Loss
- Regulators have indicated that they plan to move from using VaR to using ES for determining market risk capital
- Two portfolios with the same VaR can have very different expected shortfalls

VaR and ES with Normal Distribution

- Assume $W - W_0 \sim \mathcal{N}(\mu, \sigma)$
- $VaR = -(\mu + z(c) \times \sigma_V)$
- Can also compute expected shortfall:
  - ES $= -\mu_V + \sigma_V \frac{e^{-z(c)^2/2}}{\sqrt{2\pi}(1-c)}$
  - 95% ES $= -(\mu_V - \sigma_V \times 2.0628)$
  - 99% ES $= -(\mu_V - \sigma_V \times 2.6649)$
- For normal distributions, close relation between VaR and ES: multiple of volatility

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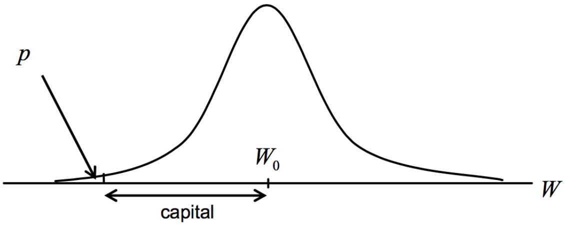
Rationale for Value-at-Risk

- It captures an important aspect of risk in a single number
- Easy to understand
- Two broad motivations
  - Measure of potential extreme loss
  - Capital to hold against possible failure

A Measure of Capital

- Example:
  - W measures value of total assets of the firm in 10 days
  - W_0 is today's value of the firm's assets
  - Firm remains solvent as long as W does not fall below $W_0 - \text{capital}$.



- 1% 10-day VaR: amount of capital to hold so that firm goes bankrupt with probability 1% in the next 10 days

Regulatory Capital

- Regulators have traditionally used VaR to calculate the capital they require banks to keep
- The market-risk capital has been based on a 10-day VaR estimated where the confidence level is 99%
- Credit risk and operational risk capital are based on a one-year 99.9% VaR

What is Special about Capital?

- If the firm or bank has limited liability, then it does not matter whether the firm goes bust just marginally, or whether it goes bust spectacularly, leaving a big shortfall

- The tail loss is not a concern for a firm with limited liability

Takeaways

- Value-at-Risk is:
  - a simple measure
  - used by regulators and practitioners to measure risk
  - which focuses on the extreme downside of a distribution
- It has some limitations
  - Does not capture the entire distribution of extreme losses
  - *Another important one next week*
- Implications
  - If you want to monitor risk, know its limitations
  - Expected shortfall is a better behaved alternative
  - If you are constrained by it, know how to game it
- Next: How do we use VaR with multiple risks? to choose portfolios?