MGMT MFE 406 – Derivative Markets (4 units)

Part 5: Time-Dependent Payoffs and Their Simulation

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Outline

1. Preamble: A Classification of Exotic Payoffs

- Two characteristics allow us to subdivide the set of exotic payoffs:
  - Path dependence/independence
  - Single/multiple underlying assets
- Number of assets part is easy; path dependence can be a little subtle:
  - Path independence:
    vanilla option payoffs don’t depend on the particular path chosen by the market
  - Path independence:
    vanilla option values are also indifferent to the underlying asset’s price history
  - Path dependence:
    value/payoff of an option does depend on the price history
  - What about American options?
- Further subdivision (by type of path dependence) is possible

1. Preamble: A Classification of Exotic Payoffs (2)

Construct a table listing examples of each type

	Path Independent	**Path Dependent**
Single Asset	Standard Binary Power	Asian Barrier Lookback
Multiple Asset	Outperformance Basket Quanto	Multi-asset Barrier Mountain Range(s)

- Working through this classification gives us an opportunity to build our analytical and computational tool kits.
- In this section of the course, focus is on path-dependent payoffs (and, more generally, time-dependent payoffs) and the simulation tools that are useful in valuing them.

1.1. Single Asset Path-Dependent Payoffs

- Path-dependent options are characterized by some measure of the asset price path from inception to maturity:
  - Maximum: $M_t = M_{+,t} = M_{+\infty,t} = \max\{S_\tau : t_0 \le \tau \le t\}$
  - Minimum: $m_t = M_{-,t} = M_{-\infty,t} = \min\{S_\tau : t_0 \le \tau \le t\}$
  - Average: $A_t = M_{1,t} = \dfrac{\int_{t_0}^{t} d\tau\, S_\tau}{t - t_0}$
- Discretely sampled versions of the above are (most) common (why?), but we will usually consider continuous limits first.
- Exposure to term structures (when they are present) is unavoidable.

2. Average-Dependent or "Asian" Options

- Standard Asian options: ubiquitious payoff structure
  - Set of sampling times: $\{t_j : j = 1, 2, \ldots, m\}$ (intraday, daily, weekly, monthly, yearly...) and a payoff date $T \geq t_m$
  - Fixings from asset price process: S_t: $\{S_{t_j}\}$
  - Set of known weights : $\{w_j\} : \sum_{j=1}^{m} w_j = \underline{\mathbf{w}}^\intercal \cdot \underline{\mathbf{1}} = 1$ (but also random weighted examples, e.g. VWAP)
  - Define arithmetic average: $A_T \doteq \sum_{j=1}^{m} w_j \, S_{t_j} = \underline{\mathbf{w}}^\intercal \cdot \underline{\mathbf{S}}$
  - (Arithmetic) Asian (end) call payoff: $AC_T \doteq [A_T - K]^+$
- Uses
  - Hedge exposures distributed through time
  - Avoid impact of abnormal prices at expiry
  - Allow smooth unwind of hedge for cash-settled options
  - Asian put: $AP_T \doteq [K - A_T]^+$ (put-call parity via average of forwards)

2. Average-Dependent or "Asian" Options (2)

Related Payoffs

- Average strike (Asian start) options: $ASC_T = [S_T - kA'_T]^+$

  (A'_T here is often a starting average with $A'_T \doteq \sum_{j'=1}^{m'} w'_{j'} S_{t'_{j'}}$, $\{t'_{j'}, S_{t'_{j'}}\}$, and $\sum_{j'=1}^{m'} w'_{j'} = 1$)
  - Smooth initial delta hedge for physically settled options
  - Change of (numéraire) measure:
    mappable into a standard Asian option: $\mathbb{E}^{\mathbf{Q}_0}[S_T - kA'_T]^+ = \mathbb{E}^{\mathbf{Q}_0}[S_T]\, \mathbb{E}^{\mathbf{Q}_1}[1 - kA^*_T]^+$,
    with $\mathbb{E}^{\mathbf{Q}_1}$ an expectation with respect to S_T as numéraire,
    and A^*_T an average over a process running "backwards" from T to each t_i
- Asian start/Asian end options: $ASAEC_T = [A_T - kA'_T]^+$
  (again, A'_T here represents a starting average, A_T an ending average)
  - Avoid impact of abnormal prices at initiation and expiry
  - Smoothes management of both initial delta hedge and final unwind
  - Valuation: integration over joint distribution of two correlated averages
  - Also "fixed notional" payoff: $\left[\frac{A_T}{A'_T} - k\right]^+$, i.e. $\left[\frac{\text{Asian end}}{\text{Asian start}} - \text{strike ratio}\right]^+ \times$ (\$ notional amt.)

2. Average-Dependent or "Asian" Options (3)

Related Payoffs (continued)

- Options on cash dividend-paying stocks
  - Consider price process modified to pay discrete cash dividends:
$$\frac{dS_t}{S_t} = \left(r_t - y_t - \sum_{j=1}^{m} \frac{D_j}{S_t} \delta(t - t_j) \right) dt + \sigma_t \, dW_t^{\mathbf{Q}_0}$$
  - Consider also the (original) process with cash dividends reinvested:
$$\frac{dR_t}{R_t} = (r_t - y_t) dt + \sigma_t \, dW_t^{\mathbf{Q}_0} \; : \; R_0 = S_0$$
  - It is easy to show that:
$$S_{T > t_m} = R_T - \sum_{j=1}^{m} D_j \frac{R_T}{R_{t_j}}, \text{ hence } [S_T - K]^+ = \left[R_T - \sum_{j=1}^{m} D_j \frac{R_T}{R_{t_j}} - K \right]^+$$
  - Change of measure: mappable into Asian put on "reciprocal" process:
$$\mathbb{E}^{\mathbf{Q}_0} [S_T - K]^+ = \mathbb{E}^{\mathbf{Q}_0}[R_T] \, \mathbb{E}^{\mathbf{Q}_1} \left[1 - \sum_{j=1}^{m} D_j R_{t_j}^* - K R_T^* \right]^+, \text{ with: } \frac{dR_t^*}{R_t^*} = (y_t - r_t) dt + \sigma_t \, dW_t^{\mathbf{Q}_1} \; : \; R_0^* = \frac{1}{S_0}$$

2. Average-Dependent or "Asian" Options (4)

Related Payoffs (continued)

- Geometric average options: $GC_T \doteq [G_T - K]^+$ with $G_T \doteq \prod_{j=1}^{m} S_{t_j}^{w_j}$
  - Geometric averages (products) of lognormal variables are lognormal
  - Of interest primarily because most payoffs may be valued in closed form (and can therefore be used as control variates)
- Harmonic average options: $HC_T \doteq [H_T - K]^+$ with $H_T \doteq \left(\sum_{j=1}^{m} \frac{w_j}{S_{t_j}}\right)^{-1}$
  - Of interest because of analogy to "fixed notional" payoffs
- Other average options: $A_pC_T \doteq [(A_p)_T - K]^+$ with $(A_p)_T \doteq \left(\sum_{j=1}^{m} w_j S_{t_j}^p\right)^{1/p}$
  - Contains all averages as subcases
    - $p = 1$: arithmetic; $p = 0$: geometric; $p = -1$: harmonic
    - $p = \infty$: maximum; $p = -\infty$: minimum

2.1. Valuation of Asian Options

- Consider discrete problem: $AC_T = \max[A_T - K, 0]$, $A_T \doteq \frac{1}{m}\sum_{j=1}^{m} S_j$
- "Black-Scholes+" framework (lognormal prices under risk-neutral measure)
- Well-known results:
  - Sum of lognormals isn't lognormal
  - Only in very special cases (continuous sampling, constant parameters) is anything like closed-form solution possible
- Continuous (or near-continuous) case:
  - Interpretation in terms of "effective forward/yield", "effective vol"
  - Effective forward: average of forwards on sample dates (exact first moment)
  - Effective vol: find variance of average, extract implied term vol. This is *approximately* $\sigma/\sqrt{3}$
  - Rule of thumb: full Asian value should be ca. 50-60% of the corresponding vanilla value
- Two traditional approaches in BS/BS+ worlds:
  - Simulation
  - Analytical approximation
- What to do about skew/smile (BS++)?

2.2. Monte Carlo Simulation

- Basic idea: brute force simulation of the (correlated) distributions of S_{t_j}
  - Need to extend our Bare-Bones algorithm to simple time stepping
- Enhancements
  - Antithetic variates – generate full antithetic path
  - Moment matching?
  - Control variates
    - Geometric average
    - Vanilla option
    - Vanilla portfolio (all with strike K or optimized for minimum portfolio value)
  - Quasi-random sequences (Faure, Sobol...)
- Properties
  - Speed: very slow; $T \sim$ (small const) $\frac{m}{\epsilon^2}$
  - Precision (ϵ): Arbitrarily high given enough time
  - Easy to implement if high precision not required
  - Applicable to any payoff
  - Difficulties computing Greeks

2.2. Monte Carlo Simulation (2)

Simple Time Stepping

- For a single time step to maturity T in the BS world, we took advantage of the log-normal representation of the asset price process:

$$dS_t = (r-y)\,S_t\,dt + \sigma\,S_t\,dW_t^{\mathbf{Q}} \Rightarrow d\ln(S_t) = (r-y-\sigma^2/2)\,dt + \sigma\,dW_t^{\mathbf{Q}}$$

to write: $S_T = S_0 \exp\left[(r-y-\sigma^2/2)T + \sigma W_T^{\mathbf{Q}}\right] = S_0 \exp\left[\tilde{\mu}^{\mathbf{Q}} T + \sigma W_T^{\mathbf{Q}}\right]$

- Then MC sampling from the terminal asset price distribution could be reduced to sampling from the standard normal distribution $z \sim N(0,1)$. E.g. for sample "path" i:

$$S_{T,i} = S_0 \exp\left[\tilde{\mu}^{\mathbf{Q}} T + \sigma\sqrt{T}\,z_i\right]$$

- Simple time stepping over a set of $\{t_j : j = 0 \ldots m\}$ with $t_0 = 0$ in the BS world extends this (path-by-path) to an iterative process over the $\{t_j\}$:

$$S_{t_j,i} = S_{t_{j-1},i} \exp\left[\tilde{\mu}^{\mathbf{Q}} \Delta t_j + \sigma\sqrt{\Delta t_j}\,z_{j,i}\right] \text{ with } \Delta t_j \doteq t_j - t_{j-1}$$

  - In practice, we will step from each time step j to the next, generating n normal samples (one for each path i) and calculating the updated asset price along each path.

- This is the most basic example in a stochastic setting of (forward-) Euler-Maruyama (丸山 [まる やま]) time stepping. If $dx = \tilde{\mu}(t)\,dt + \sigma(t)\,dW_t$ [dropping most $(\cdot)^{\mathbf{Q}}$ superscripts]:

$$x_{\Delta t} - x_0 \approx \tilde{\mu}(t{=}0)\Delta t + \sigma(t{=}0) W_{\Delta t}$$

2.2. Monte Carlo Simulation (3)

Simple Time Stepping (continued): BS+ world: time-dependent parameters

- We can do better than Euler-Maruyama if rates, yields, and vols have term structures. Using the moment properties of BM:

$$\mathbb{E}[dx] = \tilde{\mu}^{\mathbf{Q}}(t)\,dt \;\Rightarrow\; \mathbb{E}[x_{\Delta t}] = x_0 + \int_0^{\Delta t} dt\,\tilde{\mu}^{\mathbf{Q}}(t) \doteq x_0 + \tilde{\mu}^{\mathbf{Q}}_{0,\Delta t}\Delta t$$

$$\mathbb{E}[(dx)^2] = \sigma^2(t)\,dt \;\Rightarrow\; \mathrm{var}[x_{\Delta t}] = \int_0^{\Delta t} dt\,\sigma^2(t) \doteq \sigma^2_{0,\Delta t}\Delta t$$

  - These formulae serve to define the effective "term" parameters $\tilde{\mu}^{\mathbf{Q}}_{0,t}$ and $\sigma^2_{0,t}$ and, by extension, the forward rates and yields $r_{0,t}$ and $y_{0,t}$.
- A convenient (but not entirely correct) short-hand: $x_t = x_0 + \tilde{\mu}^{\mathbf{Q}}_{0,t}\,t + \sigma_{0,t}W'_t$
  - This is exact (in distribution) at time t. W'_t is *equivalent* but not *equal* to W_t (note that this representation doesn't fully preserve the covariance structure between x_t at different times).
- Time-stepping can therefore be reduced to:

$$\begin{aligned} x_{t+\Delta t} &= x_t + [\tilde{\mu}^{\mathbf{Q}}_{0,t+\Delta t}(t+\Delta t) - \tilde{\mu}^{\mathbf{Q}}_{0,t}t] + (\sigma_{0,t+\Delta t}W'_{t+\delta t} - \sigma_{0,t}W'_t) \\ &= x_t + \tilde{\mu}^{\mathbf{Q}}_{t,t+\Delta t}\Delta t + \sigma_{t,t+\Delta t}(W''_{t+\Delta t} - W''_t) \end{aligned}$$

  - These formulae serve to define the effective "forward" parameters $\tilde{\mu}^{\mathbf{Q}}_{t,t'}$ and $\sigma^2_{t,t'}$ and, by extension, the term rates and yields $r_{t,t'}$ and $y_{t,t'}$.

2.2. Monte Carlo Simulation (4)

Optimized Control Variates

- As before, for each of the n independent paths, calculate the payoff of an option C_T^* that is highly correlated with AC_T and has a reliably known expected value $\mathbb{E}[C_T^*]$.
- Average over paths to obtain an estimator $\langle AC_T \rangle_{\beta,n}$ of $\mathbb{E}[AC_T]$ in terms of an unknown coefficient β:

$$\mathbb{E}[AC_T] \simeq \langle AC_T \rangle_{\beta,n} \doteq \boldsymbol{\beta}\mathbb{E}[C_T^*] + \frac{1}{n}\sum_{i=1}^{n}(AC_{T,i} - \boldsymbol{\beta} C_{T,i}^*)$$

- Now, minimize the variance of this estimator:

$$\begin{aligned}
\operatorname{var}[\,\langle AC_T \rangle_{\beta,n}\,] &\simeq \frac{1}{n}\operatorname{var}[AC_{T,i} - \beta\, C_{T,i}^*] \\
&\simeq \frac{1}{n}(\operatorname{var}[AC_T] - 2\beta\operatorname{cov}[AC_T, C_T^*] + \beta^2\operatorname{var}[C_T^*]) \\
\Rightarrow \beta^* &= \operatorname{cov}[AC_T, C_T^*]/\operatorname{var}[C_T^*]
\end{aligned}$$

- Optimized variance of the estimator:

$$\begin{aligned}
\operatorname{var}[\,\langle AC_T \rangle_{\beta^* n}\,] &\simeq \frac{1}{n}(\operatorname{var}[AC_T] - \operatorname{cov}^2[AC_T, C_T^*]/\operatorname{var}[C_T^*]) \\
&= \frac{1}{n}(\operatorname{var}[AC_T] - (\beta^*)^2\operatorname{var}[C_T^*]) = \frac{1}{n}\operatorname{var}[AC_T]\left(1 - \rho_{AC,C^*}^2\right)
\end{aligned}$$

  - By projecting the variance of AC_T onto C_T^*, we extract all but the "idiosyncratic" contribution to the error.

2.2. Monte Carlo Simulation (5)

Geometric Option Valuation

- Consider $G_T \doteq \prod_{j=1}^m S_{t_j}^{1/m} = \left(\prod_{j=1}^m S_{t_j}\right)^{1/m}$. Define $g_T \doteq \ln(G_T/S_0)$. Then:

$$g_T \doteq \frac{1}{m}\sum_{j=1}^m \tilde{\mu}^{\mathbf{Q}}_{0,t_j} t_j + \sigma_{0,t_j} W'_{t_j}$$

$$\tilde{\mu}_g = \mathbb{E}[g_T] = \frac{1}{m}\sum_{j=1}^m \tilde{\mu}^{\mathbf{Q}}_{0,t_j} t_j$$

$$\sigma_g^2 \doteq \mathrm{var}[g_T] = \frac{1}{m^2}\sum_{j,j'=1}^m \min[\sigma^2_{0,t_j} t_j, \sigma^2_{0,t'_j} t'_j]$$

  - Note that the double variance summation over j, j' can be collapsed to a single sum over j because of the min[] property.
- Hence $G_T \sim S_0 \exp[N(\tilde{\mu}_g, \sigma_g^2)]$ and, in particular: the effective forward $F_{G,T}$ can be defined as:

$$F_{G,T} \doteq \mathbb{E}[G_T] = S_0\, e^{\tilde{\mu}_g + \sigma_g^2/2}$$

- If we define an effective vol $\sigma_{eff} \doteq \sqrt{\sigma_g^2/T}$ relative to the maturity T (why?), then Geometric average call option valuation can be reduced to a Black-Scholes call: $\mathbf{BSC}[F_{G,T}, K, T, r, r, \sigma_{eff}]$
  - Simplest (in this case, exact) example of 2-moment matching technique

2.2. Monte Carlo Simulation (6)

Optimized Control Variate Example

- 1-year monthly ($m=12$) ATM arithmetic Asian call (AC), $S_0=K=100$, $r=0.01$, $y=0.02$, $\sigma=0.20$, $n=10^8$ standard (log-)normal paths
  - BBMC $\langle AC_T\rangle_n=4.60740 \Rightarrow$ BBMC $\langle AC_t\rangle_n=4.56156$
  - exact vanilla $C_t=7.36429 \Rightarrow$ BBMC $\langle AC_t\rangle_n=61.9\% C_t$
  - $\mathrm{var}[AC_T]=57.1648 \Rightarrow \mathrm{std}[\langle AC_T\rangle_n]\approx 7.5607\times 10^{-4}$
- CV: 1-year monthly ($m=12$) ATM geometric Asian call (GC): exact $GC_t=4.40286$
  - BBMC $\langle GC_T\rangle_n=4.44736 \Rightarrow$ BBMC $\langle GC_t\rangle_n=4.40311=59.8\% C_t$
  - $\mathrm{var}[GC_T]=53.9092 \Rightarrow \mathrm{std}[\langle GC_T\rangle_n]\approx 7.3423\times 10^{-4}$
  - $\mathrm{cov}[AC_T,GC_T]=55.4940 \Rightarrow \rho_{AC,GC}=0.999656$
- Optimal $\beta^*=\mathrm{cov}[AC_T,GC_T]/\mathrm{var}[GC_T]=55.4940/53.9092=1.02940$

$$
\begin{aligned}
\text{Estimator value: } \langle AC_T\rangle_{\beta^*,n} &= \beta^*\mathbb{E}[GC_T]+(\langle AC_T\rangle_n-\beta^*\langle GC_T\rangle_n)\\
&= 1.02940\cdot e^{0.01}4.40286+(4.60740-1.02940\cdot 4.44736)\\
&= 4.57784+0.02929=4.60713 \;\Rightarrow\; \langle AC_t\rangle_{\beta^*,n}=4.56129\\
\text{Compare to: } \langle AC_T\rangle_{\beta=1,n} &= 1\mathbb{E}[GC_T]+(\langle AC_T\rangle_n-1\langle GC_T\rangle_n)\\
&= e^{0.01}4.40286+(4.60740-4.44736)\\
&= 4.44710+0.16004=4.60714 \;\Rightarrow\; \langle AC_t\rangle_{\beta=1,n}=4.56130
\end{aligned}
$$

2.2. Monte Carlo Simulation (7)

Optimized Control Variate Example, continued

- Optimized variance:

$$\begin{aligned}\mathrm{var}[AC_T-\beta^*GC_T] &= \mathrm{var}[AC_T](1-\rho^2_{AC,GC})\\ &= 57.1648(1-0.999656^2) = 57.1648(6.89\times 10^{-4})\end{aligned}$$

  - I.e. $0.999656^2 = 0.999311$ of the BBMC AC_T estimator variance is projected onto GC_T.

- Optimized estimator standard error:

$$\begin{aligned}\mathrm{std}[\langle AC_T\rangle_{\beta^*n}] &= \mathrm{std}[\langle AC_T-\beta^*GC_T\rangle_n] = \mathrm{std}[\langle AC_T\rangle_n]\sqrt{1-\rho^2_{AC,GC}}\\ &\approx 7.5607\times 10^{-4}\sqrt{1-0.999656^2} = 7.5607\times 10^{-4}(0.026) = 1.98\times 10^{-5}\end{aligned}$$

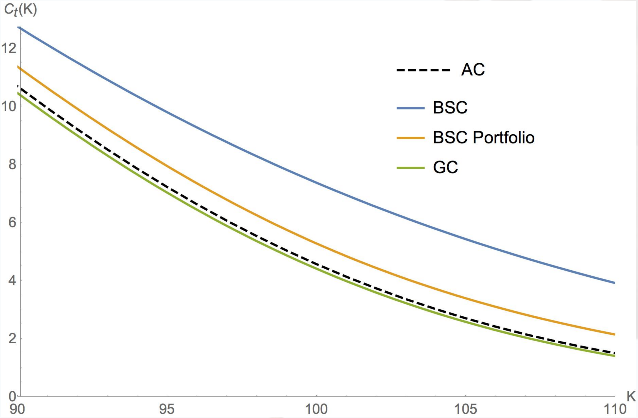
- Compare to:

$$\begin{aligned}\mathrm{std}[\langle AC_T\rangle_{\beta=1,n}] &= \mathrm{std}[\langle AC_T-1GC_T\rangle_n]\\ &= \sqrt{\mathrm{var}[\langle AC_T\rangle_n]+\mathrm{var}[\langle GC_T\rangle_n]-2\,\mathrm{cov}[\langle AC_T\rangle_n\,,\langle GC_T\rangle_n]}\\ &\approx 7.5607\times 10^{-4}\sqrt{0.0015036} = 7.5607\times 10^{-4}(0.039) = 2.93\times 10^{-5}\end{aligned}$$

  - Only $1-0.0015036 = 0.998496$ of the BBMC AC_T estimator variance is projected onto GC_T with $\beta = 1$.

2.2. Monte Carlo Simulation (8)

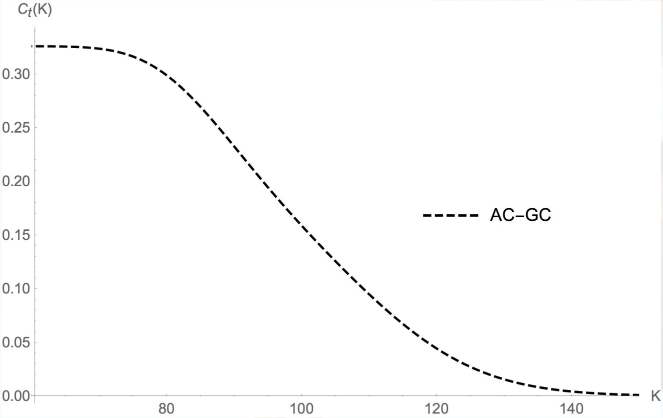
- Value vs. strike:
  - 1-year monthly ($m = 12$) Asian call, $S_0 = 100$, $r = 0.01$, $y = 0.02$, $\sigma = 0.20$
  - $n = 10^8$ standard (log-)normal paths, optimised control variate
  - Standard errors for $K = 100$, $n = 10^8$: 1.96×10^{-5}(CV), vs. 7.49×10^{-4}(BB)



- Note close similarity to Geometric Average

2.2. Monte Carlo Simulation (9)

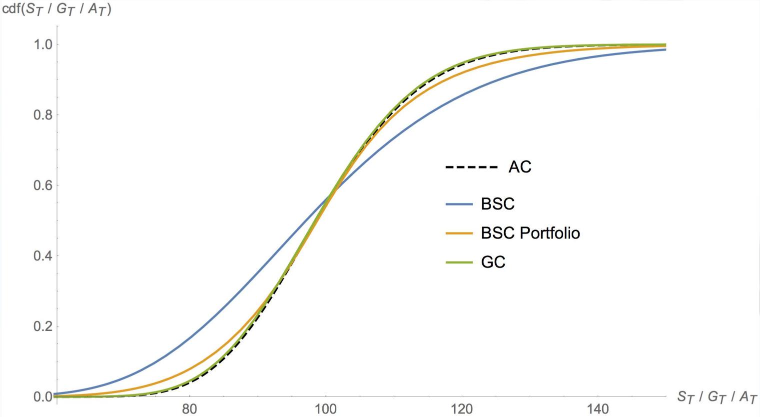
- Arithmetic – Geometric Call Values vs. strike:



- What explains the positive limiting value of ca. 32 cents for low strikes?

2.2. Monte Carlo Simulation (10)

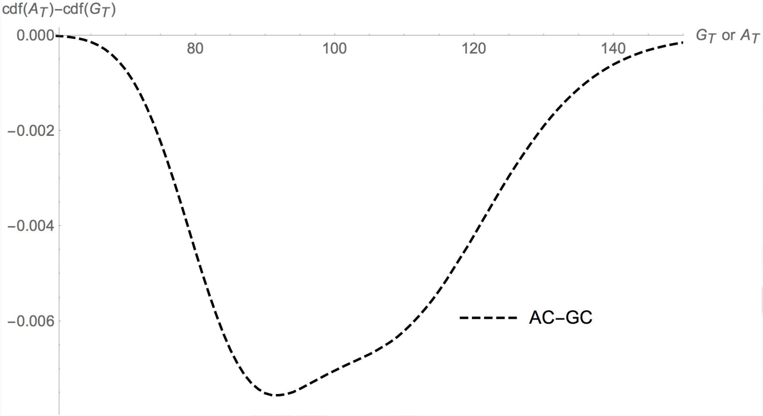
- Risk-neutral CDFs vs. terminal "asset" price:



- AC obtained via histogram of $n = 10^8$ simulated paths; analytical results for BSC, GC
- Is there another way of obtaining these (especially for AC)?

2.2. Monte Carlo Simulation (11)

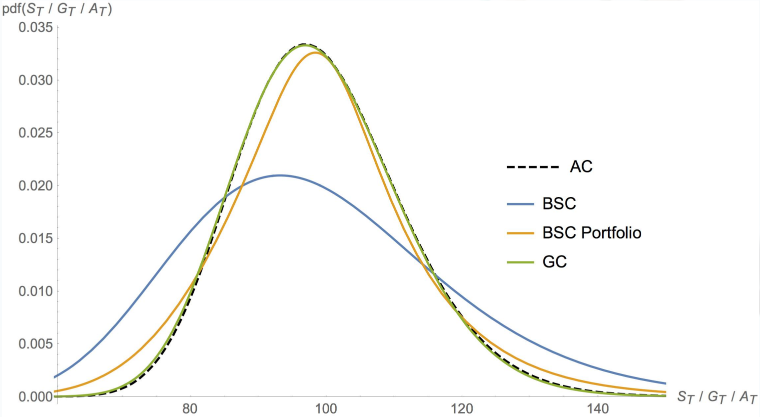
- Arithmetic – Geometric Average CDFs vs. terminal “asset” price:



- Why is the difference in CDFs always negative (implying that the CDF of the Geometric Average is always greater than that of the Arithmetic Average)?

2.2. Monte Carlo Simulation (12)

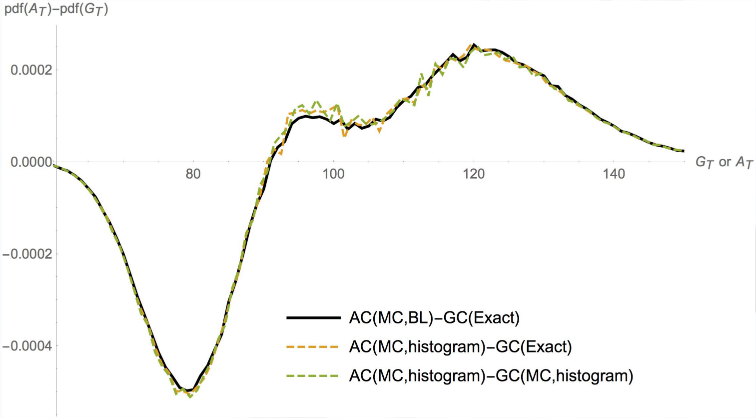
- Risk-neutral PDFs vs. terminal "asset" price:



- Obtained from histogram of 10^8 scenarios for AC, analytical results for BSC, GC
- Is there another way of obtaining these (especially for AC)?

2.2. Monte Carlo Simulation (13)

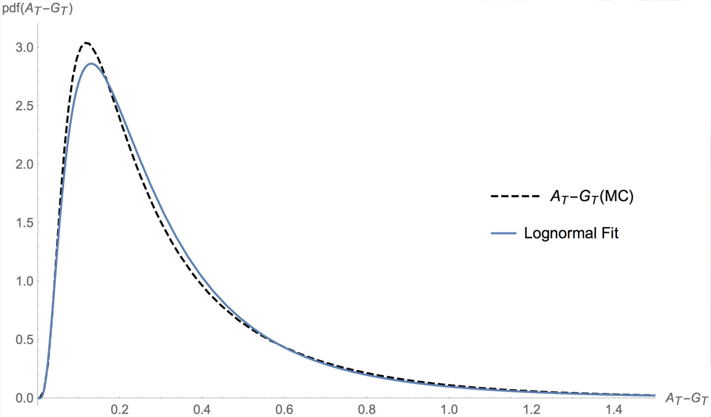
- Arithmetic – Geometric Average PDFs vs. terminal "asset" price:



- Differencing two "nearby" densities is actually a pretty rigorous test!
- Do the differences appear as we expect them to?

2.2. Monte Carlo Simulation (14)

- Risk-neutral PDF of Arithmetic − Geometric Average:



- Difference between A_T and G_T (from $n = 10^8$ simulated paths) appears close to log-normal.
- Is there a payoff whose present values we could differentiate to obtain these results?

2.2. Monte Carlo Simulation (15)

Take-aways

- Most of the variance reduction techniques we developed for path-independent payoffs can be applied to path-dependent problems
- Path-dependent options (and Asian payoffs in particular) are a setting in which the (optimized control variate approach stands out
- Some of our observations from Monte Carlo simulation give us ideas for approximation techniques for these options…

2.3. Analytical Approximations

- Oldest and most widely used set of techniques:
  Ritchken, Sankarasubramaniam, Vijh (1989); Levy (1990)
  - Choose a (tractable) family of distributions defined by k parameters
  - Fit the parameters to k moments of the target distribution
- Family almost always based on normal distribution (similarity to geometric)
  - 2 moments: lognormal
    - Simplest method: although distribution of A_T is not lognormal, approximate it as such.
    - To determine parameters of approximating distribution, fit first two moments of A_T.

$$\begin{aligned} a_T \doteq \ln(A_T) &\sim N(\tilde{\mu}_a, \sigma_a^2) \\ A_T &\sim e^{\tilde{\mu}_a + \sigma_a z}, \quad z \sim N(0,1) \\ e^{\tilde{\mu}_a + \sigma_a^2/2} &= \mu_1 = \frac{S_0}{m} \sum_{j=1}^{m} e^{(r_{0,j} - y_{0,j}) t_j} \\ e^{2\tilde{\mu}_a + 2\sigma_a^2} &= \mu_2 = \frac{S_0^2}{m^2} \sum_{j,j'=1}^{m} e^{(r_{0,j} - y_{0,j}) t_j + (r_{0,j'} - y_{0,j'}) t_{j'} + \min(\sigma_{0,j}^2 t_j, \sigma_{0,j'}^2 t_{j'})} \end{aligned}$$

    - With a little careful bookkeeping, this calculation can be made extremely fast!
    - Quick implementation, trivial solution for parameters
    - Convenient because Black-Scholes framework is retained
    - Interpretation in terms of effective forward/yield, volatility
    - Alternatively, we can define $a_T \doteq \ln(A_T/S_0)$, so that we can write the moment equations without the S_0 factors.

2.3. Analytical Approximations (2)

- 3 moments: e.g. displaced lognormal (Milevsky and Posner, 1998: also 4MM)

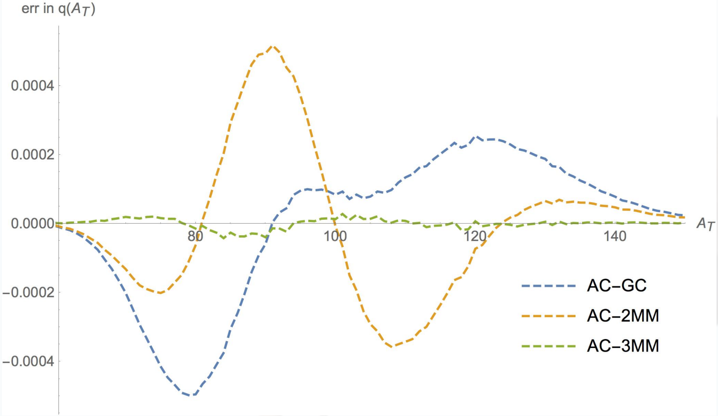
  Assume: $\ln(A_T - B) \sim N(\tilde{\mu}_a, \sigma_a^2)$ with additional parameter B

  $$\Rightarrow B + e^{\tilde{\mu}_a + \sigma_a^2/2} = \mu_1;\ e^{2\tilde{\mu}_a + \sigma_a^2}\left(e^{\sigma_a^2} - 1\right) = \mu_2^c;\ e^{3\tilde{\mu}_a + 3\sigma_a^2/2}\left(e^{3\sigma_a^2} - 3e^{\sigma_a^2} + 2\right) = \mu_3^c$$

  - Eliminate $\tilde{\mu}_a$ and solve resulting cubic equation for $e^{\sigma_a^2}$, then work downwards
  - Slight additional complication to Black-Scholes
  - Some artefacts for low strikes
  - Fitting to other distribution families is also possible
- Advantages of these methods
  - Quick to implement
  - Robust (closed form or near-closed form)
  - Rapid execution for Asians: k-th moment can be calculated in $\mathcal{O}(k!\, m)$ time
- Disadvantages
  - Inaccurate for few moments $(k = 2)$ and large volatilities/long maturity
  - For higher moments, shape error limits accuracy (non-convergent)
  - ASAE options: joint distribution required; not generally available for $k > 2$
- Alternative approximation approach: asymptotic (e.g. Cornish-Fisher/Edgeworth) expansions around (log-)normal reference distribution (Turnbull/Wakeman)

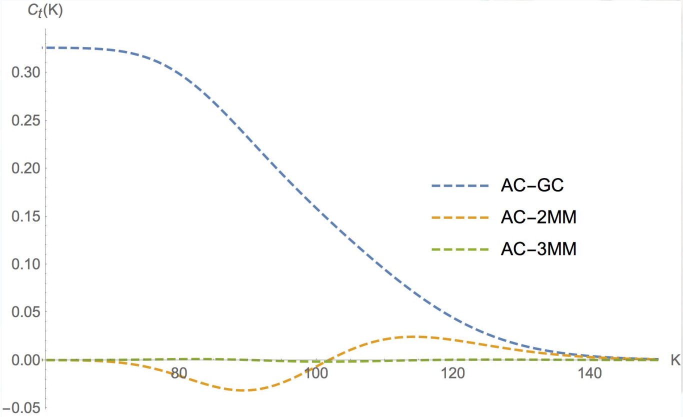
2.3. Analytical Approximations (3)

- Errors in PDF (parameters as above)



2.3. Analytical Approximations (4)

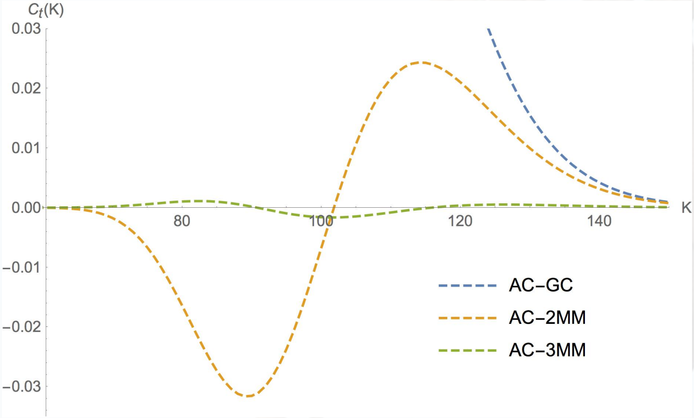
- Asian call value errors (parameters as above)



- Moment methods are clearly superior, avoiding the error in mean of A_T which dominates the Geometric approximation at low strikes

2.3. Analytical Approximations (5)

- Asian call value errors: close up



- Significant improvement of 3-moment approximation over 2-moment

2.3. Analytical Approximations (6)

Curran's Method

- Cleverer method (Curran): consider distribution of:

$$G_T \doteq \prod_{j=1}^{m} S_j^{1/m}$$

  - $A_T \geq G_T$ on any path.
  - G_T is precisely log-normal in Black-Scholes framework.
  - $A_T - G_T$ appears "close to" (well-approximated by) log-normal.
  - G_T and A_T are highly correlated.
  - Rewrite expected payoff conditional on G_T:

$$\begin{aligned} \mathbb{E}[AC_T] &= \mathbb{E}[\max(A_T - K, 0)] \\ &= \mathbb{E}[\max(A_T - K, 0)\,\mathbf{1}_{G_T \geq K}] + \mathbb{E}[\max(A_T - K, 0)\,\mathbf{1}_{G_T < K}] \\ &= \mathbb{E}[(A_T - K)\,\mathbf{1}_{G_T \geq K}] + \mathbb{E}[\max(A_T - K, 0)\,\mathbf{1}_{G_T < K}] \end{aligned}$$

    - Most of the value comes from the first term, which we can evaluate exactly!
    - We need only approximate the second term;
      in particular we can pretend $A_T - G_T$ (or even A_T) is lognormal conditional on G_T
    - Leads at most to an integral over G_T of a Black-Scholes-like formula (albeit potentially with G_T-dependent parameters).

2.3. Analytical Approximations (7)

Curran's Method (continued)

- Curran's decomposition of the Arithmetic option value conditional on the value of G_T allows for a rich variety of approximations.
- Two basic methods:
  - Simple/"naive" approximation: exchange max[] and $\mathbb{E}$[] operations in 2^{nd} term (for $G_T < K$):

$$\begin{aligned}\mathbb{E}[\max(A_T - K, 0)|G_T] &\approx \max(\mathbb{E}[A_T|G_T] - K, 0) \\ \Rightarrow \mathbb{E}[\max(A_T - K, 0)\,\mathbf{1}_{G_T<K}] &\approx \int_0^K dG_T\, q(G_T)\max(\mathbb{E}[A_T|G_T] - K, 0) \\ &\approx \int_{K^*}^K dG_T\, q(G_T)(\mathbb{E}[A_T|G_T] - K)\end{aligned}$$

    where K^* is the value of G_T for which $\mathbb{E}[A_T|G_T] = K$
    - This is a lower bound (why?).
  - Precise/"sophisticated" approximation: assume that mean and variance of $A_T - G_T$ are constant and equal to their (conditional) values for $G_T = K$

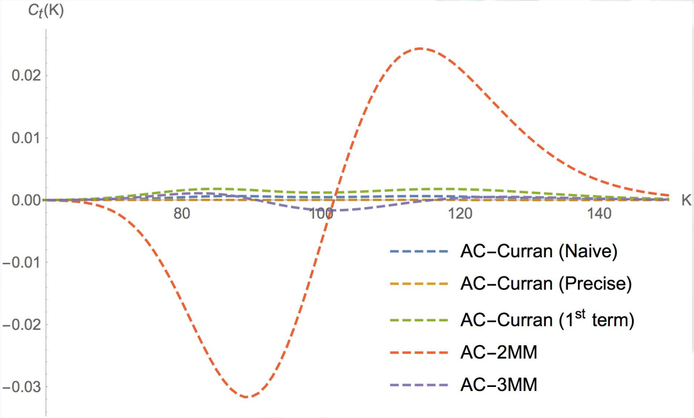
$$\mathbb{E}[\max(A_T - K, 0)\,\mathbf{1}_{G_T<K}] \approx \int_0^K dG_T\, q(G_T)\,\mathbb{E}[\max([A_T - G_T|G_T] - [K - G_T], 0)]$$

$$\text{with } \ln(A_T - G_T) \sim N(\tilde{\mu}_{a-g}, \sigma^2_{a-g})$$

    - Integral over a Black-Scholes formula with constant parameters

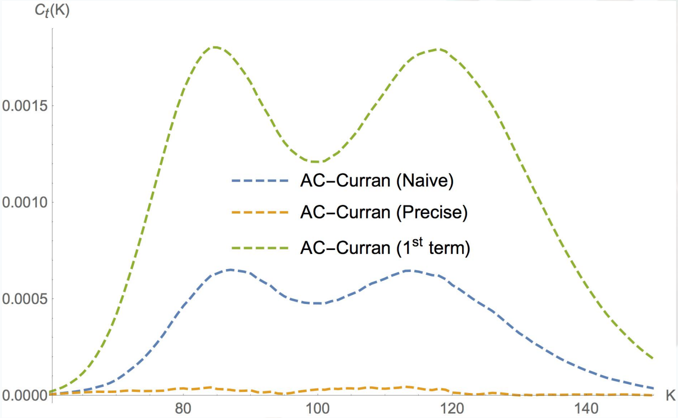
2.3. Analytical Approximations (8)

- Asian call value errors: Curran vs. Moment matching



2.3. Analytical Approximations (9)

- Asian call value errors:
  Comparison of precise versus naive forms of Curran’s method



2.3. Analytical Approximations (10)

Take-aways

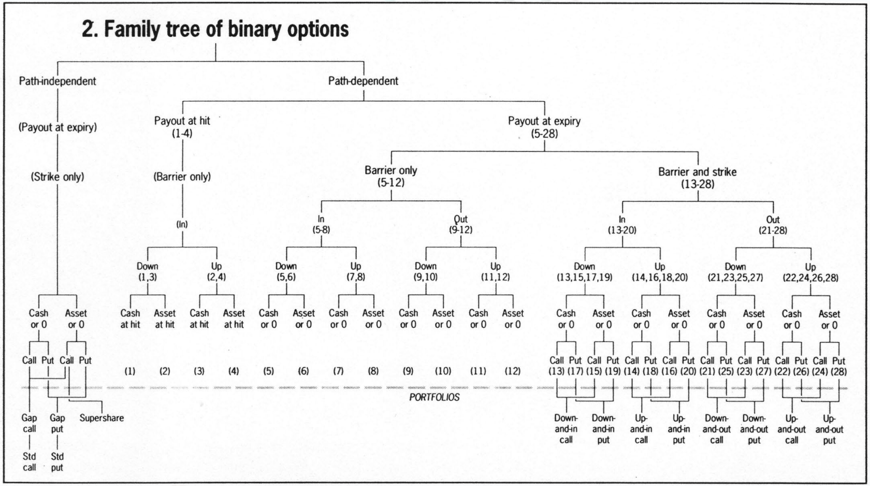
- There is a variety (to some extent, a hierarchy) of approximation techniques
  - Moment methods (e.g. 2-moments, 3-moments)
  - Curran's method ("naive", "sophisticated", and everywhere in between)
    - Underlying idea is conditioning the (unknown) distribution of the asset price on the distribution of something that is better understood.
- We can select amongst them given (the severity of) the problem parameters and our need for precision.
- These techniques are applicable to basket (and other) options as well!
- They fill an important role in any quantitative analyst's toolkit.

3. Extremum-Dependent Options: Barriers & Lookbacks

- Extremum-Dependent Options:
  payoff depends on maximum or minimum price achieved on the path.
  - $M_{+\infty} \doteq \max\left(S(t), 0 \leq t \leq T\right)$
  - $M_{-\infty} \doteq \min\left(S(t), 0 \leq t \leq T\right)$
- Single Barrier Options:
  vanilla option knocks in/out if barrier level is crossed during the option's life.
  - Calls (C) and Puts (P)
  - Down (D) and Up (U)
    - Natural and 'Reverse' Barriers
  - Knock-outs (O) and Knock-ins (I)
    - In-out parity
  - 8 basic permutations: DOC/DIC/UOP/UIP/UOC/UIC/DOP/DIC, e.g.
    - Down-and-out call: $DOC_T(S, K; M_{-\infty}, B) = (S_T - K)^+ \, \mathbf{1}_{M_{-\infty}-B}$
    - Up-and-out put: $UOP_T(S, K; M_{+\infty}, B) = (K - S_T)^+ \, \mathbf{1}_{B-M_{+\infty}}$
    - Down-and-in call: $DIC_T(S, K; M_{-\infty}, B) = (S_T - K)^+ \, \mathbf{1}_{B-M_{-\infty}}$
  - Binary (Rebate) Barriers: 'One Touch' and 'No Touch'
    - E.g.. Down-and-out binary: $DOB_T(S; M_{-\infty}, B) = \mathbf{1}_{M_{-\infty}-B}$
  - Original formulation in Merton (Bell Journal, 1973);
    classification and complete results in Rubinstein and Reiner (RISK, 1991)

3. Extremum-Dependent Options: Barriers & Lookbacks (2)

Family Tree of Binary Options



- source: Rubinstein and Reiner (RISK, 1991).

3. Extremum-Dependent Options: Barriers & Lookbacks (3)

Related Payoffs

- Double Barriers and 'Range Binaries'
  - Each option has both a down and an up barrier
  - Both barriers can potentially have in/out conditionality
  - Order of barrier crossing can can also be introduced
  - Variety of results are available: Beaglehole (working paper, 1991); Kunitomo and Ikeda (Math. Finance, 1992); Reiner (RISK lecture notes, 1994); Jamshidian (working paper, 1997)
    - Double barrier binary:
      $DBB_T(S; M_{-\infty}, B_-; M_\infty, B_+) = \left(\mathbf{1}_{M_{-\infty} - B_-}\right)\left(\mathbf{1}_{B_+ - M_\infty}\right)$
- Moving Barriers
  - Barrier level isn't constant, but drifts (or jumps) over life of option
  - Solvable in closed form if barrier level is exponential in time (i.e., log barrier level has constant drift)
- Windowed and Partial Barriers
  - Barrier is only active for a portion of the option's life
  - Barrier start and end dates
  - Sources: Heynen and Kat (J. Fin. Eng., 1994), Zhang (book, 1997)

3. Extremum-Dependent Options: Barriers & Lookbacks (4)

- Soft Barriers
  - Option knocks out fractionally as barrier range is crossed
  - Sources: Hart and Ross (RISK, 1994)
- Occupation time/Parisian Barriers
  - Option knocks out as a function of time spent across barrier
  - Results differ according to whether occupation time is required to be contiguous or not
  - Variety of results are available: Chesney *et al*. (RISK, 1997); (Linetsky, 1998)
- Discrete Barriers: Flesaker (working paper, 1992); Levy and Mantion (RISK, 1997); Reiner (lecture notes, 1991 and 1998); Broadie, Glasserman, and Kou (1996, 1997).
- Multi-asset 'Outside' Barriers: Heynen and Kat (RISK, 1994)

- Lookback Options: payoff involves extremum in place of strike or final spot. (Goldman, Sosin, & Gatto, 1979; Garman, 1989; Conze & Viswanathan, 1991)
  - Lookback call: $LC_T(S; M_{-\infty}) = S_T - M_{-\infty}$
  - Lookback put: $LP_T(S; M_{+\infty}) = M_{+\infty} - S_T$
  - Call on maximum: $C_{+,T}(S, K; M_{+\infty}) = (M_{+\infty} - K)^+$
  - Discrete Lookbacks: Reiner (lecture notes, 1991 and 1998); Broadie, Glasserman, and Kou (1996, 1997).

3.1. Valuation of Single Barrier Options & Lookbacks

- source: Financial Times, 16 Nov 1995, p. 59.

3.1. Valuation of Single Barrier Options & Lookbacks (2)

Feynman-Kac: an Extended Form

- Consider the partial differential equation:

$$\frac{\partial C}{\partial t} + \mu(x,t)\frac{\partial C}{\partial x} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2 C}{\partial x^2} = r(t)C$$

  in an open $\{\mathbb{R}^1 \times \mathbb{R}^-\} = \{x, t\}$ region $\boldsymbol{\Omega}$, subject to Dirichlet conditions:

$$C = C_{\boldsymbol{\partial\Omega}}(x_{\boldsymbol{\partial\Omega}}, t) \text{ on a boundary } \boldsymbol{\partial\Omega} = x_{\boldsymbol{\partial\Omega}}(t)$$

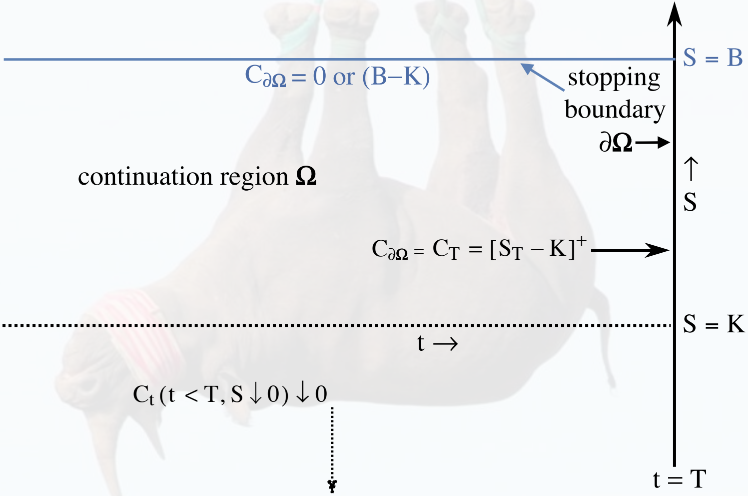
  - Basic version: take $\boldsymbol{\partial\Omega} = x(t = T)\forall x$
- The solution to this equation $C(x,t)$ is equal to the expectation:

$$\mathbb{E}_t\left[\exp\left(-\int_t^{t_{\boldsymbol{\partial\Omega}}} r(t')\,dt'\right) C_{\boldsymbol{\partial\Omega}}(x_{\boldsymbol{\partial\Omega}}, t_{\boldsymbol{\partial\Omega}})\right]$$

  for the Itô process: $dx = \mu(x,t)\,dt + \sigma(x,t)\,dW$ starting at $x(t) = x$
- The "cup-shaped" surface $\boldsymbol{\partial\Omega}$ (including, if necessary, points at ∞) is to be interpreted as a stopping boundary and $t_{\boldsymbol{\partial\Omega}}$ as a stopping (or ***hitting*** or ***killing***) time; $\boldsymbol{\Omega}$ is to be interpreted as a continuation region.
- Other extensions/generalizations are possible, including incorporation of an exogenous "source" term $f(x,t)$ in the PDE.

3.1. Valuation of Single Barrier Options & Lookbacks (3)

Illustration of Stopping Boundary (2)



3.1. Valuation of Single Barrier Options & Lookbacks (4)

Feynman-Kac Further Examples

5. Let $\partial\Omega = \{S_T : 0 < S_T \leq B; S_\tau = B : 0 < \tau < T\}$.
   The "knock-out" barrier is chosen as a constant $B > K$: $B(\tau < T) = B$.

   Define the stopping (or ***hitting***) time θ: $\theta \doteq \inf(\tau \leq T \,|\, S_\tau \notin \boldsymbol{\Omega})$.
   Equivalently, for a continuous process: $\theta \doteq \inf(\tau \leq T \,|\, S_\tau \in \partial\boldsymbol{\Omega})$.

   Let $C_{\partial\Omega} = \{0 : \theta < T;\ \max[S_\theta - K, 0] : \theta = T\}$. Then:

$$C\,(S < B, t < T; B) = \mathbb{E}_t^{\mathbf{Q}}\left[e^{-r(\theta - t)}\max[S_\theta - K, 0] \wedge (\theta = T)\right]$$

   This is a "knock-out" (barrier) call option that only pays off the intrinsic value $[S_T - K]^+$ if the stopping barrier B isn't hit before $t = T$.

6. Same as 5, but let $C_{\partial\Omega} = [S_\theta - K]^+\ \forall$ hitting times $\theta \leq T$. Then:

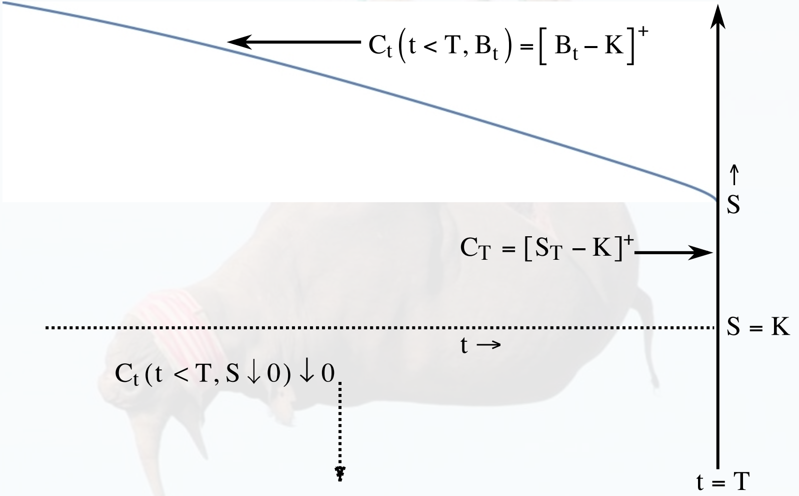
$$C\,(S < B, t < T; B) = \mathbb{E}_t^{\mathbf{Q}}\left[e^{-r(\theta - t)}[S_\theta - K]^+\right]$$

   This is a "capped" (barrier) call option that pays off the intrinsic value $B - K$ whenever the stopping barrier B is hit before maturity.
   Otherwise, it has the payoff of a regular call at maturity T.

   These were actually traded on the CBOE during the 1990s!

3.1. Valuation of Single Barrier Options & Lookbacks (5)

Illustration of Stopping Boundary (3)



3.1. Valuation of Single Barrier Options & Lookbacks (6)

7. Same as 6, but let $\partial\Omega = \{S_T\colon 0 < S_T \leq B_T;\, S_\tau = B_\tau\colon 0<\tau<T\}$, where the "knock-out" barrier is now chosen as a function $B_\tau = B(\tau) > K$.

   Again, let $C_{\partial\Omega} = [S_\theta - K]^+$ $\forall$ hitting times $\theta \leq T$. Then:

$$C\,(S < B, t < T; B) = \mathbb{E}_t^{\mathbf{Q}}\Big[e^{-r(\theta - t)}[S_\theta - K]^+\Big]$$

   This is a generalized capped barrier call that pays off the intrinsic value $B_\theta - K$ whenever the time-dependent stopping barrier B is hit before maturity.

   Otherwise, it has the payoff of a regular call at maturity T.

8. Same as 7, but where the function $B(\tau > t_0)$ is chosen to maximize:

$$C\,(S_0, t_0; B(\tau)) = \max_{B(\tau)} \mathbb{E}_{t_0}^{\mathbf{Q}}\Big[e^{-r(\theta - t_0)}[S_\theta - K]^+\Big]$$

   given an arbitrary initial starting point $S_0(t_0)$ inside the continuation region.

   Then this is an American call option.

3.1. Valuation of Single Barrier Options & Lookbacks (7)

- Consider an option that pays out (at time T) some "general" function $G_T(S_T)$ iff $S_{\theta:\, t\leq\theta\leq T}$ never crosses some boundary (barrier) B, and 0 otherwise.
- We know from Feynman-Kac that we can write the present value $G_t(S_t)$ as the solution to the backward (BS) equation:
$$\frac{\partial G}{\partial t} + (r-y)S_t\frac{\partial G}{\partial S_t} + \frac{\sigma^2 {S_t}^2}{2}\frac{\partial^2 G}{\partial {S_t}^2} = r\,G$$
subject to the "initial" condition $G(S_t, t \nearrow T) = G_T(S_T)$
as well as the boundary condition $G(S_t \to B, t < T) = 0$
- We *also* know from Feynman-Kac that we can write the present value $G_t(S_t)$ as:
the discount factor $e^{-r(T-t)}$ times an integral over S_T of:
[the payoff function $G_T(S_T)$ times a sort-of probability density $q(S_t; S_T)$]:
$$G_t(S_t) = \int_0^\infty dS_T\, G_T(S_T)[e^{-r(T-t)}q(S_t;S_T)] = e^{-r(T-t)}\int_0^\infty dS_T\, G_T(S_T)q(S_t;S_T)$$
where $e^{-r(T-t)}q(S_t;S_T)$ is the Arrow-Debreu state price density associated with the **Q**-measure process for the stock price S:
$$dS = (r-y)S\,dt + \sigma S\,dW^{\mathbf{Q}}$$
starting at S_t and reaching S_T without ever crossing the barrier B.

3.1. Valuation of Single Barrier Options & Lookbacks (8)

- $q(S_t; S_T) = q_{out}(S_t, t; S_T, T; B)$ is given by the solution to the backward (Kolmogorov) equation (without discounting term):

$$\frac{\partial q_{out}}{\partial t} + (r-y)S_t \frac{\partial q_{out}}{\partial S_t} + \frac{\sigma^2 {S_t}^2}{2} \frac{\partial^2 q_{out}}{\partial {S_t}^2} = 0$$

  subject to the "initial" condition $q_{out}(S_t, t \nearrow T) = \delta(S_t, S_T)$
  as well as the boundary condition $q_{out}(S_t = B, t) = 0$
- We'll return to the interpretation of q_{out}, but we want to do a few other things to standardize/normalize the process before that. Begin by moving to log-space:

$$dx = d\ln(S) = \tilde{\mu}^{\mathbf{Q}}\,dt + \sigma\,dW^{\mathbf{Q}} \quad \text{with: } \tilde{\mu}^{\mathbf{Q}} \doteq r - y - \sigma^2/2$$

- Returning to the ordinary BS setting (before introducing the barrier), given a starting level of S_t or $x_t = \ln(S_t)$ at time t, we typically want to find a probability density ("q") in the **Q**-measure for being at S_T or $x_T = \ln(S_T)$ at some $T > t$.
  - Let's stay in x co-ordinates from here on out; put a $\tilde{}$ on the q to keep track of that.
  - Then we can write our probability density as $\tilde{q}(x_T, T | x_t, t)$

3.1. Valuation of Single Barrier Options & Lookbacks (9)

- If we want to go down the (Kolmogorov) PDE route to solve for $\tilde{q}$, we know there are two completely equivalent ways of going about it:
  - Forward equation in $\{x_T, T\}$, holding $\{x_t, t\}$ fixed:
  $$\frac{\partial \tilde{q}}{\partial T} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial x_T^2} - \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial x_T}, \quad \text{with } \lim_{T \searrow t} \tilde{q}(x_T, T | x_t, t) = \delta(x_T - x_t)$$
  - Backward equation in $\{x_t, t\}$, holding $\{x_T, T\}$ fixed:
  $$\frac{\partial \tilde{q}}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial x_t^2} - \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial x_t}, \quad \text{with } \lim_{t \nearrow T} \tilde{q}(x_T, T | x_t, t) = \delta(x_t - x_T)$$

  The latter is the Feynman-Kac approach, i.e. it's the BS(M) PDE without discounting
- We know the solution to (both) these equations:
$$\tilde{q}(x_T, T | x_t, t) = \frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\left(\frac{-[x_T - x_t - \tilde{\mu}^{\mathbf{Q}}(T-t)]^2}{2\sigma^2(T-t)}\right)$$
- We can also re-write it as the Girsanov factor times the density for a drift-less BM:
$$\tilde{q}(x_T, T | x_t, t) = \exp\left(\frac{\tilde{\mu}^{\mathbf{Q}}}{\sigma^2}\left[x_T - x_t - \frac{\tilde{\mu}^{\mathbf{Q}}(T-t)}{2}\right]\right)\frac{\exp\left[-(x_T - x_t)^2/2\sigma^2(T-t)\right]}{\sigma\sqrt{2\pi(T-t)}}$$
- No matter how we write the solution (to BOTH equations), it's clear that it only depends on the differences $T-t$ and $x_T - x_t$.

3.1. Valuation of Single Barrier Options & Lookbacks (10)

- Let's use these facts to re-write the two PDEs.
  First, using $\tau' = T - t$:
  - The form of the forward equation doesn't change:
  $$\frac{\partial \tilde{q}}{\partial \tau'} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial x_T^2} - \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial x_T}, \quad \text{with} \lim_{\tau' \searrow 0} \tilde{q}(x_T, \tau' | x_t, 0) = \delta(x_T - x_t)$$
  - The signs of the rhs terms in the backward equation reverse:
  $$\frac{\partial \tilde{q}}{\partial \tau'} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial x_t^2} + \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial x_t}, \quad \text{with} \lim_{\tau' \searrow 0} \tilde{q}(x_T, 0 | x_t, \tau') = \delta(x_t - x_T)$$

  ...and notice that we've basically converted the backward equation into (another, almost identical) forward equation, but with x_t and x_T exchanged and the sign of the drift reversed.
- The solution (to both equations) in terms of τ' is:

$$\tilde{q} = \frac{1}{\sigma\sqrt{2\pi\tau'}} \exp\left(\frac{-[x_T - x_t - \tilde{\mu}^{\mathbf{Q}}\tau']^2}{2\sigma^2\tau'}\right)$$
$$= \exp\left(\frac{\tilde{\mu}^{\mathbf{Q}}}{\sigma^2}\left[x_T - x_t - \frac{\tilde{\mu}^{\mathbf{Q}}\tau'}{2}\right]\right) \frac{\exp\left[-(x_T - x_t)^2/2\sigma^2\tau'\right]}{\sigma\sqrt{2\pi\tau'}}$$

3.1. Valuation of Single Barrier Options & Lookbacks (11)

- Now, simplify using $\chi_+ = x_T - x_t$:
  - Again, the form of the forward equation doesn't change:

$$\frac{\partial \tilde{q}}{\partial \tau'} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial \chi_+^2} - \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial \chi_+}, \quad \text{with} \lim_{\tau' \searrow 0} \tilde{q}(\chi_+, \tau') = \delta(\chi_+)$$

  - But the sign of the drift term in the backward equation reverses:

$$\frac{\partial \tilde{q}}{\partial \tau'} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial \chi_+^2} - \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial \chi_+}, \quad \text{with} \lim_{\tau' \searrow 0} \tilde{q}(\chi_+, \tau') = \delta(\chi_+)$$

  ...and we're now identical to the forward equation.
- The solution to both equations is:

$$\tilde{q} = \frac{1}{\sigma\sqrt{2\pi\tau'}}\exp\left(\frac{-[\chi_+ - \tilde{\mu}^{\mathbf{Q}}\tau']^2}{2\sigma^2\tau'}\right) = \exp\left(\frac{\tilde{\mu}^{\mathbf{Q}}}{\sigma^2}\left[\chi_+ - \frac{\tilde{\mu}^{\mathbf{Q}}\tau'}{2}\right]\right)\frac{\exp\left[-\chi_+^2/2\sigma^2\tau'\right]}{\sigma\sqrt{2\pi\tau'}}$$

  - If we want to retain the "sense" of the backward equation, we can use $\chi_- = x_t - x_T$ instead:

$$\frac{\partial \tilde{q}}{\partial \tau'} = \frac{\sigma^2}{2}\frac{\partial^2 \tilde{q}}{\partial \chi_-^2} + \tilde{\mu}^{\mathbf{Q}}\frac{\partial \tilde{q}}{\partial \chi_-}, \quad \text{with} \lim_{\tau' \searrow 0} \tilde{q}(\chi_-, \tau') = \delta(\chi_-)$$

  but it's probably easiest to just stay with the forward equation...

3.1. Valuation of Single Barrier Options & Lookbacks (12)

- Finally, standardize (non-dimensionalize) time in the forward equation by defining $\tau = \sigma^2 \tau'$ and let $x \doteq \chi_+ = x_T - x_t$ to simplify notation:

$$\frac{\partial \tilde{q}}{\partial \tau} = \frac{1}{2}\frac{\partial^2 \tilde{q}}{\partial x^2} - \alpha \frac{\partial \tilde{q}}{\partial x}, \quad \text{with } \lim_{\tau \searrow 0} \tilde{q}(x,\tau) = \delta(x) \text{ and defining } \alpha \doteq \frac{\tilde{\mu}^{\mathbf{Q}}}{\sigma^2},$$

corresponding to the process: $dx_\tau = \alpha\, d\tau + dW_\tau^{\mathbf{Q}}$ starting at $x_0 = 0$, with solution:

$$\tilde{q}(x,\tau) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(\frac{-[x-\alpha\tau]^2}{2\tau}\right) = \exp\left(\alpha\left[x - \frac{\alpha\tau}{2}\right]\right) \frac{\exp\left[-x^2/2\tau\right]}{\sqrt{2\pi\tau}}$$

- Now, introduce a barrier level B in addition to S_t and S_T ... and therefore a log co-ordinate b in addition to x_t and x_T
  - Since we are using the forward formulation and measuring all x values relative to a starting point of x_t, so that $x(\tau = 0) = 0$, we need to define b consistently as: $b \doteq \ln(B/S_t)$.
- Looking forward in time, our state space is no longer just S_T or x_T or x, but also a binary variable indicating whether we crossed (or didn't cross) B or b.
  - It's a JOINT density: $\tilde{q}(x_T, T | x_t, t; \text{did we cross } b \text{ or not}) = \tilde{q}(x, \tau; \text{did we cross } b \text{ or not})$

3.1. Valuation of Single Barrier Options & Lookbacks (13)

- The terminology, e.g. $\tilde{q}_{out}$, reflects what we use the density for.
  - The joint event of reaching x_T *without crossing* b allows us to value payoffs contingent on *not crossing* the barrier; i.e. options which knock OUT upon touching b.

    We call that piece of the density $\tilde{q}_{out}$.
  - The joint event of reaching x_T and *having crossed* b allows us to value payoffs contingent on *having crossed* the barrier; i.e. options which knock IN upon touching b.

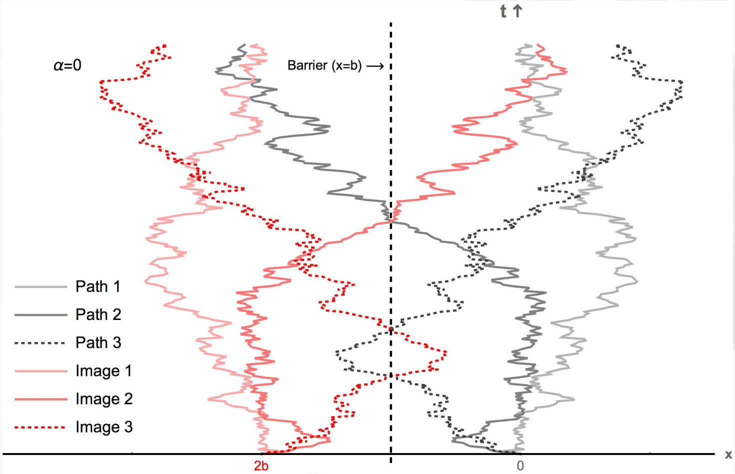
    We call that piece of the density $\tilde{q}_{in}$.
- By completeness, $\tilde{q}_{out} + \tilde{q}_{in}$ equals the BS(M) density $\tilde{q}$.
- Neither $\tilde{q}_{out}$ nor $\tilde{q}_{in}$ alone is strictly a complete density: neither usually integrates to 1.
  - Otherwise, they retain all the properties we associate with probability densities and we operate with them as if they are, with the understanding that this just means we are doing so jointly with the corresponding requirement on crossing (or not) the barrier.
  - Certain combinations of x_t, x_T, and b imply that we must cross b to get from x_t to x_T (x_T is on 'the other side' of b from x_t).

    For such combinations, $\tilde{q}_{out}$ is inherently 0 while $\tilde{q}_{in}$ inherently equals the BS(M) density $\tilde{q}$ (the barrier introduces no additional conditioning).

3.1. Valuation of Single Barrier Options & Lookbacks (14)

Illustration of Reflection Principle (Mirror Image Paths)

(driftless, standard Brownian motion)



3.1. Valuation of Single Barrier Options & Lookbacks (15)

- Consider the log-driftless ($\alpha = 0$) case. Then we can exactly cancel out the density at the barrier by subtracting off the density of a Brownian motion starting at $2b$:

$$\begin{aligned}\tilde{q}_{out}(x,\tau) &\sim \frac{\exp\left(-x^2/2\tau\right) - \exp\left(-(x-2b)^2/2\tau\right)}{\sqrt{2\pi\tau}} \\ \Rightarrow \quad \tilde{q}_{out}(x=b,\tau) &= \frac{\exp\left(-b^2/2\tau\right) - \exp\left(-(b-2b)^2/2\tau\right)}{\sqrt{2\pi\tau}} \\ &= 0\end{aligned}$$

- Use Girsanov's theorem to correct for drift when $\alpha \neq 0$:

$$\tilde{q}_{out} = \exp[\alpha(x - \alpha\tau/2)] \left\{ \frac{\exp\left(-x^2/2\tau\right) - \exp\left(-(x-2b)^2/2\tau\right)}{\sqrt{2\pi\tau}} \right\}$$

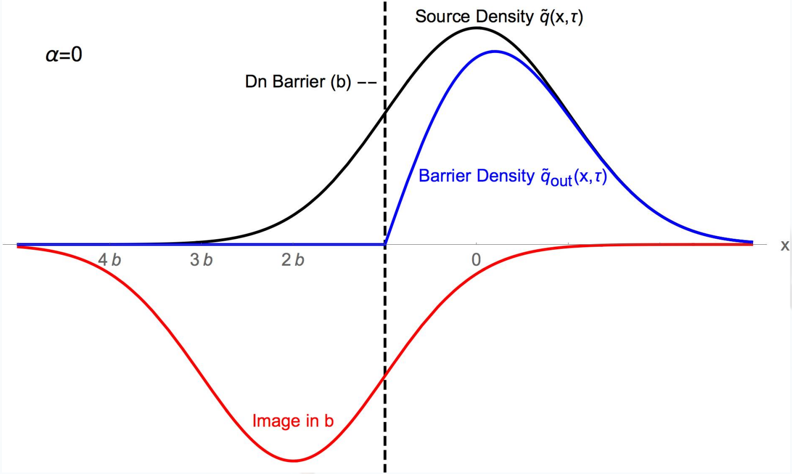
- Alternative form (fractional image):

$$\tilde{q}_{out} = \frac{\exp\left(-(x-\alpha\tau)^2/2\tau\right) - e^{2\alpha b}\exp\left(-(x-2b-\alpha\tau)^2/2\tau\right)}{\sqrt{2\pi\tau}}$$

- This holds for x within barrier (whether up or down); $\tilde{q}_{out} = 0$ outside.
- We need to be a little bit careful that the Girsanov factor works correctly with the image density... but it does as long as vol and drift are constant.

3.1. Valuation of Single Barrier Options & Lookbacks (16)

Illustration of Image Density (Reflection Principle)



3.1. Valuation of Single Barrier Options & Lookbacks (17)

Applications of the Solution

- Simple payoffs at maturity: Integrate over continuation region and discount.
- Knock-in options: use in-out parity: $\tilde{q}_{in}(x,\tau) = \frac{\exp[-(x-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} - \tilde{q}_{out}(x,\tau)$
- Use $\eta = 1$ for down barrier, $\eta = -1$ for up barrier to help keep track of whether x is inside or outside barrier. Then:

$$
\begin{aligned}
\tilde{q}_{in}(x,\tau) &= e^{\alpha x-\alpha^2\tau/2}\frac{\exp[-x^2/2\tau]}{\sqrt{2\pi\tau}} &&= \frac{\exp[-(x-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} && (\eta x < \eta b)\\
&= e^{\alpha x-\alpha^2\tau/2}\frac{\exp[-(x-2b)^2/2\tau]}{\sqrt{2\pi\tau}} &&= e^{2\alpha b}\frac{\exp[-(x-2b-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} && (\eta x > \eta b)\\
& &&= e^{2b(x-b)/\tau}\frac{\exp[-(x-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} && (\eta x > \eta b)
\end{aligned}
$$

- Valuation of rebates (binary barrier options):
  - Pay-at-maturity and no-touch options:
    Integrate $\tilde{q}_{out}$ (over $\eta x > \eta b$, see below) to find the probability of survival, then discount.
  - Pay-at-hit / one-touch options:
    Differentiate the survival probability by τ to find the first passage time density (flux), then re-integrate multiplied by discount factor.

3.1. Valuation of Single Barrier Options & Lookbacks (18)

Applications of the Solution: Lookbacks

- Options on extrema: for $\eta x \geq \eta b$, $\tilde{q}_{out}$ and $\tilde{q}_{in}$ have interpretations:

$$\tilde{q}_{out} = \tilde{q}(x_\tau = x,\ \eta x_\theta > \eta b\,\forall\theta\colon 0 \leq \theta \leq \tau) = \tilde{q}\left(x_\tau = x,\ \min_{0\leq\theta\leq\tau} \eta x_\theta > \eta b\right)$$

$$\tilde{q}_{in} = \tilde{q}(x_\tau = x,\ \exists\theta\colon 0 \leq \theta \leq \tau\colon \eta x_\theta \leq \eta b) = \tilde{q}\left(x_\tau = x,\ \min_{0\leq\theta\leq\tau} \eta x_\theta \leq \eta b\right)$$

$$\text{Differentiating:}\ \frac{\partial \tilde{q}_{in}}{\partial(\eta b)} = \tilde{q}\left(x_\tau = x,\ \min_{0\leq\theta\leq\tau} \eta x_\theta = \eta b\right) \text{(i.e. joint density of } \{x_\tau, {\min \atop \max}\})$$

$$= \eta\,\partial_b \left\{ e^{2\alpha b} \frac{\exp[-(x-2b-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} \right\}$$

$$= \frac{2\eta(x-2b)}{\tau} e^{2b(x-b)/\tau} \frac{\exp[-(x-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} = \frac{2\eta(x-2b)}{\tau}\tilde{q}_{in}$$

Note interpretation of joint density using $\tilde{q}_{in} = \min[e^{2b(x-b)/\tau}, 1] \dfrac{\exp[-(x-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}}$

$$\tilde{q}_{in} = \tilde{q}\left(x_\tau = x,\ \min_{0\leq\theta\leq\tau} \eta x_\theta \leq \eta b\right) = \tilde{q}\left(\min_{0\leq\theta\leq\tau} \eta x_\theta \leq \eta b \,\middle|\, x_\tau = x\right) \cdot \tilde{q}(x_\tau = x)$$

$$= \min[e^{2b(x-b)/\tau}, 1] \cdot \frac{e^{-(x-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$$

3.1. Valuation of Single Barrier Options & Lookbacks (19)

Applications of the Solution: Lookbacks (continued)

- Integrate $\tilde{q}_{in}$ (or $\tilde{q}_{out}$) over x to obtain CDF of minima ($\eta = 1$) or maxima ($\eta = -1$):

$$\begin{aligned}\tilde{q}\left(\min_{0\le\theta\le\tau} \eta x_\theta \le \eta b\right) &= \int_{-\infty}^{\infty} dx\, \tilde{q}\left(x_\tau = x, \min_{0\le\theta\le\tau} \eta x_\theta \le \eta b\right)\\ &= \int_{\eta x \le \eta b} dx \frac{e^{-(x-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}} + \int_{\eta x > \eta b} dx\, e^{2\alpha b} \frac{e^{-(x-2b-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}\\ &= \mathcal{N}\left(\eta \frac{b-\alpha\tau}{\sqrt{\tau}}\right) + e^{2\alpha b}\,\mathcal{N}\left(\eta\frac{b+\alpha\tau}{\sqrt{\tau}}\right) \quad \forall \eta b \le 0\end{aligned}$$

  - NB: For the CDF in the usual (b increasing) direction for $\eta = -1$, use 1 minus this result.

- Differentiate to obtain density of minima/maxima:

$$\begin{aligned}\tilde{q}\left(\min_{0\le\theta\le\tau} \eta x_\theta = \eta b\right) &= \frac{-\partial}{\partial(\eta b)}\tilde{q}\left(\min_{0\le\theta\le\tau} \eta x_\theta \ge \eta b\right) = \frac{\partial}{\partial(\eta b)}\tilde{q}\left(\min_{0\le\theta\le\tau} \eta x_\theta \le \eta b\right)\\ &= 2\left[\frac{\exp[-(b-\alpha\tau)^2/2\tau]}{\sqrt{2\pi\tau}} + \eta\,\alpha e^{2\alpha b}\,\mathcal{N}\left(\eta\frac{b+\alpha\tau}{\sqrt{\tau}}\right)\right] \quad \forall \eta b \le 0\end{aligned}$$

  - Note this proves (for $b < 0$): $\mathcal{N}\left(\frac{b-|\alpha|\tau}{\sqrt{\tau}}\right) \le \frac{\exp[-(b-|\alpha|\tau)^2/2\tau]}{\sqrt{2\pi\tau}|\alpha|}$

3.1. Valuation of Single Barrier Options & Lookbacks (20)

Summary Table of $\tilde{q}(\{x_\tau, \min_{0\le\theta\le\tau} \eta x_\theta\})$

Probability $\tilde{q}$	$\eta x < \eta b$	$\eta x > \eta b$
$\tilde{q}(x_\tau = x)$	$\frac{e^{-(x-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$	$\frac{e^{-(x-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}_{out} = \tilde{q}(x_\tau = x, \min_{0\le\theta\le\tau} \eta x_\theta \ge \eta b)$	0	$\frac{e^{-(x-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}} - e^{2\alpha b}\,\frac{e^{-(x-2b-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}(\min_{0\le\theta\le\tau} \eta x_\theta \ge \eta b \mid x_\tau = x)$	0	$1 - e^{2b(x-b)/\tau}$
$\tilde{q}_{in} = \tilde{q}(x_\tau = x, \min_{0\le\theta\le\tau} \eta x_\theta < \eta b)$	$\frac{e^{-(x-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$	$e^{2\alpha b}\,\frac{e^{-(x-2b-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}(\min_{0\le\theta\le\tau} \eta x_\theta < \eta b \mid x_\tau = x)$	1	$e^{2b(x-b)/\tau}$
$\tilde{q}(x_\tau = x, \min_{0\le\theta\le\tau} \eta x_\theta = \eta b)$	0	$\frac{2\eta(x-2b)}{\tau}\, e^{2\alpha b}\, \frac{e^{-(x-2b-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}(\min_{0\le\theta\le\tau} \eta x_\theta = \eta b \mid x_\tau = x)$	0	$\frac{2\eta(x-2b)}{\tau}\, e^{2b(x-b)/\tau}$
$\tilde{q}(\min_{0\le\theta\le\tau} \eta x_\theta \le \eta b)$	$0\ (0 < \eta b)$	$\mathcal{N}\left(\eta\,\frac{b-\alpha\tau}{\sqrt{\tau}}\right) + e^{2\alpha b}\,\mathcal{N}\left(\eta\,\frac{b+\alpha\tau}{\sqrt{\tau}}\right) \quad (0 \ge \eta b)$
$\tilde{q}(\min_{0\le\theta\le\tau} \eta x_\theta = \eta b)$	$0\ (0 < \eta b)$	$2\left[\frac{e^{-(b-\alpha\tau)^2/2\tau}}{\sqrt{2\pi\tau}} + \eta\alpha e^{2\alpha b}\,\mathcal{N}\left(\eta\,\frac{b+\alpha\tau}{\sqrt{\tau}}\right)\right]$ (" ")

3.1. Valuation of Single Barrier Options & Lookbacks (21)

Summary Table of $\tilde{q}(\{x_\tau, \min_{0\le\theta\le\tau} \eta x_\theta\})$: Limiting Cases for $\alpha = 0$

Probability $\tilde{q}$	$\eta x < \eta b$	$\eta x > \eta b$
$\tilde{q}(x_\tau = x)$	$\frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}$	$\frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}_{out} = \tilde{q}(x_\tau = x, \min\limits_{0\le\theta\le\tau} \eta x_\theta \ge \eta b)$	0	$\frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}} - \frac{e^{-(x-2b)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}(\min\limits_{0\le\theta\le\tau} \eta x_\theta \ge \eta b \,\vert\, x_\tau = x)$	0	$1 - e^{2b(x-b)/\tau}$
$\tilde{q}_{in} = \tilde{q}(x_\tau = x, \min\limits_{0\le\theta\le\tau} \eta x_\theta < \eta b)$	$\frac{e^{-x^2/2\tau}}{\sqrt{2\pi\tau}}$	$\frac{e^{-(x-2b)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}(\min\limits_{0\le\theta\le\tau} \eta x_\theta < \eta b \,\vert\, x_\tau = x)$	1	$e^{2b(x-b)/\tau}$
$\tilde{q}(x_\tau = x, \min\limits_{0\le\theta\le\tau} \eta x_\theta = \eta b)$	0	$\frac{2\eta(x-2b)}{\tau}\frac{e^{-(x-2b)^2/2\tau}}{\sqrt{2\pi\tau}}$
$\tilde{q}(\min\limits_{0\le\theta\le\tau} \eta x_\theta = \eta b \,\vert\, x_\tau = x)$	0	$\frac{2\eta(x-2b)}{\tau} e^{2b(x-b)/\tau}$
$\tilde{q}(\min\limits_{0\le\theta\le\tau} \eta x_\theta \le \eta b)$	$0\ (0 < \eta b)$	$2\,\mathcal{N}\left(\eta \frac{b}{\sqrt{\tau}}\right)\ \ (0 \ge \eta b)$
$\tilde{q}(\min\limits_{0\le\theta\le\tau} \eta x_\theta = \eta b)$	$0\ (0 < \eta b)$	$\sqrt{\frac{2}{\pi\tau}}\, e^{-b^2/2\tau}\ \ (0 \ge \eta b)$

3.1. Valuation of Single Barrier Options & Lookbacks (22)

Term Structures

- Valuing barrier & lookback options when there are term structures
  - In general, the method of images fails because a negative image source of constant strength no longer guarantees that the probability density will be zero at the barrier.
  - Resort must be had to Monte Carlo or grid methods (i.e., binomial, trinomial, finite difference)
  - Classic difficulties: poor convergence of simulation and binomial/trinomial methods

3.2. Valuing Double Barrier Options

- Consider an option that pays out some "general" function $G_T(S_T)$ iff $S_{\theta:\, t \le \theta \le T}$ never crosses either an up boundary U or a down boundary D (double barrier option).
- The same PDE for the probability density $q_t(S_t; S_T) = q_{out}(S_t, t; S_T, T; U, D)$ applies:
$$\frac{\partial q_{out}}{\partial t} + (r - y) S_t \frac{\partial q_{out}}{\partial S_t} + \frac{\sigma^2 {S_t}^2}{2} \frac{\partial^2 q_{out}}{\partial {S_t}^2} = 0$$
subject to $q_{out}(S_t, t \nearrow T) = \delta(S_t, S_T)$, $q_{out}(S_t = U, t) = 0$, and $q_{out}(S_t = D, t) = 0$.
- Changing variables (and formulation from backward to forward Kolmogorov) as before:
$$\tau = \sigma^2 (T - t), \;\; x = \ln(S_T / S_t), \;\; \tilde{q}_{out} = S_T \, q_{out}$$

Then:
$$\frac{\partial \tilde{q}_{out}}{\partial \tau} = \frac{1}{2} \frac{\partial^2 \tilde{q}_{out}}{\partial x^2} - \alpha \frac{\partial \tilde{q}_{out}}{\partial x}, \quad \text{with } \alpha = \frac{\tilde{\mu}^{\mathbf{Q}}}{\sigma^2} = \frac{r - y}{\sigma^2} - \frac{1}{2}$$

subject to:
$$\tilde{q}_{out}(x, \tau = 0) = \delta(x, 0)$$
$$\tilde{q}_{out}(x = u = \ln(U/S_t), \tau) = 0$$
$$\tilde{q}_{out}(x = b = \ln(D/S_t), \tau) = 0$$

3.2. Valuing Double Barrier Options (2)

Method of Images (Reflection Principle)

- Consider the log-driftless ($\alpha = 0$) case. Suppose as before there's only one (down) barrier at b. Then we can exactly cancel out the density at b by subtracting off the density of a Brownian motion starting at $2b$:

$$\tilde{q}_{out}(x,\tau) \sim \frac{\exp\left(-x^2/2\tau\right) - \exp\left(-(x-2b)^2/2\tau\right)}{\sqrt{2\pi\tau}}$$

$$\Rightarrow \quad \tilde{q}_{out}(x=b,\tau) = \frac{\exp\left(-b^2/2\tau\right) - \exp\left(-(b-2b)^2/2\tau\right)}{\sqrt{2\pi\tau}} = 0$$

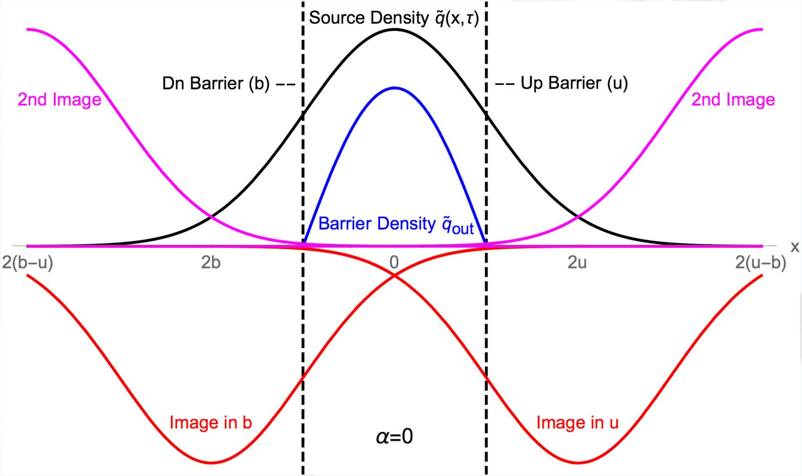
- Similarly, for log-driftless case with a single up barrier at u:

$$\tilde{q}_{out}(x,\tau) \sim \frac{\exp\left(-x^2/2\tau\right) - \exp\left(-(x-2u)^2/2\tau\right)}{\sqrt{2\pi\tau}}$$

$$\Rightarrow \quad \tilde{q}_{out}(x=u,\tau) = \frac{\exp\left(-u^2/2\tau\right) - \exp\left(-(u-2u)^2/2\tau\right)}{\sqrt{2\pi\tau}} = 0$$

3.2. Valuing Double Barrier Options (3)

Illustration of Image Density



3.2. Valuing Double Barrier Options (4)

Method of Images (Reflection Principle) (continued)

- For double barrier, have infinite set of images:

$$\tilde{q}_{out} \simeq \frac{\exp\left(-x^2/2\tau\right)}{\sqrt{2\pi\tau}} - \frac{\sum_{i=1}^{\infty} \exp\left\{-[x-2ib+2(i-1)u)]^2/2\tau\right\} + \exp\left\{-[x-2iu+2(i-1)b)]^2/2\tau\right\}}{\sqrt{2\pi\tau}} + \frac{\sum_{i=1}^{\infty} \exp\left\{-[x-2i(b-u)]^2/2\tau\right\} + \exp\left\{-[x-2i(u-b)]^2/2\tau\right\}}{\sqrt{2\pi\tau}}$$

- Final step: use Girsanov's theorem to correct for drift when $\alpha \neq 0$:

$$\tilde{q}_{out} = \exp[\alpha(x-\alpha\tau/2)] \left\{ \frac{\exp\left(-x^2/2\tau\right)}{\sqrt{2\pi\tau}} - \ldots + \ldots \right\}$$

- Must integrate between barriers and discount to obtain present values...

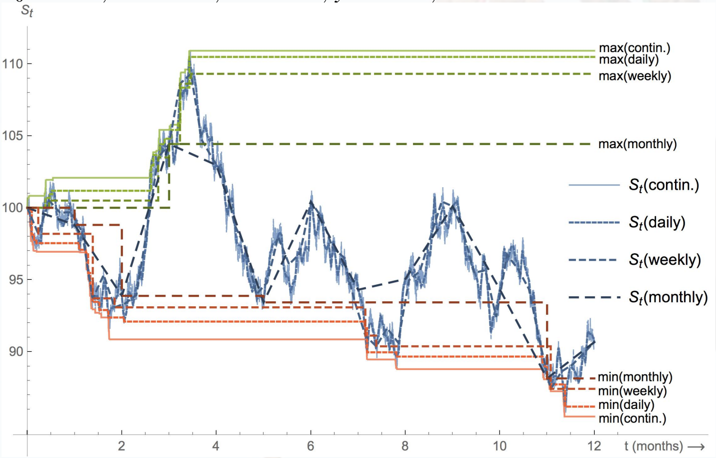
3.3. Discretely-Sampled Barrier Options & Lookbacks

- Assumptions of continuous-sampling solutions are often unrealistic and/or impractical
  - Markets aren’t always open
  - Intra-day prices may not be reliable or verifiable
  - Continuous monitoring and hedging are infeasible
  - Parameters may not be constant
- Most path-dependent option contracts are specified in terms of discrete, periodic samples
- The market has long recognized that discreteness has significant effects on value, particularly for extremum-dependent options
- There is a need for valuation tools that address this problem

3.3. Discretely-Sampled Barrier Options & Lookbacks (2)

Continuous vs. Discrete Sampling Illustration

- $S_0 = 100$, $\sigma = 0.20$, $r = 0.01$, $y = 0.02$, $T = 1.0$:

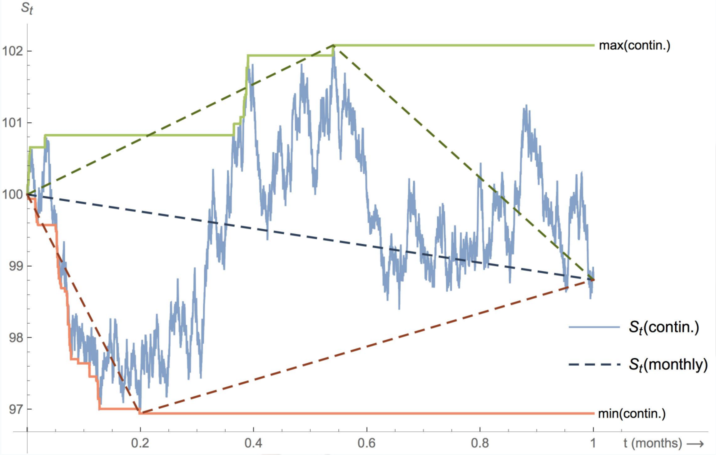


- Differences in extrema can be significant for even daily sampling!

3.3. Discretely-Sampled Barrier Options & Lookbacks (3)

Continuous vs. Discrete Sampling Illustration (continued)

- Consider the max and min processes over a single (monthly) sampling interval:



- The interval discrete vs. continuous sampling error should be related to the interval diffusion "length" $\sigma\sqrt{\Delta t}$

3.3. Discretely-Sampled Barrier Options & Lookbacks (4)

- Monte Carlo and Lattice Calculations
  - Discrete sampling substantially affects values and deltas (Flesaker, 1992; Anderson & Brotherton-Ratcliffe, 1996; Cheuk & Vorst, 1996)
  - Convergence is slow and much tweaking needs to be done to get decent numbers in reasonable time (Broadie, Glasserman, & Kou, 1996)
- Adjustment Techniques
  - Traders have long recognized that moving the barrier out by a factor $\exp(\beta\sigma\sqrt{dt})$ with $\beta \sim O(1)$ gives good agreement with simulation results
  - Broadie, Glasserman, & Kou (1996, 1997) show that $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$. This is quite accurate for frequent samples and barriers some distance from spot.
  - Term structures and fewer sample dates remain a problem.
- Exact Solutions?
  - In terms of m-variate cumulative normal distribution functions
  - Only realistic to evaluate for very small m

3.3. Discretely-Sampled Barrier Options & Lookbacks (5)

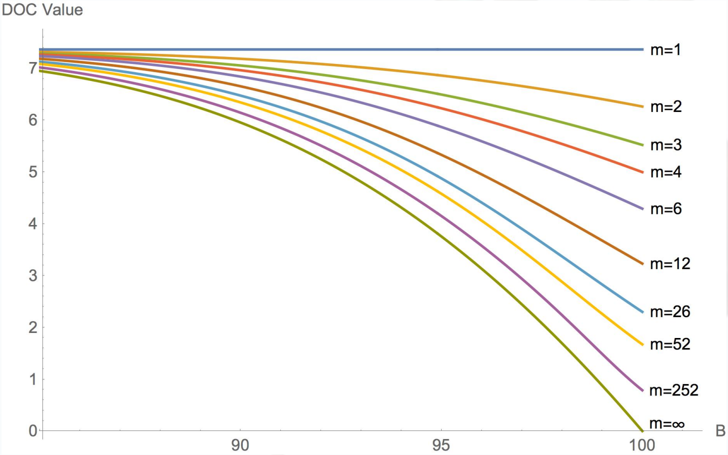
- DOC values vs. m ($S_0 = 100$, $K = 100$, $\sigma = 0.20$, $r = 0.01$, $y = 0.02$, $T = 1.0$):

Barrier	m = 1	m = 2	m = 3	m = 4	m = 6	m = 12	$m \nearrow \infty$
85	7.36429	7.31639	7.28662	7.26633	7.23745	7.18475	6.95077
86	7.36429	7.30014	7.25860	7.23013	7.19060	7.12136	6.82176
87	7.36429	7.27957	7.22280	7.18359	7.13011	7.04018	6.66215
88	7.36429	7.25390	7.17782	7.12484	7.05336	6.93765	6.46739
89	7.36429	7.22230	7.12224	7.05195	6.95766	6.80998	6.23285
90	7.36429	7.18392	7.05463	6.96303	6.84042	6.65328	5.95396
91	7.36429	7.13787	6.97361	6.85638	6.69927	6.46380	5.62633
92	7.36429	7.08331	6.87796	6.73048	6.53230	6.23828	5.24589
93	7.36429	7.01942	6.76662	6.58421	6.33818	5.97442	4.80899
94	7.36429	6.94544	6.63878	6.41684	6.11637	5.67128	4.31255
95	7.36429	6.86072	6.49393	6.22816	5.86723	5.32979	3.75412
96	7.36429	6.76471	6.33190	6.01851	5.59208	4.95304	3.13200
97	7.36429	6.65702	6.15286	5.78881	5.29326	4.54642	2.44525
98	7.36429	6.53741	5.95739	5.54055	4.97402	4.11750	1.69375
99	7.36429	6.40580	5.74641	5.27574	4.63845	3.67565	0.87818
100	7.36429	6.26232	5.52122	4.99687	4.29126	3.23135	0.00000

3.3. Discretely-Sampled Barrier Options & Lookbacks (6)

Exact Results

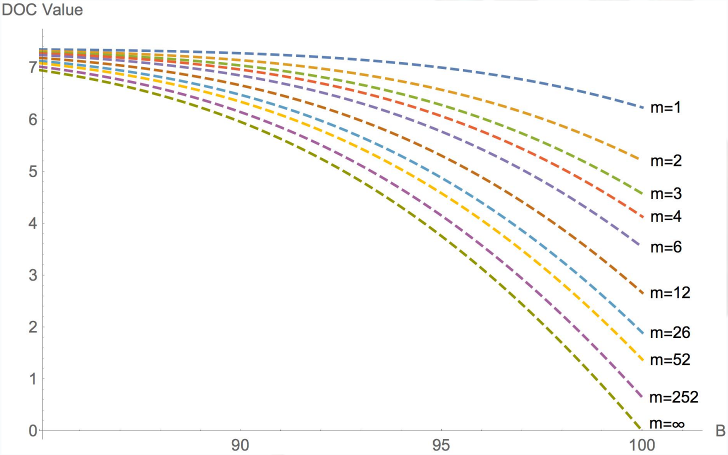
- Down-and-Out values vs. m $(S_0 = K = 100, \sigma = 0.2, r = 0.01, y = 0.02, T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (7)

BGK Approximation

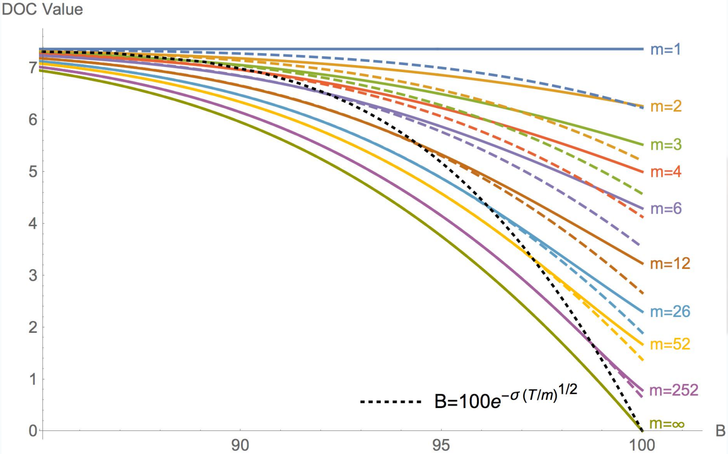
- DOC values vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (8)

Exact Results vs. BGK Approximation

- DOC values vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$

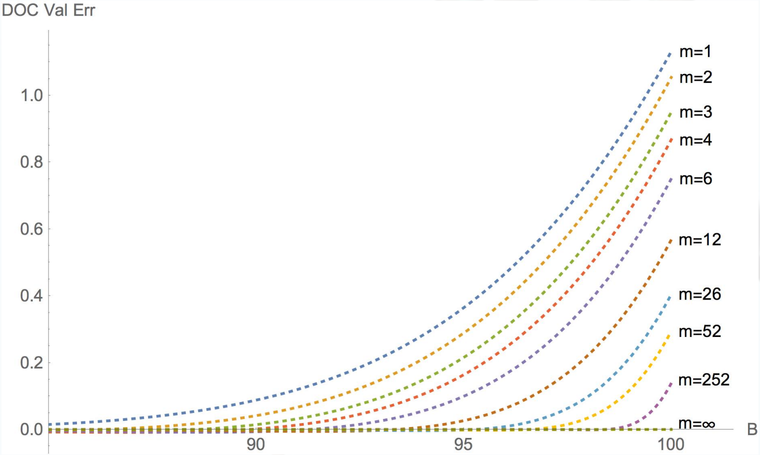


- Can you think of a simple tweak to BGK – based only on m – to improve agreement?

3.3. Discretely-Sampled Barrier Options & Lookbacks (9)

Error: Exact Results - BGK Approximation

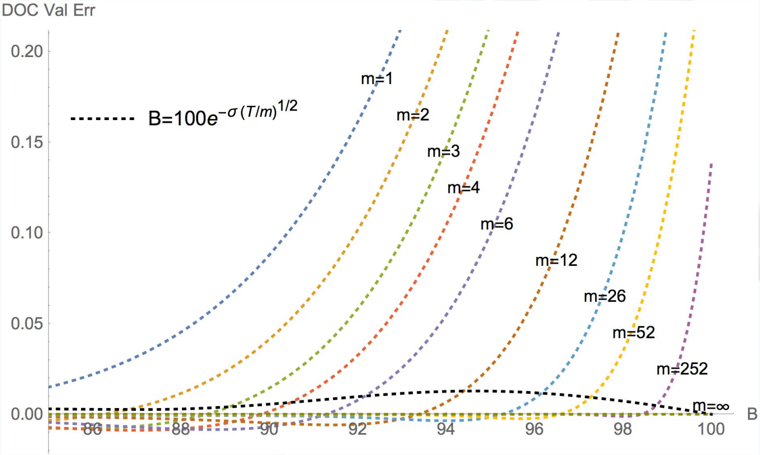
- (-)DOC errors vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (10)

Error: Exact Results - BGK Approximation (Close Up)

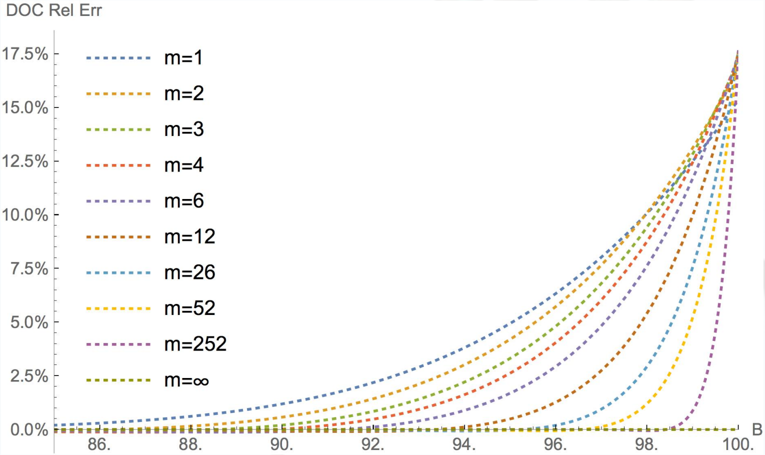
- (-)DOC errors vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (11)

Error: 1 - BGK Approximation/Exact Results

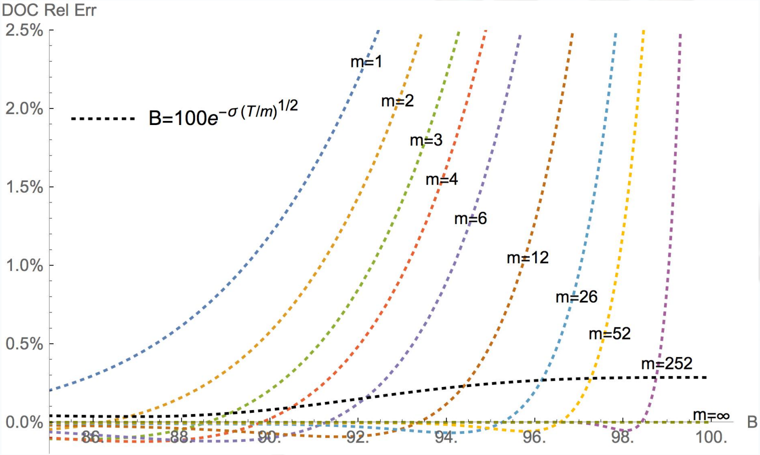
- (-)DOC relative errors vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (12)

Error: 1 - BGK Approximation/Exact Results (Close Up)

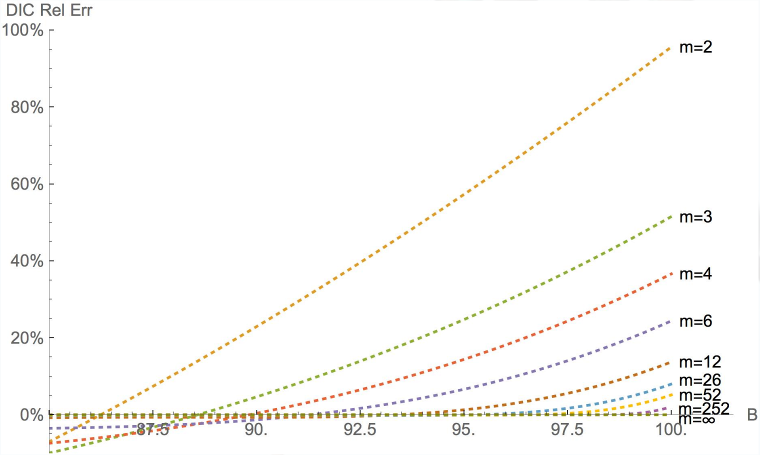
- (-)DOC relative errors vs. m $\left(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1\right)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (13)

Error: Exact Results/BGK Approximation - 1

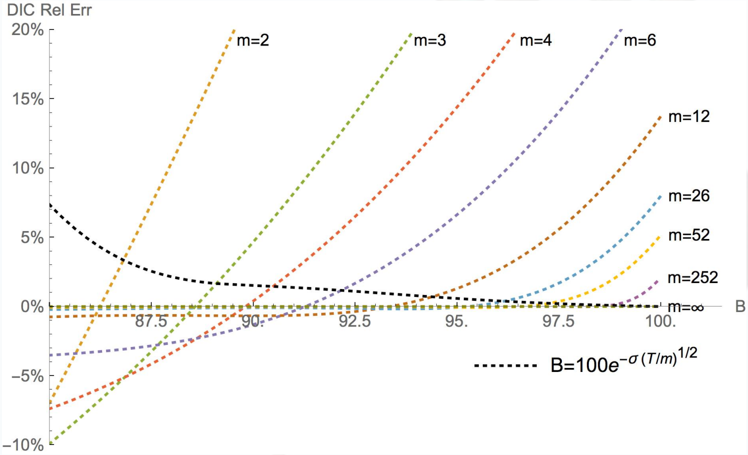
- DIC relative errors vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (14)

Error: Exact Results/BGK Approximation - 1 (Close Up)

- DIC relative errors vs. m $(S_0 = K = 100,\ \sigma = 0.2,\ r = 0.01,\ y = 0.02,\ T = 1)$



3.3. Discretely-Sampled Barrier Options & Lookbacks (15)

Simulation

- How could we value the DOC using Monte Carlo simulation?
  - Bare Bones: Simulate n paths of m time steps (just like Asian option)
  - Track running minimum over each time step, test vs. barrier at end
    - One possibility for efficiency gains: stop each path once barrier has been crossed
- Variance reduction:
  - Antithetic paths
  - Moment matching?
  - Control variates
    - Vanilla European call
    - Continuously sampled barrier call...

3.3. Discretely-Sampled Barrier Options & Lookbacks (16)

Simulation Example

- Put some of these techniques to work:
  - 1-year monthly ($m=12$) discrete DOC, $S = 100$, $K = 100$, $B = 95$
  - Parameters: $\sigma = 0.20$, $r = 0.01$, $y = 0.02$
  - Exact value: 5.32979
  - All simulations using $n = 1{,}000{,}000$ ($\pm$ represents estimator std. errs)
- Simple (Bare-Bones) Monte Carlo results:
  - Value: 5.33763 ± 0.011898
  - Time: 10.581780 seconds
- Antithetic Variate results:
  - Value: 5.33264 ± 0.007514 (variance reduction factor: 2.50682)
  - Time: 19.695893 seconds (simple $\times$ 1.86130)
  - Implied correlation of basic/antithetic path payoffs: -0.202177

3.3. Discretely-Sampled Barrier Options & Lookbacks (17)

- Simple Control Variate 1 (Continuous DOC):
  - Value: 5.32603 ± 0.006352 (variance reduction factor: 3.50846)
    - CV value: 3.76572 ± 0.0106329 (exact: 3.75412)
  - Time: 21.588675 seconds (simple $\times$ 2.04017)
  - Implied correlation of discrete/continuous barrier payoffs: 0.846855
  - Optimized value: 5.32663 ± 0.006328 (variance reduction factor: 3.53561; $\beta_1 = 0.947643$)
- Simple Control Variate 2 (European Call):
  - Value: 5.32539 ± 0.006218 (variance reduction factor: 3.66174)
    - CV value: 7.37653 ± 0.01259 (exact: 7.36429)
  - Time: 12.082632 seconds (simple $\times$ 1.14183)
  - Implied correlation of european/discrete barrier payoffs: 0.872527
  - Optimized value: 5.32754 ± 0.005813 (variance reduction factor: 4.18941; $\beta_2 = 0.824700$)

3.3. Discretely-Sampled Barrier Options & Lookbacks (18)

Multiple/Optimized Control Variates

- For each of the n independent paths, calculate the payoff of one or more options $C^*_{T,k}$.
  - Each of the $C^*_{T,k}$ should be both:
    (1) highly correlated with C_T and (2) have a reliably known expected value $\mathbb{E}[C^*_{T,k}]$.
- Average over paths to obtain an estimator $\langle C_T \rangle_{\underline{\beta},n}$ of $\mathbb{E}[C_T]$ depending on a (column) vector $\underline{\beta}$ of unknown coefficients β_k for the (column) vector $\underline{\mathbf{C}}^*_T$ of $C^*_{T,k}$:

$$\mathbb{E}[C_T] \simeq \langle C_T \rangle_{\underline{\beta},n} \doteq \underline{\beta}^\mathsf{T} \cdot \mathbb{E}[\underline{\mathbf{C}}^*_T] + \frac{1}{n} \sum_{i=1}^{n} (C_{T,i} - \underline{\beta}^\mathsf{T} \cdot \underline{\mathbf{C}}^*_{T,i})$$

- Now, minimize the variance of this estimator:

$$\begin{aligned} \mathrm{var}[\,\langle C_T \rangle_{\underline{\beta},n}\,] &\simeq \frac{1}{n} \mathrm{var}[C_{T,i} - \underline{\beta}^\mathsf{T} \cdot \underline{\mathbf{C}}^*_{T,i}] \\ &\simeq \frac{1}{n} (\mathrm{var}[C_T] - 2\underline{\beta}^\mathsf{T} \cdot \mathrm{cov}[C_T, \underline{\mathbf{C}}^*_T] + \underline{\beta}^\mathsf{T} \cdot \mathrm{cov}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \underline{\beta}) \\ \Rightarrow \underline{\beta}^* &= \mathrm{cov}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \mathrm{cov}[C_T, \underline{\mathbf{C}}^*_T] \end{aligned}$$

- Optimized variance of the estimator:

$$\mathrm{var}[\,\langle C_T \rangle_{\underline{\beta}^*,n}\,] \simeq \frac{1}{n} (\mathrm{var}[C_T] - \mathrm{cov}[C_T, \underline{\mathbf{C}}^*_T]^\mathsf{T} \cdot \mathrm{cov}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \mathrm{cov}[C_T, \underline{\mathbf{C}}^*_T])$$

  - Again, by projecting the variance of C_T onto the $\underline{\mathbf{C}}^*_T$, this approach extracts all but the "idiosyncratic" contribution to the error.

3.3. Discretely-Sampled Barrier Options & Lookbacks (19)

Multiple/Optimized Control Variates (continued)

- Note that if $\sigma_C \doteq \sqrt{\mathrm{var}[C_T]}$, $\sigma_{C^*_T kk} \doteq \sqrt{\mathrm{var}[C^*_{T,k}]}$, and we define the diagonal matrix $\underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}$ with $\{k,k\}$ element $\sigma_{C^*_T kk}$ then:

$$\mathrm{cov}[C_T, \underline{\mathbf{C}}^*_T] \doteq \sigma_C\, \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*} \cdot \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]$$
$$\mathrm{cov}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \doteq \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*} \cdot \boldsymbol{\rho}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}$$

- Re-write the vector of optimal factor loadings $\underline{\boldsymbol{\beta}}^*$ in terms of standard deviations and correlations:

$$\underline{\boldsymbol{\beta}}^* = \mathrm{cov}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \mathrm{cov}[C_T, \underline{\mathbf{C}}^*_T] = \left(\underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1} \cdot \boldsymbol{\rho}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1}\right) \cdot \left(\sigma_C\, \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*} \cdot \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]\right)$$
$$= \sigma_C\, \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1} \cdot \boldsymbol{\rho}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]$$

- We can also re-write the optimized variance of the estimator:

$$\mathrm{var}\left[\langle C_T \rangle_{\underline{\boldsymbol{\beta}}^*,\, n}\right] \simeq \frac{1}{n}\left[\sigma_C^2 - \left(\sigma_C\, \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1} \cdot \boldsymbol{\rho}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]\right)^{\mathsf{T}} \cdot \left(\sigma_C\, \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*} \cdot \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]\right)\right]$$
$$= \frac{\sigma_C^2}{n}\left[1 - \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]^{\mathsf{T}} \cdot \boldsymbol{\rho}^{-1}[\underline{\mathbf{C}}^*_T, \underline{\mathbf{C}}^{*\mathsf{T}}_T] \cdot \boldsymbol{\rho}[C_T, \underline{\mathbf{C}}^*_T]\right]$$

- Here, the variance reduction due to projection onto the $\underline{\mathbf{C}}^*_T$ is expressed in terms of correlations.

3.3. Discretely-Sampled Barrier Options & Lookbacks (20)

Multiple Control Variate Example

- Return to our 1-year monthly-sampled $(m=12)$ ATM DOC, with $B=95$ and $n=10^6$ paths
  BBMC $\langle DOC_{12,t}\rangle_n = 5.33763$, $\text{var}[DOC_{12,t}]_n = 141.5700 \Rightarrow \text{std}[\langle DOC_{12,t}\rangle_n] \approx 0.0118983$
- 2 control variates:
  1. 1-year continuously sampled DOC: exact $DOC_t = 3.75412$
     BBMC $\langle DOC_t\rangle_n = 3.76572$, $\text{var}[DOC_t] = 113.0577 \Rightarrow \text{std}[\langle DOC_t\rangle_n] \approx 0.0106329$
  2. 1-year ATM vanilla call (BSC): exact $BSC_t = 7.36429$
     BBMC $\langle BSC_t\rangle_n = 7.37653$, $\text{var}[BSC_t] = 158.4662 \Rightarrow \text{std}[\langle BSC_t\rangle_n] \approx 0.0125883$

$$\text{cov}[DOC_{12,t}, \underline{\mathbf{C}}_t^*] = \begin{pmatrix} 107.1383 \\ 130.6871 \end{pmatrix}$$

$$\text{cov}[\underline{\mathbf{C}}_t^*, \underline{\mathbf{C}}_t^{*\mathsf{T}}] = \begin{pmatrix} 113.0577 & 99.4604 \\ 99.4604 & 158.4662 \end{pmatrix}$$

$$\underline{\underline{\sigma}}_{\mathbf{C}^*} = \begin{pmatrix} \sqrt{\text{var}[DOC_t]} & 0 \\ 0 & \sqrt{\text{var}[BSC_t]} \end{pmatrix} = \begin{pmatrix} 10.6329 & 0 \\ 0 & 12.5883 \end{pmatrix}$$

$$\sigma_C = \sqrt{\text{var}[DOC_{12,t}]} = 11.8983$$

3.3. Discretely-Sampled Barrier Options & Lookbacks (21)

Multiple Control Variate Example (continued)

- Correlations:

$$\rho[DOC_{12,t}, \underline{\mathbf{C}}_t^*] = \frac{1}{\sigma_C} \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1} \cdot \operatorname{cov}[DOC_{12,t}, \underline{\mathbf{C}}_t^*]$$

$$= \frac{1}{11.8983} \begin{pmatrix} 10.6329 & 0 \\ 0 & 12.5883 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 107.1383 \\ 130.6871 \end{pmatrix} = \begin{pmatrix} 0.846855 \\ 0.872527 \end{pmatrix}$$

$$\rho[\underline{\mathbf{C}}_t^*, \underline{\mathbf{C}}_t^*] = \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1} \cdot \operatorname{cov}[\underline{\mathbf{C}}_t^*, \underline{\mathbf{C}}_t^{*\mathsf{T}}] \cdot \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1}$$

$$= \begin{pmatrix} 10.6329 & 0 \\ 0 & 12.5883 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 113.0577 & 99.4604 \\ 99.4604 & 158.4662 \end{pmatrix} \cdot \begin{pmatrix} 10.6329 & 0 \\ 0 & 12.5883 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0.743074 \\ 0.743074 & 1 \end{pmatrix}$$

$$\rho^{-1}[\underline{\mathbf{C}}_t^*, \underline{\mathbf{C}}_t^*] = \begin{pmatrix} 2.23293 & -1.65923 \\ -1.65923 & 2.23293 \end{pmatrix}$$

3.3. Discretely-Sampled Barrier Options & Lookbacks (22)

Multiple Control Variate Example (continued)

- Optimal factor loadings:

$$\begin{aligned}\underline{\beta}^* &= \sigma_C\, \underline{\underline{\boldsymbol{\sigma}}}_{\mathbf{C}^*}^{-1} \cdot \boldsymbol{\rho}^{-1}[\underline{\mathbf{C}}_t^*, \underline{\mathbf{C}}_t^{*\mathsf{T}}] \cdot \boldsymbol{\rho}[DOC_{12,t}, \underline{\mathbf{C}}_t^*] \\ &= 11.8983 \begin{pmatrix} 10.6329 & 0 \\ 0 & 12.5883 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2.23293 & -1.65923 \\ -1.65923 & 2.23293 \end{pmatrix} \cdot \begin{pmatrix} 0.846855 \\ 0.872527 \end{pmatrix} \\ &= \begin{pmatrix} 1.119014 & 0 \\ 0 & 0.945186 \end{pmatrix} \cdot \begin{pmatrix} 0.443245 \\ 0.543163 \end{pmatrix} = \begin{pmatrix} 0.495997 \\ 0.513390 \end{pmatrix}\end{aligned}$$

- Projection of variance onto control (co-)variates:

$$\begin{aligned}\boldsymbol{\rho}[DOC_{12,t}, \underline{\mathbf{C}}_t^*]^{\mathsf{T}} \cdot \boldsymbol{\rho}^{-1}[\underline{\mathbf{C}}_t^*, \underline{\mathbf{C}}_t^{*\mathsf{T}}] \cdot \boldsymbol{\rho}[DOC_{12,t}, \underline{\mathbf{C}}_t^*] & \\ = (0.846855, 0.872527) \cdot \begin{pmatrix} 2.23293 & -1.65923 \\ -1.65923 & 2.23293 \end{pmatrix} \cdot \begin{pmatrix} 0.846855 \\ 0.872527 \end{pmatrix} & \\ = 0.849289 = 1 - 0.150711 &\end{aligned}$$

3.3. Discretely-Sampled Barrier Options & Lookbacks (23)

Multiple Control Variate Example (continued)

- Residual payoff expectation and calculation of option value estimator:

$$\left\langle DOC_{12,t} - \underline{\beta}^{*\mathsf{T}} \cdot \underline{\mathbf{C}}_t^* \right\rangle_n = 5.33763 - (0.495997, 0.513390) \cdot \begin{pmatrix} 3.76572 \\ 7.37653 \end{pmatrix} = -0.317194$$

$$\begin{aligned} \langle DOC_{12,t} \rangle_{\underline{\beta}^*;\, n} &= \underline{\beta}^{*\mathsf{T}} \cdot \underline{\mathbf{C}}_t^* + \left\langle DOC_{12,t} - \underline{\beta}^{*\mathsf{T}} \cdot \underline{\mathbf{C}}_t^* \right\rangle_n \\ &= (0.495997, 0.513390) \cdot \begin{pmatrix} 3.75412 \\ 7.36429 \end{pmatrix} - 0.317194 = 5.32559 \end{aligned}$$

- Residual variance and estimator standard error:

$$\operatorname{var}[DOC_{12,t} - \underline{\beta}^* \cdot \underline{\mathbf{C}}_t^*] \approx 141.5700 \cdot 0.150711 = 21.3362$$

$$\begin{aligned} \Rightarrow \operatorname{std}[\, \langle DOC_{12,t} \rangle_{\underline{\beta}^*;\, n} \,] = \operatorname{std}[\langle DOC_{12,t} - \underline{\beta}^* \cdot \underline{\mathbf{C}}_t^* \rangle_n] &\approx 0.011898 \cdot \sqrt{0.150711} \\ &= 0.011898(0.388215) = 0.004619 \end{aligned}$$

- Compare to estimator standard error for single (continuous DOC, vanilla call) optimized CVs:

$$\operatorname{std}[\, \langle DOC_{12,t} \rangle_{\beta_1^*,\, n} \,] \approx 0.011898 \cdot \sqrt{1 - 0.846855^2} = 0.011898(0.531824) = 0.006328$$

$$\operatorname{std}[\, \langle DOC_{12,t} \rangle_{\beta_2^*,\, n} \,] \approx 0.011898 \cdot \sqrt{1 - 0.872527^2} = 0.011898(0.488566) = 0.005813$$

3.3. Discretely-Sampled Barrier Options & Lookbacks (24)

- Multiple/Optimized Control Variates:
  - Value: 5.32559 ± 0.004619 (variance reduction factor: 6.63521)
  - Time: 31.151128 seconds (simple $\times$ 2.94385)
  - Implied correlation of control variates: 0.743074
  - Factor loadings: $\underline{\beta}^\mathsf{T} = \{0.495997, 0.513390\}$
- Summary:

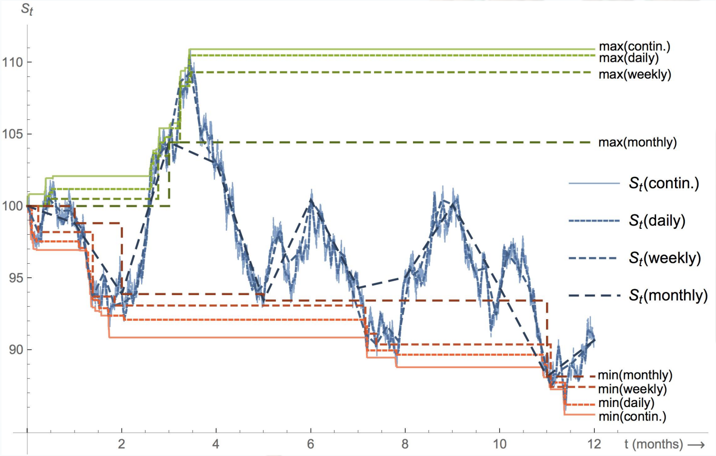
Method	DOC_{12} Value	Std. Error	Variance	Var. Ratio	Timing	Time Ratio	Efficiency
Exact	5.32979						
Bare-Bones	5.33763	0.011898	1.4157×10^{-4}	1.00000	10.581780	1.00000	1.00000
Antithetic	5.33264	0.007514	5.6474×10^{-5}	2.50682	19.695893	1.86130	1.34681
Cont. CV	5.32603	0.006352	4.0351×10^{-5}	3.50846	21.588675	2.04017	1.71969
Opt. Cont. CV	5.32663	0.006328	4.0041×10^{-5}	3.53561	25.214417	2.38281	1.48380
Euro. CV	5.32539	0.006218	3.8662×10^{-5}	3.66174	12.082632	1.14183	3.56964
Opt. Euro. CV	5.32754	0.005813	3.3792×10^{-5}	4.18941	14.167780	1.33888	3.48297
Multiple CV	5.32559	0.004619	2.1336×10^{-5}	6.63521	31.151128	2.94385	2.25393

- Particularly in combination, these techniques can be powerful!

3.3. Discretely-Sampled Barrier Options & Lookbacks (25)

Recap: Continuous vs. Discrete Sampling Illustration

- While BGK is an approximate route from continuous to discrete sampling values:

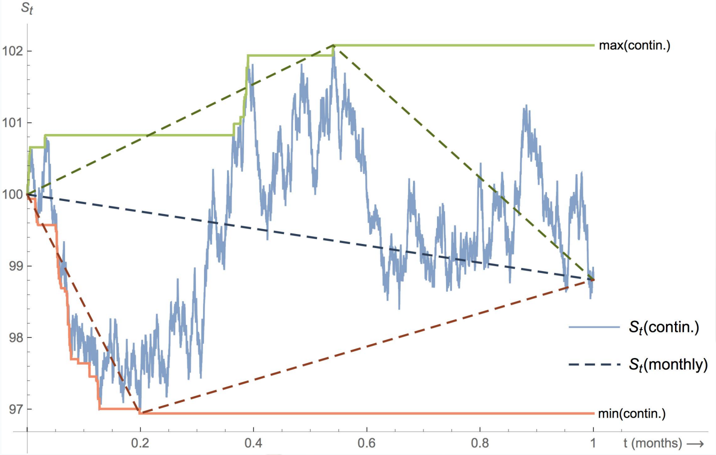


- Slow convergence $(\sim m^{-1/2})$ makes simulation of continuous extrema unrealistic.

3.3. Discretely-Sampled Barrier Options & Lookbacks (26)

Discrete Sampling: Brownian Bridge

- Yet continuous max/min are useful control variate(s) as well of intrinsic interest:



- There's an amusing irony to this...

3.4. Brownian Bridges for Path-Dependent Options

- How could we simulate crossing of a continuous barrier given only discrete samples?
  - Equivalently, how could we simulate continuous minimum discretely?

  "Down-and-in" barrier probability density ($B \leq S_0$ and $B \leq S_t$):

$$\tilde{q}_{in}[x_t = x, \tau : \min_\tau(x_\tau \leq b) \leq t] = \frac{1}{\sqrt{2\pi t\sigma}} \exp\left(\frac{2\tilde{\mu}^{\mathbf{Q}} b}{\sigma^2}\right) \exp\left(-\frac{(x - \tilde{\mu}^{\mathbf{Q}} t - 2b)^2}{2\sigma^2 t}\right)$$
$$= \exp\left(\frac{2b(x-b)}{\sigma^2 t}\right) \tilde{q}[x_t = x]$$

with $x_t = \ln(S_t/S_0)$, $b = \ln(B/S_0) \leq (x_t)^-$, $\tilde{\mu}^{\mathbf{Q}} = r - y - \sigma^2/2$.

  - The exponential factor is the probability of the barrier being hit contingent on initial and final levels (Brownian bridge).
  - A little reflection reminds us this factor is equal to the CDF for minimum at barrier!
- Hence:

$$\tilde{q}[\min_{0\leq\tau\leq t} x_\tau \leq b \mid x_t = x, x_0 = 0] = \tilde{q}_{in}[x_t = x, \tau : \min_\tau(x_\tau \leq b) \leq t] \,/\, \tilde{q}[x_t = x]$$
$$= \exp\left(2b(x-b)/\sigma^2 t\right) \; \forall b \leq (x)^-$$

3.4. Brownian Bridges for Path-Dependent Options (2)

- Incidentally, the same formula holds for the "up-and-in" barrier probability density ($B \geq S_0$ and $B \geq S_t$) and the maximum of the Brownian bridge:

$$\begin{aligned}\tilde{q}[\max_{0\leq\tau\leq t} x_\tau \geq b \mid x_t = x, x_0 = 0] &= \tilde{q}_{in}\left[x_t = x, \tau : \min_\tau(x_\tau \geq b) \leq t\right] / \tilde{q}[x_t] \\ &= \exp\left(2b(x-b)/\sigma^2 t\right) \ \forall b \geq (x)^+\end{aligned}$$

- Starting with: $\tilde{q}[\min_{0\leq\tau\leq t} x_\tau \leq b \mid x_t = x, x_0 = 0] = \exp\left(2b(x-b)/\sigma^2 t\right) \ \forall b \leq (x)^-$,
  a little further reflection tells us that (for minimum):

$$u \doteq \exp\left(2b(x-b)/\sigma^2 t\right) \sim U(0,1] \quad (u = 1 \mapsto b = (x)^-;\ u = 0 \mapsto b = -\infty)$$

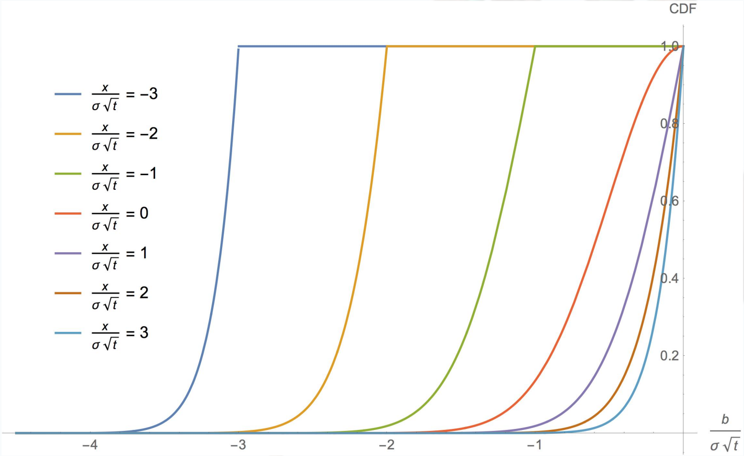
- Letting u be a random draw from $(0, 1]$:

$$\begin{aligned}\exp\left(2b(x-b)/\sigma^2 t\right) = u \ &\Rightarrow\ b^2 - xb + \frac{\sigma^2 t}{2}\ln(u) = 0 \\ &\Rightarrow\ b = \frac{x \pm \sqrt{x^2 - 2\,\sigma^2 t \ln(u)}}{2} \quad \begin{cases} + & \text{for maximum} \\ - & \text{for minimum}\end{cases}\end{aligned}$$

  - This result is exact for a constant-coefficients Ito process
  - For each interval, we need only sample one additional uniform variate to get a continuous extremum

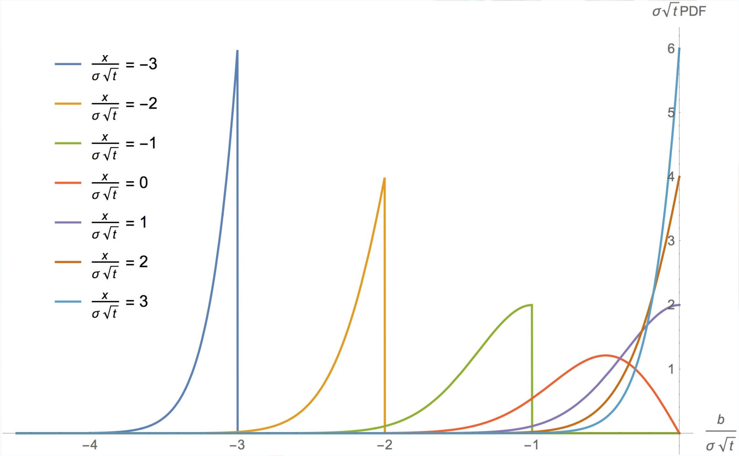
3.4. Brownian Bridges for Path-Dependent Options (3)

- CDF(minimum $\leq b$) = $\exp\left(\frac{2b(x-b)}{\sigma^2 t}\right)$ $\forall b \leq (x)^-$ plotted vs. $\frac{b}{\sigma\sqrt{t}}$



3.4. Brownian Bridges for Path-Dependent Options (4)

- PDF(minimum $= b) = \frac{2(x-2b)}{\sigma^2 t} \exp\left(\frac{2b(x-b)}{\sigma^2 t}\right)$ $\forall b \leq (x)^-$ plotted vs. $\frac{b}{\sigma\sqrt{t}}$



4. Appendices: Time Dependence: Further Topics

- Further Properties of the Brownian Bridge
- Dealing with Term Structures
- Related (Time- but not Path- Dependent) Payoffs

4.1. Further Properties of the Brownian Bridge

- Let’s look a bit more generally at $\tilde{q}[x_\tau : 0 \le \tau \le t \mid x_t = x, x_0 = 0]$.
- One way: use the joint probability density for $\{x_\tau, x_t\}$, then divide by $\tilde{q}[x_t]$ to get the conditional density.
- Easier way: use the fact that if $\{x_1, x_2\}$ are jointly normal, then:

$$x_1 | x_2 \sim N\left[\mu_1 + \frac{\rho_{1,2}\sigma_1}{\sigma_2}(x_2 - \mu_2), (1 - \rho_{1,2}^2)\sigma_1^2\right]$$

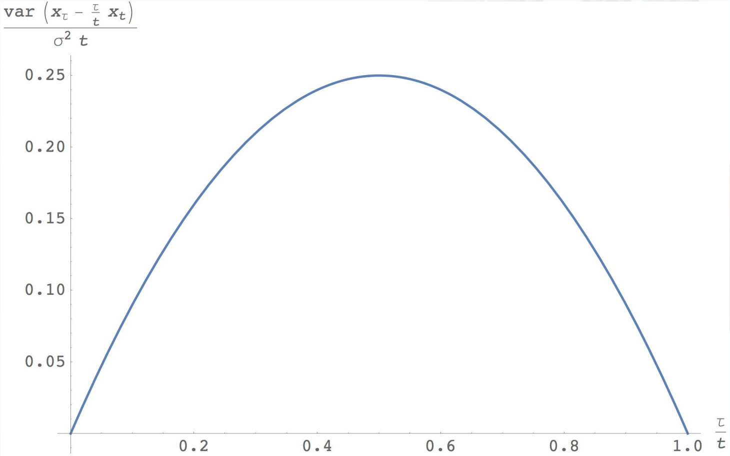
- Now, for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_\tau \\ x_t \end{pmatrix}$: $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \tilde{\mu}^{\mathbf{Q}} \begin{pmatrix} \tau \\ t \end{pmatrix}$, $\begin{pmatrix} \sigma_1^2 \\ \sigma_1^2 \end{pmatrix} = \sigma^2 \begin{pmatrix} \tau \\ t \end{pmatrix}$, and $\rho_{1,2} = \sqrt{\frac{\tau}{t}}$, so:

$$\begin{aligned} x_\tau | x_t &\sim N\left[\tilde{\mu}^{\mathbf{Q}}\tau + \frac{\sqrt{\frac{\tau}{t}}\sigma\sqrt{\tau}}{\sigma\sqrt{t}}(x_t - \tilde{\mu}^{\mathbf{Q}}t),\ \left(1 - \frac{\tau}{t}\right)\sigma^2\tau\right] \\ &= N\left[\left(\frac{\tau}{t}\right)x_t,\ \left(1 - \frac{\tau}{t}\right)\left(\frac{\tau}{t}\right)\sigma^2 t\right] \end{aligned}$$

- Drift of the process is replaced by linear accretion toward x_t;
  variance is a quadratic function of τ reaching a maximum value of $\frac{\sigma^2 t}{4}$ at $\tau = \frac{t}{2}$.
- Convention is to define the Brownian bridge as $x_\tau - \left(\frac{\tau}{t}\right)x_t$ so that its mean is $0\,\forall\tau$.
  - But don’t forget the drift toward x_t when using the conditional properties of x_τ...

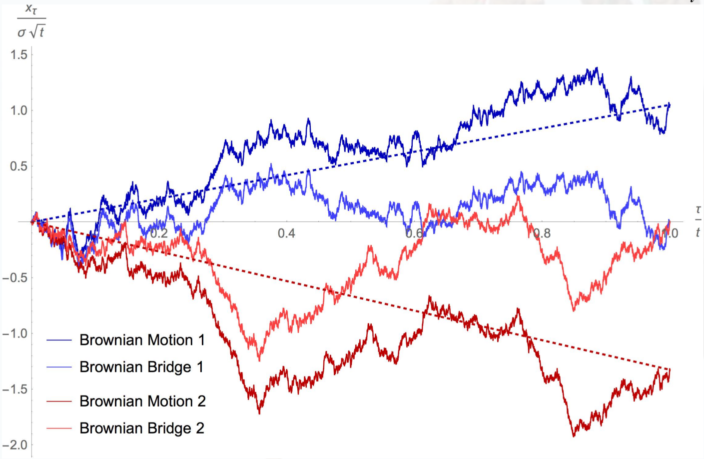
4.1. Further Properties of the Brownian Bridge (2)

- Brownian bridge $(x_\tau - \frac{\tau}{t} x_t)$ variance vs. $\frac{\tau}{t}$:



4.1. Further Properties of the Brownian Bridge (3)

- Illustration of Brownian motion (x_τ) and Brownian bridge $(x_\tau - \frac{\tau}{t} x_t)$ paths vs. $\frac{\tau}{t}$:



4.2. Dealing with Term Structures

- To use constant coefficients continuous results we need to transform problem to a similar one with flat term structures of volatilities and asset drift rates
  - Alternative 1: hold sample times constant and flatten all term structures
  - Alternative 2: hold sample times constant and reallocate drift proportional to variance (equivalent to flattening term structures, moving sample times in proportion to variance)
  - Alternative 3: hold sample times constant and reallocate variance proportional to drift (equivalent to flattening term structures, moving sample times in proportion to drift)
- Want to maximize correlation with payoff under real discrete process: choice may depend on the problem!
- Simulating "parallel universe" is the major contribution to extra computation time
- Brownian bridge is very general, very powerful method for turning discrete processes into continuous ones
  - Example: Lognormal approximation to (continuous) arithmetic average may not be very good, but discrete sum of lognormal approximations over smaller continuous sub-intervals may be much better!

4.2. Dealing with Term Structures (2)

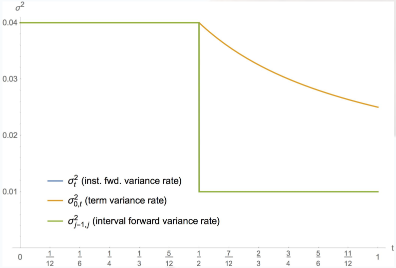
Illustration: Flattening Term Structures

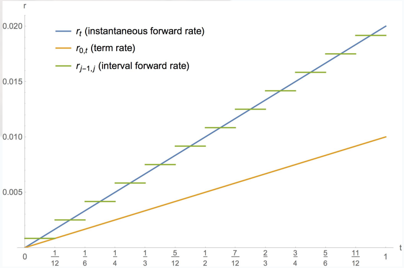
- Suppose we want to value a 1-year maximum-dependent option (e.g. a lookback) with discrete, monthly samples i.e. $t_j = \frac{j}{12}$.
  - We are faced with the following term structures ($y = 0$ for simplicity):

$$\sigma_t = \begin{cases} \sigma_1 = 0.2 & t \le \frac{1}{2} \\ \sigma_2 = 0.1 & t > \frac{1}{2} \end{cases} \quad \Rightarrow \quad \sigma_{0,t} = \begin{cases} \sigma_1 & t \le \frac{1}{2} \\ \sqrt{\sigma_1^2 \frac{1}{2} + \sigma_2^2 (t - \frac{1}{2})} & t > \frac{1}{2} \end{cases}$$

$$r_t = \begin{cases} 0.02\, t & \forall t \end{cases} \quad \Rightarrow \quad r_{0,t} = \begin{cases} 0.01\, t & \forall t \end{cases}$$

$$y_t = \begin{cases} 0 & \forall t \end{cases} \quad \Rightarrow \quad y_{0,t} = \begin{cases} 0 & \forall t \end{cases}$$

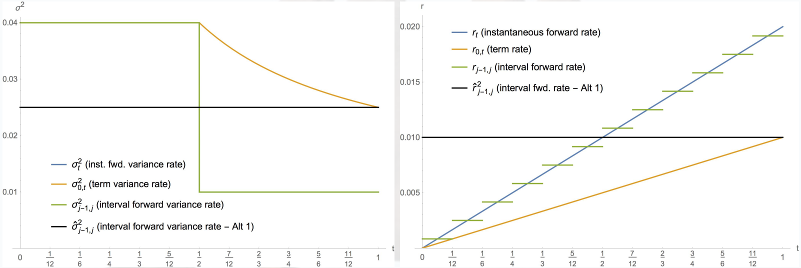




4.2. Dealing with Term Structures (3)

Illustration: Flattening Term Structures

- Problem is that exact solutions for continuous lookbacks assume constant σ, r, y.
- To use exact results as control variate, need to transform to a problem with constant parameters and simulate that alongside our real process.
  - Alternative 1: hold sample times constant and flatten all term structures

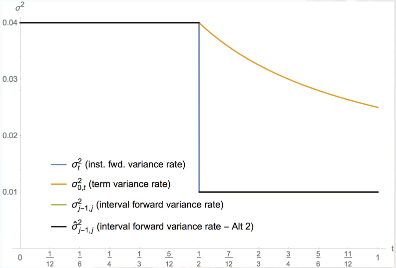


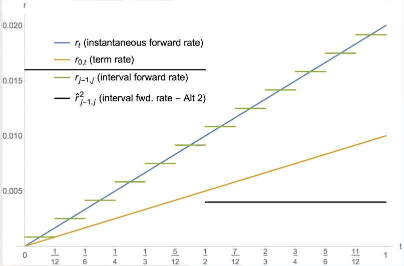
  - Simulate parallel control variate process with $\hat{\sigma}_{j-1,j} = \sqrt{0.025}, \hat{r}_{j-1,j} = 0.01\ \forall j$
  - Probably won't be very effective for payoffs that are sensitive to timing of volatility and drift.

4.2. Dealing with Term Structures (4)

Illustration: Flattening Term Structures (continued)

- Control variate alternatives (continued)
  - Alternative 2: hold sample times constant and reallocate drift proportional to variance



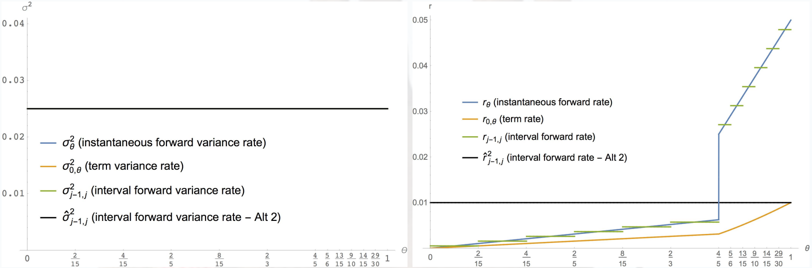


  - Simulate parallel control variate process with:
    $\hat{\sigma}_{j-1,j} = 0.04, \hat{r}_{j-1,j} = 0.016\, \forall j \leq 6,$
    $\hat{\sigma}_{j-1,j} = 0.01, \hat{r}_{j-1,j} = 0.004\, \forall j > 6$

4.2. Dealing with Term Structures (5)

Illustration: Flattening Term Structures (continued)

- Control variate alternatives (continued)
  - Alternative 2 (continued)
    - How/why does this work? Consider time change: $\theta \doteq \int_0^t d\tau \sigma_\tau^2 / \sigma_{0,1}^2, \theta \in [0, 1]$. In θ co-ordinates:



    - Equivalent to flattening term structures, moving sample times in proportion to variance
    - Likely to be effective for payoffs more sensitive to timing of volatility than drift.
  - Alternative 3: hold sample times constant and reallocate variance proportional to drift (equivalent to flattening term structures, moving sample times in proportion to drift)
    - Left as an exercise ...

4.3. Related (Time- but not Path- Dependent) Payoffs

- American (Bermuda) Options
  - Discrete sampling effects also long-known (Geske, 1979).
    "Exact" solutions available, but require self-consistent solution of stopping boundary (Sheikh, 1992) as for compound options.
  - Adjustment techniques (Ait-Sahlia, 1995) inspired stream of research on discrete barriers and lookbacks.
- Shout/Strike reset options
  - All structures in which optionality is added through purchaser's ability to reset strike
  - E.g., call with reset down or reset up with return lock-in
  - Often mappable into a similar American option problem: early exercise boundary
- Options on a (hypothetical) traded account
  - "Passport" or "Perfect Trader" options
  - Underlying assets include: equity indices, currencies, funds-of-(hedge-)funds
  - Variations include: gearing (leveraged or short), discrete rebalancing, transactions costs
  - Often mappable into a lookback option

4.3. Related (Time- but not Path- Dependent) Payoffs (2)

- Options with Intrinsically Discrete Path Dependence
  - Compound and Chooser Options
    1. Call on Call: $COC_t(S_t) = (C_t(S_t, K_T) - K_t)^+$
    2. Call on Put: $COP_t(S_t) = (P_t(S_t, K_T) - K_t)^+$
    3. Put on Call: $POC_t(S_t) = (K_t - C_t(S_t, K_T))^+$
    4. Call on Put: $POP_t(S_t) = (K_t - P_t(S_t, K_T))^+$
    5. Chooser: $CorP_t(S_t) = \max(C_t(S_t, KC_T) - KC_t, P_t(S_t, KP_T) - KP_t, 0)$
    - All can be valued exactly using $\mathcal{N}_2(\bullet, \bullet; \bullet)$ and self-consistent solution for exercise points. E.g.:

$$COC_0(S_0) = S_0 e^{-yT} N_2(x_+, y_+; \rho) - K_T e^{-rT} N_2(x_-, y_-; \rho) - K_t e^{-rt} N(x_-)$$

$$\text{with: } x_\pm = \frac{\ln(S_0/S_t^*) + (r-y)t}{\sigma\sqrt{t}} \pm \frac{\sigma\sqrt{t}}{2},\ y_\pm = \frac{\ln(S_0/K_T) + (r-y)T}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}, \text{ and } \rho = \sqrt{\frac{t}{T}}$$

$$\text{for: } S_t^* \ :\ S_t^* e^{-y(T-t)} N(z_+) - K_T e^{-r(T-t)} N(z_-) = K_t,\ z_\pm = \frac{\ln(S_t^*/K_T) + (r-y)(T-t)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2}$$

  - Installment Options
    - Installment Call: $INC_j(S_{t_j}; t_j) = (INC_{j+1}(S_{t_j}; t_j) - K_j)^+$
    - Installment Put: $INP_j(S_{t_j}; t_j) = (INP_{j+1}(S_{t_j}; t_j) - K_j)^+$
    - Natural extensions of compound calls to m mandatory call dates.
    - All can be valued exactly using $\mathcal{N}_m(\underline{\bullet}; \underline{\bullet})$ and self-consistent solution for exercise points.

4.3. Related (Time- but not Path- Dependent) Payoffs (3)

- Puttable Cliquets
  - Consider the payoff of a series of forward starting options:

$$CLCFU_T = \sum_{j=1}^{m} (S_j - k\, S_{j-1})^+$$

$$CLCFN_T = \sum_{j=1}^{m} (S_j / S_{j-1} - k)^+$$

  - Final puttability:

$$PCLCFU_T = (CLCFU_T - K)^+$$

$$PCLCFN_T = (CLCFN_T - K)^+$$

  - Although optionlet returns are independent in Black-Scholes framework, valuation is analogous to arithmetic average option problem.
  - Puttability can be made “Bermudian,” i.e., K_j can be introduced for each reset date.