Week 2: Stochastic Integrals

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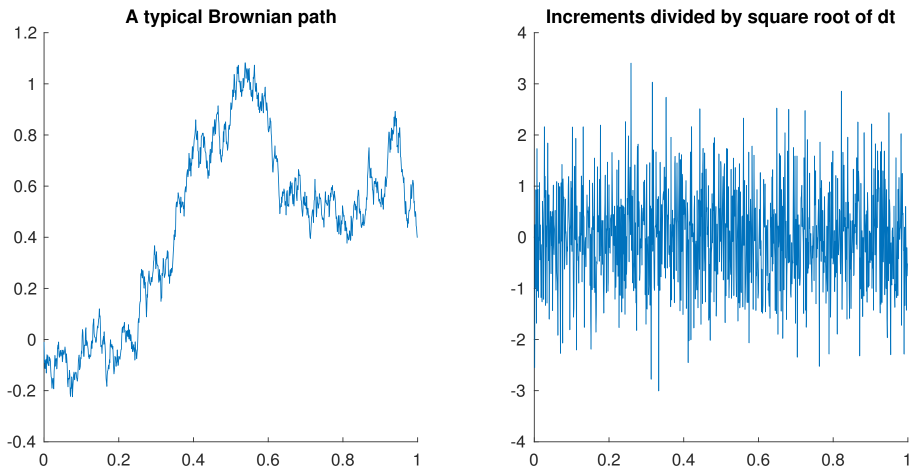
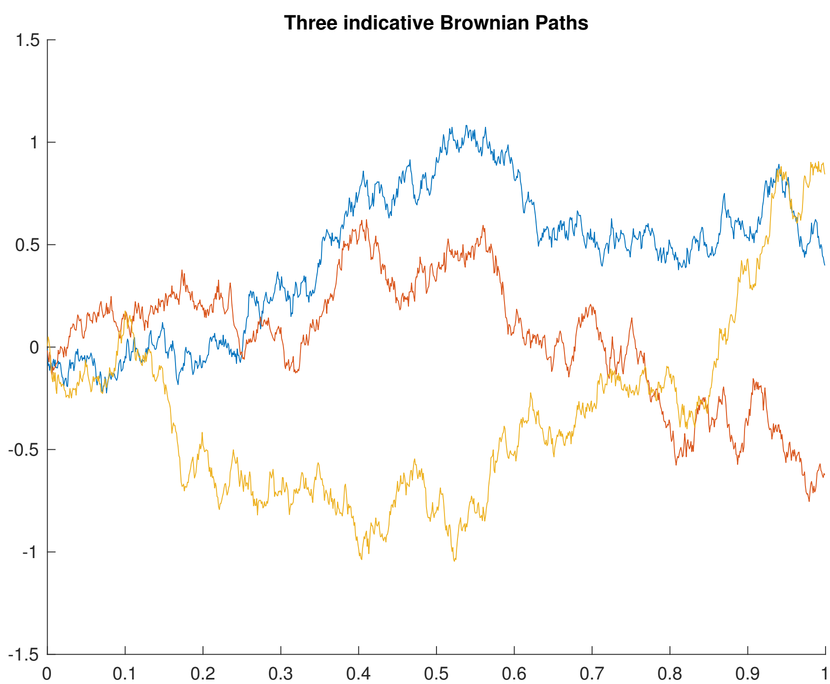
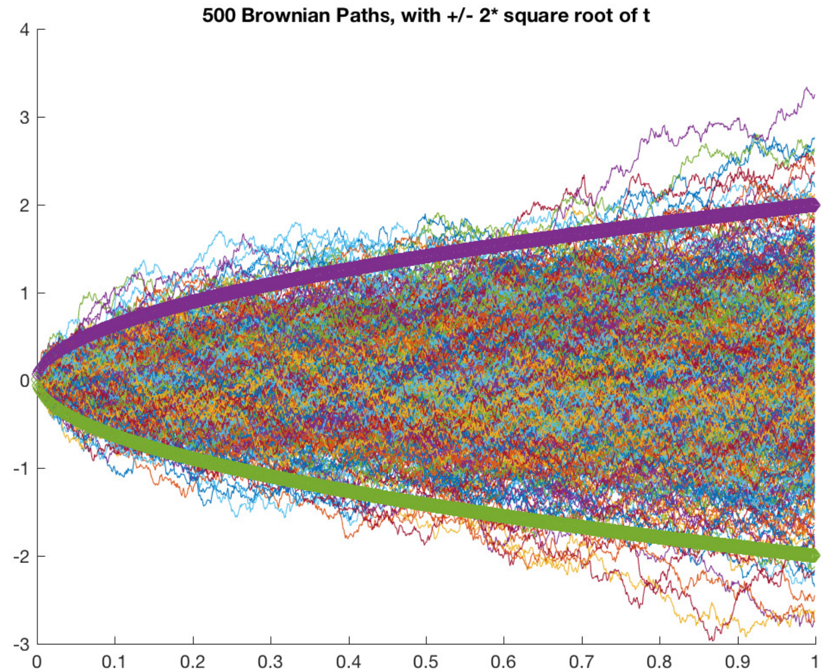
I. Diffusion and Wiener process

- In this class, we want to generalize the simple binomial dynamics of the previous class.
- Towards that end we introduce the notion of a stochastic process (a function of time that is drawn randomly), which we will refer to as a Wiener process (or Brownian Motion)
- A stochastic process W is called a **Wiener process** (or equivalently Brownian Motion) if
  1. $W(0) = 0$
  2. W has independent increments, i.e., if $r < s \leq t < u$, then $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables
  3. For $s < t$ the stochastic process $W(t) - W(s)$ has the Gaussian distribution $N\left[0, \sqrt{t-s}\right]$.
  4. W has continuous trajectories

Illustration of a Wiener process on the time interval $[0, 1]$







This figure plots 1000 Brownian paths, together with the functions $2 \times \sqrt(t)$ and $-2 \times \sqrt(t)$

- In many applications we are interested in the dynamics of processes of the form

$$X(t + \Delta t) - X(t) = \mu(t, X(t))\,\Delta t + \sigma(t, X(t))\,\Delta W(t) \qquad (1)$$

where

$$\Delta W(t) = W(t + \Delta t) - W(t)$$

and Δt becomes small.
  - The "drift" term $\mu(t, X(t))$ captures the expected change of $X(t + \Delta t) - X(t)$ over the next interval Δt.
  - The "diffusion" term $\sigma(t, X(t))\,\Delta W(t)$ captures the impact of noise.

- In what sense is there a "solution" to (1)? It looks like a differential equation. So, it is tempting (but terribly wrong, as it turns out) to write it as

$$\dot{X}(t) = \mu(t, X(t)) + \sigma(t, X(t))\,u(t),$$

where

$$u(t) = \frac{dW_t}{dt}$$

- The next figure shows what is the problem with this idea, namely that the Brownian paths are never differentiable

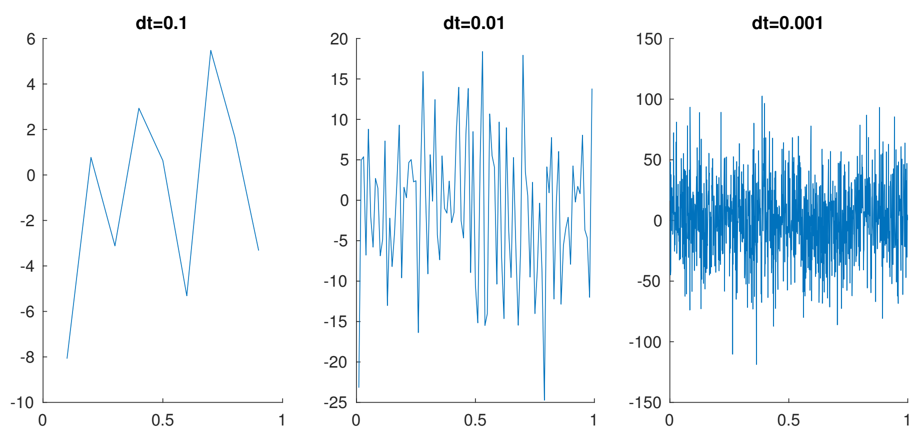


Illustration of the ratio $\frac{W(t+dt)-W(t)}{dt}$when $dt = 0.1$, $dt = 0.01$, $dt = 0.01$. Note that the paths do not converge to a constant, but instead become increasingly more volatile as dt shrinks.

- Second attempt. How about re-writing the differential equation in integral form

$$X(t) = \alpha + \int_0^t \mu(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s)$$

and interpreting $\int_0^t \sigma(s, X(s))\, dW(s)$ as a Rieman-Stieltjes integral?

- (Aside: A Rieman-Stieltjes integral is defined as an integral against a function such as –say a distribution function– $\int_0^t \sigma(s, X(s))\, dF(s)$, which may not be differentiable)

- The problem with this idea is that Brownian motion as a function of time:

$$\sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |W(x_{i+1}) - W(x_i)| = \infty$$

where $\mathcal{P}$=$\{P = \{x_0, ..., x_{n_P}\} | P$ is a partition of a finite interval $[a, b]\}$

- Key idea: Let's give up on trying to give a path-wise meaning to "$\int_0^t \sigma(s, X(s))\, dW(s)$". Instead let's think of it as a probabilistic limit of a sequence of certain random variables.

II. Information, Filtration, Adaptedness

- The symbol $\mathcal{F}_t^X$ will be used to denote the "information generated by observing a path of the stochastic process X over the interval $[0, t]$

- If, based on the observation of the trajectory of X_t up to the point t we can determine if a given event A has occurred, then we will call that event $\mathcal{F}_t^X-$ "measurable"

- Examples: Are the following events in $\mathcal{F}_1^X$?
  1. $X_{0.5} > 5$
  2. $X_{1.5} > 6$

- If Y is a stochastic process such that we have

$$Y(t) \in \mathcal{F}_t^X$$

then we will call Y **adapted to the filtration** $\mathcal{F}_t^X$

- Examples:
  1. $S(t) = \sup_{s \le t} W(s)$ is adapted to $\mathcal{F}_t^X$
  2. $S(t) = \sup_{s \le t+1} W(s)$ is not adapted to $\mathcal{F}_t^X$

III. Stochastic Integrals

- The process g belongs to the class $\mathcal{L}^2[a,b]$ if the following conditions are satisfied
  1. $\int_a^b E\left[g^2(s)\right] ds < \infty$
  2. The process g is adapted to the Filtration $\mathcal{F}_t^W$
- The process g belongs to the class $\mathcal{L}^2$ if it belongs to the class $\mathcal{L}^2[0,t]$ for all t.
- A process g is **simple** if there exist deterministic points in time $a = t_0 < t_1 < ... < t_n = b$, such that g is constant on that subinterval. In other words $g(s) = g(t_k)$ for $s \in [t_k, t_{k+1})$.
- For simple g, it is straightforward to define the stochastic integral

$$\int_a^b g(s)\, dW(s) = \sum_{k=0}^{n-1} g(t_k)\left[W\left(t_{k+1}\right) - W(t_k)\right]$$

- The strategy from now on is to approximate g with a sequence of of simple processes g_n such that

$$\int_a^b E\left[\{g_n(s) - g(s)\}^2\right] ds \to 0$$

- For each n the intergral $\int_a^b g_n(s)\, dW(s)$ is a well defined stochastic variable
- We are now in a position to define the stochastic integral by

$$\int_a^b g(s)\, dW(s) = \lim_{n\to\infty} \int_a^b g_n(s)\, dW(s)$$

- It can be shown that the limit exists in a probabilistic sense (mean square error convergence).

Properties of Stochastic Integrals

- The following properties hold for the stochastic integral

$$E \int_a^b g(s)\, dW(s) = 0$$

- The Ito isometry

$$E\left[\left(\int_a^b g(s)\, dW(s)\right)^2\right] = E \int_a^b g^2(s)\, ds$$

- Let's prove these identities for simple g

- Using that fact that g is simple

$$\begin{aligned} E\left[\int_a^b g(s)\, dW(s)\right] &= E\left[\sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right] \\ &= \sum_{k=0}^{n-1} E\left[g(t_k)\, E_{t_k}\left[W(t_{k+1}) - W(t_k)\right]\right] \\ &= 0 \end{aligned}$$

- Similarly,

$$\begin{aligned} E\left(\int_a^b g(s)\, dW(s)\right)^2 &= E\left[\begin{array}{c} \left(\sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right) \\ \times\left(\sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right) \end{array}\right] \\ &= \sum_{k=0}^{n-1} E\left(g^2(t_k)\, E_{t_k}\left[W(t_{k+1}) - W(t_k)\right]^2\right) \\ &= \sum_{k=0}^{n-1} E\left(g^2(t_k)\right)\Delta_{t_k} = \int_a^b E\left(g^2(s)\right) ds \end{aligned}$$

IV. Martingales

- A stochastic process X is called an $\mathcal{F}_t$ martingale if the following conditions hold

1. X is adapted to the filtration $\mathcal{F}_t$ for $t \geq 0$.
2. For all t
$$E\left[|X_t|\right] < \infty$$
3. For all s and t with $s \leq t$, the following relation holds
$$E\left[X\left(t\right)|\mathcal{F}_s\right] = X\left(s\right)$$

A process satisfying for all s and t with $s \leq t$ the inequality

$$E\left[X\left(t\right)|\mathcal{F}_s\right] \leq X\left(s\right)$$

is called a supermartingale and a process satisfying the reverse inequality is called a submartingale.

- Let X_t be given by
$$X\left(t\right) = \int_0^t g\left(s\right) dW\left(s\right)$$
- Leaving integrability aside, every stochastic integral is a martingale.
- Proof: Fix any $u < t$. We have
$$\begin{aligned} E\left[X\left(t\right)|\mathcal{F}_u\right] &= E\left[\int_0^t g\left(s\right) dW\left(s\right)|\mathcal{F}_u\right] \\ &= \int_0^u g\left(s\right) dW\left(s\right) + E\left[\int_u^t g\left(s\right) dW\left(s\right)|\mathcal{F}_u\right] \\ &= X\left(u\right) + 0 \end{aligned}$$
- Implication: A process dX_t can only be a martingale if it does not have a "dt" term, that is if
$$dX\left(t\right) = g\left(t\right) dW\left(t\right).$$

V. Quadratic Variation

- In preparation for the development of Ito's formula, we next introduce the notion of "infill asymptotics" by introducing the concept of quadratic variation
- Let us divide the interval $[0, t]$ into n equally large subintervals of the form $\left[k\frac{t}{n}, (k+1)\frac{t}{n}\right]$ where $k = 0, 1, ..., n-1$. Given this subdivision, we now define the quadratic variations of the Wiener process S_n

$$S_n = \sum_{i=1}^{n} \left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2$$

- The computation of the expectation of S_n is easy, and indeed is independent of n :

$$\begin{aligned} E\left(S_n\right) &= \sum_{i=1}^{n} E\left\{\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2\right\} \\ &= \sum_{i=1}^{n} E\left\{\left(i\frac{t}{n}\right) - (i-1)\frac{t}{n}\right\} = t \end{aligned}$$

- Remarkably, the variance of S_n disappears as $n \to \infty$, as we show next. The fact that W_t has independent increments implies that

$$\begin{aligned} Var\left[S_n\right] &= \sum_{i=1}^{n} Var\left\{\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2\right\} \\ &= \sum_{i=1}^{n} 2\left[\frac{t^2}{n^2}\right] = 2\frac{t^2}{n} \end{aligned}$$

- (Hint: To prove that $Var\left\{\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2\right\} = 2\left[\frac{t^2}{n^2}\right]$ use the fact that $\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2$ is –up to scaling– Chi-square distributed)
- As $n \to \infty$, the variance of S_n disappears.
- Hence

$$\int_0^t [dW]^2 = t$$

which is sometimes written in short-hand form as

$$[dW]^2 = dt$$

VI. Ito's formula

- Assume that the process X is given by

$$dX\,(t) = \mu\,(t)\,dt + \sigma\,(t)\,dW\,(t)$$

- where μ and σ are adapted processes and let f be a $C^{1,2}$ function. Define the process Z by $Z\,(t) = f\,(t, X\,(t))$. Then Z has a stochastic differential given by

$$df\,(t\,, X\,(t)) = \left\{\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial x^2}\right\} dt + \sigma(t)\frac{\partial f}{\partial x} dW\,(t)$$

- Let's start with a heuristic proof. Using Taylor's theorem, we obtain

$$df = \left\{\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX\right\} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,(dX)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\,(dt)^2 + \frac{\partial^2 f}{\partial x \partial t}dX\,dt$$

- Heuristically, we can write

$$(dX)^2 = \mu^2\,(dt)^2 + 2\mu\sigma\,(dt)\,dW + \sigma^2\,(dW)^2$$

where we have used the short-hand notation μ, σ for $\mu(t), \sigma(t)$.

- The terms of order $(dt)^2$ are tiny compared to dt as $dt \to 0$, so we can ignore them.
- It can also be shown that the $dtdW-$ term is negligible compared to the dt term (in a sense we will make more precise shortly). Finally we have $(dW)^2 = dt$ and hence plugging back into the expression for df we get

$$df = \left\{\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial x^2}\right\} dt + \sigma(t)\frac{\partial f}{\partial x} dW\,(t) + o\,(dt)\,.$$

- It is sometimes more convenient to remember the Ito Formula as

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,(dX)^2$$

and use the following formal multiplication rules

$$\begin{aligned} (dt)^2 &= 0 \\ dtdW &= 0 \\ (dW)^2 &= dt \end{aligned}$$

VII. Applications of Ito's formula - Example 1

- Compute the moment generating function of brownian motion $E\left[e^{aW(t)}\right]$
- Let $Z(t) = e^{aW(t)}$
- Apply Ito's formula to compute

$$dZ(t) = aZ(t)\,dW(t) + \frac{a^2}{2}Z(t)\,dt,\ \ Z(0) = 1$$

- Write this in intergral form

$$Z(t) = 1 + \frac{a^2}{2}\int_0^t Z(s)\,ds + a\int_0^t Z(s)\,dW(s) \qquad (2)$$

- Let $m(t) = E\left[e^{aW(t)}\right]$. Then taking expectations on both sides of (2) leads to

$$m(t) = 1 + \frac{a^2}{2}\int_0^t m(s)\,ds.$$

- or in differential form

$$\dot{m}(t) = \frac{a^2}{2}m(t), \quad m(0) = 1$$

Hence

$$m(t) = e^{\frac{a^2}{2}t}$$

Example 2

- Compute the distribution of

$$\int_0^t W(s)\, dW(s)$$

- Let $Z(t) = W^2(t)$. Applying Ito's Lemma gives

$$dZ(t) = dt + 2W(t)\, dW(t)$$

Integrating gives

$$W^2(t) = \int_0^t dZ(s) = t + 2\int_0^t W(s)\, dW(s)$$

- Therefore

$$\int_0^t W(s)\, dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}$$

- Therefore (up to scaling by $\frac{1}{2}$ and translation by $\frac{t}{2}$) the stochastic integral $\int_0^t W(s)\, dW(s)$ is distributed as a (scaled) chi-squared variable

Example 3

- Compute the distribution of

$$dX(t) = \alpha X(t)\, dt + \sigma X(t)\, dW(t), \quad X(0) = 1$$

- Let $Z(t) = \log(X_t)$. Applying Ito's Lemma gives

$$\begin{aligned} dZ(t) &= \frac{1}{X_t}\left(\alpha X(t)\, dt + \sigma X(t)\, dW(t)\right) - \frac{\sigma^2}{2}\frac{X_t^2}{X_t^2}dt \\ &= \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dW(t). \end{aligned}$$

Integrating gives

$$\log(X_t) = \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t)$$

- Hence,

$$X_t = \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right\}$$

is log-normally distributed with mean $e^{\alpha t}$.

VIII. The multi-dimensional Ito's Formula

- Take a vector $X = (X_1, ..., X_n)$, where the componenct X_i has a stochastic differential of the form

$$dX_i(t) = \mu_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)$$

and $W_1 ... W_d$ are d independent Wiener processes.

- Defining the drift vector μ by

$$\mu = \begin{bmatrix} \mu_1 \\ .. \\ \mu_n \end{bmatrix},$$

the $d-$dimensional vector Wiener process W by

$$W = \begin{bmatrix} W_1 \\ .. \\ W_d \end{bmatrix}$$

and the $n \times d$ dimensional diffusion matrix by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & .. & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & .. & .. \\ .. & .. & .. & .. \\ \sigma_{n1} & .. & .. & \sigma_{nd} \end{bmatrix}.$$

We can then write the dynamics of the $X(t)$ process as

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)$$

Now letting $f : R_+ \times R^n \to R$ be $C^{1,2}$. Then

$$df(t, X(t)) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dX_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j$$

using the formal multiplication rules

$$\begin{aligned} (dt)^2 &= 0 \\ dtdW &= 0 \\ (dW_i)^2 &= dt,\ \ i = 1, ..., d \\ dW_i dW_j &= 0,\ \ \text{for } i \neq j \end{aligned}$$

Example 4

- Suppose that

$$\begin{aligned} dX_1 &= \mu_1 X_1 dt + \sigma_{11} X_1 dW_1(t) + \sigma_{12} X_1 dW_2(t) \\ dX_2 &= \mu_2 X_2 dt + \sigma_{22} X_2 dW_2(t) \end{aligned}$$

- Compute

$$d\left(X_1(t) \cdot X_2(t)\right)$$

- Solution: Let $Z(t) = X_1(t) \cdot X_2(t)$. Applying Ito's Lemma we have

$$\begin{aligned} dZ(t) &= Z(t)\left[(\mu_1 + \mu_2) dt + \sigma_{11} dW_1(t) + (\sigma_{12} + \sigma_{22}) dW_2(t)\right] \\ &\quad + Z(t)\, \sigma_{12}\sigma_{22} dt \end{aligned}$$

Example 5

- Suppose that

$$
\begin{aligned}
dX_1 &= \mu_1 X_1 dt + \sigma_{11} X_1 dW_1(t) + \sigma_{12} X_1 dW_2(t) \\
dX_2 &= \mu_2 X_2 dt + \sigma_{22} X_2 dW_2(t)
\end{aligned}
$$

- Compute

$$
d\left(\frac{X_1(t)}{X_2(t)}\right)
$$

- Solution: Let $Z(t) = \frac{X_1(t)}{X_2(t)}$. Then –applying Ito's Lemma– we have

$$
\begin{aligned}
dZ(t) &= Z(t)\left[(\mu_1 - \mu_2)dt + \sigma_{11} dW_1(t) + (\sigma_{12} - \sigma_{22}) dW_2(t)\right] \\
&\quad - Z(t)\sigma_{12}\sigma_{22} dt + Z(t)\sigma_{22}^2 dt
\end{aligned}
$$

Appendix: A more rigorous derivation of Ito's Formula

- Let's start by writing

$$
f(t, X(t)) - f(0, X(0)) = \sum_{k=0}^{n-1} f\left(t_{k+1}, X\left(t_{k+1}\right)\right) - f\left(t_k, X\left(t_k\right)\right).
$$

Taylor's theorem implies that

$$
f\left(t_{k+1}, X\left(t_{k+1}\right)\right) - f\left(t_k, X\left(t_k\right)\right) = f_t \Delta t + f_x \Delta X_k + \frac{1}{2} f_{xx}\left(\Delta X_k\right)^2 + Q_k
$$

where Q_k is a remainder term and $f_t = \frac{\partial f\left(t_k, X_{t_k}\right)}{\partial t}$, $f_X = \frac{\partial f\left(t_k, X_{t_k}\right)}{\partial X}$, etc.

- Moreover, we have that

$$
\begin{aligned}
\Delta X_k &= X\left(t_{k+1}\right) - X\left(t_k\right) = \int_{t_k}^{t_{k+1}} \mu(s)\, ds + \int_{t_k}^{t_{k+1}} \sigma(s)\, dW(s) \\
&= \mu\left(t_k\right)\Delta t + \sigma\left(t_k\right)\Delta W_{t_k} + S_{t_k}
\end{aligned}
$$

- Hence

$$(\Delta X_k)^2 = \mu^2(t_k)(\Delta t)^2 + 2\mu(t_k)\sigma(t_k)\Delta t \Delta W_{t_k} + \sigma^2(t_k)\left(\Delta W_{t_k}\right)^2 + P_k,$$

where P_k is a remainder term.

- Collecting all the above equations into one gives

$$f(t, X(t)) - f(0, X(0)) = I_1 + I_2 + I_3 + \frac{1}{2}I_4 + \frac{1}{2}K_1 + K_2 + R$$

where

$$\begin{aligned} I_1 &= \sum_k f_t \Delta t,\ \ I_2 = \sum_k f_x(t_k)\mu(t_k)\Delta t,\ \ I_3 = \sum_k f_x \sigma(t_k)\Delta W_k \\ I_4 &= \sum_k f_{xx}\sigma^2(t_k)(\Delta W_k)^2, K_1 = \sum_k f_{xx}(t_k)\mu^2(t_k)(\Delta t)^2, \\ K_2 &= \sum_k f_{xx}(t_k)\mu(t_k)\sigma(t_k)(\Delta t)(\Delta W_k), \\ R &= \sum_k \{Q_k + S_k + P_k\}, \end{aligned}$$

- Letting $n \to \infty$, we have by definition

$$\begin{aligned} I_1 &\to \int_0^t f_t(s,\ X(s))\, ds,\ \ I_2 \to \int_0^t f_x(s\, X(s))\,\mu(s)\, ds, \\ I_3 &\to \int_0^t f_x(s\, X(s))\,\sigma(s)\, dW(s) \end{aligned}$$

- Using essentially the same proof as the one we used earlier to show that $\sum_k (\Delta W_k)^2 \to t$, we can show that

$$I_4 \to \int_0^t f_{xx}(s, X(s))\,\sigma^2(s)\, ds$$

- K_1 converges to $(\Delta t) \times \left(\int_0^t \mu^2(s)\, ds\right) \to 0$, and K_2 has expectation 0 and variance that goes to zero as $\Delta t \to 0$:

$$\begin{aligned} Var(K_2) &= \sum_k f_{xx}^2(t_k)\,\mu^2(t_k)\,\sigma^2(t_k)\,(\Delta t)^2\, Var(\Delta W_k) \\ &= \sum_k f_{xx}^2(t_k)\,\mu^2(t_k)\,\sigma^2(t_k)\,(\Delta t)^3 \to 0 \end{aligned}$$

- The more difficult part of the proof is to show that the term R goes to zero, which we ommit here. However, the fact that the sums $I_1 - I_4$ go to well-defined limits, while the sums K_1 and K_2 vanish as Δt goes to zero shows why we use the formal multiplication rules

$$\begin{aligned} (dt)^2 &= 0 \\ dt dW &= 0 \\ (dW)^2 &= dt \end{aligned}$$