MFE 409 Lecture 2
Value-at-Risk in Portfolios

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Spring 2025

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Interview Question

Lecture Objectives

- How to choose among various strategies when using VaR?
- How to identify which assets drive the VaR of your portfolio?
- What to expect when computing the VaR of a portfolio?

Outline

Using VaR for Capital Allocation

- You are the head of prop trading for an investment bank. You have to allocate capital between investing in FX or in fixed income. Last year FX invested \$100m and made 10% profits, while fixed income invested \$200m and made 5% profit. What do you do?
- If you have access to leverage quantity of assets is not so important, rather quantity of capital mobilized.
- **Risk Adjusted Rate of Return on Capital (RAROC)**: profit per unit of necessary capital, i.e. profit per unit of VaR

$$\text{RAROC} = \frac{\text{Profit}}{\text{VaR}}$$

- Developed in the 1980s by Bankers Trust (taken over by Deutsche Bank) to develop internal capital budgeting system

Example: RAROC

- Let us compute RAROC for the two positions
  - Assume normal distribution with annual volatility 10% for FX and 4% for fixed income
  - We want to use annual 99.97% VaR ($z(99.97\%) = -3.4$)
- FX
  - VaR $= 3.4 \times 0.1 \times 100\text{m} = \$34\text{m}$
  - RAROC $= 10\text{m}/34\text{m} = 29.4\%$
- Fixed Income
  - VaR $= 3.4 \times 0.04 \times 200\text{m} = \$27.2\text{m}$
  - RAROC $= 10\text{m}/27.2\text{m} = 36.8\%$
- Should tilt allocation towards Fixed Income even if it has lower return
- Similar to Sharpe Ratio, but with focus on downside

Outline

Decomposing the VaR of a Portfolio

- Assume you have a portfolio with positions $x_1, ..., x_n$ in various assets
  - Example: you invest \$2,000 in Gamestop, \$5,000 in Tesla, and \$1,000 in Google

- You have a program to compute the VaR for this portfolio as a function of the positions

- How can you tell which asset is responsible for the risk of the overall portfolio?

The contribution of each position

- Marginal VaR: how much does my VaR increase if I increase position x_i by 1$?

$$\mathrm{DVaR}_i = \frac{\partial \mathrm{VaR}}{\partial x_i}$$

- Component VaR: 100 times how much does my VaR increase if I increase position x_i by 1%?

$$\mathrm{CVaR}_i = x_i \frac{\partial \mathrm{VaR}}{\partial x_i}$$

A Useful Decomposition

- Decomposition (Euler Theorem):

$$\mathsf{VaR} = \sum_i \mathsf{CVaR}_i$$

- Warning: you cannot start by computing CVaR_i and add them up to get VaR

Outline

1. Capital allocation with a VaR constraint
2. Decomposing the VaR of a portfolio: where does the risk come from?
3. Computing the VaR of a portfolio and when it misbehaves

Example: VaR for a Portfolio

- Positions: 10mil EUR, 1bil Yen
  - $J_t =$ USD/JPY $= 0.01078749$; USD/EUR $= 1.436$
  - Assume $R_{M,t+1}$ and $R_{J,t+1}$ jointly normal with
  - $E(R_M) = E(R_J) \approx 0$, $\sigma_M = 0.65\%$, $\sigma_J = 0.69\%$
  - $\text{Corr}(R_M, R_J) = \rho_{MJ} = 0.2775$
- The change in portfolio value is:

$$W - W_0 = \$14.36\text{m} \times R_{M,t+1} + \$10.78\text{m} \times R_{J,t+1} \sim \mathcal{N}(0, \sigma_V)$$

  - $\sigma_V = \$134,445.20$
- 99% 1-day VaR$= 2.326 \times \sigma_V = \$312,719.40$

VaR for a Portfolio: Approximate approach

- An approximate approach that seems to works well is
$$\text{VaR}_{\text{total}} = \sqrt{\sum_i \sum_j \text{VaR}_i \text{VaR}_j \rho_{ij}}$$
where VaR_i is the VaR for the i-th segment, $\text{VaR}_{\text{total}}$ is the total VaR, and ρ_{ij} is the coefficient of correlation between losses from the i-th and j-th segments

- Exact formula for normal distributions

VaR and Diversification

- Diversification: 2 investments x_1 and x_2 with same mean and variance, correlation ρ

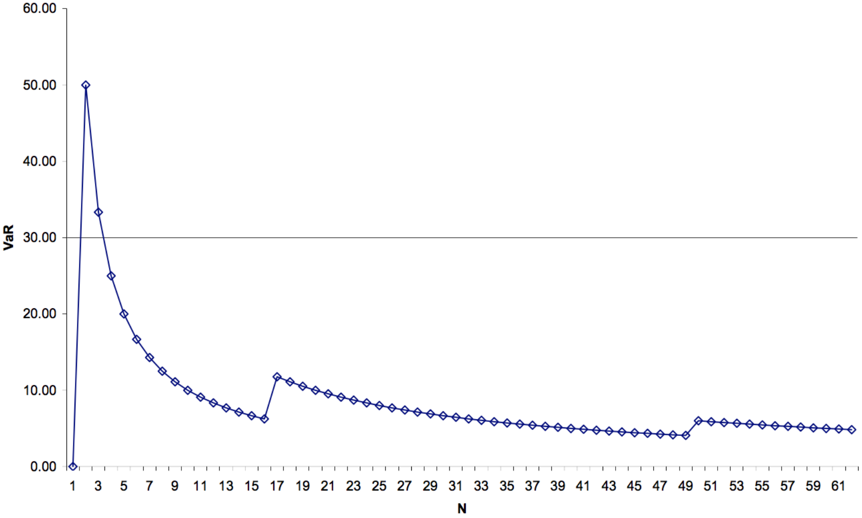
$$\begin{aligned}
\mathsf{Var}\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) &= \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 + 2\mathsf{Cov}\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right) \\
&= \frac{1}{2}\left(\sigma^2 + \mathsf{Cov}\left(x_1, x_2\right)\right) \\
&= \frac{1}{2}\sigma^2(1+\rho) \\
&\leq \sigma^2
\end{aligned}$$

- With normal distribution, also applies to Value-at-Risk: diversification reduces risk

VaR and Diversification

- Consider bonds with face value of 100 and default probability of 0.9% and 0 recovery. Assume defaults are independent across bonds and that the baseline level is $W_0 = 100$.
- What is the 99% VaR for one bond?
  - VaR$= 0$
- What is the 99% VaR for two bonds?
  - Both pay, loss 0, with proba $0.991^2 = 0.982$
  - Both default, loss 100 with proba $0.009^2 = 8 \times 10^{-5}$
  - Last case, loss 50
  - VaR$= 50$
- What is the 99% VaR for n bonds?

VaR and Diversification



Coherent Risk Measures

- Define a coherent risk measure as the amount of cash that has to be added to a portfolio to make its risk acceptable
- Properties of coherent risk measure
  - If one portfolio always produces a worse outcome than another its risk measure should be greater
  - If we add an amount of cash K to a portfolio its risk measure should go down by K
  - Changing the size of a portfolio by a factor λ should result in the risk measure being multiplied by λ
  - The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged

Coherent Risk Measures

- Value-at-Risk ✗
- Expected Shortfall ✓
- Spectral measures
  - Spectral measures assigns weight to quantiles of the loss distribution
  - VaR assigns all weight to c-th percentile of the loss distribution
  - Expected shortfall assigns equal weight to all percentiles greater than the c-th percentile
  - For a coherent risk measure weights must be a non-decreasing function of the percentiles