Week 1: The Binomial Model

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- **I.** The Binomial Model
- **II.** The Value Process, Arbitrage
- **III.** Martingale Measure
- **IV.** Contingent Claims, Replication
- **V.** Risk Neutral Valuation
- **VI.** The Multi-Period Model, The Binomial pricing formula

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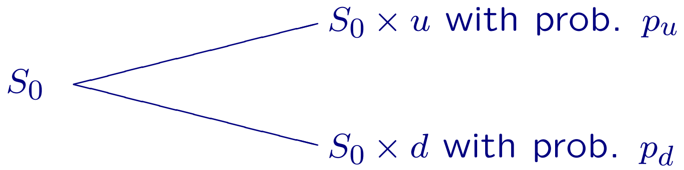
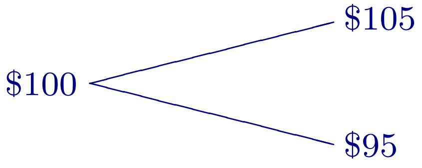
I. The Binomial Model

- Simplest possible asset price dynamics
  - Two time periods $t = 0$ and $t = 1$
  - Bond: $B_0 = 1$ today and $B_1 = 1 + R$, where R is the spot rate
  - Stock price:



  - Sometimes we will write $S_1 = S_0 \times Z$, where $Z \in \{u, d\}$.
  - Example: Stock evolution of company A over one day:



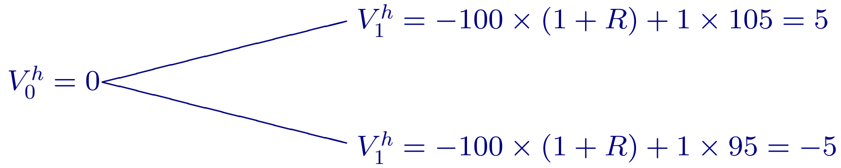
In this example $S_0 = 100$, $u = 1.05$ and $d = 0.95$.

II. The Value Process and Arbitrage

- A **portfolio** is a vector $h \equiv (x, y)$, where x is the number of bonds and y is the number of stocks. Assumptions throughout the class
  - Availability of borrowing and short selling ($h \in R^2$)
  - No other trading frictions (Bid Ask spread, transactions costs, illiquidity, etc.)

- A **value process** is

$$V_t^h = xB_t + yS_t$$

- Example: Same example as before and $R = 0$. The value process associated with borrowing 100 dollars, and purchasing one unit of the stock is
  - $V_0^h = -100 \times 1 + 1 \times 100 = 0$
  - 

$V_0^h = 0$

$V_1^h = -100 \times (1 + R) + 1 \times 105 = 5$

$V_1^h = -100 \times (1 + R) + 1 \times 95 = -5$

- An **arbitrage** is a portfolio with the properties
  - $V_0^h = 0$
  - $V_1^h > 0$ with probability one

- The binomial model **is free of arbitrage** if and only if $d \leq 1 + R \leq u$

- Example: In our example, show that if $u = 1.05$, $d = 1.03$ and $R = 0$, then the portfolio $h = (-100, 1)$ is an arbitrage

- Intuitively, the condition $d \leq 1 + R \leq u$ states that the bond cannot dominate the stock and the stock cannot dominate the bond in all states of the world.

III. Martingale Measure Risk Neutral Valuation

- The arbitrage condition $d \leq 1+R \leq u$ implies that $1+R$ can be expressed as a convex combination of d and u with weights $q_u \in [0,1]$ and $q_d = 1 - q_u$

$$1 + R = q_u \times u + q_d \times d$$

- The weights q_u and q_d can be interpreted as probability weights. An implication of the above equation is that if we build expectations under this measure we obtain

$$\frac{E^Q(S_1)}{1+R} = \frac{S_0 \times (q_u \times u + q_d \times d)}{1+R} = S_0$$

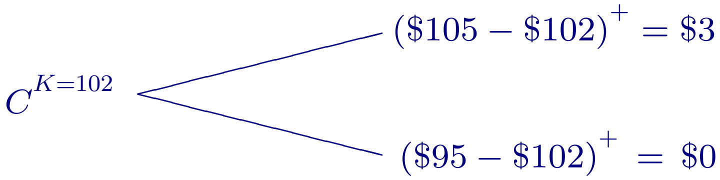
- We will refer to Q as a **risk neutral (or martingale) measure**. It is a probability measure such that the price today is equal to the expected (under Q) price tomorrow discounted to the present.
- The market is arbitrage free if and only if there exists a martingale measure Q.
- To see this, compute explicitly the probabilities associated with the martingale measure Q

$$\begin{aligned} q_u &= \frac{(1+R)-d}{u-d} \\ q_d &= \frac{u-(1+R)}{u-d} \end{aligned}$$

- Note that these are between 0 and 1 when and only when $d \leq 1+R \leq u$

IV. Contingent Claims and Replication

- A contingent claim is any stochastic variable of the form $X = \Phi(Z)$.
- Example: A European Call Option: $\Phi(Z) = (S_0 \times Z - K)^+ \equiv \max(0, S_0 \times Z - K)$



$C^{K=102}$

$(\$105 - \$102)^+ = \$3$

$(\$95 - \$102)^+ = \$0$

- How to determine the "fair price" of such an option?
- A given contingent claim X will be called **reachable** if there exists a portfolio h such that

$$V_1^h = X$$

with probability one.

- The portfolio that replicates the claim X will be called the **replicating portfolio**.
- If all claims can be replicated, then the market is **complete**.
- Pricing by the **absence of arbitrage**: The price of the contingent claim X at time t, denoted by $\Pi(t, X)$ must satisfy

$$\Pi(t, X) = V_t^h, \text{ for all } t = 0, 1$$

where h is the replicating portfolio of claim X. Otherwise there is an arbitrage.

- We next show that in the binomial model $u > d$ implies completeness. To see this take any claim with payoffs $\Phi(u)$ and $\Phi(d)$. Then replicability requires that we should be able to find x, y that solve the following linear system of equations

$$\begin{aligned}(1+R)x + (S_0 \times u)y &= \Phi(u) \\ (1+R)x + (S_0 \times d)y &= \Phi(d),\end{aligned}$$

As long as $u > d$, the above system of equations has a unique solution given by

$$\begin{aligned}x &= \frac{1}{1+R} \times \frac{u\Phi(d) - d\Phi(u)}{u-d} \\ y &= \frac{1}{S_0} \times \frac{\Phi(u) - \Phi(d)}{u-d}\end{aligned}$$

V. Risk neutral Valuation

- The value process associated with the replicating portfolio at time 0 is

$$\begin{aligned}V_0^h &= x + S_0 y \\ &= \frac{1}{1+R}\left\{\frac{(1+R)-d}{u-d}\Phi(u) + \frac{u-(1+R)}{u-d}\Phi(d)\right\} \\ &= \frac{1}{1+R}\{q_u\Phi(u) + q_d\Phi(d)\} \\ &= \frac{1}{1+R}E^Q[X]\end{aligned}$$

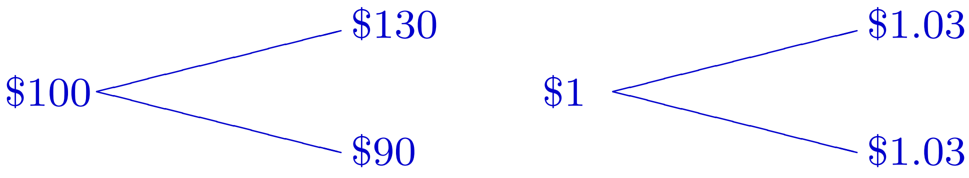
- By the absence of arbitrage

$$V_0^h = \Pi(0; X) = \frac{1}{1+R}E^Q[X]$$

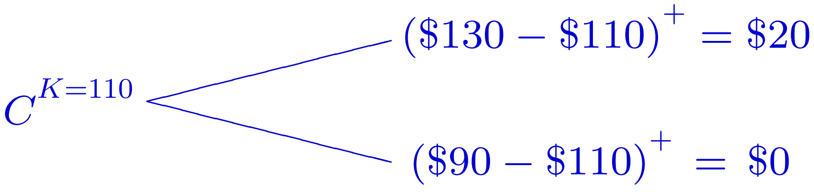
- The contingent claim price is an expectation (under Q) of the the random variable X. In particular, the probabilities of an up or down move don't enter the calculation!
- Why is that?
- The initial price already reflects the probabilities beliefs, risk aversion etc.
  - Consider the following example.
  - Fix the payoffs $105\$$ and $95\$$.
  - Suppose that markets are very optimistic, so that the initial price S_0 becomes $102\$$. What is u, d, q_u, q_d in this case?
  - Similarly, what happens to u, d, q_u, q_d if market participants become pessimistic and S_0 becomes $97\$$?

A worked out example

- The payoff diagram for IBM and the risk-free asset are:



- Consider the payoff of a Call option with a strike price at 110

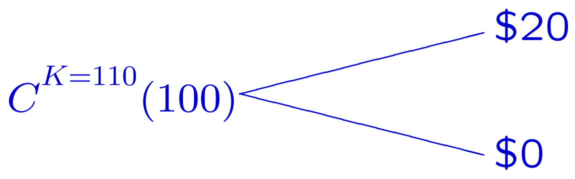


where we use X^+ to denote max$(X, 0)$

Form the **replicating portfolio** for the option

- The replicating portfolio is a portfolio of
  - The underlying stock (IBM), and
  - The risk-free asset

  that has exactly the *same payoffs* as the option



- That is, we need a portfolio that solves

$$\begin{aligned} 1.03x + 130y &= 20 \\ 1.03x + 90y &= 0 \end{aligned}$$

- Solving the two equations

$$\begin{aligned} x &= -43.69\$ \\ y &= 0.5 \end{aligned}$$

- That is, to perfectly replicate the option payoff
  - Buy 0.5 shares of IBM
  - Borrow $43.69 at the risk-free rate
- We would have arrived at the same conclusion using our general formula

$$\begin{aligned} x &= \frac{1}{1+R} \times \frac{u\Phi(d) - d\Phi(u)}{u-d} = \frac{1}{1.03} \times \frac{1.3 \times 0 - 0.9 \times 20}{1.3 - 0.9} = -43.69 \\ y &= \frac{1}{S_0} \times \frac{\Phi(u) - \Phi(d)}{u-d} = \frac{1}{100} \times \frac{20-0}{1.3-0.9} = 0.5 \end{aligned}$$

Price the replicating portfolio

- Under *no-arbitrage* the option and its replicating portfolio must have the same price, since they have the same payoffs in all states of the world

$$\begin{aligned}\text{Price of option} &= \text{Price of replicating portfolio}\\ \Longrightarrow \quad C^{K=110}(100) &= x + y \times \$100\\ &= 0.5 \times \$100 - \$43.69\\ &= \$50 - \$43.69\\ &= \$6.31\end{aligned}$$

- The call option is therefore worth $\$6.31$

Confirm that the risk-neutral approach would have given the same answer

- Solve for $q_u = \frac{1.03-0.9}{1.3-0.9} = 0.325$ and thus $q_d = 1 - 0.325 = 0.675$
- Use the risk-neutral probabilities to price the asset
- The call price is the expected payoff under the risk-neutral probabilities, discounted at the risk-free rate:

$$C^{110}(100) = \frac{0.325 \times \$20 + 0.675 \times 0}{1.03} = \$6.31$$

VI. The multi-period model, $t = 0, ..., T$

- Bond prices

$$\begin{aligned} B_{n+1} &= (1+R)B_n \\ B_0 &= 1 \end{aligned}$$

- Stock prices

$$S_{n+1} = S_n \times Z_n$$

  where $Z_0, ...Z_{T-1}$ are i.i.d random variables that take the value either u or d.

- A **portfolio strategy** is a stochastic process $(h_t = (x_t, y_t); t = 1, ..., T)$ such that h_t is a function of $S_0, ..., S_{t-1}$.
- Note that a portfolio strategy depends on information up to time $t-1$.
- The **value process** corresponding to h is defined by

$$V_t^h = x_t(1+R) + y_t S_t$$

- A portfolio strategy is **self-financing** if the following condition holds for all $t = 0, ..., T-1$

$$x_t(1+R) + y_t S_t = x_{t+1} + y_{t+1} S_t$$

- An **arbitrage** possibility is a self-financing portfolio strategy h with the properties

$$\begin{aligned} V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0, \end{aligned}$$

- If the model is free of arbitrage then $d \leq (1+R) \leq u$.
- The martingale probabilities q_d, and q_u are defined as the probabilities that

$$S_t = \frac{1}{1+R} E_t^Q[S_{t+1}]$$

- The martingale probabilities are

$$\begin{aligned} q_u &= \frac{(1+R)-d}{u-d} \\ q_d &= \frac{u-(1+R)}{u-d} \end{aligned}$$

- A **contingent claim** is a stochastic variable X of the form

$$X = \Phi(S_T)$$

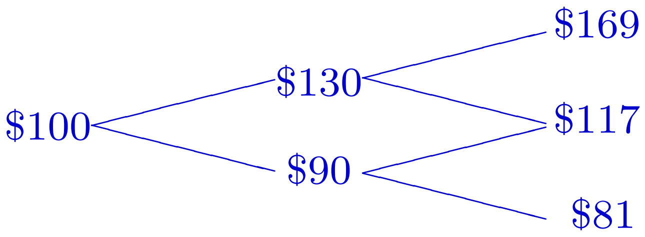
- A given contingent claim X is **reachable** if there exists a self-financing portfolio h such that $V_T^h = X$ with probability 1. The portfolio h will be called a **replicating** portfolio. If all claims can be replicated, then the market is (dynamically) **complete**.
- If a claim X js reachable with replicating portfolio h, then the **arbitrage-free** price of the claim is given by

$$\Pi(t; X) = V_t^h$$

- We will next show that the multi period binomial model is complete. We illustrate how to prove this with an example.

A Call option pricing example: Two Periods

- **Example:** Consider a European Call Option on IBM stock
    - Currently IBM trades at \$100/share. Suppose its price for the next two years has a binomial distribution, and that every year it either goes up by 30% or down by 10%



$\$100$

$\$130$

$\$90$

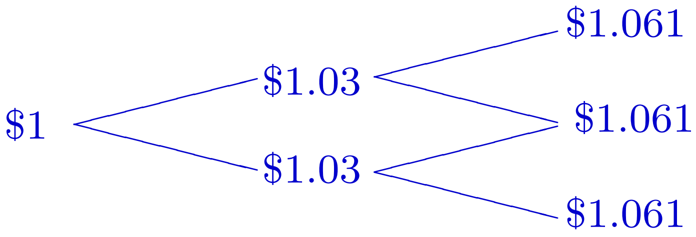
$\$169$

$\$117$

$\$81$

-
    - The risk-free rate over the whole period is $r = 3\%$
    - Find the price of a European call option which
        - Expires in *two* years
        - Has a strike price of $K = \$110$

- **Solution:** The payoff diagram for the bond:



$\$1$

$\$1.03$

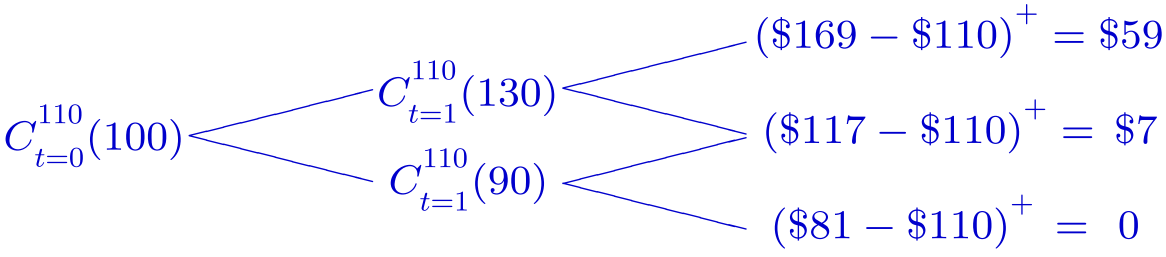
$\$1.03$

$\$1.061$

$\$1.061$

$\$1.061$

And the payoff diagram for the call option:



$C^{110}_{t=0}(100)$

$C^{110}_{t=1}(130)$

$C^{110}_{t=1}(90)$

$(\$169 - \$110)^+ = \$59$

$(\$117 - \$110)^+ = \$7$

$(\$81 - \$110)^+ = 0$

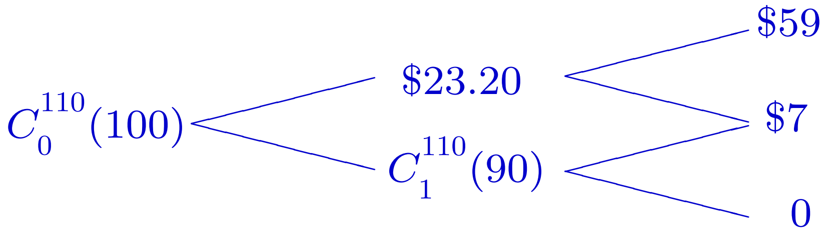
- We want to find the replicating portfolio at each node of the tree
- Start with $C^{110}_{t=1}(130)$
  - This is the replicating portfolio for the call if the stock goes up (remember that we are working backwards)
- For this node, we need to find x and y. We can use the formulas

$$\begin{aligned} x &= \frac{1}{1+R}\frac{S_1u \times \Phi(d) - S_1d \times \Phi(u)}{S_1u - S_1d} = \frac{1}{1.03}\frac{169 \times 7 - 117 \times 59}{169 - 117} = -106.80 \\ y &= \frac{\Phi(u) - \Phi(d)}{S_1u - S_1d} = \frac{59 - 7}{169 - 117} = 1 \end{aligned}$$

- The value of the call is just the value of the replicating portfolio (by no arbitrage)
- So when the stock price goes to \$130 we have

$$C^{110}_1(130) = x + y \times 130 = 130 - 106.80 = \$23.20$$

- The payoff diagram for the call options so far is:



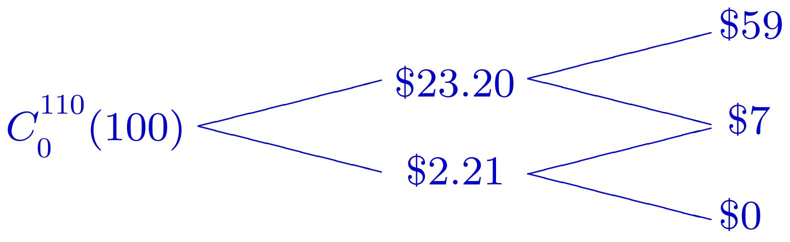
- To find the replicating portfolio for $C^{110}_1(90)$, we compute

$$\begin{aligned} x &= \frac{1}{1.03}\frac{117 \times 0 - 81 \times 7}{117 - 81} = -15.29 \\ y &= \frac{7 - 0}{117 - 81} = 0.1944 \end{aligned}$$

- So the value of the call when the stock price goes to \$90 is

$$C^{110}_1(90) = 0.1944 \times \$90 - \$15.29 = \$2.21$$

- The payoff diagram so far is:



$C_0^{110}(100)$

$23.20

$2.21

$59

$7

$0

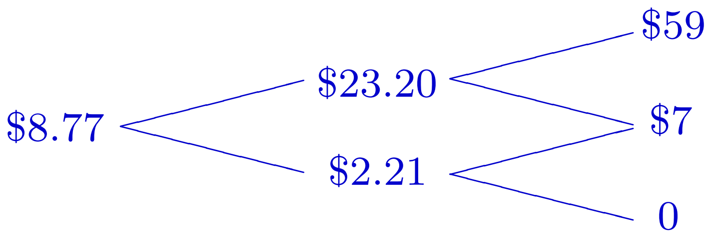
- Finally, to find $C_0^{110}(100)$, solve

$$\begin{aligned} x &= \frac{1}{1.03}\,\frac{130 \times 2.21 - 90 \times 23.20}{130 - 90} = -43.70 \\ y &= \frac{23.20 - 2.21}{130 - 90} = 0.5247 \end{aligned}$$

- So the $t = 0$ value of the call when the stock price is \$100 is

$$C_0^{110}(100) = 0.5247 \times \$100 - \$43.70 = \$8.77$$

- The complete payoff diagram for the call looks like



$8.77

$23.20

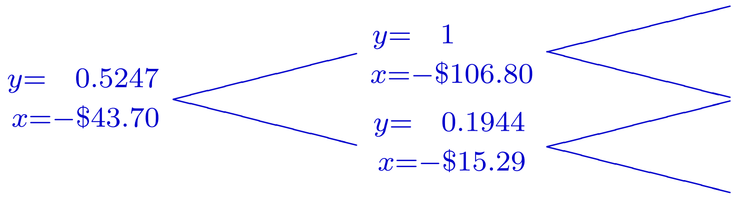
$2.21

$59

$7

0

- How does the replicating portfolio develop dynamically over time?



$y=$ 0.5247
$x=-\$43.70$

$y=$ 1
$x=-\$106.80$

$y=$ 0.1944
$x=-\$15.29$

- Calculating the replicating portfolio at each node of the tree is tedious
- **Question:** Is there an easier way?

- By the results of the one-period model, we know that each node of the tree we can write V_t as

$$\begin{aligned} V_t &= \frac{1}{1+R}[q_u V_{t+1}^u + q_d V_{t+1}^d] = \frac{1}{1+R} E_t^Q (V_{t+1}) \\ V_T &= \Phi(S_T), \end{aligned}$$

where V_{t+1}^u is the value of the replicating portfolio (and hence of the claim) conditional on an "up move" and V_{t+1}^d is the value of the claim conditional on a "down" move.

- The law of the iterated expectation then implies that

$$\begin{aligned} V_0 &= \frac{1}{1+R} E_0^Q [V_1] = \frac{1}{1+R} E_0^Q \left[\frac{1}{1+R} E_1^Q [V_2] \right] = \\ &= \frac{1}{(1+R)^2} E_0^Q [V_2] = \frac{1}{(1+R)^2} E_0^Q \left[\frac{1}{1+R} E_2^Q [V_3] \right] = \\ &= \dots \\ &= \frac{1}{(1+R)^T} E_0^Q [V_T] \\ &= \frac{1}{(1+R)^T} E_0^Q [\Phi(S_T)] \end{aligned}$$

- Hence, we obtain that

$$\Pi(0; X) = V_0 = \frac{1}{(1+R)^T} E_0^Q [\Phi(S_T)]$$

- The law of the iterated expectation then implies that

$$\begin{aligned}
V_0 &= \frac{1}{1+R}E_0^Q[V_1] = \frac{1}{1+R}E_0^Q\left[\frac{1}{1+R}E_1^Q[V_2]\right] = \\
&= \frac{1}{(1+R)^2}E_0^Q[V_2] = \frac{1}{(1+R)^2}E_0^Q\left[\frac{1}{1+R}E_2^Q[V_3]\right] = \\
&= \dots \\
&= \frac{1}{(1+R)^T}E_0^Q[V_T] \\
&= \frac{1}{(1+R)^T}E_0^Q[\Phi(S_T)]
\end{aligned}$$

- Hence, we obtain that

$$\Pi(0;X) = V_0 = \frac{1}{(1+R)^T}E_0^Q[\Phi(S_T)] = \frac{1}{(1+R)^T}E_0^Q[X]$$

- For the binomial model, we can even use the fact that the stock market is distributed according to a binomial distribution to write

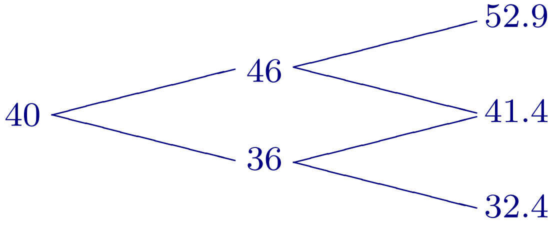
$$\Pi(0;X) \quad = \quad \frac{1}{(1+R)^T}\sum_{k=0}^{T}\binom{T}{k}q_u^k q_d^{T-k}\Phi(S_0u^kd^{T-k})$$

- Let's confirm that in the context of our example (recall $q_u = 0.325$)

$$\Pi(0;X) = \frac{(0.325)^2 \times 59\$ + 2 \times (0.325) \times (1-0.325) \times 7\$ + (1-0.325)^2 \times 0}{(1.03)^2} = 8.77\$$$

The example of a Look-Back Put Option

- You are considering HPQ stock over a period of 2 years. Today HPQ trades at \$40. Assume that every year HPQ can go up by 15% or down by 10%:



40

46

36

52.9

41.4

32.4

The interest rates are stable at 5%

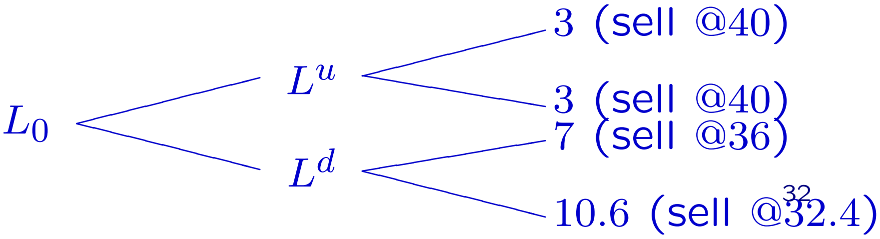
- Find the value of an American **look-back** put option on HPQ, with a strike of \$43, and which expires in 2 years
  - A look-back put uses the *minimum* share price that has occurred over the life of the option
  - and since it is American it can be exercised at any point in time.

Solution

- First, compute the risk-neutral probabilities of the "up" and "down" movements on HPQ

$$
\begin{aligned}
q_u &= \frac{(1+R)-d}{u-d} = \frac{40 \times 1.05 - 36}{46 - 36} = 0.6 \\
q^d &= 0.4
\end{aligned}
$$

- Consider first the *European* look-back put
  - Its payoff is determined by the difference between \$43 (the strike price) and the minimum stock price along the path
  - Since the payoff depends on the price path, we have to separate the paths in the binary tree



L_0

L^u

L^d

3 (sell @40)

3 (sell @40)

7 (sell @36)

10.6 (sell @32.4)

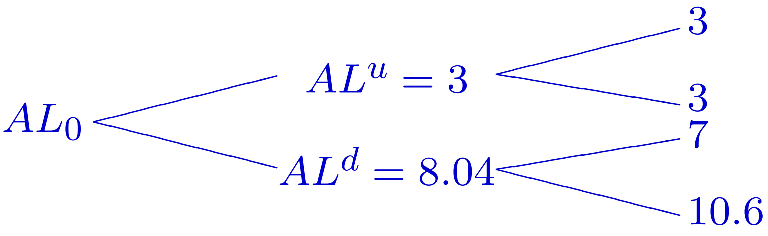
- We can use the risk-neutral probability $\pi = 0.6$ to compute L^u, L^d:

$$\begin{aligned} L^u &= \frac{0.6 \times \$3 + 0.4 \times \$3}{1.05} = \$2.86 \\ L^d &= \frac{0.6 \times \$7 \ + \ 0.4 \times \$10.6}{1.05} = \$8.04 \end{aligned}$$

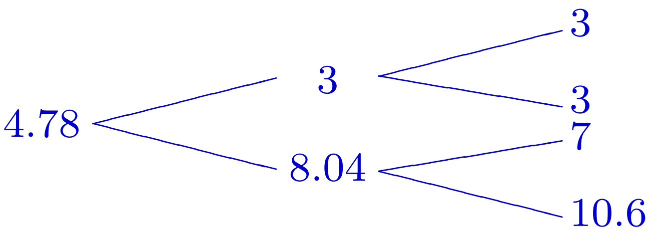
- But the *American* look-back gives us the option to exercise early, at the "up" or "down" nodes
  - Exercising early at the "up" node gives a payoff of $\$43 - \$40 = \$3$, which is more than $L^u = \$2.86$ (the value of the look-back put if one doesn't exercise)
  - Exercising early at the "down" node gives a payoff of $\$43 - \$36 = \$7$, which is less than $L^d = \$8.04$
- So the American look-back put has payoffs

$$AL^u \ = \ \max\{2.86, 3\} = \$3, \qquad AL^u \ = \ \max\{7, 8.04\} = \$8.04$$

- So far we have the following tree for the American look-back put option



- The payoff at t=0 from not exercising is $\frac{0.6\times\$3+0.4\times\$8.04}{1.05} = \$4.78$. Exercising early gives a (lower) payoff of $\$43 - \$40 = \$3$
- Thus we get the following payoff tree for the American look-back



- So one should exercise early only at $t = 1$, at the "up" node.