Notes for Stanford EE364a - Convex Optimization I

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Intro

Notes are from both course lectures and from the course textbook *Convex Optimization* by Stephen Boyd and Lieven Vandenberghe.

In addition, I added my own notes wherever I saw fit (e.g., p-norms in section Norm Ball and Norm Cone).

$\forall x$ means "for every x."

Lecture 2

Affine Set

A line through two points x_1 and x_2 can be represented by

$$x = \theta x_1 + (1-\theta)x_2 \qquad \forall \theta \in \mathbb{R}.$$

An affine set contains all points on the line connecting any two distinct points in the set.

The solution set $\{x | Ax = b\}$ of a linear system of equations is an affine set.

Conversely, any affine set is a solution set of some linear system of equations.

Convex Set

A line segment connecting two points x_1 and x_2 can be represented by

$$x = \theta x_1 + (1-\theta)x_2, \qquad 0 \le \theta \le 1.$$

A convex set contains the line segment connecting any two points in the set:

$$x_1, x_2 \in C \implies \theta x_1 + (1-\theta)x_2 \in C \qquad \forall \theta \in [0,1].$$

So, an affine set is a convex set.

Also, the null set is a convex set, as it is not non-convex.

Convex Combination

Convex combination of set $S = \{x_1, ..., x_k\}$:

$$x = \theta_1 x_1 + ... + \theta_k x_k, \quad 0 \leq \theta_i \leq 1, \;\; \theta_i \geq 0 \quad \forall i = 1, ..., k).$$

Convex hull of S $convS = \{x | x$ is convex combination of S $\}$.

In $\mathbb{R}^n$, $convS =$ set of points within or on boundary of border line segments.

Convex hull of a convex set is the convex set.

Convex combinations differ from affine combinations only by the constraints above.

Convex Cone

A set S is a cone if $\forall x \in S$ and $\forall \theta \geq 0$, we have $\theta x \in S$.

A set S is a convex cone if S is convex and a cone, which means $\forall x_1, x_2 \in S$ and $\forall \theta \geq 0$, we have

$$\theta_1 x_1 + (1 - \theta) x_2 \in S.$$

Conic (non-neg) combination of $S = \{x_1, ..., x_k\}$:

$$x = \theta_1 x_1 + ... + \theta_k x_k, \quad \theta_i \geq 0 \quad \forall i = 1, ..., k.$$

Convex hull: (1) set of all conic combinations of points in S, (2) smallest convex cone that contains S.

Convex cone is convex because the definition of a convex set is a subset of the definition of a convex cone (similar to why an affine set is a convex set).

Hyperplane

A hyperplane is a set of points of the form

$$\{x | a^T x = b\}$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$.

This is simply the solution set to a non-trivial linear equation.

$a^T x = b$ is equivalent to $a_1 x_1 + ... + a_n x_n = b$, so a is the normal vector to the hyperplane.

A hyperplane need not pass through the origin.
$\implies$ A hyperplane need not be a vector space.

A hyperplane in $\mathbb{R}^n$ is an affine subspace with codimension n.
$\implies$ A hyperplane is affine and convex.

Halfspace

A halfspace is a set of points of the form

$$\{x | a^T x \square b\}$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and $b \in \mathbb{R}$ and where the square can be $<, >$ (open halfspace), $\leq$, or $\geq$ (closed halfspace).

A halfspace is not a vector space.

A halfspace is convex but not affine.

A hyperplane splits the surrounding space into two halfspaces.

Euclidean Ball and Ellipsoid

A euclidean ball $B(x_c, r)$ is a set of points of the form

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\}.$$

Alternatively, this can be written as

$$B(x_c, r) = \{x_c + ru \mid \|u\|_2 \leq 1\}.$$

An ellipsoid is a set of the form

$$\mathcal{E} = \{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

with $P \in \boldsymbol{S}^n_{++}$ (i.e., P is symmetric positive definite).

Alternatively, this can be written as

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 < 1\}$$

where $A = P^{1/2} \implies A \in \boldsymbol{S}^n_{++}$.

The lengths of the semi-major axes of the ellipsoid are equal to square roots of the eigenvalues of P.

A ball is an ellipsoid with $P = r^2 I$.

When A is positive semidefinite but singular, the ellipsoid is called degenerate, and the affine dimension equals the rank of A.
$\implies$ Degenerate ellipsoids are convex.

Norm Ball and Norm Cone

A norm is a function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ that satisfies

- (Non-negativity) $\|x\| \geq 0$ and 0 iff $x = 0$
- (Absolute homogeneity) $\|tx\| = |t|\|x\|$

- (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$

$\forall x, y \in \mathbb{R}^n$ and $t \in R$.

We treat $\|\cdot\|$ as a general (unspecified) norm. Only $\|\cdot\|_{\text{symb}}$ is a specific norm.

The absolute value function is an L1 norm over $\mathbb{R}$ (or $\mathbb{C}$).

For $p \geq 1$, the p-norm of a vector $x \in \mathbb{R}^n$ is

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i| \right)^{1/p}.$$

A norm ball of radius r and center x_c is the set of points

$$\{x \mid \|x - x_c\| \leq r\}.$$

The norm cone C associated with $\|\cdot\|$ is the set of points

$$C = \{(x, t) \mid \|x\| \leq t\}.$$

Unit norm ball in R^n is cross section (level set at $t = 1$) of corresponding norm cone.

All norm balls and norm cones are convex.

Polyhedra

A polyhedron $\mathcal{P}$ is defined as the solution set of a finite number of linear inequalities and equalities

$$\mathcal{P} = \{x \mid Ax \preceq b, \;\; Cx = d\}.$$

$\implies$ a polyhedron is an intersection of halfspaces and hyperplanes.

$\preceq$ can be another component-wise inequality.

Affine sets, rays, line segments, and halfspaces are all polyhedra.

Polyhedra are convex.

A bounded polyhedron is sometimes called a polytope.

The nonnegative orthant $\mathbb{R}^n_+$ is

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \succeq 0\}.$$

$\mathbb{R}^n_+$ is a polyhedron and a cone, sometimes called a polyhedral cone.

Positive Semidefinite Cone

$\boldsymbol{S}^n$, the set of all symmetric n x n matrices is convex, affine, and linear.

$\boldsymbol{S}^n_+ = \{X \in S \mid X \succeq 0\}$, the set of all positive semidefinite n x n matrices is a convex cone.

- Note, here and for matrix inequalities in general, $\succeq$ denotes definiteness.

$\boldsymbol{S}^n_{++}$ also a convex cone.

Can use quadratic forms of $X, Y \in$ either $\boldsymbol{S}^n_{+}$ or $\boldsymbol{S}^n_{++}$ that each set is a convex cone.