Review Questions for EE364a – Convex Optimization I

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Intro

From my experience, asking yourself questions and answering them is one of the best ways to test how well you understand something.

Most of the questions here are simple and test for only a basic understanding. Some questions require some insight, but almost none of them should be "challenging."

Chapter 2 – Convex Sets

Affine Set

1. What is an affine set?
2. Describe an affine set using set notation.
3. What are some examples of affine sets?
4. Is any affine set the solution to a linear system of equations?

Convex Set

5. What is a convex set?
6. Describe a convex set using set notation.
7. Are affine sets also convex sets? Why or why not?
8. Are null sets also convex sets? Why or why not?
9. What do convex sets look like in $\mathbb{R}^n$?
10. How does the definition of convexity apply to a set $S \nsubseteq \mathbb{R}^n$? Say, for $S = \boldsymbol{S}^n_{++}$?
11. What is the interior of a set?
12. Give examples of convex sets with an interior equal to the empty set.

Convex Combination

13. What is a convex combination?
14. What is a convex hull?
15. What is the convex hull of a convex set?
16. How do convex combinations differ from affine combinations?

Convex Cone

17. What is a cone?
18. What is a convex cone?
19. Give an example of a cone that is not convex.
20. Why is a convex cone considered to be convex even though the constraints on weights are different than what is required by the definition of convexity?
21. What is a conic combination?
22. What is a conic hull?
23. What convex cone that contains a set S is bigger than the conic hull of S?

Hyperplane

24. What is a hyperplane?
25. Describe a hyperplane using set notation.
26. Describe how the variables in the definition of a hyperplane determine the hyperplane.
27. Is a hyperplane a vector space?
28. Is a hyperplane a subspace?
29. Is a hyperplane an affine subspace?
30. How does the definition of an affine subspace differ from the definition of a vector space? How do the relate?
31. Does every affine subspace contain a vector space?
32. Does every vector space contain an affine subspace?
33. What is the dimension of a hyperplane?
34. What is the codimension of a hyperplane?
35. What is meant by codimension in this context? When does is the codimension defined in this context?
36. Is a hyperplane convex? Affine?

Halfspace

37. What is a halfspace?
38. If a halfspace contains the origin, is it a vector space?
39. Is a halfspace convex?
40. Is a halfspace affine?
41. How does the definition of a halfspace relate to the definition of a hyperplane?

Euclidean Ball and Ellipsoid

42. What is a euclidean ball?
43. Give two representations of a euclidean ball using set notation.
44. What is an ellipsoid?
45. Give two representations of an ellipsoid using set notation.
46. If P (in the first representation) is positive definite, what does this imply about A (in the second representation)?
47. What are the lengths of the semi-major axes of the ellipsoid equal to?
48. What if P is not positive definite?
49. What is the affine dimension of such an ellipsoid? **I'm unsure why Boyd's book makes the claim it does regarding this. It doesn't seem like a degenerate ellipsoid has an affine dimension because the there doesn't exist a translation that makes the set a vector space.**

50. Which of the following are not convex: euclidean ball, non-degenerate ellipsoid, degenerate ellipsoid?

Norm Ball and Norm Cone

51. What is a norm?
52. What is a p-norm?
53. Describe the absolute value function as a norm.
54. What is a norm ball?
55. What is a norm cone?
56. How do the norm balls of a particular norm relate to the norm cone of the same norm?
57. What is the unit norm ball?
58. What is the p-"norm?"
59. What does the unit p-"norm" ball look like for $0 < p < 1$? For $p = 0$?
60. Are norm balls convex?
61. Are norm cones convex?
62. Why are p-"norms" with $0 \leq p < 1$ not true norms?
63. When are such norms useful?

Polyhedra

64. What is a polyhedra in set notation? What does this imply about polyhedra in terms of other convex sets?
65. Are rays polyhedra? Are line segments?
66. Are polyhedra also convex?
67. What is a bounded polyhedron sometimes called?
68. What is the nonnegative orthant? Is it a polyhedron? Is it a cone?
69. What is the positive orthant? Is it a polyhedron? Is it a cone?
70. What are polyhedrons that are also cones sometimes called?

Positive Semidefinite Cone

71. What is the positive semidefinite cone?
72. Is $\boldsymbol{S}_+^n$ convex? Affine? Linear?
73. Is $\boldsymbol{S}_{++}^n$ also a convex cone?

Operations that Preserve Convexity

74. What are two ways to show that a set is convex?
75. What is an easy way to show that a complicated set is not convex?
76. What are pros and cons of each of these approaches?
77. Name four convexity-preserving types of operations.
78. Explicity describe each of these operations.
79. Is the inverse image of a convex set under f also convex? Is the converse true?
80. State examples of types of operations that can be shown to be convex through affine functions.
81. What is a linear matrix inequality? Is its solution set convex?
82. What is the perspective function on a vector?
83. What is the generalization of the perspective function called? Describe it.

Generalized Inequalities

84. What is a proper cone?
85. Give examples of sets that are proper cones.
86. Give examples of sets that are cones but not proper cones.
87. Are all convex cones also proper cones?
88. Are all proper cones convex?
89. Is a line in $\mathbb{R}^n$ a proper cone? A ray?
90. Is a vector space a proper cone? An affine subspace?
91. Is a hyperplane a proper cone? A halfspace? A polyhedron?
92. Is $\boldsymbol{S}^n$ a proper cone? $\boldsymbol{S}^n_+$? $\boldsymbol{S}^n_{++}$?
93. Is $\mathbb{R}^n$ a proper cone? $\boldsymbol{R}^n_+$? $\boldsymbol{R}^n_{++}$?
94. What is a generalized inequality?
95. What does the generalized inequality imply when $K = \boldsymbol{R}^n_+$? When $K = \boldsymbol{S}^n_+$?
96. What is the interior of $\boldsymbol{S}^n_+$ called?
97. Name a property of $\preceq_K$ that has an analog on $\mathbb{R}$.
98. Name a property of $\preceq_K$ that does not have an analog on $\mathbb{R}$.

Minimum and Minimal Elements

99. What is a minimum element of a set wrt a generalized inequality? Is it unique? Does one always exist?
100. What is a minimal element of a set wrt a generalized inequality? Is it unique? Does one always exist?
101. When does every minimum equal every minimal element?
102. In what region is the ordering wrt a proper cone K unambiguous? In what region is it ambiguous?
103. If all other points in a set are more than a point x by $\preceq_K$, then is x a minimum element? A minimal element? Both?
104. If no other point in a set is less than a point x by $\preceq_K$, then is x a minimum element? A minimal element? Both?

Separating Hyperplane Theorem

105. State the separating hyperplane theorem.
106. What is required for strict separation?
107. Give an example of two disjoint convex sets for which there does not exist a strict separating hyperplane.

Supporting Hyperplane Theorem

108. What is a supporting hyperplane?
109. Can nonconvex sets have a supporting hyperplane?
110. What sets are separated by a supporting hyperplane?
111. Geometrically, in Euclidean space, what is the supporting hyperplane at a particular point?
112. State the supporting hyperplane theorem.
113. Do all convex sets have at least one supporting hyperplane?

Dual Cones and Generalized Inequalities

114. What is a dual cone?
115. What is the maximum angle between any vector in a cone and any vector in its corresponding dual? Minimum angle?
116. How does the dual cone relate to supporting hyperplanes?
117. When is a dual cone nonconvex?
118. When is a cone nonconvex?
119. What is the dual cone of a vector space? Why is the dual cone of a vector space defined?
120. What are some examples of cones and their duals?
121. What are some examples of self-dual cones?
122. What are some examples of cones that are not self-dual?
123. Is K^* always closed?
124. If $K_1 \subseteq K_2$, then what is the relationship between K_1^* and K_2^*?
125. Is K^* always pointed?
126. When does the $(K^*)^* = K$? Otherwise, what does $(K^*)^*$ equal?
127. If K is a proper cone, then is K^* always a proper cone? If not, give a counterexample.

Dual Generalized Inequality

128. What is the dual of a generalized inequality?
129. What is a property relating a generalized inequality and its dual?
130. How does this property change when we set $K = K^*$?

Minimum and Minimal Elements via Dual Inequalities

131. State the dual characterization of the minimum element of a set wrt a generalized inequality.
132. How does this relate to supporting hyperplanes?
133. Does the set have to be convex? If not, give an example of a nonconvex set for which the dual characterization of the minimum element holds.
134. State the dual characterization of minimal elements of a set wrt a generalized inequality.
135. How does this relate to supporting hyperplanes?
136. Does the set have to be convex? If not, give an example of a nonconvex set for which the dual characterization of the minimum element holds.
137. If the set is convex, what is implied about the minimal elements of the set (note: minimal, not minimal wrt a generalized equality)?

Optimal Production Frontier

138. Describe the production–possibility frontier.
139. Describe its regions.
140. What solutions are considered efficient?
141. What is another name for an efficient solution?

Chapter 3 – Convex Functions

Definition