Chapter 4 – Convex Optimization Problems Summary

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Key Takeaways

Convex problems are easy to optimize.

Many families of problems are convex or can be transformed into convex problems.

Problems have equivalent representations that are sometimes easier to solve.

Can convert quasiconvex optimization problem into convex feasibility problem. Can use bisection approach to solve this.

Sometimes can transform quasiconvex opt. problem into a convex opt. problem (e.g., linear-fractional). Sometimes cannot (e.g., generalized linear-fraction).

Vector optimization generalizes the objective to vector-valued outputs wrt a particular cone K.

Can find Pareto optimal points through scalarization

Notes Summary

Basic

A feasible x is optimal if $f_0(x) = p^*$. So, the x must be in the domain of f.

x is locally optimal if it is optimal in some neighborhood centered at x.

For a feasibility problem, every x that satisfies the constraints is equally good.

Convex optimization problem has a convex objective, convex inequality constraints, and affine equality constraints.

The feasible set of a convex optimization problem is itself convex.

Any local optimum of a convex problem is the global optimum. This is not generally true for quasiconvex problems.

Optimality criterion for differentiable functions

x is optimal iff $\nabla f_0(x)^\top y \geq \nabla f_0(x)^\top x$ for all feasible y.

- The gradient is normal to the level curve at x.
- The gradient points in the direction of max increase of $f(x)$.
- The RHS determines a supporting hyperplane at x, which determines a halfspace. All feasible y must be on the "greater than or equal to" side of that halfspace. By the supporting hyperplane theorem, this is possible for any convex feasible set.

This is used to make the optimality criterion in the following cases more specific:

- Unconstrained problem
- Equality constrained problem
- Minimization over nonnegative orthant

Equivalent convex problems

We say two problems are equivalent if by solving either one, we can construct the solution of the other with little effort.

There are many equivalence transforms that preserve convexity:

- Eliminating equality constraints
- Introducing equality constaints
- Introducing slack variables for linear inequalities
- Epigraph form (i.e., the epi trick)
- Minimizing over a subset of variables

Quasiconvex optimization

A local optimum is not generally a global optimum.

We can transform the quasiconvex optimization problem into a convex feasibility problem.

- Iterative approach (e.g., bisection) to find the global optimum.
- Each step requires solving convex feasibility problem.
- Easier with lower and upper bounds on p^*, but usually still easy without bounds.

Linear program

Affine objective and affine constraints.

Feasible set is a polyhedron.

Level sets of a affine function are hyperplanes, so there always exists an optimal point at a vertex of the feasible set.

Linear-fractional program

Linear-fractional objective and affine constraints.

A quasiconvex optimization problem but is equivalent to an LP. After transforming, can solve as an LP.

Generalized linear-fractional program

Objective is the max of multiple linear-fractional functions. Affine constraints.

Quasiconvex problem but cannot transform into LP.

Quadratic program

Quadratic convex objective ($P \in §^n_+$) and affine constraints.

Level sets of convex quadratic are ellipsoids.

QP $\supset$ LP.

Least-squares (even with linear constraints on x) is a QP.

Quadratically-constrained quadratic program

Quadratic convex objective and inequality constraints.

Feasible set is an intersection of ellipsoids and an affine set.

If P_i in inequality constraints aren't full rank, then the corresponding ellipsoid is degenerate.

QCQP $\supset$ QP $\supset$ LP

Second-order cone program

Linear objective, second-order cone inequality constraints, and affine equality constraints.

Inequalities are called second-order cone constraints bc $(Ax_i + b_i, c_i \top x + d_i) \in$ the second-order cone $C = \{(z,t) | \|z\|_2 \leq t\}$.

SOC constraints are differentiable everywhere except at 0, but 0 is often where the solution is (because pointed at 0).

SOCP $\supset$ QCQP $\supset$ QP $\supset$ LP

Robust linear program

Parameters in optimization problems are often uncertain.

Two common approaches to handle uncertainty:

- Deterministic approach: constraints must hold for parameters in certain ellipsoids
- Stochastic approach: parameters are treated as RVs, and constraints must hold with certain probabilities
- Both approaches are SOCPs

Geometric program

Objective and inequality constraints are posynomials, and affine constraints are monomials.

Monomial: $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ with $c > 0, a_i \in \mathbf{R}$

Posynomial: sum of monomials

Domain of both is $\mathbf{R}^n_{++}$.

Can transform GP into convex problem by taking logs of objective and constraint functions.

Generalized inequality constraints

Convex problem wrt generalized inequality constraints: convex objective, K_i-convex inequality constraints, affine equality constraints.

Same properties as standard convex problems.

Cone program:

- Affine objective and constraints
- Inequality constraints wrt a cone K
- Extends LP to nonpolyhedral cones (e.g., union of a narrow cone in $\mathbf{R}^n_+$ and its mirror image in $\mathbf{R}^n_-$.

Semidefinite program

Linear objective, LMI inequality constraint, and affine equality constraint.

Cone program $\supset$ SDP $\supset$ SOCP $\supset$ QCQP $\supset$ QP $\supset$ LP

A single SDP solver can solve SOCP...LP equally well.

Vector optimization

Generalizes objective to vectors (outputs) wrt a particular cone K.

Objective is K-convex, inequality constraints are convex, and equality constraints are affine.

Multicriterion approach

Minimize multiple objective functions simultaneously.

x is optimal if it minimizes all objectives simultaneously (no trade-off exists and objectives are noncompeting).

x is Pareto optimal if no other feasible point is unambiguously better (but trade-off may exist).

Scalarization

Can use to find Pareto optimal points of vector optimization problem.

Linear objective function. Changing λ changes slope. Optimization finds point where $\lambda\top f_0(x)$ supports the feasible set. Each of these are Pareto optimal points of vector optimization problem.

In general, not all Pareto optimal points can be found this way.

Scalarization for multicriterion problems

Minimizes positive weighted sum of $\lambda\top f_0(x)$ where $f_0(x)$ is the multicriterion objective.

Adjusting weights moves you along Pareto optimal surface.