Chapter 5 – Duality Summary

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Key Takeaways

Dual function is always convex and gives lower bound on p^*.

Dual problem is always convex and gives best lower bound possible from the dual function.

When original problem is convex, and Slater's is satisfied, strong duality holds.

Weak duality always holds and can be used to find useful lower bounds for difficult problems.

Dual function/problem have an interesting geometric interpretation.

Strong duality and $x^*, \lambda^*, \nu^* \rightarrow$ complementary slackness.

Some theorems relating optimal values and KKT conditions for convex and nonconvex problems.

Can perturb original problem to get info on sensitivity of p^* wrt changes in inequality constraints.

Reformulation of primal problem may have simpler dual than original primal.

Can solve problems with generalized inequality constraints through SDP.

Notes Summary

Lagrangian

Before, optimization problems had a soft cost function f_0 and hard cost constraints $f_1, ..., f_m \leq 0$ and $h_1, ..., h_p = 0$.

Now for any optimization problem (not necessary convex), the Lagrangian is an overall soft cost function that is a weighted sum of the objective and constraint functions.

Lagrangian dual function

g is the infimum of the Lagrangian over all x in the domain of the primal problem.

g is concave because it is affine in λ and ν.

Lower bound property: if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^*$.
$\rightarrow$ so, if $g(\lambda, \nu)$ and λ depend on the same variable, say ν, then any ν that results in $\lambda \succeq 0$ also results in a lower bound of p^*.

Some problems, such as the standard form LP, always have the same form lower bound property.

Lagrange dual and conjugate function

Sometimes the Lagrange dual of a problem and the conjugate of the objective are related.

For the case in which the objective is any function and the constraints are all affine, the dual function has a known form that includes the conjugate of the objective.

Knowing the conjugate may make finding the dual easier.

The dual problem

Maximizes the dual function subject to $\lambda \succeq 0$.

Finds d^*, the best lower bound of p^*.

Always a convex problem.

λ, ν dual feasible if $\lambda \succeq 0$ and $(\lambda, \nu) \in \textbf{dom}\, g$.

Weak and strong duality

Weak duality $(p^* \leq d^*)$

- Always holds

- Can use to find nontrivial lower bounds for difficult problems

Strong duality ($p^* = d^*$)

- (Usually) holds for convex problems

Slater's constraint qualification

For a convex problem, strong duality holds if the problem is strictly feasible (there exists x in relative interior of inequality constraint polyhedron).

- Guarantees dual optimum is attained (if $p > -\inf$)
- Linear inequalities can hold with equality

Slater's holds for almost everything practical.

Geometric interpretation of dual function

Let λ (and ν?) be the normal vector to a hyperplane. Shift that hyperplane until it supports the domain of G. The point $(g(\lambda), 0, 0, ..., 0)$ is where the hyperplane's Lagrange multiplier coefficients all $= 0$.

d^* is the max value attained over all λ.

$p^* - d^* =$ duality gap.

Complementary slackness

If strong duality holds, λ_i^* and $f_i(x^*)$ can't both be > 0 for same i.

KKT conditions

For differentiable f_i, h_i:

1. primal constraints
2. dual constraints
3. complementary slackness
4. gradient of Lagrangian evaluated at x vanishes

If strong duality holds and x, λ, ν are optimal, KKT conditions must hold.

KKT conditions for convex problem

If $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfy KKT for a convex problem, then they are (primal and dual) optimal.
$\rightarrow f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$.

If Slater's holds, x optimal iff there exists λ, ν that satisfy KKT.

Perturbation and sensitivity analysis

Perturbed problem changes RHS of constraints (which equal 0 in standard form) by u_i, v_i for inequality, equality.

For inequality constraints, decrease = tighten, increase = loosen. For equality constraints, decrease/increase just shifts.

Perturbed problem has a known dual in terms of the original problem's dual.

Global sensitivity result gives:

- a lower bound on the increase of $p*$ when a constraint is tightened
- an idea of how $p*$ is affected by tightening/loosing constraints
- an idea of how "tight" the tight constraints are when evaluated at x^*

Solvers provide sensitivities for no additional cost.

Duality and problem reformulations

Equivalent (even trivially) formulations of a problem can have very different duals that have no obvious relationship with one another.

Reformulating primal problem may be helpful when the original dual is difficult.

Some trivial ways to reformulate:

- new variables and equality constraints so that new variables equal stuff in the original objective
- capturing constraints through the domain of the objective (explicit constraints $\rightarrow$ implicit constraints)

Problems with generalized inequalities

Each inequality is now wrt cone K_i.

Dual feasible if $\lambda \succeq_{K_i^*} 0$ (dual cones).

Everything that applied to regular inequality case also applies to generalized inequality case.

This is just a semidefinite program.