$rt - s^2 = 12 - 9$
$= 3 > 0.$

2] $f(x,y) = 2x^4 + 3x^2y - y^2$

$f_x = 8x^3 + 6xy$
$f_y = 3x^2 - 2y$

Now,
$f_x = 0$
$\therefore 8x^3 + 6xy = 0$
$2x(4x^2 + 3y) = 0$
$4x^2 + 3y = 0$ — ①

$f_y = 0$
$3x^2 - 2y = 0$
$3x^2 - 2y = 0$ — ②

① × 3
② × 4
① − ②

$12x^2 + 18y = 0$
$- 12x^2 - 8y = 0$

$24y = 0$
$y = 0$ — ③

Substituting ③ in ②
$3x^2 - 2(0) = 0$
$3x^2 = 0$
$x = 0$ — ④

Critical points are $(0,0)$
Now,
$r = f_{xx} = 24x^2 + 6y$
$t = f_{yy} = -2$
$s = f_{xy} = 6x$

$rt - s^2 = (24x^2 + 6y)(-2) - (6x)^2$
$= -48x^2 - 12y - 36x^2$
$= -84x^2 - 12y$

3] $f(x,y) = x^2 - y^2 + 2x + 8y - 70$
$f_x = 2x + 2$
$f_y = -2y + 8$

$f_x = 0$
$2x + 2 = 0$
$2(x+1) = 0$
$\boxed{x = -1}$

$f_y = 0$
$-2(y-4) = 0$
$\boxed{y = 4}$

Critical points are $(-1,4)$

$\therefore r = f_{xx} = 2$
$t = f_{yy} = -2$
$s = f_{xy} = 0$

Calculus

Practical 01

1] $\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

$\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]\times\left[\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+2\sqrt{x}}\right]$

$\lim_{x\to a}\frac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a-3x)(\sqrt{a+2x}+\sqrt{3x})}$

$\frac{1}{3}\lim_{x\to a}\frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(a-x)(\sqrt{a+2x}+\sqrt{3x})}$

$\frac{1}{3}\times\frac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$\frac{1}{3}\,\frac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$\frac{1}{3}\,\frac{\cancel{4}^{2}\sqrt{a}}{\cancel{2}\sqrt{3a}}$

$=\frac{2}{3\sqrt{3}}$

2] $\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$=\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\times\frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$

$=\lim_{y\to 0}\frac{y}{y\sqrt{a+y}\,(\sqrt{a+y}+\sqrt{a})}$

$=\frac{1}{\sqrt{a+0}\,(\sqrt{a+0}+\sqrt{a})}$

$=\frac{1}{\sqrt{a}\,(2\sqrt{a})}$

$=\frac{1}{2a}$

3] $\lim_{x\to \pi/6}\frac{\cos x-\sqrt{3}\sin x}{\pi-6x}$

By substituting $x-\frac{\pi}{6}=h$

$x=\frac{\pi}{6}+h$

where $h\to 0$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{6}\right) - \sqrt{3}\sin\left(h+\frac{\pi}{6}\right)}{\pi - 6\left(h+\frac{\pi}{6}\right)}$$

$$\lim_{h\to 0} \frac{\cos h \cdot \cos\frac{\pi}{6} - \sin h \sin\frac{\pi}{6} - \sqrt{3}\left(\sin h \cos\frac{\pi}{6} + \cos h \sin\frac{\pi}{6}\right)}{\pi - 6\left(\frac{6h+\pi}{6}\right)}$$

$$\lim_{h\to 0} \frac{\cos h \cdot \frac{\sqrt{3}}{2} - \frac{\sin h}{\sqrt{2}} - \sqrt{3}\left(\sin h \frac{\sqrt{3}}{2} + \cos h \frac{1}{\sqrt{2}}\right)}{\pi - (h+\pi)}$$

$$\lim_{h\to 0} \frac{\cos \frac{\sqrt{3}h}{2} - \frac{\sin h}{2} - \sin\frac{3h}{2} - \cos\frac{\sqrt{3}h}{2}}{-6h}$$

$$\lim_{h\to 0} \frac{-\sin\frac{4h}{2}}{-6h}$$

$$\lim_{h\to 0} \frac{\sin 4h}{\sin h}$$

$$\frac{1}{3} \lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{3}$$

(3) $f(x) = \frac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \le \frac{\pi}{2}$
$= \frac{\cos x}{\pi - 2x}$, for $\frac{\pi}{2} < x < \pi$ } at $x = \frac{\pi}{2}$

$$f\left(\tfrac{\pi}{2}\right) = \frac{\sin 2\left(\frac{\pi}{2}\right)}{\sqrt{1-\cos 2\left(\frac{\pi}{2}\right)}} \qquad \therefore f\left(\tfrac{\pi}{2}\right) = 0$$

f at $x = \frac{\pi}{2}$ define

(ii) $\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\pi/2} \frac{\cos x}{\pi - 2x}$

By substituting method,

[illegible] $x - \frac{\pi}{2} = h$

$x = h + \frac{\pi}{2}$

where $h \to 0$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{\pi - 2\left(h+\frac{\pi}{2}\right)}$$

$$\lim_{h\to 0} \frac{\cos\left(h+\frac{\pi}{2}\right)}{-2h}$$

$$\lim_{h\to 0} \frac{\cos h \cdot \cos\frac{\pi}{2} - \sin h \sin\frac{\pi}{2}}{-2h}$$

$$\lim_{h \to 0} \frac{\cos h \cdot 0 - \sin h}{-2h}$$

$$\lim_{h \to 0} \frac{-\sin h}{-2h}$$

$$= \frac{1}{2}$$

4) $\lim_{x \to \pi/2}$ $f(x) = \lim_{x \to \pi/2} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$

$$\lim_{x \to \pi/2} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$$

$$\lim_{x \to \pi/2^-} \frac{2\cos x \sin x}{\sqrt{2}\sin x}$$

$$\lim_{x \to \pi/2^-} \frac{2\cos x}{\sqrt{2}}$$

$$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2} \cos x$$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ f is not continuous at $x = \pi/2$

5) $f(x) = \frac{x^2-9}{x-3}$ $\quad 0 < x < 3$

$= x+3$ $\quad 3 \le x < 6$

$= \frac{x^2-9}{x-3}$ $\quad 6 \le x < 9$.

For $f(x) = 3$ check continuity at $x = 3$

L.H.L = R.H.L

L.H.L $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)}$

$= \lim_{x \to 3^-} x+3 = 3+3 = 6$

L.H.L = 6.

R.H.L $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} x+3 = 3+3 = 6$

L.H.L = R.H.L

$\therefore$ Funct^n is continuous at $x = 3$.

for $f(x) = 6$ to check the continuous at $x = 6$.

L.H.L $\Rightarrow \lim_{x \to 6^-} f(x) = \lim_{x \to 6} (x+3) = 6+3 = 9$

L.H.L = 9

R.H.L $= \lim_{x \to 6^+} f(x) = \lim_{x \to 6} \frac{x^2-9}{x-3} = \frac{36-9}{6-3}$

$= \frac{27}{3} = 9$

R.H.L = 9.

L.H.L = R.H.L

The funct^n continues at $x = 9$.

8] $f(x) = \dfrac{1-\cos 4x}{x^2}$, $x < 0$ } at $x = 0$

$= k$, $x = 0$

$\rightarrow$ f(x) is continuous at $x = 0$

$\therefore f(0) = \lim_{x \to 0}$

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} \dfrac{1-\cos 4x}{x^2} = k$

$\lim_{x \to 0} \dfrac{2\sin^2 2x}{x^2} = k$

$2\lim_{x \to 0} \dfrac{\sin^2 x}{x^2} = k$

$2\lim_{x \to 0} \left(\dfrac{\sin 2x}{x}\right)^2 = k$

$2(2)^2 = k$

$\therefore k = 8$

9] $f(x) = (\sec^2 x)^{\cot^2 x}$, $x \neq 0$ } at $x = 0$

$= k$, $x = 0$

$\rightarrow f(x) = (\sec^2 x)^{\cot^2 x}$

Using $\tan^2 x - \sec^2 x = 1$

$\therefore \sec^2 x = 1 + \tan^2 x$

$\cot^2 x = \dfrac{1}{\tan^2 x}$

$\therefore \lim_{x \to 0} (1 + \tan^2 x)^{\frac{1}{\tan^2 x}}$

Using $\lim_{x \to 0} (1 + px)^{\frac{1}{px}} = e$

$\therefore k = e$

10] $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$, $x \neq \pi/3$ } $x = \frac{\pi}{3}$

$= k$, $x = \pi/3$

$x - \frac{\pi}{3} = h$

$x = h + \frac{\pi}{3}$

when $h \to 0$

$f\left(\frac{\pi}{3} + h\right) = \dfrac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3\left(\tan\frac{\pi}{3} + h\right)}$

$\lim_{h \to 0} \dfrac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$

$$\lim_{h\to 0} \frac{\dfrac{\sqrt{3} - \tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3}\cdot\tan h}}{\cancel{\pi} - \cancel{\pi} - 3h}$$

$$\lim_{h\to 0} \frac{\dfrac{\sqrt{3}(1 - \tan\frac{\pi}{3}\tan h) - (\tan\frac{\pi}{3} + \tan h)}{1 - \tan\frac{\pi}{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3}\times\sqrt{3} - \tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3 - \tan h) - (\sqrt{3} + \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\dfrac{(\sqrt{3} - 3\tan h - \sqrt{3} - \tan h)}{1 - \sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3h(1 - \sqrt{3}\tan h)}$$

33

$$\lim_{h\to 0} \frac{4\tan h}{3h(1 - \sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h} \lim_{h\to 0} \frac{1}{(1 - \sqrt{3}\tan h)}$$

$$= \frac{4}{3} \frac{1}{(1 - \sqrt{3}(0))}$$

$$= \frac{4}{3}$$

Q) $f(x) = \dfrac{1 - \cos 3x}{x\tan x}$, $x \neq 0$ $\Big\}$ $x = 0$

$\quad\;\; = 9$, $x = 0$

$$f(x) = \frac{1 - \cos 3x}{x\tan x}$$

$$\lim_{x\to 0} \frac{2\sin^2 3/2\, x}{x\tan x}$$

$$\lim_{x\to 0} \frac{\dfrac{2\sin^2 \frac{3x}{2}}{x^2} \times x^2}{\dfrac{x\cdot\tan x}{x^2} \times x^2}$$

$$= 2\lim_{x\to 0} \left(\frac{3}{2}\right)^2 = \frac{2\times 9}{2\times 2} = \frac{9}{2}$$

$f(x) = \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$, $x \neq \pi/2$

$f(x)$ is continuous at $x = \pi/2$

$$= \lim_{x\to\pi/2} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$= \lim_{x\to\pi/2} \frac{2-1-\sin x}{\cos^2 x\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to\pi/2} \frac{\cancel{\ }\ (1-\sin x)}{\cos^2 x\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to\pi/2} \frac{1-\sin x}{(1-\sin^2 x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to\pi/2} \frac{\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}(1+\sin x)\left(\sqrt{2}+\sqrt{1+\sin x}\right)}$$

$$= \lim_{x\to\pi/2} \frac{1}{1+\sin x} \times \lim_{x\to\pi/2} \frac{1}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$= \lim_{x\to\pi/2} \frac{1}{1+\sin 90} \times \frac{1}{\sqrt{2}+\sqrt{1+\sin\pi/2}}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2(2\sqrt{2})}$$

$\cot x$

$f(x) = \cot x$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\cot x - \cot a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x-a}$$

$= \lim_{x\to a}$ $x - a = h$

$x = a + h$ $\because x \to a,\ h \to 0$

$$= \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{(a+h-a)\tan(a+h)\tan a}$$

$$= \lim_{h\to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h\to 0} \frac{(\tan a - h) - (1+\tan a\tan(a+h))}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h\to 0} -\frac{\tan h}{h} \times \frac{1+\tan a\tan(a+h)}{\tan(a+h)\tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a} = \frac{-\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cos^2 a} \times \frac{\cos^2 a}{\sin^2 a}$$

$$= -\operatorname{cosec}^2 a$$

$$Df(a) = -\operatorname{cosec}^2 a$$

$\therefore f$ is differentiable $\forall a \in \mathbb{R}$.

(ii) cosec x

$$f(x) = \operatorname{cosec} x$$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sin a - \sin x}{(x-a) \sin a \sin x}$$

Put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$

$$Df(a) = \lim_{h \to 0} \frac{\sin a - \sin x}{(x-a) \sin a \sin x}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{a+a+h}{2}\right) \times \sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \sin(a+h)}$$

$$= \lim_{h \to 0} \frac{-\sin\frac{h}{2}}{\frac{h}{2}} \times \frac{1}{2} \times \frac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \sin(a+h)}$$

$$= -1/2 \times \frac{2\cos\left(\frac{2a+0}{2}\right)}{\sin(a+0)}$$

$$= \frac{-\cos a}{\sin^2 a} = -\cot a \operatorname{cosec} a.$$

(iii) sec x.

$$f(x) = \sec x$$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a) \cos a \cos x}$$

$x - a = h$

$x = a + h$, $\because x \to a$, $h \to 0$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right) \cdot \sin\left(\frac{a-a-h}{2}\right)}{h \cos a \cos(a+h)}$$

$$= \frac{[illegible] \sin\left(-\frac{h}{2}\right) \cdot -\frac{1}{2}}{\cos a \cos(a+h)\left(-\frac{h}{2}\right)}$$

$$= \frac{-2\sin\left(\frac{2a+0}{2}\right)}{\cos a \cos(a+h)\left(-\frac{h}{2}\right)}$$

$$= -\frac{1}{2} \times -2 \frac{\sin\left(\frac{2a}{2}\right)}{\cos a \cos(a)}$$

$$= -\frac{1}{2} \times -2 \frac{\sin a}{\cos a \cos a}$$

$= \tan a \sec a.$

Q 2] $f(x) = 4x+1, \quad x \leq 2$

$= x^2+5, \quad x > 2 \quad$ at $x = 2.$

L.H.D

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-(4 \times 2+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4(x-2)}{x-2} = 4$$

R.H.D.

$$Df(2^+) = \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} (x+2)$$

$= 4.$

L.H.D = R.H.D

f is differentiable at $x = 2.$

Q3] $f(x) = 4x+7, \quad x < 3$

$= x^2+3x+1, \quad x \geq 3 \quad$ at $x = 2.$

R.H.D

$$= \lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x+1-(9+9+1)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x(x+6)-3(x+6)}{(x-3)}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}} = 9.$$

38

$= \lim_{x \to 2^+} \dfrac{3x(x-2) + 2(x-2)}{x-2}$

$= \lim_{x \to 2^+} \dfrac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= \lim_{x \to 2^+} 8.$

L.H.D = R.H.D

$\therefore$ f is diff. at $x = 2$.

Q8
(i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

let $f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0$

$\therefore 2x^4 - 32 = 0$

$\frac{32}{2} = x^4$

$\therefore x^4 = 16$

$\boxed{x = \pm 2}$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{16}$

$f''(2) = 8 > 0$

$\therefore f$ has min. value at $x = 2$.

$f''(-2) = 2 + \frac{96}{x^4} = 2 + \frac{96}{16}$

$= 8$

$\therefore f$ has min values at $x = -2$.

$f(2) = 4 + \frac{16}{4}$

$= 8$

$f(-2) = 4 + \frac{16}{4}$

$= 8.$

i) $f(x) = 2 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Let $f'(x) = 0$

$-15x^2 + 15x^4 = 0$

$x^2 = x^4$

$x^4 - x^2 = 0$

$x^2(x^2 - 1) = 0$

$\therefore x^2 = 0$ or $x^2 = 1$

$x = 0$ or $x = \pm 1$

$\therefore f''(1) = -30 + 60$; $f''(x) = -30x + 60x^3$

$= 30 > 0$

$\therefore f$ is min. at $x = 1$

$\therefore f''(-1) = -30(-1) + 60(-1)^3$

$= -30 < 0$

f is min at $x = -1$.

$\therefore f(1) = 2 - 5 + 3$

$\boxed{f(1) = 0}$ — Minimum value

$\therefore f(-1) = 2 + 5 - 3$

$\boxed{f(-1) = 4}$ — Max. value

ii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x = 0$

Let $f'(x) = 0$

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$ — Maximum at 0.

$f''(2) = 12 - 6$

$f''(2) = 6 > 0$ — Minimum at 2.

$f(0) = 1$ — Maximum value.

$f(2) = 8 - 3(4) + 1$

$f(2) = -3$ — Minimum value.

iii) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

Let $f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$\therefore x^2 - x - 2 = 0$

$\therefore x^2 - 2x + x - 2 = 0$

$\therefore x(x - 2) + (x - 2) = 0$

$\therefore (x - 2)(x + 1) = 0$.

$\therefore x = 2$ or $x = -1$

$f''(x) = 12x - 6$

$f''(2) = 18 > 0$ Min. at 2.

$f''(-1) = -18 < 0$ Max. at -1.

$f(2) = 16 - 12 - 24 + 1$

$= 16 - 36 + 1$

$= -19$ — Minimum value.

$f(-1) = -2 - 3 + 12 + 1$

$= 8$ — Max. value.

Q5]

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ $x_0 = 0$ --- given

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 + \frac{9.5}{55}$$

$\therefore x_1 = 0.1727$

$f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0896 - 9.4985 + 9.5$

$= -0.0829$

$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0896 - 1.0362 - 55$

$= -55.9467$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{55.9467}$$

$= 0.1712$.

3. $f(x) = x^3 - 4x - 9$ $[2,3]$

$f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ be the initial approx^n

$\therefore$ By newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$= 2.7392$

$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$

$= 20.5528 - 10.9568 - 9$

$= 0.596$

$f'(x_1) = 3(2.7392)^2 - 4$

$= 22.5096 - 4$

$= 18.5096$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7392 - \frac{0.596}{18.5096}$$

$= 2.7071$

$f(x_0) = (2.7071)^3 - 4(2.7071) - 9$
$= 19.8256 - 10.8284 - 9$
$= 0.0102$

$f'(x_0) = 3(2.7071)^2 - 4$
$= 21.9851 - 4$
$= 17.9851$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
$= 2 - \dfrac{2.2}{5.2}$
$= 2 - 0.4230$
$= 1.577$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$
$= 3.9219 - 4.4764 - 15.77 + 17$
$= 0.6785$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$
$= 7.4608 - 5.6772 - 10$
$= -8.21644$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1.577 + 0.0822$
$= 1.6592$

$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$
$= 4.5677 - 4.9553$
$= 0.0204$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$
$= -7.7163$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$
$= 1.6592 + \dfrac{0.0204}{7.7163}$
$= 1.6592 + 0.0026$
$= 1.6618$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$
$= 4.5892 - 4.9708 - 16.618 + 17$
$= 0.00004$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$
$= 1.6618 + \dfrac{0.0004}{7.6977}$
$= 1.6618$

$\therefore$ The root of eqn is 1.6618

Practical 6.

Answers.

i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$I = \int \frac{dx}{\sqrt{x^2+2x-3}}$

$= \int \frac{dx}{\sqrt{x^2+2x+1-4}}$

$I = \int \frac{dx}{\sqrt{(x+1)^2-(2)^2}}$

Comparing with $\int \frac{dx}{\sqrt{x^2-a^2}}$; $x^2 = (x+1)^2$, $a^2 = 2^2$

$I = \log|x+\sqrt{x^2-a^2}| + C$

$= \log|x+1+\sqrt{(x+1)^2-2^2}| + C$

(ii) $\int (4e^{3x}+1)\,dx$

$I = \int (4e^{3x}\,dx) + \int dx$

$= \int 4e^{3x}\,dx + \int 1\,dx$

$= \frac{4e^{3x}}{3} + x + C$

iii) $\int (3x^2 - 3\sin x + 5\sqrt{x})\,dx$

$I = \int (3x^2 - 3\sin x + 5\sqrt{x})\,dx$

$= 3\int x^2\,dx - 3\int \sin x + 5\int \sqrt{x}\,dx$

$= \frac{3x^3}{3} + 3\cos x + \frac{5x^{3/2}}{3/2}$

$= \frac{3}{3}x^3 + 3\cos x + \frac{10}{3}x^{3/2} + C$

iv) $\int \frac{x^2+3x+4}{\sqrt{x}}\,dx$

$= \int \frac{x^2+3x+4}{\sqrt{x}}\,dx$

$= \int \left(\frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right) dx$

$= \int x^2x^{-1/2} + 3x \cdot x^{-1/2} + 4x^{-1/2}\,dx$

$= \int x^{3/2}\,dx + 3\int x^{1/2} + 4\int x^{-1/2}\,dx$

$= \frac{2}{5}x^{5/2} + 3 \cdot \frac{2}{3}x^{3/2} + 4x^{1/2} \cdot 2 + C$

x) $\int t^7 \sin(2t^4)\, dt$

$I = \int t^7 \sin(2t^4)\, dt$

$t^4 = x$

$4t^3 \frac{dt}{dx} = dx$

$I = \frac{1}{4}\int 4t^3 \cdot t^4 \cdot \sin(2t^4)\, dt$

$= \frac{1}{4}\int x \cdot \sin(2x)\, dx$

$= \frac{1}{4}\left[x\int \sin 2x - \int\left[\int \sin 2x \frac{dx}{dx}\right]\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x - 1\right]$

$= -\frac{1}{8} x\cos 2x + \frac{1}{16}\sin 2x + C$

$x = t^4$

$\therefore I = -\frac{1}{8} t^4\cos(2t^4) + \frac{1}{16}\sin(2t^4) + C$

xi) $\int \sqrt{x}\,(x^2 - 1)\, dx$

$I = \int \sqrt{x}\,(x^2 - 1)\, dx$

$= \int (\sqrt{x} \cdot x^2 - \sqrt{x})\, dx$

$= \int x^{1/2} \cdot x^2\, dx - \int x^{1/2}\, dx$

$= \int x^{5/2} - \frac{2}{3}(x^{3/2})$

$= \frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2}$

xii) $I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$= \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

Let,

$\frac{1}{x^2} = t \quad \therefore t = x^{-2}$

$\therefore -\frac{2}{x^3}\, dx = dt$

$I = -\frac{1}{2}\int -\frac{2}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$= -\frac{1}{2}\int \sin t\, dt$

$= -\frac{1}{2}\cos t + C$

$\because \boxed{t = 1/x^2}$

$\therefore I = 1/2 \cos\left(\frac{1}{x^2}\right) + C$

vii) $I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

$\sin x = t$

$dt = \cos x \, dx$

$I = \int \frac{dt}{t^{2/3}}$

$= \int t^{-2/3} \, dt$

$= 3t^{1/3} + C$

$= 3\sqrt[3]{\sin x} + C.$

(ix) ~~[illegible]~~ $\int e^{\cos^2 x} \sin 2x \, dx$

$\cos^2 x = t$

$dt = -2\cos x \sin x$

$dt = -2\sin 2x \, dx$

$I = -\frac{1}{2}\int e^t \, dt$

$= -\frac{1}{2} e^t + C$

$\therefore \underline{I} = -\frac{e^{\cos^2 x}}{2} + c$

x) $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

$x^3 - 3x^2 + 1 = t$

$\therefore (3x^2 - 6x)dx = dt$

$3(x^2 - 2x)dx = dt$

$(x^2 - 2x)dx = \frac{dt}{3}$

$\therefore I = \int \frac{1}{3t} \, dt$

$= \frac{1}{3}\int \frac{dt}{t}$

$= \frac{1}{3}\log t + c$

$= \frac{1}{3}\log(x^3 - 3x^2 + 1) + C$

Practical No. 3.

Topic: Applicatⁿ of derivative

Q)

1) $f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

For interval of increasing funcⁿ

$f'(x) > 0$

$3x^2 - 5 > 0$

$(\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5}) > 0$

$+ \quad - \quad +$

$-\frac{\sqrt{5}}{\sqrt{3}} \qquad \frac{\sqrt{5}}{\sqrt{3}}$

$x \in \left(-\infty, -\frac{\sqrt{5}}{\sqrt{3}}\right) \cup \left(\frac{\sqrt{5}}{\sqrt{3}}, \infty\right)$

For interval when f is decreasing

$f'(x) < 0$

$3x^2 - 5 < 0$

$(\sqrt{3}x - \sqrt{5})(\sqrt{3}x + \sqrt{5}) < 0$

$+ \quad - \quad +$

$-\frac{\sqrt{5}}{\sqrt{3}} \qquad \frac{\sqrt{5}}{\sqrt{3}}$

$x \in \left(-\frac{\sqrt{5}}{\sqrt{3}}, \frac{\sqrt{5}}{\sqrt{3}}\right)$

2) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$.

For interval of f is increasing funcⁿ.

$f'(x) > 0$

$2x - 4 > 0$

$2(x-2) > 0$

$x - 2 > 0$

$-\infty \quad 2 \quad \infty$

f is increasing when $x \in (2, \infty)$

for f is a decreasing funcⁿ.

$f'(x) < 0$

$2x - 4 < 0$

$x - 2 < 0$

where $x \in (-\infty, 2)$ where funcⁿ is decreasing.

$f(x) = 2x^3 + x^2 - 20x + 4.$

$f'(x) = 6x^2 + 2x - 20.$

for f when it is an increasing funct^n

$f'(x) > 0.$

$6x^2 + 2x - 20 > 0.$

$3x^2 - x - 10 > 0$

$3x^2 - 6x + 5x - 10 > 0$

$3x(x-2) + 5(x-2) > 0$

$(3x+5)(x-2) > 0$

$+ \quad -\frac{5}{3} \quad - \quad 2 \quad +$

$x \in \left(-\infty, -\frac{5}{3}\right) \cup (2, \infty)$

for f when its decreasing.

$f'(x) < 0$

$6x^2 - 2x - 20 < 0$

$3x^2 - x - 20 < 0$

$3x(x-2) + 5(x-2) < 0$

$(3x+5)(x-2) < 0$

where $x \in \left(-5/3, 2\right)$

$f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

when f is an increasing f

$f'(x) > 0$

$3x^2 - 27 > 0$

$(x-3)(x+3) > 0$

$+ \quad -3 \quad - \quad 3$

∴ for inc. funct^n $x \in (-\infty, -3) \cup (3, \infty)$

when f is an decreasing funct^n.

$f'(x) < 0$

$3x^2 - 27 < 0$

$(x-3)(x+3) < 0$

$-3 \quad 3$

f is decreasing when $x \in (-3, 3)$

Q.3) 5) $y = 3x^2 - 2x^3$

$f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

Interval of concave upward

$f''(x) > 0$

$-12x + 6 > 0$

$-2x + 1 > 0$

$x > 1/2$

$\therefore x \in (1/2, \infty)$

For concave downward.

$f''(x) < 0$

$-12x + 6 < 0$

$x < 1/2$

$x \in (-\infty, 1/2)$.

6) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

Let $f(x) = y$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

Interval of concave upward

$f''(x) > 0$

$12x^2 - 36x + 24 > 0$

$x^2 - 3x + 2 > 0$

$x^2 - 2x - x + 2 > 0$

$x(x-2) - 1(x-2) > 0$

$(x-1)(x-2) > 0$

+ − +

1 2

$x \in (-\infty, 1) \cup (2, \infty)$

For concave downward

$f''(x) < 0$

$(x-1)(x-2) < 0$

1 − 2

$x \in (1, 2)$

7) $y = x^3 - 27x + 5$

Let $f(x) = y$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

Concave upward,

$f''(x) > 0$

$6x > 0$

$x > 0$

$x \in (0, \infty)$ in concave upward.

For concave downward

$f''(x) < 0$

$6x < 0$

$x < 0$

$x \in (-\infty, 0)$.

4) $y = 2x^3 + x^2 - 20x + 4$

Let $f(x) = y$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

Concave upward

$f''(x) > 0$

$12x + 2 > 0$

$x > 1/6$

$\therefore x \in (1/6, \infty)$

& for concave downward

$f''(x) < 0$

$12x + 2 < 0$

$x < -1/6$

$\therefore x \in (-\infty, -1/6)$ for concave downwards.

Practical No. 6.

Answers:

1) $x = t - \sin t$ $\quad y = 1 - \cos t$ $\quad t \in [0, 2\pi]$

$$I = \int_0^{b} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$\frac{dy}{dt} = \sin t$ $\qquad \frac{dx}{dt} = 1 - \cos t$

$$I = \int_0^{2\pi} \sqrt{(\sin t)^2 + (1 - \cos t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + 1 + \cos^2 t - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - \cos t}\, dt$$

$$= \sqrt{2} \int_0^{2\pi} \sqrt{\cancel{1 - \cos t}\ 2\sin^2\left(\frac{t}{2}\right)}\, dt$$

$= \sqrt{2} \cdot \sqrt{2}$

$$= 2 \times 2 \left[-\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$$

$= -4[-1 - 1]$

$= 8$

④ $y = \sqrt{4-x^2}$

$$I = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1(-2x)}{2\sqrt{4-x^2}}$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$x \in [-2, 2]$

$$\therefore I = \int_{-2}^{2} \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$$

$$= \int_{-2}^{2} \sqrt{1+\frac{x^2}{4-x^2}}$$

$$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\int_{-2}^{2} \sqrt{\frac{1}{(2)^2-(x)^2}}\, dx$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 2\left[\pi/2 - (-\pi/2)\right]$$

$$= 2\pi$$

⑤ $y = x^{3/2}$

$$I = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

$$I = \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx$$

$$= 1/2 \left[\frac{4+9x}{1/2+1}\right]_0^4 dx$$

$$= \frac{1}{27}\left[(4+9x)^{3/2}\right]_0^4$$

$$= 1/27\left[(4+0)^{3/2} - (4+36)^{3/2}\right]$$

$$= \frac{1}{27}\left[4^{3/2} - 40^{3/2}\right]$$

4] $x = 3\sin t \qquad y = 3\cos t$

$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t$

$I = \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$

$= 3\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\ dt$

$= 3\int_0^{2\pi} 1$

$= 3\,[x]_0^{2\pi}$

$= 3(2\pi - 0)$

$= 6\pi$

5] $x = \frac{1}{6}y^3 + \frac{1}{2y} \qquad y \in [1, 2]$

$I = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$

$\frac{dx}{dy} = \frac{3y^2}{6} - \frac{1}{2y^2}$

$I = \int_1^2 \sqrt{1 + \left(\frac{1}{2}y^2 - \frac{1}{2y^2}\right)^2}\, dy$

$= \int_1^2 \sqrt{1 + \left(\frac{y^4 - 1}{2y^2}\right)^2}\ dy$

$= \int_1^2 \sqrt{\frac{(y^4-1)^2 + 4y^4}{4y^4}}\ dy$

$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\ dy$

$= \int_1^2 \frac{y^4+1}{(2y^2)}\ dy$

$= \int_1^2 \frac{y^2}{2}\ dy + \int_1^2 \frac{1}{2y^2}\ dy$

$= \frac{1}{2}\left[\frac{y^3}{3}\right]_1^2 + \int_1^2 \frac{1}{2y^2}\ dy$

$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{3}\right] + \frac{1}{2}\left[\frac{1}{2} - 1\right]$

$= 7/6 - 1/4$

$= \frac{14 - 3}{12}$

$= \frac{11}{12}$

Q.1)
(a) $\int_0^2 e^{x^2} dx$ with $n=4$

$a=0, b=2, n=4.$

$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

x	0	0.5	1	1.5	2
y	1	1.284	2.718	9.48	54.5981

By ~~Simpson~~ Simpson's Rule.

$$\int_0^2 e^{x^2} dx = \frac{0.5}{3}\left[(1+54.5981)+4(1.2840+9.4877) + 2(2.7182)\right]$$

$= 17.3535$

(b) $\int_0^4 x^2\, dx$ with $n=4$

$a=0, b=4, n=4.$

$h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

x	0	1	2	3	4
y	0	1	4	9	16

By Simpson's Rule.

$$\int_0^4 x^2 dx = \frac{1}{3}\left[(0+16)+4(1+9)+2(4)\right]$$

$= \frac{1}{3}(16+40+8)$

$= \frac{1}{3} \times 64$

$= \frac{64}{3}$

(c) $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ $n=6$

$a=0, b=\frac{\pi}{3}, n=6$

$h = \frac{\pi}{3\times6} = \frac{\pi}{18}$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$
y	0	0.4167	0.5848	0.7071	0.8017	0.8752

By Simpson's Rule.

$$\int_0^{\pi/3} \sqrt{\sin x} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)\right]$$

$= \frac{\pi}{54} \times 11.6996$

$= 0.6806$

$\int_0^{\pi/3} \sqrt{\sin x} = 0.6806$

Practical No. 7

(i) $x\frac{dy}{dx} + y = e^x \quad \therefore \frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$P(x) = \frac{1}{x} \quad ; \quad Q(x) = \frac{e^x}{x}$

$I.F. = e^{\int 1/x \, dx}$

$= e^{\log x}$

$I.F. = x$

$\therefore yx = \int \frac{e^x}{x} x \, dx$

$\boxed{yx = e^x + c}$

ii) $e^x\frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$P(x) = 2 \; ; \; Q(x) = \frac{1}{e^x}$

$\therefore I.F. = e^{\int 2 dx}$

$= e^{2x}$

$y e^{2x} = \int \frac{1}{e^x} e^{2x} \, dx$

$ye^{2x} = \int e^x dx$

$ye^{2x} = e^x + C$

$x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$

$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{\cos x}{x^2}$

$\therefore P(x) = 2/x \; ; \; Q(x) = \cos x / x^2$

$I.F. = e^{\int \frac{2}{x} dx}$

$= e^{2\log x}$

$I.F. = x^2$

$yx^2 = \int \frac{\cos x}{x^2} \cdot x^2 \, dx$

$yx^2 = \sin x + c$

④ $x\frac{dy}{dx} + 2y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^3}$

$\therefore I.F. = e^{\int \frac{2}{x} dx}$

$= e^{3\log x}$
$= x^3$

$yx^3 = \int \frac{\sin x}{x^3} \cdot x^3 \, dx$

$yx^3 = -\cos x + C$

vi) $\sec^2 x \cdot \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$\frac{\sec^2 x}{\tan x} dx = \frac{-\sec^2 y}{\tan y}$

$\therefore \int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y}$

$\therefore \log|\tan x| = -\log|\tan y| + C$

$\log|\tan x| + \log|\tan y| = C$

$\log|\tan x \cdot \tan y| = C$

$\boxed{\tan x \cdot \tan y = e^C}$

51

vii) $\frac{dy}{dx} = \sin^2(x-y+1)$

$x-y+1 = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = 1 - \frac{dv}{dx}$

$1 - \frac{dv}{dx} = \sin^2 v$

$dx = \frac{dv}{1-\sin^2 v}$

$\int x = \int \sec^2 v \, dv$

$x = \tan v + c$

But $v = x+y-1$

$\therefore x = \tan(x+y-1) + c$

viii) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

$\frac{dy}{dx} = \frac{2x+3y-1}{3(2x+3y+2)}$

Put $2x+3y = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{v-1}{3(v+2)}$

$\frac{dv}{dx} = \frac{3v+3}{v+2}$

$\frac{v+2\,dv}{3(v+1)} = dx$

$\frac{1}{3}\int \frac{v+2}{v+1}\,dv = \int dx$

$\frac{1}{3}\int 1 + \frac{1}{v+1}\,dv = \int dx$

$\frac{1}{3}\left(v + \log(v+1)\right) = x + c$

$v = 2x + 3y$

$\therefore 2x + 3y + \log|2x+3y+1| = 3x + c$

$\therefore 3y = x - \log|2x+3y+1| + c$

10/01/2022

8 Practical no 8. 51

Answers

1) $\frac{dy}{dx} = y + e^x - 2$

$f(x, y) = y + e^x - 2$, $y_0 = 2$, $x_0 = 0$, $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.4	3.5743
2	1	3.574	4.2	5.3616

Using Euler's Formula

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
3	1.5	5.3	7.8	9.2
4	2	9.2		

$\therefore$ By Euler's Formula,

$y(2) = 9.2531$

2) $\frac{dy}{dx} = 1 + y^2$

$f(x, y) = 1 + y^2$, $y_0 = 0$, $x_0 = 0$, $h = 0.2$

Using Euler's Formula,

2

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.16	[illegible]
3	0.6	0.64	1.0489	0.9[illegible]
4	0.8	0.929	1.85	1.2[illegible]
5	1	1.129		

∴ By Euler's Formula,
$y(1) = 1.296$

3] $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0)=1$, $x_0=0$, $h=0.2$

Using Euler's iterat^n formula,

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	0
1	0.2	0		
2	0.4			
3	0.6			
4	0.8			
5	1			

6) $\frac{dy}{dx} = 3x^2 + 1$ $y_0 = 2$, $x_0 = 1$, $h = 0.5$

for $h = 0.5$

Using Euler's iterat^n formula,

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	49	28.5
2	2	28.5		

By Euler's Formula,
$y(2) = 28.5$

for $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.68	4.42
2	1.5	4.42	7.75	6.35
3	1.75	6.35	10.18	8.90
4	2	8.90		

∴ By Euler's Formula,
$y(2) = 8.0948$

7] $\frac{dy}{dx} = \sqrt{xy} + 2$ $y_0 = 1$, $x_0 = 1$, $h = 0.2$

Using Euler's iterat^n formula,

$y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	1.6
1	1.2	1.6		

By Euler Formula,

$y(1.2) = 1.6$

Practical No:-9

87

Topic: Limits & Partial order derivatives

Q1. $\lim_{(x,y) \to (-4,-1)} \frac{x^3 - 3y + y^2 - 1}{xy + 5}$

At $(-4,-1)$, Denominator ≠ 0.

$\therefore = \frac{(-4)^3 - 3(-1) + (-1)^2 - 1}{-4(-1) + 5}$

$= \frac{-64 + 3 + 1 - 1}{4 + 5}$

$= \frac{-61}{9}$

Q2. $\lim_{(x,y) \to (2,0)} \frac{(y+1)(x^2 + y^2 - 4x)}{x + 3y}$

At $(2,0)$, Denominator ≠ 0

$\therefore$ By applying limit,

$= \frac{(0+1)(2^2 + 0 - 4(2))}{2}$

$= -4/2$

$= -2$

Q3. $\lim_{(x,y,z) \to (1,1,1)} \frac{x^2 - y^2}{[illegible]}$

At $(1,1,1)$, Denominator ≠ 0.

$\therefore \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2}{x^2-y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2}{x^2-y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{(x+y)(x-y-z)}{(x-y-z)}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{xyz}{x^2}$

On applying limit,

$= \frac{1+1(1)}{1^2}$

$= 2$

Q.2)

i) $f(x,y) = xye^{x^2+y^2}$

$\therefore f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(xye^{x^2+y^2})$

$= ye^{x^2+y^2}(2x)$

$\therefore f_x = 2xye^{x^2+y^2}$

$\therefore f_y = \frac{\partial}{\partial y}(f(x,y))$

5

$= \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(xye^{x^2+y^2})$

$= xe^{x^2+y^2}(2y)$

$f_y = 2yxe^{x^2+y^2}$

ii) $f(x,y) = e^x \cos y$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(e^x \cos y)$

$f_y = xe^{x^2+y^2}(2y)$

iii) $f(x,y) = e^x \cos y$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(e^x \cos y)$

iv) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = 3x^2y^2 - 6xy$

$f_y = \frac{\partial}{\partial y}(x^3y^2 - 3x^2y + y^3 + 1)$

$f_y = 2x^3y - 3x^2 + 3y^2$

Q.3

i) $f(x,y) = \frac{2x}{1+y^2}$

$f(x) = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$

$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$

$= \frac{2+2y^2}{(1+y^2)^2}$

$= \frac{2(1+y^2)}{(1+y^2)(1+y^2)}$

$= \frac{2}{1+y^2}$

At $(0,0)$

$= 2$

$f_y = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$

$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)}$

53

$= \frac{(1+y^2)(0) - 2x(2y)}{(1+y^2)^2}$

$= \frac{1+y^2(0) - 2x(2y)}{}$

$= \frac{-4xy}{(1+y^2)^2}$

At $(0,0)$

$= \frac{-4}{(1+0)^2}$

$= -4$

ii) $f(x,y) = \frac{y^2 - xy}{x^2}$

$f(x) = \frac{x^2 \frac{\partial}{\partial x}(y^2 - xy) - (y^2 - xy)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$

$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$

$f_y = \frac{2y - x^2}{x^2}$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-x^2 y - 2x(y^2 - xy)}{x^4}\right)$

$= x^2/y$

$f_{yy} = \frac{\partial}{\partial y} \frac{(2y - x)}{x^2}$

$f_{yy} = \frac{2}{x^2}$

$f_{xy} = \frac{\partial}{\partial y} \left(\frac{-x^2 y - 2xy^2 + 2x^2 y}{x^4} \right)$

$f_{xy} = \frac{-x^2 y - x^2 - 2xy + 2x^2}{x^4}$

$f_{yx} = \frac{-x^2 - 4xy + 2x^2}{x^4}$

$\therefore f_{xy} = f_{yx}$

(x) $f(x,y) = x^3 + 3x^2 y^2 - \log(x^2+1)$

$f_x = 3x^2 + 6xy - \frac{2x}{x^2+1}$

$f_y = \frac{\partial}{\partial y}\left(x^3 + 3x^2 y^2 - \log(x^2+1)\right)$

$f_y = 6x^2 y$

$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1 \frac{\partial(2x)}{\partial x} - 2x \frac{\partial(x^2+1)}{\partial x}}{(x^2+1)^2} \right)$

$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2} \right)$ — ①

$f_{yy} = 6x^2$

$f_{xy} = 12xy$

$f_{yx} = 12xy$

$\therefore f_{xy} = f_{yx}$

(xi) $f(x,y) = \sin(x\,y) + e^{x+y}$

$f(x) = y\cos(xy) + e^{x+y}(1)$

$= y\cos(xy) + e^{x+y}$

$\therefore f(xx) = \frac{\partial}{\partial x}\left(y\cos(xy) + e^{x+y}\right)$

$= -y\sin(xy)\cdot y + e^{x+y}(1)$

$= -y^2\sin(xy) + e^{x+y}$

$f_{yy} = \frac{\partial}{\partial y}\left(x\cos(xy) + e^{x+y}\right)$

$= -x^2\sin(xy) + e^{x+y}$ — ②

$f_{xy} = -y^2\sin(xy) + \cos(xy) + e^{x+y}$ — ③

$f_{yx} = \frac{\partial}{\partial x}\left(x\cos(xy) + e^{x+y}\right)$

$f_{yx} = -x^2\sin(x\,y) + \cos(xy) + e^{x+y}$.

$\therefore f_{xy} \neq f_{yx}$

Q.3

1) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f(1,1) = \sqrt{1^2+1^2} = \sqrt{2}$

$f_x = \frac{x}{\sqrt{x^2+y^2}}$ $f_y = \frac{y}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$ f_y at $(1,1) = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \cancel{\sqrt{2} + \frac{1}{\sqrt{2}}}(x+y-2)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{2}{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

61

2) $f(x,y) = 1 - x + y\sin x$ at $(\frac{\pi}{2}, 0)$

$f(\frac{\pi}{2}, 0) = 1 - \frac{\pi}{2}$

$f_x = 0 - 1 + y\cos x$ $f_y = \sin x$

f_x at $(\frac{\pi}{2}, 0) = -1$ f_y at $(\frac{\pi}{2}, 0) = \sin\frac{\pi}{2} = 1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1 - \frac{\pi}{2} + (-1)(x - \frac{\pi}{2}) + 1(y)$

$L(x,y) = 1 - x + y$

3) $f(x,y) = \log x + \log y$

$f(1,1) = \log(1) + \log(1)$

$f_x = \frac{1}{x}$ $f_y = \frac{1}{y}$

f_x at $(1,1) = 1$ f_y at $(1,1) = 1$

$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= x - 1 + y - 1$

$= x + y - 2$

Practical No. 10

Q.1)

i) $f(x,y) = x+2y-3$, $a=(1,-1)$, $u=3i-j$

→ Here, $u = 3i - j$ is not a unit vector

$\bar{u} = 3i - j$

$|u| = \sqrt{10}$

∴ Unit vector along u is $\frac{\bar{u}}{|u|} = \frac{1}{\sqrt{10}}(3i - j)$

$= \frac{1}{\sqrt{10}}(3, -1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

Now,

$f(a+hu) = f\left((1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= f\left(1+\frac{3h}{\sqrt{10}}, -1-\frac{h}{\sqrt{10}}\right)$

$= 1+\frac{3h}{\sqrt{10}} + 2\left(-1-\frac{h}{\sqrt{10}}\right) - 3$

$= 1-2-3+\frac{3h}{\sqrt{10}} - \frac{2h}{\sqrt{10}}$

$= -4 + \frac{h}{\sqrt{10}}$

∴ $D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h\to 0} \frac{-4+\frac{h}{\sqrt{10}} - (-4)}{h}$

$= \lim_{h\to 0} \frac{h}{h\sqrt{10}}$

$= \frac{1}{\sqrt{10}}$

ii) $f(x,y) = y^2 - 4x + 1$, $a=(3,4)$, $u=i+5j$

→ Here, $u = i+5j$ is not a unit vector

$\bar{u} = i + 5j$

$|u| = \sqrt{26}$

∴ Unit vector along u is $\frac{\bar{u}}{|u|} = \frac{1}{\sqrt{26}}(i+5j)$

$= \frac{1}{\sqrt{26}}(1,5)$

$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

Now,

$f(a+hu) = f\left((3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right)$

$= f\left(3+\frac{h}{\sqrt{26}}, 4+\frac{5h}{\sqrt{26}}\right)$

$= 2x^2 - 3y + 1 = \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 3\left(3 + \frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{3h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$

$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5 - 5}{h}$

$= \lim_{h \to 0} \frac{\frac{25h^2}{26}}{h} + \frac{\frac{36h}{\sqrt{26}}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$

$= \lim_{h \to 0} \frac{25(0)}{26} + \frac{36}{\sqrt{26}}$

$= \frac{36}{\sqrt{26}}$

Q $f(x,y) = 2x + 3y$ $a = (1, 2)$, $u = 3i + 4j$

$u = 3i + 4j$ is not a unit vector

$|u| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

Unit vector along $u = \frac{u}{|u|} = \frac{(3i + 4j)}{\sqrt{25}}$

$= \frac{3i + 4j}{5}$

$= \frac{1}{5}(3, 4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

Now,

$f(a + hu) = f\left((1, 2) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right)$

$= f\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$

$= f\left(\frac{5 + 3h}{5}, \frac{10 + 4h}{5}\right)$

$f(x, y) = 2x + 3y$

$= 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= \frac{18h}{5} + 8$

$\therefore D_u f(a) = \lim_{h\to 0} \dfrac{f(a+hu) - f(a)}{h}$

$= \lim_{h\to 0} \dfrac{8 + \frac{12h}{5} - 8}{h}$

$= \lim_{h\to 0} \dfrac{12h}{5h}$

$\therefore D_u f(a) = \dfrac{12}{5}$

Q-4)

i) $f(x,y) = x^y + y^x$ $\quad a = (1,1)$

$f_x = y(x^{y-1}) + y^x \log y$

$f_y = x(y^{x-1}) + x^y \log x$

$\nabla f(x,y) = (f_x, f_y)$

$= (yx^{y-1} + y^x \log y,\ xy^{x-1} + x^y \log x)$

$\nabla f(x,y)$ at $(1,1)$

$= (1(1)^0 + 1^1 \log(1),\ 1(1)^{(1-1)} + 1^1 \log 1)$

$= (1,1)$

ii) $f(x,y) = (\tan^{-1}x)\cdot y^2$ $\quad a = (1,-1)$

$f_x = y^2\left(\dfrac{1}{1+x^2}\right)$

$f_y = 2y\tan^{-1}x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\dfrac{y^2}{1+x^2},\ 2y\tan^{-1}x\right)$

51

$\nabla f(x,y)$ at $(1,-1)$

$= \left(\dfrac{(-1)^2}{1+1^2},\ 2(-1)\tan^{-1}(1)\right)$

$= \left(\dfrac{1}{2},\ \dfrac{-2\pi}{4}\right)$

$= \left(\dfrac{1}{2},\ \dfrac{-\pi}{2}\right)$

iii) $f(x,y,z) = xyz - e^{x+y+z}$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (yz - e^{x+y+z},\ xz - e^{x+y+z},\ xy - e^{x+y+z})$

$\nabla f(x,y,z)$ at $(1,-1,0)$

$= (-1(0) - e^{1-1+0},\ 1(0) - e^{1-1},\ 1(-1) - e^{1-1})$

$= (0-1,\ 0-1,\ -1-1)$

$= (-1,-1,-2)$

Q.1)

i) $x^2\cos y + e^{xy} = 2$ at $(1,0)$

$f(x,y) = x^2\cos y + e^{xy} - 2$

$f_x = 2x\cos y + ye^{xy}$

$f_y = -x^2\sin y + xe^{xy}$

$(x_0, y_0) = (1,0)$

f_x at $(1,0) = 2(1)\cos 0 + 0$
$= 2$

f_y at $(1,0) = -(1)^2\sin(0) + 1(e)^0$
$= 1$

$f_x(x-x_0) + f_y(y-y_0) = 0$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$2x + y - 2 = 0$ — Eq^n of tangent

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$

$f(x,y) = x^2 + y^2 - 2x + 3y + 2$

$f_x = 2x + 0 - 2$
$= 2x - 2$

$f(x)$ at $(2,-2) = 2$.

$f_y = 2y + 3$

f_y at $(2,-2) = -1$.

Equat^n of tangent

$f_x(x-x_0) + f_y(y-y_0) = 0$

$2(x-2) + (-1)(y+2) = 0$

$2x - 4 - y - 2 = 0$

$2x - y - 6 = 0$ — Eq^n of tangent

iii) $x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$

$f(x,y,z) = x^2 - 2yz + 3y + xz - 7$

$f_x = 2x + z$ $\quad$ f_x at $(2,1,0) = 4$

$f_y = -2z + 3$ $\quad$ f_y at $(2,1,0) = 3$

$f_z = -2y + x$ $\quad$ f_z at $(2,1,0) = 0$

Eq^n of tangent

$f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$

$4(x-2) + 3(y-1) + 0(z-0) = 0$

$4x - 8 + 3y - 3 = 0$

$4x + 3y - 11 = 0$ — Eq^n of Tangent.

Equat^n of normal,

$\frac{x-x_0}{f_x} = \frac{y-y_0}{f_y} = \frac{z-z_0}{f_z}$

$\therefore \frac{x-2}{4} = \frac{y-1}{3} = \frac{z}{0}$ — Eq^n of tangent.

(ii) $3xy^2 - x - y + z = -4$

$f(x,y,z) = 3xy^2 - x - y + z + 4$

$f_x = 3y^2 - 1 - 0$, f_x at $(1,-1,2) = 3(1)^2 - 1$ $= 2$

$= 3y^2 - 1$

$f_y = 3x^2 - 1$, f_y at $(1,-1,2) = 5$

$f_z = 3xy + 1$, f_z at $(1,-1,2) = -2$

Equat^n of tangent,

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$\therefore -7(x-1) + 5(y+1) + (-2)(z-2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$ — Equat^n of Tangent.

$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$ — Equat^n of normal.

$\therefore \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$

Q.5]

i] $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$\therefore f_x = 6x + 0 - 3y + 6$

$= 6x - 3y + 6$ — ①

$f_y = 2y - 3x - 4$

$= 2y - 3x - 4$ — ②

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$

$2x - y = -2$ — ③

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4$ — ④

Multiplying ③ by 2 & subtracting ④ from ③

$4x - 2y = -4$

$-2y - 3x = 4$

$7x = 0$

$\boxed{x = 0}$

$2(0) - y = -2$

$\boxed{y = 2}$

$\therefore$ Critical points are $(0,2)$

Now,

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$