20CYS301
Digital Communications
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DSP Review

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I. DSP Review

DSP Review [1/12]

Time Shifting: Shifting, or delaying, a signal $x(t)$ by some constant, t_0 yields $x(t - t_0)$.

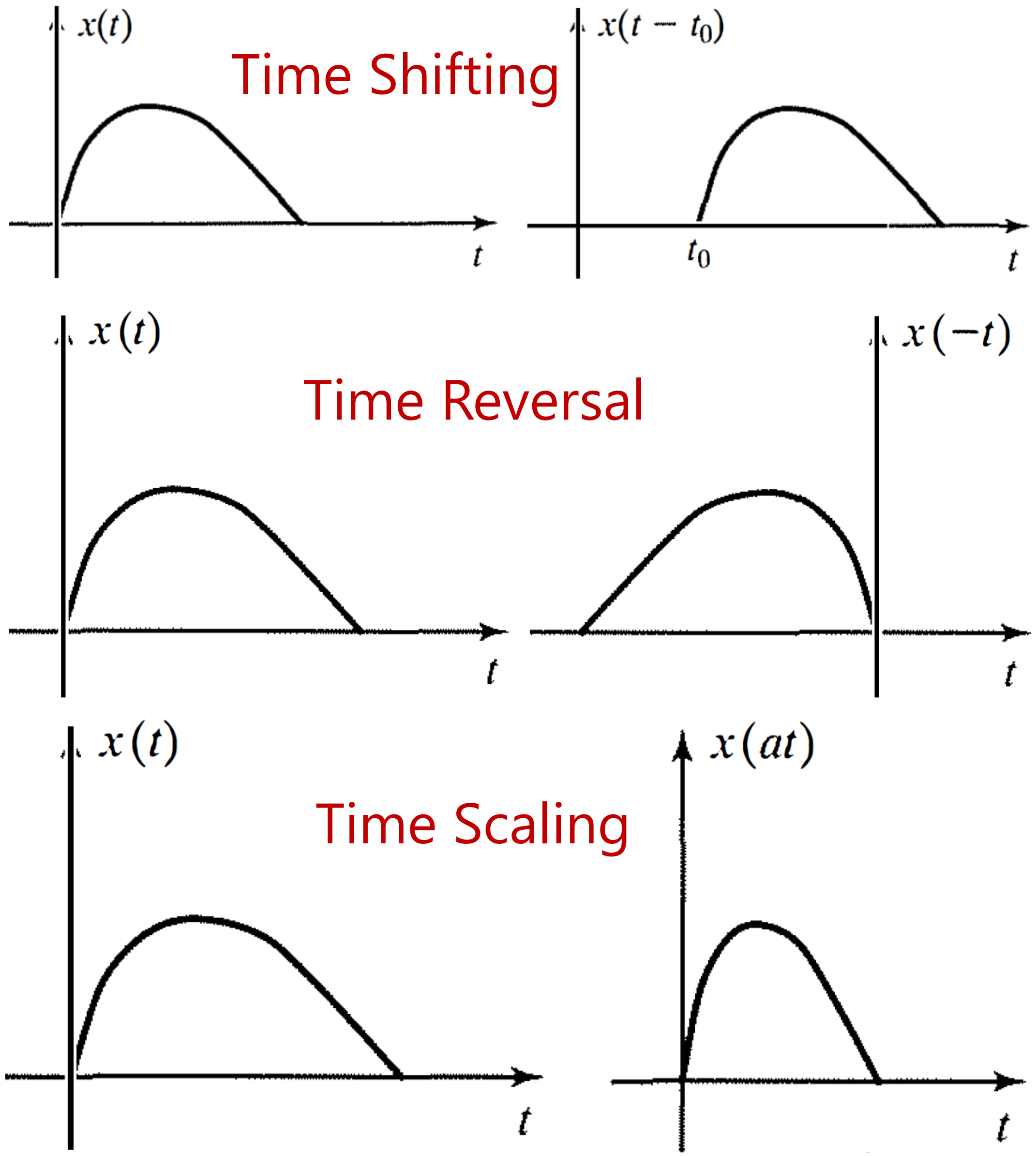
Time Reversal: results in flipping the signal around the vertical axis. If $x(t) = x(-t)$ then the function is even; if $x(t) = -x(t)$ then odd. Else neither.

Time Scaling: the signal, $x(at)$ appears contracted if scaling factor, a, $a > 1$.

Combo example: $x(2t - 5) = x[2(t - 2.5)]$, i.e., right-shift by 2.5 and contract by 2.

Problem: Simplify $x(-2t - 5)$. Does this result in the time axis actually reversing?

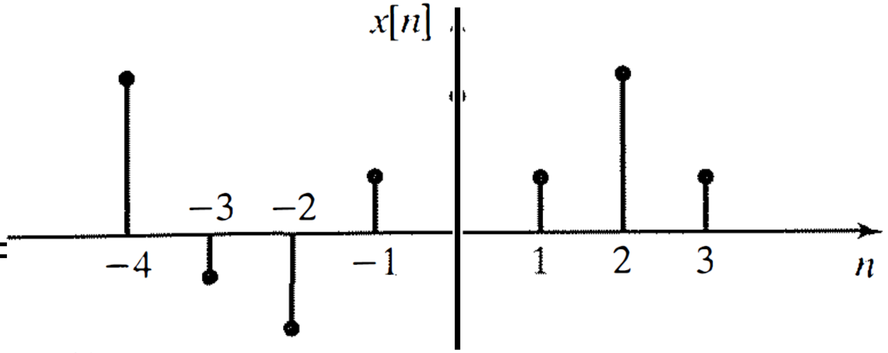
Solution: $x(-2t - 5) = x[-2(t + 2.5)]$. This is $x(t)$ shifted to -2.5, then contracted by 2 and finally, the time axis is reversed. The time axis is NOT reversed as the signal is flipped about vertical axis twice.



DSP Review [2/12]

Discrete-time Signal

$x[n] = Acos(2\pi f_0 n + \theta)$



Complex Representations: $x(t) = Ae^{j(2\pi f_0 t+\theta)}$. Then, $x_R(t) = Acos(2\pi f_0 t + \theta)$ and $x_I(t) = Asin(2\pi f_0 t + \theta)$.

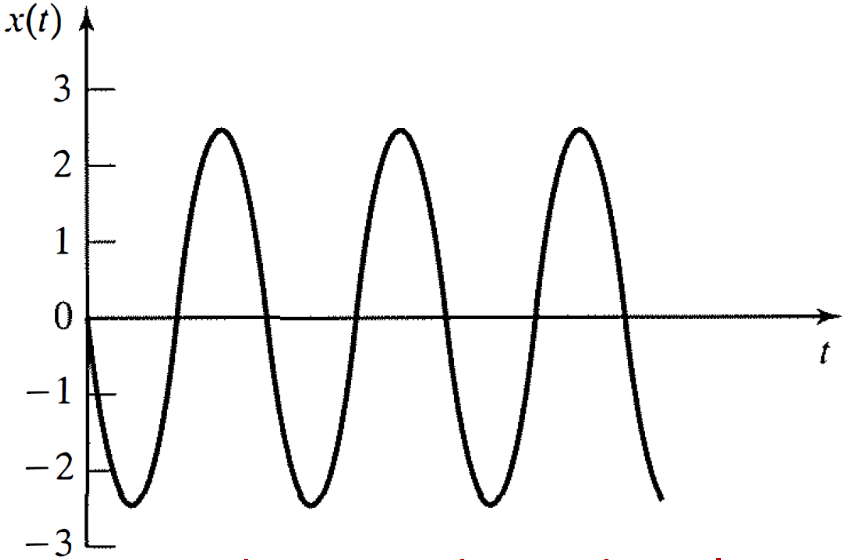
$|x(t)| = \sqrt{x_I(t)^2 + x_R(t)^2} = |A|$ &

$phase = \angle x(t) = \left[\frac{x_R(t)}{x_I(t)}\right]$ which is $tan^{-1}\left[\frac{Asin(2\pi f_0 t+\theta)}{Acos(2\pi f_0 t+\theta)}\right] = tan^{-1}[tan(2\pi f_0 t + \theta)] = (2\pi f_0 t + \theta)$

Deterministic vs Random Signals: Deterministic signals have a specific value at any given time. Random signals do not have a specific value; its value is chosen from a probability density function (pdf).

Periodic vs Non-periodic Signals: $e^{j\theta t}$ & $Acos(2\pi f_0 t + \theta)$ are periodic. $e^{\theta t}$, $e^{j\theta}$ & $Acos\left(\sqrt{2}\pi f_0 t + \theta\right)$ are NOT periodic.

Why?



Continuous-time Signal

$x[t] = Acos(2\pi f_0 t + \theta)$

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DSP Review [3/12]

Odd & Even Signal: Any signal can be written as a sum of its odd and even components, i.e., $x(t) = x_E(t) + x_O(t)$. Here, $x_E(t) = \frac{x(t)+x(-t)}{2}$ & $x_O(t) = \frac{x(t)-x(-t)}{2}$.

Problem: Is $Acos(2\pi f_0 t + \theta)$ odd or even?

Solution: $f(t) = Acos(2\pi f_0 t + \theta) = A \cdot [C(2\pi f_0 t) \cdot C(\theta) - S(2\pi f_0 t) \cdot S(\theta)]$ ... (I)

$f(-t) = Acos(2\pi f_0(-t) + \theta) = A \cdot \left[C\left(2\pi f_0(-t)\right) \cdot C(\theta) - S\left(2\pi f_0(-t)\right) \cdot S(\theta)\right]$

$\therefore\ f(-t) = A \cdot [C(2\pi f_0 t) \cdot C(\theta) + S(2\pi f_0 t) \cdot S(\theta)] = Acos(2\pi f_0 t - \theta)$... (II)

From (I) & (II)

$f(-t) \neq f(t)$. Thus, $f(t)$ is NOT even.

Also, $f(-t) \neq -f(t)$. Thus, $f(t)$ is NOT odd.

Thus, $f(t) = Acos(2\pi f_0 t + \theta)$ is neither even nor odd.

DSP Review [4/12]

Hermitian Signal: A complex signal has Hermitian symmetry if $x_R(t)\ is\ even$ and $x_I(t)\ is\ odd$. For a Hermitian signal, $|x(t)|\ is\ even$ and $\angle x(t)\ is\ odd.$

Problem: How much is the energy of the signal $A cos(2\pi f_0 t + \theta)$?

Solution: It is ∞. Hint: use the formula $\int_{-\infty}^{\infty} |x(t)|^2 dt$ and $\cos(2t) = cos^2 t - sin^2 t$.

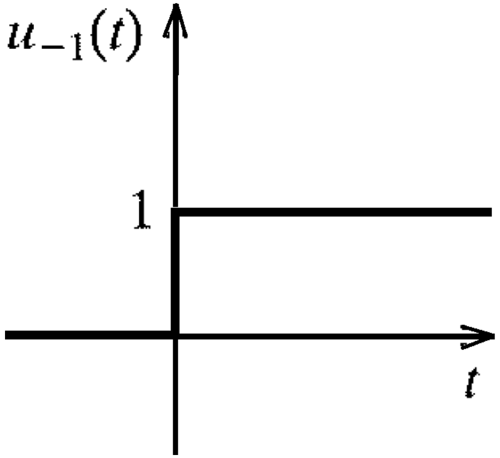
Problem: How much is the power of the signal?

Solution:

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(2\pi f_0 t + \theta)\, dt \quad = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} [1 + \cos(4\pi f_0 t + 2\theta)]\, dt$$

$$= \lim_{T\to\infty} \left[\frac{A^2 T}{2T} + \left[\frac{A^2}{8\pi f_0 T} \sin(4\pi f_0 t + 2\theta) \right]_{-T/2}^{T/2} \right] = \frac{A^2}{2} < \infty$$

DSP Review [5/12]



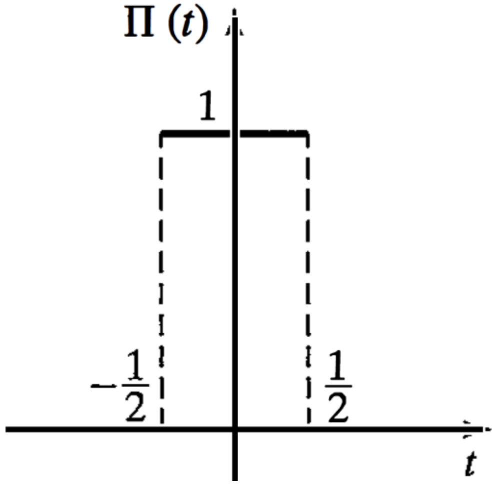
Unit Step Signal: $u(t) = 1 \; if \; t \geq 0 \; else, 0.$

Any signal multiplied by $u(t)$ produces a causal version of that signal.

$u(|a|t) = u(t)$. Is $u(|a|t)$ an odd or an even function or neither?

Is $u(at)$ an odd or an even function or neither?

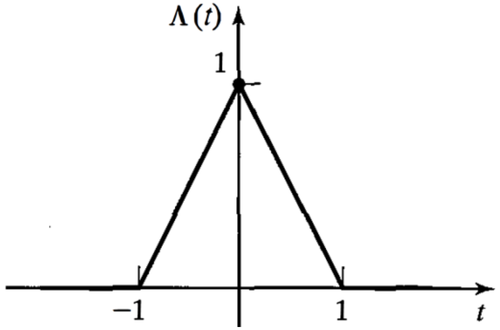
Plot $u(t) + 2u(3t - 3) + 0.5 \times \delta(t-2) - 0.5 \times u(t-3)$



Rectangular Pulse: $\sqcap(t) = 1 \; if \; -\frac{1}{2} \leq t \leq \frac{1}{2} \; else, 0.$

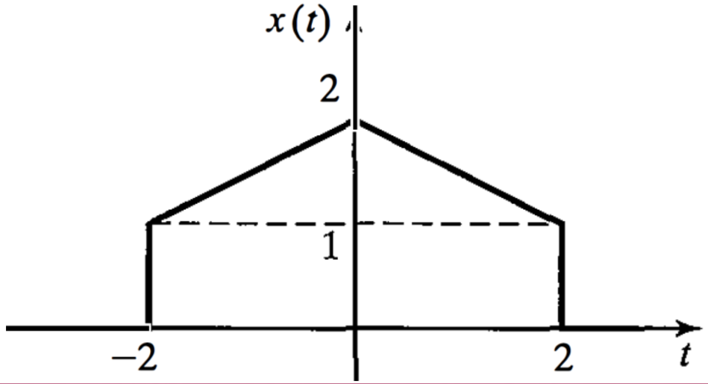
Triangular Pulse: $\Lambda(t) = t + 1 \; if \; -1 \leq t \leq 0; \; -t + 1 \; if \; 0 \leq t \leq 1; \; else, 0.$

It is the convolution of two rectangular pulses.

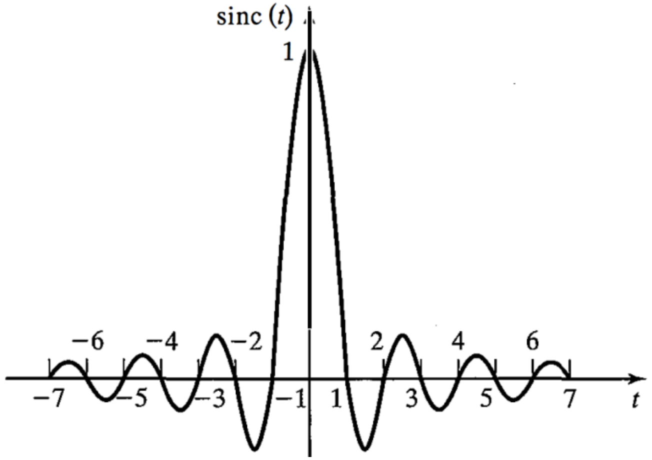


Problem: Plot $\Lambda\left(\frac{t}{2}\right) + \sqcap\left(\frac{t}{4}\right)$

Solution:



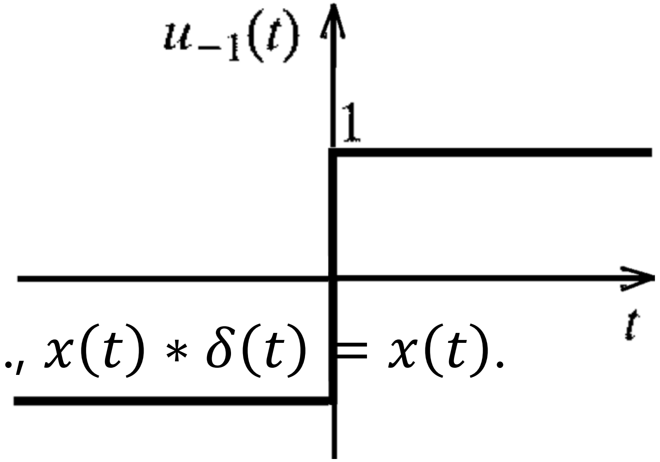
DSP Review [6/12]



Sinc Signal: $sinc(\pi t) = 1\ \frac{sin\pi t}{\pi t} if\ t \neq 0\ else, 1.$

Signum Signal: $sign(t) = 1\ if\ t \geq 0; \quad -1\ if\ t < 0.$

Note: $sign(t) = 2u(t) - 1$



Delta Signal: $\delta(t) = 0\ \forall\ t \neq 0, else\ (or\ \infty)$

$x(t)\delta(t - t_0) = \int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0) \cdot 1 = x(t_0)$

If $a \neq 0$, $\delta(at) = \frac{1}{|a|}\delta(t)$

Convolution of any signal with an impulse function is the signal itself, i.e., $x(t) * \delta(t) = x(t)$.

Note: $x(t) * \delta(t - t_0) = x(t - t_0)$ and $\delta(t) = \frac{d[u(t)]}{dt}$

Problem: Find $\cos(t) \cdot \delta(2t - 3)$

Solution: $= \cos(t) \cdot \delta[2(t - 1.5)] = \cos(t) \cdot \frac{1}{|2|}\delta[(t - 1.5)] = \frac{cos1.5}{2} \cdot 1 = 0.5 \cdot \cos(1.5)$

DSP Review [7/12]

Problem: Find $\int_{\pi}^{\infty} e^{x(t)}\delta(t-2)dt$

Solution: $\delta(t-2)$ is 0 everywhere except for $t=2$. However, 2 is outside of the limits of integration which is $[\pi,\infty]$. $\therefore \int_{\pi}^{\infty} e^{x(t)}\delta(t-2)dt = 0$.

Problem: Find $\int_{-\infty}^{\infty} e^{x(t)}\delta(t-2)\delta(t)dt$

Linear & Non-linear systems: $\mathfrak{J}$ is a linear operator, i.e., it represents a linear system, IFF for two input signals, $x_1(t)$ and $x_2(t)$ and two scalars, α and β:

$$\mathfrak{J}[\alpha x_1(t)+\beta x_2(t)] = \alpha\mathfrak{J}[x_1(t)] + \beta\mathfrak{J}[x_2(t)]$$

$\mathfrak{J}$, represents Fourier transform. It is thus a linear operator.

Problem: Find fourier transform of 0.

Solution: $\mathfrak{J}[0] = \mathfrak{J}[0\cdot x(t)] = 0\cdot\mathfrak{J}[x(t)] = 0.$

If a system is a linear time-invariant (LTI) system, then the system response to a signal can be derived by simply convolving the input signal with the impulse response of the system.

DSP Review [8/12]

Problem: Why it is useful to express arbitrary signals in terms of complex exponentials?

Solution: Consider an LTI system with impulse response $h(t)$. It receives a $x(t) = Ae^{j(2\pi f_o t+\theta)}$.

The output of the receiver is $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$

$= \int_{-\infty}^{\infty} Ae^{j(2\pi f_o[t-\tau]+\theta)}h(\tau)d\tau = Ae^{j\theta}e^{j2\pi f_o t}\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_o \tau}d\tau = Ae^{j(2\pi f_o t+\theta)}\mathfrak{I}[h(t)] =$ $Ae^{j(2\pi f_o t+\theta)}H(f_o)$ where $\mathfrak{I}[h(t)] = H(f_o)$ which is the Fourier transformation of the LTI system, i.e., the receiver. This can be pre-computed and kept.

$H(f_o)$ is a function of frequency of input signal and the receiver's impulse response.

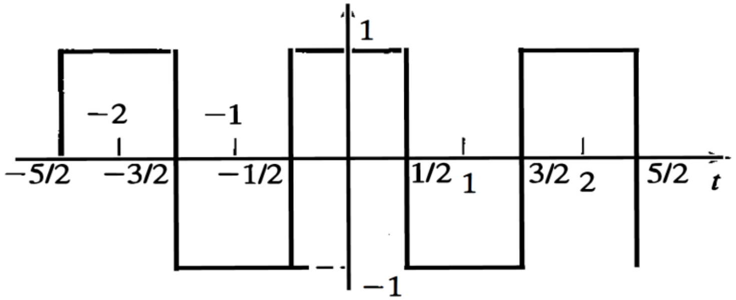
As $H(f_o)$ is a linear transformation, the output is a linear scaling of the input. $\therefore$ the response to exponential functions to an LTI systems can be easily computed.

Thus, it is useful to express arbitrary signals in terms of complex exponentials.

DSP Review [9/12]

Why Study Fourier Transformations? Basic blocks of a communications system – transmitters, receivers, filters, amplifiers, and equalizers are all LTI systems. We need tools to analyze LTI systems so that system output can be computed in an energy, time and memory efficient manner. This requires, where feasible, to offload some tasks that apparently must be performed online, to offline mode.

It turns out that when inputs are represented as linear combination of some 'basic signals', both the requirements are more easily met than if convolution was performed directly on the signal or system representation.



Problem: Find $\mathfrak{F}[x(t)]\ where\ x(t) = \sum_{n=-\infty}^{\infty}(-1)^n \Pi(t-n)$

Solution: $x(t)$ is a periodic signal with $T_0 = 2$.

Thus, $X(n) = \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{3}{2}} x(t)e^{-jn\pi t}d\tau = \frac{1}{2}\int_{-1/2}^{1/2}(1)e^{-jn\pi t}d\tau + \frac{1}{2}\int_{1/2}^{3/2}(-1)e^{-jn\pi t}d\tau$

$= \frac{1}{-2jn\pi}\left[e^{-jn\pi/2} - e^{jn\pi/2}\right] - \frac{1}{-2jn\pi}\left[e^{-jn3\pi/2} - e^{-jn\pi/2}\right]$

$e^{-jn\pi/2} = C(-n\,\pi/2) + jS(-n\,\pi/2)$ $\therefore$ 1st term is $\frac{1}{-2jn\pi}\left[\cancel{C(\theta)} - jS(\theta) - \cancel{C(\theta)} - jS(\theta)\right] = \frac{S(n\pi/2)}{n\pi}$

DSP Review [10/12]

2^{nd} term is $-\frac{1}{-2jn\pi}\left[e^{-jn3\pi/2} - e^{-jn\pi/2}\right] = \frac{1}{2jn\pi}\left[e^{-jn\pi}e^{-jn\pi/2} - e^{-jn\pi/2}\right]$

$$= \frac{e^{-jn\pi}}{2jn\pi}\left[e^{-jn\pi/2} - e^{+jn\pi/2}\right] = \frac{e^{-jn\pi}}{2jn\pi}[-2jS(n\,\pi/2)] = \frac{-e^{-jn\pi}}{n\pi}[S(n\,\pi/2)]$$

Thus, total term is $\frac{S(n\pi/2)}{n\pi}$ - $\frac{e^{-jn\pi}[S(n\pi/2)]}{n\pi} = \frac{S(n\pi/2)}{n\pi}\left[1 - e^{-jn\pi}\right]$

$$= \frac{S(n\pi/2)}{n\pi}\left[1 - \left(C(-n\pi) + jS(-\cancel{n}\pi)\right)\right] = \frac{S(n\pi/2)}{n\pi}[1 - C(n\pi)]$$

If n is even then, 0

If n is 1 then, $\frac{2}{\pi}$If n is 3 then, $\frac{-2}{2\cdot\pi}$, etc.

If n is 0 then, in the limits, $\frac{(n\pi/2-(n\pi/2)^2/2!+\cdots-\cdots)}{n\pi}[1 - 1 + (n\pi)^3/3! - \cdots + \cdots]$

$= [1 - n\,\pi/2/2! + \cdots - \cdots]\cdot[(n\pi)^2/3! - \cdots + \cdots]$ which, in the limits, is 0 because of 2^{nd} term.

DSP Review [11/12]

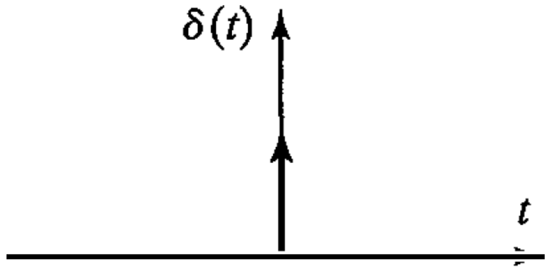
Fourier Representations for Real Signals: For a real signal $x(n)$, $x(-n) = x^*(n)$. $\therefore$ $|x(n)| = |x(-n)|$ & $\angle x(n) = -\angle x(-n)$, i.e., magnitude has even symmetry and phase has odd symmetry.

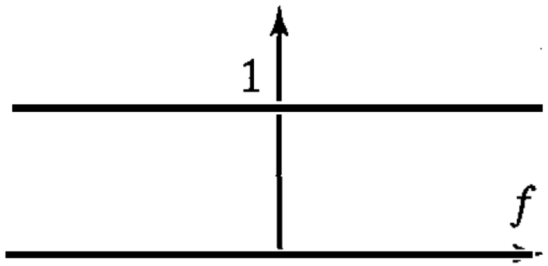
Fourier Representations for Real Even Signals: Fourier representation of a real, even signal has only a DC and cosine terms. It is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\left(2\pi \frac{n}{t_0} t\right)$

Fourier Representations for Real Odd Signals: Fourier representation of a real, odd signal has only sine terms. It is $\sum_{n=1}^{\infty} b_n sin\left(2\pi \frac{n}{t_0} t\right)$

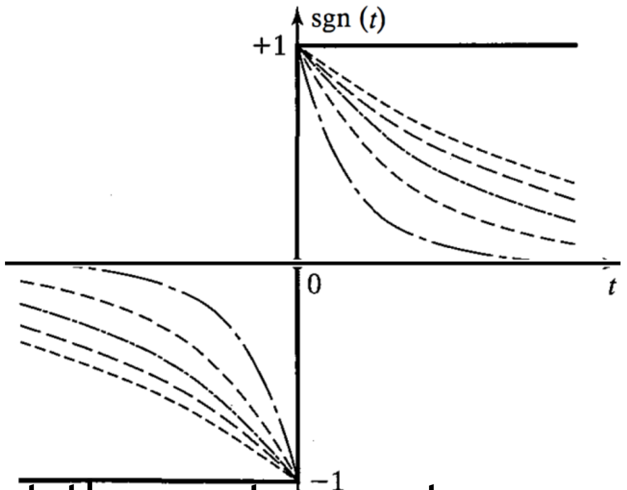
LTI Response to a Periodic Signal: An LTI system cannot introduce new frequencies in the output. If output frequencies are different from those already present at the input then the system is either non-linear or, time-varying.

$\Im[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = 1 \cdot e^{-j2\pi f \cdot 0} = 1$ Thus, all frequencies are present in delta function with magnitude 1 and phase 0.





DSP Review [12/12]



Problem: $x_n(t)\begin{cases} e^{-\frac{t}{n}} & ; t > 0 \\ -e^{\frac{t}{n}} & ; t < 0 \\ 0 & ; t = 0 \end{cases}$ which, tends to $sgn(t)$ as $n \to \infty$.

Find $X_n(f) = \Im[x_n(t)] = \int_{-\infty}^{\infty} x_n(t) e^{-j2\pi f t} dt$. Show that $x_n(t)$ is such that the system acts as a blocker for both very low and very high frequencies. What happens if $n \to \infty$?

Solution: $\int_{-\infty}^{\infty} x_n(t) e^{-j2\pi f t} dt = \int_{-\infty}^{0} -e^{\frac{t}{n}} e^{-j2\pi f t} dt + \int_{0}^{\infty} e^{-\frac{t}{n}} e^{-j2\pi f t} dt$

$= -\int_{-\infty}^{0} e^{\left(\frac{1}{n} - j2\pi f\right)t} dt + \int_{0}^{\infty} e^{-\left(\frac{1}{n} + j2\pi f\right)t} dt = -\frac{1}{\frac{1}{n} - j2\pi f} + \frac{1}{\frac{1}{n} + j2\pi f} = -\frac{j4\pi f}{\frac{1}{n^2} + 4\pi^2 f^2}$ which is $-\frac{j4\pi}{\frac{1}{fn^2} + 4\pi^2 f}$.

$f \to 0$: The 1st denominator term $\to \infty$ & the 2nd $\to 0$. Thus, the system that generates $x_n(t)$ acts as a blocker for very low frequencies.

$f \to \infty$: The 1st denominator term $\to 0$ & the second $\to \infty$. Thus, the system that generates $x_n(t)$ acts as a blocker for very high frequencies also.

The answer to the unanswered Q has 2 parts: magnitude & frequency … find it out.