Projective Transformation

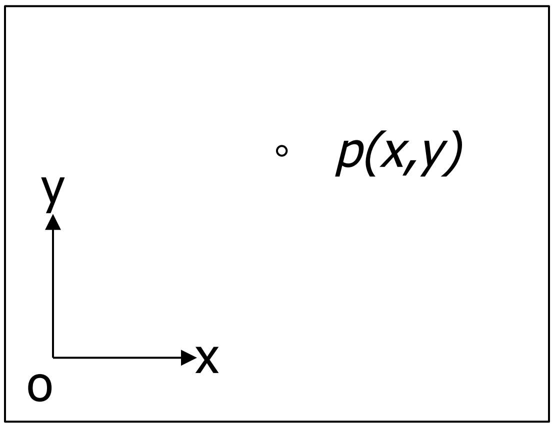
Jayanta Mukhopadhyay
Dept. of Computer Science and Engg.

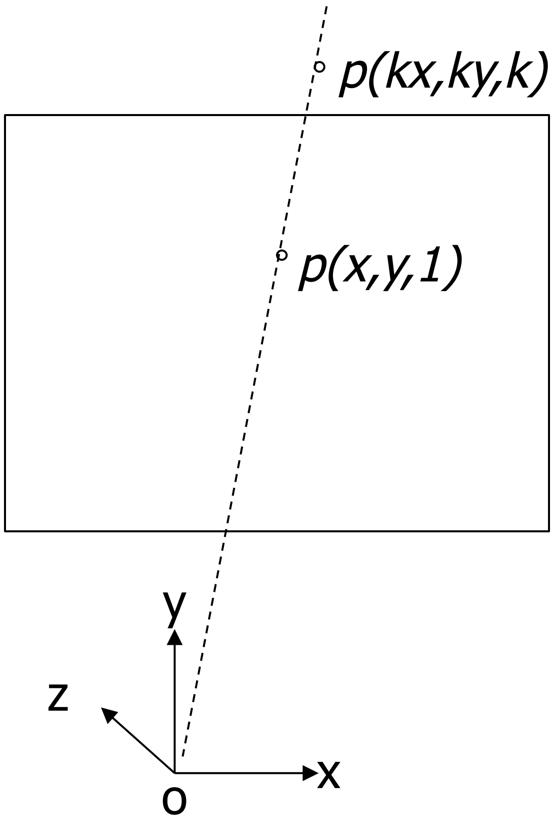
Projective transformation

- h: $P^2 \rightarrow P^2$
- Invertible
- Collinearity of every three points to be preserved, i.e. three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Real Space and Projective Space (2D)

$p \in R^2$



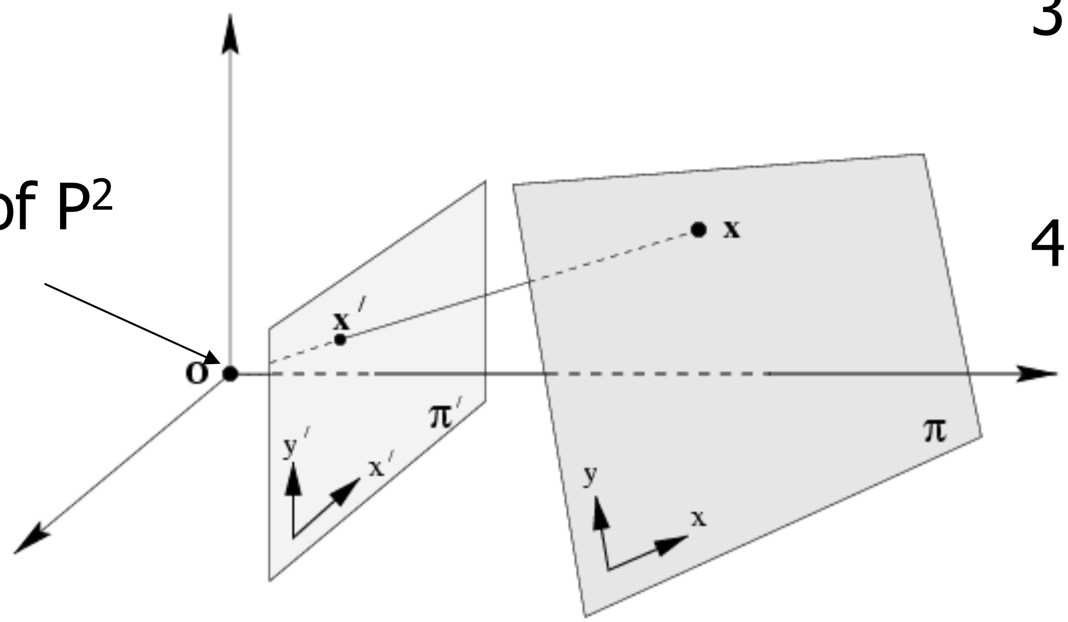


$p \in P^2$

$p(x,y)$ ↔ $p(kx,ky,k)$

Change of coordinate convention

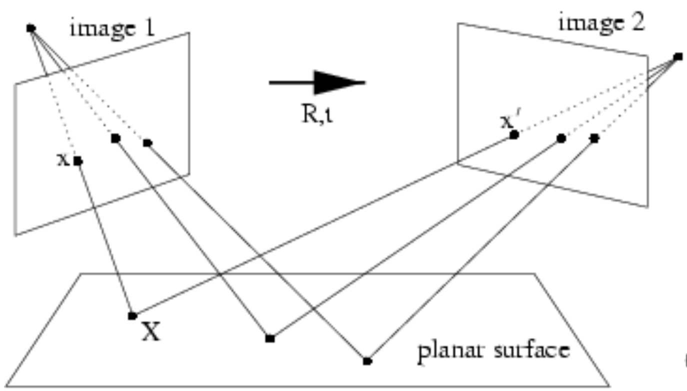
Origin of P^2 fixed.

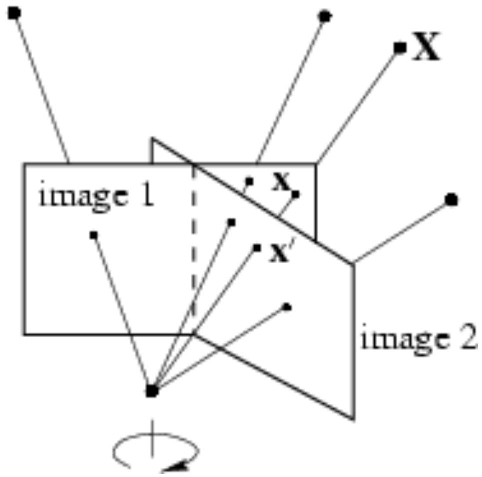


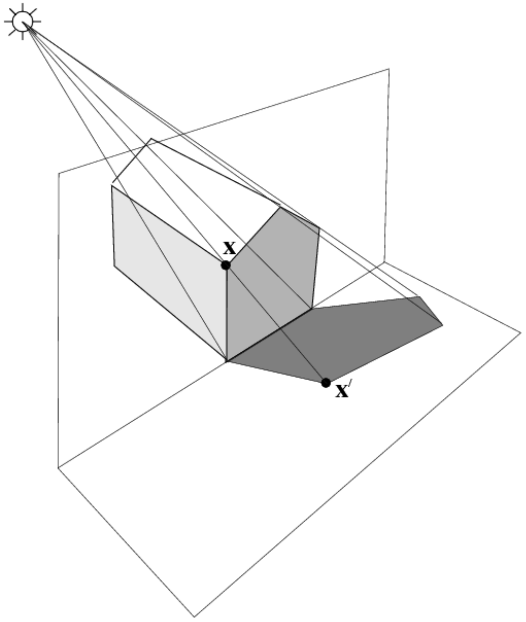
1. Rotation of axes.
2. Change of scale.
3. Translation of origin in planar coordinate system.
4. Use of Affine coordinate system in plane.

More examples

Shift of origin of P^2







Form of h

- Only one form possible.
- It is linear and invertible.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

8DOF

$$\mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

Also called homography and **H** is the homography matrix.

Show that **Hx** preserves collinearity.

- Let **l** be a line in P^2.
- A point **x** on **l** satisfies

$$\mathbf{l}^T\mathbf{x}=\mathbf{0}$$

$$\rightarrow \mathbf{l}^T\mathbf{H}^{-1}\mathbf{H}\mathbf{x}=\mathbf{0}$$

$$\rightarrow (\mathbf{H}^{-T}\mathbf{l})^T\mathbf{x}=\mathbf{0}$$

- $\mathbf{H}^{-T}\mathbf{l}$ is the transformed line of **l**.

Harder to show that **H** is the only form of homography.

Implications

- If there is a homography, there exists a unique **H**, which is a 3x3 invertible matrix.
- Functional form known, so easier to estimate.
- **H** and k**H** are equivalent, where k is a scalar constant.
- Number of unknowns in **H** = 8.

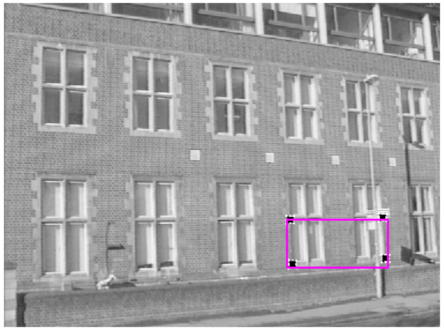
Estimation of **H**

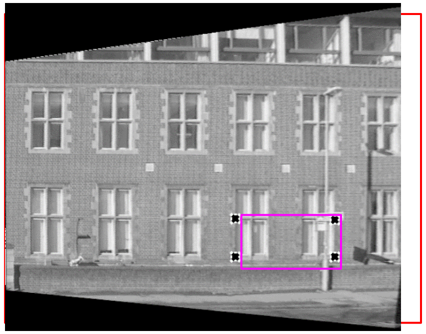
- Given point correspondences ($\mathbf{x}_i$ ↔ $\mathbf{x}_i'$) estimate **H** such that $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$.
- There are 8 unknowns.
- $\mathbf{x}' = \mathbf{H}\mathbf{x}$ → Two independent equations.

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

- Minimum 4 point correspondences needed.

Removing projective distortion





1. Select four points in a plane with known coordinates.
2. Form equations.

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

(linear in h_{ij})

$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$

$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

Remark: no calibration at all necessary. Does not work if $h_{33}=0$ in **H**. It happens when the plane of transformation passes through origin.

3. Setting h_{33} at *1* solve them.

Direct Linear Transformation (DLT)

$$\mathrm{x}'_i = \left(x'_i, y'_i, w'_i\right)^{\mathsf{T}}$$

$$\mathrm{x}'_i = \mathrm{H}\mathrm{x}_i$$

$$\mathrm{H}\mathrm{x}_i = \begin{pmatrix} \mathrm{h}^{1^{\mathsf{T}}}\mathrm{x}_i \\ \mathrm{h}^{2^{\mathsf{T}}}\mathrm{x}_i \\ \mathrm{h}^{3^{\mathsf{T}}}\mathrm{x}_i \end{pmatrix}$$

$$\mathrm{x}'_i \times \mathrm{H}\mathrm{x}_i = 0$$

$$H = \begin{bmatrix} h^{1^T} \\ h^{2^T} \\ h^{3^T} \end{bmatrix}$$

$$\mathrm{x}'_i \times \mathrm{H}\mathrm{x}_i = \begin{pmatrix} y'_i\mathrm{h}^{3^{\mathsf{T}}}\mathrm{x}_i - w'_i\mathrm{h}^{2^{\mathsf{T}}}\mathrm{x}_i \\ w'_i\mathrm{h}^{1^{\mathsf{T}}}\mathrm{x}_i - x'_i\mathrm{h}^{3^{\mathsf{T}}}\mathrm{x}_i \\ x'_i\mathrm{h}^{2^{\mathsf{T}}}\mathrm{x}_i - y'_i\mathrm{h}^{1^{\mathsf{T}}}\mathrm{x}_i \end{pmatrix} = 0$$

Redundant: x_i'(1)+y_i' (2)=(3)

Direct Linear Transformation (DLT)

$$\begin{bmatrix} 0^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & y'_i\mathbf{x}_i^\mathsf{T} \\ w'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \\ -y'_i\mathbf{x}_i^\mathsf{T} & x'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

$\mathrm{A}_i\mathbf{h} = 0$ where $A_i = \begin{bmatrix} 0^T & -w'_i x_i^T & y'_i x_i^T \\ w'_i x_i^T & 0^T & -x'_i x_i^T \end{bmatrix}$

Dimension of A_i : 2 x 9.

Direct Linear Transformation (DLT): Non-homogeneous Equations

- Solving for H by setting h_{33}=1. $h = \begin{bmatrix} \tilde{h} \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w'_i & -y_i w'_i & -w_i w'_i & x_i y'_i & y_i y'_i \\ x_i w'_i & y_i w'_i & w_i w'_i & 0 & 0 & 0 & -x_i x'_i & -y_i x'_i \end{bmatrix} \tilde{h} = \begin{bmatrix} -w_i y'_i \\ w_i x'_i \end{bmatrix}$$

$$\tilde{A}_i \tilde{h} = b_i$$

$$A\tilde{h} = b$$

$$Minimize \left\| A\tilde{h} - b \right\|$$

Solution: $\tilde{h} = (A^T A)^{-1} A^T b$

Dimension of A: 2nx8

Rank: 8

Dimension of h: 8x1

Dimension of b: 2nx1

Caution: If h_{33}=0, no multiplication scale exists, and no solution obtained. It happens if the origin of the plane lies on the vanishing line.

Direct Linear Transformation (DLT): Homogeneous Equations

- Solving for **H**: $\mathrm{Ah} = 0$

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}$$

Dimension of *A*: 2nx9

Rank: 8

Dimension of *h*: 9x1

Dimension of *Ah*: 2nx1

Minimize $||Ah||$ such that $\|\mathrm{h}\| = 1$

Solution: Unit eigen vector of smallest eigen value of A^TA.

Other error criteria

- Algebraic error: Error term in DLT.
- Geometric error: $\sum d_e^2\left(\mathbf{x}', \mathbf{H}\mathbf{x}\right)$ Euclidean distance
- Geometric error with reprojection:

$$\sum \left(d_e^2\left(\mathbf{x}', \mathbf{H}\mathbf{x}\right) + d_e^2\left(\mathbf{H}^{-1}\mathbf{x}', \mathbf{x}\right)\right)$$

- Use of nonlinear iterative optimization techniques such as Newton iteration, Levenberg-Marquardt (LM) method, etc.

Transformation invariance and normalization

- Problem: To estimate **H** given a set of ($\mathbf{x}_i$ ↔ $\mathbf{x}_i'$) .
- Consider, $\mathbf{y}_i = \mathbf{T}\mathbf{x}_i$ and $\mathbf{y}'_i = \mathbf{T}'\mathbf{x}_i'$ for known **T** and **T'**, which are invertible.
- Now estimate homography **G** from ($\mathbf{y}_i$ ↔ $\mathbf{y}'_i$).
- Can you estimate **H** from **G**?

$$
\begin{aligned}
\mathbf{x}' &= \mathbf{H}\mathbf{x} \\
\Rightarrow \mathbf{T}'^{-1}\mathbf{y}' &= \mathbf{H}\mathbf{T}^{-1}\mathbf{y} \\
\Rightarrow \mathbf{y}' &= \underbrace{\mathbf{T}'\mathbf{H}\mathbf{T}^{-1}}_{\mathbf{G}}\,\mathbf{y}
\end{aligned}
$$

Caution: For DLT it is not equivalent.
As the constraint ||g||=1 is not equivalent to ||h||=1.

Robust computation through Normalization of data

- Transform the point set so that its center becomes origin (in the plane) and avg. distance from it is $\sqrt{2}$.

$$x_i^{(n)} = \frac{x_i - \bar{x}}{\sigma_x} \qquad y_i^{(n)} = \frac{y_i - \bar{y}}{\sigma_y}$$

- Apply DLT on transformed point.
- Recover homography from the homography of transformed point sets.

Thank you!