# Homography: Properties

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# Projective transformation

- h: $P^2 \rightarrow P^2$.
- Invertible.
- Collinearity of every three points to be preserved, i.e. three points $x_1, x_2, x_3$ lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.
- Only in the form of non-singular 3x3 matrix.

# Point and line transformation

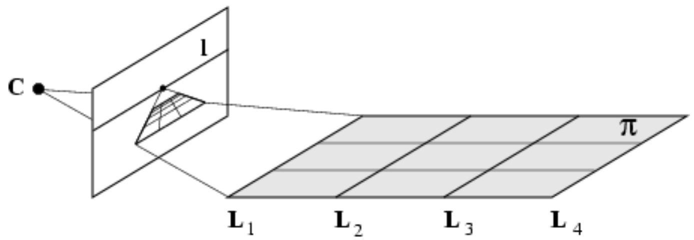
- Point: $\mathbf{x}'=\mathbf{H}\mathbf{x}$
- Line: $\mathbf{l}'=\mathbf{H}^{-T}\mathbf{l}$
- Vanishing point for lines parallel to $\mathbf{l}=(a,b,c)^T$:

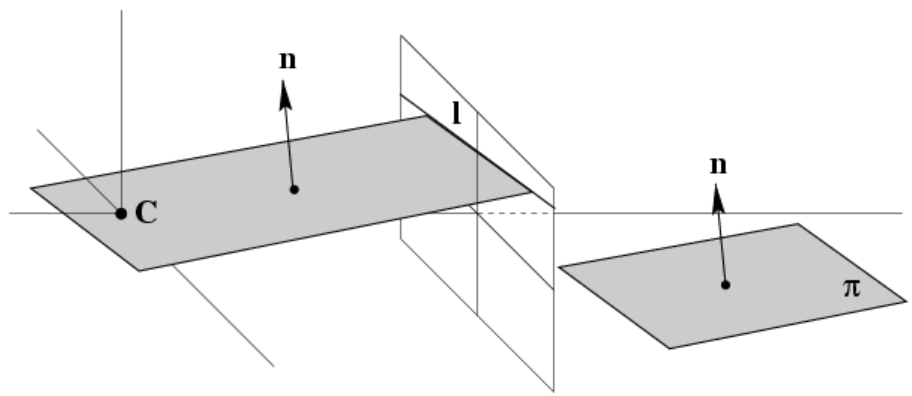
$$\mathbf{v}_\mathbf{l}=\mathbf{H}\,(b,\,-a,\,0)^T$$

- Vanishing line:

$$\mathbf{l}_\mathbf{H}=\mathbf{H}^{-T}\mathbf{l}_\alpha$$
$$=\mathbf{H}^{-T}\,(0,\,0,1)^T$$

# Vanishing line : Geometric Interpretation



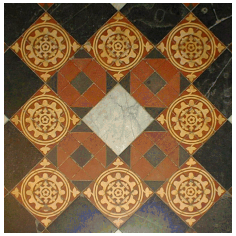

The vanishing line **l** of a plane **π** is obtained by intersecting the image plane with a plane through the camera center C and parallel to **π**.

# A hierarchy of transformations

Projective linear group

Affine group (last row (0,0,1))

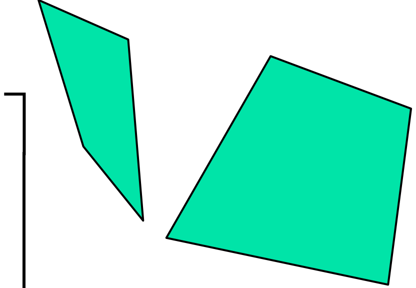
Euclidean group (upper left 2x2 orthogonal)



Oriented Euclidean group (upper left 2x2 det 1)

# Projective Group

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



$$\mathbf{x}' = \mathbf{H}_P \, \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix} \mathbf{x}$$

$$\mathbf{v} = (v_1, v_2)^{\mathsf{T}}$$

dof=8: 2 scale, 2 rotation, 2 translation, 2 line at infinity)

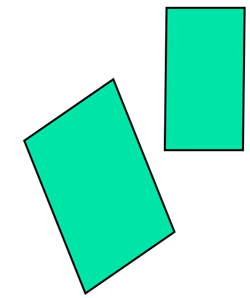
$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon.

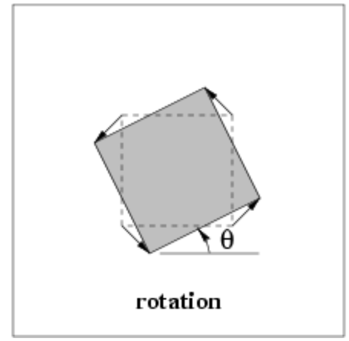
Concurrency, collinearity, order of contacts (intersection, tangency, inflection, etc.), cross ratio (ratio of ratio).

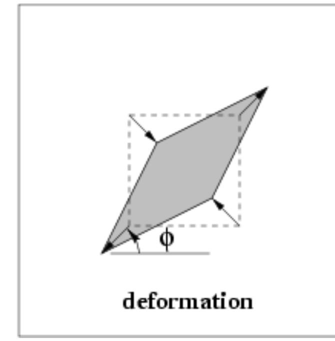
# Affine group

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{x}' = \mathbf{H}_A\,\mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ 0^\mathsf{T} & 1 \end{bmatrix}\mathbf{x}$$




$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

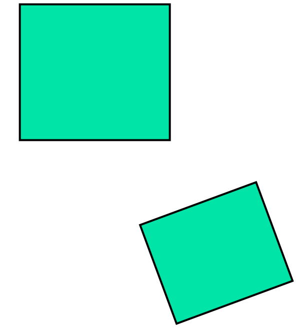
dof=6

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ 0^\mathsf{T} & v \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along line.

Parallelism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity $\mathbf{l}_\infty$.**

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



# Similarity Group

$$\mathbf{x}' = \mathbf{H}_S\,\mathbf{x} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$

$$\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

dof=4 (1 scale, 1 rotation, 2 translation)

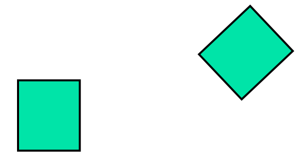
$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

Ratios of lengths, angles. **The circular points I,J.**

# Isometry

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon \cos\theta & -\sin\theta & t_x \\ \varepsilon \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

Orientation preserving: $\varepsilon = 1$

Orientation reversing: $\varepsilon = -1$

dof=3 (1 rotation, 2 translation)

**Invariants:** length, angle, area

# Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S\mathbf{H}_A\mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

$\mathbf{K}$ Upper-triangular

$\det \mathbf{K} = 1$ $\quad v \neq 0$

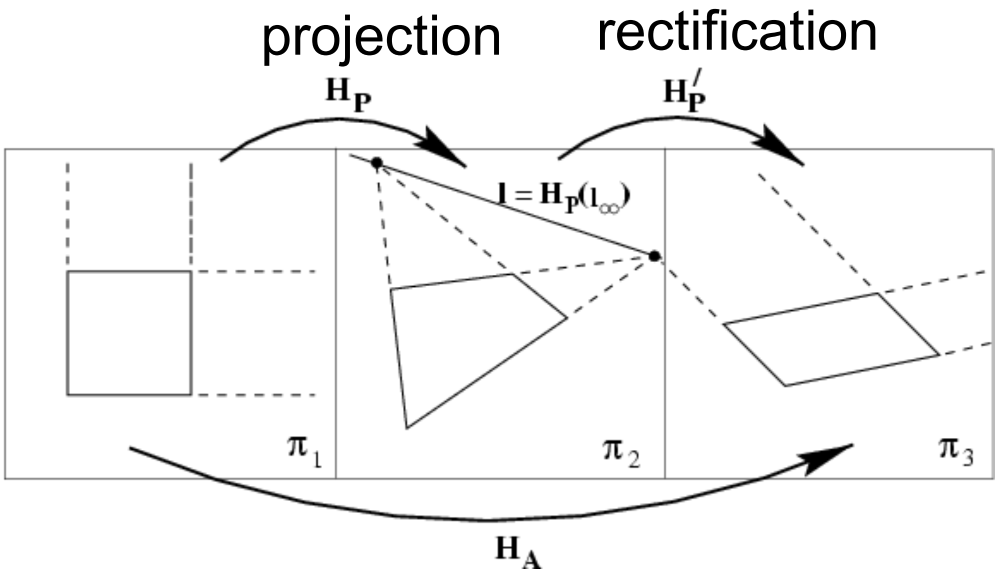
<u>Example</u>:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

decomposition unique (if chosen s>0)

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

# Affine properties from images



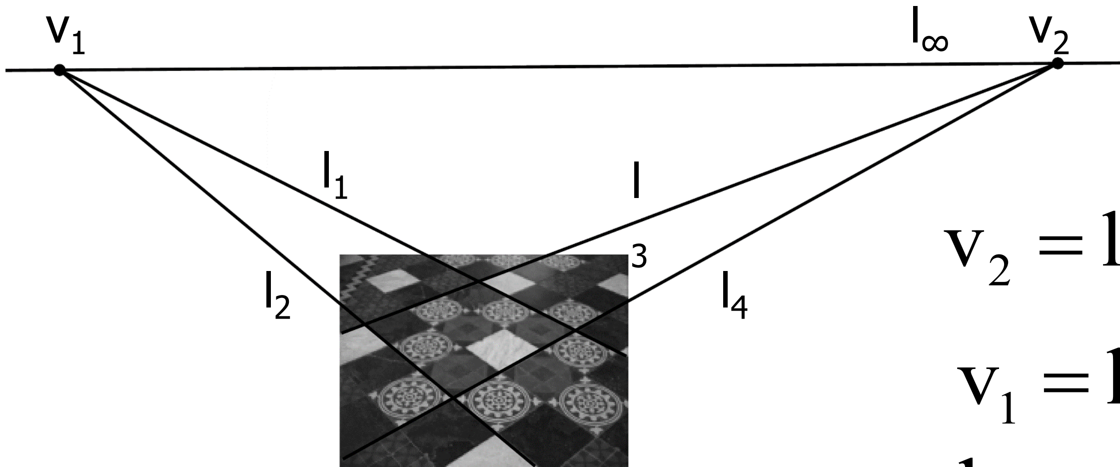

$$\mathbf{l}_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\mathsf{T}, l_3 \neq 0$$

$$H'_p = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

For any affine $H_A$.

$$H'^{-T}_p \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

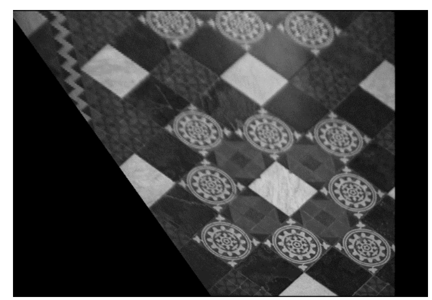
# Affine rectification



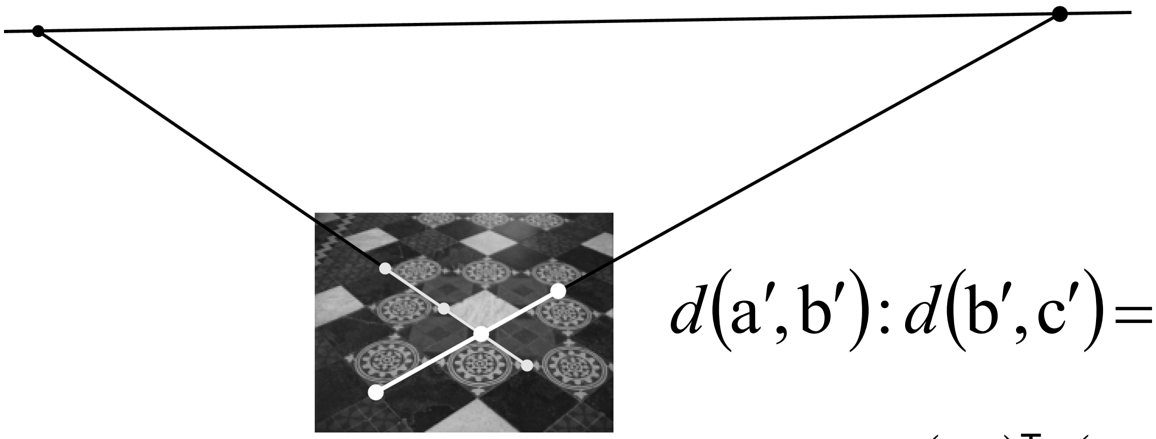

$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

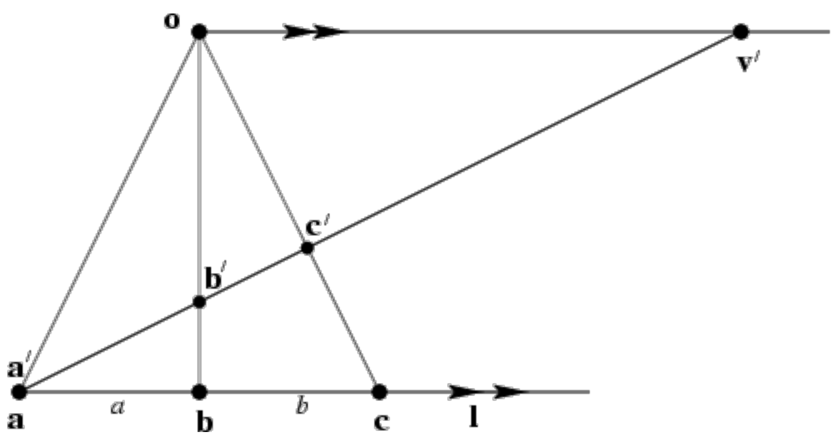
$$l_\infty = v_1 \times v_2$$



# Distance ratios



$$d(\mathrm{a}',\mathrm{b}'):d(\mathrm{b}',\mathrm{c}')=a':b'$$




$$(0,1)^{\mathsf{T}},(a,1)^{\mathsf{T}},(a+b,1)^{\mathsf{T}}$$

$$\Big\downarrow \mathbf{H}$$

$$\mathrm{a}',\mathrm{b}',\mathrm{c}'$$

**H** is a 2x2 matrix, requiring 3 equations.

$$\mathrm{v}'=\mathbf{H}(1,0)^{\mathsf{T}}$$

$a{:}b$ known from world coordinate.

# Conics in $P^2$

- Curves described by 2nd degree equation in the plane.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- In homogeneous coordinate: $(x,y) \rightarrow (x_1,x_2,x_3) = (x_1/x_3, x_2/x_3)$

$$a\left(\frac{x_1}{x_3}\right)^2 + \mathrm{b}\left(\frac{x_1}{x_3}\right)\left(\frac{x_2}{x_3}\right) + \mathrm{c}\left(\frac{x_2}{x_3}\right)^2 + \mathrm{d}\left(\frac{x_1}{x_3}\right) + \mathrm{e}\left(\frac{x_2}{x_3}\right) + \mathrm{f} = 0$$

$$\Longrightarrow ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

# Conics in $P^2$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$\Rightarrow X^T C X = 0$$

Where

$$C = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

Conics identified by $C$ with 5 d.o.f. ($a{:}b{:}c{:}d{:}e{:}f$)

# Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

$$\left(x_i^2, x_iy_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

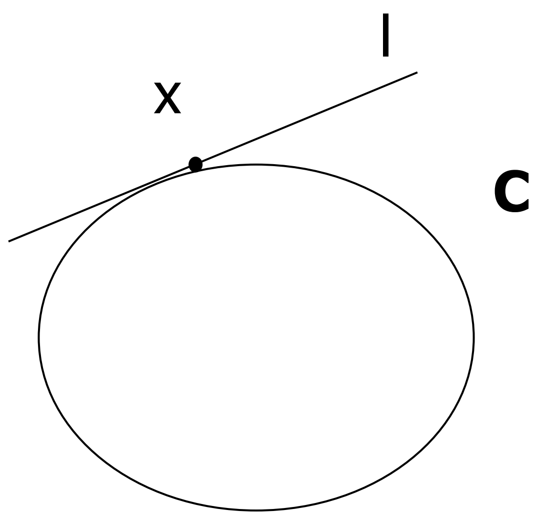
Stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

Rank deficient **C** → degenerate conic (e.g. two lines (of rank 2) or a repeated line (of rank 1).

# Tangent lines to conics

The line **l** tangent to **C** at point x on **C** is given by **l**=**C**x



# Dual conics

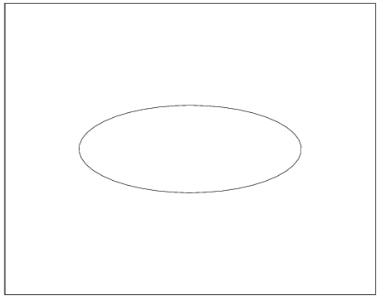
A line tangent to the conic **C** satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

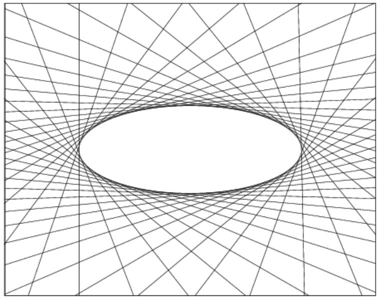
$$X = C^{-1}l$$

$$X^T CX = 0 \Longrightarrow (C^{-1}l)^T C(C^{-1}l) \Longrightarrow l^T\ (C^{-1})^T CC^{-1}l = 0$$

$\Longrightarrow l^T C^* l = 0$ where $C^* = (C^{-1})^T CC^{-1} = C^{-1}$ (as $C$ is symmetric).

Dual conics = line conics = conic envelopes





# Degenerate Conics

- Rank of C <3
- Rank 2 → Two lines / points
- Rank 1 → One repeated lines / points
- Degenerate point conic:

  $\mathbf{C}=\mathbf{l}.\mathbf{m}^T+\mathbf{m}.\mathbf{l}^T$ rank 2, if $\mathbf{l} <> \mathbf{m}$
- Degenerate dual line conic:

  $\mathbf{C}^*=\mathbf{x}.\mathbf{y}^T+\mathbf{y}.\mathbf{x}^T$ rank 2, if $\mathbf{x} <> \mathbf{y}$

# Transformation of conics under homography **H**

- $\mathbf{X'}=\mathbf{HX}$
- $\mathbf{X}^{T}\mathbf{CX}=0$

  $\rightarrow (\mathbf{H}^{-1}\mathbf{X'})^{T}\mathbf{C}(\mathbf{H}^{-1}\mathbf{X'})=0$

  $\rightarrow \mathbf{X'}^{T}\,\mathbf{H}^{-T}\mathbf{CH}^{-1}\,\mathbf{X'}=0$

  $\rightarrow \mathbf{X'}^{T}\,\mathbf{C'}\,\mathbf{X'}=0$

A conic remains a conic under homography.

where transformed conics $\mathbf{C'}=\mathbf{H}^{-T}\mathbf{CH}^{-1}$

- $\mathbf{C'}^{*}=\mathbf{C'}^{-1}=(\mathbf{H}^{-T}\mathbf{CH}^{-1})^{-1}=\mathbf{HC}^{-1}\mathbf{H}^{T}$

# The circular points

$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \mathrm{I}' = \mathbf{H}_S \mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity. They are also on $\mathbf{l}_\alpha$.

Every circle intersects $\mathbf{l}_\alpha$ at I and J.

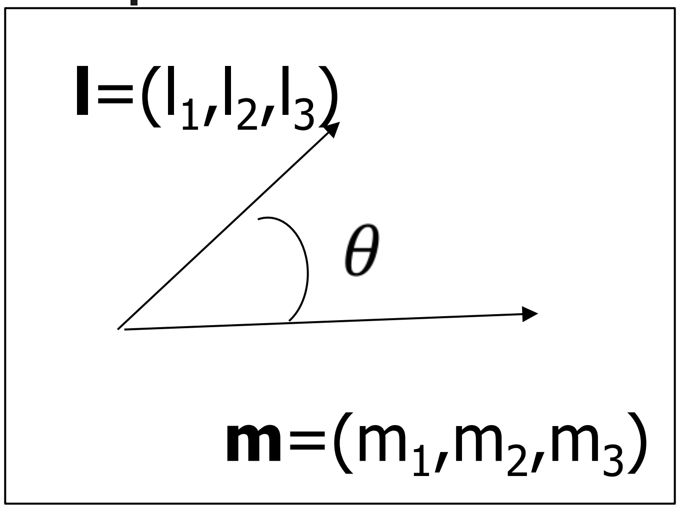
Circle: $x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$

Setting $x_3$=0, $x_1^2 + x_2^2$=0. (I and J satisfies it)

# Conic dual to the circular points ($C_\alpha{}^*$)

- $C_\alpha{}^* = I.J^T + J.I^T$ (line conic)
- $C_\alpha{}^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- As I and J are fixed under similarity $C_\alpha{}^*$ is also fixed, i.e. $C_\alpha{}^{*\prime} = H_s\, C_\alpha{}^* H^T = C_\alpha{}^*$
- $C_\alpha{}^*$ is fixed iff H is a similarity.
- D.o.f. of transformed $C_\alpha{}^*$ is 4 and det. =0.
- $l_\alpha$ is the NULL vector of $C_\alpha{}^*$.

# Measurement of angle under homography




$$\cos(\theta) = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}}$$

Invariant under homography

$$\cos(\theta) = \frac{\boldsymbol{l}^T C_\infty^* \boldsymbol{m}}{\sqrt{(\boldsymbol{l}^T C_\infty^* \boldsymbol{l})(\boldsymbol{m}^T C_\infty^* \boldsymbol{m})}}$$

$C_\infty^{*\prime} = H C_\infty^* H^T$ and $\boldsymbol{l}' = H^{-T} \boldsymbol{l}$

$$\boldsymbol{l}'^T C_\infty^{*\prime} \boldsymbol{m}'$$
$$= \boldsymbol{l}^T H^{-1} H C_\infty^* H^T H^{-T} \boldsymbol{m}$$
$$= \boldsymbol{l}^T \boldsymbol{C}_\infty^* \boldsymbol{m}$$

Once $C_\alpha^{*\prime}$ is obtained Euclidean angle could be recovered.

If $\boldsymbol{l}$ and $\boldsymbol{m}$ orthogonal, $\boldsymbol{l}^T C_\alpha^{*\prime} \boldsymbol{m} = 0$.

# Estimation of $C_\alpha^{*\prime}$

- Use the property of orthogonal lines:
- $\mathbf{l}^T C_\alpha^{*\prime} \mathbf{m}^T = 0$
- 5 such orthogonal pairs needed. A typical equation:

$$\begin{bmatrix} l_1 m_1 & \frac{l_1 m_2 + l_2 m_1}{2} & l_2 m_2 & \frac{l_1 m_3 + l_3 m_1}{2} & \frac{l_1 m_3 + l_3 m_1}{2} & l_2 m_2 \end{bmatrix} C = 0$$

Where $C$ is represented by $(a,b,c,d,e,f)^T$.

- Apply SVD and take unit singular vector of minimum singular value as the solution.
- Make $C_\alpha^{*\prime}$ a rank 4 matrix by SVD again.

# Recovery of metric properties

- Compute H from $C_\alpha^{*\prime}$ upto similarity.
  - Matrix decomposition method
- Apply $H^{-1}$ to the image.
- Method is also called ”stratification”.

Thank you!