Projective Geometry

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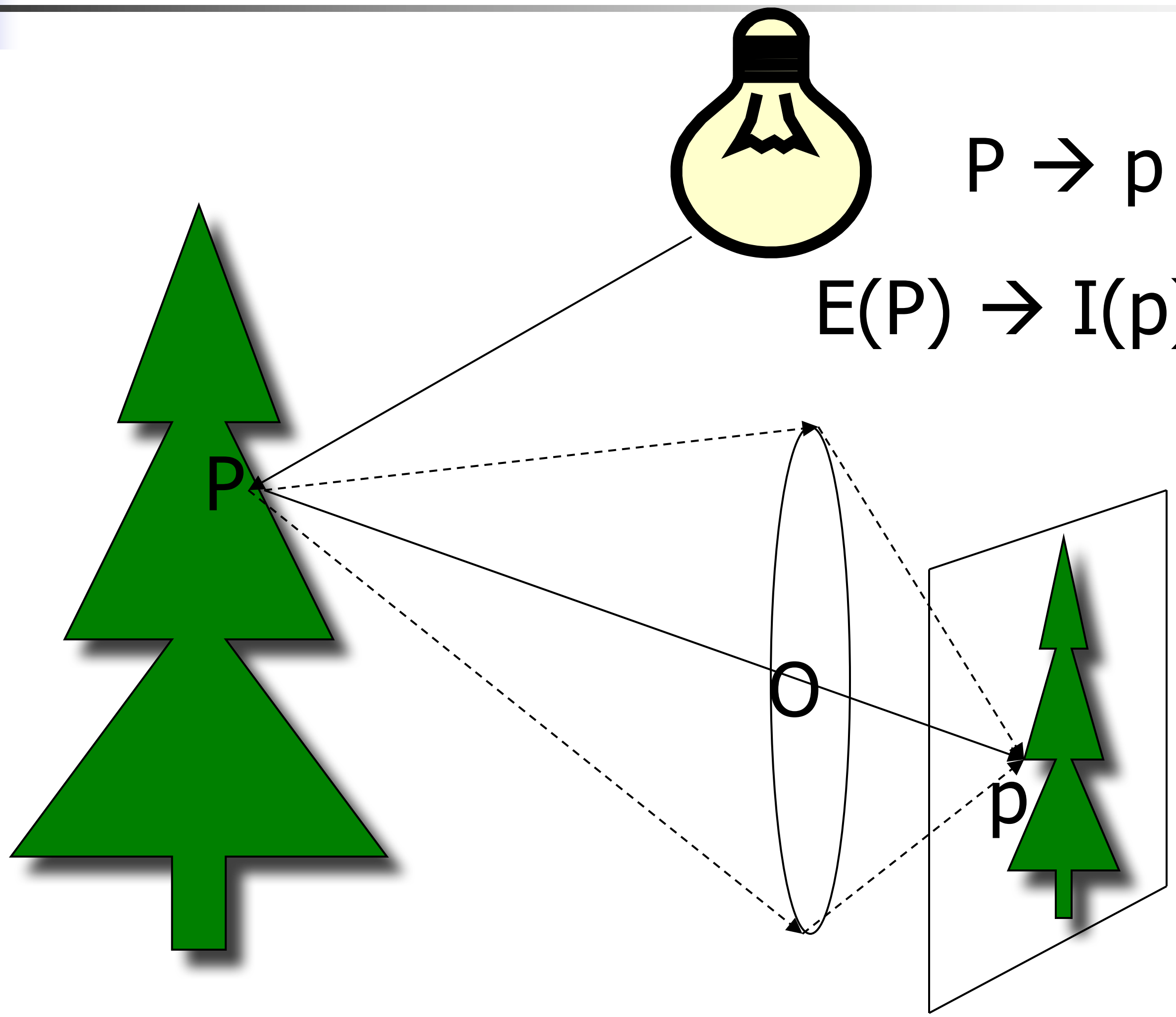
Image formation in optical camera

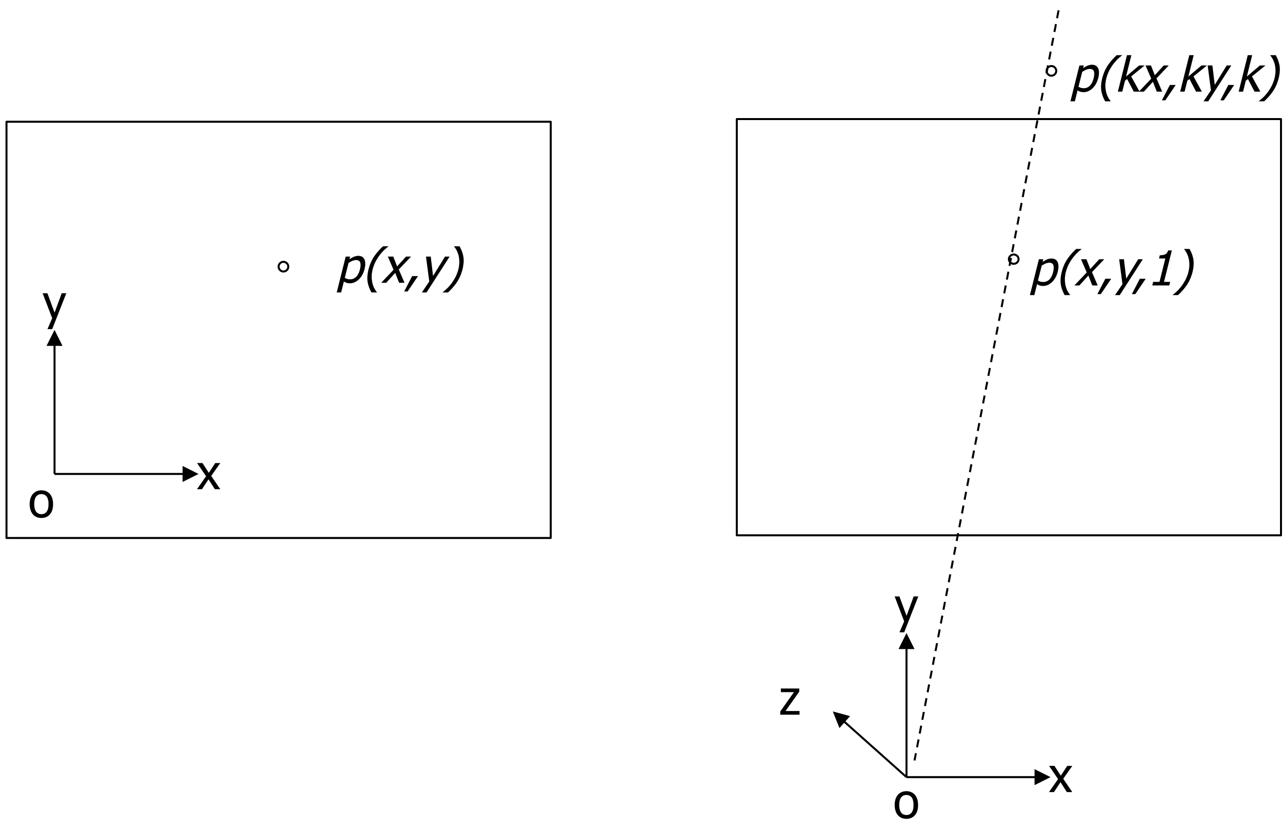
$P \rightarrow p$: Projection

$E(P) \rightarrow I(p)$: Stimulation



Real Space and Projective Space (2D)

$p \in R^2$



$p \in P^2$

$p(x,y)$ ↔ $p(kx,ky,k)$

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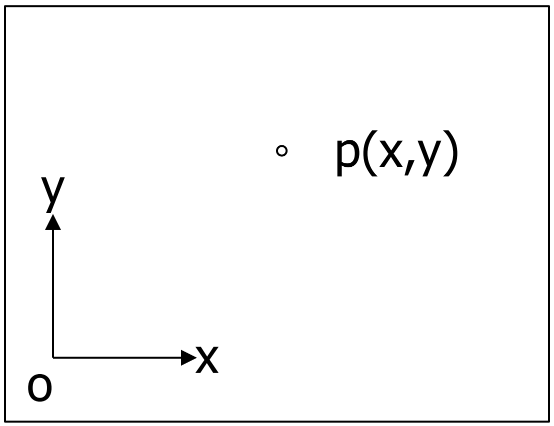
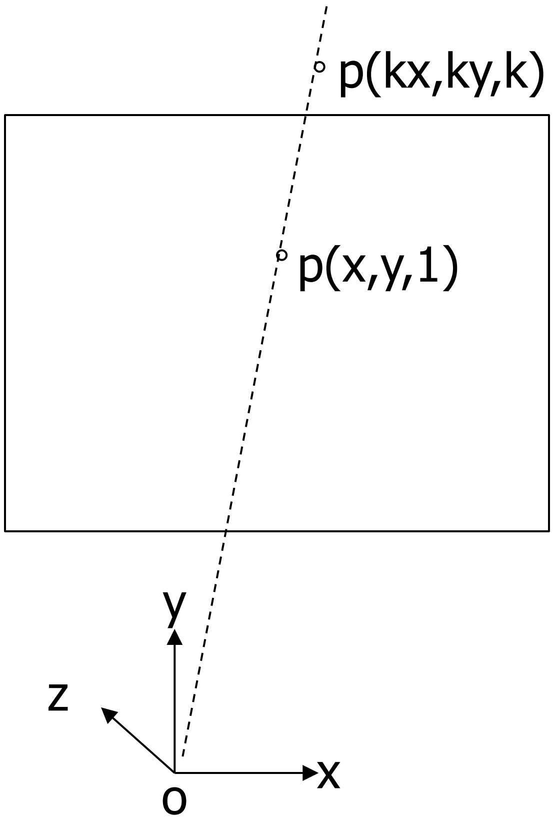




$p \in P^2$

p(x,y) ↔ p(kx,ky,k)

Homogeneous Coordinate system

Homogeneous Representation

A point in R^2: $\vec{x} \equiv \begin{bmatrix} x \\ y \end{bmatrix}$ $\Leftrightarrow$ A point in P^2: $\vec{X} \equiv \begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$

$$P^2 = R^3 - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Singular point in the projective space.

Homogeneous Representation

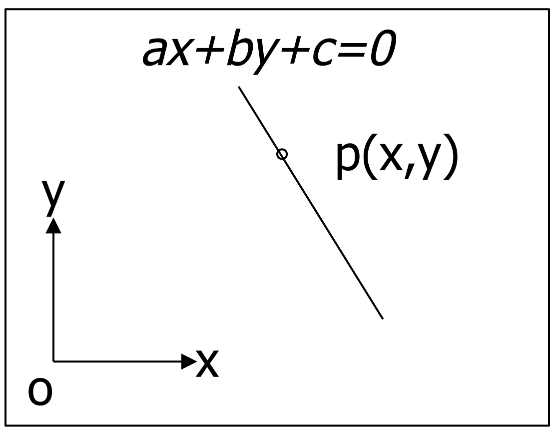
$$\text{In } R^2\text{: } ? \quad \Longleftrightarrow \quad \text{In } P^2 : \vec{X} \equiv \begin{bmatrix} 25 \\ 30 \\ 5 \end{bmatrix}$$

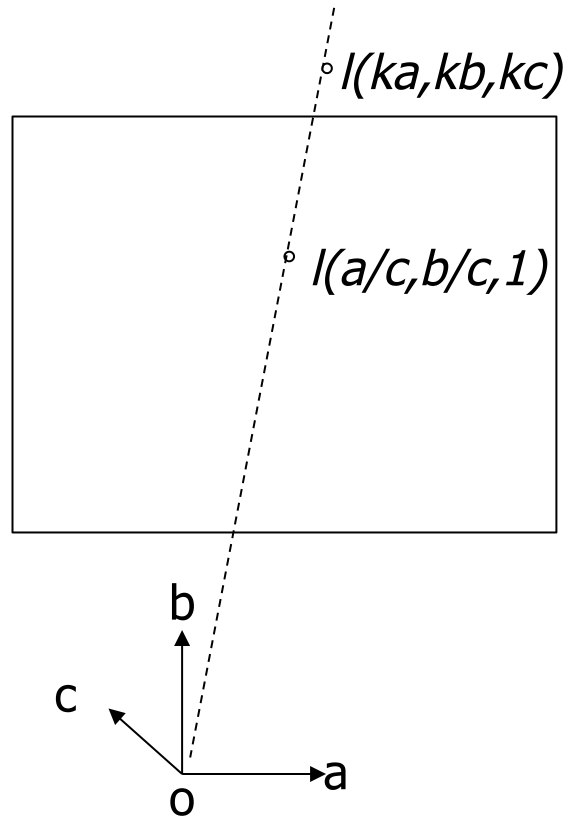
$$\text{The point in } R^2\text{: } \vec{x} \equiv \begin{bmatrix} \frac{25}{5} \\ \frac{30}{5} \end{bmatrix} \equiv \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\text{In } P^2 : \vec{X} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longleftrightarrow \quad \text{In } R^2\text{: } ?$$

Homogeneous representation of a line in a plane

$p \in R^2$



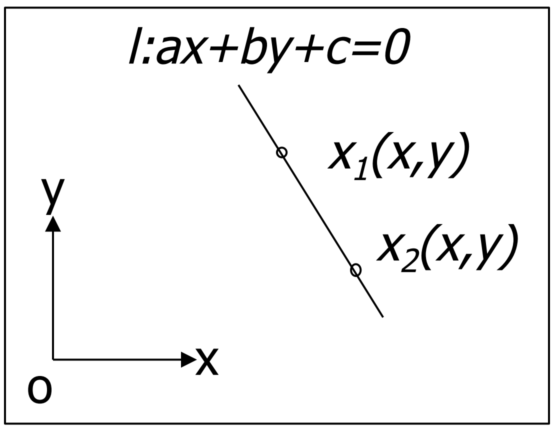


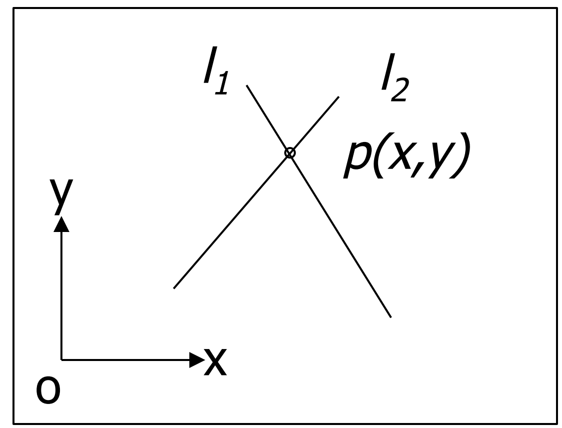
$l \in P^2$

Point containment in P^2 $\Rightarrow$ $\vec{X}^T.\vec{l}=0 \Leftrightarrow \vec{l}^T.\vec{X}=0$

Points and lines in P^2

$x_1 \in R^2$





$$\vec{l} = \vec{X}_1 \mathrm{x}\, \vec{X}_2$$

$$\vec{P} = \vec{l}_1 \mathrm{x}\, \vec{l}_2$$

Exactly one line through two points.

Exactly one point at intersection of two lines.

Examples

1. Compute the line passing through (3,5) and (5,0) in a plane.

$$\vec{l} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -25 \end{bmatrix}$$

2. Compute the point of intersection of the lines: 5x-2y+4=0 and 6x-7y-3=0.

$$\vec{P} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \times \begin{bmatrix} 6 \\ -7 \\ -3 \end{bmatrix} = \begin{bmatrix} 34 \\ 35 \\ -23 \end{bmatrix}$$

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

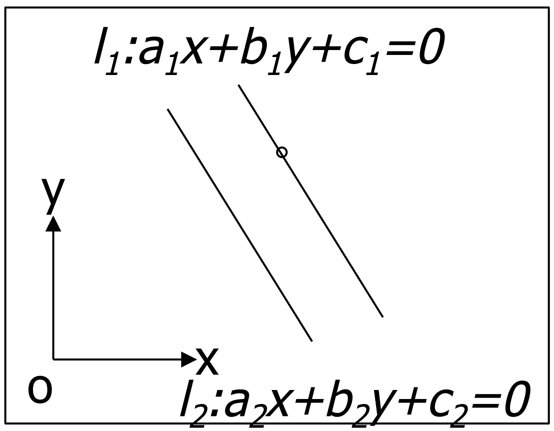
$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

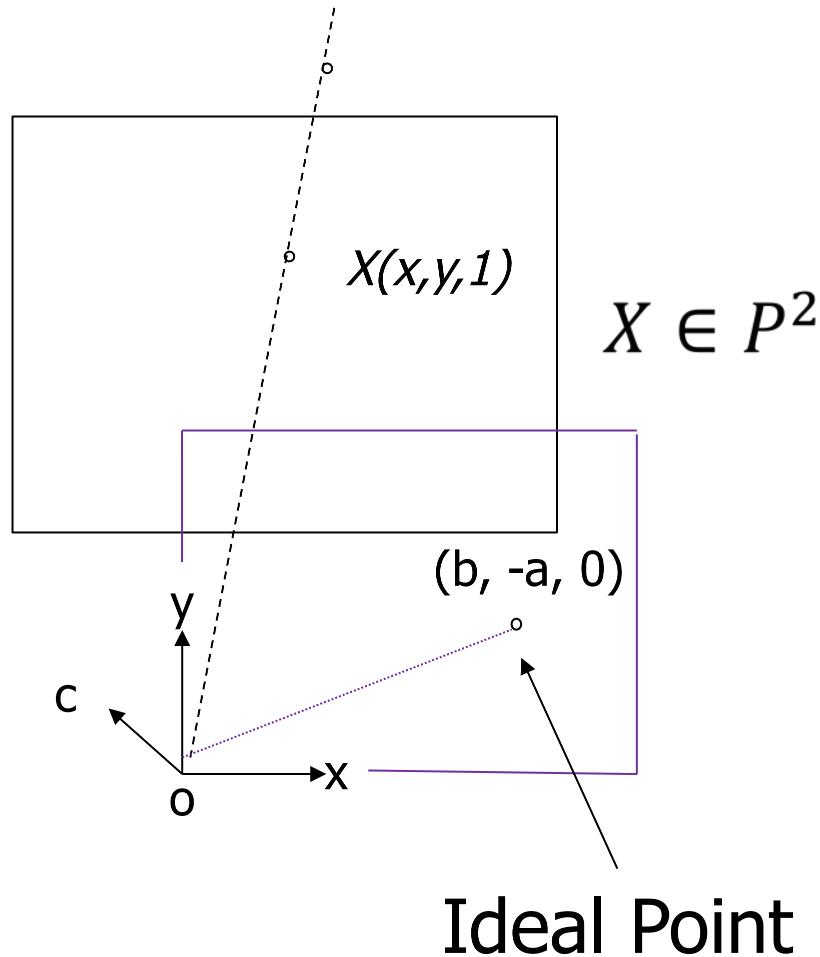
Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem.

Intersection of parallel lines



$$\vec{l}_1 \times \vec{l}_2 = (c_2 - c_1)\begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$



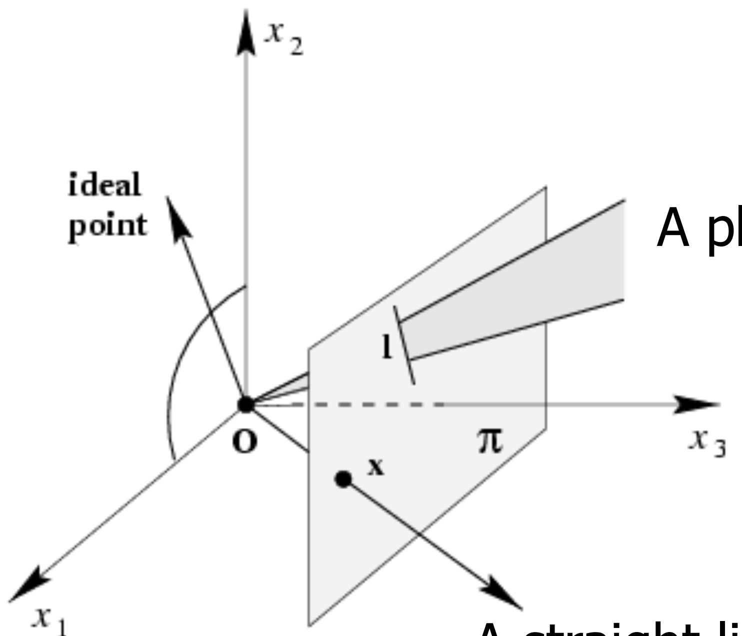
Ideal points and line at infinity

Ideal points: Points on the X-Y plane or principal plane parallel to projection plane.
For canonical coordinate system, they are of the form:

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Line at infinity (l_∞): Line containing every ideal point.
In canonical system, it is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

A model for the projective plane



A plane passing through the origin.

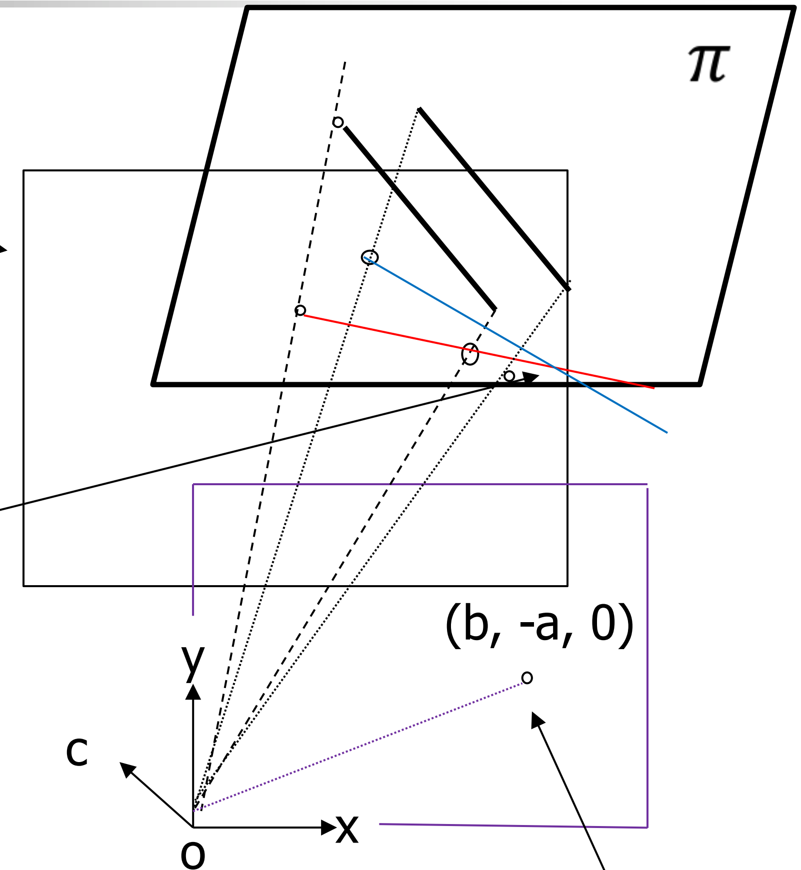
l_∞

A straight line passing through the origin.

$$P^2 = R^3 - \begin{bmatrix}0\\0\\0\end{bmatrix} = R^2 \cup l_\infty$$

Intersection of parallel lines on any arbitrary plane

Canonical projection plane (CPP)

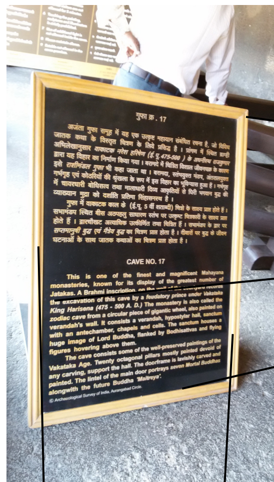


Vanishing Point

Point of intersection of parallel lines on π.

Ideal Plane

Examples of vanishing points



Can you use this property to cluster sets of parallel lines in an image?

Vanishing points

Thank you!