Camera Geometry

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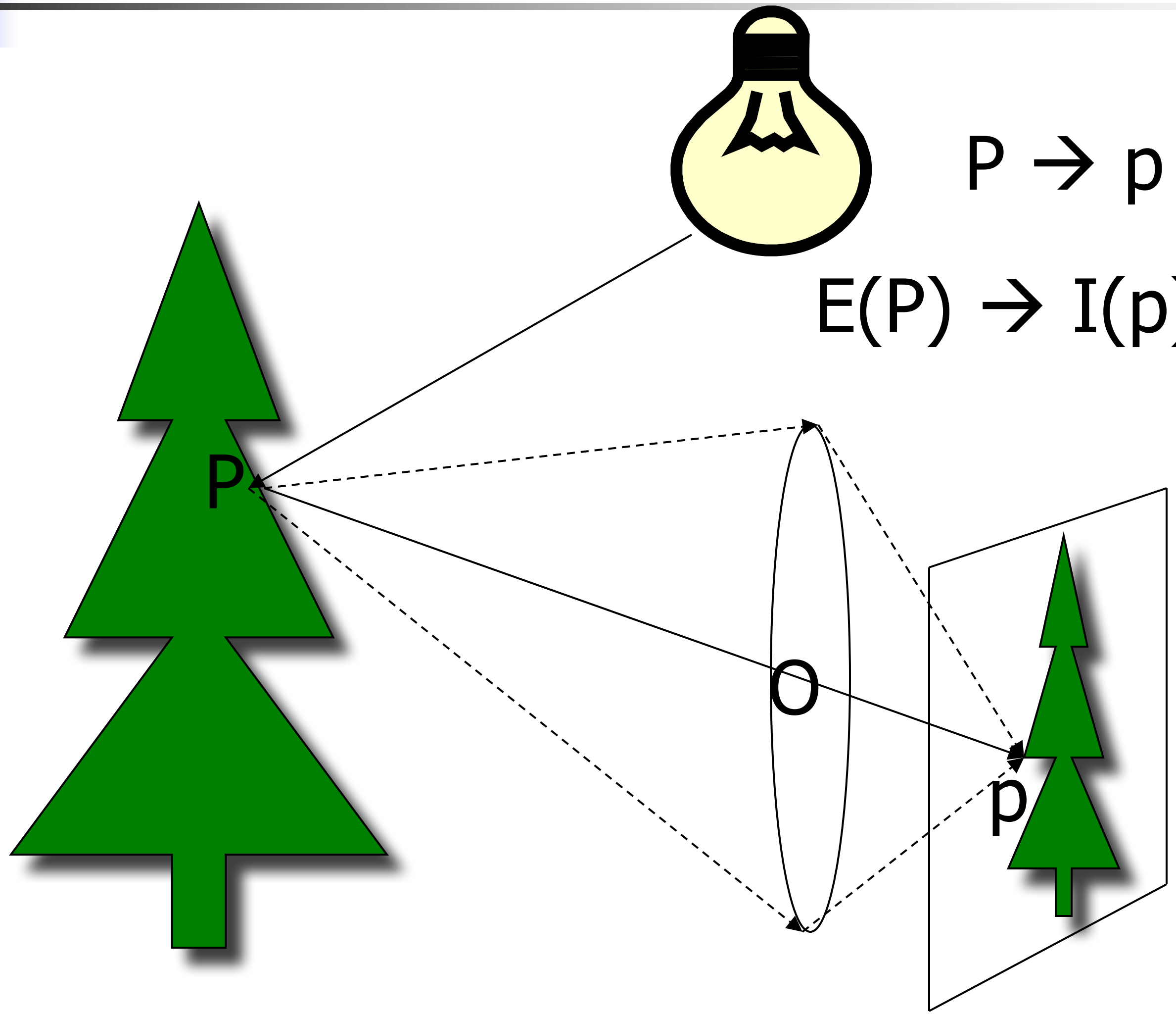
Image formation in optical camera



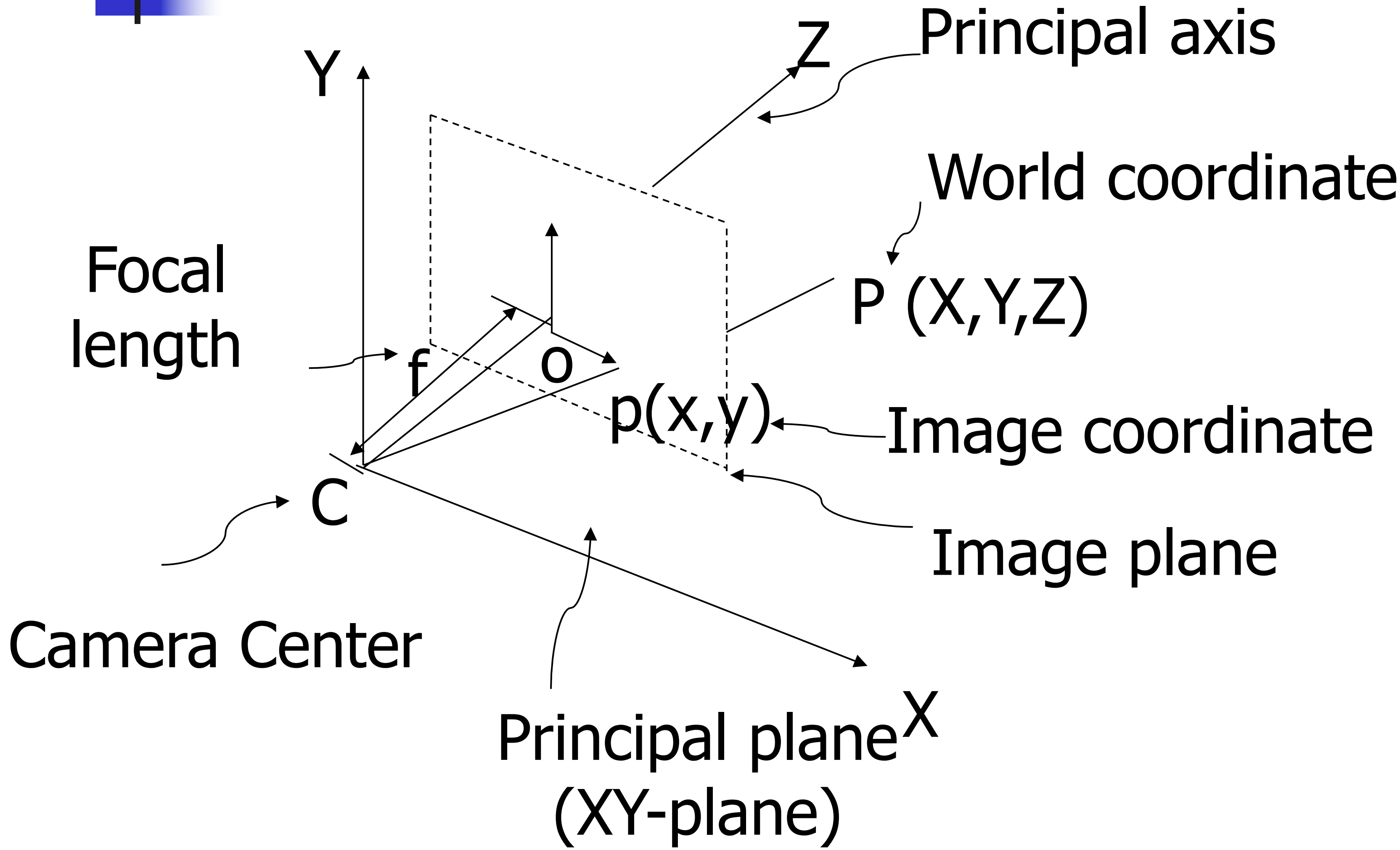
P → p : Projection

E(P) → I(p) : Stimulation

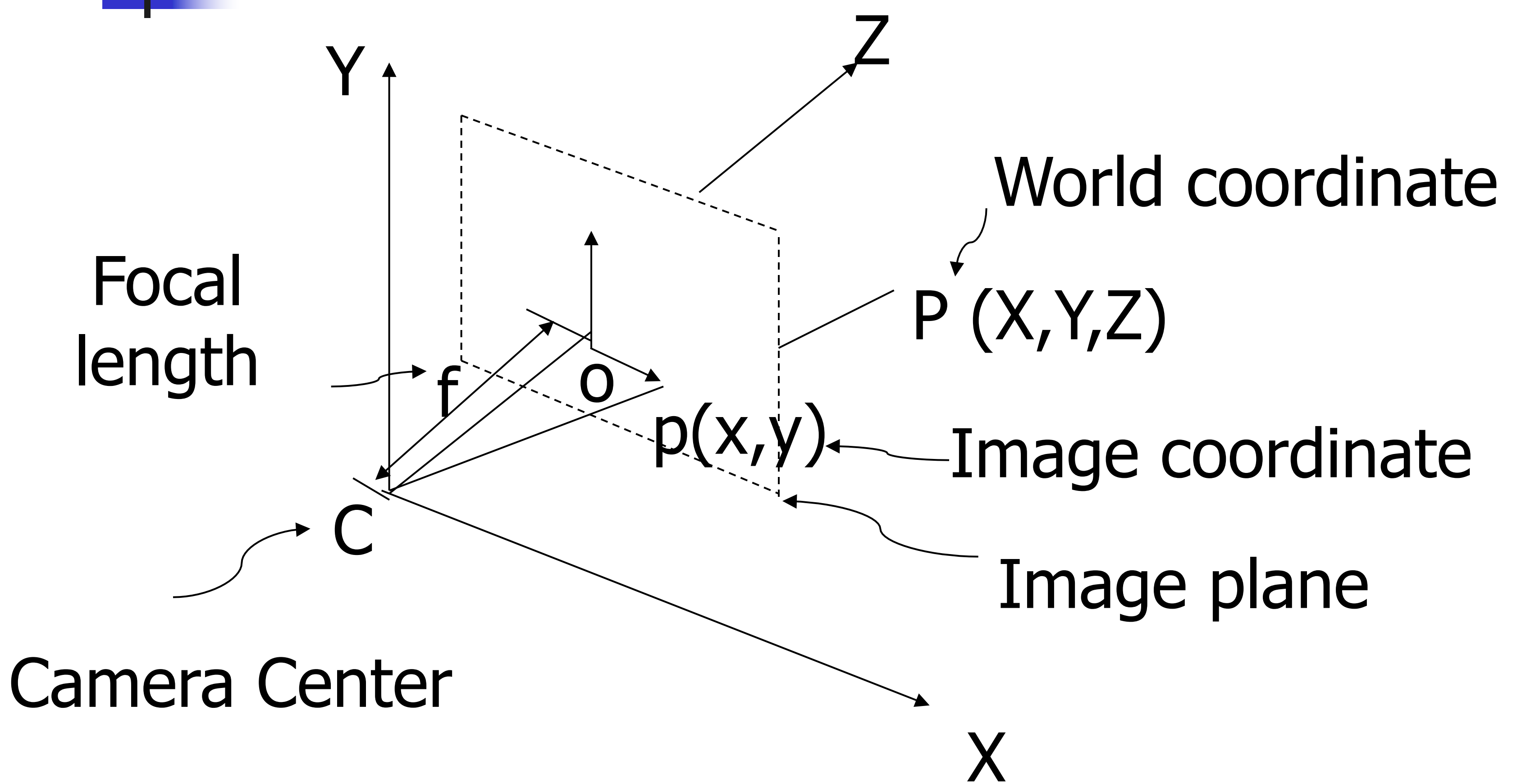
Pinhole camera

$$x = \frac{fX}{Z}$$

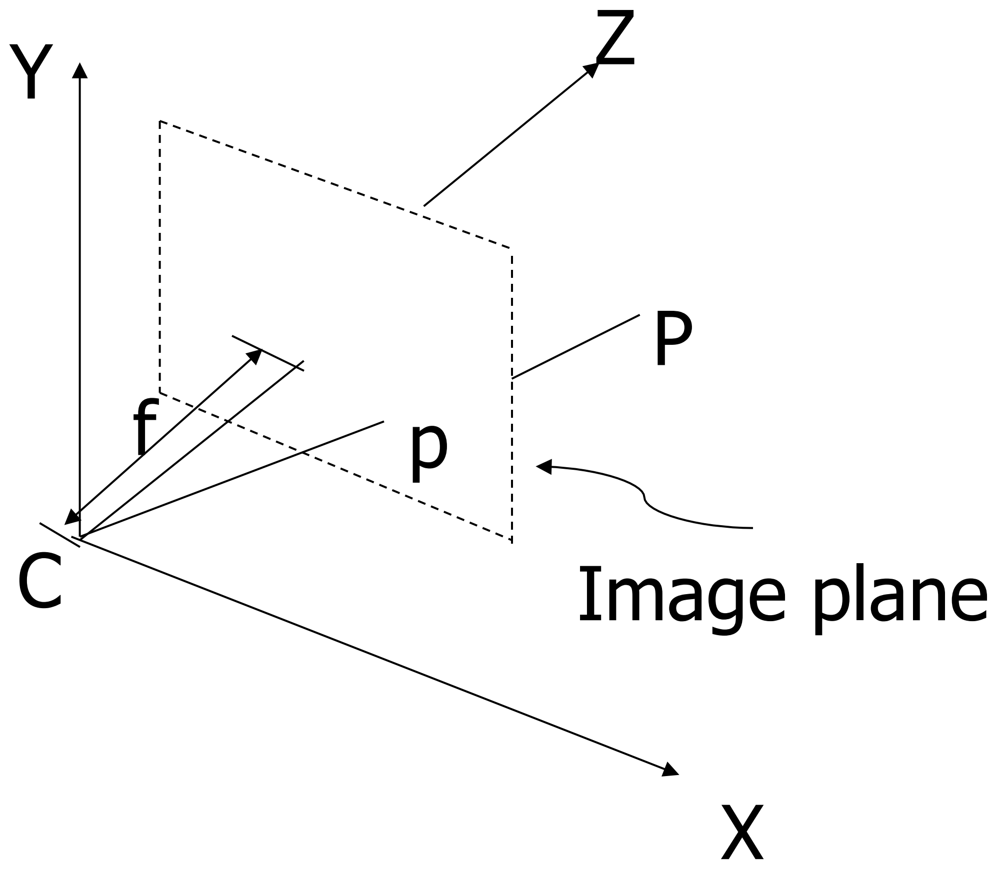
$$y = \frac{fY}{Z}$$



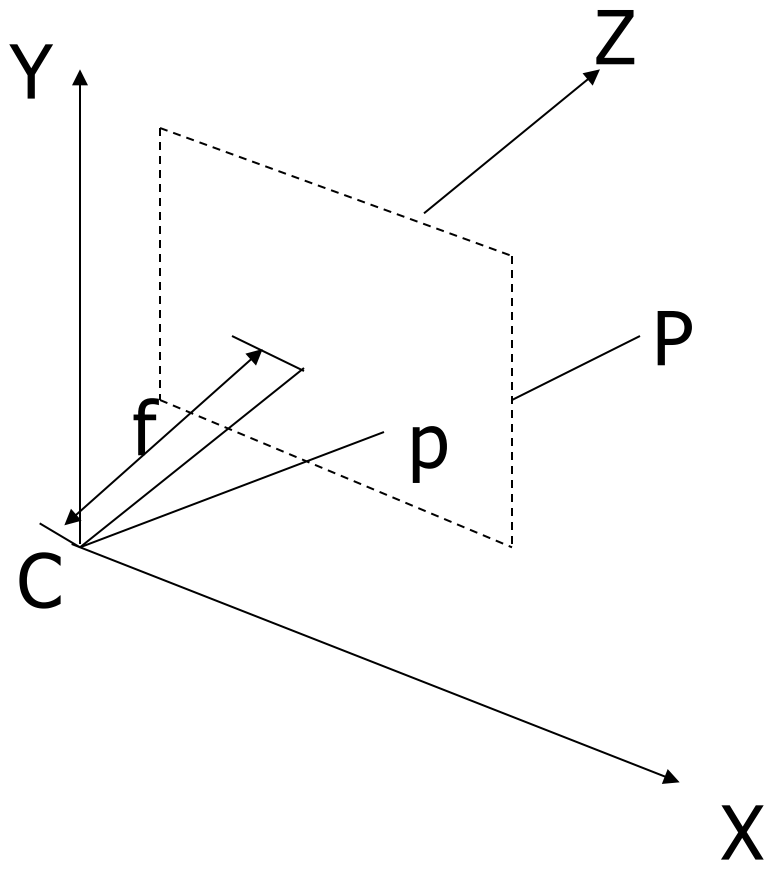
Pinhole camera



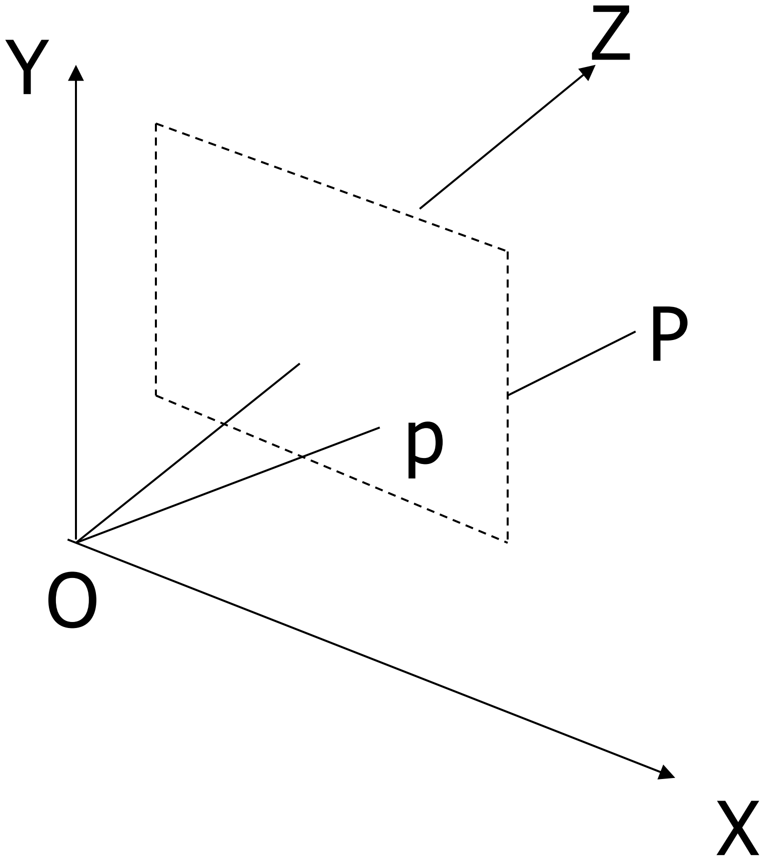
Pinhole camera



Pinhole camera



Pinhole camera



Pinhole Camera: Mapping from P³→P²

$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv \begin{bmatrix} fX \\ fY \\ Z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

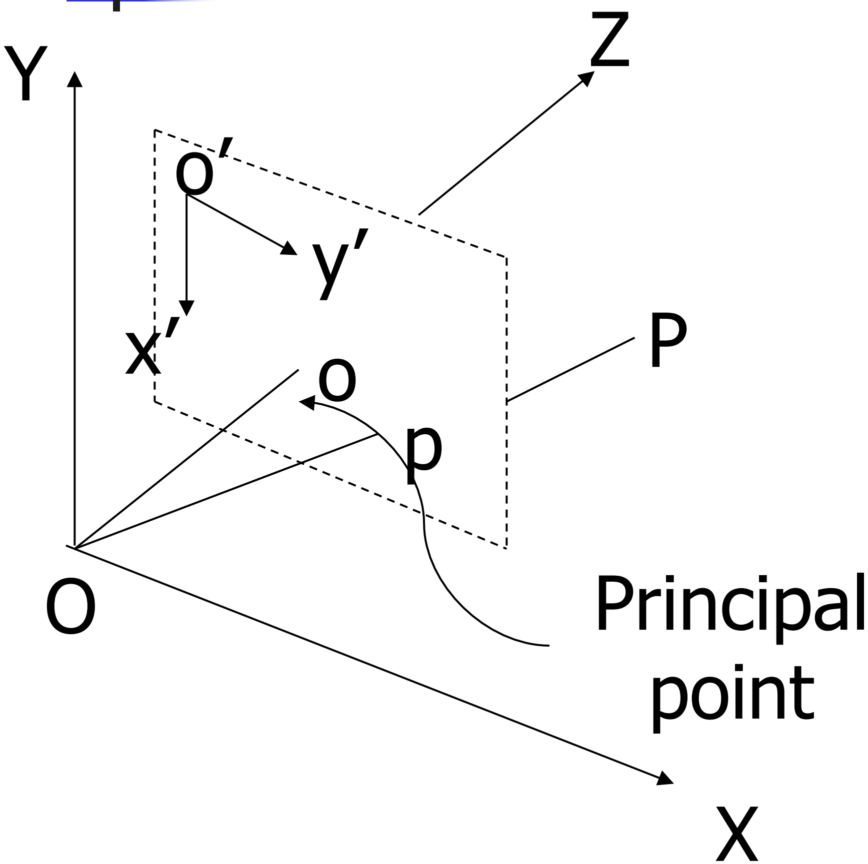
Projection Matrix (P)

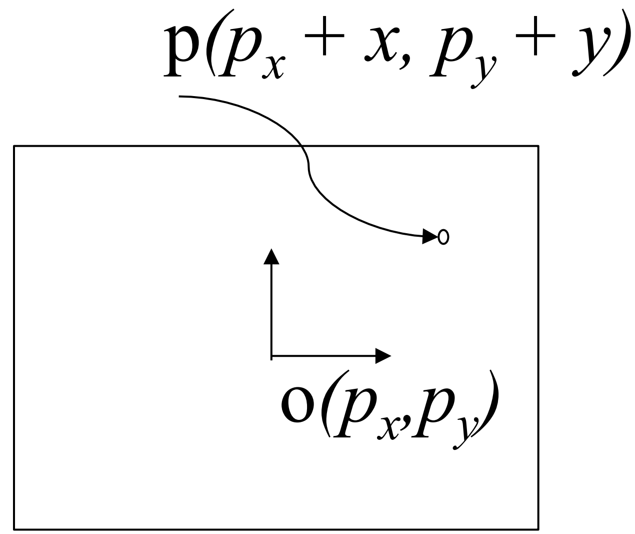
$$P = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = diag(f, f, 1)[I \quad | \quad 0]$$

Offset of principal point

$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$





$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Projection Matrix under the offset

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = K[I \quad | \quad 0]$$

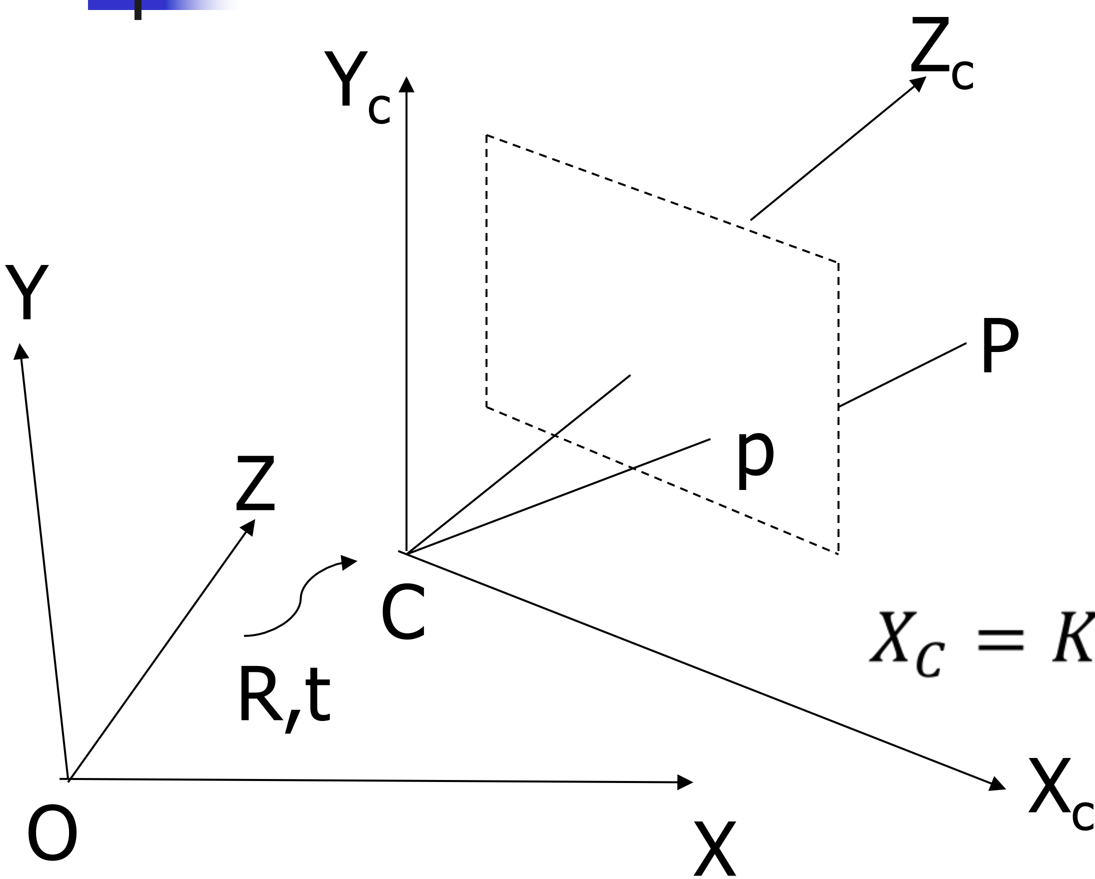
$$\boldsymbol{x} = K[I \quad | \quad 0]\boldsymbol{X}$$

K (Camera Calibration Matrix)

$$\tilde{X} \equiv \mathit{Inhomogeneous\ Coordinate}$$

$$X = \begin{bmatrix} \tilde{X} \\ 1 \end{bmatrix} \equiv \mathit{Homogeneous\ Coordinate}$$

Shifting of world coordinate



$$\widetilde{X_c} = R\left(\tilde{X} - \tilde{C}\right)$$

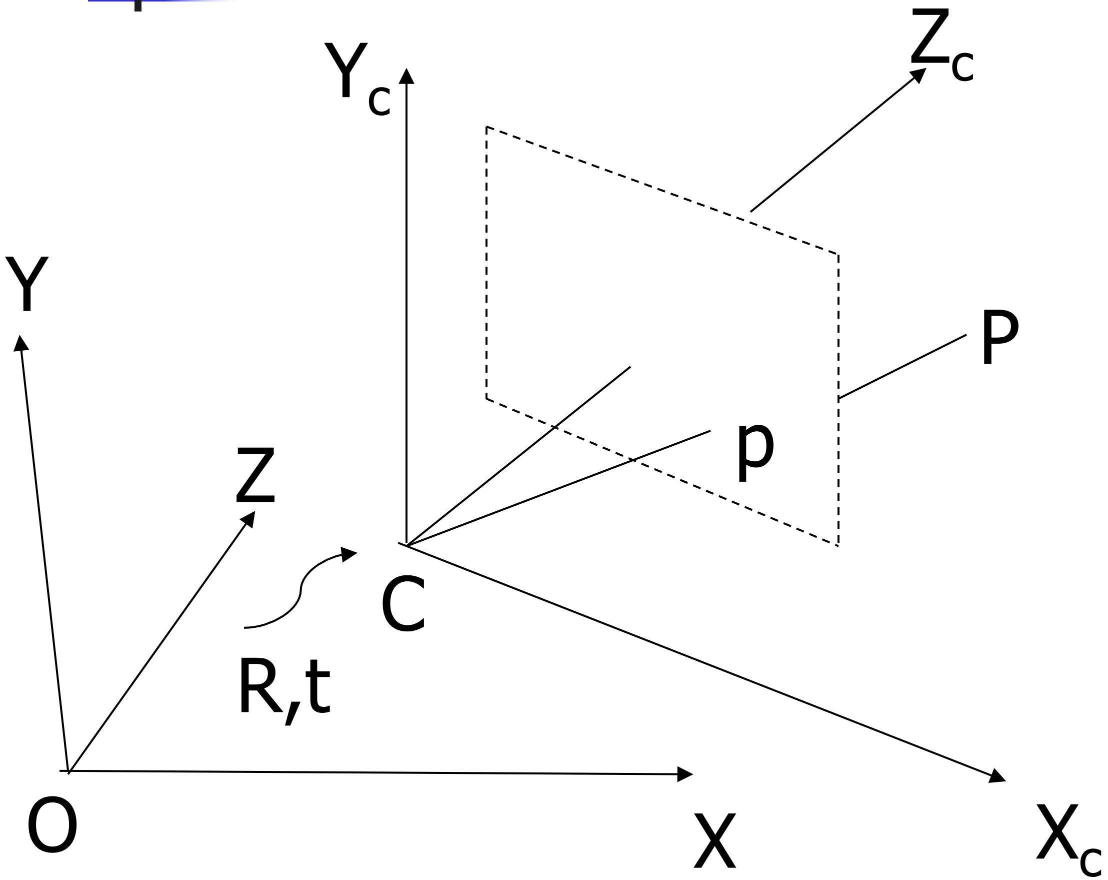
$$X_C = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = K[I \quad | \quad 0]\boldsymbol{X_c}$$

$$X_C = K[I \quad | \quad 0] \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$P = KR[I \quad | \quad -\tilde{C}] = K[R \quad | \quad t]$$

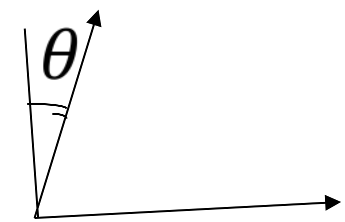
Shifting of world coordnate



CCD Camera model

$$P = KR[I \quad | \quad -\tilde{C}] = K[R \quad | \quad t]$$

$$where \;\; K = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$$



$Let \; \alpha_x = f.m_x,$

$\alpha_y = f.m_y$

No. of pixels per unit length

$s = \tan\theta$

$$K = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

General Projective Camera

$$P = KR[I \quad | \quad -\tilde{C}] = K[R \quad | \quad t]$$

$$where\ K = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

3

3

11 d.o.f

5

Extrinsic parameters: R, t

Intrinsic parameters: K

$$|K| = \alpha_x\ \alpha_y > 0$$

$$P = [M \quad | \quad p_4] = M[I \quad | \quad M^{-1}p_4] = KR[I \quad | \quad -\tilde{C}]$$

$$where\ M = KR\ and\ p_4\ is\ the\ last\ column\ of\ P.$$

Properties of projective camera $P = [M \mid p_4]$

Rank of P: 3; Size: 3x4; d.o.f.=11;
# of extrinsic params: 6, # of intrinsic params: 5

$\boldsymbol{x} = P\boldsymbol{X}$ ← Two independent equations

Minimum # of point correspondences between world and image coordinates required to estimate P: 6

Rank: 2 → Range of matrix mapping: **line**.
Rank: 1 → Range of mapping: **point**.

Estimation of the camera matrix (P)

$$P = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

$$\boldsymbol{X_i} \leftrightarrow \boldsymbol{x_i} = (x_i \quad y_i \quad w_i)^T \ for\ i = 1,2,\ldots,n \geq 6$$

$$P\boldsymbol{X_i} \equiv \boldsymbol{x_i}$$

$$\Longrightarrow P\boldsymbol{X_i} \times \boldsymbol{x_i} = 0$$

$$\Longrightarrow \begin{bmatrix} 0^T & -w_i\boldsymbol{X_i^T} & y_i\boldsymbol{X_i^T} \\ w_i\boldsymbol{X_i^T} & 0^T & -x_i\boldsymbol{X_i^T} \\ -y_i\boldsymbol{X_i^T} & x_i\boldsymbol{X_i^T} & 0^T \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0$$

Redundant, as x_i x (1) + y_i x (2)= w_i x (3)

Estimation of the camera matrix (P)

$$P = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

$$\boldsymbol{X_i} \leftrightarrow \boldsymbol{x_i} = (x_i \quad y_i \quad w_i)^T \;\; for\; i = 1,2,\ldots,n \geq 6$$

$$\begin{bmatrix} 0^T & -w_i\boldsymbol{X_i^T} & y_i\boldsymbol{X_i^T} \\ w_i\boldsymbol{X_i^T} & 0^T & -x_i\boldsymbol{X_i^T} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0$$

For n correspondences $$A_{2n\times 12} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0$$

Minimize $||A\boldsymbol{p}||$ subject to $||\boldsymbol{p}||=1$

Use similar techniques, such as DLT.

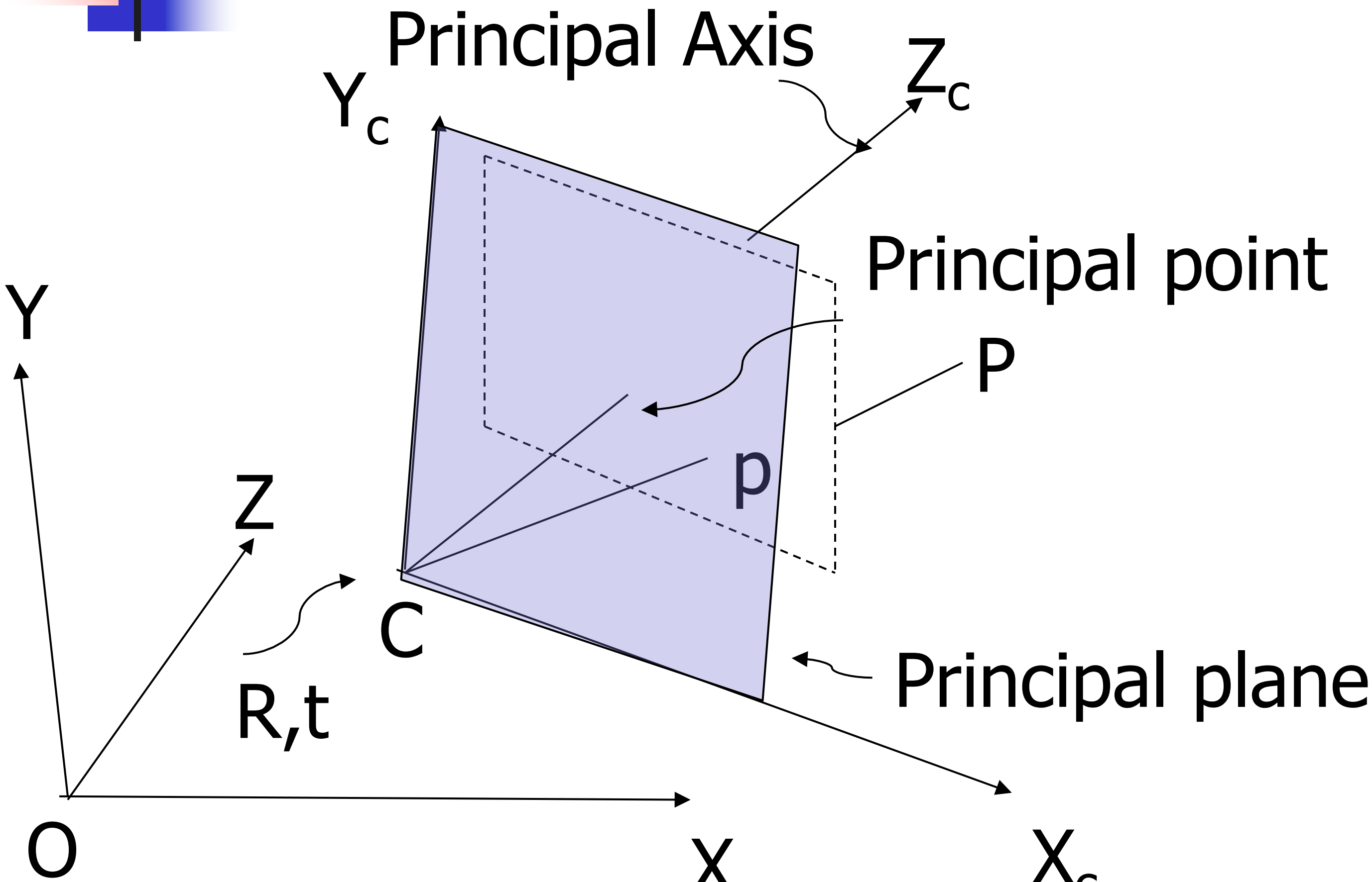
Properties of projective camera $P = [M \mid p_4]$

$$P \equiv [p_1 \quad p_2 \quad p_3 \quad p_4] \equiv \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

1. Camera Center (C): 1-D right null space of P, i.e. $PC=0$.
   1. Finite camera: M non-singular.
   2. Camera at infinity: M singular C= $\begin{bmatrix} d \\ 0 \end{bmatrix}$
2. Column points: p_1, p_2, and p_3 are vanishing points of X, Y and Z axes. p_4 is the image of coordinate origin.

$$p_1 = [p_1 \quad p_2 \quad p_3 \quad p_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad p_4 = [p_1 \quad p_2 \quad p_3 \quad p_4] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Principal plane, axis, and point



Properties of projective camera $P = [M \quad | \quad p_4]$

$$P \equiv [p_1 \quad p_2 \quad p_3 \quad p_4] \equiv \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

3. Principal plane: Plane parallel to image plane: r_3; As any point belonging to this plane should be imaged at $[x\ y\ 0]^T$, $r_3^T \boldsymbol{X}$=0.
4. Axes plane: $r_1^T \boldsymbol{X}$=0 → Imaged at y-axis of the image coordinate, i.e. plane containing camera center ($r_1^T \boldsymbol{C}=0$) and y-axis of image plane.
5. Similarly, $r_2^T \boldsymbol{X}$=0 → Plane defined by camera center ($r_2^T \boldsymbol{C}=0$) and x-axis of image plane.
6. Principal point: $M.\ \boldsymbol{mr_3}$; $\boldsymbol{mr_3}$ is third row of M.

Properties of projective camera $P = [M \quad | \quad p_4]$

$$P \equiv [p_1 \quad p_2 \quad p_3 \quad p_4] \equiv \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

6. Principal point: $M.\,\boldsymbol{mr_3}$; $\boldsymbol{mr_3}$ is third row of M. A point at infinity along the normal of $r_3^T X$=0 plane is projected to the principal point ($\boldsymbol{x_0}$).

$$\boldsymbol{x_0} = P \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \\ 0 \end{bmatrix} = M.\,\boldsymbol{mr_3}$$

7. Principal Ray: $\boldsymbol{mr_3}$; $\boldsymbol{mr_3}$ is the third row of M. A point at infinity along the normal of $r_3^T X$=0 plane is projected to the principal point ($\boldsymbol{x_0}$). det(M). $\boldsymbol{mr_3}$ directed towards front of camera.

Projective camera on points

Forward projection: Mapping of vanishing points $(\boldsymbol{d}, 0)^T$ on the plane at infinity (π_∞):

$$\boldsymbol{x} = [M \mid p_4] \begin{bmatrix} \boldsymbol{d} \\ 0 \end{bmatrix} = M\boldsymbol{d}$$

Only affected by M.

Back Projection:

$$[M \mid p_4] \begin{bmatrix} M^{-1}\boldsymbol{x} \\ 0 \end{bmatrix} = \boldsymbol{x}$$

$$D = \begin{bmatrix} M^{-1}\boldsymbol{x} \\ 0 \end{bmatrix}$$

$$X(\mu) = \mu D + C$$

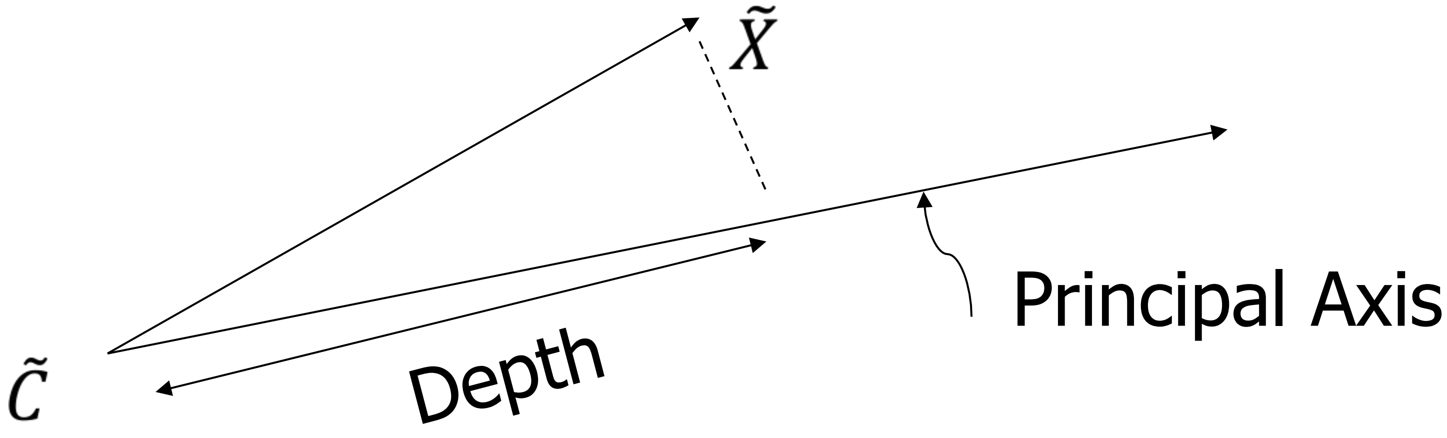
$$= \mu \begin{bmatrix} M^{-1}\boldsymbol{x} \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{C} \\ 1 \end{bmatrix}$$

$$= \mu \begin{bmatrix} M^{-1}\boldsymbol{x} \\ 0 \end{bmatrix} + \begin{bmatrix} -M^{-1}p_4 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \tilde{C} \\ 1 \end{bmatrix}$$

$$X(\mu) = \begin{bmatrix} M^{-1}(\mu\boldsymbol{x} - p_4) \\ 1 \end{bmatrix}$$

Depth of points



$$Depth = \widehat{mr_3^T}.(\tilde{X} - \tilde{C})$$

Computing camera center for $P = [M \quad | \quad p_4]$

$$M = [p_1 \quad p_2 \quad p_3] \qquad \tilde{C} = [X_c \quad Y_c \quad Z_c]^{\mathsf{T}}$$

$$PC = 0 \implies [M \quad | \quad p_4]\begin{bmatrix}\tilde{C}\\1\end{bmatrix} = 0$$
$$\implies M\tilde{C} = -p_4$$

$$X_c = \frac{|-p_4 \quad p_2 \quad p_3|}{|p_1 \quad p_2 \quad p_3|} \qquad Y_c = \frac{|p_1 \quad -p_4 \quad p_3|}{|p_1 \quad p_2 \quad p_3|}$$

$$Z_c = \frac{|p_1 \quad p_2 \quad -p_4|}{|p_1 \quad p_2 \quad p_3|}$$

Camera parameters from P

$$
\begin{aligned}
P &= [M \mid p_4] \\
&= [M \mid -M\tilde{C}] \\
&= K[R \mid -R\tilde{C}]
\end{aligned}
$$

1. RQ-decomposition of M s.t. $M=KR$, where K is an upper-triangular matrix and R is an orthogonal matrix.
2. Obtain camera center using $M\tilde{C} = -p_4$.
3. From R get the orientation of camera.
4. From K get elements of calibration matrix.

Cameras at ∞

$P = [M \quad | \quad p_4]$ where M is singular.

Affine: Last row of P
$[0 \quad 0 \quad 0 \quad 1]$

Non-affine: Otherwise

Affine Camera:

1. Principal plane → Plane at ∞ (π_∞).
2. Camera center lies on π_∞.
3. Points at ∞ are mapped to points at ∞.
4. Parallel lines remain parallel after projection.

$$P\begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Affine projection

$$\begin{bmatrix} \widetilde{\boldsymbol{x}} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{X}} \\ 1 \end{bmatrix}$$

$$[\widetilde{\boldsymbol{x}}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} [\widetilde{\boldsymbol{X}}] + \boldsymbol{t}$$

$$\widetilde{\boldsymbol{x}} = M_{2\times 3}\widetilde{\boldsymbol{X}} + \boldsymbol{t}$$

- Affine projection matrix: 8 d.o.f.
- For estimating the matrix, it requires four point correspondences.

$$[\tilde{\boldsymbol{x}}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} [\tilde{\boldsymbol{X}}] + \boldsymbol{t}$$

Affine Camera

$$\tilde{\boldsymbol{x}} = M_{2\times3}\tilde{\boldsymbol{X}} + \boldsymbol{t}$$

Camera Center→ Direction of parallel rays ($\boldsymbol{d}$)

$$M_{2\times3}\boldsymbol{d} = 0$$

- Image of the world origin: $\boldsymbol{t}$
- Principal plane for projection matrix P_A is the plane at ∞.
- Parallel world lines remain parallel in image.
- $M_{2\times3}$ should be of rank 2, to ensure P_A to be of rank 3.

$$\begin{bmatrix}\tilde{\boldsymbol{x}}\\1\end{bmatrix}=\begin{bmatrix}m_{11} & m_{12} & m_{13} & t_1\\ m_{21} & m_{22} & m_{23} & t_2\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{X}}\\1\end{bmatrix}$$

Estimation of an affine camera

$$\boldsymbol{X_i} \leftrightarrow \boldsymbol{x_i}=(x_i, y_i, 1),\ for\ i=1,2,3,\ldots,n$$

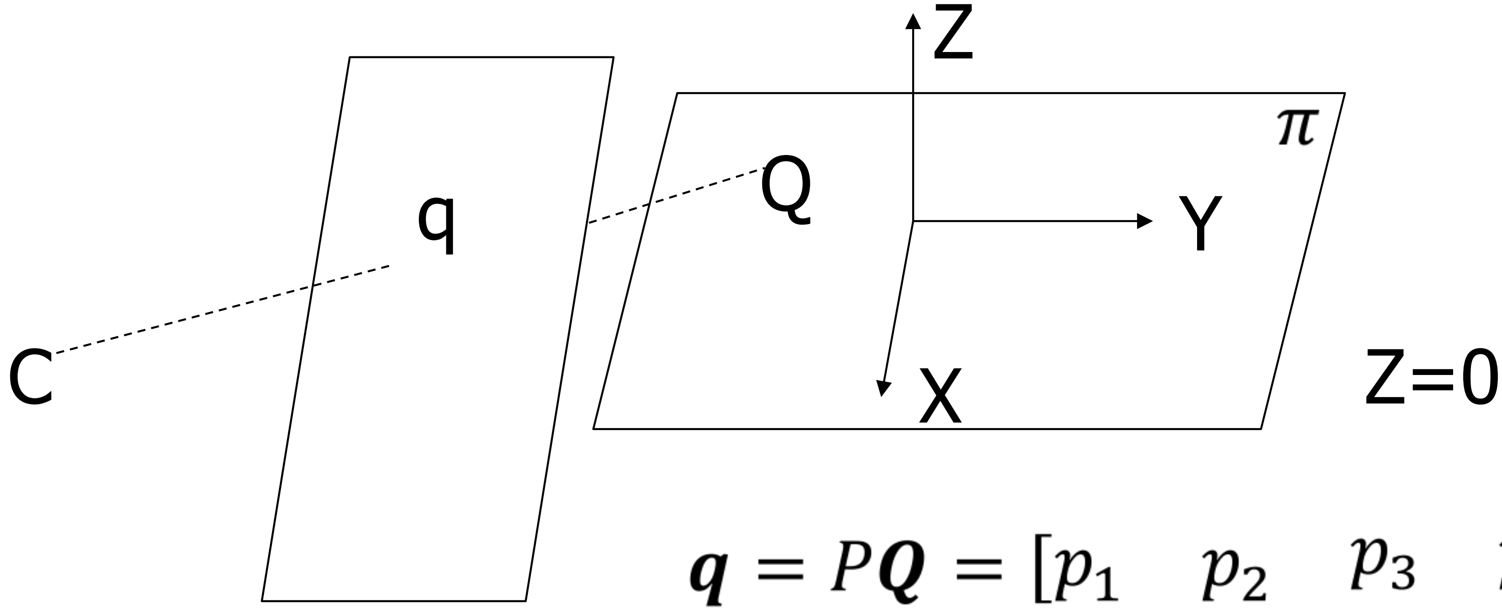
$$\boldsymbol{r_3}^T=[0 \quad 0 \quad 0 \quad 1]$$

$$\begin{bmatrix}\boldsymbol{X_i} & 0^T\\ 0^T & \boldsymbol{X_i}\end{bmatrix}\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\end{bmatrix}=\begin{bmatrix}x_i\\ y_i\end{bmatrix}$$

For n points $$A_{2n\times 8}\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\end{bmatrix}=b_{2n\times 1}$$

$$\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\end{bmatrix}=[A^T A]^{-1}A^T b$$

Projective Camera on plane

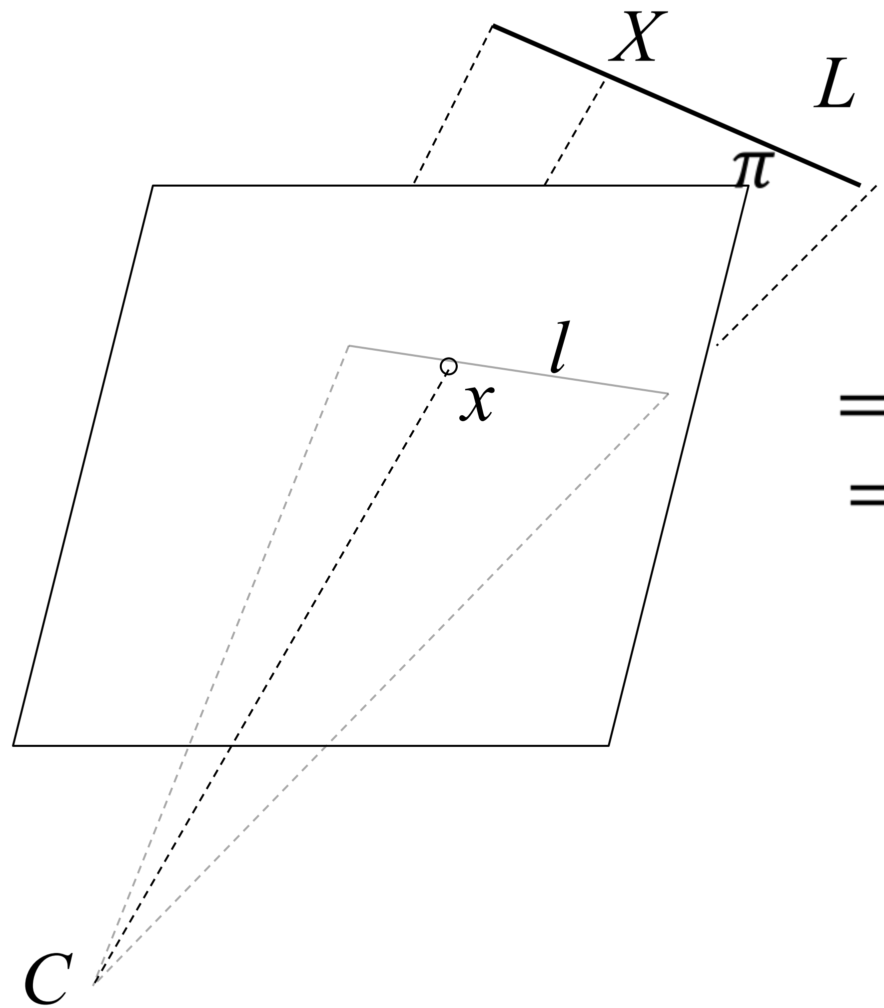


$$\boldsymbol{q} = P\boldsymbol{Q} = [p_1 \quad p_2 \quad p_3 \quad p_4] \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$= [p_1 \quad p_2 \quad p_4] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = H\,Q_\pi$$

Perspective projection of scene plane is a projective transformation.

Projective camera on a line



$$x^T l = 0$$

$$\Longrightarrow (PX)^T l = 0$$

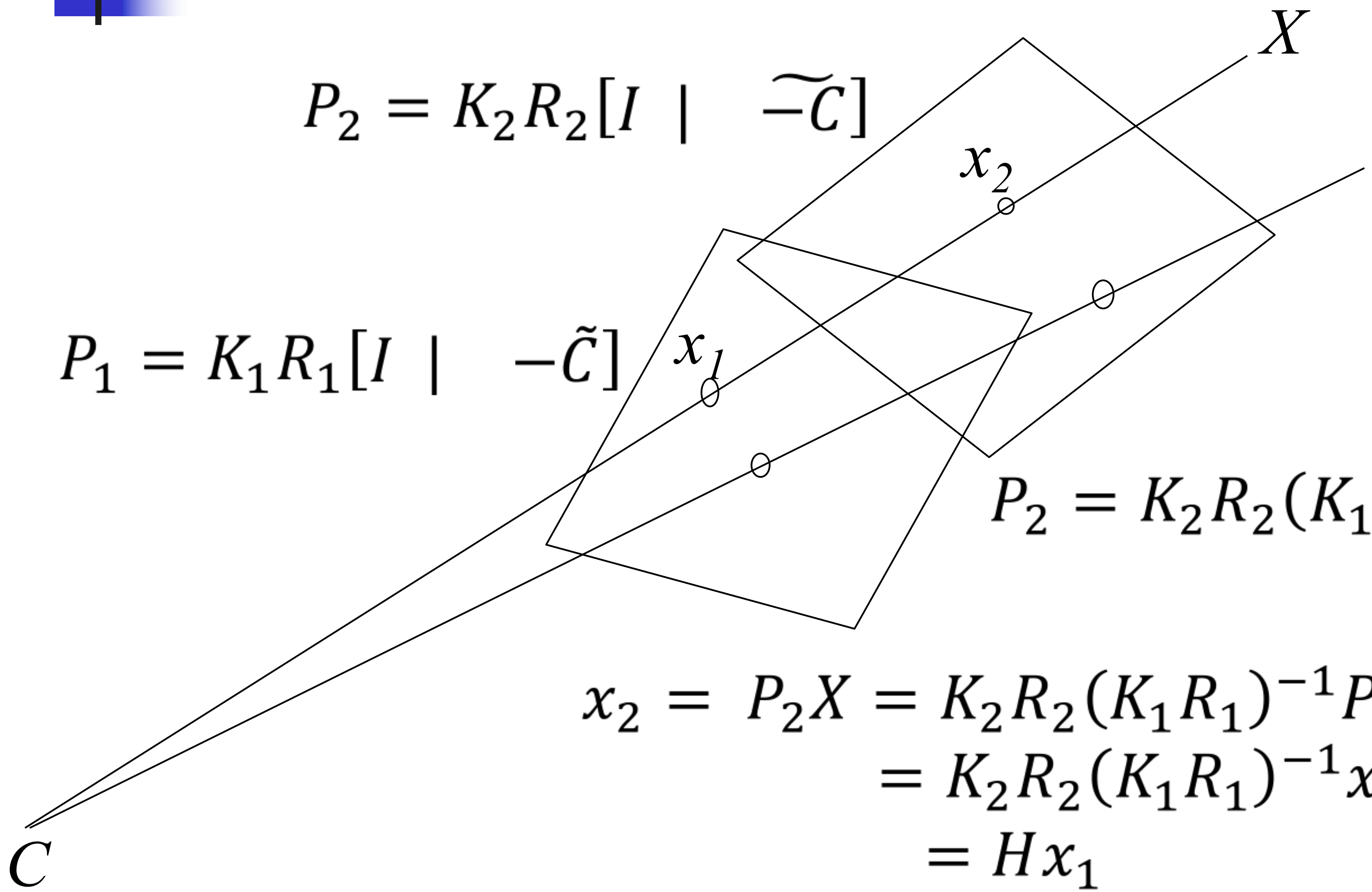
$$\Longrightarrow X^T \underbrace{P^T l}_{\pi} = 0$$

$$\pi \equiv P^T l$$

Fixed camera center and moving image plane

$$P_2 = K_2 R_2 [I \mid -\tilde{C}]$$

$$P_1 = K_1 R_1 [I \mid -\tilde{C}]$$

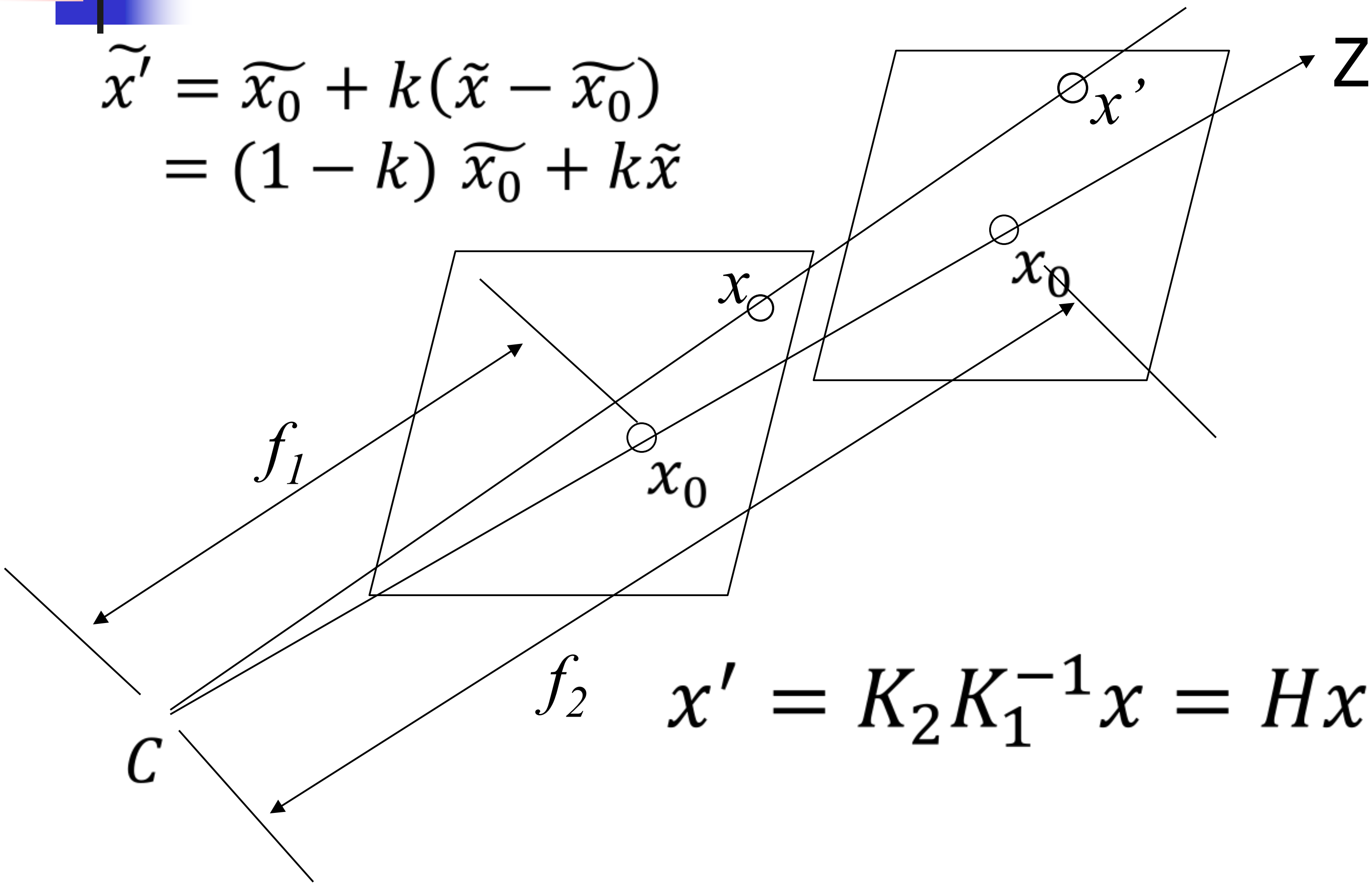


$$P_2 = K_2 R_2 (K_1 R_1)^{-1} P_1$$

$$\begin{aligned} x_2 = P_2 X &= K_2 R_2 (K_1 R_1)^{-1} P_1 X \\ &= K_2 R_2 (K_1 R_1)^{-1} x_1 \\ &= H x_1 \end{aligned}$$

Simple zooming ($k=f_2/f_1,\ R=I$)

$$\widetilde{x'} = \widetilde{x_0} + k(\tilde{x} - \widetilde{x_0})$$
$$= (1-k)\,\widetilde{x_0} + k\tilde{x}$$



$$x' = K_2 K_1^{-1} x = Hx$$

Simple Zooming

$$x' = K_2 K_1^{-1} x = Hx$$

$$\Rightarrow H = \begin{bmatrix} kI & (1-k)\ \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix} = K_2 K_1^{-1}$$

$$\Rightarrow K_2 = \begin{bmatrix} kI & (1-k)\ \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix} K_1$$

$$= \begin{bmatrix} kI & (1-k)\ \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} A & \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} kA & k\widetilde{x_0} + (1-k)\ \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} kA & \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= K_1 \begin{bmatrix} kI & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = K_1 \ .\ diag(k,k,1)$$

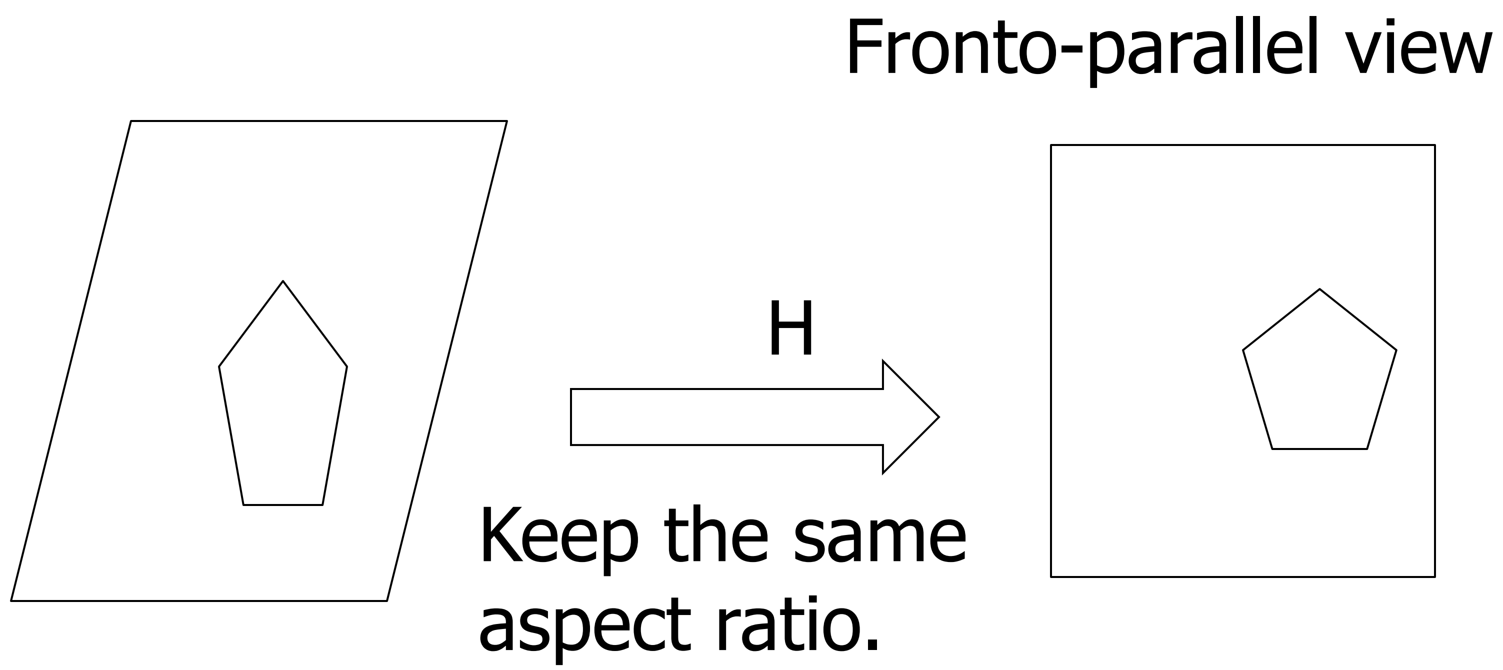
The effect of zooming by a factor k is to multiply the calibration matrix K on the right by $diag(k,k,1)$.

Rotation about an axis passing through the camera center (assuming at origin)

$$x = K[I \mid 0]X \qquad x' = K[R \mid 0]X$$
$$= KRK^{-1}K[I \mid 0]X$$
$$= KRK^{-1}x$$
$$\Longrightarrow H = KRK^{-1}$$

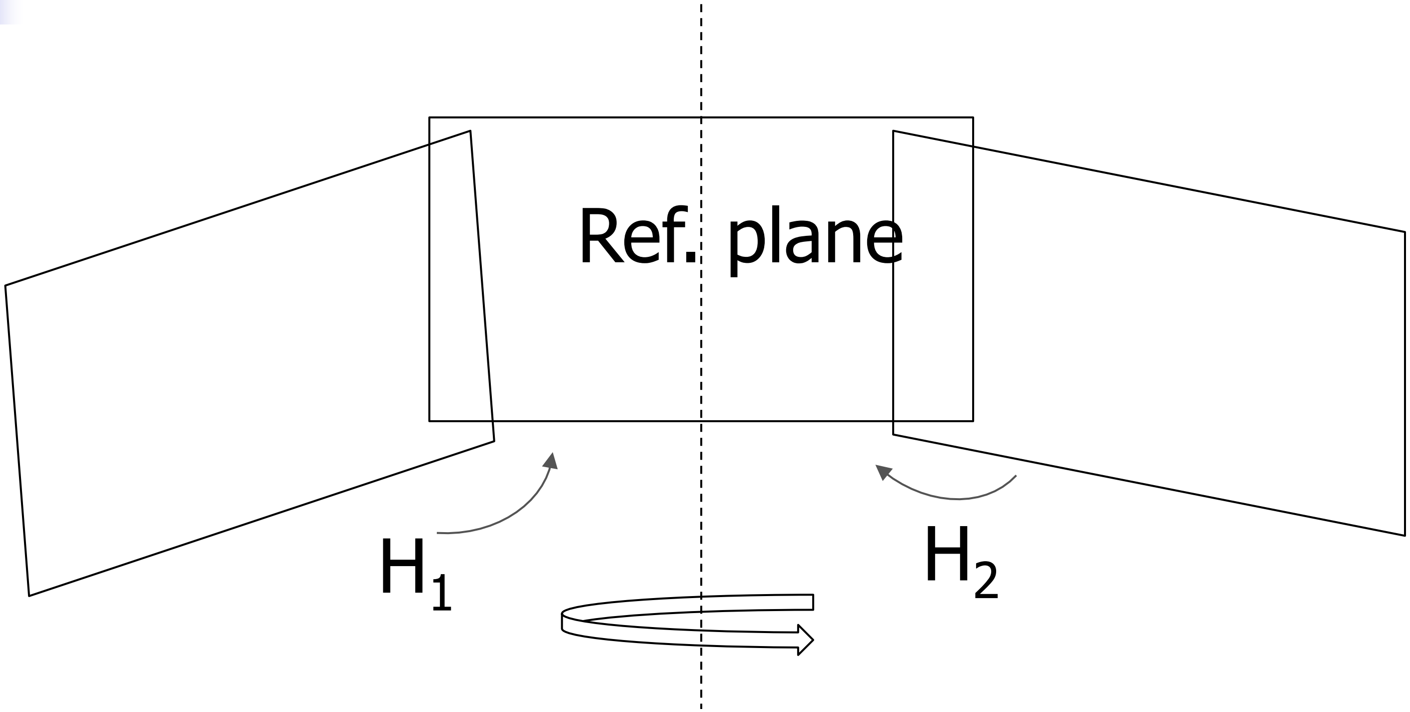
- H has the same eigen values (upto scale) as R, namely $\mu, \mu e^{i\theta}$, and $\mu e^{-i\theta}$, where μ is the scale factor.
- H is also known as *conjugate rotation* homography and can be used to measure the angle of rotation of two views.
- The eigen vector corresponding to the real eigen value (i.e. μ) is the vanishing point of the rotation axis.

Application-I: Generation of synthetic view



1. Compute H.
2. Warp the source image with H.

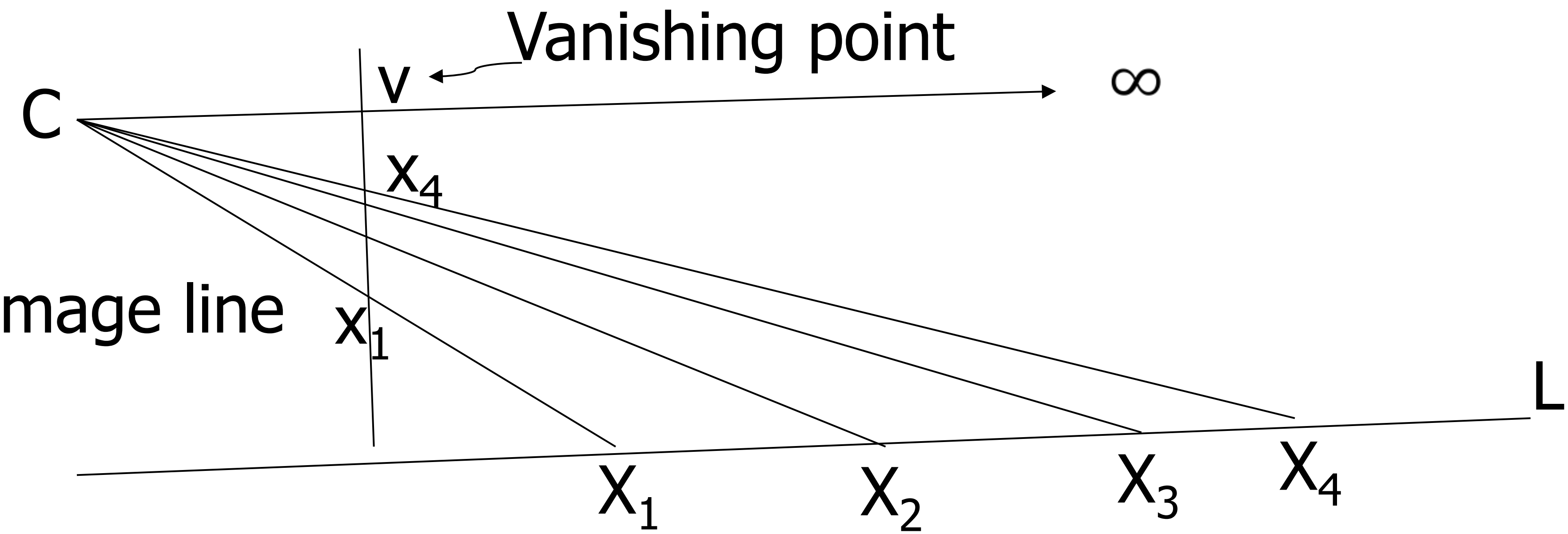
Planar panoramic mosaicing



Axis of rotation for imaging

Vanishing points

Vanishing points: images at points at ∞.

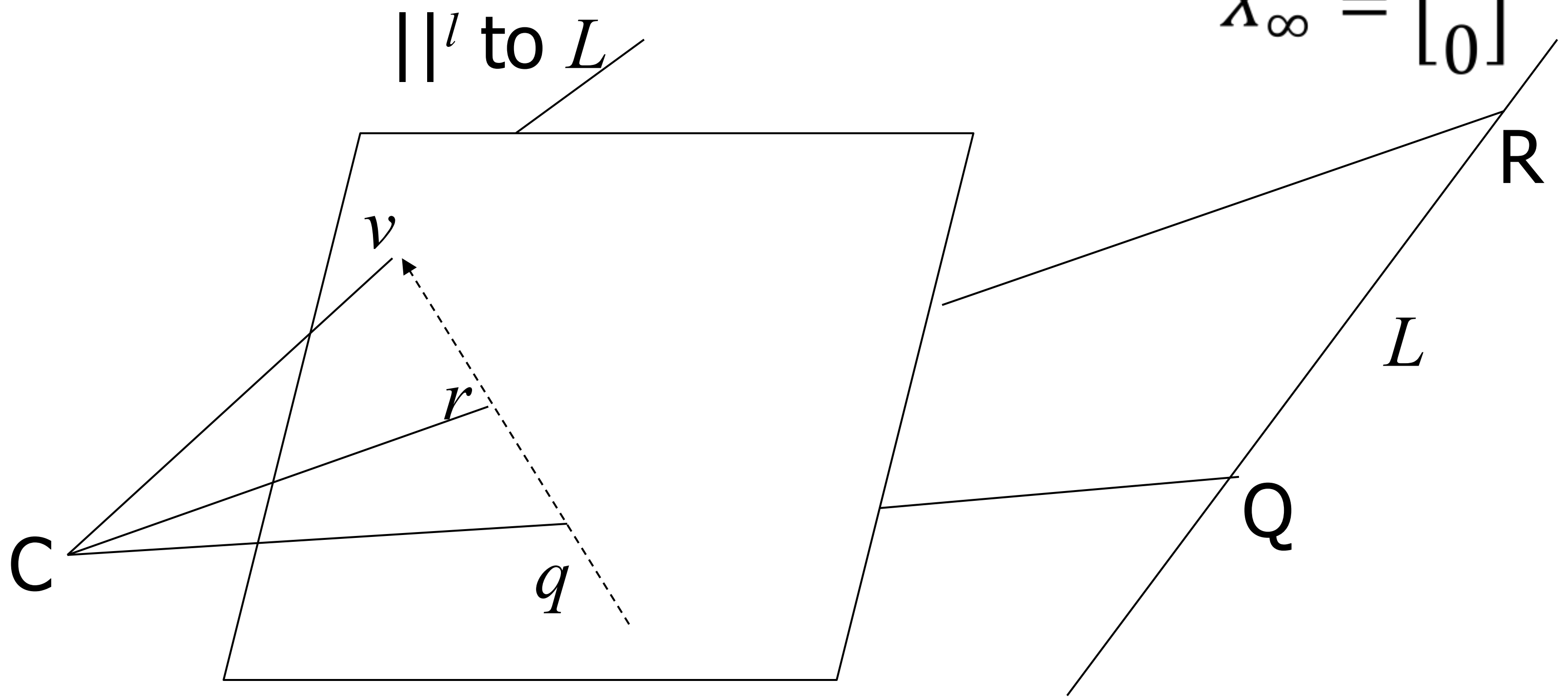


Vanishing point of a line L is the intersecting point in the image plane parallel to L and passing through the camera center C.

$$v = PX_\infty = K[I \quad | \quad 0]\begin{bmatrix} \boldsymbol{d} \\ 0 \end{bmatrix} = K\boldsymbol{d}$$

Vanishing points

$$X_\infty = \begin{bmatrix} \boldsymbol{d} \\ 0 \end{bmatrix}$$



Vanishing points are independent of camera position, if it is not rotated.

$$v = PX_\infty = K[I \quad | \quad 0] \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = K\mathbf{d}$$

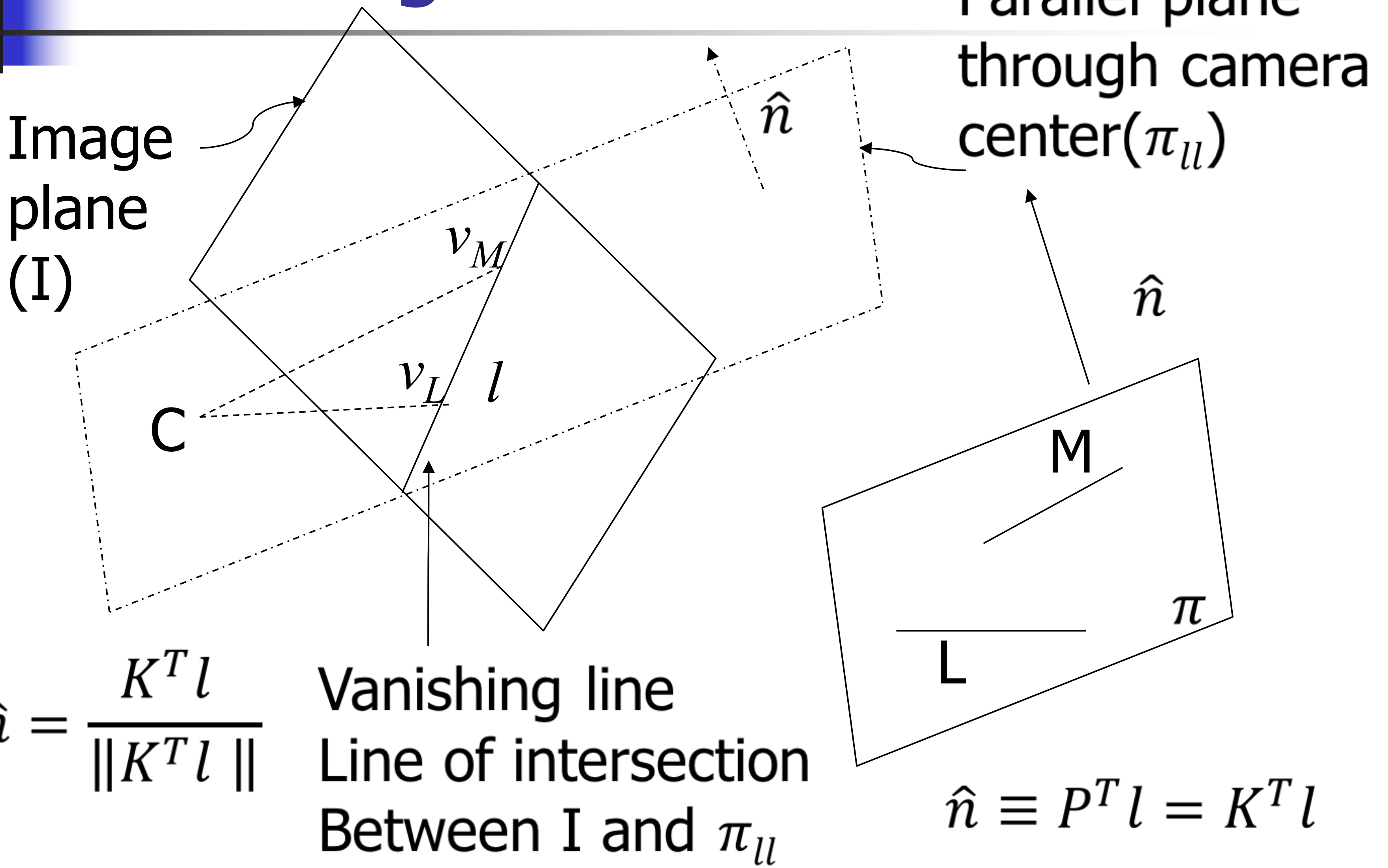
Vanishing points

Vanishing points are independent of camera position, if it is not rotated.
With rotation R it becomes $v' = KR\mathbf{d}$.
If we know v, v', and K, we can compute R.

$$\hat{\mathbf{d}} = \frac{K^{-1}v}{\|K^{-1}v\|} \qquad \hat{\mathbf{d}}' = \frac{K^{-1}v'}{\|K^{-1}v'\|} \qquad \hat{\mathbf{d}}' = R\mathbf{d}$$

Two independent constraints on R and it can be computed.

Vanishing lines



$$\hat{n} = \frac{K^T l}{\|K^T l\|}$$

$$\hat{n} \equiv P^T l = K^T l$$

Computing vanishing line

- Identify groups of sets of parallel lines in a plane at different directions.
- Obtain their vanishing points.
- Get the line among them.

Exercise

Consider the following projection matrix.

$$P = \begin{bmatrix} -9 & 2 & 3 & 1 \\ 3 & -9 & 6 & 1 \\ 2 & 6 & -10 & 1 \end{bmatrix}$$

Compute the following:

(i) Camera center

(ii) Vanishing point of X-axis.

(iii) Image point of origin.

(iv) Vanishing point of the line with the direction cosines 2:3:4.

Solution

$$P = \begin{bmatrix} -9 & 2 & 3 & 1 \\ 3 & -9 & 6 & 1 \\ 2 & 6 & -10 & 1 \end{bmatrix}$$

M

p_4

$$\tilde{C} = -M^{-1}p_4$$

$$Cofactor(M) = \begin{bmatrix} 54 & 42 & 36 \\ 38 & 84 & 58 \\ 39 & 63 & 75 \end{bmatrix} \qquad M^{-1} = -\frac{1}{294}\begin{bmatrix} 54 & 38 & 39 \\ 42 & 84 & 63 \\ 36 & 58 & 75 \end{bmatrix}$$

$$\det(M) = -9(90 - 36) + 2(12 + 30) + 3(18 + 18) = -294$$

$$\tilde{C} = \frac{1}{294}\begin{bmatrix} 131 \\ 189 \\ 169 \end{bmatrix}$$

Thank you!