PH3500 Project

Torus Geodesics

ME17B129
Praneeth Avasarala

Let take a torus with major radius 'c' and minor radius 'a'. Then, using parameters θ and ϕ, the parametric equation of the position vector becomes,

$$\boxed{\vec{R} = (c+a\cos\phi)\cos\theta\,\hat{i} + (c+a\cos\phi)\sin\theta\,\hat{j} + a\sin\phi\,\hat{k}}$$

Now, Geodesic equations are

$$\boxed{\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0} \qquad \boxed{\begin{matrix} u^1 = \theta \\ u^2 = \phi. \end{matrix}}$$

where, Christoffel Symbols.

$$\boxed{\Gamma^k_{ij} = \frac{\partial^2\vec{R}}{\partial u^i\,\partial u^j}\cdot\frac{\partial\vec{R}}{\partial u^l}\,g^{lk}}$$

inverse metric tensor

$$\boxed{g_{kl} = \frac{\partial\vec{R}}{\partial u^k}\cdot\frac{\partial\vec{R}}{\partial u^l}} \qquad \boxed{g_{kl}\,g^{lm} = \delta^m_k}$$

$$g\cdot g^{-1} = I$$

Now,

$$\frac{\partial\vec{R}}{\partial\theta} = (c+a\cos\phi)(-\sin\theta)\hat{i} + (c+a\cos\phi)\cos\theta\,\hat{j} + 0\,\hat{k}$$

$$\frac{\partial\vec{R}}{\partial\phi} = (0-a\sin\phi)\cos\theta\,\hat{i} + (0-a\sin\phi)\sin\theta\,\hat{j} + a\cos\phi\,\hat{k}$$

$$\frac{\partial^2\vec{R}}{\partial\theta^2} = (c+a\cos\phi)(-\cos\theta)\hat{i} + (c+a\cos\phi)(-\sin\theta)\hat{j}$$

$$\frac{\partial^2\vec{R}}{\partial\phi^2} = -a\cos\phi\cos\theta\,\hat{i} - a\cos\phi\sin\theta\,\hat{j} - a\sin\phi\,\hat{k}$$

$$\frac{\partial^2 \vec{R}}{\partial\theta\cdot\partial\phi} = a\sin\phi\sin\theta\,\hat{i} - a\sin\phi\cos\theta\,\hat{j} = \frac{\partial \vec{R}}{\partial\phi\cdot\partial\theta}$$

→ Now we compute metric tensor, inverse metric tensor and christoffel symbol matrices.

$$g_{ij} = \begin{bmatrix} (c+a\cos\phi)^2 & 0 \\ 0 & a^2 \end{bmatrix}. \qquad g^{ij} = g^{-1}_{ij} = \begin{bmatrix} \frac{1}{(c+a\cos\phi)^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}$$

$$\Gamma^{\theta}_{ij} = \begin{bmatrix} 0 & \frac{-a\sin\phi}{c+a\cos\phi} \\ \frac{-a\sin\phi}{c+a\cos\phi} & 0 \end{bmatrix} \qquad \Gamma^{\phi}_{ij} = \begin{bmatrix} \frac{(c+a\cos\phi)\sin\phi}{a} & 0 \\ 0 & 0 \end{bmatrix}$$

→ Equations of Geodesics

$$\Rightarrow \quad \frac{d^2\theta}{d\lambda^2} + \Gamma^{\theta}_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0 \quad -①$$

and,

$$\frac{d^2\phi}{d\lambda^2} + \Gamma^{\phi}_{ij}\frac{du^i}{d\lambda}\cdot\frac{du^j}{d\lambda} = 0 \quad -②$$

On substituting christoffel symbols, the equations become,

$$\boxed{\ddot{\theta} - \frac{2a\sin\phi}{c+a\cos v}\cdot\dot{\theta}\dot{\phi} = 0} \quad -③ \qquad \dot{\theta} = \frac{d\theta}{d\lambda},\ \dot{\phi} = \frac{d\phi}{d\lambda}$$

$$\boxed{\ddot{\phi} + \frac{1}{a}\sin\phi(c+a\cos\phi)(\dot{\theta})^2 = 0} \quad -④$$

We can consider, $\phi' = \frac{d\phi}{d\lambda}$: and eliminate 'λ' from the two equations to get one equation.

$$\Rightarrow \quad \phi' = \frac{d\phi}{d\theta} = \frac{d\phi/d\lambda}{d\theta/d\lambda} = \frac{\dot{\phi}}{\dot{\theta}}$$

Now, $$\phi'' = \frac{d}{d\theta}(\phi') = \frac{d}{d\theta}\left(\frac{\dot{\phi}}{\dot{\theta}}\right)$$

$$= \frac{d}{d\theta}\left(\frac{1}{\dot{\theta}}\right)\cdot\dot{\phi} + \frac{d}{d\theta}(\dot{\phi})\cdot\frac{1}{\dot{\theta}}$$

$$= \frac{-1}{(\dot{\theta})^2}\times\ddot{\theta}\times\frac{d\lambda}{d\theta}\times\dot{\phi} + \ddot{\phi}\times\frac{d\lambda}{d\theta}\times\frac{1}{\dot{\theta}}$$

$$= \frac{-\ddot{\theta}\dot{\phi}}{(\dot{\theta})^2}\times\frac{1}{\dot{\theta}} + \frac{\ddot{\phi}}{\dot{\theta}}\times\frac{1}{\dot{\theta}}$$

$$\therefore \quad \phi'' = \frac{-\ddot{\theta}\dot{\phi} + \dot{\theta}\ddot{\phi}}{(\dot{\theta})^3}$$

$$\Rightarrow \quad \phi'' = \frac{-\ddot{\theta}\,\dot{\theta}\cdot\phi' + \dot{\theta}\cdot\ddot{\phi}}{(\dot{\theta})^3}$$

On substituting $\ddot{\theta}$ and $\ddot{\phi}$ from eqn (3) and eqn (4),

we get,

$$\boxed{\phi'' + \frac{2a\sin\phi}{c+a\cos\phi}\times(\phi')^2 + \frac{\sin\phi\,(c+a\cos\phi)}{a} = 0}$$

$$\Downarrow$$

$\boxed{\text{Geodesic Eq}^n \text{ of Torus}}$