Automata Theory

Theory Assignment 2

Moida Praneeth Jain, 2022010193

Question 1

Pumping lemma for context free languages states that if L is a context-free language then $\exists p$ (pumping length) such that $\forall s \in L$, s can be divided into 5 substrings $s = uvxyz$ such that

- $uv^t xy^t z \in L \ \forall t \geq 0$
- $|vy| > 0$
- $|vxy| \leq p$

(a)

Given: $L = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$

To Prove: L is not a context-free language

Proof: Assume L is a context-free language.

$\therefore$ L satisfies the pumping lemma for context-free languages. Let p be its pumping length.

Consider the string $s = a^p b^p c^p$. $|s| = 3p > p$, $\therefore s = uvxyz$

Define a function $f : L \to \mathbb{N}$ that takes in a string from the language and outputs the number of unique characters in it. For example, $f(\text{aabc}) = 3$, $f(\text{aaa}) = 1$, $f(\varepsilon) = 0$.

- Case 1: $f(v) \leq 1 \wedge f(y) \leq 1$

  The maximum number of unique symbols in both v and y combined is $\max(f(v) + f(y)) = 2$ in this case. Since we have 3 symbols (a, b, c), atleast one symbol never occurs in both v and y. Let this symbol be m.

  - Subcase 1: $m = a$

    We pump down with $t = 0$, i.e, $s' = uv^0 xy^0 z = uxz$.

    v and y contained either b or c (or both). Before pumping, all three characters had equal count p. After pumping, the count of either b or c (or both) is lower than the count of a.

    $\therefore s' \notin L$
  - Subcase 2: $m = b$

    If a is present in u or v, then we pump up with $t = 2$, i.e, $s' = uv^2 xy^2 z$. The number of b remains same as p while the number of a has increased.

    Otherweise, if c is present in u or v, then we pump down with $t = 0$, i.e, $s' = uv^0 xy^0 z$. The number of b remains same as p while the number of c has decreased.

    $\therefore s' \notin L$
  - Subcase 3: $m = c$

    We pump up with $t = 2$, i.e, $s' = uv^2 xy^2 z$. The number of c remains the same as p while the number of a or b (or both) increases.

    $\therefore s' \notin L$

  In all the subcases, the pumping lemma does not hold.

- Case 2: $f(v) > 1 \lor f(y) > 1$

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$.

  WLOG assume $f(v) > 1$, i.e, $v = l_1 l_2 l_3$ in the correct order (l_3 may be ε).

  $v^2 = l_1 l_2 l_3 l_1 l_2 l_3$. But, l_1 cannot appear after l_2. This is a contradiction.

  Pumping lemma does not hold.

In all possible cases, pumping lemma does not hold. This is a contradiction.

$\therefore$ L is not a context free language.

QED

(b)

Given: $L = \{ww \mid w \in \{0,1\}^*\}$

To Prove: L is not a context-free language

Proof: Assume L is a context-free language.

$\therefore$ L satisfies the pumping lemma for context-free languages. Let p be its pumping length.

Consider the string $s = 0^p1^p0^p1^p$. $|s| = 4p > p$, $\therefore$ $s = uvxyz$

- Case 1: vxy occurs in the left half of the string, i.e, in the first w.

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$. This pushes a 1 onto the first position of the second half, while the first position of the first half has a 0. Therefore, this string is not of the form ww.

  $\therefore$ $s' \notin L$
- Case 2: vxy occurs in the right half of the string, i.e, in the second w.

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$. This pushes a 0 onto the last position of the first half, while the last position of the second half has a 1. Therefore, this string is not of the form ww.

  $\therefore$ $s' \notin L$
- Case 3: vxy contains the midpoint of the string

  We pump down with t = 0, i.e, $s' = uv^0xy^0z = uxz$.

  Clearly, $s' = 0^p1^{k_1}0^{k_2}1^p$. Note that the initial 0^p and final 1^p remain unaffected because $|vxy| \leq p$. For s' to belong in L, $k_1 = p \land k_2 = p$. This is not possible as the string has been pumped down and its length can no longer be $4p$. This is a contradiction.

  $s' \notin L$

In all possible cases, pumping lemma does not hold. This is a contradiction.

$\therefore$ L is not a context free language.

QED

(c)

Given: $L = \{a^{n!} \mid n \geq 0\}$

To Prove: L is not a context-free language

Proof: Assume L is a context-free language.

$\therefore$ L satisfies the pumping lemma for context-free languages. Let p be its pumping length.

Consider the string $s = a^{p!}$. $|s| = p! \geq p$, $\therefore s = uvxyz$

On pumping the string, we get $s_i = uv^i xy^i z$

Clearly, $|s_i| = p! + (i - 1)|vy|$ (Note that $|vy| > 0$ according to pumping lemma).

$\forall i \geq 0\ s_i \in L$

$\forall i \geq 0 |s_i|$ is a factorial

$\forall i \geq 0\ p! + (i - 1)|vy|$ is a factorial

This implies that there exists an arithmetic progression of factorials with common difference $|vy|$.

Since Γ (The gamma function) is convex for positive inputs, no infinite sequence with linear slope can fit it. So, there can exist no arithmetic progression of factorials with non zero common difference,

This is a contradiction. L does not satisfy the pumping lemma.

$\therefore$ L is not a context free language.

QED