Automata Theory

Theory Assignment 2

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Question 1

Pumping lemma for context free languages states that if L is a context-free language then $\exists p$ (pumping length) such that $\forall s \in L$, s can be divided into 5 substrings $s = uvxyz$ such that
- $uv^t xy^t z \in L \ \forall t \geq 0$
- $|vy| > 0$
- $|vxy| \leq p$

(a)

Given: $L = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$

To Prove: L is not a context-free language

Proof: Assume L is a context-free language.

$\therefore$ L satisfies the pumping lemma for context-free languages. Let p be its pumping length.

Consider the string $s = a^p b^p c^p$. $|s| = 3p > p$, $\therefore s = uvxyz$

Define a function $f : L \to \mathbb{N}$ that takes in a string from the language and outputs the number of unique characters in it. For example, $f(\text{aabc}) = 3, f(\text{aaa}) = 1, f(\varepsilon) = 0$.

- Case 1: $f(v) \leq 1 \land f(y) \leq 1$

  The maximum number of unique symbols in both v and y combined is $\max(f(v) + f(y)) = 2$ in this case. Since we have 3 symbols (a, b, c), atleast one symbol never occurs in both v and y. Let this symbol be m.
  - Subcase 1: $m = a$

    We pump down with $t = 0$, i.e, $s' = uv^0 xy^0 z = uxz$.

    v and y contained either b or c (or both). Before pumping, all three characters had equal count p. After pumping, the count of either b or c (or both) is lower than the count of a.

    $\therefore s' \notin L$
  - Subcase 2: $m = b$

    If a is present in u or v, then we pump up with $t = 2$, i.e, $s' = uv^2 xy^2 z$. The number of b remains same as p while the number of a has increased.

    Otherweise, if c is present in u or v, then we pump down with $t = 0$, i.e, $s' = uv^0 xy^0 z$. The number of b remains same as p while the number of c has decreased.

    $\therefore s' \notin L$
  - Subcase 3: $m = c$

    We pump up with $t = 2$, i.e, $s' = uv^2 xy^2 z$. The number of c remains the same as p while the number of a or b (or both) increases.

    $\therefore s' \notin L$

  In all the subcases, the pumping lemma does not hold.

- Case 2: $f(v) > 1 \lor f(y) > 1$

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$.

  WLOG assume $f(v) > 1$, i.e, $v = l_1 l_2 l_3$ in the correct order (l_3 may be ε).

  $v^2 = l_1 l_2 l_3 l_1 l_2 l_3$. But, l_1 cannot appear after l_2. This is a contradiction.

  Pumping lemma does not hold.

In all possible cases, pumping lemma does not hold. This is a contradiction.

$\therefore L$ is not a context free language.

QED

(b)

Given: $L = \{ww \mid w \in \{0,1\}^*\}$

To Prove: L is not a context-free language

Proof: Assume L is a context-free language.

$\therefore$ L satisfies the pumping lemma for context-free languages. Let p be its pumping length.

Consider the string $s = 0^p1^p0^p1^p$. $|s| = 4p > p$, $\therefore s = uvxyz$

- Case 1: vxy occurs in the left half of the string, i.e, in the first w.

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$. This pushes a 1 onto the first position of the second half, while the first position of the first half has a 0. Therefore, this string is not of the form ww.

  $\therefore s' \notin L$
- Case 2: vxy occurs in the right half of the string, i.e, in the second w.

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$. This pushes a 0 onto the last position of the first half, while the last position of the second half has a 1. Therefore, this string is not of the form ww.

  $\therefore s' \notin L$
- Case 3: vxy contains the midpoint of the string

  We pump down with t = 0, i.e, $s' = uv^0xy^0z = uxz$.

  Clearly, $s' = 0^p1^{k_1}0^{k_2}1^p$. Note that the initial 0^p and final 1^p remain unaffected because $|vxy| \leq p$. For s' to belong in L, $k_1 = p \land k_2 = p$. This is not possible as the string has been pumped down and its length can no longer be $4p$. This is a contradiction.

  $s' \notin L$

In all possible cases, pumping lemma does not hold. This is a contradiction.

$\therefore L$ is not a context free language.

QED