Automata Theory

Theory Assignment 2

Moida Praneeth Jain, 2022010193

Question 1

Pumping lemma for context free languages states that if L is a context-free language then $\exists p$ (pumping length) such that $\forall s \in L$, s can be divided into 5 substrings $s = uvxyz$ such that

- $uv^t xy^t z \in L \ \forall t \geq 0$
- $|vy| > 0$
- $|vxy| \leq p$

(a)

Given: $L = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$

To Prove: L is not a context-free language

Proof: Assume L is a context-free language.

$\therefore$ L satisfies the pumping lemma for context-free languages. Let p be its pumping length.

Consider the string $s = a^p b^p c^p$. $|s| = 3p > p$, $\therefore s = uvxyz$

Define a function $f : L \to \mathbb{N}$ that takes in a string from the language and outputs the number of unique characters in it. For example, $f(\text{aabc}) = 3, f(\text{aaa}) = 1, f(\varepsilon) = 0$.

- Case 1: $f(v) \leq 1 \land f(y) \leq 1$

  The maximum number of unique symbols in both v and y combined is $\max(f(v) + f(y)) = 2$ in this case. Since we have 3 symbols (a, b, c), atleast one symbol never occurs in both v and y. Let this symbol be m.

  - Subcase 1: $m = a$

    We pump down with $t = 0$, i.e, $s' = uv^0 xy^0 z = uxz$.

    v and y contained either b or c (or both). Before pumping, all three characters had equal count p. After pumping, the count of either b or c (or both) is lower than the count of a.

    $\therefore s' \notin L$
  - Subcase 2: $m = b$

    If a is present in u or v, then we pump up with $t = 2$, i.e, $s' = uv^2 xy^2 z$. The number of b remains same as p while the number of a has increased.

    Otherweise, if c is present in u or v, then we pump down with $t = 0$, i.e, $s' = uv^0 xy^0 z$. The number of b remains same as p while the number of c has decreased.

    $\therefore s' \notin L$
  - Subcase 3: $m = c$

    We pump up with $t = 2$, i.e, $s' = uv^2 xy^2 z$. The number of c remains the same as p while the number of a or b (or both) increases.

    $\therefore s' \notin L$

  In all the subcases, the pumping lemma does not hold.

- Case 2: $f(v) > 1 \lor f(y) > 1$

  We pump up with $t = 2$, i.e, $s' = uv^2xy^2z$.

  WLOG assume $f(v) > 1$, i.e, $v = l_1l_2l_3$ in the correct order ($l_3 \text{ may be } \varepsilon$).

  $v^2 = l_1l_2l_3l_1l_2l_3$. But, l_1 cannot appear after l_2. This is a contradiction.

  Pumping lemma does not hold.

In all possible cases, pumping lemma does not hold. This is a contradiction.

$\therefore$ L is not a context free language.

QED