Question & Solution

Question

Prove that the given function is discontinuous $\forall x \in \mathbb{Q}$
$f(x) = 1/q$ if $x = p/q \ with \ p \in \mathbb{Z} \ and \ q \in \mathbb{N}, \ \ gcd(p,q) = 1$
$f(x) = 0$ if x is irrational

Solution

Given:

$f(x) = 1/q$ if $x = p/q \ with \ p \in \mathbb{Z} \ and \ q \in \mathbb{N}, \ \ gcd(p,q) = 1$
$f(x) = 0$ if x is irrational

To Prove:

f is discontinuous $\forall x \in \mathbb{Q}$

Proof:

Choose arbitrary $x_0 \in \mathbb{Q}$ such that $x_0 = p/q \ with \ p \in \mathbb{Z} \ and \ q \in \mathbb{N}, \ \ gcd(p,q) = 1$
$f(x_0) = 1/q$
Let y be an irrational number, i.e, $y \in \mathbb{R} \setminus \mathbb{Q}$
Define $x_n = x_0 + y/n \ \forall n \in \mathbb{N}$

$$\begin{aligned} x_n - x_0 &= y/n \ \forall n \in \mathbb{N} \\ |x_n - x_0| &= y/n \ \forall n \in \mathbb{N} \\ |x_0 - x_n| &= y/n \ \forall n \in \mathbb{N} \end{aligned}$$

Since x_n is irrational for all n

$$\begin{aligned} f(x_n) &= 0 \ \forall n \in \mathbb{N} \\ f(x_0) &= 1/q \\ f(x_0) - f(x_n) &= 1/q \\ |f(x_0) - f(x_n)| &= 1/q \end{aligned}$$

Now, choose $\epsilon = 1/q$, and for some $\delta > 0$, choose $n = 1 + \lceil y/\delta \rceil$ ($\lceil x \rceil$ is the least integer $\geq x$)

We have

$$\begin{aligned} |x_0 - x_n| = \frac{y}{1 + \lceil y/\delta \rceil} < \frac{y}{\lceil y/\delta \rceil} &\leq \delta \\ |f(x_0) - f(x_n)| = 1/q &\geq \epsilon \end{aligned}$$

Therefore, from the definition of continuity, f is discontinuous at x_0 $\forall x_0 \in \mathbb{Q}$
Hence, f is discontinuous $\forall x \in \mathbb{Q}$

$$QED$$