Real Analysis Project

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Question & Solution

Question

Prove that the given function is discontinuous $\forall x \in \mathbb{Q}$
$f(x) = 1/q$ if $x = p/q \ with \ p \in \mathbb{Z} \ and \ q \in \mathbb{N}, \ gcd(p,q) = 1$
$f(x) = 0$ if x is irrational

Solution

Given:

$f(x) = 1/q$ if $x = p/q \ with \ p \in \mathbb{Z} \ and \ q \in \mathbb{N}, \ gcd(p,q) = 1$
$f(x) = 0$ if x is irrational

To Prove:

f is discontinuous $\forall x \in \mathbb{Q}$

Proof:

Choose arbitrary $x_0 \in \mathbb{Q}$ such that $x_0 = p/q \ with \ p \in \mathbb{Z} \ and \ q \in \mathbb{N}, \ gcd(p,q) = 1$
$f(x_0) = 1/q$
Let y be an irrational number, i.e, $y \in \mathbb{R} \setminus \mathbb{Q}$
Define $x_n = x_0 + y/n \ \forall n \in \mathbb{N}$

$$\begin{aligned} x_n - x_0 &= y/n \ \forall n \in \mathbb{N} \\ |x_n - x_0| &= y/n \ \forall n \in \mathbb{N} \\ |x_0 - x_n| &= y/n \ \forall n \in \mathbb{N} \end{aligned}$$

Since x_n is irrational for all n

$$\begin{aligned} f(x_n) &= 0 \ \forall n \in \mathbb{N} \\ f(x_0) &= 1/q \\ f(x_0) - f(x_n) &= 1/q \\ |f(x_0) - f(x_n)| &= 1/q \end{aligned}$$

Now, choose $\epsilon = 1/q$, and for some $\delta > 0$, choose $n = 1 + \lceil y/\delta \rceil$ ($\lceil x \rceil$ is the least integer $\geq x$)

We have

$$\begin{aligned} |x_0 - x_n| = \frac{y}{1 + \lceil y/\delta \rceil} &< \frac{y}{\lceil y/\delta \rceil} \leq \delta \\ |f(x_0) - f(x_n)| &= 1/q \geq \epsilon \end{aligned}$$

Therefore, from the definition of continuity, f is discontinuous at x_0 $\forall x_0 \in \mathbb{Q}$
Hence, f is discontinuous $\forall x \in \mathbb{Q}$

$$QED$$

Gödel's First Incompleteness Theorem (Application 1)

Definitions

Complete

A formal system F is complete if for any statement (proposition) p in F, either p or $\neg p$ can be proven using F's axioms.

F is incomplete if it is not complete.

Consistent

A formal system F is consistent if for any statement (proposition) p in F, both p and $\neg p$ cannot be proven together using F's axioms.

F is inconsistent if it is not consistent.

Elementary Arithmetic

A formal system F is said to be able to carry out elementary arithmetic if the set of theorems of Robinson Arithmetic can be proven using F's axioms.

Theorem Statement

Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete.

Implications

For any mathematical system (**group, ring or field**) with any amount of axioms (with which elementary arithmetic can be performed), there will always exist theorems whose truth can not be determined.

Even in the field of real numbers, there exist propositions which can neither be proven true or false. For example, the continuum hypothesis is independent of ZFC, i.e, it is proven that the continuum hypothesis can neither be proven nor disproven with our current set of axioms.

Various philosophical arguments stem from the results of this theorem regarding the pursuit of rigorous proofs.

Proof Sketch

Assumptions

We assume our formal system to be **Peano Arithmetic** (P), based on the Peano Axioms.
We assume that P is consistent.
This formal system consists of 15 symbols:

- 0
- S for increment
- - and x for addition and multiplication
- $\wedge$, $\vee$, and $\neg$ for logical and, logical or and logical negation.
- $\forall$, $\exists$ for universal and existential quantifiers.
- $=$ and $<$ for equality and less than.
- (and) for order precedence.
- x as a variable, and $*$ for creating more variables. (x^*, x^{**}, ...)
  Let the set of these symbols be S
  (This system is capable of performing elementary arithmetic.)

Proof

- Every formula in P can be expressed as a combination of the 15 symbols.
- Let J_n be the n^{th} segment of $\mathbb{N}$ = $\{1, 2, 3, \ldots, n\}$.
- We assign a natural number to each $s \in S$ from 0 to $n-1$, $\mathbf{card}(S) = n$ with a function $f : S \to J_n$.
- Let the set of all permutations of collections of $s \in S$ be T. We can construct T by
  $$T = \bigcup_{k \in \mathbb{N}} \{f : J_n \to S\}$$
- T is the set of all propositions of P.
- Define a function G:
  $$G : T \to \mathbb{N}$$
  $$G(t) = \sum_{i=0}^{len(t)-1} n^i f(t_i)$$
- This function G is trivially injective. Since G is an injective function from T to $\mathbb{N}$, T is a countably infinite set.
- $G(t)$ is called the Gödel number of t. Such a numbering is called **Gödel numbering**

- Every provable proposition is either an axiom or deduced in finite steps from axioms. Therefore, using the same function G, we can map entire proofs to $n \in \mathbb{N}$.
- Let any arbitrary proposition be Q. Define a formula $Proof(x, y)$ such that $Proof(x, y)$ is provable $\iff x = G(\text{proof of } Q)$ and $y = G(Q)$
- Therefore, the provability of a proposition is a proposition itself, and it also has a corresponding Gödel number.
- $\forall n \in \mathbb{N}\ \forall F \text{ such that } F(x) \text{ is a formula, let } q(n, G(F)) \text{ be a binary relation that is true } \iff n$
- Define $K(x) = \forall y\ q(y, x)$
- Now, $K(G(K)) = \forall y\ q(y, G(k))$
  i.e, For all y, y is not the Gödel number of G(K)
  If no Gödel number correspongs to G(K), then there is no proof of K(G(K))
- Therefore, there always exists a formula in any consistent system which cannot be proven.
- **Hence, any such consistent system is incomplete**

QED

Zermelo's Theorem in Game Theory (Application 2)

Definitions

Sequential Game

A game where one player chooses their action before the other choose theirs, and the other players have information on the first player's choice so that the difference in time has no strategic effect.

Perfect Information

A sequential game has perfect information if each player is perfectly informed of all events that have previously occurred when making any decision, including the start of the game.

Theorem Statement

In a finite two-person game of **perfect information**, either player 1 can force a win, or player 1 can force a tie, or player 2 can force a win.

Implications

- In chess, either white can always win, black can always win or they can force a draw.
- In tic-tac-toe, either the first player always wins, or the second player can force a draw.

Proof Sketch (by induction)

- Since the game is finite, assume the maximum length of the game is n.

Base Case

For $n = 1$, only player 1 gets to move. After player 1 moves, an outcome has to be reached. If player 1 loses, then player 2 can force a win. If player 1 wins, then player 1 can force a win. If the game ends in a draw, then player 1 can force a draw.
Therefore, the hypothesis holds true for the base case.

Inductive Step

Assume that the hypothesis holds true for $n = k$
Now, it's either player 1's turn or player 2's turn.
If it is player 1's turn, the hypothesis holds true trivially, just like the base case.
If it is player 2's turn, after player 2 moves, the game has to end. So player 2 can either force a win, or end the game in a draw.

Therefore, if the hypothesis holds true for $n = k$, then it also holds true for $n = k + 1$

Hence, in any such game, either player 1 can force a win, or player 1 can force a tie, or player 2 can force a win.

$$QED$$