Introduction to Quantum Information and Communication

Theory Assignment-1

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Question 1

To Prove: Any $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent

Proof:

Let V be an n dimensional vector space

Assume $A = \{v_1, v_2, v_3, ..., v_{n+1}\}$ is a set of linearly independent vectors where $v_i \in V$

Let $B = A \setminus \{v_{n+1}\} = \{v_1, v_2, v_3, ..., v_n\}$. Since $B \subset A$, B is also a set of linearly independent vectors.

Now, since V is n dimensional and $|B| = n$, $\text{span}(B) = V$ by the definition of n dimensional vector space.

Therefore, every vector $v \in V$ can be expressed as a linear combination of vectors in B

$\therefore v_{n+1} = a_1 v_1 + a_2 v_2 + a_3 v_3 + ... + a_n v_n$, where $a_i \in \mathbb{F}(\text{field over which } V \text{ is defined})$

$\therefore V$ is not linearly dependent. This is a contradiction

Any set A of $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent.

Question 2

Given: $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

To Find: square root of matrix A

Solution:

Note that $A^\dagger = A$. Thus, by the spectral theorem, A can be decomposed into an orthonormal eigenbasis. Now, we find this eigenbasis.

$$|A - \lambda I| = 0$$

$$|\begin{pmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{pmatrix}| = 0$$

$$\lambda_1 = 2, \lambda_2 = -3$$

Let their corresponding normalized eigenvectors be $|2\rangle$ and $|-3\rangle$

$$A|2\rangle = 2|2\rangle \text{ and } A|-3\rangle = 2|-3\rangle$$

On solving, we get

$$|2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } |-3\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now, by the spectral theorem, we have

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$A = 2|2\rangle\langle 2| - 3|{-3}\rangle\langle{-3}|$$

We know that

$$f(A) = \sum_i f(\lambda_i)|\lambda_i\rangle\langle\lambda_i|$$

So

$$\sqrt{A} = \sqrt{2}|2\rangle\langle 2| + \sqrt{-3}|{-3}\rangle\langle{-3}|$$

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$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}(2 \ \ 1) + \sqrt{-3}\begin{pmatrix} 1 \\ -2 \end{pmatrix}(1 \ \ -2)\right)$$

$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} + \sqrt{-3}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\right)$$

$$\sqrt{A} = \frac{1}{5}\begin{pmatrix} 4\sqrt{2} + i\sqrt{3} & 2\sqrt{2} - 2i\sqrt{3} \\ 2\sqrt{2} - 2i\sqrt{3} & \sqrt{2} + 4i\sqrt{3} \end{pmatrix}$$

Question 3

Given: A is an $n \times n$ matrix and B is an $m \times m$ matrix

To Prove: $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \times \mathrm{tr}(B)$

Proof:

$$A \otimes B = \begin{pmatrix} A_{1,1}B & A_{1,2}B & \ldots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \ldots & A_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}B & A_{n,2}B & \ldots & A_{n,n}B \end{pmatrix}$$

where each $A_{i,j}B$ is an $m \times m$ matrix expanded.

$$\mathrm{tr}(A \otimes B) = \sum_{i=1}^{n} \mathrm{tr}\left(A_{i,i}B\right)$$

$$\mathrm{tr}(A \otimes B) = \sum_{i=1}^{n} A_{i,i}\,\mathrm{tr}(B)$$

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(B) \times \sum_{i=1}^{n} A_{i,i}$$

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \times \mathrm{tr}(B)$$

Question 4

Given: $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

To Prove: states are diametrically opposite on Bloch sphere $\Leftrightarrow$ states are orthogonal

Proof: Let state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Now, its diametrically opposite state is given by adding π to θ

$$|\psi'\rangle = \cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta+\pi}{2}\right)|1\rangle$$

$$|\psi'\rangle = \cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)|1\rangle$$

$$|\psi'\rangle = -\sin\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\cos\left(\frac{\theta}{2}\right)|1\rangle$$

Now, consider

$$\langle\psi|\psi'\rangle = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\begin{pmatrix}-\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$\langle\psi|\psi'\rangle = -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\langle\psi|\psi'\rangle = 0$$

Since the inner product of any two diametrically opposite states is 0, we can conclude that diametrically opposite states on the Bloch sphere are orthogonal

states are diametrically opposite on Bloch sphere $\Rightarrow$ states are orthogonal

Now, assume two orthogonal states

$$|\psi_1\rangle = \cos\left(\frac{\theta_1}{2}\right)|0\rangle + e^{i\varphi_1}\sin\left(\frac{\theta_1}{2}\right)|1\rangle \text{ and } |\psi_2\rangle = \cos\left(\frac{\theta_2}{2}\right)|0\rangle + e^{i\varphi_2}\sin\left(\frac{\theta_2}{2}\right)|1\rangle$$

$$\langle\psi_1|\psi_2\rangle = 0$$

$$\begin{pmatrix}\cos\left(\frac{\theta_1}{2}\right) & e^{-i\varphi_1}\sin\left(\frac{\theta_1}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta_2}{2}\right) \\ e^{i\varphi_2}\sin\left(\frac{\theta_2}{2}\right)\end{pmatrix} = 0$$

$$\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + e^{i(\varphi_2-\varphi_1)}\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) = 0$$

$$\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + (\cos(\varphi_2-\varphi_1) + i\sin(\varphi_2-\varphi_1))\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) = 0$$

Since the imaginary part is 0 on RHS, we have $\sin(\varphi_2 - \varphi_1) = 0 \Rightarrow \varphi_2 = \varphi_1$

$$\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + \cos(0)\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) = 0$$

$$\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right) = 0$$

$$\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = 0$$

$$\frac{\theta_1 - \theta_2}{2} = \frac{\pi}{2}$$

$$\theta_1 = \pi + \theta_2$$

states are orthogonal $\Rightarrow$ states are diametrically opposite on Bloch sphere

Since we have proven both sides, we can assert

states are diametrically opposite on Bloch sphere $\Leftrightarrow$ states are orthogonal

Question 5

Given: $|\psi\rangle = \sum_{i=1}^{n} \alpha_i |u_i\rangle$ for some basis set $\{|u_i\rangle\}_{i=1}^{n}$ and probability amplitudes $\alpha_i \in \mathbb{C}$

To Prove: $|\psi\rangle$ collapses to $|u_k\rangle$ after measurement in the basis $\{|u_i\rangle\}_{i=1}^{n}$ with probability $|\alpha_k|^2$

Proof: Born rule states that the probability of a density operator ρ collapsing to state $|u_k\rangle\langle u_k|$ is

$$P = \operatorname{tr}(|u_k\rangle\langle u_k|\rho)$$

For the vector ψ, we have state $\rho = |\psi\rangle\langle\psi|$

Now, we find the probability of ρ collapsing to $|u_k\rangle\langle u_k|$

$$P = \operatorname{tr}(|u_k\rangle\langle u_k||\psi\rangle\langle\psi|)$$

On using the cyclicity of trace

$$P = \operatorname{tr}(\langle\psi|u_k\rangle\langle u_k|\psi\rangle)$$

Since the matrix inside trace is 1×1

$$P = \langle\psi|u_k\rangle\langle u_k|\psi\rangle$$

$$P = \overline{\langle u_k|\psi\rangle}\langle u_k|\psi\rangle$$

$$P = |\langle u_k|\psi\rangle|^2$$

$$P = |\langle u_k| \sum_{i=1}^{n} \alpha_i |u_i\rangle|^2$$

$$P = |\sum_{i=1}^{n} \alpha_i \langle u_k||u_i\rangle|^2$$

$$P = |\sum_{i=1}^{n} \alpha_i \langle u_k|u_i\rangle|^2$$

Since $\langle u_i|u_j\rangle = \delta_{ij}$

$$P = |\sum_{i=1}^{n} \alpha_i \delta_{ki}|^2$$

$$P = |\alpha_k|^2$$

$\therefore$ $|\psi\rangle$ collapses to $|u_k\rangle$ after measurement in the basis $\{|u_i\rangle\}_{i=1}^{n}$ with probability $|\alpha_k|^2$

Question 6

(a)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho = \frac{1}{2}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix}$$

$$\rho = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$$

Now, we find the probability of the state collapsing to $|1\rangle$ in both formalisms

State Vector Formalism

$$\Pr[\text{state collapsing to } |1\rangle] = |\langle 1|\psi\rangle|^2$$

$$P = |\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix}|^2$$

$$P = \frac{1}{2}|i|^2$$

$$P = \frac{1}{2}$$

Density Matrix Formalism

$$\Pr[\text{state collapsing to } |1\rangle\langle 1|] = \operatorname{tr}(|1\rangle\langle 1|\rho)$$

$$P = \operatorname{tr}\left(\frac{1}{2}\begin{pmatrix}0\\ 1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & 0\\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{2}$$

In both the formalisms, we get the required probability to be $\frac{1}{2}$

(b)

State Vector Formalism

$$\Pr[\text{state collapsing to } |+i\rangle] = |\langle +i|\psi\rangle|^2$$

$$P = |\frac{1}{2}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix}|^2$$

$$P = |\frac{1}{2} * 2|^2$$

$$P = 1$$

Density Matrix Formalism

$$\Pr[\text{state collapsing to } |{+i}\rangle\langle{+i}|] = \operatorname{tr}(|{+i}\rangle\langle{+1}|\rho)$$

$$P = \operatorname{tr}\left(\frac{1}{4}\begin{pmatrix}1\\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{4}\operatorname{tr}\left(\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{4}\operatorname{tr}\left(\begin{pmatrix}2 & -2i\\ 2i & 2\end{pmatrix}\right)$$

$$P = 1$$

$\therefore$ the probability of getting $|{+\psi}\rangle$ when measuring in the basis $\{|{+\psi}\rangle, |{-\psi}\rangle\}$ is 1

Question 7

$$|\Psi-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ -1\\ 0\end{pmatrix}$$

$$\mathbb{B} = \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

(a)

For a pure state to be separable, it must be a product state.

Let $|\Psi-\rangle = \binom{a}{b} \otimes \binom{c}{d}$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ -1\\ 0\end{pmatrix} = \begin{pmatrix}ac\\ ad\\ bc\\ bd\end{pmatrix}$$

$$ac = 0 \Rightarrow a = 0 \vee c = 0$$

$$bd = 0 \Rightarrow b = 0 \vee d = 0$$

If any one of these are zero, then $ad \neq 1 \wedge bc \neq -1$. Thus, no such states exist whose tensor product is $|\Psi-\rangle$

Thus, $|\Psi-\rangle$ is entangled.

(b)

$$\Pr[\text{state collapsing to } ||1\rangle \otimes |0\rangle\rangle] = |\langle|1\rangle \otimes |0\rangle|\Psi-\rangle|^2$$

$$P = |\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\\ -1\\ 0\end{pmatrix}|^2$$

$$P = \frac{1}{2}\,|-1|^2$$

$$P = \frac{1}{2}$$

(c)

$$\Pr[\text{state collapsing to } ||0\rangle \otimes |0\rangle\rangle] = |\langle|0\rangle \otimes |0\rangle|\Psi -\rangle|^2$$

$$P = |\frac{1}{\sqrt{2}}(1\ \ 0\ \ 0\ \ 0)\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}|^2$$

$$P = \frac{1}{2}\ |0|^2$$

$$P = 0$$

(d)

Z is the Pauli Z matrix

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$O = Z \otimes Z$$

$$O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$O = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\langle O \rangle = \langle \Psi -|O|\Psi -\rangle$$

$$\langle O \rangle = \frac{1}{2}(0\ \ 1\ \ -1\ \ 0)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\langle O \rangle = \frac{1}{2}(0\ \ 1\ \ -1\ \ 0)\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\langle O \rangle = \frac{1}{2}(-2)$$

$$\langle O \rangle = -1$$

The expected value of operator $O = Z \otimes Z$ for the state $|\Psi -\rangle$ is -1