# Introduction to Quantum Information and Communication

## Take Home Mid-Sem

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## Question 5

**Given**:

$$f : \{0,1\}^n \mapsto \{0,1\}^n$$

$$U_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := |x\rangle_Q \otimes |y \oplus f(x)\rangle_R$$

$$V_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := (-1)^{y \cdot f(x)} |x\rangle_Q \otimes |y\rangle_R$$

### (a)

**To Prove**:

$$\sum_{z \in \{0,1\}^n} (-1)^{(x \oplus y) \cdot z} = 2^n \delta(x, y)$$

**Proof**:

Case 1: $x = y$

$$\sum_{z \in \{0,1\}^n} (-1)^{(x \oplus y) \cdot z}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{0 \cdot z}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{0}$$

$$\sum_{z \in \{0,1\}^n} 1$$

$$2^n$$

$$2^n \times 1$$

$$2^n \delta(x, y)$$

Case 2: $x \neq y$

Let $k$ be the number of digits different between $x$ and $y$, and let the corresponding indices be $\alpha = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k\}$

$$\forall i \in \{1, 2, ..., k\} \; x_{\alpha_i} \neq y_{\alpha_i}$$

$$\forall i \notin \alpha \; x_i = y_i$$

.

$$\sum_{z \in \{0,1\}^n} (-1)^{(x \oplus y) \cdot z}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{\bigoplus_{i=1}^{n} (x_i \oplus y_i) z_i}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{\bigoplus_{i=1}^{k} z_{\alpha_i}}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{z_{\alpha_1} \oplus z_{\alpha_2} \oplus \ldots \oplus z_{\alpha_k}}$$

Now, since $z$ is looping through all possible bitstrings of length $n$, the parity of any subset of its bits will be odd half the times and even half the times.

$$-1 + 1 - 1 + 1 \ldots - 1 + 1$$

$$0$$

$$2^n \times 0$$

$$2^n \delta(x, y)$$

Now, from both the cases we get

$$\sum_{z \in \{0,1\}^n} (-1)^{(x \oplus y) \cdot z} = 2^n \delta(x, y)$$

Hence, proven.