Introduction to Quantum Information and Communication

Theory Assignment-1

Moida Praneeth Jain, 2022101093

Question 1

To Prove: Any $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent

Proof:

Let V be an n dimensional vector space

Assume $A = \{v_1, v_2, v_3, ..., v_{n+1}\}$ is a set of linearly independent vectors where $v_i \in V$

Let $B = A \setminus \{v_{n+1}\} = \{v_1, v_2, v_3, ..., v_n\}$. Since $B \subset A$, B is also a set of linearly independent vectors.

Now, since V is n dimensional and $|B| = n$, $\text{span}(B) = V$ by the definition of n dimensional vector space.

Therefore, every vector $v \in V$ can be expressed as a linear combination of vectors in B

$\therefore v_{n+1} = a_1 v_1 + a_2 v_2 + a_3 v_3 + ... + a_n v_n$, where $a_i \in \mathbb{F}$(field over which V is defined)

$\therefore V$ is not linearly dependent. This is a contradiction

Any set A of $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent.

Question 2

Given: $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

To Find: square root of matrix A

Solution:

Note that $A^\dagger = A$. Thus, by the spectral theorem, A can be decomposed into an orthonormal eigenbasis. Now, we find this eigenbasis.

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{pmatrix} \right| = 0$$

$$\lambda_1 = 2, \lambda_2 = -3$$

Let their corresponding normalized eigenvectors be $|2\rangle$ and $|-3\rangle$

$$A|2\rangle = 2|2\rangle \text{ and } A|-3\rangle = 2|-3\rangle$$

On solving, we get

$$|2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } |-3\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now, by the spectral theorem, we have

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$A = 2|2\rangle\langle 2| - 3|-3\rangle\langle -3|$$

We know that

$$f(A) = \sum_i f(\lambda_i)|\lambda_i\rangle\langle\lambda_i|$$

So

$$\sqrt{A} = \sqrt{2}|2\rangle\langle 2| + \sqrt{-3}|-3\rangle\langle -3|$$

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$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix}2\\1\end{pmatrix}(2\ \ 1) + \sqrt{-3}\begin{pmatrix}1\\-2\end{pmatrix}(1\ \ -2)\right)$$

$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix} + \sqrt{-3}\begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}\right)$$

$$\sqrt{A} = \frac{1}{5}\begin{pmatrix}4\sqrt{2} + i\sqrt{3} & 2\sqrt{2} - 2i\sqrt{3}\\2\sqrt{2} - 2i\sqrt{3} & \sqrt{2} + 4i\sqrt{3}\end{pmatrix}$$

Question 3

Given: A is an $n \times n$ matrix and B is an $m \times m$ matrix

To Prove: $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \times \mathrm{tr}(B)$

Proof:

$$A \otimes B = \begin{pmatrix} A_{1,1}B & A_{1,2}B & \ldots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \ldots & A_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}B & A_{n,2}B & \ldots & A_{n,n}B \end{pmatrix}$$

where each $A_{i,j}B$ is an $m \times m$ matrix expanded.

$$\mathrm{tr}(A \otimes B) = \sum_{i=1}^{n} \mathrm{tr}\left(A_{i,i}B\right)$$

$$\mathrm{tr}(A \otimes B) = \sum_{i=1}^{n} A_{i,i}\,\mathrm{tr}(B)$$

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(B) \times \sum_{i=1}^{n} A_{i,i}$$

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \times \mathrm{tr}(B)$$

Question 4

Given: $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

To Prove: Diametrically opposite states on the Bloch sphere are orthogonal

Proof: Let state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Now, its diametrically opposite state is given by adding π to θ

$$|\psi'\rangle = \cos\left(\frac{\theta + \pi}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta + \pi}{2}\right)|1\rangle$$

$$|\psi'\rangle = \cos\left(\frac{\pi}{2} + \frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)|1\rangle$$

$$|\psi'\rangle = -\sin\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\cos\left(\frac{\theta}{2}\right)|1\rangle$$

Now, consider

$$\langle\psi|\psi'\rangle = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\langle\psi|\psi'\rangle = -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\langle\psi|\psi'\rangle = 0$$

Since the inner product of any two diametrically opposite states is 0, we can conclude that diametrically opposite states on the Bloch sphere are orthogonal