Introduction to Quantum Information and Communication

Theory Assignment-1

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Question 1

To Prove: Any $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent

Proof:

Let V be an n dimensional vector space

Assume $A = \{v_1, v_2, v_3, ..., v_{n+1}\}$ is a set of linearly independent vectors where $v_i \in V$

Let $B = A \setminus \{v_{n+1}\} = \{v_1, v_2, v_3, ..., v_n\}$. Since $B \subset A$, B is also a set of linearly independent vectors.

Now, since V is n dimensional and $|B| = n$, span$(B) = V$ by the definition of n dimensional vector space.

Therefore, every vector $v \in V$ can be expressed as a linear combination of vectors in B

$\therefore v_{n+1} = a_1 v_1 + a_2 v_2 + a_3 v_3 + ... + a_n v_n$, where $a_i \in \mathbb{F}(\text{field over which } V \text{ is defined})$

$\therefore V$ is not linearly dependent. This is a contradiction

Any set A of $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent.

Question 2

Given: $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

To Find: square root of matrix A

Solution:

Note that $A^\dagger = A$. Thus, by the spectral theorem, A can be decomposed into an orthonormal eigenbasis. Now, we find this eigenbasis.

$$|A - \lambda I| = 0$$

$$|\begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix}| = 0$$

$$\lambda_1 = 2, \lambda_2 = -3$$

Let their corresponding normalized eigenvectors be $|2\rangle$ and $|-3\rangle$

$$A|2\rangle = 2|2\rangle \text{ and } A|-3\rangle = 2|-3\rangle$$

On solving, we get

$$|2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } |-3\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now, by the spectral theorem, we have

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$A = 2|2\rangle\langle 2| - 3|{-3}\rangle\langle{-3}|$$

We know that

$$f(A) = \sum_i f(\lambda_i)|\lambda_i\rangle\langle\lambda_i|$$

So

$$\sqrt{A} = \sqrt{2}|2\rangle\langle 2| + \sqrt{-3}|{-3}\rangle\langle{-3}|$$

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$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}(2 \;\; 1) + \sqrt{-3}\begin{pmatrix} 1 \\ -2 \end{pmatrix}(1 \;\; -2)\right)$$

$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} + \sqrt{-3}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\right)$$

$$\sqrt{A} = \frac{1}{5}\begin{pmatrix} 4\sqrt{2} + i\sqrt{3} & 2\sqrt{2} - 2i\sqrt{3} \\ 2\sqrt{2} - 2i\sqrt{3} & \sqrt{2} + 4i\sqrt{3} \end{pmatrix}$$

Question 3

Given: A is an $n \times n$ matrix and B is an $m \times m$ matrix

To Prove: $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \times \operatorname{tr}(B)$

Proof:

$$A \otimes B = \begin{pmatrix} A_{1,1}B & A_{1,2}B & \ldots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \ldots & A_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}B & A_{n,2}B & \ldots & A_{n,n}B \end{pmatrix}$$

where each $A_{i,j}B$ is an $m \times m$ matrix expanded.

$$\operatorname{tr}(A \otimes B) = \sum_{i=1}^{n} \operatorname{tr}(A_{i,i}B)$$

$$\operatorname{tr}(A \otimes B) = \sum_{i=1}^{n} A_{i,i} \operatorname{tr}(B)$$

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(B) \times \sum_{i=1}^{n} A_{i,i}$$

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \times \operatorname{tr}(B)$$

Question 4

Given: $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

To Prove: Diametrically opposite states on the Bloch sphere are orthogonal

Proof: Let state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Now, its diametrically opposite state is given by adding π to θ

$$|\psi'\rangle = \cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta+\pi}{2}\right)|1\rangle$$

$$|\psi'\rangle = \cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)|1\rangle$$

$$|\psi'\rangle = -\sin\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\cos\left(\frac{\theta}{2}\right)|1\rangle$$

Now, consider

$$\langle\psi|\psi'\rangle = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\begin{pmatrix}-\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$\langle\psi|\psi'\rangle = -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\langle\psi|\psi'\rangle = 0$$

Since the inner product of any two diametrically opposite states is 0, we can conclude that diametrically opposite states on the Bloch sphere are orthogonal

Question 5

Given: $|\psi\rangle = \sum_{i=1}^{n} \alpha_i |u_i\rangle$ for some basis set $\{|u_i\rangle\}_{i=1}^{n}$ and probability amplitudes $\alpha_i \in \mathbb{C}$

To Prove: $|\psi\rangle$ collapses to $|u_k\rangle$ after measurement in the basis $\{|u_i\rangle\}_{i=1}^{n}$ with probability $|\alpha_k|^2$

Proof: Born rule states that the probability of a density operator ρ collapsing to state $|u_k\rangle\langle u_k|$ is

$$P = \operatorname{tr}(|u_k\rangle\langle u_k|\rho)$$

For the vector ψ, we have state $\rho = |\psi\rangle\langle\psi|$

Now, we find the probability of ρ collapsing to $|u_k\rangle\langle u_k|$

$$P = \operatorname{tr}(|u_k\rangle\langle u_k||\psi\rangle\langle\psi|)$$

On using the cyclicity of trace

$$P = \operatorname{tr}(\langle\psi|u_k\rangle\langle u_k|\psi\rangle)$$

Since the matrix inside trace is 1×1

$$P = \langle\psi|u_k\rangle\langle u_k|\psi\rangle$$

$$P = \overline{\langle u_k|\psi\rangle}\langle u_k|\psi\rangle$$

$$P = |\langle u_k|\psi\rangle|^2$$

$$P = |\langle u_k| \sum_{i=1}^{n} \alpha_i |u_i\rangle|^2$$

$$P = |\sum_{i=1}^{n} \alpha_i \langle u_k||u_i\rangle|^2$$

$$P = |\sum_{i=1}^{n} \alpha_i \langle u_k|u_i\rangle|^2$$

Since $\langle u_i|u_j\rangle = \delta_{ij}$

$$P = |\sum_{i=1}^{n} \alpha_i \delta_{ki}|^2$$

$$P = |\alpha_k|^2$$

$\therefore$ $|\psi\rangle$ collapses to $|u_k\rangle$ after measurement in the basis $\{|u_i\rangle\}_{i=1}^{n}$ with probability $|\alpha_k|^2$

Question 6

(a)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho = \frac{1}{2}\begin{pmatrix}1 \\ i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix}$$

$$\rho = \frac{1}{2}\begin{pmatrix}1 & -i \\ i & 1\end{pmatrix}$$

Now, we find the probability of the state collapsing to $|1\rangle$ in both formalisms

State Vector Formalism

$$\Pr[\text{state collapsing to } |1\rangle] = |\langle 1|\psi\rangle|^2$$

$$P = |\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 \\ i\end{pmatrix}|^2$$

$$P = \frac{1}{2}|i|^2$$

$$P = \frac{1}{2}$$

Density Matrix Formalism

$$\Pr[\text{state collapsing to } |1\rangle\langle 1|] = \operatorname{tr}(|1\rangle\langle 1|\rho)$$

$$P = \operatorname{tr}\left(\frac{1}{2}\begin{pmatrix}0 \\ 1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & -i \\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -i \\ i & 1\end{pmatrix}\right)$$

$$P = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & 0 \\ i & 1 \end{pmatrix}\right)$$

$$P = \frac{1}{2}$$

In both the formalisms, we get the required probability to be $\frac{1}{2}$

(b)

State Vector Formalism

$$\Pr[\text{state collapsing to } |{+i}\rangle] = |\langle +i|\psi\rangle|^2$$

$$P = |\frac{1}{2}(1 \;\; -i)\begin{pmatrix} 1 \\ i \end{pmatrix}|^2$$

$$P = |\frac{1}{2} * 2|^2$$

$$P = 1$$

Density Matrix Formalism

$$\Pr[\text{state collapsing to } |{+i}\rangle\langle +i|] = \operatorname{tr}(|{+i}\rangle\langle +1|\rho)$$

$$P = \operatorname{tr}\left(\frac{1}{4}\begin{pmatrix} 1 \\ i \end{pmatrix}(1 \;\; -i)\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\right)$$

$$P = \frac{1}{4}\operatorname{tr}\left(\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\right)$$

$$P = \frac{1}{4}\operatorname{tr}\left(\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix}\right)$$

$$P = 1$$

$\therefore$ the probability of getting $|{+\psi}\rangle$ when measuring in the basis $\{|{+\psi}\rangle, |{-\psi}\rangle\}$ is 1