Introduction to Quantum Information and Communication

Theory Assignment-2

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Exercise 4.1.3

Given:

- A is a square operator acting on Hilbert space $\mathcal{H}_S$
- I_R is the identity operator acting on a Hilbert space $\mathcal{H}_R$ isomorphic to $\mathcal{H}_S$
- $|\Gamma\rangle_{RS}$ is the unnormalized maximally entangled vector.

To Prove:

$$\mathrm{Tr}\{A\} = \langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS}$$

Proof:

In the computational basis

$$|\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_S$$

$$\langle\Gamma|_{RS} = \sum_{i=0}^{d-1} \langle i|_R \langle i|_S$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) (I_R \otimes A_S) \left(\sum_{j=0}^{d-1} |j\rangle_R |j\rangle_S\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) \left(\sum_{j=0}^{d-1} (I_R \otimes A_S)\big(|j\rangle_R \otimes |j\rangle_S\big)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) \left(\sum_{j=0}^{d-1} \big(I_R |j\rangle_R\big) \otimes \big(A_S |j\rangle_S\big)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) \left(\sum_{j=0}^{d-1} |j\rangle_R \otimes \big(A_S |j\rangle_S\big)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i,j=0}^{d-1} \big(\langle i|_R \otimes \langle i|_S\big)\big(|j\rangle_R \otimes \big(A_S |j\rangle_S\big)\big)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i,j=0}^{d-1} \big(\langle i|j\rangle_R \otimes \langle i|_S A_S |j\rangle_S\big)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i,j=0}^{d-1} \big(\delta_{i,j} \otimes \langle i|_S A_S |j\rangle_S\big)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} \langle i|_S A_S |i\rangle_S$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \mathrm{Tr}\{A\}$$

Hence, proven.

Exercise 4.1.16

Given:

- Commutating projectors Π_1 and Π_2
- $0 \leq \Pi_1, \Pi_2 \leq I$

To Prove:

For arbitrary density operator ρ

$$\mathrm{Tr}\{(I - \Pi_1 \Pi_2)\rho\} \leq \mathrm{Tr}\{(I - \Pi_1)\rho\} + \mathrm{Tr}\{(I - \Pi_2)\rho\}$$

Proof:

TO DO

Exercise 4.2.2

Given:

- Ensemble $\{p_X(x), \rho_x\}$ of density operators
- POVM with elements $\{\Lambda_x\}$
- Operator τ such that $\tau \geq p_X(x)\rho_x$

To Prove:

$$\mathrm{Tr}\{\tau\} \geq \sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\}$$

Proof:

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} = \sum_x \mathrm{Tr}\{\Lambda_x p_X(x) \rho_x\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \sum_x \mathrm{Tr}\{\Lambda_x \tau\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\left\{\sum_x \Lambda_x \tau\right\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\left\{\tau \sum_x \Lambda_x\right\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\{\tau I\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\{\tau\}$$

Hence, proven.

Now for the case of encoding n bits into a d-dimensional subspace.

$$\{2^{-n}, \rho_i\}_{i \in \{0,1\}^n}$$

Consider

$$p_X(x)\rho_x = 2^{-n}\rho_i$$

$$p_X(x)\rho_x = 2^{-n}\sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}I - 2^{-n}\sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}\sum_j |j\rangle\langle j| - 2^{-n}\sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}\sum_j (1-\lambda_j)|j\rangle\langle j|$$

Since $0 \le \lambda_j \le 1\ \forall j$, $1 - \lambda_j \ge 0\ \forall j$. All the eigenvalues of the matrix in LHS are non-negative.

$$2^{-n}I - p_X(x)\rho_x \ge 0$$

$$2^{-n}I \ge p_X(x)\rho_x$$

∴ We consider $\tau = 2^{-n}I$

Now, we know that the probability of success is upper bounded by $\mathrm{Tr}\{\tau\}$

$$\mathrm{Tr}\{\tau\} = \mathrm{Tr}\{2^{-n}I\}$$

$$\mathrm{Tr}\{\tau\} = 2^{-n}\ \mathrm{Tr}\{I\}$$

Since I is d-dimensional,

$$\mathrm{Tr}\{\tau\} = d2^{-n}$$

Thus, the expected success probability is bounded above by $d2^{-n}$

Exercise 4.3.1

Given:

- A' has a Hilbert space structure isomorphic to that of system A
- $\forall x, y\ F_{AA'}|x\rangle_A|y\rangle_{A'} = |y\rangle_A|x\rangle_{A'}$

To Prove:

$$P(\rho_A) = \mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\}$$

Proof:

$$\rho_A = \sum_i \lambda_i |i\rangle_A \langle i|_A$$

$$\rho_{A'} = \sum_j \lambda_j |j\rangle_{A'} \langle j|_{A'}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\{F_{AA'}(\rho_A \otimes \rho_{A'})\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ F_{AA'} \left(\left(\sum_i \lambda_i |i\rangle_A \langle i|_A \right) \otimes \left(\sum_j \lambda_j |j\rangle_{A'} \langle j|_{A'} \right) \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ F_{AA'} \left(\sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle i|_A \otimes |j\rangle_{A'} \langle j|_{A'} \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ F_{AA'} \left(\sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle i|_A \otimes |j\rangle_{A'} \langle j|_{A'} \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ F_{AA'} \left(\sum_{i,j} \lambda_i \lambda_j \left(|i\rangle_A |j\rangle_{A'} \right) \left(\langle i|_A \langle j|_{A'} \right) \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ \left(\sum_{i,j} \lambda_i \lambda_j \left(F_{AA'} |i\rangle_A |j\rangle_{A'} \right) \left(\langle i|_A \langle j|_{A'} \right) \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ \sum_{i,j} \lambda_i \lambda_j \left(|j\rangle_A |i\rangle_{A'} \right) \left(\langle i|_A \langle j|_{A'} \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \operatorname{Tr}\left\{ \left(|j\rangle_A |i\rangle_{A'} \right) \left(\langle i|_A \langle j|_{A'} \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \operatorname{Tr}\left\{ \left(\langle i|_A \langle j|_{A'} \right) \left(|j\rangle_A |i\rangle_{A'} \right) \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \operatorname{Tr}\left\{ \langle i|j\rangle_A \otimes \langle j|i\rangle_{A'} \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \langle i|j\rangle_A \langle j|i\rangle_{A'}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \delta_{i,j}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_i \lambda_i^2$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \operatorname{Tr}\left\{ \rho_A^2 \right\}$$

$$\operatorname{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = P(\rho_A)$$

Hence, proven.