Introduction to Quantum Information and Communication

Take Home Mid-Sem

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Question 5

(a)

To Prove:

$$\sum_{z\in\{0,1\}^n} (-1)^{(x\oplus y)\cdot z} = 2^n \delta(x, y)$$

Proof:

Case 1: $x = y$

$$\sum_{z\in\{0,1\}^n} (-1)^{(x\oplus y)\cdot z}$$

$$\sum_{z\in\{0,1\}^n} (-1)^{0\cdot z}$$

$$\sum_{z\in\{0,1\}^n} (-1)^{0}$$

$$\sum_{z\in\{0,1\}^n} 1$$

$$2^n$$

$$2^n \times 1$$

$$2^n \delta(x, y)$$

Case 2: $x \neq y$

Let k be the number of digits different between x and y, and let the corresponding indices be $\alpha = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k\}$

$$\forall i \in \{1, 2, ..., k\}\ x_{\alpha_i} \neq y_{\alpha_i}$$

$$\forall i \notin \alpha\ x_i = y_i$$

.

$$\sum_{z\in\{0,1\}^n} (-1)^{(x\oplus y)\cdot z}$$

$$\sum_{z\in\{0,1\}^n} (-1)^{\oplus_{i=1}^{n} (x_i \oplus y_i) z_i}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{\oplus_{i=1}^{k} z_{\alpha_i}}$$

$$\sum_{z \in \{0,1\}^n} (-1)^{z_{\alpha_1} \oplus z_{\alpha_2} \oplus \ldots \oplus z_{\alpha_k}}$$

Now, since z is looping through all possible bitstrings of length n, the parity of any subset of its bits will be odd half the times and even half the times.

$$-1 + 1 - 1 + 1 \ldots - 1 + 1$$

$$0$$

$$2^n \times 0$$

$$2^n \delta(x, y)$$

Now, from both the cases we get

$$\sum_{z \in \{0,1\}^n} (-1)^{(x \oplus y) \cdot z} = 2^n \delta(x, y)$$

Hence, proven

(b)

Given:

$$f : \{0,1\}^n \mapsto \{0,1\}^n$$

$$U_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := |x\rangle_Q \otimes |y \oplus f(x)\rangle_R$$

$$V_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := (-1)^{y \cdot f(x)} |x\rangle_Q \otimes |y\rangle_R$$

To Prove:

$$V_f\left(|x\rangle_Q \otimes |y\rangle_R\right) = (\mathbb{I}_Q \otimes H^{\otimes n}) U_f (\mathbb{I}_Q \otimes H^{\otimes n})\left(|x\rangle_Q \otimes |y\rangle_R\right)$$

Proof: We will be using the identity $H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{x \cdot z} |z\rangle$

$$(\mathbb{I}_Q \otimes H^{\otimes n}) U_f (\mathbb{I}_Q \otimes H^{\otimes n})\left(|x\rangle_Q \otimes |y\rangle_R\right)$$

$$(\mathbb{I}_Q \otimes H^{\otimes n}) U_f \left(\mathbb{I}_Q |x\rangle_Q \otimes H^{\otimes n} |y\rangle_R\right)$$

$$(\mathbb{I}_Q \otimes H^{\otimes n}) U_f \left(|x\rangle_Q \otimes \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} |z\rangle_R\right)$$

$$\frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} (\mathbb{I}_Q \otimes H^{\otimes n}) U_f \left(|x\rangle_Q \otimes |z\rangle_R\right)$$

$$\frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} (\mathbb{I}_Q \otimes H^{\otimes n}) \left(|x\rangle_Q \otimes |z \oplus f(x)\rangle_R\right)$$

$$\frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} \left(\mathbb{I}_Q |x\rangle_Q \otimes H^{\otimes n} |z \oplus f(x)\rangle_R \right)$$

$$\frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{y \cdot z} |x\rangle_Q \otimes \left(\frac{1}{\sqrt{2^n}} \sum_{w \in \{0,1\}^n} (-1)^{(z \oplus f(x)) \cdot w} |w\rangle_R \right)$$

$$\frac{1}{2^n} \sum_{z, w \in \{0,1\}^n} (-1)^{(y \cdot z)} (-1)^{(z \oplus f(x)) \cdot w} |x\rangle_Q \otimes |w\rangle_R$$

$$\frac{1}{2^n} \sum_{w \in \{0,1\}^n} (-1)^{w \cdot f(x)} |x\rangle_Q \otimes |w\rangle_R \sum_{z \in \{0,1\}^n} (-1)^{(y \oplus w) \cdot z}$$

$$\frac{1}{2^n} \sum_{w \in \{0,1\}^n} (-1)^{w \cdot f(x)} |x\rangle_Q \otimes |w\rangle_R 2^n \delta(w, y)$$

$$\sum_{w \in \{0,1\}^n} (-1)^{w \cdot f(x)} |x\rangle_Q \otimes |w\rangle_R \delta(w, y)$$

$$(-1)^{y \cdot f(x)} |x\rangle_Q \otimes |y\rangle_R$$

$$V_f \left(|x\rangle_Q \otimes |y\rangle_R \right)$$

Hence, proven

Question 6

(a)

Before the first Hadamard, the state is

$$|0\rangle_A |\psi\rangle_B |\varphi\rangle_C$$

After the first Hadamard, the state is

$$H_A |0\rangle_A |\psi\rangle_B |\varphi\rangle_C$$

$$\frac{1}{\sqrt{2}} \left(|0\rangle_A + |1\rangle_A \right) |\psi\rangle_B |\varphi\rangle_C$$

$$\frac{1}{\sqrt{2}} |0\rangle_A |\psi\rangle_B |\varphi\rangle_C + \frac{1}{\sqrt{2}} |1\rangle_A |\psi\rangle_B |\varphi\rangle_C$$

After the Controlled-SWAP, the state is

$$\frac{1}{\sqrt{2}} |0\rangle_A |\psi\rangle_B |\varphi\rangle_C + \frac{1}{\sqrt{2}} |1\rangle_A |\varphi\rangle_B |\psi\rangle_C$$

After the second Hadamard, we get the required state

$$|\psi'\rangle_{ABC} = H_A \left(\frac{1}{\sqrt{2}} |0\rangle_A |\psi\rangle_B |\varphi\rangle_C + \frac{1}{\sqrt{2}} |1\rangle_A |\varphi\rangle_B |\psi\rangle_C \right)$$

$$|\psi'\rangle_{ABC} = \frac{1}{\sqrt{2}} H_A |0\rangle_A |\psi\rangle_B |\varphi\rangle_C + \frac{1}{\sqrt{2}} H_A |1\rangle_A |\varphi\rangle_B |\psi\rangle_C$$

$$|\psi'\rangle_{ABC} = \frac{1}{2}\left(|0\rangle_A + |1\rangle_A\right)|\psi\rangle_B |\varphi\rangle_C + \frac{1}{2}\left(|0\rangle_A - |1\rangle_A\right)|\varphi\rangle_B |\psi\rangle_C$$

$$|\psi'\rangle_{ABC} = \frac{1}{2}|0\rangle_A\left(|\psi\rangle_B |\varphi\rangle_C + |\varphi\rangle_B |\psi\rangle_C\right) + \frac{1}{2}|1\rangle_A\left(|\psi\rangle_B |\varphi\rangle_C - |\varphi\rangle_B |\psi\rangle_C\right)$$

This is the required tripartite state

(b)

$$p_0 = \frac{1}{2}\left(\langle\psi|_B \langle\varphi|_C + \langle\varphi|_B \langle\psi|_C\right)\frac{1}{2}\left(|\psi\rangle_B |\varphi\rangle_C + |\varphi\rangle_B |\psi\rangle_C\right)$$

$$p_0 = \frac{1}{4}\left(\langle\psi|_B \langle\varphi|_C |\psi\rangle_B |\varphi\rangle_C + \langle\psi|_B \langle\varphi|_C |\varphi\rangle_B |\psi\rangle_C + \langle\varphi|_B \langle\psi|_C |\psi\rangle_B |\varphi\rangle_C + \langle\varphi|_B \langle\psi|_C |\varphi\rangle_B |\psi\rangle_C\right)$$

$$p_0 = \frac{1}{4}\left(\langle\psi|\psi\rangle_B \otimes \langle\varphi|\varphi\rangle_C + \langle\psi|\varphi\rangle_B \otimes \langle\varphi|\psi\rangle_C + \langle\varphi|\psi\rangle_B \otimes \langle\psi|\varphi\rangle_C + \langle\varphi|\varphi\rangle_B \otimes \langle\psi|\psi\rangle_C\right)$$

$$p_0 = \frac{1}{4}(1 + |\langle\psi|\varphi\rangle|^2 + |\langle\psi|\varphi\rangle|^2 + 1)$$

$$p_0 = \frac{1}{2} + \frac{1}{2}|\langle\psi|\varphi\rangle|^2$$

Since $p_0 + p_1 = 1$,

$$p_1 = \frac{1}{2} - \frac{1}{2}|\langle\psi|\varphi\rangle|^2$$

(c)

Since $|\psi\rangle_A$ and $|\varphi\rangle_B$ are pure states, their fidelity is $|\langle\psi|\varphi\rangle|^2$

The probability of measuring a 0 is p_0, so we get

$$p_0 = \frac{m}{N}$$

$$\frac{1}{2} + \frac{1}{2}|\langle\psi|\varphi\rangle|^2 = \frac{m}{N}$$

$$1 + |\langle\psi|\varphi\rangle|^2 = 2\frac{m}{N}$$

$$|\langle\psi|\varphi\rangle|^2 = 2\frac{m}{N} - 1$$

This is the required fidelity

Question 7

Given:

$$f : \{0,1\}^n \mapsto \{0,1\}^n$$

$$\forall x, y \in \{0,1\}^n \ f(x) = f(y) \leftrightarrow x \oplus y \in \{0^n, d\}$$

$$U_f\left(|x\rangle_Q \otimes |y\rangle_R\right) := |x\rangle_Q \otimes |y \oplus f(x)\rangle_R$$

(a)

To Prove: f is one-to-one when $d = 0^n$ and two-to-one otherwise

Proof:

Case 1: $d = 0^n$

$$\forall x, y \in \{0,1\}^n \ f(x) = f(y) \leftrightarrow x \oplus y \in \{0^n, 0^n\}$$

$$\forall x, y \in \{0,1\}^n \ f(x) = f(y) \leftrightarrow x \oplus y \in \{0^n\}$$

$$\forall x, y \in \{0,1\}^n \ f(x) = f(y) \leftrightarrow x \oplus y = 0^n$$

$$\forall x, y \in \{0,1\}^n \ f(x) = f(y) \leftrightarrow x = y$$

Thus, f is one-one in this case

Case 2: $d \neq 0^n$

To prove that f is two-one, we need to show that $\forall z \in \text{range}(f)$, we have exactly two elements x, y such that $f(x) = f(y) = z$

$$\forall x, y \in \{0,1\}^n \ f(x) = f(y) \leftrightarrow x \oplus y \in \{0^n, d\}$$

(i) $x \oplus y = 0^n$

$x = y$, thus $f(x) = f(y)$

(ii) $x \oplus y = d$ with $d \neq 0^n$

$$y = d \oplus x$$

Since $d \neq 0^n$, we get $y \neq x$, and $f(x) = f(y)$

Clearly, two distinct values x and y give the same output. Now, we need to prove that no more than two distinct inputs give the same output.

Consider distinct $a, b, c \in \{0,1\}^n$ such that $f(a) = f(b) = f(c)$

Since a, b, c are distinct, their xor cannot be 0^d, thus we have

$$a \oplus b = b \oplus c = d$$

$$a = d \oplus b, c = d \oplus b$$

$$a = c$$

This is a contradiction. Thus, there only exist exactly two input values for each output value.

Thus, f is a two-one function in this case

Hence, proven

(b)

To Find: $|\psi'\rangle_{QR}$

Solution:

Initially, the state is

$$|0^n\rangle_Q \otimes |O^n\rangle_R$$

After the first Hadamard, the state is

$$H^{\otimes n}|0^n\rangle_Q \otimes |0^n\rangle_R$$

$$\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle_Q \otimes |0^n\rangle_R$$

After the oracle, the state is

$$U_f \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle_Q \otimes |0^n\rangle_R$$

$$\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} U_f|x\rangle_Q \otimes |0^n\rangle_R$$

$$\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle_Q \otimes |0^n \oplus f(x)\rangle_R$$

$$\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle_Q \otimes |f(x)\rangle_R$$

After the second Hadamard, the required state is

$$|\psi'\rangle_{QR} = H^{\otimes n} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle_Q \otimes |f(x)\rangle_R$$

$$|\psi'\rangle_{QR} = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} H^{\otimes n}|x\rangle_Q \otimes |f(x)\rangle_R$$

$$|\psi'\rangle_{QR} = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle_Q \otimes |f(x)\rangle_R$$

$$|\psi'\rangle_{QR} = \frac{1}{2^n} \sum_{x,z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle_Q \otimes |f(x)\rangle_R$$

(c)

To Prove: Probability of getting outcome $j = j_1 ... j_n$ is given by

$$p(j) = \| \frac{1}{2^n} \sum_{z\in \text{ range}(f)} \left(1 + (-1)^{j\cdot d}\right)|z\rangle\|^2$$

Proof:

$$|\psi'\rangle_{QR} = \frac{1}{2^n} \sum_{x,z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle_Q \otimes |f(x)\rangle_R$$

The coefficient of $|j\rangle$ is

$$|\varphi\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot j} |f(x)\rangle$$

The probability of measuring $|j\rangle$ is thus

$$p(j) = \| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot j} |f(x)\rangle \|^2$$

Case 1: $d = 0^n$

The function is one-one, so $\text{range}(f) = \{0,1\}^n$, the summation is over every basis vector, and we get

$$p(j) = \frac{1}{2^n}$$

Case 2: $d \neq 0^n$

The function is two-one, thus the range is half of the co-domain (as $f(x) = f(x \oplus d) = z$)

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left((-1)^{x \cdot j} + (-1)^{(x \oplus d) \cdot j}\right) |z\rangle \|^2$$

where x is any one of the two pre-images of z

If $x \cdot j = 0$, we have

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \|^2$$

If $x \cdot j = 1$, we have

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(-1 + (-1)^{1 + j \cdot d}\right) |z\rangle \|^2$$

$$p(j) = \| \frac{1}{2^n}(-1) \sum_{z \in \text{ range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \|^2$$

$$p(j) = \left((-1) \ \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \ \| \right)^2$$

$$p(j) = (-1)^2 \ \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \|^2$$

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \|^2$$

In all cases, we get

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^{j \cdot d}\right) |z\rangle \|^2$$

Hence, proven.

(d)

To Prove: $p(j)$ is nonzero only if $j \cdot z = 0$

Proof:

We know that

$$j \cdot z = \oplus_{i=1}^{n} j_i z_i$$

Thus, either $j \cdot z = 0$ or $j \cdot z = 1$, since the xor of bits can only be a bit.

If $j \cdot z = 0$,

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^0\right)|z\rangle \|^2$$

$$p(j) = \| \frac{1}{2^{n-1}} \sum_{z \in \text{ range}(f)} |z\rangle \|^2$$

If otherwise, i.e, $j \cdot z = 1$

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} \left(1 + (-1)^1\right)|z\rangle \|^2$$

$$p(j) = \| \frac{1}{2^n} \sum_{z \in \text{ range}(f)} 0|z\rangle \|^2$$

$$p(j) = 0$$

Clearly, if $j \cdot z = 0$, only then $p(j)$ can be non-zero.

Hence, proven.

(e)

To Find: The number of queries to f to determine d classically

Solution: We can use the fact that the function is either one-one or two-one depending on the choice of d.

If we perform $2^{n-1} + 1$ queries, there are two cases

1. If all the outputs are distinct, the function can't be two-one, as one of the outputs must have been repeated if it was. Thus, the function is one-one. Thus, d is 0^n.
2. If any two outputs are same, say $f(x) = f(y) = z$, then we have $d = x \oplus y$

Thus, always within $2^{n-1} + 1$ queries to f, one can determine d classically.

Question 8

Given:

- Arbitrary pure state $|\Psi\rangle_{AB}$
- Measurement operators $\left\{Q_y^B\right\}_y$

To Prove: There exists as set of measurement operators $\left\{P_y^A\right\}_y$ such that

$$|\Psi'\rangle_{AB} = \left(\mathbb{I}_A \otimes Q_y^A\right)|\Psi\rangle_{AB} = \left(U_y^A \otimes V_y^B\right)\left(P_y^A \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

Proof:

Consider the Schmidt decomposition of $|\Psi\rangle_{AB}$

$$|\Psi\rangle_{AB} = \sum_k \alpha_k |u_k\rangle_A \otimes |v_k\rangle_B$$

where α_i are real and non-negative. We will be working in this basis.

The probabilities of measurements in both the Hilbert spaces must be the same, so on applying the measurement operators to the respective states, we get the same outcome probabilties

$$\langle\Psi|\mathbb{I}_A \otimes Q_y|\Psi\rangle = \langle\Psi|P_y \otimes \mathbb{I}_B|\Psi\rangle$$

$$\left(\sum_k \alpha_k \langle u_k|_A \otimes \langle v_k|_B\right) \mathbb{I}_A \otimes Q_y \left(\sum_k \alpha_k |u_k\rangle_A \otimes |v_k\rangle_B\right) = \left(\sum_k \alpha_k \langle u_k|_A \otimes \langle v_k|_B\right) P_y \otimes \mathbb{I}_B \left(\sum_k \alpha_k |u_k\rangle_A \otimes |v_k\rangle_B\right)$$

$$\left(\sum_k \alpha_k \langle u_k|_A \otimes \langle v_k|_B\right)\left(\sum_k \alpha_k \mathbb{I}_A |u_k\rangle_A \otimes Q_y |v_k\rangle_B\right) = \left(\sum_k \alpha_k \langle u_k|_A \otimes \langle v_k|_B\right)\left(\sum_k \alpha_k P_y |u_k\rangle_A \otimes \mathbb{I}_B |v_k\rangle_B\right)$$

$$\left(\sum_k \alpha_k \langle u_k|_A \otimes \langle v_k|_B\right)\left(\sum_k \alpha_k |u_k\rangle_A \otimes Q_y |v_k\rangle_B\right) = \left(\sum_k \alpha_k \langle u_k|_A \otimes \langle v_k|_B\right)\left(\sum_k \alpha_k P_y |u_k\rangle_A \otimes |v_k\rangle_B\right)$$

$$\sum_{k_1,k_2} \alpha_{k_1}\alpha_{k_2}\left(\left\langle u_{k_1}\right|_A \otimes \left\langle v_{k_1}\right|_B\right)\left(\left|u_{k_2}\right\rangle_A \otimes Q_y\left|v_{k_2}\right\rangle_B\right) = \sum_{k_1,k_2} \alpha_{k_1}\alpha_{k_2}\left(\left\langle u_{k_1}\right|_A \otimes \left\langle v_{k_1}\right|_B\right)\left(P_y\left|u_{k_2}\right\rangle_A \otimes \left|v_{k_2}\right\rangle\right)$$

$$\sum_{k_1,k_2} \alpha_{k_1}\alpha_{k_2}\left\langle u_{k_1}|u_{k_2}\right\rangle_A \left\langle v_{k_1}\right|_B Q_y \left|v_{k_2}\right\rangle_B = \sum_{k_1,k_2} \alpha_{k_1}\alpha_{k_2}\left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A \left\langle v_{k_1}|v_{k_2}\right\rangle_B$$

$$\sum_{k_1,k_2} \alpha_{k_1}\alpha_{k_2}\delta_{k_1,k_2}\left\langle v_{k_1}\right|_B Q_y \left|v_{k_2}\right\rangle_B = \sum_{k_1,k_2} \alpha_{k_1}\alpha_{k_2}\left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A \delta_{k_1,k_2}$$

$$\sum_k \alpha_k^2 \langle v_k|_B Q_y |v_k\rangle_B = \sum_{k_1,k_2} \alpha_k^2 \left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A$$

On expressing Q_y in the Schmidt basis

$$\sum_k \alpha_k^2 \langle v_k|_B \left(\sum_{i,j} q_{ij} |v_i\rangle\langle v_j|\right) |v_k\rangle_B = \sum_{k_1,k_2} \alpha_k^2 \left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A$$

$$\sum_{i,j,k} \alpha_k^2 q_{ij} \langle v_k|v_i\rangle\langle v_j|v_k\rangle = \sum_{k_1,k_2} \alpha_k^2 \left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A$$

$$\sum_{i,j,k} \alpha_k^2 q_{ij} \delta_{i,k}\delta_{j,k} = \sum_{k_1,k_2} \alpha_k^2 \left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A$$

$$\sum_k \alpha_k^2 q_{kk} = \sum_{k_1,k_2} \alpha_k^2 \left\langle u_{k_1}\right|_A P_y \left|u_{k_2}\right\rangle_A$$

Thus, we get

$$P_y = \sum_{i,j} q_{ij} |u_i\rangle\langle u_j|$$

Now, on applying the measurement, we get

$$|\Psi'\rangle_{AB} = \sum_k \alpha_k P_y |u_k\rangle_A \otimes |v_k\rangle_B$$

$$|\Psi'\rangle_{AB} = \sum_k \alpha_k \left(\sum_{i,j} q_{ij} |u_i\rangle \right) \langle u_j || u_k\rangle_A \otimes |v_k\rangle_B$$

$$\sum_{i,j,k} \alpha_k q_{ij} |u_i\rangle \langle u_j | u_k\rangle \otimes |v_k\rangle$$

$$|\Psi'\rangle_{AB} = \sum_{i,j,k} \alpha_k q_{ij} \delta_{j,k} |u_i\rangle \otimes |v_k\rangle$$

$$|\Psi'\rangle_{AB} = \sum_{i,k} \alpha_k q_{ik} |u_i\rangle \otimes |v_k\rangle$$

and

$$|\Psi\rangle_{AB} = \sum_k \alpha_k |u_k\rangle_A \otimes Q_y |v_k\rangle_B$$

$$|\Psi\rangle_{AB} = \sum_k \alpha_k |u_k\rangle_A \otimes \left(\sum_{i,j} q_{ij} |v_i\rangle \langle v_j| \right) |v_k\rangle_B$$

$$|\Psi\rangle_{AB} = \sum_{i,j,k} \alpha_k q_{i,j} |u_k\rangle \otimes |v_i\rangle \langle v_j | v_k\rangle$$

$$|\Psi\rangle_{AB} = \sum_{i,j,k} \alpha_k q_{i,j} \delta_{j,k} |u_k\rangle \otimes |v_i\rangle$$

$$|\Psi\rangle_{AB} = \sum_{i,k} \alpha_k q_{i,k} |u_k\rangle \otimes |v_i\rangle$$

Consider unitaries $U'_y \otimes V'_y$ that transform $|\Psi\rangle_{AB}$ into Schmidt form

$$U'_y \otimes V'_y |\Psi\rangle_{AB} = \sum_k \alpha_k |u_k\rangle_A \otimes |v_k\rangle_B$$

Clearly, $V'_y \otimes U'_y$ transforms $|\Psi'\rangle_{AB}$ into Schmidt form

$$\left(U'_y \otimes V'_y\right)|\Psi\rangle_{AB} = \left(V'_y \otimes U'_y\right)|\Psi'\rangle_{AB}$$

$$\left(U'_y \otimes V'_y\right)\left(\mathbb{I}_A \otimes Q_y\right)|\Psi\rangle_{AB} = \left(V'_y \otimes U'_y\right)\left(P_y \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

$$\left(\mathbb{I}_A \otimes Q_y\right)|\Psi\rangle_{AB} = \left(U'_y \otimes V'_y\right)^\dagger \left(V'_y \otimes U'_y\right)\left(P_y \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

$$\left(\mathbb{I}_A \otimes Q_y\right)|\Psi\rangle_{AB} = \left({U'_y}^\dagger \otimes {V'_y}^\dagger\right) \left(V'_y \otimes U'_y\right)\left(P_y \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

$$\left(\mathbb{I}_A \otimes Q_y\right)|\Psi\rangle_{AB} = \left({U'_y}^\dagger V'_y \otimes {V'_y}^\dagger U'_y\right)\left(P_y \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

Let $U^A_y = {U'_y}^\dagger V'_y$ and $V^B_y = {V'_y}^\dagger U'_y$

$$\left(\mathbb{I}_A \otimes Q_y\right)|\Psi\rangle_{AB} = \left(U^A_y \otimes V^B_y\right)\left(P_y \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

Thus, we have

$$|\Psi'\rangle_{AB} = \left(\mathbb{I}_A \otimes Q_y\right)|\Psi\rangle_{AB} = \left(U_y^A \otimes V_y^B\right)\left(P_y \otimes \mathbb{I}_B\right)|\Psi\rangle_{AB}$$

Hence, proven.