

Introduction to Quantum Information and Communication

Theory Assignment-2

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Exercise 4.1.3

Given:

- A is a square operator acting on Hilbert space \mathcal{H}_S
- I_R is the identity operator acting on a Hilbert space \mathcal{H}_R isomorphic to \mathcal{H}_S
- $|\Gamma\rangle_{RS}$ is the unnormalized maximally entangled vector.

To Prove:

$$\text{Tr}\{A\} = \langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS}$$

Proof:

In the computational basis

$$|\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_S$$

$$\langle \Gamma |_{RS} = \sum_{i=0}^{d-1} \langle i |_R \langle i |_S$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i |_R \langle i |_S \right) (I_R \otimes A_S) \left(\sum_{j=0}^{d-1} |j\rangle_R |j\rangle_S \right)$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i |_R \langle i |_S \right) \left(\sum_{j=0}^{d-1} (I_R \otimes A_S) (|j\rangle_R \otimes |j\rangle_S) \right)$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i |_R \langle i |_S \right) \left(\sum_{j=0}^{d-1} (I_R |j\rangle_R) \otimes (A_S |j\rangle_S) \right)$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i |_R \langle i |_S \right) \left(\sum_{j=0}^{d-1} |j\rangle_R \otimes (A_S |j\rangle_S) \right)$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \sum_{i,j=0}^{d-1} (\langle i |_R \otimes \langle i |_S) (|j\rangle_R \otimes (A_S |j\rangle_S))$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \sum_{i,j=0}^{d-1} (\langle i | j \rangle_R \otimes \langle i |_S A_S | j \rangle_S)$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \sum_{i,j=0}^{d-1} (\delta_{i,j} \otimes \langle i |_S A_S | j \rangle_S)$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \sum_{i=0}^{d-1} \langle i |_S A_S | i \rangle_S$$

$$\langle \Gamma |_{RS} I_R \otimes A_S | \Gamma \rangle_{RS} = \text{Tr}\{A\}$$

Hence, proven.

Exercise 4.1.16

Given:

- Commuting projectors Π_1 and Π_2
- $0 \leq \Pi_1, \Pi_2 \leq I$

To Prove:

For arbitrary density operator ρ

$$\text{Tr}\{(I - \Pi_1 \Pi_2)\rho\} \leq \text{Tr}\{(I - \Pi_1)\rho\} + \text{Tr}\{(I - \Pi_2)\rho\}$$

Proof:

TO DO

Exercise 4.2.2

Given:

- Ensemble $\{p_X(x), \rho_x\}$ of density operators
- POVM with elements $\{\Lambda_x\}$
- Operator τ such that $\tau \geq p_X(x)\rho_x$

To Prove:

$$\text{Tr}\{\tau\} \geq \sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\}$$

Proof:

$$\sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\} = \sum_x \text{Tr}\{\Lambda_x p_X(x) \rho_x\}$$

$$\sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\} \leq \sum_x \text{Tr}\{\Lambda_x \tau\}$$

$$\sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\} \leq \text{Tr}\left\{\sum_x \Lambda_x \tau\right\}$$

$$\sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\} \leq \text{Tr}\left\{\tau \sum_x \Lambda_x\right\}$$

$$\sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\} \leq \text{Tr}\{\tau I\}$$

$$\sum_x p_X(x) \text{Tr}\{\Lambda_x \rho_x\} \leq \text{Tr}\{\tau\}$$

Hence, proven.

Now for the case of encoding n bits into a d -dimensional subspace.

$$\{2^{-n}, \rho_i\}_{i \in \{0,1\}^n}$$

Consider

$$p_X(x)\rho_x = 2^{-n}\rho_i$$

$$p_X(x)\rho_x = 2^{-n} \sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}I - 2^{-n} \sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n} \sum_j |j\rangle\langle j| - 2^{-n} \sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n} \sum_j (1 - \lambda_j) |j\rangle\langle j|$$

Since $0 \leq \lambda_j \leq 1 \forall j$, $1 - \lambda_j \geq 0 \forall j$. All the eigenvalues of the matrix in LHS are non-negative.

$$2^{-n}I - p_X(x)\rho_x \geq 0$$

$$2^{-n}I \geq p_X(x)\rho_x$$

\therefore We consider $\tau = 2^{-n}I$

Now, we know that the probability of success is upper bounded by $\text{Tr}\{\tau\}$

$$\text{Tr}\{\tau\} = \text{Tr}\{2^{-n}I\}$$

$$\text{Tr}\{\tau\} = 2^{-n} \text{Tr}\{I\}$$

Since I is d -dimensional,

$$\text{Tr}\{\tau\} = d2^{-n}$$

Thus, the expected success probability is bounded above by $d2^{-n}$

Exercise 4.3.1

Given:

- A' has a Hilbert space structure isomorphic to that of system A
- $\forall x, y \ F_{AA'} |x\rangle_A |y\rangle_{A'} = |y\rangle_A |x\rangle_{A'}$

To Prove:

$$P(\rho_A) = \text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\}$$

Proof:

$$\rho_A = \sum_i \lambda_i |i\rangle_A \langle i|_A$$

$$\rho_{A'} = \sum_j \lambda_j |j\rangle_{A'} \langle j|_{A'}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\{F_{AA'}(\rho_A \otimes \rho_{A'})\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\left\{F_{AA'}\left(\left(\sum_i \lambda_i |i\rangle_A \langle i|_A\right) \otimes \left(\sum_j \lambda_j |j\rangle_{A'} \langle j|_{A'}\right)\right)\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\left\{F_{AA'}\left(\sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle i|_A \otimes |j\rangle_{A'} \langle j|_{A'}\right)\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\left\{F_{AA'}\left(\sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle i|_A \otimes |j\rangle_{A'} \langle j|_{A'}\right)\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\left\{F_{AA'}\left(\sum_{i,j} \lambda_i \lambda_j (|i\rangle_A \langle j|_{A'}) (\langle i|_A \langle j|_{A'})\right)\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\left\{\left(\sum_{i,j} \lambda_i \lambda_j (F_{AA'} |i\rangle_A \langle j|_{A'}) (\langle i|_A \langle j|_{A'})\right)\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\left\{\sum_{i,j} \lambda_i \lambda_j (|j\rangle_A \langle i|_{A'}) (\langle i|_A \langle j|_{A'})\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \text{Tr}\left\{(|j\rangle_A \langle i|_{A'}) (\langle i|_A \langle j|_{A'})\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \text{Tr}\left\{(\langle i|_A \langle j|_{A'}) (|j\rangle_A \langle i|_{A'})\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \text{Tr}\left\{\langle i|j\rangle_A \otimes \langle j|i\rangle_{A'}\right\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \langle i|j\rangle_A \langle j|i\rangle_{A'}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \delta_{i,j}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_i \lambda_i^2$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \text{Tr}\{\rho_A^2\}$$

$$\text{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = P(\rho_A)$$

Hence, proven.

Exercise 4.3.6

Given:

$$\Pi_{\text{even}} = \frac{1}{2}(I_A \otimes I_B + Z_A \otimes Z_B) = |00\rangle_{AB} \langle 00|_{AB} + |11\rangle_{AB} \langle 11|_{AB}$$

$$\Pi_{\text{odd}} = \frac{1}{2}(I_A \otimes I_B - Z_A \otimes Z_B) = |01\rangle_{AB} \langle 01|_{AB} + |10\rangle_{AB} \langle 10|_{AB}$$

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

$$\pi_A = \frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A)$$

$$\pi_B = \frac{1}{2}(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B)$$

To Prove:

- $|\Phi^+\rangle_{AB}$ returns an even parity result with probability 1
- $\pi_A \otimes \pi_B$ returns even or odd parity with equal probability

Proof:

First we find the density matrix of the bell state

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}$$

Now, the probability of the bell state collapsing to Π_{even} is

$$P = \text{Tr}\{\rho_{AB}\Pi_{\text{even}}\}$$

$$P = \text{Tr}\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB})\}$$

$$P = \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||00\rangle\langle 00| + |\Phi^+\rangle\langle\Phi^+||11\rangle\langle 11|\}$$

$$P = \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||00\rangle\langle 00|\} + \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||11\rangle\langle 11|\}$$

$$P = \text{Tr}\{\langle 00|\Phi^+\rangle\langle\Phi^+|00\rangle\} + \text{Tr}\{\langle 11|\Phi^+\rangle\langle\Phi^+|11\rangle\}$$

$$P = \frac{1}{2} + \frac{1}{2}$$

$$P = 1$$

$\therefore |\Phi^+\rangle_{AB}$ returns an even parity result with probability 1

Now, we find the probability of $\pi_A \otimes \pi_B$ returning even parity

$$P = \text{Tr}\{(\pi_A \otimes \pi_B)\Pi_{\text{even}}\}$$

$$P = \frac{1}{4} \text{Tr}\{(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A) \otimes (|0\rangle\langle 0|_B + |1\rangle\langle 1|_B)(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB})\}$$

$$P = \frac{1}{4} \text{Tr}\{(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B)(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB})\}$$

$$P = \frac{1}{4} \text{Tr}\{(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|)(|00\rangle\langle 00| + |11\rangle\langle 11|)\}$$

$$\begin{aligned} P = \frac{1}{4} \text{Tr}\{ & |00\rangle\langle 00||00\rangle\langle 00| + |00\rangle\langle 00||11\rangle\langle 11| + \\ & |01\rangle\langle 01||00\rangle\langle 00| + |01\rangle\langle 01||11\rangle\langle 11| + \\ & |10\rangle\langle 10||00\rangle\langle 00| + |10\rangle\langle 10||11\rangle\langle 11| + \\ & |11\rangle\langle 11||00\rangle\langle 00| + |11\rangle\langle 11||11\rangle\langle 11|\} \end{aligned}$$

$$P = \frac{1}{4}(\text{Tr}\{|00\rangle\langle 00|\} + \text{Tr}\{|11\rangle\langle 11|\})$$

$$P = \frac{1}{4}(1 + 1)$$

$$P = \frac{1}{2}$$

The probability of $\pi_A \otimes \pi_B$ returning an odd parity is $1 - P = 1 - \frac{1}{2} = \frac{1}{2}$ (As the measurements are orthogonal)

$\therefore \pi_A \otimes \pi_B$ returns even or odd parity with equal probability

Now, we perform the same calculations for the phase parity measurement

$$\Pi_{\text{even}}^X = \frac{1}{2}(I_A \otimes I_B + X_A \otimes X_B)$$

$$\Pi_{\text{odd}}^X = \frac{1}{2}(I_A \otimes I_B - X_A \otimes X_B)$$

The probability of the bell state collapsing to Π_{even}^X is

$$P = \text{Tr}\{\rho_{AB}\Pi_{\text{even}}^X\}$$

$$P = \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} (I_A \otimes I_B + X_A \otimes X_B)\}$$

$$P = \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} I_A \otimes I_B\} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} X_A \otimes X_B\}$$

$$P = \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB}\} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} X_A \otimes X_B\}$$

$$P = \frac{1}{2} \text{Tr}\{\langle \Phi^+ |_{AB} | \Phi^+ \rangle_{AB}\} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} X_A \otimes X_B\}$$

$$P = \frac{1}{2} \text{Tr}\{\langle \Phi^+ | \Phi^+ \rangle_{AB}\} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} X_A \otimes X_B\}$$

$$P = \frac{1}{2} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} X_A \otimes X_B\}$$

$$P = \frac{1}{2} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} (|0\rangle\langle 1|_A + |1\rangle\langle 0|_A) \otimes (|0\rangle\langle 1|_B + |1\rangle\langle 0|_B)\}$$

$$P = \frac{1}{2} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle_{AB} \langle \Phi^+ |_{AB} (|0\rangle\langle 1|_A \otimes |0\rangle\langle 1|_B + |0\rangle\langle 1|_A \otimes |1\rangle\langle 0|_B + |1\rangle\langle 0|_A \otimes |0\rangle\langle 1|_B + |1\rangle\langle 0|_A \otimes |1\rangle\langle 0|_B)\}$$

$$P = \frac{1}{2} + \frac{1}{2} \text{Tr}\{| \Phi^+ \rangle \langle \Phi^+ | (|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|)\}$$

$$P = \frac{1}{2}(1 + \text{Tr}\{| \Phi^+ \rangle \langle \Phi^+ | |00\rangle\langle 11|\} + \text{Tr}\{| \Phi^+ \rangle \langle \Phi^+ | |01\rangle\langle 10|\} + \text{Tr}\{| \Phi^+ \rangle \langle \Phi^+ | |10\rangle\langle 01|\} + \text{Tr}\{| \Phi^+ \rangle \langle \Phi^+ | |11\rangle\langle 00|\})$$

$$P = \frac{1}{2}(1 + \text{Tr}\{\langle 11 | \Phi^+ \rangle \langle \Phi^+ | 00 \rangle\} + \text{Tr}\{\langle 10 | \Phi^+ \rangle \langle \Phi^+ | 01 \rangle\} + \text{Tr}\{\langle 01 | \Phi^+ \rangle \langle \Phi^+ | 10 \rangle\} + \text{Tr}\{\langle 00 | \Phi^+ \rangle \langle \Phi^+ | 11 \rangle\})$$

$$P = \frac{1}{2} \left(1 + \text{Tr} \left\{ \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} \right\} + \text{Tr}\{0 * 0\} + \text{Tr}\{0 * 0\} + \text{Tr} \left\{ \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} \right\} \right)$$

$$P = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{2} \right)$$

$$P = \frac{1}{2}(2)$$

$$P = 1$$

$\therefore |\Phi^+\rangle_{AB}$ returns an even phase parity result with probability 1

Now, we find the probability of $\pi_A \otimes \pi_B$ returning even phase parity

$$P = \text{Tr}\{(\pi_A \otimes \pi_B) \Pi_{\text{even}}^X\}$$

$$P = \frac{1}{2} \text{Tr}\{(\pi_A \otimes \pi_B)(I_A \otimes I_B + X_A \otimes X_B)\}$$

$$P = \frac{1}{2} \text{Tr}\{(\pi_A \otimes \pi_B)(I_A \otimes I_B)\} + \frac{1}{2} \text{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2} \text{Tr}\{\pi_A \otimes \pi_B\} + \frac{1}{2} \text{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2} \text{Tr}\{\pi_A\} \text{Tr}\{\pi_B\} + \frac{1}{2} \text{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2} + \frac{1}{2} \text{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2}(1 + \text{Tr}\{\pi_A X_A \otimes \pi_B X_B\})$$

$$P = \frac{1}{2}(1 + \text{Tr}\{\pi_A X_A\} \text{Tr}\{\pi_B X_B\})$$

$$P = \frac{1}{2}(1 + \text{Tr}\{X_A \pi_A\} \text{Tr}\{X_B \pi_B\})$$

$$P = \frac{1}{2}(1 + \text{Tr}\{X \pi\}^2)$$

$$P = \frac{1}{2}(1 + \text{Tr}\{X(|0\rangle\langle 0| + |1\rangle\langle 1|)\}^2)$$

$$P = \frac{1}{2}(1 + \text{Tr}\{|1\rangle\langle 0| + |0\rangle\langle 1|\}^2)$$

$$P = \frac{1}{2} \left(1 + \text{Tr} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}^2 \right)$$

$$P = \frac{1}{2}(1 + 0)$$

$$P = \frac{1}{2}$$

The probability of $\pi_A \otimes \pi_B$ returning an odd phase parity is $1 - P = 1 - \frac{1}{2} = \frac{1}{2}$ (As the measurements are orthogonal)

$\therefore \pi_A \otimes \pi_B$ returns even or odd phase parity with equal probability

The same is true for the phase parity measurement. Hence, proven.

Exercise 4.3.18

Given:

$$\rho_A = \sum_{x \in X} p_X(x) \rho_A^x$$

$$\rho_{XA} = \sum_{x \in X} p_X(x) |x\rangle\langle x| \otimes \rho_A^x$$

Measurement operators $\{\Lambda_A^j\}$

To Prove:

$$\text{Tr}\{\rho_A \Lambda_A^j\} = \text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\}$$

Proof:

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \text{Tr}\left\{\left(\sum_{x \in X} p_X(x) |x\rangle\langle x| \otimes \rho_A^x\right) (I_X \otimes \Lambda_A^j)\right\}$$

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \text{Tr}\left\{\sum_{x \in X} (p_X(x) (|x\rangle\langle x| \otimes \rho_A^x) (I_X \otimes \Lambda_A^j))\right\}$$

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \text{Tr}\left\{\sum_{x \in X} (p_X(x) (|x\rangle\langle x| I_X \otimes \rho_A^x \Lambda_A^j))\right\}$$

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \text{Tr}\left\{\sum_{x \in X} (p_X(x) (|x\rangle\langle x|_X \otimes \rho_A^x \Lambda_A^j))\right\}$$

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \sum_{x \in X} p_X(x) \text{Tr}\{|x\rangle\langle x|_X \otimes \rho_A^x \Lambda_A^j\}$$

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \sum_{x \in X} p_X(x) \text{Tr}\{|x\rangle\langle x|_X\} \text{Tr}\{\rho_A^x \Lambda_A^j\}$$

Since trace of a density operator is 1,

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \sum_{x \in X} p_X(x) \text{Tr}\{\rho_A^x \Lambda_A^j\}$$

$$\text{Tr}\{\rho_{XA} (I_X \otimes \Lambda_A^j)\} = \text{Tr}\left\{\sum_{x \in X} p_X(x) \rho_A^x \Lambda_A^j\right\}$$

$$\text{Tr}\{\rho_{XA}(I_X \otimes \Lambda_A^j)\} = \text{Tr}\{\rho_A \Lambda_A^j\}$$

Hence, proven.

Exercise 4.4.1

Given:

- Linear Map \mathcal{N}
- Choi operator $\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}) = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j|_R \otimes \mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A)$ is PSD

To Prove:

- \mathcal{N} is completely positive

Proof:

Consider an arbitrary bipartite vector $|\varphi\rangle_{RB}$

$$\varphi_{RB} = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} \alpha_{i,j} |i\rangle_R \otimes |j\rangle_B$$

and let $V_{A \rightarrow B}$ be a linear operator such that

$$V_{A \rightarrow B} = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} \alpha_{i,j} |j\rangle_B \langle i|_A$$

Note that

$$(I_R \otimes V_{A \rightarrow B})|\Gamma\rangle_{RA} = |\varphi\rangle_{RB}$$

Thus, for every bipartite vector $|\varphi_l\rangle_{RB}$, we can find a corresponding V_l such that

$$|\varphi_l\rangle_{RB} = I_R \otimes (V_l)_{A \rightarrow B} |\Gamma\rangle_{RA}$$

Now, for the choi operator, we have

$$\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}) = \sum_{p,q=0}^{d_A-1} |p\rangle\langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|p\rangle\langle q|_A)$$

$$(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA})) = (\langle i|_R \otimes I_B) \sum_{p,q=0}^{d_A-1} |p\rangle\langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|p\rangle\langle q|_A)$$

$$(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA})) = \sum_{p,q=0}^{d_A-1} (\langle i|_R \otimes I_B)(|p\rangle\langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|p\rangle\langle q|_A))$$

$$(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA})) = \sum_{p,q=0}^{d_A-1} \langle i|_R |p\rangle_R \langle q|_R \otimes I_B \mathcal{N}_{A \rightarrow B}(|p\rangle\langle q|_A)$$

$$(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA})) = \sum_{p,q=0}^{d_A-1} \langle i|p\rangle_R \langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|p\rangle\langle q|_A)$$

$$(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA})) = \sum_{p,q=0}^{d_A-1} \delta_{i,p} \langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|p\rangle\langle q|_A)$$

$$\begin{aligned}
(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA})) &= \sum_{q=0}^{d_A-1} \langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|i\rangle\langle q|_A) \\
(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) &= \sum_{q=0}^{d_A-1} \langle q|_R \otimes \mathcal{N}_{A \rightarrow B}(|i\rangle\langle q|_A)(|j\rangle_R \otimes I_B) \\
(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) &= \sum_{q=0}^{d_A-1} \langle q|_j \otimes I_B \mathcal{N}_{A \rightarrow B}(|i\rangle\langle q|_A) \\
(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) &= \sum_{q=0}^{d_A-1} \delta_{q,j} \otimes \mathcal{N}_{A \rightarrow B}(|i\rangle\langle q|_A) \\
(\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) &= \mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= (\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= (\langle i|_R \otimes I_B)(\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B)
\end{aligned}$$

Since the Choi operator is PSD, we diagonalize it

$$\begin{aligned}
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= (\langle i|_R \otimes I_B) \left(\sum_{l=0}^{d-1} |\varphi_l\rangle\langle\varphi_l|_{RB} \right) (|j\rangle_R \otimes I_B) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} (\langle i|_R \otimes I_B) (|\varphi_l\rangle\langle\varphi_l|_{RB}) (|j\rangle_R \otimes I_B) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} ((\langle i|_R \otimes I_B) |\varphi_l\rangle_{RB}) (\langle\varphi_l|_{RB} (|j\rangle_R \otimes I_B)) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} (\langle i|_R \otimes I_B) (I_R \otimes V_l |\Gamma\rangle_{RA}) (I_R \otimes \langle\Gamma|_{RA} V_l^\dagger) (|j\rangle_R \otimes I_B) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} (\langle i|_R I_R \otimes I_B V_l |\Gamma\rangle_{RA}) (I_R |j\rangle_R \otimes \langle\Gamma|_{RA} V_l^\dagger I_B) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} (\langle i|_R \otimes V_l |\Gamma\rangle_{RA}) (|j\rangle_R \otimes \langle\Gamma|_{RA} V_l^\dagger) \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} \langle i|j\rangle_R \otimes V_l |\Gamma\rangle_{RA} \langle\Gamma|_{RA} V_l^\dagger \\
\mathcal{N}_{A \rightarrow B}(|i\rangle\langle j|_A) &= \sum_{l=0}^{d-1} \langle i|j\rangle_R \otimes V_l |\Gamma\rangle_{RA} \langle\Gamma|_{RA} V_l^\dagger
\end{aligned}$$

To prove that \mathcal{N} is completely positive, we need to show that $\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(X_{RA})$ is PSD for all X_{RA} that are PSD

$$\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(X_{RA}) = \text{id}_R \otimes \mathcal{N}_{A \rightarrow B} \left(\sum |\varphi_i\rangle\langle\varphi_i|_{RA} \right)$$

$$\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(X_{RA}) = \sum_i \text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\varphi_i\rangle\langle\varphi_i|_{RA})$$

Consider $M_i = \text{id}_R \otimes \mathcal{N}_{A \rightarrow B}(|\varphi_i\rangle\langle\varphi_i|_{RA})$

We have $|\varphi_i\rangle = \sum_{j,k=0}^{d_A-1} \alpha_{jk} |j\rangle_R |k\rangle_A$

$$M_i = \text{id}_R \otimes \mathcal{N}_{A \rightarrow B} \left(\sum_{j_1, k_1=0}^{d_A-1} \alpha_{j_1 k_1} |j_1\rangle_R |k_1\rangle_A \sum_{j_2, k_2=0}^{d_A-1} \alpha_{j_2 k_2}^* \langle j_2|_R \langle k_2|_A \right)$$

$$M_i = \text{id}_R \otimes \mathcal{N}_{A \rightarrow B} \left(\sum_{j_1, k_1, j_2, k_2=0}^{d_A-1} \alpha_{j_1 k_1} \alpha_{j_2 k_2}^* |j_1\rangle_R |k_1\rangle_A \langle j_2|_R \langle k_2|_A \right)$$

$$M_i = \text{id}_R \otimes \mathcal{N}_{A \rightarrow B} \left(\sum_{j_1, k_1, j_2, k_2=0}^{d_A-1} \alpha_{j_1 k_1} \alpha_{j_2 k_2}^* |j_1\rangle\langle j_2|_R \otimes |k_1\rangle\langle k_2|_A \right)$$

$$M_i = \sum_{j_1, k_1, j_2, k_2=0}^{d_A-1} \alpha_{j_1 k_1} \alpha_{j_2 k_2}^* (\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}) (|j_1\rangle\langle j_2|_R \otimes |k_1\rangle\langle k_2|_A)$$

$$M_i = \sum_{j_1, k_1, j_2, k_2=0}^{d_A-1} \alpha_{j_1 k_1} \alpha_{j_2 k_2}^* (\text{id}_R |j_1\rangle\langle j_2|_R) \otimes (\mathcal{N}_{A \rightarrow B} |k_1\rangle\langle k_2|_A)$$

$$M_i = \sum_{j_1, k_1, j_2, k_2=0}^{d_A-1} \alpha_{j_1 k_1} \alpha_{j_2 k_2}^* |j_1\rangle\langle j_2|_R \otimes \mathcal{N}_{A \rightarrow B} |k_1\rangle\langle k_2|_A$$