Introduction to Quantum Information and Communication

Theory Assignment-2

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Exercise 4.1.3

Given:

- A is a square operator acting on Hilbert space $\mathcal{H}_S$
- I_R is the identity operator acting on a Hilbert space $\mathcal{H}_R$ isomorphic to $\mathcal{H}_S$
- $|\Gamma\rangle_{RS}$ is the unnormalized maximally entangled vector.

To Prove:

$$\mathrm{Tr}\{A\} = \langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS}$$

Proof:

In the computational basis

$$|\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_S$$

$$\langle\Gamma|_{RS} = \sum_{i=0}^{d-1} \langle i|_R \langle i|_S$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) (I_R \otimes A_S) \left(\sum_{j=0}^{d-1} |j\rangle_R |j\rangle_S\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) \left(\sum_{j=0}^{d-1} (I_R \otimes A_S)\left(|j\rangle_R \otimes |j\rangle_S\right)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) \left(\sum_{j=0}^{d-1} \left(I_R |j\rangle_R\right) \otimes \left(A_S |j\rangle_S\right)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \left(\sum_{i=0}^{d-1} \langle i|_R \langle i|_S\right) \left(\sum_{j=0}^{d-1} |j\rangle_R \otimes \left(A_S |j\rangle_S\right)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i,j=0}^{d-1} \left(\langle i|_R \otimes \langle i|_S\right)\left(|j\rangle_R \otimes \left(A_S |j\rangle_S\right)\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i,j=0}^{d-1} \left(\langle i|j\rangle_R \otimes \langle i|_S A_S |j\rangle_S\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i,j=0}^{d-1} \left(\delta_{i,j} \otimes \langle i|_S A_S |j\rangle_S\right)$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \sum_{i=0}^{d-1} \langle i|_S A_S |i\rangle_S$$

$$\langle\Gamma|_{RS} I_R \otimes A_S |\Gamma\rangle_{RS} = \mathrm{Tr}\{A\}$$

Hence, proven.

Exercise 4.1.16

Given:

- Commutating projectors Π_1 and Π_2
- $0 \leq \Pi_1, \Pi_2 \leq I$

To Prove:

For arbitrary density operator ρ

$$\mathrm{Tr}\{(I - \Pi_1 \Pi_2)\rho\} \leq \mathrm{Tr}\{(I - \Pi_1)\rho\} + \mathrm{Tr}\{(I - \Pi_2)\rho\}$$

Proof:

TO DO

Exercise 4.2.2

Given:

- Ensemble $\{p_X(x), \rho_x\}$ of density operators
- POVM with elements $\{\Lambda_x\}$
- Operator τ such that $\tau \geq p_X(x)\rho_x$

To Prove:

$$\mathrm{Tr}\{\tau\} \geq \sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\}$$

Proof:

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} = \sum_x \mathrm{Tr}\{\Lambda_x p_X(x) \rho_x\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \sum_x \mathrm{Tr}\{\Lambda_x \tau\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\left\{\sum_x \Lambda_x \tau\right\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\left\{\tau \sum_x \Lambda_x\right\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\{\tau I\}$$

$$\sum_x p_X(x)\ \mathrm{Tr}\{\Lambda_x \rho_x\} \leq \mathrm{Tr}\{\tau\}$$

Hence, proven.

Now for the case of encoding n bits into a d-dimensional subspace.

$$\{2^{-n}, \rho_i\}_{i \in \{0,1\}^n}$$

Consider

$$p_X(x)\rho_x = 2^{-n}\rho_i$$

$$p_X(x)\rho_x = 2^{-n}\sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}I - 2^{-n}\sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}\sum_j |j\rangle\langle j| - 2^{-n}\sum_j \lambda_j |j\rangle\langle j|$$

$$2^{-n}I - p_X(x)\rho_x = 2^{-n}\sum_j (1 - \lambda_j)|j\rangle\langle j|$$

Since $0 \le \lambda_j \le 1 \ \forall j, 1 - \lambda_j \ge 0 \ \forall j$. All the eigenvalues of the matrix in LHS are non-negative.

$$2^{-n}I - p_X(x)\rho_x \ge 0$$

$$2^{-n}I \ge p_X(x)\rho_x$$

$\therefore$ We consider $\tau = 2^{-n}I$

Now, we know that the probability of success is upper bounded by $\mathrm{Tr}\{\tau\}$

$$\mathrm{Tr}\{\tau\} = \mathrm{Tr}\{2^{-n}I\}$$

$$\mathrm{Tr}\{\tau\} = 2^{-n}\ \mathrm{Tr}\{I\}$$

Since I is d-dimensional,

$$\mathrm{Tr}\{\tau\} = d2^{-n}$$

Thus, the expected success probability is bounded above by $d2^{-n}$

Exercise 4.3.1

Given:

- A' has a Hilbert space structure isomorphic to that of system A
- $\forall x, y\ F_{AA'}|x\rangle_A|y\rangle_{A'} = |y\rangle_A|x\rangle_{A'}$

To Prove:

$$P(\rho_A) = \mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\}$$

Proof:

$$\rho_A = \sum_i \lambda_i |i\rangle_A \langle i|_A$$

$$\rho_{A'} = \sum_j \lambda_j |j\rangle_{A'} \langle j|_{A'}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\{F_{AA'}(\rho_A \otimes \rho_{A'})\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\left\{F_{AA'}\left(\left(\sum_i \lambda_i |i\rangle_A \langle i|_A\right) \otimes \left(\sum_j \lambda_j |j\rangle_{A'} \langle j|_{A'}\right)\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\left\{F_{AA'}\left(\sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle i|_A \otimes |j\rangle_{A'} \langle j|_{A'}\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\left\{F_{AA'}\left(\sum_{i,j} \lambda_i \lambda_j |i\rangle_A \langle i|_A \otimes |j\rangle_{A'} \langle j|_{A'}\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\left\{F_{AA'}\left(\sum_{i,j} \lambda_i \lambda_j \left(|i\rangle_A |j\rangle_{A'}\right)\left(\langle i|_A \langle j|_{A'}\right)\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\left\{\left(\sum_{i,j} \lambda_i \lambda_j \left(F_{AA'} |i\rangle_A |j\rangle_{A'}\right)\left(\langle i|_A \langle j|_{A'}\right)\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\left\{\sum_{i,j} \lambda_i \lambda_j \left(|j\rangle_A |i\rangle_{A'}\right)\left(\langle i|_A \langle j|_{A'}\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \mathrm{Tr}\left\{\left(|j\rangle_A |i\rangle_{A'}\right)\left(\langle i|_A \langle j|_{A'}\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \mathrm{Tr}\left\{\left(\langle i|_A \langle j|_{A'}\right)\left(|j\rangle_A |i\rangle_{A'}\right)\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \mathrm{Tr}\left\{\langle i|j\rangle_A \otimes \langle j|i\rangle_{A'}\right\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \langle i|j\rangle_A \langle j|i\rangle_{A'}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_{i,j} \lambda_i \lambda_j \delta_{i,j}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \sum_i \lambda_i^2$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = \mathrm{Tr}\{\rho_A^2\}$$

$$\mathrm{Tr}\{(\rho_A \otimes \rho_{A'})F_{AA'}\} = P(\rho_A)$$

Hence, proven.

Exercise 4.3.6

Given:

$$\Pi_{\text{even}} = \frac{1}{2}(I_A \otimes I_B + Z_A \otimes Z_B) = |00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}$$

$$\Pi_{\text{odd}} = \frac{1}{2}(I_A \otimes I_B - Z_A \otimes Z_B) = |01\rangle\langle 01|_{AB} + |10\rangle\langle 10|_{AB}$$

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

$$\pi_A = \frac{1}{2}\left(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\right)$$

$$\pi_B = \frac{1}{2}\left(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\right)$$

To Prove:

- $|\Phi^+\rangle_{AB}$ returns an even parity result with probabilty 1
- $\pi_A \otimes \pi_B$ returns even or odd parity with equal probability

Proof:

First we find the density matrix of the bell state

$$\rho_{AB} = |\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}$$

Now, the probability of the bell state collapsing to Π_{even} is

$$P = \operatorname{Tr}\{\rho_{AB}\Pi_{\text{even}}\}$$

$$P = \operatorname{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}\left(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}\right)\right\}$$

$$P = \operatorname{Tr}\{|\Phi^+\rangle\langle\Phi^+||00\rangle\langle 00| + |\Phi^+\rangle\langle\Phi^+||11\rangle\langle 11|\}$$

$$P = \operatorname{Tr}\{|\Phi^+\rangle\langle\Phi^+||00\rangle\langle 00|\} + \operatorname{Tr}\{|\Phi^+\rangle\langle\Phi^+||11\rangle\langle 11|\}$$

$$P = \operatorname{Tr}\{\langle 00|\Phi^+\rangle\langle\Phi^+|00\rangle\} + \operatorname{Tr}\{\langle 11|\Phi^+\rangle\langle\Phi^+|11\rangle\}$$

$$P = \frac{1}{2} + \frac{1}{2}$$

$$P = 1$$

$\therefore$ $|\Phi^+\rangle_{AB}$ returns an even parity result with probabilty 1

Now, we find the probability of $\pi_A \otimes \pi_B$ returning even parity

$$P = \operatorname{Tr}\{(\pi_A \otimes \pi_B)\Pi_{\text{even}}\}$$

$$P = \frac{1}{4}\operatorname{Tr}\left\{\left(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\right) \otimes \left(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\right)\left(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}\right)\right\}$$

$$P = \frac{1}{4}\operatorname{Tr}\left\{\left(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\right)\left(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}\right)\right\}$$

$$P = \frac{1}{4}\operatorname{Tr}\{(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|)(|00\rangle\langle 00| + |11\rangle\langle 11|)\}$$

$$\begin{gathered}P = \frac{1}{4}\operatorname{Tr}\{|00\rangle\langle 00||00\rangle\langle 00| + |00\rangle\langle 00||11\rangle\langle 11| + \\ |01\rangle\langle 01||00\rangle\langle 00| + |01\rangle\langle 01||11\rangle\langle 11| + \\ |10\rangle\langle 10||00\rangle\langle 00| + |10\rangle\langle 10||11\rangle\langle 11| + \\ |11\rangle\langle 11||00\rangle\langle 00| + |11\rangle\langle 11||11\rangle\langle 11|\}\end{gathered}$$

$$P = \frac{1}{4}(\text{Tr}\{|00\rangle\langle 00|\} + \text{Tr}\{|11\rangle\langle 11|\})$$

$$P = \frac{1}{4}(1 + 1)$$

$$P = \frac{1}{2}$$

The probability of $\pi_A \otimes \pi_B$ returning an odd parity is $1 - P = 1 - \frac{1}{2} = \frac{1}{2}$ (As the measurements are orthogonal)

$\therefore \pi_A \otimes \pi_B$ returns even or odd parity with equal probability

Now, we perform the same calculations for the phase parity measurement

$$\Pi^X_{\text{even}} = \frac{1}{2}(I_A \otimes I_B + X_A \otimes X_B)$$

$$\Pi^X_{\text{odd}} = \frac{1}{2}(I_A \otimes I_B - X_A \otimes X_B)$$

The probability of the bell state collapsing to Π^X_{even} is

$$P = \text{Tr}\{\rho_{AB}\Pi^X_{\text{even}}\}$$

$$P = \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}(I_A \otimes I_B + X_A \otimes X_B)\right\}$$

$$P = \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} I_A \otimes I_B\right\} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} X_A \otimes X_B\right\}$$

$$P = \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}\right\} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} X_A \otimes X_B\right\}$$

$$P = \frac{1}{2}\ \text{Tr}\left\{\langle\Phi^+|_{AB}|\Phi^+\rangle_{AB}\right\} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} X_A \otimes X_B\right\}$$

$$P = \frac{1}{2}\ \text{Tr}\left\{\langle\Phi^+|\Phi^+\rangle_{AB}\right\} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} X_A \otimes X_B\right\}$$

$$P = \frac{1}{2} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB} X_A \otimes X_B\right\}$$

$$P = \frac{1}{2} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}\left(|0\rangle\langle 1|_A + |1\rangle\langle 0|_A\right) \otimes \left(|0\rangle\langle 1|_B + |1\rangle\langle 0|_B\right)\right\}$$

$$P = \frac{1}{2} + \frac{1}{2}\ \text{Tr}\left\{|\Phi^+\rangle_{AB}\langle\Phi^+|_{AB}\left(|0\rangle\langle 1|_A \otimes |0\rangle\langle 1|_B + |0\rangle\langle 1|_A \otimes |1\rangle\langle 0|_B + |1\rangle\langle 0|_A \otimes |0\rangle\langle 1|_B + |1\rangle\langle 0|_A \otimes |1\rangle\langle 0|_B\right)\right\}$$

$$P = \frac{1}{2} + \frac{1}{2}\ \text{Tr}\{|\Phi^+\rangle\langle\Phi^+|(|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|\}$$

$$P = \frac{1}{2}(1 + \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||00\rangle\langle 11|\} + \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||01\rangle\langle 10|\} + \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||10\rangle\langle 01|\} + \text{Tr}\{|\Phi^+\rangle\langle\Phi^+||11\rangle\langle 00|\})$$

$$P = \frac{1}{2}(1 + \text{Tr}\{\langle 11|\Phi^+\rangle\langle\Phi^+|00\rangle\} + \text{Tr}\{\langle 10|\Phi^+\rangle\langle\Phi^+|01\rangle\} + \text{Tr}\{\langle 01|\Phi^+\rangle\langle\Phi^+|10\rangle\} + \text{Tr}\{\langle 00|\Phi^+\rangle\langle\Phi^+|11\rangle\})$$

$$P = \frac{1}{2}\left(1 + \mathrm{Tr}\left\{\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}\right\} + \mathrm{Tr}\{0 * 0\} + \mathrm{Tr}\{0 * 0\} + \mathrm{Tr}\left\{\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}\right\}\right)$$

$$P = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2}\right)$$

$$P = \frac{1}{2}(2)$$

$$P = 1$$

$\therefore |\Phi^+\rangle_{AB}$ returns an even phase parity result with probabilty 1

Now, we find the probability of $\pi_A \otimes \pi_B$ returning even phase parity

$$P = \mathrm{Tr}\{(\pi_A \otimes \pi_B)\Pi^X_{\text{even}}\}$$

$$P = \frac{1}{2}\ \mathrm{Tr}\{(\pi_A \otimes \pi_B)(I_A \otimes I_B + X_A \otimes X_B)\}$$

$$P = \frac{1}{2}\ \mathrm{Tr}\{(\pi_A \otimes \pi_B)(I_A \otimes I_B)\} + \frac{1}{2}\ \mathrm{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2}\ \mathrm{Tr}\{\pi_A \otimes \pi_B\} + \frac{1}{2}\ \mathrm{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2}\ \mathrm{Tr}\{\pi_A\}\mathrm{Tr}\{\pi_B\} + \frac{1}{2}\ \mathrm{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2} + \frac{1}{2}\ \mathrm{Tr}\{(\pi_A \otimes \pi_B)(X_A \otimes X_B)\}$$

$$P = \frac{1}{2}(1 + \mathrm{Tr}\{\pi_A X_A \otimes \pi_B X_B\})$$

$$P = \frac{1}{2}(1 + \mathrm{Tr}\{\pi_A X_A\}\ \mathrm{Tr}\{\pi_B X_B\})$$

$$P = \frac{1}{2}(1 + \mathrm{Tr}\{X_A \pi_A\}\ \mathrm{Tr}\{X_B \pi_B\})$$

$$P = \frac{1}{2}\left(1 + \mathrm{Tr}\{X\pi\}^2\right)$$

$$P = \frac{1}{2}\left(1 + \mathrm{Tr}\{X(|0\rangle\langle 0| + |1\rangle\langle 1|)\}^2\right)$$

$$P = \frac{1}{2}\left(1 + \mathrm{Tr}\{|1\rangle\langle 0| + |0\rangle\langle 1|\}^2\right)$$

$$P = \frac{1}{2}\left(1 + \mathrm{Tr}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}^2\right)$$

$$P = \frac{1}{2}(1 + 0)$$

$$P = \frac{1}{2}$$

The probability of $\pi_A \otimes \pi_B$ returning an odd phase parity is $1 - P = 1 - \frac{1}{2} = \frac{1}{2}$ (As the measurements are orthogonal)

$\therefore \pi_A \otimes \pi_B$ returns even or odd phase parity with equal probability

The same is true for the phase parity measurement. Hence, proven.

Exercise 4.3.18

Given:

$$\rho_A = \sum_{x \in X} p_X(x) \rho_A^x$$

$$\rho_{XA} = \sum_{x \in X} p_X(x) |x\rangle\langle x| \otimes \rho_A^x$$

$$\text{Measurement operators } \left\{\Lambda_A^j\right\}$$

To Prove:

$$\mathrm{Tr}\left\{\rho_A \Lambda_A^j\right\} = \mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\}$$

Proof:

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \mathrm{Tr}\left\{\left(\sum_{x \in X} p_X(x) |x\rangle\langle x| \otimes \rho_A^x\right)\left(I_X \otimes \Lambda_A^j\right)\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \mathrm{Tr}\left\{\sum_{x \in X}\left(p_X(x)(|x\rangle\langle x| \otimes \rho_A^x)\left(I_X \otimes \Lambda_A^j\right)\right)\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \mathrm{Tr}\left\{\sum_{x \in X}\left(p_X(x)\left(|x\rangle\langle x| I_X \otimes \rho_A^x \Lambda_A^j\right)\right)\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \mathrm{Tr}\left\{\sum_{x \in X}\left(p_X(x)\left(|x\rangle\langle x|_X \otimes \rho_A^x \Lambda_A^j\right)\right)\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \sum_{x \in X} p_X(x) \mathrm{Tr}\left\{|x\rangle\langle x|_X \otimes \rho_A^x \Lambda_A^j\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \sum_{x \in X} p_X(x) \mathrm{Tr}\left\{|x\rangle\langle x|_X\right\} \mathrm{Tr}\left\{\rho_A^x \Lambda_A^j\right\}$$

Since trace of a density operator is 1,

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \sum_{x \in X} p_X(x) \mathrm{Tr}\left\{\rho_A^x \Lambda_A^j\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \mathrm{Tr}\left\{\sum_{x \in X} p_X(x) \rho_A^x \Lambda_A^j\right\}$$

$$\mathrm{Tr}\left\{\rho_{XA}\left(I_X \otimes \Lambda_A^j\right)\right\} = \mathrm{Tr}\left\{\rho_A \Lambda_A^j\right\}$$

Hence, proven.

Exercise 4.4.1

Given:

- Linear Map $\mathcal{N}$
- Choi operator $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right) = \sum_{i,j=0}^{d-1} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}\left(|i\rangle\langle j|_A\right)$ is PSD

To Prove:

- $\mathcal{N}$ is completely positive

Proof:

To prove that $\mathcal{N}$ is completely positive, we need to show that $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(X_{RA})$ is PSD for all X_{RA} that are PSD

$$\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(X_{RA}) = \mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(\sum |\varphi_i\rangle\langle\varphi_i|_{RA}\right)$$

$$\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(X_{RA}) = \sum_i \mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(|\varphi_i\rangle\langle\varphi_i|_{RA}\right)$$

Consider $M_i = \mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(|\varphi_i\rangle\langle\varphi_i|_{RA}\right)$

We have $|\varphi_i\rangle = \sum_{j,k=0}^{d-1} \alpha_{jk} |j\rangle_R |k\rangle_A$

$$M_i = \mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(\sum_{j_1,k_1=0}^{d-1} \alpha_{j_1 k_1} |j_1\rangle_R |k_1\rangle_A \sum_{j_2,k_2=0}^{d-1} \alpha^*_{j_2 k_2} \langle j_2|_R \langle k_2|_A\right)$$

$$M_i = \mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(\sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} |j_1\rangle_R |k_1\rangle_A \langle j_2|_R \langle k_2|_A\right)$$

$$M_i = \mathrm{id}_R \otimes \mathcal{N}_{A\to B}\left(\sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} |j_1\rangle\langle j_2|_R \otimes |k_1\rangle\langle k_2|_A\right)$$

$$M_i = \sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})\left(|j_1\rangle\langle j_2|_R \otimes |k_1\rangle\langle k_2|_A\right)$$

$$M_i = \sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} \left(\mathrm{id}_R |j_1\rangle\langle j_2|_R\right) \otimes \left(\mathcal{N}_{A\to B} |k_1\rangle\langle k_2|_A\right)$$

$$M_i = \sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} |j_1\rangle\langle j_2|_R \otimes \mathcal{N}_{A\to B} |k_1\rangle\langle k_2|_A$$

Now, using 4.198-4.212 of the book, we have

$$M_i = \sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} |j_1\rangle\langle j_2|_R \otimes \sum_{l=0}^{d-1} V_l |k_1\rangle\langle k_2| V_l^\dagger$$

$$M_i = \sum_{l=0}^{d-1} \sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1}\alpha^*_{j_2 k_2} |j_1\rangle\langle j_2|_R \otimes V_l |k_1\rangle\langle k_2| V_l^\dagger$$

$$M_i = \sum_{l=0}^{d-1} \sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1} \alpha^*_{j_2 k_2} I_R |j_1\rangle\langle j_2|_R I_R \otimes V_l |k_1\rangle\langle k_2| V_l^\dagger$$

$$M_i = \sum_{l=0}^{d-1} I_R \otimes V_l \left(\sum_{j_1,k_1,j_2,k_2=0}^{d-1} \alpha_{j_1 k_1} \alpha^*_{j_2 k_2} |j_1\rangle\langle j_2|_R \otimes |k_1\rangle\langle k_2| \right) I_R \otimes V_l^\dagger$$

$$M_i = \sum_{l=0}^{d-1} I_R \otimes V_l \left(|\varphi_l\rangle\langle\varphi_l|_{RA} \right) I_R \otimes V_l^\dagger$$

Now, from the Choi-Kraus theorem, M_i is a completely positive.

$$\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(X_{RA}) = \sum_i M_i$$

Since sum of completely positive maps is also completely positive, $\mathcal{N}$ is completely positive.

Hence, proven