Introduction to Quantum Information and Communication

Theory Assignment-1

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Question 1

To Prove: Any $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent

Proof:

Let V be an n dimensional vector space

Assume $A = \{v_1, v_2, v_3, ..., v_{n+1}\}$ is a set of linearly independent vectors where $v_i \in V$

Let $B = A \setminus \{v_{n+1}\} = \{v_1, v_2, v_3, ..., v_n\}$. Since $B \subset A$, B is also a set of linearly independent vectors.

Now, since V is n dimensional and $|B| = n$, $\text{span}(B) = V$ by the definition of n dimensional vector space.

Therefore, every vector $v \in V$ can be expressed as a linear combination of vectors in B

$\therefore v_{n+1} = a_1 v_1 + a_2 v_2 + a_3 v_3 + ... + a_n v_n$, where $a_i \in \mathbb{F}(\text{field over which } V \text{ is defined})$

$\therefore V$ is not linearly dependent. This is a contradiction

Any set A of $n + 1$ vectors belonging to an n dimensional vector space must be linearly dependent.

Question 2

Given: $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

To Find: square root of matrix A

Solution:

Note that $A^\dagger = A$. Thus, by the spectral theorem, A can be decomposed into an orthonormal eigenbasis. Now, we find this eigenbasis.

$$|A - \lambda I| = 0$$

$$|\begin{pmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{pmatrix}| = 0$$

$$\lambda_1 = 2, \lambda_2 = -3$$

Let their corresponding normalized eigenvectors be $|2\rangle$ and $|-3\rangle$

$$A|2\rangle = 2|2\rangle \text{ and } A|-3\rangle = 2|-3\rangle$$

On solving, we get

$$|2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } |-3\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now, by the spectral theorem, we have

$$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$A = 2|2\rangle\langle 2| - 3|{-3}\rangle\langle{-3}|$$

We know that

$$f(A) = \sum_i f(\lambda_i) |\lambda_i\rangle\langle\lambda_i|$$

So

$$\sqrt{A} = \sqrt{2}|2\rangle\langle 2| + \sqrt{-3}|{-3}\rangle\langle{-3}|$$

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$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix} + \sqrt{-3}\begin{pmatrix} 1 \\ -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \end{pmatrix}\right)$$

$$\sqrt{A} = \frac{1}{5}\left(\sqrt{2}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} + \sqrt{-3}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\right)$$

$$\sqrt{A} = \frac{1}{5}\begin{pmatrix} 4\sqrt{2} + i\sqrt{3} & 2\sqrt{2} - 2i\sqrt{3} \\ 2\sqrt{2} - 2i\sqrt{3} & \sqrt{2} + 4i\sqrt{3} \end{pmatrix}$$

Question 3

Given: A is an $n \times n$ matrix and B is an $m \times m$ matrix

To Prove: $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \times \operatorname{tr}(B)$

Proof:

$$A \otimes B = \begin{pmatrix} A_{1,1}B & A_{1,2}B & \dots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \dots & A_{2,n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1}B & A_{n,2}B & \dots & A_{n,n}B \end{pmatrix}$$

where each $A_{i,j}B$ is an $m \times m$ matrix expanded.

$$\operatorname{tr}(A \otimes B) = \sum_{i=1}^{n} \operatorname{tr}\left(A_{i,i}B\right)$$

$$\operatorname{tr}(A \otimes B) = \sum_{i=1}^{n} A_{i,i} \operatorname{tr}(B)$$

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(B) \times \sum_{i=1}^{n} A_{i,i}$$

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \times \operatorname{tr}(B)$$