Problem 1

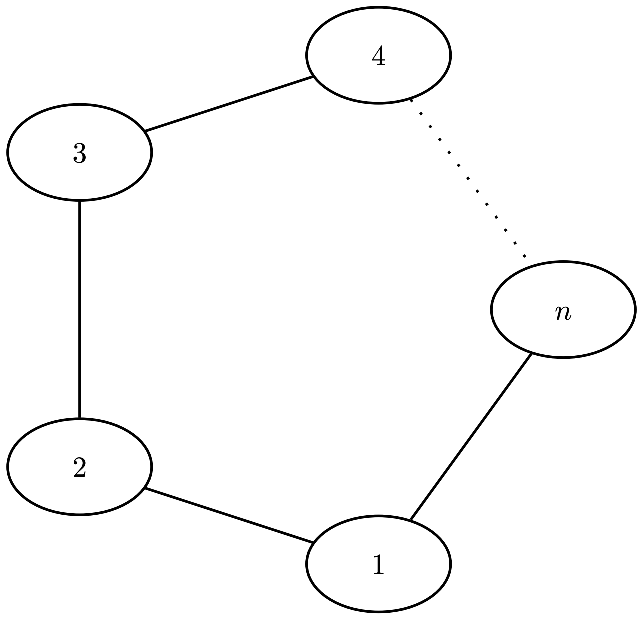
Assuming the graph is connected

(i)

Let the graph be $G = (V, E)$ such that $|V| = n$ and $|E| = m$

Lower Bound

The lower bound for the number of articulation points is trivially 0, by considering the graph where every vertex is part of the same cycle and has degree 2.



In the above graph, there are no articulation points, as every vertex has two vertex disjoint paths to every other vertex, thus removing any vertex would not cause the graph to become disconnected.

Thus, the lower bound is 0.

Upper Bound

To find an upper bound on the number of articulation points, we first prove the following claims

Claim 1: Adding edges to a graph can never increase the number of articulation points

Proof: The number of articulation points can either stay the same if the edge was added between nodes in the same biconnected component, or it can decrease by one if the edge was added between two different biconnected components

Claim 2: In a connected graph with every node being part of a cycle, all articulation points must have degree greater than or equal to 3.

Proof: Since every graph is part of a cycle, every node has degree greater than or equal to 2. A node of degree 2 can never be an articulation point, as both the edges of that node are part of the cycle it belongs to. Due to this node, the cycle ensures connectivity does not change.

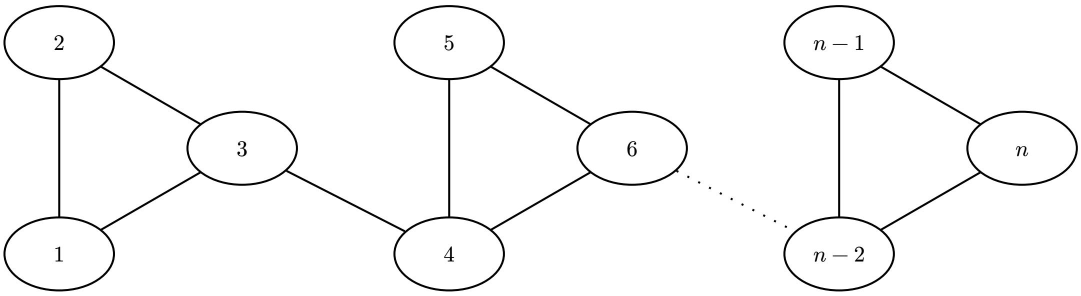
Claim 3: A graph containing cycles of exactly size three will have the maximum number of articulation points.

Proof: Since articulation points result in forming biconnected components, and biconnected components are cyclic, the larger the number of biconnected components we have, the larger the number of articulation points. In order to increase the number of biconnected components for a fixed number of nodes, we increase the number of cycles, thus minimizing the size of each cycle.

Now, we will be constructing a graph using these cycles of size three. The construction will result in a graph with all nodes of degree 3 being articulation points, and thus no node would be of degree greater than or equal to 4.

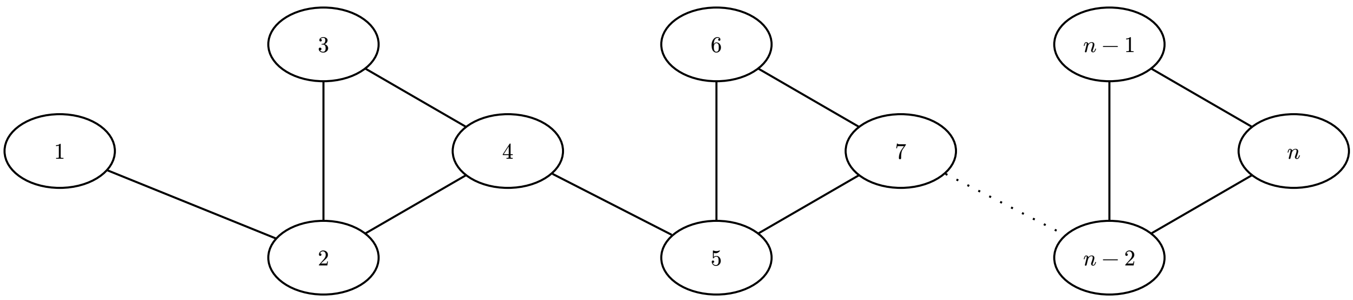
Note that we will be joining the cycles with bridges wherever possible because each bridge produces two articulation points.

Case 1: $n\%3 == 0$



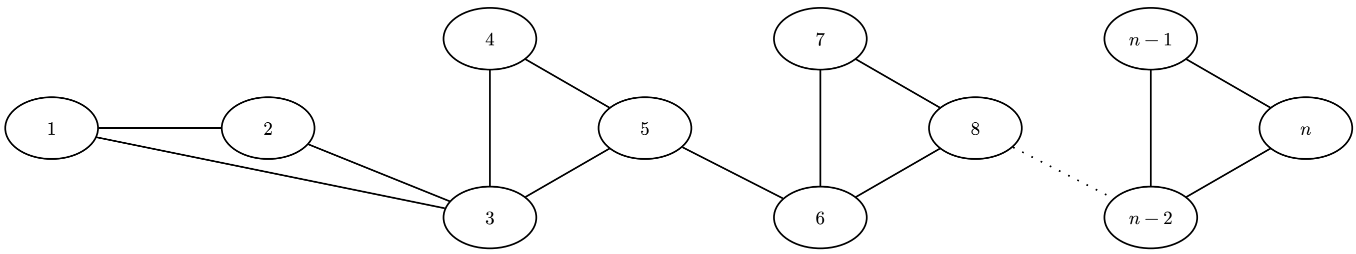
The articulation points are the nodes numbered $3i$ and $3i+1$, with a total of $\frac{2n}{3}-2$ articulation points

Case 2: $n\%3 == 1$



$\frac{2(n+2)}{3}-3$ articulation points

Case 3: $n\%3 == 2$



$\frac{2(n+1)}{3}-3$ articulation points, assuming $n>2$

Thus, the maximum number of articulation points are of the order of $\frac{2n}{3}$, with exact values depicted above.

(ii)

The APSP algorithm requires a matrix of size $a \times a$, where a is the number of articulation points. From the above question, the required memory is $\frac{2n}{3} * \frac{2n}{3}$ integers, i.e, $O\left(\frac{4n^2}{9}\right)$.

Problem 2

To Prove: All $(u,v) \notin E(H)$ with u and v in different clusters must have been removed from E' in Phase 2 of the spanner algorithm

Proof:

Let $(u,v) \notin E(H)$ be an arbitrary edge between different clusters. Since the edge does not belong to the t-spanner, it was discarded. Since the edge was discarded, it was either discarded in phase 1 or phase 2.

Assume the edge was discarded in phase 1. The only edges discarded in phase 1 are those belonding to the same cluster. This means the edge (u,v) is in the same cluster. This is a contradiction. Thus, the edge must have been discared in phase 2.

Hence, proven.

Problem 3

To Prove: If an edge $(u,v) \in E$ is not present in $E(H)$ at the end of the first phase, then the weight of edge (u,v) is greater than or equal to the weight of the edge (v,w), where w is the center of the cluster to which v belongs.

Proof:

Let $(u,v) \in E$ and $(u,v) \notin E(H)$ (after phase 1) be an arbitrary edge

Case 1: v is sampled

v is the center of the cluster, thus w = v, and weight of $(v,w) = 0$.

Thus, weight of (u,v) is greater than or equal to 0.

Case 2: v is not sampled

Assume the weight of edge (u,v) is less than weight of edge (v,w). Then, the edge (u,v) would be moved to $E(H)$. This is a contradiction, since $(u,v) \notin E(H)$.

Thus, for all edges $(u,v) \in E$ such that $(u,v) \notin E(H)$ (after phase 1), the weight of edge (u,v) is greater than or equal to the weight of the edge (v,w)

Hence, proven.