Analysis of Adjacency Matrices by Spectral Graph Theory

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1 Introduction

1.1 Eigenvalue and Eigenvector

For a matrix A, a vector which when multiplied by A gives a scalar multiple of itself is called an eigenvector of A. This scalar multiple is called the corresponding eigenvalue.

$$Ax = \lambda x$$

x is an eigenvector and λ is the corresponding eigenvalue

1.2 Graph

A graph is a set of vertices V connected by a set of edges E.

$$G = (V, E)$$

1.2.1 Directed Graph

A directed graph is a graph where each edge represents an ordered pair of vertices

1.2.2 Undirected Graph

An undirected graph is a graph where each edge represents an unordered pair of vertices

1.2.3 Path

A sequence of vertices

$$V_1 V_2 V_3 ... V_n$$

is a path if

$$\forall i \in \{1, 2, 3, .., n-1\} \qquad (V_i, V_{i+1}) \in E$$

The length of such a path is $n-1$

1.3 Adjacency Matrix of a graph

For a graph G with n vertices, its adjacency matrix $A(G)$ is an n by n square matrix with $a_{i,j}$ equal to the number of edges connecting V_i to V_j

1.3.1 Properties

For an undirected graph G

- $A(G)$ is symmetric
- $A(G)$ has real eigenvalues
- (i, j) entry of $A(G)^k$ represents the number of paths of length k from V_i to $V_j \quad \forall k \geq 1$

2 Connectivity

We will analyse the connectivity of a graph using its adjacency matrix and its eigenvalues.

2.1 Connected Components

Consider a regular undirected graph G (each vertex having degree d), and some eigenvector x of $A(G)$ with its corresponding eigenvalue λ

$$A(G) * x = \lambda * x$$
$$\lambda x_i = \sum_j A_{ij} x_j$$
$$|\lambda||x_i| \leq \sum_j |A_{ij}||x_i|$$
$$|\lambda||x_i| \leq d|x_i| \qquad (Since \quad |x_j| \leq |x_i| \forall j)$$
$$|\lambda| \leq d$$

All the eigenvalues of this graph are smaller than or equal to d.
Upon setting all x_i to 1, we get $Ax = dx$, which gives

$$\lambda_{max} = d$$

The multiplicity of the maximum eigenvalue (equal to d) gives the number of connected components of G. The number of connected components of a graph have various applications in different fields

- Social Network Analysis: Connected components are used to identify a set of individuals as a community. By finding these components, insights on information flow and social interactions are studied by researchers.

- Image Processing: Segmentation of objects and finding regions of interest is done by finding the connected components of an image. The image is converted into a graph, and then pixels sharing similar characteristics are grouped together. This is useful for object recognition and feature extraction.

- Network Routing: In large-scale computer networks, connected components are used by routing algorithms to determine the best path that the data packets should take in order to reach their destination.

2.2 Number of paths in a graph

Consider an undirected graph G with adjacency matrix $A(G)$. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the eigenvalues of $A(G)$

$$\forall i, j \quad \exists c_1, c_2, c_3, ..., c_n \quad (A(G)^k)_{ij} = c_1\lambda_1^k + c_2\lambda_2^k + c_3\lambda_3^k + ... + c_n\lambda_n^k \quad \forall k \geq 1$$

For each index (i, j), upon identifying its corresponding c_i values, the number of paths from V_i to V_j of any length can be calculated in constant time using this equation.
Similarly, the total number of self loops of length k is given by

$$\lambda_1^k + \lambda_2^k + \lambda_3^k + ... + \lambda_n^k$$

The number of paths of given length between any two vertices is a very useful metric to have

- Network Analysis: The number of paths between any two access points in a network informs us of its reliability. We can figure out the number of alternate paths the network can take, and how to distribute the packets between all the paths in order to optimally transmit data.

- Reachability Analysis: The presence of a direct or indirect path between any two vertices gives insights on information flow and communication possibilities. The number of paths tells us how easily accessible any vertex is from any other vertex.

- Disease Spread: While modelling the spread of a disease, individuals and locations act as vertices in the graph, and the number of paths quantify how likely it is that the disease spread to the other end of that path. This informs us regarding control strategies and which locations require quarantine measures the most.