Question 2

Given:

- Heads probability p
- Tails probability $1-p$
- Even number of heads after n tosses probability q_n

To Find: q_n

Solution

If after $n-1$ tosses we get even number of heads, we require a tail for even number of heads after n tosses. If we get an odd number of heads, we require a head for even number of heads. Thus, we get the recurrence relation

$$q_n = (1-p)q_{n-1} + p(1-q_{n-1}), \text{with } q_1 = 1-p$$

Now, we solve this recurrence relation

$$q_n = (1-2p)q_{n-1} + p$$

$$(1-2p)q_{n-1} = (1-2p)^2 q_{n-2} + (1-2p)p$$

$$(1-2p)^2 q_{n-2} = (1-2p)^3 q_{n-3} + (1-2p)^2 p$$

$$\vdots$$

$$(1-2p)^{n-2} q_2 = (1-2p)^{n-1} q_1 + (1-2p)^{n-2} p$$

On adding the above equations, we get

$$q_n = (1-2p)^{n-1} q_1 + p\left(1 + (1-2p) + (1-2p)^2 + \ldots + (1-2p)^{n-2}\right)$$

$$q_n = (1-2p)^{n-1}(1-p) + \frac{p\left((1-2p)^{n-1} - 1\right)}{1-2p-1}$$

$$q_n = (1-2p)^{n-1} - (1-2p)^{n-1}p + \frac{1}{2} - \frac{(1-2p)^{n-1}}{2}$$

$$q_n = \frac{(1-2p)^{n-1}}{2} - (1-2p)^{n-1}p + \frac{1}{2}$$

$$q_n = (1-2p)^{n-1}\frac{1-2p}{2} + \frac{1}{2}$$

$$q_n = \frac{1 + (1-2p)^n}{2}$$

This is the required formula.