

# Open Quantum Systems and Quantum Thermodynamics Assignment

## Question 1

### Part (a)

The quantum depolarising channel is defined as

$$\Delta_\lambda(\rho) = (1 - \lambda)\rho + \frac{\lambda}{d} \text{tr}(\rho)I \quad \text{with the constraint } 0 \leq \lambda \leq 1 + \frac{1}{d^2 - 1}$$

To find its fixed point, we have

$$\Delta_\lambda(\rho) = \rho$$

$$(1 - \lambda)\rho + \frac{\lambda}{d} \text{tr}(\rho)I = \rho$$

$$\rho - \rho\lambda + \frac{\lambda}{d}I = \rho$$

$$\rho\lambda = \frac{\lambda}{d}I$$

$$\rho = \frac{I}{d}$$

$\therefore \frac{I}{d}$  is the fixed point for the quantum depolarising channel.

### Part (b)

For the case of qubits, we have  $d = 2$ , and thus

$$\Delta_\lambda(\rho) = (1 - \lambda)\rho + \frac{\lambda}{2} \text{tr}(\rho)I \quad \text{with the constraint } 0 \leq \lambda \leq \frac{4}{3}$$

First, we find the Choi state for this map

$$I \otimes \Delta_\lambda(|\psi\rangle\langle\psi|)$$

where  $|\psi\rangle$  is the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

We have

$$\begin{aligned} & I \otimes \Delta_\lambda\left(\frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)\right) \\ &= \frac{1}{2}I \otimes \Delta_\lambda(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\ &= \frac{1}{2}I \otimes \Delta_\lambda(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) \end{aligned}$$

Since  $(a \otimes b)(c \otimes d) = ac \otimes bd$ , we get

$$= \frac{1}{2}(|0\rangle\langle 0| \otimes \Delta_\lambda(|0\rangle\langle 0|) + |0\rangle\langle 1| \otimes \Delta_\lambda(|0\rangle\langle 1|) + |1\rangle\langle 0| \otimes \Delta_\lambda(|1\rangle\langle 0|) + |1\rangle\langle 1| \otimes \Delta_\lambda(|1\rangle\langle 1|))$$

$$\begin{aligned}
&= \frac{1}{2} \left( |0\rangle\langle 0| \otimes \left( (1-\lambda)|0\rangle\langle 0| + \frac{\lambda}{2} * 1 * I \right) + |0\rangle\langle 1| \otimes \left( (1-\lambda)|0\rangle\langle 1| + \frac{\lambda}{2} * 0 * I \right) \right. \\
&\quad \left. + |1\rangle\langle 0| \otimes \left( (1-\lambda)|1\rangle\langle 0| + \frac{\lambda}{2} * 0 * I \right) + |1\rangle\langle 1| \otimes \left( (1-\lambda)|1\rangle\langle 1| + \frac{\lambda}{2} * 1 * I \right) \right) \\
&= \frac{1}{2} \left( |0\rangle\langle 0| \otimes \left( (1-\lambda)|0\rangle\langle 0| + \frac{\lambda}{2} I \right) + (1-\lambda)|0\rangle\langle 1| \otimes |0\rangle\langle 1| \right. \\
&\quad \left. + (1-\lambda)|1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes \left( (1-\lambda)|1\rangle\langle 1| + \frac{\lambda}{2} I \right) \right) \\
&= \frac{1}{2} \left( (1-\lambda)|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{\lambda}{2}|0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + (1-\lambda)|0\rangle\langle 1| \otimes |0\rangle\langle 1| \right. \\
&\quad \left. + (1-\lambda)|1\rangle\langle 0| \otimes |1\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{\lambda}{2}|1\rangle\langle 1| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \right) \\
&= \frac{1}{2} \left( (1-\lambda)|00\rangle\langle 00| + \frac{\lambda}{2}(|00\rangle\langle 00| + |01\rangle\langle 01|) + (1-\lambda)|00\rangle\langle 11| \right. \\
&\quad \left. + (1-\lambda)|11\rangle\langle 00| + (1-\lambda)|11\rangle\langle 11| + \frac{\lambda}{2}(|10\rangle\langle 10| + |11\rangle\langle 11|) \right) \\
&= \left( \frac{1}{2} - \frac{\lambda}{4} \right) (|00\rangle\langle 00| + |11\rangle\langle 11|) + \left( \frac{1}{2} - \frac{\lambda}{2} \right) (|00\rangle\langle 11| + |11\rangle\langle 00|) + \frac{\lambda}{4} (|01\rangle\langle 01| + |10\rangle\langle 10|)
\end{aligned}$$

Thus, we get the Choi state

$$C_d = \begin{pmatrix} \frac{1}{2} - \frac{\lambda}{4} & 0 & 0 & \frac{1}{2} - \frac{\lambda}{2} \\ 0 & \frac{\lambda}{4} & 0 & 0 \\ 0 & 0 & \frac{\lambda}{4} & 0 \\ \frac{1}{2} - \frac{\lambda}{2} & 0 & 0 & \frac{1}{2} - \frac{\lambda}{4} \end{pmatrix}$$

On diagonalizing  $C_d$ , we get

$$C_d = \sum_{i=0}^3 \lambda_i |\alpha_i\rangle\langle \alpha_i|$$

where

$$\lambda_0 = \frac{1}{4}(4 - 3\lambda), \lambda_1 = \lambda_2 = \lambda_3 = \frac{\lambda}{4}$$

and

$$\alpha_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \alpha_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, we construct  $A_i$  from the  $\alpha_i$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, we get a set of Kraus operators  $K_i = \sqrt{\lambda_i} A_i$

$$K_0 = \left( \sqrt{1 - \frac{3}{4}\lambda} \right) I, K_1 = \sqrt{\frac{\lambda}{4}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, K_2 = \sqrt{\frac{\lambda}{4}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, K_3 = \sqrt{\frac{\lambda}{4}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that the Kraus representation for a map is not unique. We can construct the canonical Kraus operators, which are

$$K_0 = \left( \sqrt{1 - \frac{3}{4}\lambda} \right) I, K_1 = \sqrt{\frac{\lambda}{4}} \sigma_x, K_2 = \sqrt{\frac{\lambda}{4}} \sigma_y, K_3 = \sqrt{\frac{\lambda}{4}} \sigma_z$$

### Part (c)

We have

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_k \gamma_k \left( L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right)$$

Since the depolarising channel is purely dissipative, we have  $H = 0$

The parameter  $\lambda$  is interpreted as  $\lambda = \gamma \Delta t$

Thus, we get the Lindblad operators

$$L_1 = \sqrt{\gamma} \sigma_x, L_2 = \sqrt{\gamma} \sigma_y, L_3 = \sqrt{\gamma} \sigma_z$$

and

$$\frac{d\rho}{dt} = \sum_{k=1}^3 \gamma_k (\sigma_k \rho \sigma_k - \rho)$$

is the required Lindblad type master equation

### Question 2

$$\rho_{11}(t) = \rho_{11}(0) e^{-\gamma t}$$

$$\rho_{12}(t) = \rho_{12}(0) e^{-2\gamma t}$$

Since  $\rho$  is hermitian, we have  $\rho_{21} = \rho_{12}^*$ , and assuming  $\gamma$  is real

$$\rho_{21}(t) = \rho_{12}(0)^* e^{-2\gamma t}$$

Since  $\text{tr}(\rho) = 1$ ,  $\rho_{22} = 1 - \rho_{11}$

$$\rho_{22}(t) = 1 - \rho_{11}(0) e^{-\gamma t}$$

Thus, we have

$$\rho(t) = \phi(\rho(0)) = \begin{pmatrix} \rho_{11}(0) e^{-\gamma t} & \rho_{12}(0) e^{-2\gamma t} \\ \rho_{12}(0)^* e^{-2\gamma t} & 1 - \rho_{11}(0) e^{-\gamma t} \end{pmatrix}$$

We now find the  $F$ -matrix, defined by  $F_{ij} = \text{tr}(G_i \phi(G_j))$ , where  $\{G_i\}$  is the matrix basis  $\left\{ \frac{\mathbb{I}}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\}$

$$F_{00} = \frac{1}{2} \text{tr}(I \phi(I)) = \frac{1}{2} \text{tr}(\phi(I)) = \frac{1}{2} \text{tr} \left( \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix} \right) = \frac{1}{2}$$

$$F_{01} = \frac{1}{2} \text{tr}(I \phi(\sigma_x)) = \frac{1}{2} \text{tr}(\phi(\sigma_x)) = \frac{1}{2} \text{tr} \left( \begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 1 \end{pmatrix} \right) = \frac{1}{2}$$

$$\begin{aligned}
F_{02} &= \frac{1}{2} \text{tr}(I\phi(\sigma_y)) = \frac{1}{2} \text{tr}(\phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \\
F_{03} &= \frac{1}{2} \text{tr}(I\phi(\sigma_z)) = \frac{1}{2} \text{tr}(\phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \\
F_{10} &= \frac{1}{2} \text{tr}(\sigma_x \phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & 1 - e^{-\gamma t} \\ e^{-\gamma t} & 0 \end{pmatrix}\right) = 0 \\
F_{11} &= \frac{1}{2} \text{tr}(\sigma_x \phi(\sigma_x)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} e^{-2\gamma t} & 1 \\ 0 & e^{-2\gamma t} \end{pmatrix}\right) = e^{-2\gamma t} \\
F_{12} &= \frac{1}{2} \text{tr}(\sigma_x \phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} ie^{-2\gamma t} & 1 \\ 0 & -ie^{-2\gamma t} \end{pmatrix}\right) = 0 \\
F_{13} &= \frac{1}{2} \text{tr}(\sigma_x \phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & 1 - e^{-\gamma t} \\ e^{-\gamma t} & 0 \end{pmatrix}\right) = 0 \\
F_{20} &= \frac{1}{2} \text{tr}(\sigma_y \phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i + ie^{-\gamma t} \\ ie^{-\gamma t} & 0 \end{pmatrix}\right) = 0 \\
F_{21} &= \frac{1}{2} \text{tr}(\sigma_y \phi(\sigma_x)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} -ie^{-2\gamma t} & -i \\ 0 & ie^{-2\gamma t} \end{pmatrix}\right) = 0 \\
F_{22} &= \frac{1}{2} \text{tr}(\sigma_y \phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} e^{-2\gamma t} & -i \\ 0 & e^{-2\gamma t} \end{pmatrix}\right) = e^{-2\gamma t} \\
F_{23} &= \frac{1}{2} \text{tr}(\sigma_y \phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i + ie^{-\gamma t} \\ ie^{-\gamma t} & 0 \end{pmatrix}\right) = 0 \\
F_{30} &= \frac{1}{2} \text{tr}(\sigma_z \phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -1 + e^{-\gamma t} \end{pmatrix}\right) = e^{-\gamma t} - \frac{1}{2} \\
F_{31} &= \frac{1}{2} \text{tr}(\sigma_z \phi(\sigma_x)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & e^{-2\gamma t} \\ -e^{-2\gamma t} & -1 \end{pmatrix}\right) = -\frac{1}{2} \\
F_{32} &= \frac{1}{2} \text{tr}(\sigma_z \phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ -ie^{-2\gamma t} & -1 \end{pmatrix}\right) = -\frac{1}{2} \\
F_{33} &= \frac{1}{2} \text{tr}(\sigma_z \phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -1 + e^{-\gamma t} \end{pmatrix}\right) = e^{-\gamma t} - \frac{1}{2} \\
F &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & e^{-2\gamma t} & 0 & 0 \\ 0 & 0 & e^{-2\gamma t} & 0 \\ e^{-\gamma t} - \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & e^{-\gamma t} - \frac{1}{2} \end{pmatrix}
\end{aligned}$$

Clearly, the matrix is not full rank, and thus its determinant is 0.

Thus, the  $F$  matrix corresponding to this map is **not** invertible.

Also, since this operation cannot be represented by a single unitary, it is also **not** reversible.

### Question 3

The generalised amplitude damping channel for a qubit is defined as

$$\Lambda\left(\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\right) = \begin{pmatrix} (1-p_1)\rho_{00} + p_2\rho_{11} & \sqrt{1-p_2}\sqrt{1-p_1}\rho_{01} \\ \sqrt{1-p_2}\sqrt{1-p_1}\rho_{10} & (1-p_2)\rho_{11} + p_1\rho_{00} \end{pmatrix}$$

where  $p_\uparrow$  represents qubit excitation probability, and  $p_\downarrow$  represents qubit relaxation probability.

We now find the  $F$ -matrix, defined by  $F_{ij} = \text{tr}(G_i \phi(G_j))$ , where  $\{G_i\}$  is the matrix basis  $\left\{\frac{\mathbb{I}}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\right\}$

$$F_{00} = \frac{1}{2} \text{tr}(I\phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} -p_1 + p_2 + 1 & 0 \\ 0 & p_1 - p_2 + 1 \end{pmatrix}\right) = 1$$

$$F_{01} = \frac{1}{2} \text{tr}(I\phi(\sigma_x)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & \sqrt{1-p_1} * \sqrt{1-p_2} \\ \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \end{pmatrix}\right) = 0$$

$$F_{02} = \frac{1}{2} \text{tr}(I\phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i * \sqrt{1-p_1} * \sqrt{1-p_2} \\ i * \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \end{pmatrix}\right) = 0$$

$$F_{03} = \frac{1}{2} \text{tr}(I\phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} -p_1 - p_2 + 1 & 0 \\ 0 & p_1 + p_2 - 1 \end{pmatrix}\right) = 0$$

$$F_{10} = \frac{1}{2} \text{tr}(\sigma_x \phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & p_1 - p_2 + 1 \\ -p_1 + p_2 + 1 & 0 \end{pmatrix}\right) = 0$$

$$F_{11} = \frac{1}{2} \text{tr}(\sigma_x \phi(\sigma_x)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \\ 0 & \sqrt{1-p_1} * \sqrt{1-p_2} \end{pmatrix}\right) = \sqrt{1-p_1} \sqrt{1-p_2}$$

$$F_{12} = \frac{1}{2} \text{tr}(\sigma_x \phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} i * \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \\ 0 & -i * \sqrt{1-p_1} * \sqrt{1-p_2} \end{pmatrix}\right) = 0$$

$$F_{13} = \frac{1}{2} \text{tr}(\sigma_x \phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & p_1 + p_2 - 1 \\ -p_1 - p_2 + 1 & 0 \end{pmatrix}\right) = 0$$

$$F_{20} = \frac{1}{2} \text{tr}(\sigma_y \phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i * (p_1 - p_2 + 1) \\ i * (-p_1 + p_2 + 1) & 0 \end{pmatrix}\right) = 0$$

$$F_{21} = \frac{1}{2} \text{tr}(\sigma_y \phi(\sigma_x)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} -i * \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \\ 0 & i * \sqrt{1-p_1} * \sqrt{1-p_2} \end{pmatrix}\right) = 0$$

$$F_{22} = \frac{1}{2} \text{tr}(\sigma_y \phi(\sigma_y)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \\ 0 & \sqrt{1-p_1} * \sqrt{1-p_2} \end{pmatrix}\right) = \sqrt{1-p_1} \sqrt{1-p_2}$$

$$F_{23} = \frac{1}{2} \text{tr}(\sigma_y \phi(\sigma_z)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} 0 & -i * (p_1 + p_2 - 1) \\ i * (-p_1 - p_2 + 1) & 0 \end{pmatrix}\right) = 0$$

$$F_{30} = \frac{1}{2} \text{tr}(\sigma_z \phi(I)) = \frac{1}{2} \text{tr}\left(\begin{pmatrix} -p_1 + p_2 + 1 & 0 \\ 0 & -p_1 + p_2 - 1 \end{pmatrix}\right) = p_2 - p_1$$

$$F_{31} = \frac{1}{2} \text{tr}(\sigma_z \phi(\sigma_x)) = \frac{1}{2} \text{tr} \left( \begin{pmatrix} 0 & \sqrt{1-p_1} * \sqrt{1-p_2} \\ -\sqrt{1-p_1} * \sqrt{1-p_2} & 0 \end{pmatrix} \right) = 0$$

$$F_{32} = \frac{1}{2} \text{tr}(\sigma_z \phi(\sigma_y)) = \frac{1}{2} \text{tr} \left( \begin{pmatrix} 0 & -i * \sqrt{1-p_1} * \sqrt{1-p_2} \\ -i * \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \end{pmatrix} \right) = 0$$

$$F_{33} = \frac{1}{2} \text{tr}(\sigma_z \phi(\sigma_z)) = \frac{1}{2} \text{tr} \left( \begin{pmatrix} -p_1 - p_2 + 1 & 0 \\ 0 & -p_1 - p_2 + 1 \end{pmatrix} \right) = 1 - p_1 - p_2$$

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p_1} \sqrt{1-p_2} & 0 & 0 \\ 0 & 0 & \sqrt{1-p_1} \sqrt{1-p_2} & 0 \\ p_2 - p_1 & 0 & 0 & 1 - p_1 - p_2 \end{pmatrix}$$

Now, we know that

$$L = \dot{F} F^{-1}$$

$$\dot{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-(1-p_1)\dot{p}_2 - (1-p_2)\dot{p}_1}{2\sqrt{1-p_1}\sqrt{1-p_2}} & 0 & 0 \\ 0 & 0 & \frac{-(1-p_1)\dot{p}_2 - (1-p_2)\dot{p}_1}{2\sqrt{1-p_1}\sqrt{1-p_2}} & 0 \\ \dot{p}_2 - \dot{p}_1 & 0 & 0 & -\dot{p}_1 - \dot{p}_2 \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-p_1}\sqrt{1-p_2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-p_1}\sqrt{1-p_2}} & 0 \\ \frac{p_2-p_1}{p_1+p_2-1} & 0 & 0 & \frac{1}{1-p_1-p_2} \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(p_1-1)\dot{p}_2 + (p_2-1)\dot{p}_1}{2(p_1-1)(p_2-1)} & 0 & 0 \\ 0 & 0 & \frac{(p_1-1)\dot{p}_2 + (p_2-1)\dot{p}_1}{2(p_1-1)(p_2-1)} & 0 \\ \frac{2p_1\dot{p}_2 - 2p_2\dot{p}_1 + \dot{p}_1 - \dot{p}_2}{p_1+p_2-1} & 0 & 0 & \frac{\dot{p}_1 + \dot{p}_2}{p_1+p_2-1} \end{pmatrix}$$