Open Quantum Systems and Quantum Thermodynamics Assignment

Question 1

Part (a)
The quantum depolarising channel is defined as

$$\Delta_\lambda(\rho) = (1-\lambda)\rho + \frac{\lambda}{d}\operatorname{tr}(\rho)I \quad \text{with the constraint} \quad 0 \le \lambda \le 1 + \frac{1}{d^2-1}$$

To find its fixed point, we have

$$\Delta_\lambda(\rho) = \rho$$

$$(1-\lambda)\rho + \frac{\lambda}{d}\operatorname{tr}(\rho)I = \rho$$

$$\rho - \rho\lambda + \frac{\lambda}{d}I = \rho$$

$$\rho\lambda = \frac{\lambda}{d}I$$

$$\rho = \frac{I}{d}$$

$\therefore \frac{I}{d}$ is the fixed point for the quantum depolarising channel.

Part (b)
For the case of qubits, we have $d = 2$, and thus

$$\Delta_\lambda(\rho) = (1-\lambda)\rho + \frac{\lambda}{2}\operatorname{tr}(\rho)I \text{ with the constraint } 0 \le \lambda \le \frac{4}{3}$$

First, we find the Choi state for this map

$$I \otimes \Delta_\lambda(|\psi\rangle\langle\psi|)$$

where $|\psi\rangle$ is the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

We have

$$I \otimes \Delta_\lambda\left(\frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)\right)$$

$$= \frac{1}{2} I \otimes \Delta_\lambda(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$= \frac{1}{2} I \otimes \Delta_\lambda(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

Since $(a \otimes b)(c \otimes d) = ac \otimes bd$, we get

$$= \frac{1}{2}(|0\rangle\langle 0| \otimes \Delta_\lambda(|0\rangle\langle 0|) + |0\rangle\langle 1| \otimes \Delta_\lambda(|0\rangle\langle 1|) + |1\rangle\langle 0| \otimes \Delta_\lambda(|1\rangle\langle 0|) + |1\rangle\langle 1| \otimes \Delta_\lambda(|1\rangle\langle 1|))$$

$$= \frac{1}{2}\Bigg(|0\rangle\langle 0| \otimes \left((1-\lambda)|0\rangle\langle 0| + \frac{\lambda}{2} * 1 * I\right) + |0\rangle\langle 1| \otimes \left((1-\lambda)|0\rangle\langle 1| + \frac{\lambda}{2} * 0 * I\right)$$

$$+|1\rangle\langle 0| \otimes \left((1-\lambda)|1\rangle\langle 0| + \frac{\lambda}{2} * 0 * I\right) + |1\rangle\langle 1| \otimes \left((1-\lambda)|1\rangle\langle 1| + \frac{\lambda}{2} * 1 * I\right)\Bigg)$$

$$= \frac{1}{2}\Bigg(|0\rangle\langle 0| \otimes \left((1-\lambda)|0\rangle\langle 0| + \frac{\lambda}{2} I\right) + (1-\lambda)|0\rangle\langle 1| \otimes |0\rangle\langle 1|$$

$$+(1-\lambda)|1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes \left((1-\lambda)|1\rangle\langle 1| + \frac{\lambda}{2} I\right)\Bigg)$$

$$= \frac{1}{2}\Bigg((1-\lambda)|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{\lambda}{2}|0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + (1-\lambda)|0\rangle\langle 1| \otimes |0\rangle\langle 1|$$

$$+(1-\lambda)|1\rangle\langle 0| \otimes |1\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{\lambda}{2}|1\rangle\langle 1| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)\Bigg)$$

$$= \frac{1}{2}\Bigg((1-\lambda)|00\rangle\langle 00| + \frac{\lambda}{2}(|00\rangle\langle 00| + |01\rangle\langle 01|) + (1-\lambda)|00\rangle\langle 11|$$

$$+(1-\lambda)|11\rangle\langle 00| + (1-\lambda)|11\rangle\langle 11| + \frac{\lambda}{2}(|10\rangle\langle 10| + |11\rangle\langle 11|)\Bigg)$$

$$= \left(\frac{1}{2} - \frac{\lambda}{4}\right)(|00\rangle\langle 00| + |11\rangle\langle 11|) + \left(\frac{1}{2} - \frac{\lambda}{2}\right)(|00\rangle\langle 11| + |11\rangle\langle 00|) + \frac{\lambda}{4}(|01\rangle\langle 01| + |10\rangle\langle 10|)$$

Thus, we get the Choi state

$$C_d = \begin{pmatrix} \frac{1}{2} - \frac{\lambda}{4} & 0 & 0 & \frac{1}{2} - \frac{\lambda}{2} \\ 0 & \frac{\lambda}{4} & 0 & 0 \\ 0 & 0 & \frac{\lambda}{4} & 0 \\ \frac{1}{2} - \frac{\lambda}{2} & 0 & 0 & \frac{1}{2} - \frac{\lambda}{4} \end{pmatrix}$$

On diagonalizing C_d, we get

$$C_d = \Sigma_{i=0}^{3} \lambda_i |\alpha_i\rangle\langle\alpha_i|$$

where

$$\lambda_0 = \frac{1}{4}(4 - 3\lambda), \lambda_1 = \lambda_2 = \lambda_3 = \frac{\lambda}{4}$$

and

$$\alpha_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \alpha_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, we construct A_i from the α_i

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So, we get a set of Kraus operators $K_i = \sqrt{\lambda_i} A_i$

$$K_0 = \left(\sqrt{1-\frac{3}{4}\lambda}\right)I, K_1 = \sqrt{\frac{\lambda}{4}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, K_2 = \sqrt{\frac{\lambda}{4}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, K_3 = \sqrt{\frac{\lambda}{4}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

Note that the Kraus representation for a map is not unique. We can construct the canonical Kraus operators, which are

$$K_0 = \left(\sqrt{1-\frac{3}{4}\lambda}\right)I, K_1 = \sqrt{\frac{\lambda}{4}}\sigma_x, K_2 = \sqrt{\frac{\lambda}{4}}\sigma_y, K_3 = \sqrt{\frac{\lambda}{4}}\sigma_z$$

Part (c)

We have

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_k \gamma_k\left(L_k\rho L_k^\dagger - \frac{1}{2}\left\{L_k^\dagger L_k, \rho\right\}\right)$$

Since the depolarising channel is purely dissipative, we have H = 0

The parameter λ is interpreted as $\lambda = \gamma\Delta t$

Thus, we get the Lindblad operators

$$L_1 = \sqrt{\gamma}\sigma_x, L_2 = \sqrt{\gamma}\sigma_y, L_3 = \sqrt{\gamma}\sigma_Z$$

and

$$\frac{d\rho}{dt} = \sum_{k=1}^{3}\gamma_k(\sigma_k\rho\sigma_k - \rho)$$

is the required Lindblad type master equation

Question 2

$$\rho_{11}(t) = \rho_{11}(0)e^{-\gamma t}$$

$$\rho_{12}(t) = \rho_{12}(0)e^{-2\gamma t}$$

Since ρ is hermitian, we have $\rho_{21} = \rho_{12}^*$, and assuming γ is real

$$\rho_{21}(t) = \rho_{12}(0)^* e^{-2\gamma t} = \rho_{22}(0)e^{-2\gamma t}$$

Since $\mathrm{tr}(\rho) = 1$, $\rho_{22} = 1 - \rho_{11}$

$$\rho_{22}(t) = 1 - \rho_{11}(0)e^{-\gamma t} = \rho_{11}(0) + \rho_{22}(0) - \rho_{11}(0)e^{-\gamma t} = (1-e^{-\gamma t})\rho_{11}(0) + \rho_{22}(0)$$

Thus, we have

$$\rho(t) = \phi(\rho(0)) = \begin{pmatrix}\rho_{11}(0)e^{-\gamma t} & \rho_{12}(0)e^{-2\gamma t}\\ \rho_{22}(0)e^{-2\gamma t} & (1-e^{-\gamma t})\rho_{11}(0) + \rho_{22}(0)\end{pmatrix}$$

We now find the F-matrix, defined by $F_{ij} = \mathrm{tr}\big(G_i\phi(G_j)\big)$, where $\{G_i\}$ is the matrix basis $\left\{\frac{\mathbb{I}}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\right\}$

$$F_{00} = \frac{1}{2}\mathrm{tr}(I\phi(I)) = \frac{1}{2}\mathrm{tr}(\phi(I)) = \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix}e^{-\gamma t} & 0\\ 0 & 2-e^{-\gamma t}\end{pmatrix}\right) = 1$$

$$F_{01} = \frac{1}{2}\mathrm{tr}(I\phi(\sigma_x)) = \frac{1}{2}\mathrm{tr}(\phi(\sigma_x)) = \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix}0 & e^{-2\gamma t}\\ e^{-2\gamma t} & 0\end{pmatrix}\right) = 0$$

$$F_{02} = \frac{1}{2}\operatorname{tr}\big(I\phi(\sigma_y)\big) = \frac{1}{2}\operatorname{tr}\big(\phi(\sigma_y)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{03} = \frac{1}{2}\operatorname{tr}(I\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}(\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -e^{-\gamma t} \end{pmatrix}\right) = 0$$

$$F_{10} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(I)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 2-e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & 2-e^{-\gamma t} \\ e^{-\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{11} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(\sigma_x)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 0 \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} e^{-2\gamma t} & 1 \\ 0 & e^{-2\gamma t} \end{pmatrix}\right) = e^{-2\gamma t}$$

$$F_{12} = \frac{1}{2}\operatorname{tr}\big(\sigma_x\phi(\sigma_y)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 0 \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} ie^{-2\gamma t} & 1 \\ 0 & -ie^{-2\gamma t} \end{pmatrix}\right) = 0$$

$$F_{13} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -e^{-\gamma t} \\ e^{-\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{20} = \frac{1}{2}\operatorname{tr}\big(\sigma_y\phi(I)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 2-e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -2i+ie^{-\gamma t} \\ ie^{-\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{21} = \frac{1}{2}\operatorname{tr}\big(\sigma_y\phi(\sigma_x)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 1 \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} -ie^{-2\gamma t} & -i \\ 0 & ie^{-2\gamma t} \end{pmatrix}\right) = 0$$

$$F_{22} = \frac{1}{2}\operatorname{tr}\big(\sigma_y\phi(\sigma_y)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 0 \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} e^{-2\gamma t} & -i \\ 0 & e^{-2\gamma t} \end{pmatrix}\right) = e^{-2\gamma t}$$

$$F_{23} = \frac{1}{2}\operatorname{tr}\big(\sigma_y\phi(\sigma_z)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & ie^{-\gamma t} \\ ie^{-\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{30} = \frac{1}{2}\operatorname{tr}(\sigma_z\phi(I)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 2-e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -2+e^{-\gamma t} \end{pmatrix}\right) = e^{-\gamma t}-1$$

$$F_{31} = \frac{1}{2}\operatorname{tr}(\sigma_z\phi(\sigma_x)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & e^{-2\gamma t} \\ e^{-2\gamma t} & 0 \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & e^{-2\gamma t} \\ -e^{-2\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{32} = \frac{1}{2}\operatorname{tr}\big(\sigma_z\phi(\sigma_y)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ ie^{-2\gamma t} & 0 \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -ie^{-2\gamma t} \\ -ie^{-2\gamma t} & 0 \end{pmatrix}\right) = 0$$

$$F_{33} = \frac{1}{2}\operatorname{tr}(\sigma_z\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & -e^{-\gamma t} \end{pmatrix}\right) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & e^{-\gamma t} \end{pmatrix}\right) = e^{-\gamma t}$$

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\gamma t} & 0 & 0 \\ 0 & 0 & e^{-2\gamma t} & 0 \\ e^{-\gamma t}-1 & 0 & 0 & e^{-\gamma t} \end{pmatrix}$$

Clearly, the matrix has determinant $\neq 0$.

Thus, the F matrix corresponding to this map , and thus the map, **is** invertible.

Also, since this operation cannot be represented by a single unitary, it is **not** reversible.

Question 3

The generalised amplitude damping channel for a qubit is defined as

$$\Lambda\left(\begin{pmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{pmatrix}\right) = \begin{pmatrix}(1-p_1)\rho_{00}+p_2\rho_{11} & \sqrt{1-p_2}\sqrt{1-p_1}\rho_{01}\\ \sqrt{1-p_2}\sqrt{1-p_1}\rho_{10} & (1-p_2)\rho_{11}+p_1\rho_{00}\end{pmatrix}$$

where $p_\uparrow$ represents qubit excitation probability, and $p_\downarrow$ represents qubit relaxation probability.

We now find the F-matrix, defined by $F_{ij} = \operatorname{tr}(G_i\phi(G_j))$, where $\{G_i\}$ is the matrix basis $\left\{\frac{\mathbb{I}}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\right\}$

$$F_{00} = \frac{1}{2}\operatorname{tr}(I\phi(I)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}-p_1+p_2+1 & 0\\ 0 & p_1-p_2+1\end{pmatrix}\right) = 1$$

$$F_{01} = \frac{1}{2}\operatorname{tr}(I\phi(\sigma_x)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & \sqrt{1-p_1}*\sqrt{1-p_2}\\ \sqrt{1-p_1}*\sqrt{1-p_2} & 0\end{pmatrix}\right) = 0$$

$$F_{02} = \frac{1}{2}\operatorname{tr}(I\phi(\sigma_y)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & -i*\sqrt{1-p_1}*\sqrt{1-p_2}\\ i*\sqrt{1-p_1}*\sqrt{1-p_2} & 0\end{pmatrix}\right) = 0$$

$$F_{03} = \frac{1}{2}\operatorname{tr}(I\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}-p_1-p_2+1 & 0\\ 0 & p_1+p_2-1\end{pmatrix}\right) = 0$$

$$F_{10} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(I)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & p_1-p_2+1\\ -p_1+p_2+1 & 0\end{pmatrix}\right) = 0$$

$$F_{11} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(\sigma_x)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}\sqrt{1-p_1}*\sqrt{1-p_2} & 0\\ 0 & \sqrt{1-p_1}*\sqrt{1-p_2}\end{pmatrix}\right) = \sqrt{1-p_1}\sqrt{1-p_2}$$

$$F_{12} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(\sigma_y)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}i*\sqrt{1-p_1}*\sqrt{1-p_2} & 0\\ 0 & -i*\sqrt{1-p_1}*\sqrt{1-p_2}\end{pmatrix}\right) = 0$$

$$F_{13} = \frac{1}{2}\operatorname{tr}(\sigma_x\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & p_1+p_2-1\\ -p_1-p_2+1 & 0\end{pmatrix}\right) = 0$$

$$F_{20} = \frac{1}{2}\operatorname{tr}(\sigma_y\phi(I)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & -i*(p_1-p_2+1)\\ i*(-p_1+p_2+1) & 0\end{pmatrix}\right) = 0$$

$$F_{21} = \frac{1}{2}\operatorname{tr}(\sigma_y\phi(\sigma_x)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}-i*\sqrt{1-p_1}*\sqrt{1-p_2} & 0\\ 0 & i*\sqrt{1-p_1}*\sqrt{1-p_2}\end{pmatrix}\right) = 0$$

$$F_{22} = \frac{1}{2}\operatorname{tr}(\sigma_y\phi(\sigma_y)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}\sqrt{1-p_1}*\sqrt{1-p_2} & 0\\ 0 & \sqrt{1-p_1}*\sqrt{1-p_2}\end{pmatrix}\right) = \sqrt{1-p_1}\sqrt{1-p_2}$$

$$F_{23} = \frac{1}{2}\operatorname{tr}(\sigma_y\phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}0 & -i*(p_1+p_2-1)\\ i*(-p_1-p_2+1) & 0\end{pmatrix}\right) = 0$$

$$F_{30} = \frac{1}{2}\operatorname{tr}(\sigma_z\phi(I)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix}-p_1+p_2+1 & 0\\ 0 & -p_1+p_2-1\end{pmatrix}\right) = p_2-p_1$$

$$F_{31} = \frac{1}{2}\operatorname{tr}(\sigma_z \phi(\sigma_x)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & \sqrt{1-p_1} * \sqrt{1-p_2} \\ -\sqrt{1-p_1} * \sqrt{1-p_2} & 0 \end{pmatrix}\right) = 0$$

$$F_{32} = \frac{1}{2}\operatorname{tr}\big(\sigma_z \phi(\sigma_y)\big) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} 0 & -i * \sqrt{1-p_1} * \sqrt{1-p_2} \\ -i * \sqrt{1-p_1} * \sqrt{1-p_2} & 0 \end{pmatrix}\right) = 0$$

$$F_{33} = \frac{1}{2}\operatorname{tr}(\sigma_z \phi(\sigma_z)) = \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} -p_1 - p_2 + 1 & 0 \\ 0 & -p_1 - p_2 + 1 \end{pmatrix}\right) = 1 - p_1 - p_2$$

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p_1}\sqrt{1-p_2} & 0 & 0 \\ 0 & 0 & \sqrt{1-p_1}\sqrt{1-p_2} & 0 \\ p_2 - p_1 & 0 & 0 & 1 - p_1 - p_2 \end{pmatrix}$$

Now, we know that

$$L = \dot{F}F^{-1}$$

$$\dot{F} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{-(1-p_1)\dot{p_2} - (1-p_2)\dot{p_1}}{2\sqrt{1-p_1}\sqrt{1-p_2}} & 0 & 0 \\ 0 & 0 & \frac{-(1-p_1)\dot{p_2} - (1-p_2)\dot{p_1}}{2\sqrt{1-p_1}\sqrt{1-p_2}} & 0 \\ \dot{p_2} - \dot{p_1} & 0 & 0 & -\dot{p_1} - \dot{p_2} \end{pmatrix}$$

$$F^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-p_1}\sqrt{1-p_2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-p_1}\sqrt{1-p_2}} & 0 \\ \frac{p_2 - p_1}{p_1 + p_2 - 1} & 0 & 0 & \frac{1}{1 - p_1 - p_2} \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(p_1-1)\dot{p_2} + (p_2-1)\dot{p_1}}{2(p_1-1)(p_2-1)} & 0 & 0 \\ 0 & 0 & \frac{(p_1-1)\dot{p_2} + (p_2-1)\dot{p_1}}{2(p_1-1)(p_2-1)} & 0 \\ \frac{2p_1\dot{p_2} - 2p_2\dot{p_1} + \dot{p_1} - \dot{p_2}}{p_1 + p_2 - 1} & 0 & 0 & \frac{\dot{p_1} + \dot{p_2}}{p_1 + p_2 - 1} \end{pmatrix}$$