Science-1

Assignment-4

Moida Praneeth Jain, 2022010193

Question 1

(1)

Kinetic Energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_1^2$$

Potential Energy

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$

Lagrangian

$$L = T - V$$

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2$$

(2)

For x_1

$$\frac{\partial L}{\partial x_1} = -(k_1 + k_2)x_1 + k_2x_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m_1\ddot{x}_1$$

Lagrange equation for x_1

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1}$$

$$m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2$$

For x_2

$$\frac{\partial L}{\partial x_2} = -k_2x_2 + k_2x_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = m_2\ddot{x}_2$$

Lagrange equation for x_2

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_2}$$

$$m_2\ddot{x}_2 = -k_2x_2 + k_2x_1$$

(3)

$$\ddot{x}_1 = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\ddot{x}_2 = \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_2$$

In matrix form

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We now find eigenvalues of

$$A = \begin{pmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$| \begin{pmatrix} -\frac{k_1+k_2}{m_1} - \lambda & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} - \lambda \end{pmatrix} | = 0$$

$$\left(\frac{k_1 + k_2}{m_1} + \lambda \right) \left(\frac{k_2}{m_2} + \lambda \right) = \frac{k_2^2}{m_1 m_2}$$

On solving the quadratic equation, we get

$$\lambda_1 = \frac{\sqrt{(k_1 m_2 + k_2 m_1 + k_2 m_2)^2 - 4 k_1 k_2 m_1 m_2} - k_1 m_2 - k_2 m_1 - k_2 m_2}{2 m_1 m_2}$$

$$\lambda_2 = \frac{-\sqrt{(k_1 m_2 + k_2 m_1 + k_2 m_2)^2 - 4 k_1 k_2 m_1 m_2} - k_1 m_2 - k_2 m_1 - k_2 m_2}{2 m_1 m_2}$$

$$w_1^2 = -\lambda_1, w_2^2 = -\lambda_2$$

The normal mode frequencies are

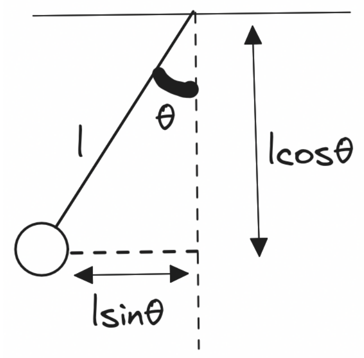
$$w_1 = \sqrt{-\lambda_1}$$

$$w_2 = \sqrt{-\lambda_2}$$

$$w_1 = \sqrt{-\frac{\sqrt{(k_1 m_2 + k_2 m_1 + k_2 m_2)^2 - 4 k_1 k_2 m_1 m_2} - k_1 m_2 - k_2 m_1 - k_2 m_2}{2 m_1 m_2}}$$

$$w_2 = \sqrt{-\frac{-\sqrt{(k_1 m_2 + k_2 m_1 + k_2 m_2)^2 - 4 k_1 k_2 m_1 m_2} - k_1 m_2 - k_2 m_1 - k_2 m_2}{2 m_1 m_2}}$$

Question 2



(1)

$$x = l\sin(\theta)$$

$$\dot{x} = l\cos(\theta)\dot{\theta}$$

$$y = l(1 - \cos(\theta))$$

$$\dot{y} = l\sin(\theta)\dot{\theta}$$

Kinetic Energy

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

Potential Energy with reference to bottom-most position

$$V = mgl(1 - \cos(\theta))$$

Lagrangian

$$L = T - V$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos(\theta))$$

Generalised momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$p_\theta = ml^2\dot{\theta}$$

(2)

Hamiltonian

$$H = \sum p_i \dot{q}_i - L$$

$$H = p_\theta \dot{\theta} - L$$

We know that $\dot{\theta} = \frac{p_\theta}{ml^2}$

$$L = \frac{1}{2}\frac{p_\theta^2}{ml^2} - mgl(1 - \cos(\theta))$$

$$H = \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl(1 - \cos(\theta))$$

$$H = \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl(1 - \cos(\theta))$$

$$H = \frac{1}{2}\frac{p_\theta^2}{ml^2} + 2mgl\sin^2\left(\frac{\theta}{2}\right)$$

(3)

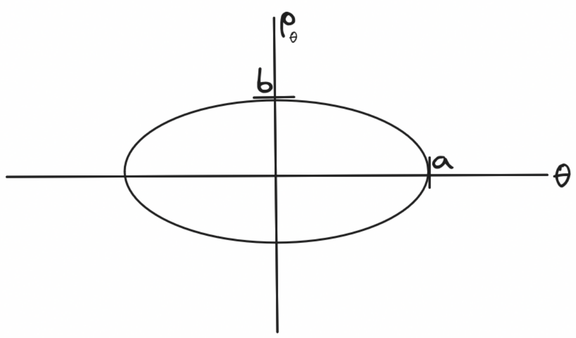
when θ is small, $\sin(\theta) \to \theta$

$$H = \frac{1}{2}\frac{p_\theta^2}{ml^2} + 2mgl\left(\frac{\theta}{2}\right)^2$$

$$H = \frac{p_\theta^2}{2ml^2} + \frac{\theta^2}{\frac{2}{mgl}}$$

$$\frac{p_\theta^2}{2ml^2 H} + \frac{\theta^2}{\frac{2H}{mgl}} = 1$$

Since the total energy is constant (no energy loss), and the hamiltonian is equal to the total energy in this case, H is also constant.



with $a = \frac{2H}{mgl}$ and $b = 2ml^2 H$

Question 3

(1)

In polar form,

$$x = r\cos(\theta)$$

$$\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta}$$

$$y = r\sin(\theta)$$

$$\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta}$$

Kinetic Energy

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Lagrangian

$$L = T - V$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\alpha}{r}$$

Generalised momentum corresponding to r

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

Generalised momentum corresponding to θ

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = A$$

$$\dot{\theta} = \frac{A}{mr^2}$$

Lagrange equation for θ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}mr^2\dot{\theta} = 0$$

$\therefore$ Angular momentum A remains conserved throughout.

Lagrange equation for r

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

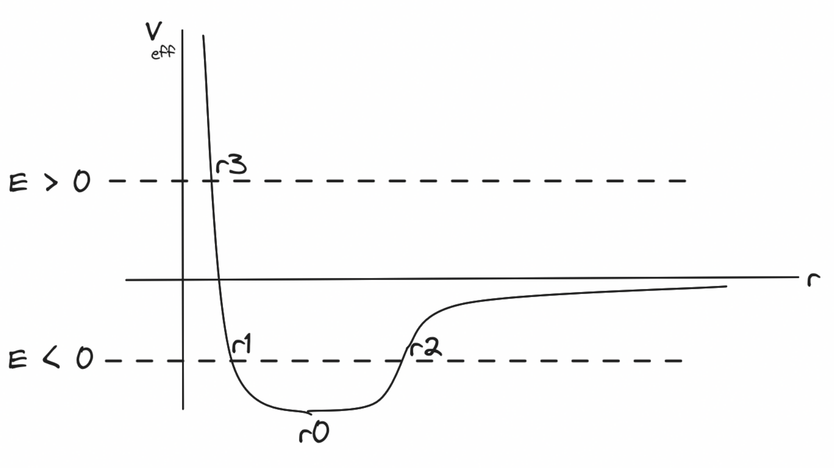
$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\alpha}{r^2}$$

$$m\ddot{r} = \frac{A^2}{mr^3} - \frac{\alpha}{r^2}$$

$$m\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{A^2}{2mr^2} - \frac{\alpha}{r}\right)$$

$$m\ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r}$$

$$\therefore V_{\text{eff}} = \frac{A^2}{2mr^2} - \frac{\alpha}{r}$$



Since the kinetic energy T is always positive, the total energy E is always greater than the potential energy V.

$\therefore$ Orbit will be bound for negative total energies as it is between r_1 and r_2. Otherwise, orbit will be unbound.

Condition for orbit to be bound is $E < 0$

(2)

For a circular orbit, $r_1 = r_2 = r_0$. To find r_0, since it is minima

$$\frac{dV_{\text{eff}}}{dr}\Big|_{r=r_0} = 0$$

$$\frac{\alpha}{r_0^2} = \frac{A^2}{mr_0^3}$$

$$r_0 = \frac{A^2}{m\alpha}$$

The corresponding energy is

$$E = \frac{A^2}{2mr_0^2} - \frac{\alpha}{r_0}$$

$$E = \frac{A^2 m^2 \alpha^2}{2mA^4} - \frac{\alpha m \alpha}{A^2}$$

$$E = \frac{m\alpha^2}{2A^2} - \frac{m\alpha^2}{A^2}$$

$$E = -\frac{m\alpha^2}{2A^2}$$

(3)

The minimum and maximum radius r_1 and r_2 are the roots of the equation $E = V_{\text{eff}}$ where $E < 0$, as can be seen from the graph above.

$$E = V_{\text{eff}}$$

$$E = \frac{A^2}{2mr^2} - \frac{\alpha}{r}$$

$$2Emr^2 = A^2 - 2\alpha mr$$

On solving the quadratic, we get

$$r_{\text{min}} = \frac{-2\alpha m + \sqrt{4\alpha^2 m^2 + 8A^2 Em}}{4Em}$$

$$r_{\text{max}} = \frac{-2\alpha m - \sqrt{4\alpha^2 m^2 + 8A^2 Em}}{4Em}$$

In case $E > 0$, we only get a minimum radius r_{min}