Science-1

Assignment-3

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Question 1

For plot (i), $y = x$ for $0 \le x \le 1, 0 \le y \le 1$

For plot (ii), $y = 0$ for $0 \le x < 1$ and $x = 1$ for $0 < y \le 1$

(a)

$$df = d(xy)$$

$$df = (dx)y + x(dy)$$

(i)

$$df = xdx + ydy$$

$$\int df = \int_0^1 xdx + \int_0^1 ydy$$

$$\int df = \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1$$

$$\int df = \frac{1}{2} + \frac{1}{2}$$

$$\int df = 1$$

(ii)

$$\int df = \int_0^1 xdx + \int_0^0 ydy + \int_1^1 xdx + \int_0^1 ydy$$

$$\int df = \left[\frac{x^2}{2}\right]_0^1 + 0 + 0 + \left[\frac{y^2}{2}\right]_0^1$$

$$\int df = \frac{1}{2} + \frac{1}{2}$$

$$\int df = 1$$

(b)

$$df = d(xy)$$

Note that the integral is path independent.

Since df is equal to the differential of a differentiable function xy, it is an exact differential.

(c)

(i)

$$dg = ydx$$

$$dg = xdx$$

$$\int dg = \int_0^1 xdx$$

$$\int dg = \left[\frac{x^2}{2}\right]_0^1$$

$$\int dg = \frac{1}{2}$$

(ii)

$$dg = ydx$$

$$\int dg = \int_0^1 0dx + \int_1^1 1dx$$

$$\int dg = 0$$

(d)

$$dg = ydx$$

Note that the integral is path dependent

Since dg is not equal to the differential of a differentiable function, it is an inexact differential.

Question 2

Assuming 1 mole of gas as $U = C_V T$

Let ρ be the mass density, p be the pressure, V be the volume, M be the molar mass.

We know that $C_v = \frac{R}{1-\gamma}$, where R is the gas constant.

(i)

$$U = C_V T$$

$$\frac{U}{M} = \frac{C_V T}{M}$$

Since $pV = RT$

$$\frac{U}{M} = \frac{C_V pV}{MR}$$

Since $\rho = \frac{M}{V}$

$$\frac{U}{M} = \frac{C_V p}{\rho R}$$

$$\frac{U}{M} = \frac{\cancel{R}p}{\rho\cancel{R}(\gamma - 1)}$$

$$\frac{U}{M} = \frac{p}{\rho(\gamma - 1)}$$

This is the required internal energy per unit mass.

(ii)

$$U = \frac{RT}{\gamma - 1}$$

$$\frac{U}{V} = \frac{RT}{V(\gamma - 1)}$$

Since $pV = RT$

$$\frac{U}{V} = \frac{p\cancel{V}}{\cancel{V}(\gamma - 1)}$$

$$\frac{U}{V} = \frac{p}{\gamma - 1}$$

This is the required internal energy per unit volume.

Question 3

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{\left(\frac{\partial T}{\partial V}\right)_p}$$

For a van der Waals gas,

$$RT = \left(p + \frac{a}{V^2}\right)(V - b)$$

$$R\left(\frac{\partial T}{\partial V}\right)_p = \left(p + \frac{a}{V^2}\right) - \frac{2a}{V^3}(V - b)$$

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{RV^3}{V(pV^2 + a) - 2a(V - b)}$$

$$\alpha = \frac{RV^2}{V(pV^2 + a) - 2a(V - b)}$$

This is the coefficient of thermal expansion α for a van der Waals gas.

Now, for isothermal compressibility:

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

At a constant temperature, on increasing the pressure, the volume of a gas decreases. This will result in $\left(\frac{\partial V}{\partial p}\right)_T$ being negative. Hence, for the constant κ_T to be positive, the RHS is multiplied by -1.

Now, to prove the required relation:

$$\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\left(\frac{\partial p}{\partial V}\right)_T}$$

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

$$\left(\frac{\partial p}{\partial V}\right)_T = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3}$$

On substituting RT from the van der Waals gas equation,

$$\left(\frac{\partial p}{\partial V}\right)_T = -\left(p + \frac{a}{V^2}\right)\frac{\cancel{V-b}}{(V-b)^{\cancel{2}}} + \frac{2a}{V^3}$$

$$\left(\frac{\partial V}{\partial p}\right)_T = \frac{V^3(V-b)}{2a(V-b) - V(pV^2 + a)}$$

$$\kappa_T = \frac{V^2(V-b)}{V(pV^2 + a) - 2a(V-b)}$$

$$\kappa_T R = \frac{RV^2}{V(pV^2 + a) - 2a(V-b)}(V-b)$$

$$\kappa_T R = \alpha(V-b)$$

For one mole of gas, $V = V_m$

$$\kappa_T R = \alpha(V_m - b)$$

Hence, proven.