Science-1

Assignment-5

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Question 1

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a)

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1$$

$$[\sigma_1, \sigma_2] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$[\sigma_2, \sigma_1] = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$$

Therefore, σ_1 and σ_2 do not commute.

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_2] = \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix}$$

Therefore, σ_2 and σ_3 do not commute.

$$[\sigma_1, \sigma_3] = \sigma_1\sigma_3 - \sigma_3\sigma_1$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_1] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Therefore, σ_1 and σ_3 do not commute.

Note that trivially, σ_1 commutes with σ_1, σ_2 commutes with σ_2 and σ_3 commutes with σ_3 ($[\sigma_i, \sigma_i] = \mathbf{0}$)

(b)

For σ_1, $\det(\sigma_1 - \lambda I) = 0$

$$\det\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_1 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix} = \begin{pmatrix}-x_{11}\\ -x_{12}\end{pmatrix}$$

$$x_{12} = -x_{11}$$

$$x_1 = \begin{pmatrix}\alpha\\ -\alpha\end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1\rangle} = \sqrt{2}\alpha$

$$|x_1\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\sigma_1 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_{21}\\ x_{22}\end{pmatrix} = \begin{pmatrix}x_{21}\\ x_{22}\end{pmatrix}$$

$$x_{22} = x_{21}$$

$$x_2 = \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2|x_2\rangle} = \sqrt{2}\alpha$

$$|x_2\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$\therefore$ for σ_1, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\begin{pmatrix}\alpha\\ -\alpha\end{pmatrix}, \begin{pmatrix}\beta\\ \beta\end{pmatrix}$, which upon normalizing give $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$.

For σ_2, $\det(\sigma_2 - \lambda I) = 0$

$$\det\begin{pmatrix}-\lambda & -i\\ i & -\lambda\end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_2 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix} = \begin{pmatrix}-x_{11}\\ -x_{12}\end{pmatrix}$$

$$x_{12} = -ix_{11}$$

$$x_1 = \begin{pmatrix} \alpha \\ -i\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \sqrt{2\alpha}$

$$|x_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_2 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = i x_{21}$$

$$x_2 = \begin{pmatrix} \alpha \\ i\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \sqrt{2\alpha}$

$$|x_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ for σ_2, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\begin{pmatrix} \alpha \\ -i\alpha \end{pmatrix}$, $\begin{pmatrix} \beta \\ i\beta \end{pmatrix}$, which upon normalizing give $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

For σ_3, $\det(\sigma_3 - \lambda I) = 0$

$$\det \begin{pmatrix} 1 - \lambda & 0 \\ 0 & -1 - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_3 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{11} = 0$$

$$x_1 = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \alpha$

$$|x_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_3 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = 0$$

$$x_2 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \alpha$

$$|x_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\therefore$ for σ_3, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\left(\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta \\ 0 \end{smallmatrix}\right)$, which upon normalizing give $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

(c)

Since $|\varphi\rangle = \sum_i c_i |\lambda_i\rangle, \therefore P(|\lambda_i\rangle) = |c_i|^2$

$P\left(\hat{A}\right)$ in $|\varphi\rangle$ is $|\langle x_1| \ \varphi\rangle|^2$ for state $|x_1\rangle$

Here, we consider the eigenstate of σ_2

$$\varphi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$P(\sigma_1) = \begin{cases} |\langle x_1| \ \varphi\rangle|^2 & \text{for state } |x_1\rangle \\ |\langle x_2| \ \varphi\rangle|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} |\left(\frac{1}{\sqrt{2}} \ -\frac{1}{\sqrt{2}}\right) * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_1\rangle \\ |\left(\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right) * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} |\frac{1-i}{2}|^2 & \text{for state } |x_1\rangle \\ |\frac{1+i}{2}|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_1\rangle \\ \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} \frac{1}{2} & \text{for state } |x_1\rangle \\ \frac{1}{2} & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} |\langle x_1| \ \varphi\rangle|^2 & \text{for state } |x_1\rangle \\ |\langle x_2| \ \varphi\rangle|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} |\begin{pmatrix} 0 & 1 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_1\rangle \\ |\begin{pmatrix} 1 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_1\rangle \\ \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} \frac{1}{2} & \text{for state } |x_1\rangle \\ \frac{1}{2} & \text{for state } |x_2\rangle \end{cases}$$

(d)

For $\Delta\sigma_1$

$$\Delta\sigma_1^2 = \langle\sigma_1^2\rangle - \langle\sigma_1\rangle^2$$

$$\Delta\sigma_1^2 = \langle\varphi|\sigma_1^2|\varphi\rangle - \langle\varphi|\sigma_1|\varphi\rangle^2$$

$$\Delta\sigma_1^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} - \left(\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}\right)^2$$

$$\Delta\sigma_1^2 = 1 - 0$$

$$\Delta\sigma_1^2 = 1$$

For $\Delta\sigma_3$

$$\Delta\sigma_3^2 = \langle\sigma_3^2\rangle - \langle\sigma_3\rangle^2$$

$$\Delta\sigma_3^2 = \langle\varphi|\sigma_3^2|\varphi\rangle - \langle\varphi|\sigma_3|\varphi\rangle^2$$

$$\Delta\sigma_3^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} - \left(\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}\right)^2$$

$$\Delta\sigma_3^2 = 1 - 0$$

$$\Delta\sigma_3^2 = 1$$

So, the uncertainty $\Delta\sigma_1 = 1, \Delta\sigma_3 = 1, \Delta\sigma_1\Delta\sigma_3 = 1.$

According to the uncertainty principle

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2}\ |\langle[\sigma_1, \sigma_3]\rangle|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2}\ |\langle\varphi|\sigma_1\sigma_3|\varphi\rangle|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2}\ |\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2}\ |2i|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq 1$$

These are the corresponding uncertainty relations.