Science-1

Assignment-1

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Question 1

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a)

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1$$

$$[\sigma_1, \sigma_2] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$[\sigma_2, \sigma_1] = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$$

Therefore, σ_1 and σ_2 do not commute.

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_2] = \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix}$$

Therefore, σ_2 and σ_3 do not commute.

$$[\sigma_1, \sigma_3] = \sigma_1\sigma_3 - \sigma_3\sigma_1$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_1] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Therefore, σ_1 and σ_3 do not commute.

Note that trivially, σ_1 commutes with σ_1, σ_2 commutes with σ_2 and σ_3 commutes with σ_3 ($[\sigma_i, \sigma_i] = \mathbf{0}$)

(b)

For σ_1, $\det(\sigma_1 - \lambda I) = 0$

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_1 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{12} = -x_{11}$$

$$x_1 = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \sqrt{2\alpha}$

$$\mid x_1 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_1 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = x_{21}$$

$$x_2 = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \sqrt{2\alpha}$

$$\mid x_2 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ for σ_1, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \end{pmatrix}$, which upon normalizing give $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

For σ_2, $\det(\sigma_2 - \lambda I) = 0$

$$\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_2 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{12} = -i x_{11}$$

$$x_1 = \begin{pmatrix} \alpha \\ -i\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \sqrt{2\alpha}$

$$\mid x_1 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_2 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = i x_{21}$$

$$x_2 = \begin{pmatrix} \alpha \\ i\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \sqrt{2\alpha}$

$$\mid x_2 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ for σ_2, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\begin{pmatrix} \alpha \\ -i\alpha \end{pmatrix}, \begin{pmatrix} \beta \\ i\beta \end{pmatrix}$, which upon normalizing give $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

For σ_3, $\det(\sigma_3 - \lambda I) = 0$

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_3 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{11} = 0$$

$$x_1 = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \alpha$

$$\mid x_1 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_3 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = 0$$

$$x_2 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \alpha$

$$\mid x_2 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\therefore$ for σ_3, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \begin{pmatrix} \beta \\ 0 \end{pmatrix}$, which upon normalizing give $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.