Science-1

Assignment-5

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Question 1

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a)

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1$$

$$[\sigma_1, \sigma_2] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[\sigma_1, \sigma_2] = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$[\sigma_2, \sigma_1] = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$$

Therefore, σ_1 and σ_2 do not commute.

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$[\sigma_2, \sigma_3] = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_2] = \begin{pmatrix} 0 & -2i \\ -2i & 0 \end{pmatrix}$$

Therefore, σ_2 and σ_3 do not commute.

$$[\sigma_1, \sigma_3] = \sigma_1\sigma_3 - \sigma_3\sigma_1$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$[\sigma_1, \sigma_3] = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_1] = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Therefore, σ_1 and σ_3 do not commute.

Note that trivially, σ_1 commutes with σ_1, σ_2 commutes with σ_2 and σ_3 commutes with σ_3 ($[\sigma_i, \sigma_i] = \mathbf{0}$)

(b)

For σ_1, $\det(\sigma_1 - \lambda I) = 0$

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_1 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{12} = -x_{11}$$

$$x_1 = \binom{\alpha}{-\alpha}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \sqrt{2}\alpha$

$$|x_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_1 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = x_{21}$$

$$x_2 = \binom{\alpha}{\alpha}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 | x_2 \rangle} = \sqrt{2}\alpha$

$$|x_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ for σ_1, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\binom{\alpha}{-\alpha}, \binom{\beta}{\beta}$, which upon normalizing give $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

For σ_2, $\det(\sigma_2 - \lambda I) = 0$

$$\det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_2 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{12} = -i x_{11}$$

$$x_1 = \begin{pmatrix} \alpha \\ -i\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \sqrt{2}\alpha$

$$|x_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_2 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = i x_{21}$$

$$x_2 = \begin{pmatrix} \alpha \\ i\alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \sqrt{2}\alpha$

$$|x_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$\therefore$ for σ_2, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\begin{pmatrix} \alpha \\ -i\alpha \end{pmatrix}, \begin{pmatrix} \beta \\ i\beta \end{pmatrix}$, which upon normalizing give $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

For σ_3, $\det(\sigma_3 - \lambda I) = 0$

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = -1, \lambda_2 = 1$$

$$\sigma_3 x_1 = \lambda_1 x_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} -x_{11} \\ -x_{12} \end{pmatrix}$$

$$x_{11} = 0$$

$$x_1 = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_1 \mid x_1 \rangle} = \alpha$

$$|x_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma_3 x_2 = \lambda_2 x_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$x_{22} = 0$$

$$x_2 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

For orthonormal basis, we divide by its magnitude $\sqrt{\langle x_2 \mid x_2 \rangle} = \alpha$

$$|x_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\therefore$ for σ_3, eigenvalues are $-1, 1$, and corresponding orthogonal eigenvectors are $\binom{0}{\alpha}, \binom{\beta}{0}$, which upon normalizing give $\binom{0}{1}, \binom{1}{0}$.

(c)

Since $|\varphi\rangle = \sum_i c_i |\lambda_i\rangle, \therefore P(|\lambda_i\rangle) = |c_i|^2$

$P\left(\hat{A}\right)$ in $|\varphi\rangle$ is $|\langle x_1 | \ \varphi\rangle|^2$ for state $|x_1\rangle$

Here, we consider the eigenstate of σ_2

$$\varphi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$P(\sigma_1) = \begin{cases} |\langle x_1 | \ \varphi\rangle|^2 & \text{for state } |x_1\rangle \\ |\langle x_2 | \ \varphi\rangle|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} |\left(\frac{1}{\sqrt{2}} \ -\frac{1}{\sqrt{2}}\right) * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_1\rangle \\ |\left(\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right) * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} |\frac{1-i}{2}|^2 & \text{for state } |x_1\rangle \\ |\frac{1+i}{2}|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_1\rangle \\ \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_1) = \begin{cases} \frac{1}{2} & \text{for state } |x_1\rangle \\ \frac{1}{2} & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} |\langle x_1 | \ \varphi\rangle|^2 & \text{for state } |x_1\rangle \\ |\langle x_2 | \ \varphi\rangle|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} |\begin{pmatrix} 0 & 1 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_1\rangle \\ |\begin{pmatrix} 1 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_1\rangle \\ \left(\frac{1}{\sqrt{2}}\right)^2 & \text{for state } |x_2\rangle \end{cases}$$

$$P(\sigma_3) = \begin{cases} \frac{1}{2} & \text{for state } |x_1\rangle \\ \frac{1}{2} & \text{for state } |x_2\rangle \end{cases}$$

(d)

For $\Delta\sigma_1$

$$\Delta\sigma_1^2 = \langle\sigma_1^2\rangle - \langle\sigma_1\rangle^2$$

$$\Delta\sigma_1^2 = \langle\varphi|\sigma_1^2|\varphi\rangle - \langle\varphi|\sigma_1|\varphi\rangle^2$$

$$\Delta\sigma_1^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} - \left(\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \right)^2$$

$$\Delta\sigma_1^2 = 1 - 0$$

$$\Delta\sigma_1^2 = 1$$

For $\Delta\sigma_3$

$$\Delta\sigma_3^2 = \langle\sigma_3^2\rangle - \langle\sigma_3\rangle^2$$

$$\Delta\sigma_3^2 = \langle\varphi|\sigma_3^2|\varphi\rangle - \langle\varphi|\sigma_3|\varphi\rangle^2$$

$$\Delta\sigma_3^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} - \left(\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \right)^2$$

$$\Delta\sigma_3^2 = 1 - 0$$

$$\Delta\sigma_3^2 = 1$$

So, the uncertainty $\Delta\sigma_1 = 1, \Delta\sigma_3 = 1, \Delta\sigma_1\Delta\sigma_3 = 1$.

According to the uncertainty principle

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2} \, |\langle[\sigma_1, \sigma_3]\rangle|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2} \, |\langle\varphi|\sigma_1\sigma_3|\varphi\rangle|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2}\ |\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} * \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} * \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq \frac{1}{2}\ |2i|$$

$$\Delta\sigma_1\Delta\sigma_3 \geq 1$$

These are the corresponding uncertainty relations.

Question 2

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{for } x \leq 0 \quad \text{or } x \geq L \end{cases}$$

$$\psi = \frac{1}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle$$

We use the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

For $x \leq 0$ or $x \geq L$, we have

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x) + (\infty)\psi(x) = E\psi(x)$$

which implies $\psi(x) = 0\ \forall x \notin (0, L)$

For regions inside the wall, we have

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x) + 0 * \psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}\psi(x) = E\psi(x)$$

$$\frac{d}{dx^2}\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

We know the general solution to this differential equation is

$$\psi(x) = A\sin(kx) + B\cos(kx) \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

At $x = 0$

$$\psi(0) = A\sin(0) + B\cos(0)$$

$$B = 0$$

At $x = L$

$$\psi(L) = A\sin(kL) + 0\cos(kL)$$

$$0 = \sin(kL)$$

$$k = \frac{n\pi}{L}$$

$$\sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L}$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$$

$$E = n^2\frac{\pi^2\hbar^2}{2mL^2}$$

The first two lowest energies are $E_1 = \frac{\pi^2\hbar^2}{2mL^2}, E_2 = \frac{2\pi^2\hbar^2}{mL^2}$

Since the total probability of the particle existing at any position is 1

$$\int_0^L \psi^2(x)dx = 1$$

$$\int_0^L A^2\sin^2(kx)dx = 1$$

$$\int_0^L A^2\sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

$$A^2\frac{L}{2} = 1$$

$$A = \sqrt{\frac{2}{L}}$$

Now, we know that

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t}|\varphi_1\rangle + \frac{1}{\sqrt{2}}e^{-iE_2t}|\varphi_2\rangle$$

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{L}}e^{-i\frac{\pi^2\hbar^2}{2mL^2}t}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{L}}e^{-i\frac{2\pi^2\hbar^2}{mL^2}t}\sin\left(\frac{2\pi x}{L}\right)$$

(a)

(b)

(c)

(d)