Science-2

Assignment-1

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Question 1

(a)

Singular Value Decomposition (SVD) of an $m \times n$ matrix X is given by $X = U\Sigma V^T$ where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix and Σ is $m \times n$ diagonal matrix.

The steps to find SVD of a matrix A are as follows:

- Calculate the eigenvalues and eigenvectors of AA^T and A^TA. Sort them in descending order of their eigenvalues
- Both of these will have the same eigenvalues (may have different eigenvectors)
- Say $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_k > 0$ are the non-zero eigenvalues
- Construct the matrix

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \sqrt{\lambda_r} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

  by placing the square roots of the positive eigenvalues as the diagonal entries, and the rest of the values as 0.
- Construct U: The columns of U are the normalized eigenvectors of AA^T.
- Construct V: The columns of V are the normalized eigenvectors of A^TA

Following these steps for the required matrix

$$A = \begin{pmatrix} 9 & 3 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$B = AA^T = \begin{pmatrix} 99 & 69 & 114 & 159 \\ 69 & 77 & 122 & 167 \\ 114 & 122 & 194 & 266 \\ 159 & 167 & 266 & 365 \end{pmatrix}$$

$$C = A^TA = \begin{pmatrix} 246 & 213 & 234 \\ 213 & 219 & 243 \\ 234 & 243 & 270 \end{pmatrix}$$

$$\lambda_1 = 706.335, \lambda_2 = 28.569, \lambda_3 = 0.096, \lambda_4 = 0$$

With corresponding eigenvectors

$$b_1 = \begin{pmatrix} -0.32 \\ -0.33 \\ -0.52 \\ -0.72 \end{pmatrix}, b_2 = \begin{pmatrix} 0.94 \\ -0.21 \\ -0.2 \\ -0.18 \end{pmatrix}, b_3 = \begin{pmatrix} 0.11 \\ 0.83 \\ 0.15 \\ -0.53 \end{pmatrix}, b_4 = \begin{pmatrix} 0 \\ 0.41 \\ -0.82 \\ 0.41 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} -0.57 \\ -0.55 \\ -0.61 \end{pmatrix}, c_2 = \begin{pmatrix} 0.82 \\ -0.34 \\ -0.45 \end{pmatrix}, c_3 = \begin{pmatrix} 0.04 \\ -0.76 \\ 0.65 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 26.57695941 & 0 & 0 \\ 0 & 5.34498755 & 0 \\ 0 & 0 & 0.31038172 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.32 & 0.94 & 0.11 & 0 \\ -0.33 & -0.21 & 0.83 & 0.41 \\ -0.52 & -0.2 & 0.15 & -0.82 \\ -0.72 & -0.18 & -0.53 & 0.41 \end{pmatrix}$$

$$V^T = \begin{pmatrix} -0.57 & -0.55 & -0.61 \\ 0.82 & -0.34 & -0.45 \\ 0.04 & -0.76 & 0.65 \end{pmatrix}$$

The calculation can alternatively be performed through the following code snipped:

```
import numpy as np
inp = np.matrix([[9, 3, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]])

U, d, V_T = np.linalg.svd(inp)
D = np.vstack(
    (*np.diag(d), *[np.zeros(V_T.shape[0]) for _ in range(U.shape[0] -
V_T.shape[0])])
)

print(U)
print(D)
print(V_T)
print(U.dot(D).dot(V_T))
```

(b)

Consider a matrix A that has a standard diagonalization (A is diagonalizable). A must be a square matrix.

$$A = PDP^{-1}$$

$$A = U\Sigma V^T$$

For this matrix to have the same decompositions, we must have

$$P = U \quad D = \Sigma \quad P^{-1} = V^T$$

Now, consider

$$A^T = \left(U\Sigma V^T\right)^T$$

$$A^T = V^{T^T}\Sigma^T U^T$$

Since Σ is a diagonal matrix, we have $\Sigma = \Sigma^T$

$$A^T = V^{T^T}\Sigma U^T$$

$$A^T = \left(P^{-1}\right)^T DP^T$$

Since U is orthogonal, we have $U^T = U^{-1}$, and since $U = P$, $P^T = P^{-1}$

$$A^T = P^{T^T}DP^{-1}$$

$$A^T = PDP^{-1}$$

$$A^T = A$$

Therefore, for SVD and standard diagonalization of a matrix to give the same results, the matrix must by **symmetric**

Now, consider a symmetric matrix A. The spectral theorem states implies A is orthogonally diagonalizable.

$$A = PDP^T$$

with $P^T = P^{-1}$. Since A is symmetric, for its SVD, we have $U = V$,

$$A = U\Sigma U^T$$

Assume the SVD decomposition is different from the diagonalization. The spectral theorem guarantees a unique diagonalization for a symmetric matrix. But if the SVD is not the same, then it means there are multiple diagonalizations. This is a contradiction. Therefore, both the decompositions are the same.

Therefore, for a symmetric matrix, the SVD and standard diagonalization of a matrix give the same results.

Since we have proven both ways, we can conclude:

The SVD and Standard Diagonalization are same if and only if the matrix is symmetric.

Question 2

(a)

Kinetic Energy T = $\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2$

Potential Energy V = $\frac{1}{2}k_1x^2 + \frac{1}{2}k_1y^2 + \frac{1}{2}k_2(x-y)^2$

Lagragian $L = T - V$

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - \frac{1}{2}k_1x^2 - \frac{1}{2}k_1y^2 - \frac{1}{2}k_2(x-y)^2$$

Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For $q_i = x$

$$m_1\ddot{x} - (-k_1 x - k_2(x-y)) = 0$$

$$m_1\ddot{x} = (-k_1 - k_2)x + k_2 y$$

For $q_i = y$

$$m_2\ddot{y} - (-k_1 y + k_2(x-y)) = 0$$

$$m_2\ddot{y} = (-k_1 - k_2)y + k_2 x$$

These are the equations of motion for the system.

(b)

Yes, the solutions of this system can be mapped through eigenvalue analysis.

We can represent the above system of equations in matrix form as follows:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_1 - k_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_1-k_2}{m_2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

let $A = \begin{pmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_1-k_2}{m_2} \end{pmatrix}$. The modal frequencies

We now find eigenvalues of A

$$A = \begin{pmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_1+k_2}{m_2} \end{pmatrix}$$

$$|A - \lambda I| = 0$$

By calculating eigenvalues, we find the frequencies as

$$\omega_1 = \sqrt{-\lambda_1} \quad w_2 = \sqrt{-\lambda_2}$$

For the case of $k_1 = k_2 = k, m_1 = m_2 = m$

$$|\begin{pmatrix} -\frac{2k}{m} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} - \lambda \end{pmatrix}| = 0$$

$$\left(\frac{2k}{m} + \lambda\right)^2 = \frac{k^2}{m^2}$$

$$\frac{2k}{m} + \lambda = \pm\frac{k}{m}$$

$$\lambda_1 = -\frac{k}{m} \quad \lambda_2 = -\frac{3k}{m}$$

Therefore, the frequencies are

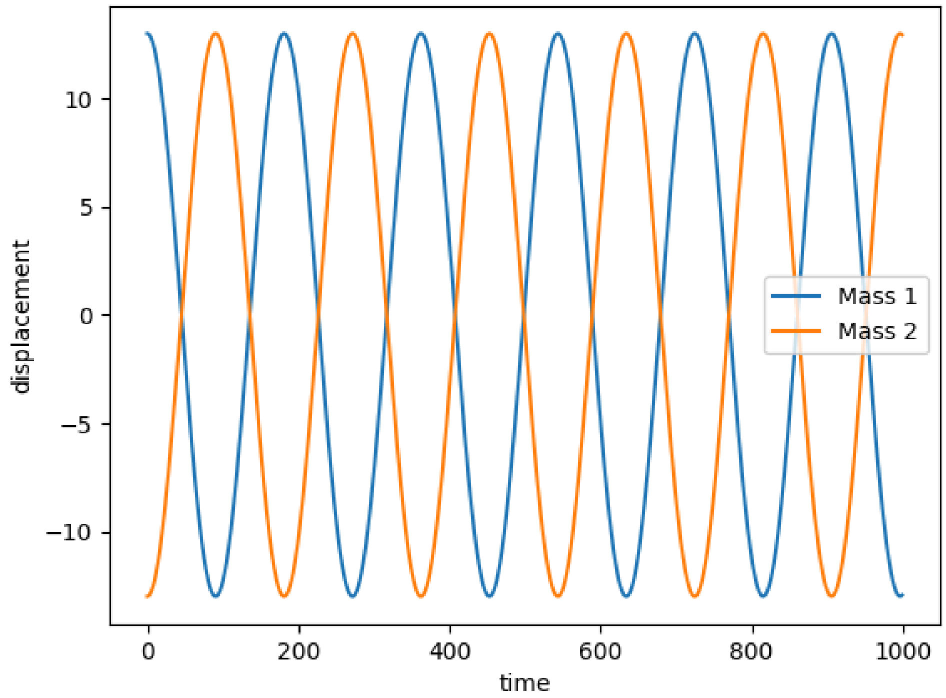
$$\omega_1 = \sqrt{\frac{k}{m}} \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

(c)

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from scipy.integrate._ivp.ivp import OdeResult

class Plotter:
    def __init__(self, r: range, x_start: float, y_start: float) -> None:
        self.t_span = (r.start, r.stop)
        self.t_eval = np.linspace(r.start, r.stop, r.step)
        self.initial_conditions = [x_start, 0.0, y_start, 0.0]

    @staticmethod
    def equations(_, y) -> list[np.float64]:
        x, v1, y, v2 = y
        return [
            v1,
            ((-2 * k) * x + k * y) / m,
            v2,
            ((-2 * k) * y + k * x) / m,
        ]

    def solve(self) -> OdeResult:
        return solve_ivp(
            self.equations,
            self.t_span,
            self.initial_conditions,
            args=(),
            t_eval=self.t_eval,
        )

    def plot(self) -> None:
        sol = self.solve()
        x1, _, x2, _ = sol.y
        plt.plot(x1, label="Mass 1")
        plt.plot(x2, label="Mass 2")
        plt.xlabel("time")
        plt.ylabel("displacement")
        plt.legend()
        plt.show()


if __name__ == "__main__":
    m = 1.0
    k = 1.0
    C = 13.0
    p = Plotter(range(0, 20, 1000), -C / 2, -C)
    p.plot()
```

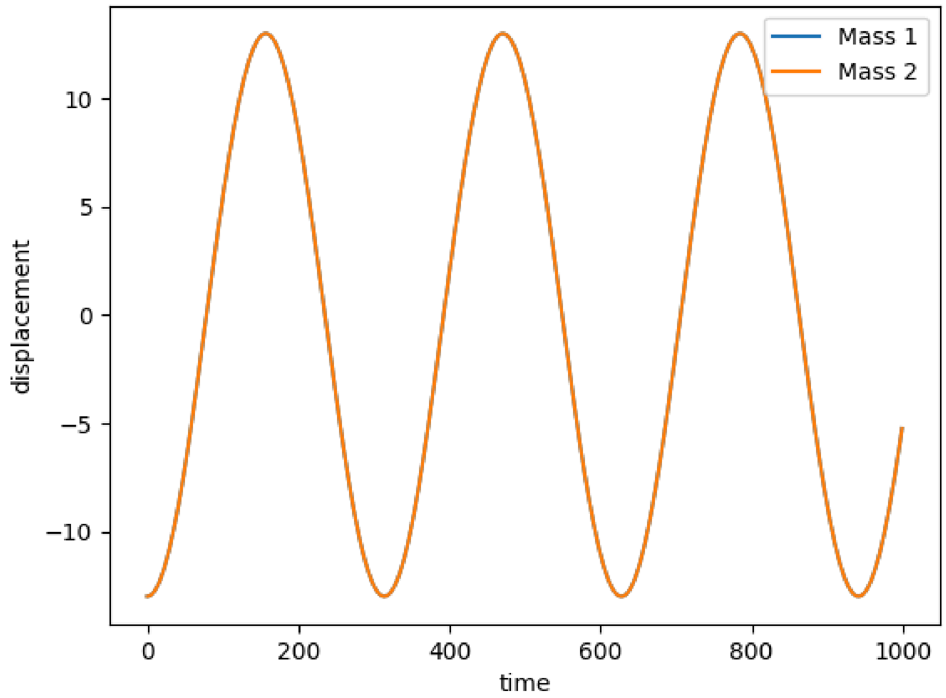
(i)

Since both the masses start from opposite ends, they have opposite phase

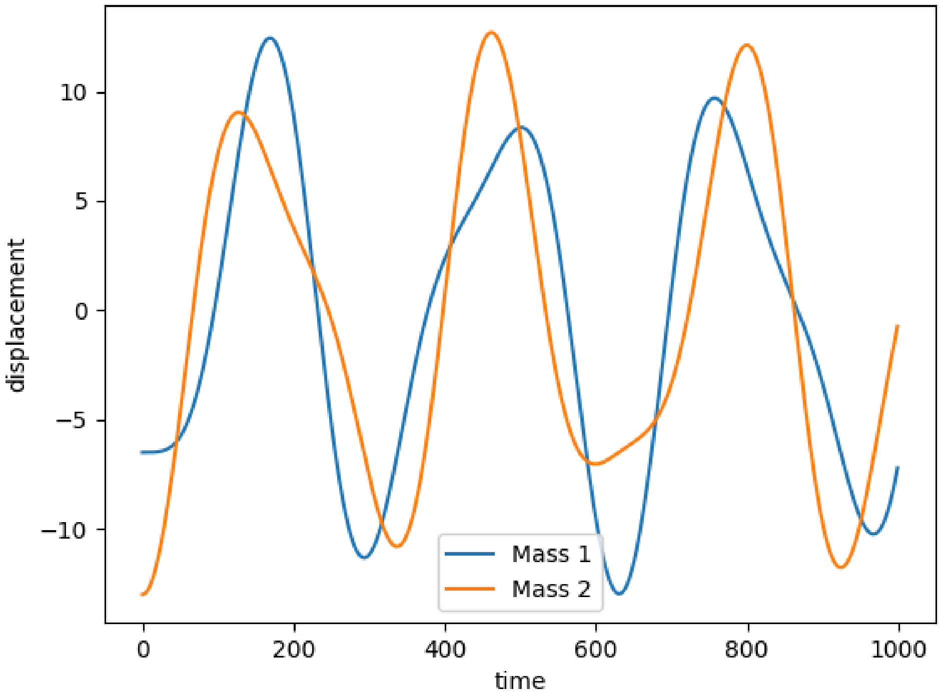


(ii)

Both the masses have the same displacement at the start, so their graphs overlap



(iii)



10
5
0
−5
−10
displacement
0
200
400
600
800
1000
time
Mass 1
Mass 2