

# Science-2

## Assignment-1

Moida Praneeth Jain, 2022101093

### Question 1

(a)

Singular Value Decomposition (SVD) of an  $m \times n$  matrix  $X$  is given by  $X = U\Sigma V^T$  where  $U$  is  $m \times m$  orthogonal matrix,  $V$  is  $n \times n$  orthogonal matrix and  $\Sigma$  is  $m \times n$  diagonal matrix.

The steps to find SVD of a matrix  $A$  are as follows:

- Calculate the eigenvalues and eigenvectors of  $AA^T$  and  $A^T A$ . Sort them in descending order of their eigenvalues
- Both of these will have the same eigenvalues (may have different eigenvectors)
- Say  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k > 0$  are the non-zero eigenvalues
- Construct the matrix

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & \dots & 0 & 0 \\ 0 & 0 & \dots & \sqrt{\lambda_r} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

by placing the square roots of the positive eigenvalues as the diagonal entries, and the rest of the values as 0.

- Construct  $U$ : The columns of  $U$  are the normalized eigenvectors of  $AA^T$ .
- Construct  $V$ : The columns of  $V$  are the normalized eigenvectors of  $A^T A$

Following these steps for the required matrix

$$A = \begin{pmatrix} 9 & 3 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$B = AA^T = \begin{pmatrix} 99 & 69 & 114 & 159 \\ 69 & 77 & 122 & 167 \\ 114 & 122 & 194 & 266 \\ 159 & 167 & 266 & 365 \end{pmatrix}$$

$$C = A^T A = \begin{pmatrix} 246 & 213 & 234 \\ 213 & 219 & 243 \\ 234 & 243 & 270 \end{pmatrix}$$

$$\lambda_1 = 706.335, \lambda_2 = 28.569, \lambda_3 = 0.096, \lambda_4 = 0$$

With corresponding eigenvectors

$$b_1 = \begin{pmatrix} -0.32 \\ -0.33 \\ -0.52 \\ -0.72 \end{pmatrix}, b_2 = \begin{pmatrix} 0.94 \\ -0.21 \\ -0.2 \\ -0.18 \end{pmatrix}, b_3 = \begin{pmatrix} 0.11 \\ 0.83 \\ 0.15 \\ -0.53 \end{pmatrix}, b_4 = \begin{pmatrix} 0 \\ 0.41 \\ -0.82 \\ 0.41 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} -0.57 \\ -0.55 \\ -0.61 \end{pmatrix}, c_2 = \begin{pmatrix} 0.82 \\ -0.34 \\ -0.45 \end{pmatrix}, c_3 = \begin{pmatrix} 0.04 \\ -0.76 \\ 0.65 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 26.57695941 & 0 & 0 \\ 0 & 5.34498755 & 0 \\ 0 & 0 & 0.31038172 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.32 & 0.94 & 0.11 & 0 \\ -0.33 & -0.21 & 0.83 & 0.41 \\ -0.52 & -0.2 & 0.15 & -0.82 \\ -0.72 & -0.18 & -0.53 & 0.41 \end{pmatrix}$$

$$V^T = \begin{pmatrix} -0.57 & -0.55 & -0.61 \\ 0.82 & -0.34 & -0.45 \\ 0.04 & -0.76 & 0.65 \end{pmatrix}$$

The calculation can alternatively be performed through the following code snippet:

```
import numpy as np
inp = np.matrix([[9, 3, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]])

U, d, V_T = np.linalg.svd(inp)
D = np.vstack(
    (*np.diag(d), *[np.zeros(V_T.shape[0]) for _ in range(U.shape[0] -
V_T.shape[0])])
)

print(U)
print(D)
print(V_T)
print(U.dot(D).dot(V_T))
```

**(b)**

Consider a matrix  $A$  that has a standard diagonalization ( $A$  is diagonalizable).  $A$  must be a square matrix.

$$A = PDP^{-1}$$

$$A = U\Sigma V^T$$

For this matrix to have the same decompositions, we must have

$$P = U \quad D = \Sigma \quad P^{-1} = V^T$$

Now, consider

$$A^T = (U\Sigma V^T)^T$$

$$A^T = V^{TT} \Sigma^T U^T$$

Since  $\Sigma$  is a diagonal matrix, we have  $\Sigma = \Sigma^T$

$$A^T = V^{TT} \Sigma U^T$$

$$A^T = (P^{-1})^T D P^T$$

Since  $U$  is orthogonal, we have  $U^T = U^{-1}$ , and since  $U = P$ ,  $P^T = P^{-1}$

$$A^T = P^{TT} D P^{-1}$$

$$A^T = P D P^{-1}$$

$$A^T = A$$

Therefore, for SVD and standard diagonalization of a matrix to give the same results, the matrix must be **symmetric**

Now, consider a symmetric matrix  $A$ . The spectral theorem states implies  $A$  is orthogonally diagonalizable.

$$A = P D P^T$$

with  $P^T = P^{-1}$ . Since  $A$  is symmetric, for its SVD, we have  $U = V$ ,

$$A = U \Sigma U^T$$

Assume the SVD decomposition is different from the diagonalization. The spectral theorem guarantees a unique diagonalization for a symmetric matrix. But if the SVD is not the same, then it means there are multiple diagonalizations. This is a contradiction. Therefore, both the decompositions are the same.

Therefore, for a symmetric matrix, the SVD and standard diagonalization of a matrix give the same results.

Since we have proven both ways, we can conclude:

**The SVD and Standard Diagonalization are same if and only if the matrix is symmetric.**

## Question 2

(a)

$$\text{Kinetic Energy } T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$\text{Potential Energy } V = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_1 y^2 + \frac{1}{2} k_2 (x - y)^2$$

$$\text{Lagrangian } L = T - V$$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k_1 x^2 - \frac{1}{2} k_1 y^2 - \frac{1}{2} k_2 (x - y)^2$$

Lagrange Equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For  $q_i = x$

$$m\ddot{x} - (-k_1x - k_2(x - y)) = 0$$

$$m\ddot{x} = (-k_1 - k_2)x + k_2y$$

For  $q_i = y$

$$m\ddot{y} - (-k_1y + k_2(x - y)) = 0$$

$$m\ddot{y} = (-k_1 - k_2)y + k_2x$$

These are the equations of motion for the system.