Science-2

Assignment-1

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Question 1

(a)

Singular Value Decomposition (SVD) of an $m \times n$ matrix X is given by $X = U\Sigma V^T$ where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix and Σ is $m \times n$ diagonal matrix.

The steps to find SVD of a matrix A are as follows:

- Calculate the eigenvalues and eigenvectors of AA^T and A^TA. Sort them in descending order of their eigenvalues
- Both of these will have the same eigenvalues (may have different eigenvectors)
- Say $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_k > 0$ are the non-zero eigenvalues
- Construct the matrix

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \sqrt{\lambda_r} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

  by placing the non-zero eigenvalues as the diagonal entries, and the rest of the values as 0.
- Construct U: The columns of U are the eigenvectors of AA^T divided by the root of their corresponding eigenvalues.
- Construct V: The columns of V are the eigenvectors of A^TA

Following these steps for the required matrix

$$A = \begin{pmatrix} 9 & 3 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

We get the following decomposition:

$$U = \begin{pmatrix} -0.32321119 & 0.93939789 & 0.11430716 & 0 \\ -0.32723941 & -0.21299656 & 0.82515463 & 0.40824829 \\ -0.522567 & -0.19752899 & 0.14573731 & -0.81649658 \\ -0.71789458 & -0.18206142 & -0.53368 & 0.40824829 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 26.57695941 & 0 & 0 \\ 0 & 5.34498755 & 0 \\ 0 & 0 & 0.31038172 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V^T = \begin{pmatrix} -0.56645958 & -0.55247881 & -0.61146603 \\ 0.82306601 & -0.34232011 & -0.45318791 \\ 0.0410596 & -0.75998954 & 0.64863704 \end{pmatrix}$$

The calculation can be performed through the following code snipped:

```
import numpy as np
inp = np.matrix([[9, 3, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]])

U, d, V_T = np.linalg.svd(inp)
D = np.vstack(
    (*np.diag(d), *[np.zeros(V_T.shape[0]) for _ in range(U.shape[0] -
V_T.shape[0])])
)

print(U)
print(D)
print(V_T)
print(U.dot(D).dot(V_T))
```

(b)

Consider a matrix A that has a standard diagonalization (A is diagonalizable). A must be a square matrix.

$$A = PDP^{-1}$$

$$A = U\Sigma V^T$$

For this matrix to have the same decompositions, we must have

$$P = U \quad D = \Sigma \quad P^{-1} = V^T$$

Now, consider

$$A^T = \left(U\Sigma V^T\right)^T$$

$$A^T = V^{T^T}\Sigma^T U^T$$

Since Σ is a diagonal matrix, we have $\Sigma = \Sigma^T$

$$A^T = V^{T^T}\Sigma U^T$$

$$A^T = \left(P^{-1}\right)^T DP^T$$

Since U is orthogonal, we have $U^T = U^{-1}$, and since $U = P$, $P^T = P^{-1}$

$$A^T = P^{T^T} DP^{-1}$$

$$A^T = PDP^{-1}$$

$$A^T = A$$

Therefore, for SVD and standard diagonalization of a matrix to give the same results, the matrix must by **symmetric**