Science-2

Assignment-1

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Question 1

(a)

Singular Value Decomposition (SVD) of an $m \times n$ matrix X is given by $X = U\Sigma V^T$ where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix and Σ is $m \times n$ diagonal matrix.

The steps to find SVD of a matrix A are as follows:

- Calculate the eigenvalues and eigenvectors of AA^T and A^TA. Sort them in descending order of their eigenvalues
- Both of these will have the same eigenvalues (may have different eigenvectors)
- Say $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_k > 0$ are the non-zero eigenvalues
- Construct the matrix

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & ... & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & ... & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & ... & 0 & 0 \\ 0 & 0 & ... & \sqrt{\lambda_r} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix}$$

  by placing the square roots of the positive eigenvalues as the diagonal entries, and the rest of the values as 0.

- Construct U: The columns of U are the normalized eigenvectors of AA^T.
- Construct V: The columns of V are the normalized eigenvectors of A^TA

Following these steps for the required matrix

$$A = \begin{pmatrix} 9 & 3 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$B = AA^T = \begin{pmatrix} 99 & 69 & 114 & 159 \\ 69 & 77 & 122 & 167 \\ 114 & 122 & 194 & 266 \\ 159 & 167 & 266 & 365 \end{pmatrix}$$

$$C = A^TA = \begin{pmatrix} 246 & 213 & 234 \\ 213 & 219 & 243 \\ 234 & 243 & 270 \end{pmatrix}$$

$$\lambda_1 = 706.335, \lambda_2 = 28.569, \lambda_3 = 0.096, \lambda_4 = 0$$

With corresponding eigenvectors

$$b_1 = \begin{pmatrix} -0.32 \\ -0.33 \\ -0.52 \\ -0.72 \end{pmatrix}, b_2 = \begin{pmatrix} 0.94 \\ -0.21 \\ -0.2 \\ -0.18 \end{pmatrix}, b_3 = \begin{pmatrix} 0.11 \\ 0.83 \\ 0.15 \\ -0.53 \end{pmatrix}, b_4 = \begin{pmatrix} 0 \\ 0.41 \\ -0.82 \\ 0.41 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} -0.57 \\ -0.55 \\ -0.61 \end{pmatrix}, c_2 = \begin{pmatrix} 0.82 \\ -0.34 \\ -0.45 \end{pmatrix}, c_3 = \begin{pmatrix} 0.04 \\ -0.76 \\ 0.65 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 26.57695941 & 0 & 0 \\ 0 & 5.34498755 & 0 \\ 0 & 0 & 0.31038172 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.32 & 0.94 & 0.11 & 0 \\ -0.33 & -0.21 & 0.83 & 0.41 \\ -0.52 & -0.2 & 0.15 & -0.82 \\ -0.72 & -0.18 & -0.53 & 0.41 \end{pmatrix}$$

$$V^T = \begin{pmatrix} -0.57 & -0.55 & -0.61 \\ 0.82 & -0.34 & -0.45 \\ 0.04 & -0.76 & 0.65 \end{pmatrix}$$

The calculation can alternatively be performed through the following code snipped:

```
import numpy as np
inp = np.matrix([[9, 3, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]])

U, d, V_T = np.linalg.svd(inp)
D = np.vstack(
    (*np.diag(d), *[np.zeros(V_T.shape[0]) for _ in range(U.shape[0] -
V_T.shape[0])])
)

print(U)
print(D)
print(V_T)
print(U.dot(D).dot(V_T))
```

(b)

Consider a matrix A that has a standard diagonalization (A is diagonalizable). A must be a square matrix.

$$A = PDP^{-1}$$

$$A = U\Sigma V^T$$

For this matrix to have the same decompositions, we must have

$$P = U \quad D = \Sigma \quad P^{-1} = V^T$$

Now, consider

$$A^T = (U\Sigma V^T)^T$$

$$A^T = V^{T^T}\Sigma^T U^T$$

Since Σ is a diagonal matrix, we have $\Sigma = \Sigma^T$

$$A^T = V^{T^T}\Sigma U^T$$

$$A^T = (P^{-1})^T DP^T$$

Since U is orthogonal, we have $U^T = U^{-1}$, and since $U = P$, $P^T = P^{-1}$

$$A^T = P^{T^T}DP^{-1}$$

$$A^T = PDP^{-1}$$

$$A^T = A$$

Therefore, for SVD and standard diagonalization of a matrix to give the same results, the matrix must by **symmetric**

Question 2

(a)

Kinetic Energy T = $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$

Potential Energy V = $\frac{1}{2}k_1x^2 + \frac{1}{2}k_1y^2 + \frac{1}{2}k_2(x-y)^2$

Lagragian $L = T - V$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k_1x^2 - \frac{1}{2}k_1y^2 - \frac{1}{2}k_2(x-y)^2$$

Lagrange Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For $q_i = x$