

# Science-2

## Assignment-1

Moida Praneeth Jain, 2022101093

### Question 1

(a)

Singular Value Decomposition (SVD) of an  $m \times n$  matrix  $X$  is given by  $X = U\Sigma V^T$  where  $U$  is  $m \times m$  orthogonal matrix,  $V$  is  $n \times n$  orthogonal matrix and  $\Sigma$  is  $m \times n$  diagonal matrix.

The steps to find SVD of a matrix  $A$  are as follows:

- Calculate the eigenvalues and eigenvectors of  $AA^T$  and  $A^T A$ . Sort them in descending order of their eigenvalues
- Both of these will have the same eigenvalues (may have different eigenvectors)
- Say  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_k > 0$  are the non-zero eigenvalues
- Construct the matrix

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & \dots & 0 & 0 \\ 0 & 0 & \dots & \sqrt{\lambda_r} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

by placing the square roots of the positive eigenvalues as the diagonal entries, and the rest of the values as 0.

- Construct  $U$ : The columns of  $U$  are the normalized eigenvectors of  $AA^T$ .
- Construct  $V$ : The columns of  $V$  are the normalized eigenvectors of  $A^T A$

Following these steps for the required matrix

$$A = \begin{pmatrix} 9 & 3 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$B = AA^T = \begin{pmatrix} 99 & 69 & 114 & 159 \\ 69 & 77 & 122 & 167 \\ 114 & 122 & 194 & 266 \\ 159 & 167 & 266 & 365 \end{pmatrix}$$

$$C = A^T A = \begin{pmatrix} 246 & 213 & 234 \\ 213 & 219 & 243 \\ 234 & 243 & 270 \end{pmatrix}$$

$$\lambda_1 = 706.335, \lambda_2 = 28.569, \lambda_3 = 0.096, \lambda_4 = 0$$

With corresponding eigenvectors

$$b_1 = \begin{pmatrix} -0.32 \\ -0.33 \\ -0.52 \\ -0.72 \end{pmatrix}, b_2 = \begin{pmatrix} 0.94 \\ -0.21 \\ -0.2 \\ -0.18 \end{pmatrix}, b_3 = \begin{pmatrix} 0.11 \\ 0.83 \\ 0.15 \\ -0.53 \end{pmatrix}, b_4 = \begin{pmatrix} 0 \\ 0.41 \\ -0.82 \\ 0.41 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} -0.57 \\ -0.55 \\ -0.61 \end{pmatrix}, c_2 = \begin{pmatrix} 0.82 \\ -0.34 \\ -0.45 \end{pmatrix}, c_3 = \begin{pmatrix} 0.04 \\ -0.76 \\ 0.65 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 26.57695941 & 0 & 0 \\ 0 & 5.34498755 & 0 \\ 0 & 0 & 0.31038172 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -0.32 & 0.94 & 0.11 & 0 \\ -0.33 & -0.21 & 0.83 & 0.41 \\ -0.52 & -0.2 & 0.15 & -0.82 \\ -0.72 & -0.18 & -0.53 & 0.41 \end{pmatrix}$$

$$V^T = \begin{pmatrix} -0.57 & -0.55 & -0.61 \\ 0.82 & -0.34 & -0.45 \\ 0.04 & -0.76 & 0.65 \end{pmatrix}$$

The calculation can alternatively be performed through the following code snippet:

```
import numpy as np
inp = np.matrix([[9, 3, 3], [4, 5, 6], [7, 8, 9], [10, 11, 12]])

U, d, V_T = np.linalg.svd(inp)
D = np.vstack(
    (*np.diag(d), *[np.zeros(V_T.shape[0]) for _ in range(U.shape[0] -
V_T.shape[0])])
)

print(U)
print(D)
print(V_T)
print(U.dot(D).dot(V_T))
```

**(b)**

Consider a matrix  $A$  that has a standard diagonalization ( $A$  is diagonalizable).  $A$  must be a square matrix.

$$A = PDP^{-1}$$

$$A = U\Sigma V^T$$

For this matrix to have the same decompositions, we must have

$$P = U \quad D = \Sigma \quad P^{-1} = V^T$$

Now, consider

$$A^T = (U\Sigma V^T)^T$$

$$A^T = V^{TT} \Sigma^T U^T$$

Since  $\Sigma$  is a diagonal matrix, we have  $\Sigma = \Sigma^T$

$$A^T = V^{TT} \Sigma U^T$$

$$A^T = (P^{-1})^T D P^T$$

Since  $U$  is orthogonal, we have  $U^T = U^{-1}$ , and since  $U = P$ ,  $P^T = P^{-1}$

$$A^T = P^{TT} D P^{-1}$$

$$A^T = P D P^{-1}$$

$$A^T = A$$

Therefore, for SVD and standard diagonalization of a matrix to give the same results, the matrix must be **symmetric**

Now, consider a symmetric matrix  $A$ . The spectral theorem states implies  $A$  is orthogonally diagonalizable.

$$A = P D P^T$$

with  $P^T = P^{-1}$ . Since  $A$  is symmetric, for its SVD, we have  $U = V$ ,

$$A = U \Sigma U^T$$

Assume the SVD decomposition is different from the diagonalization. The spectral theorem guarantees a unique diagonalization for a symmetric matrix. But if the SVD is not the same, then it means there are multiple diagonalizations. This is a contradiction. Therefore, both the decompositions are the same.

Therefore, for a symmetric matrix, the SVD and standard diagonalization of a matrix give the same results.

Since we have proven both ways, we can conclude:

**The SVD and Standard Diagonalization are same if and only if the matrix is symmetric.**

## Question 2

(a)

$$\text{Kinetic Energy } T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2$$

$$\text{Potential Energy } V = \frac{1}{2}k_1x^2 + \frac{1}{2}k_1y^2 + \frac{1}{2}k_2(x-y)^2$$

$$\text{Lagrangian } L = T - V$$

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - \frac{1}{2}k_1x^2 - \frac{1}{2}k_1y^2 - \frac{1}{2}k_2(x-y)^2$$

Lagrange Equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

For  $q_i = x$

$$m_1 \ddot{x} - (-k_1 x - k_2(x - y)) = 0$$

$$m_1 \ddot{x} = (-k_1 - k_2)x + k_2 y$$

For  $q_i = y$

$$m_2 \ddot{y} - (-k_1 y + k_2(x - y)) = 0$$

$$m_2 \ddot{y} = (-k_1 - k_2)y + k_2 x$$

These are the equations of motion for the system.

**(b)**

Yes, the solutions of this system can be mapped through eigenvalue analysis.

We can represent the above system of equations in matrix form as follows:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_1 - k_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_1 - k_2}{m_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

let  $A = \begin{pmatrix} \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_1 - k_2}{m_2} \end{pmatrix}$ . The modal frequencies

We now find eigenvalues of A

$$A = \begin{pmatrix} -\frac{k_1 + k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_1 + k_2}{m_2} \end{pmatrix}$$

$$|A - \lambda I| = 0$$

By calculating eigenvalues, we find the frequencies as

$$\omega_1 = \sqrt{-\lambda_1} \quad \omega_2 = \sqrt{-\lambda_2}$$

For the case of  $k_1 = k_2 = k, m_1 = m_2 = m$

$$\left| \begin{pmatrix} -\frac{2k}{m} - \lambda & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} - \lambda \end{pmatrix} \right| = 0$$

$$\left( \frac{2k}{m} + \lambda \right)^2 = \frac{k^2}{m^2}$$

$$\frac{2k}{m} + \lambda = \pm \frac{k}{m}$$

$$\lambda_1 = -\frac{k}{m} \quad \lambda_2 = -\frac{3k}{m}$$

Therefore, the frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

(c)

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
from scipy.integrate._ivp.ivp import OdeResult

class Plotter:
    def __init__(self, r: range, x_start: float, y_start: float) -> None:
        self.t_span = (r.start, r.stop)
        self.t_eval = np.linspace(r.start, r.stop, r.step)
        self.initial_conditions = [x_start, 0.0, y_start, 0.0]

    @staticmethod
    def equations(_, y) -> list[np.float64]:
        x, v1, y, v2 = y
        return [
            v1,
            ((-2 * k) * x + k * y) / m,
            v2,
            ((-2 * k) * y + k * x) / m,
        ]

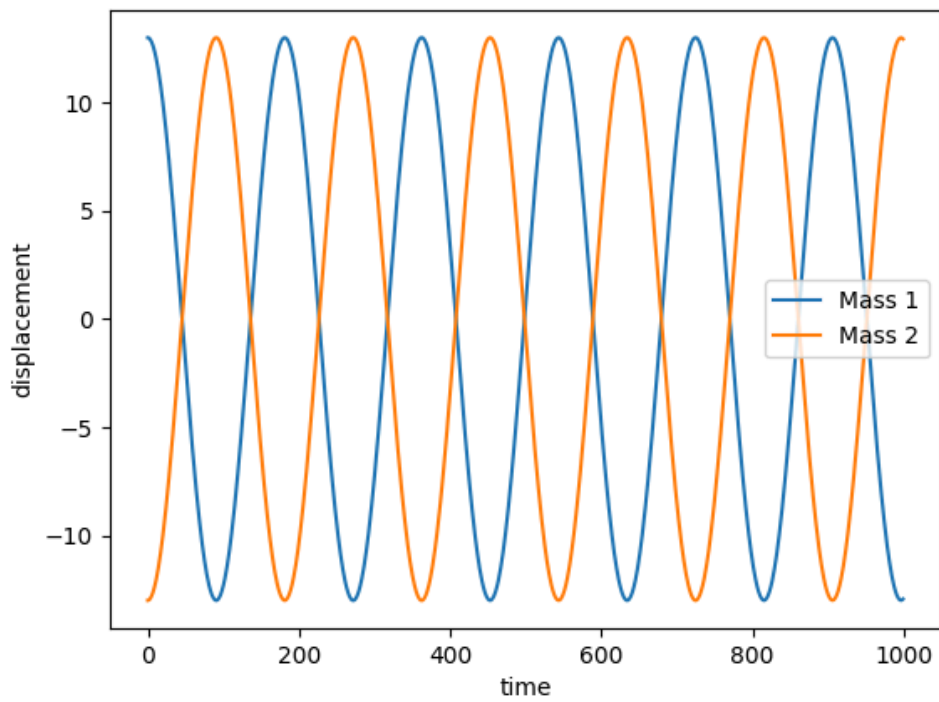
    def solve(self) -> OdeResult:
        return solve_ivp(
            self.equations,
            self.t_span,
            self.initial_conditions,
            args=(),
            t_eval=self.t_eval,
        )

    def plot(self) -> None:
        sol = self.solve()
        x1, _, x2, _ = sol.y
        plt.plot(x1, label="Mass 1")
        plt.plot(x2, label="Mass 2")
        plt.xlabel("time")
        plt.ylabel("displacement")
        plt.legend()
        plt.show()

if __name__ == "__main__":
    m = 1.0
    k = 1.0
    C = 13.0
    p = Plotter(range(0, 20, 1000), -C / 2, -C)
    p.plot()
```

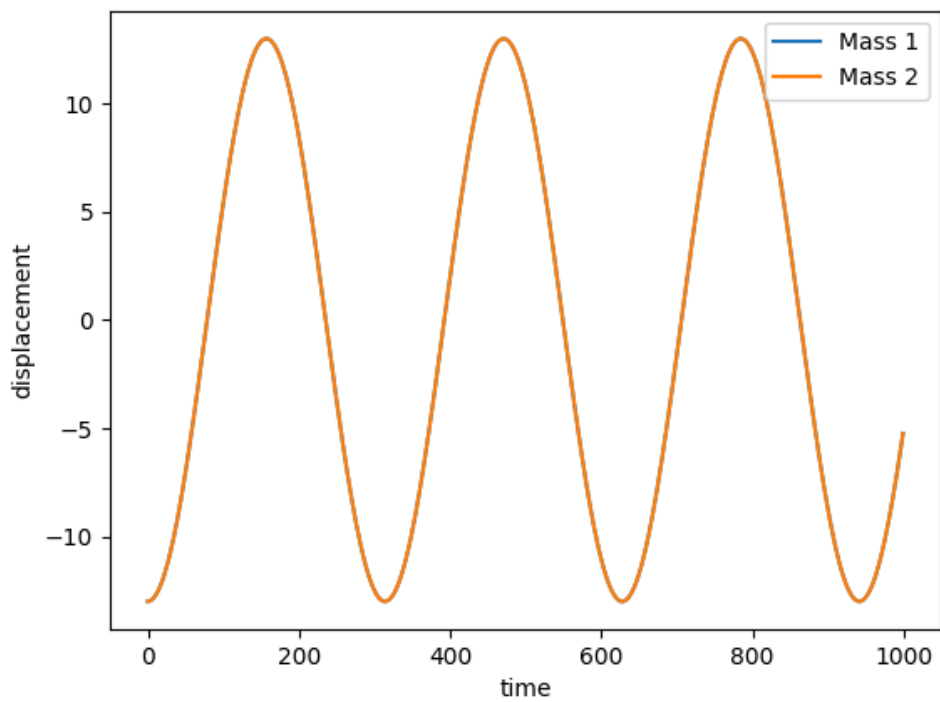
(i)

Since both the masses start from opposite ends, they have opposite phase

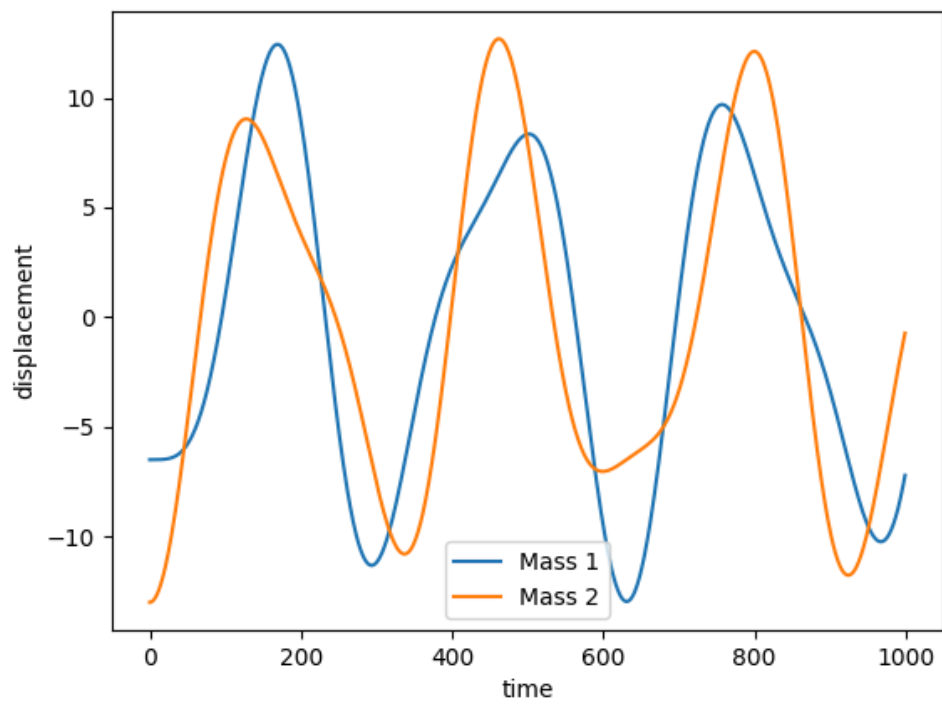


(ii)

Both the masses have the same displacement at the start, so their graphs overlap



(iii)

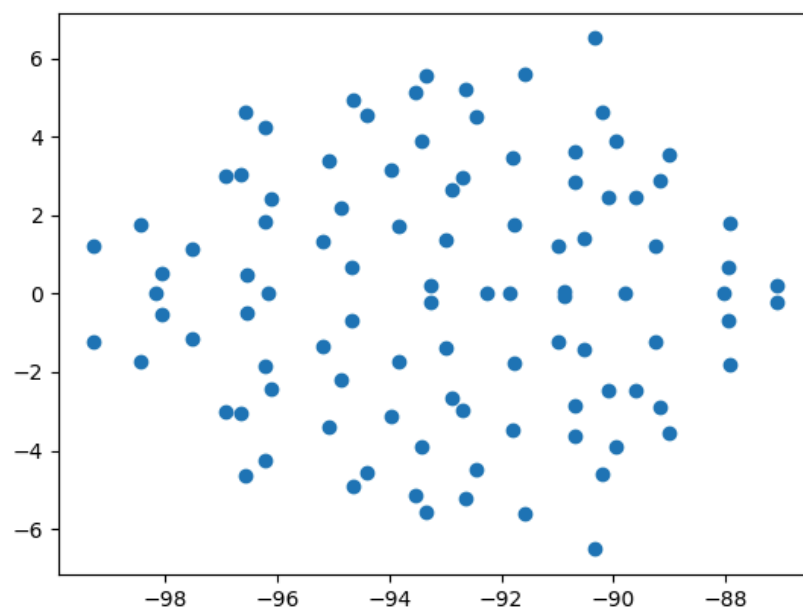


### Question 3

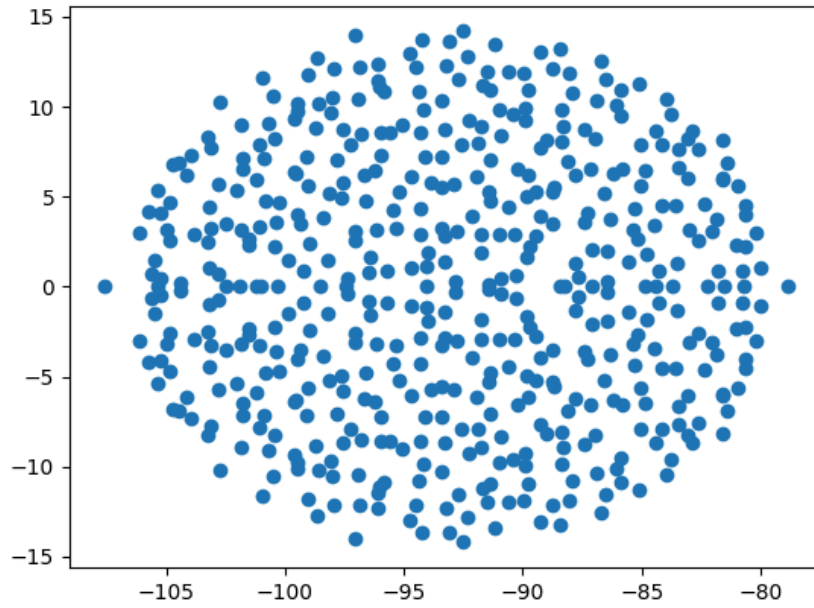
(i)

(ii) (iii)

$S = 100$



**S = 500**



**(iv)**

- $d$  is responsible for shifting the entire figure in the  $x$  axis. The diagonal values represent the center of the figure on the  $x$  axis. In this case, since  $d = 93$ , the image is centered at  $x = -93$
- $C$  is responsible for stretching the image along the  $y$ -axis. A larger  $C$  value leads to the image being stretched along the  $y$ -axis (larger maximum  $y$  and smaller minimum  $y$ ), while a smaller  $C$  value leads to compression in the  $y$ -axis.
- $S$  controls the density of the points. As  $S$  increases, the number of points in the matrix increases, so the image appears denser. As  $S$  decreases, the points appear less dense.
- $\sigma$  controls the spread of the points. Larger  $\sigma$  causes the values to spread out (equally) in all directions, causing the image to zoom out, while smaller  $\sigma$  causes the image to zoom in.