1. BASIC ARITHMETIC

Common Math Symbols

- **Number Terms**

Number	Definition	Example
Whole numbers	The set of counting numbers, including zero	0, 1, 2, 3
Natural numbers	The set of whole positive numbers except zero	1, 2, 3, 4
Integers	The set of all positive and negative whole numbers, including zero, not including fractions and decimals. Integers in a sequence, such as those in the example to the right, are called consecutive integers.	–3, –2, –1, 0, 1, 2, 3
Rational numbers	The set of all numbers that can be expressed as integers in fractions—that is, any number that can be expressed in the form $\frac{m}{n}$, where *m* and *n* are integers	$\frac{9}{10}, \frac{7}{8}, \frac{1}{2}$
Irrational numbers	The set of all numbers that cannot be expressed as integers in a fraction	π, $\sqrt{3}$, 1.010100001000110000
Real numbers	Every number on the number line, including all rational and irrational numbers	Every number you can think of

- **Even and Odd Numbers**

An <u>even number</u> is an integer that is divisible by 2 with no remainder, including zero.
Even numbers: –10, –4, 0, 4, 10
An <u>odd number</u> is an integer that leaves a remainder of 1 when divided by 2.
Odd numbers: –9, –3, –1, 1, 3, 9

The following chart shows the rules for addition, subtraction, and multiplication (multiplication and division are the same in terms of even and odd).

Addition	Subtraction	Multiplication / Division
even + even = even	even – even = even	even × even = even
even + odd = odd	even – odd = odd	even × odd = even
odd + odd = even	odd – odd = even	odd × odd = odd

Zero, as we've mentioned, is even, but it has its own special properties when used in calculations. Anything multiplied by 0 is 0, and 0 divided by anything is 0. However, anything **divided by 0 is undefined**.

- **Positive and Negative Numbers**

A positive number is greater than 0. Examples include $\frac{1}{2}$, 15, and 83.4.
A negative number is less than 0. Examples include –0.2, –1, and –100.
One tip-off is the negative sign (–) that precedes negative numbers.

- **Zero is neither positive nor negative**.

On a number line, positive numbers appear to the right of zero, and negative numbers appear to the left:

–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5

Positive and negative numbers act differently when you add, subtract, multiply, or divide them. Adding a negative number is the same as *subtracting* a positive number: 5 + (–3) = 2, just as 5 – 3 = 2

Subtracting a negative number is the same as *adding* a positive number: 7 – (–2) = 9, just as 7 + 2 = 9

To determine the sign of a number that results from multiplication or division of positive and negative numbers, memorize the following rules.

Multiplication	Division
positive × positive = positive	positive ÷ positive = positive
positive × negative = negative	positive ÷ negative = negative
negative × negative = positive	negative ÷ negative = positive

Here's a **helpful trick** when dealing with a series of multiplied or divided positive and negative numbers: If there's an even number of negative numbers in the series, the outcome will be positive. If there's an odd number, the outcome will be negative.

When negative signs and parentheses collide, it can get pretty ugly. However, the principle is simple: A negative sign outside parentheses is distributed across the parentheses. Take this question: 3 + 4 – (3 + 1 – 8) = ?

We first work out the parentheses, which gives us: 3 + 4 – (4 – 8). This can be simplified to: 3 + 4 – (– 4)

As discussed earlier, subtracting a negative number is the same as adding a positive number, so our equation further simplifies to: 3 + 4 + 4 = 11

- **Remainders**

A remainder is the integer left over after one number has been divided by another. Take, for example, 92 ÷ 6. Performing the division we see that 6 goes into 92 a total of 15 times, but 6 × 15 = 90, so there's 2 left over. In other words, the remainder is 2.

- **Divisibility**

Integer *x* is said to be divisible by integer *y* when *x* divided by *y* yields a remainder of zero.

Divisibility Rules

1	All whole numbers are divisible by 1.
2	A number is divisible by 2 if it's even.
3	A number is divisible by 3 if the sum of its digits is divisible by 3. This means you add up all the digits of the original number. If that total is divisible by 3, then so is the number. For example, to see whether 83,503 is divisible by 3, we calculate 8 + 3 + 5 + 0 + 3 = 19. 19 is not divisible by 3, so neither is 83,503. NOTE: (Short cut) If you want to reduce the addition, strike out the multiples of 3 from the number. For example, in the number 9726311, 9, 72, 63 are all multiples of 3, so no need to add them. Simply add the remaining digits: 1+1 =2. Since, 2 is not divisible by 3, the number 9726311 is also not divisible by 3.
4	A number is divisible by 4 if its last two digits, taken as a single number, are divisible by 4. For example, 179,316 is divisible by 4 because 16 is divisible by 4.

5	A number is divisible by 5 if its last digit is 0 or 5. Examples include 0, 430, and –20.
6	A number is divisible by 6 if it's divisible by both 2 and 3. For example, 663 is not divisible by 6 because it's not divisible by 2. But 570 is divisible by 6 because it's divisible by both 2 and 3 (5 + 7 + 0 = 12, and 12 is divisible by 3).
7	7 may be a lucky number in general, but it's unlucky when it comes to divisibility. Although a divisibility rule for 7 does exist, it's much harder than dividing the original number by 7 and seeing if the result is an integer. So if the GRE happens to throw a "divisible by 7" question at you, you'll just have to do the math i.e. divide the number by 7 and find the remainder.
8	A number is divisible by 8 if its last three digits, taken as a single number, are divisible by 8. For example, 179,128 is divisible by 8 because 128 is divisible by 8.
9	A number is divisible by 9 if the sum of its digits is divisible by 9. This means you add up all the digits of the original number. If that total is divisible by 9, then so is the number. For example, to see whether 531 is divisible by 9, we calculate 5 + 3 + 1 = 9. Since 9 is divisible by 9, 531 is as well. **NOTE**: The shortcut applicable to the divisibility test of 3 could be applied to the divisibility test of 9 as well, the only difference being that the multiples have to be of 9 and the sum of the remaining digits has to be divisible by 9.
10	A number is divisible by 10 if the units digit is a 0. For example, 0, 490, and –20 are all divisible by 10.
11	This one's a bit involved but worth knowing. Here's how to tell if a number is divisible by 11: Add every other digit starting with the leftmost digit and write their sum. Then add all the numbers that you *didn't* add in the first step and write their sum. If the difference between the two sums is divisible by 11, then so is the original number. For example, to test whether 803,715 is divisible by 11, we first add 8 + 3 + 1 = 12. To do this, we just started with the leftmost digit and added alternating digits. Now we add the numbers that we didn't add in the first step: 0 + 7 + 5 = 12. Finally, we take the difference between these two sums: 12 – 12 = 0. Zero is divisible by all numbers, including 11, so 803,715 is divisible by 11.
12	A number is divisible by 12 if it's divisible by both 3 and 4. For example, 663 is not divisible by 12 because it's not divisible by 4. 162,480 is divisible by 12 because it's divisible by both 4 (the last two digits, 80, are divisible by 4) and 3 (1 + 6 + 2 + 4 + 8 + 0 = 21, and 21 is divisible by 3).

- **Factors**

A factor is an integer that divides into another integer evenly, with no remainder. In other words, if $\frac{a}{b}$ is an integer, then *b* is a factor of *a*. For example, 1, 2, 4, 7, 14, and 28 are all factors of 28, because they go into 28 without having anything left over. Likewise, 3 is *not* a factor of 28 since dividing 28 by 3 yields a remainder of 1. The number 1 is a factor of every number.

Some GRE problems may require you to determine the factors of a number. To do this, write down all the factors of the given number in pairs, beginning with 1 and the number you're factoring. For example, to factor 24:

- 1 and 24 (1 × 24 = 24)
- 2 and 12 (2 × 12 = 24)
- 3 and 8 (3 × 8 = 24)
- 4 and 6 (4 × 6 = 24)

Five doesn't go into 24, so you'd move on to 6. But we've already included 6 as part of the 4 × 6 equation, and there's no need to repeat. If you find yourself beginning to repeat numbers, then the factorization's complete. The factors of 24 are therefore 1, 2, 3, 4, 6, 8, 12, and 24.

- **Prime Numbers**

They are the only numbers whose sole factors are 1 and themselves. More precisely, a prime number is a number that has exactly two positive factors, 1 and itself. For example, 3, 5, and 13 are all prime, because each is only divisible by 1 and itself. In contrast, 6 is *not* prime, because, in addition to being divisible by 1 and itself, 6 is also divisible by 2 and 3. Here are a couple of points about primes that are worth memorizing:

- All prime numbers are positive. This is because every negative number has –1 as a factor in addition to 1 and itself.
- The **number 1 is *not* prime**. Prime numbers must have two positive factors, and 1 has only one positive factor, itself. It is co-prime.
- The **number 2 is prime**. It is the only even prime number. All prime numbers besides 2 are odd.

Here's a list of the prime numbers less than 100:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97

If the number is divisible by anything other than 1 and itself, it's not prime.

If a number under consideration is larger than the ones in the list above, or if you've gone and ignored our advice to memorize that list, here's a quick way to figure out whether a number is prime:

1. Estimate the square root of the number.
2. Check all the prime numbers that fall below your estimate to see if they are factors of the number. If no prime below your estimate is a factor of the number, then the number is prime.

Let's see how this works using the number 97.

1. Estimate the square root of the number: $\sqrt{97} \approx 10$
2. Check all the prime numbers that fall below 10 to see if they are factors of 97: Is 97 divisible by 2? No, it does not end with an even number. Is 97 divisible by 2? No, it does not end with an even number. Is 97 divisible by 3? No, 9 + 7 = 16, and 16 is not divisible by 3. Is 97 divisible by 5? No, 97 does not end with 0 or 5. Is 97 divisible by 7? No, 97 ÷ 7 = 13, with a remainder of 6.

Therefore, 97 is prime.

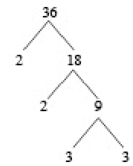
- **Prime Factorization**

A math problem may ask you to directly calculate the prime factorization of a number. Other problems, such as those involving greatest common factors or least common multiples (which we'll discuss soon), are easier to solve if you know how to calculate the prime factorization. Either way, it's good to know how to do it.

To find the prime factorization of a number, divide it and all its factors until every remaining integer is prime. The resulting group of prime numbers is the prime factorization of the original integer. Want to find the prime factorization of 36? We thought so:

$36 = 2 \times 18 = 2 \times 2 \times 9 = 2 \times 2 \times 3 \times 3$

That's two prime 2s, and two prime 3s, for those of you keeping track at home.

It can be helpful to think of prime factorization in the form of a tree: 

As you may already have noticed, there's more than one way to find the prime factorization of a number. Instead of cutting 36 into 2 and 18, you could have factored it into 6 × 6, and then continued from there. As long as you don't screw up the math, there's no wrong path—you'll always get the same result.

Let's try one more example. The prime factorization of 220 could be found like so: **220 = 10 × 22**

10 is not prime, so we replace it with **5 × 2: 10 × 22 = 2 × 5 × 22**

22 is not prime, so we replace it with **2 × 11: 2 × 5 × 22 = 2 × 2 × 5 × 11**

2, 5, and 11 are all prime, so we're done. The prime factorization of 220 is thus **2 × 2 × 5 × 11**.

- **<u>Greatest Common Factor / Divisor</u>**

The greatest common factor (GCF) of two numbers is the largest number that is a factor of both numbers—that is, the GCF is the largest factor that both numbers have in common. For example, the GCF of 12 and 18 is 6, because 6 is the largest number that divides evenly into 12 and 18. Put another way, 6 is the largest number that is a factor of both 12 and 18.

To find the GCF of two numbers, you can use their prime factorizations. The GCF is the product of all the numbers that appear in both prime factorizations. In other words, the GCF is the overlap of the two factorizations.

For example, let's calculate the GCF of 24 and 150. First, we figure out their prime factorizations:

24 = 2 × 2 × 2 × 3

150 = 2 × 3 × 5 × 5

Both factorizations contain 2 × 3. The overlap of the two factorizations is 2 and 3. The product of the overlap is the GCF. Therefore, the GCF of 24 and 150 is 2 × 3 = 6.

- **<u>Multiples</u>**

A multiple can be thought of as the opposite of a factor: If $\frac{x}{y}$ is an integer, then *x* is a multiple of *y*. Less formally, a multiple is what you get when you multiply an integer by another integer. For example, 7, 14, 21, 28, 70, and 700 are all multiples of 7, because they each result from multiplying 7 by an integer. Similarly, the numbers 12, 20, and 96 are all multiples of 4 because 12 = 4 × 3, 20 = 4 × 5, and 96 = 4 × 24. Keep in mind that zero is a multiple of every number. Also, note that any integer, *n*, is a multiple of 1 and *n*, because 1 × *n* = *n*.

- **<u>Least Common Multiple</u>**

The least common multiple (LCM) of two integers is the smallest number that is divisible by the two original integers. As with the GCF, you can use prime factorization as a shortcut to find the LCM. For example, to find the least common multiple of 10 and 15, we begin with their prime factorizations:

10 = 5 × 2

15 = 5 × 3

The LCM is equal to the product of each factor by the maximum number of times it appears in either number. Since 5 appears once in both factorizations, we need to include it once in our final product. The same goes for the 2 and the 3, since each of these numbers appears one time in each factorization. The LCM of 10 and 15, then, is 5 × 3 × 2 = 30. In other words, 30 is the smallest number that is divisible by both 10 and 15. **Remember that the LCM is the *least* common multiple—you have to choose the smallest number that is a multiple of each original number**. So, even though 60 is a multiple of both 10 and 15, 60 is not the LCM, because it's not the smallest multiple of those two numbers.

This is a bit tricky, so let's try it again with two more numbers. What's the LCM of 60 and 100?

First, find the prime factorizations:

$$60 = 2 \times 2 \times 3 \times 5$$
$$100 = 2 \times 2 \times 5 \times 5$$

So, 2 occurs twice in each of these factorizations, so we'll need to include two 2s in our final product. We have one 5 in our factorization of 60, but *two* 5s in our factorization of 100. Since we're looking to include the *maximum* number of appearances of each factor, we'll include two 5s in our product. There's also one 3 in the first factorization, and no 3s in the second, so we have to add one 3 to the mix. This results in an LCM of 2 × 2 × 3 × 5 × 5 = 300.

LCM of two numbers, n1 and n2 could be found as: ————————

- **Order of Operations**

What if you see something like this on the test: $\frac{(18-3)\times 2^1}{5} - 7 + (6\times 3 - 1) =$

PEMDAS is an acronym for the order in which mathematical operations should be performed as you move from left to right through an expression or equation. It stands for:

Parentheses **E**xponents **M**ultiplication **D**ivision **A**ddition **S**ubtraction

You may have had PEMDAS introduced to you as "Please Excuse My Dear Aunt Sally." It is also called the **BODMAS** rule (**B**rackets **O**pen **D**ivision **M**ultiplication **A**ddition **S**ubtraction)

If an equation contains any or all of these PEMDAS elements, first carry out the math within the parentheses, then work out the exponents, then the multiplication, and the division. Addition and subtraction are actually a bit more complicated. When you have an equation to the point that it only contains addition and subtraction, **perform each operation moving from left to right across the equation.** Let's see how this all plays out in the context of the example above: $\frac{(18-3)\times 2^1}{5} - 7 + (6\times 3 - 1) =$

First work out the math in the parentheses, following PEMDAS even *within* the parentheses. So here we focus on the second parentheses and do the multiplication before the subtraction: $\frac{(18-3)\times 2^1}{5} - 7 + (18 - 1)$

Now taking care of the subtraction in both sets of parentheses: $\frac{15\times 2^1}{5} - 7 + 17$

Now work out the exponent (more on those later): $\frac{15\times4}{5}-7+17$

Then do the multiplication: $\frac{60}{5}-7+17$

Then the division: 12 – 7 + 17

We're left with just addition and subtraction, so we simply work from left to right: 5 + 17

And finally: 22

PEMDAS is the way to crunch down the most difficult-looking equations or expressions. Take it one step at a time, and you'll do just fine.

- **Fractions**

A fraction is a part of a whole. It's composed of two expressions, a **numerator and a denominator**. The numerator of a fraction is the quantity above the fraction bar, and the denominator is the quantity below the fraction bar. For example, in the fraction $\frac{1}{2}$, 1 is the numerator and 2 is the denominator. The denominator tells us how many units there are in all, while the numerator tells us how many units out of that total are specified in a given instance. The general concept of fractions isn't difficult, but things can get dicey when you have to do things with them. Hence, the following subtopics that you need to have under your belt.

- **Fraction Equivalencies**

Fractions represent a part of a whole, so if you increase both the part and whole by the same multiple, you will not change the relationship between the part and the whole.

To determine if two fractions are equivalent, multiply the denominator and numerator of one fraction so that the denominators of the two fractions are equal (this is one place where knowing how to calculate LCM and GCF comes in handy). For example, because if you multiply the numerator and denominator of $\frac{1}{2}$ by 3, you get: $\frac{1\times3}{2\times3}=\frac{3}{6}$. As long as you multiply or divide both the numerator and denominator of a fraction by the same nonzero number, you will not change the overall value of the fraction.

- **Reducing Fractions**

Reducing fractions makes life simpler, and we all know life is complicated enough without crazy fractions weighing us down. Reducing takes unwieldy monsters like and makes them into smaller, friendlier critters. To reduce a fraction to its lowest terms, divide the numerator and denominator by their GCF. For example, for $\frac{450}{600}$, the GCF of 450 and 600 is 150. So the fraction reduces down to $\frac{3}{4}$, since $450\div150=3$ and $600\div150=4$.

A fraction is in its simplest, totally reduced form when the GCF of its numerator and denominator is 1. There is no number but 1, for instance, that can divide into both 3 and 4, so $\frac{3}{4}$ is a fraction in its lowest form, reduced as far as it can go. The same goes for the fraction $\frac{3}{5}$, but $\frac{3}{6}$ is a different story because 3 is a common factor of both the numerator and denominator. Dividing each by this common factor yields $\frac{1}{2}$, the fraction in its most reduced form.

- **Adding, Subtracting, and Comparing Fractions**

To add fractions with the same denominators, all you have to do is add up the numerators and keep the denominator the same: $\frac{1}{20}+\frac{3}{20}+\frac{13}{20}=\frac{17}{20}$

Subtraction works similarly. If the denominators of the fractions are equal, just subtract one numerator from the other and keep the denominator the same: $\frac{13}{20}-\frac{2}{20}=\frac{11}{20}$

Remember that fractions can be negative too: $\frac{2}{20}-\frac{13}{20}=\frac{-11}{20}$

Some questions require you to compare fractions. Again, this is relatively straightforward when the denominators are the same. The fraction with the greater numerator will be the larger fraction. For example, $\frac{13}{27}$ is greater than $\frac{5}{27}$, while $\frac{-5}{27}$ is greater than $\frac{-13}{27}$. (Be careful of those negative numbers! Since –5 is less *negative* than –13, –5 is greater than –13.)

Working with fractions with the same denominators is one thing, but working with fractions with different denominators is quite another. So we came up with an easy alternative: the Magic X. For adding, subtracting, and comparing fractions with different denominators, the Magic X is a lifesaver. ***Adding.*** Consider the following equation: $\frac{3}{7}+\frac{2}{9}=?$

You could try to find the common denominator by multiplying $\frac{3}{7}$ by 9 and $\frac{2}{9}$ by 7, but then you'd be working with some pretty big numbers. Keep things simple, and use the Magic X. The key is to multiply *diagonally and up*, which in this case means from the 9 to the 3 and also from the 7 to the 2:

$$\begin{array}{ccc} 9\times 3=27 & & 7\times 2=14 \\ \frac{3}{7} & \times & \frac{2}{9} \end{array}$$

In an addition problem, we add the products to get our numerator: 27 + 14 = 41. For the denominator, we simply multiply the two denominators to get:

$$\begin{array}{c} \frac{3}{7} \longrightarrow \frac{2}{9} \\ 7\times 9=63 \end{array}$$

The numerator is 41, and the denominator is 63, which results in a final answer of $\frac{41}{63}$.

Subtracting. Same basic deal, except this time we subtract the products that we get when we multiply diagonally and up. See if you can feel the magic in this one: $\frac{4}{5}-\frac{5}{6}=?$

Multiplying diagonally and up gives:

$$\begin{array}{ccc} 6\times 4=24 & & 5\times 5=25 \\ \frac{4}{5} & \times & \frac{5}{6} \end{array}$$

The problem asks us to subtract fractions, so this means we need to *subtract* these numbers to get our numerator: 24 – 25 = –1. Just like in the case of addition, we multiply across the denominators to get the denominator of our answer:

$$\begin{array}{c} \frac{4}{5} \longrightarrow \frac{5}{6} \\ 5\times 6=30 \end{array}$$

That's it! The numerator is –1 and the denominator is 30, giving us an answer of $\frac{-1}{30}$. Not the prettiest number you'll ever see, but it'll do.

Comparing. The Magic X is so magical that it can also be used to compare two fractions, with just a slight modification: **omitting the step where we multiply the denominators**. Say you're given the following Quantitative Comparison problem.

Column A	Column B
$\frac{5}{23}$	$\frac{7}{30}$

Now, if you were a mere mortal with no magic at your fingertips, this would be quite a drag. But the Magic X makes it a pleasure. Again, begin by multiplying diagonally and up:

$$30 \times 5 = 150 \qquad 23 \times 7 = 161$$
$$\frac{5}{23} \quad \frac{7}{30}$$

Now compare the numbers you get: 161 is larger than 150, so $\frac{7}{30}$ is greater than $\frac{5}{23}$. Done!

Learn how to employ the Magic X in these three circumstances, and you're likely to save yourself some time and effort.

- **<u>Multiplying Fractions</u>**

Multiplying fractions is a breeze, whether the denominators are equal or not. The product of two fractions is merely the product of their numerators over the product of their denominators: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

Want an example with numbers? You got one: $\frac{3}{7} \times \frac{2}{5} = \frac{3 \times 2}{7 \times 5} = \frac{6}{35}$

Canceling Out. You can make multiplying fractions even easier by canceling out. If the numerator and denominator of any of the fractions you need to multiply share a common factor, you can divide by the common factor to reduce both numerator and denominator before multiplying. For example, consider this fraction multiplication problem:

$$\frac{4}{5} \times \frac{1}{8} \times \frac{10}{11}$$

You could simply multiply the numerators and denominators and then reduce, but that would take some time. Canceling out provides a shortcut. We can cancel out the numerator 4 with the denominator 8 and the numerator 10 with the denominator 5, like this:

$$\frac{\cancel{4}^{1}}{\cancel{5}_{1}} \times \frac{1}{\cancel{8}_{2}} \times \frac{\cancel{10}^{2}}{11} = \frac{1}{1} \times \frac{1}{2} \times \frac{2}{11}$$

Then, canceling the 2s, you get: $\frac{1}{1} \times \frac{1}{\cancel{2}_{1}} \times \frac{\cancel{2}^{1}}{11} = \frac{1}{1} \times \frac{1}{1} \times \frac{1}{11} = \frac{1}{11}$

Canceling out can dramatically cut the amount of time you need to spend working with big numbers. When dealing with fractions, whether they're filled with numbers or variables, always be on the lookout for chances to cancel out.

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- **Dividing Fractions**

Multiplication and division are inverse operations. It makes sense, then, that to perform division with fractions, all you have to do is flip the second fraction and then multiply. Check it out: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

Here's a numerical example:

$$\frac{1}{2} \div \frac{4}{5} = \frac{1}{2} \times \frac{5}{4} = \frac{5}{8}$$

Compound Fractions. Compound fractions are nothing more than division problems in disguise. Here's an example of a compound fraction: $\frac{\frac{9}{5}}{\frac{3}{10}}$

It looks intimidating, sure, but it's really only another way of writing $\frac{9}{5} \div \frac{3}{10}$, which now looks just like the previous example. Again, the rule is to invert and multiply. Take whichever fraction appears on the bottom of the compound fraction, or whichever fraction appears second if they're written in a single line, and flip it over. Then multiply by the other fraction. In this case, we get $\frac{9}{5} \times \frac{10}{3}$. Now we can use our trusty canceling technique to reduce this to $\frac{3}{1} \times \frac{2}{1}$, or plain old 6. A far cry from the original!

- **Mixed Numbers**

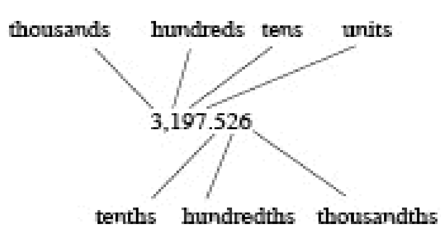
It concerns fractions mixed with integers. Specifically, a mixed number is an integer followed by a fraction, like $1\frac{2}{3}$. But operations such as addition, subtraction, multiplication, and division can't be performed on mixed numbers, so you have to know how to convert them into standard fraction form.

The method is easy: Multiply the integer (the 1) of the mixed number by the denominator of the fraction part, and add that product to the numerator: 1 × 3 + 2 = 5. This will be the numerator. Now, put that over the original denominator, 3, to finalize the converted fraction: $\frac{5}{3}$.

Let's try a more complicated example: $3\frac{2}{13} = \frac{(3 \times 13) + 2}{13} = \frac{39 + 2}{13} = \frac{41}{13}$

- **Decimals**

A decimal is any number with a nonzero digit to the right of the decimal point. Like fractions, decimals are a way of writing parts of wholes. Some GRE questions ask you to identify specific digits in a decimal, so you need to know the names of these different digits. In this case, a picture is worth a thousand (that is, 1000.00) words:



Notice that all of the digits to the right of the decimal point have a *th* in their names.

In the number 839.401, for example, here are the values of the different digits.

Left of the decimal point	Right of the decimal point
Units: 9	Tenths: 4
Tens: 3	Hundredths: 0
Hundreds: 8	Thousandths: 1

- **Converting Fractions to Decimals**

So, what if a problem contains fractions, but the answer choices are all decimals? In that case, you'll have to convert whatever fractional answer you get to a decimal. A fraction is really just shorthand for division. For example, $\frac{6}{15}$ is exactly the same as 6 ÷ 15. Dividing this out on your scratch paper results in its decimal equivalent, .4.

- **Converting Decimals to Fractions**

What comes around goes around. If we can convert fractions to decimals, it stands to reason that we can also convert decimals to fractions. Here's how:

1. Remove the decimal point and make the decimal number the numerator.
2. Let the denominator be the number 1 followed by as many zeros as there are decimal places in the original decimal number.
3. Reduce this fraction if possible.

Let's see this in action. To convert .3875 into a fraction, first eliminate the decimal point and place 3875 as the numerator: $.3875 = \frac{3{,}875}{?}$

Since .3875 has four digits after the decimal point, put four zeros in the denominator following the number 1:

$$.3875 = \frac{3{,}875}{10{,}000}$$

We can reduce this fraction by dividing the numerator and denominator by the GCF, which is 125, or, if it's too difficult to find the GCF right off the bat, we can divide the numerator and denominator by common factors such as 5 or 25.

- **Exponents**

An exponent is a shorthand way of saying, "**Multiply this number by itself this number of times**." In a^b, a is multiplied by itself b times. Here's a numerical example: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$. An exponent can also be referred to as a power: 25 is "two to the fifth power" or simply "two raised to five." Before jumping into the exponent nitty-gritty, learn these five terms:

- **Base:** The base refers to the 3 in 3^5. In other words, the base is the number multiplied by itself, however many times specified by the exponent.
- **Exponent:** The exponent is the 5 in 3^5. The exponent tells how many times the base is to be multiplied by itself.
- **Squared:** Saying that a number is squared is a common code word to indicate that it has an exponent of 2. In the expression 6^2, 6 has been squared.
- **Cubed:** Saying that a number is cubed means it has an exponent of 3. In the expression 4^3, 4 has been cubed.

- **Power:** The term power is another way to talk about a number being raised to an exponent. A number raised to the third power has an exponent of 3. So 6 raised to the third power is 6^3.

❖ Common Exponents

It can be very helpful and a real time saver on the GRE if you can easily translate back and forth between a number and its exponential form. For instance, if you can easily see that $36 = 6^2$, it can really come in handy when you're dealing with binomials, quadratic equations, and a number of other algebraic topics we'll cover later. Below are some lists of common exponents.

Squares	Cubes	Powers of 2
We'll start with the squares of the first ten integers:	Here are the first ten cubes:	Finally, the powers of 2 up to 10 are useful to know for various applications:
$1^2 = 1$	$1^3 = 1$	$2^0 = 1$
$2^2 = 4$	$2^3 = 8$	$2^1 = 2$
$3^2 = 9$	$3^3 = 27$	$2^2 = 4$
$4^2 = 16$	$4^3 = 64$	$2^3 = 8$
$5^2 = 25$	$5^3 = 125$	$2^4 = 16$
$6^2 = 36$	$6^3 = 216$	$2^5 = 32$
$7^2 = 49$	$7^3 = 343$	$2^6 = 64$
$8^2 = 64$	$8^3 = 512$	$2^7 = 128$
$9^2 = 81$	$9^3 = 729$	$2^8 = 256$
$10^2 = 100$	$10^3 = 1000$	$2^9 = 512$
		$2^{10} = 1{,}024$

❖ Adding and Subtracting Exponents

The rule for adding and subtracting values with exponents is pretty simple : **Just *Don't* Do It.**

This doesn't mean that you won't see such addition and subtraction problems; it just means that you can't simplify them. For example, the expression **$2^{15} + 2^7$ does *not* equal 2^{22}**. The expression $2^{15} + 2^7$ is written as simply as possible, so don't make the mistake of trying to simplify it further. If the problem is simple enough, then work out each exponent to find its value, then add the two numbers. For example, to add $3^3 + 4^2$, work out the exponents to get $(3 \times 3 \times 3) + (4 \times 4) = 27 + 16 = 43$.
However, if you're dealing with algebraic expressions that have the same base variable and exponents, then you can add or subtract them. For example, $3x^4 + 5x^4 = 8x^4$. The base variables are both x, and the exponents are both 4, so we can add them. **Just remember that expressions that have different bases or exponents cannot be added or subtracted**.

❖ Multiplying and Dividing Exponents with Equal Bases

Multiplying or dividing exponential numbers or terms that have the same base is so quick and easy it's like a little math oasis. When multiplying, just add the exponents together. This is known as the Product Rule:

$$3^6 \times 3^2 = 3^{(6+2)} = 3^8$$

$$x^4 \times x^3 = x^{(4+3)} = x^7$$

To divide two same-base exponential numbers or terms, subtract the exponents. This is known as the Quotient Rule:

$$\frac{3^6}{3^2} = 3^{(6-2)} = 3^4$$
$$\frac{x^4}{x^3} = x^{(4-3)} = x^1$$

- **Multiplying and Dividing Exponents with Unequal Bases**

The same isn't true if you need to multiply or divide two exponential numbers that *don't* have the same base, such as, say, $3^3 \times 4^2$. When two exponents have different bases, you just have to do your work the old-fashioned way: Multiply the numbers out and multiply or divide the result accordingly: $3^3 \times 4^2 = 27 \times 16 = 432$.

There is, however, one trick you should know. Sometimes when the bases aren't the same, it's still possible to simplify an expression or equation if one base can be expressed in terms of the other. For example: $\mathbf{2^5 \times 8^4}$

Even though 2 and 8 are different bases, 8 can be rewritten as a power of 2; namely, $8 = 2^3$. This means that we can replace 8 with 2^3 in the original expression: $\mathbf{2^5 \times (2^3)^4}$

Since the base is the same for both values, we can simplify this further, but first we're going to need another rule to deal with the $(2^3)^4$ term. This is called . . .

- **Raising an Exponent to an Exponent**

To raise one exponent to another exponent (also called taking the power of a power), simply multiply the exponents.

This is known as the Power Rule:
$$(3^2)^4 = 3^{(2\times4)} = 3^8$$
$$(x^4)^3 = x^{(4\times3)} = x^{12}$$

Let's use the Power Rule to simplify the expression that we were just working on: $2^5 \times (2^3)^9 = 2^5 \times 2^{3\times9} = 2^5 \times 2^{27}$

Our "multiplication with equal bases rule" tells us to now add the exponents, which yields: $2^5 \times 2^{27} = 2^{32}$

2^{32} is a pretty huge number, and the GRE would *never* have you calculate out something this large. This means that you can leave it as 2^{32}, because that's how it would appear in the answer choices.

To Recap: Multiply the exponents when raising one exponent to another, and add the exponents when multiplying two identical bases with exponents. The test makers expect lots of people to mix these operations up, and they're usually not disappointed.

- **Fractions Raised to an Exponent**

To raise a fraction to an exponent, raise both the numerator and denominator to that exponent:

$$\left(\frac{1}{3}\right)^3 = \frac{1^3}{3^3} = \frac{1\times1\times1}{3\times3\times3} = \frac{1}{27}$$

- **Negative Numbers Raised to an Exponent**

When you multiply a negative number by another negative number, you get a positive number, and when you multiply a negative number by a positive number, you get a negative number. Since exponents result in multiplication, a negative number raised to an exponent follows these rules:

- **A negative number raised to an even exponent will be positive.** For example, $(-2)^4 = 16$. Why? Because $(-2)^4$ means $-2 \times -2 \times -2 \times -2$. When you multiply the first two –2s together, you get positive 4 because you're multiplying two negative numbers. When you multiply the +4 by the next –2, you get –8, since you're multiplying a positive number by a negative number. Finally, you multiply the –8 by the last –2 and get +16, since you're once again multiplying two negative numbers. The negatives cancel themselves out and vanish.
- **A negative number raised to an odd exponent will be negative.** To see why, just look at the example above, but stop the process at -2^3, which equals –8.

❖ Special Exponents

It's helpful to know a few special types of exponents for the GRE.

▪ ZERO

Any base raised to the power of zero is equal to 1.

$123^0 = 1$; $0.8775^0 = 1$; a million trillion gazillion$^0 = 1$

You should also know that 0 raised to any positive power is 0. For example: **$0^1 = 0$** and **$0^{73} = 0$**

▪ ONE

Any base raised to the power of 1 is equal to itself: $2^1 = 2$, $-67^1 = -67$, and $x^1 = x$. This fact is important to know when you have to multiply or divide exponential terms with the same base: $3x^6 \times x = 3x^6 \times x^1 = 3x^{(6+1)} = 3x^7$

The number 1 raised to any power is 1. Be it **$1^2 = 1$** or **$1^{4,000} = 1$**

▪ NEGATIVE EXPONENTS

Any number or term raised to a negative power is equal to the reciprocal of that base raised to the opposite power. An example will make it clearer: $x^{-5} = \frac{1}{x^5}$

Here's a more complicated example: $\left(\frac{2}{3}\right)^{-3} = \left(\frac{1}{\frac{2}{3}}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

Here's an English translation of the rule: If you see a base raised to a negative exponent, put the base as the denominator under a numerator of 1 and then drop the negative from the exponent. From there, just simplify.

▪ FRACTIONAL EXPONENTS

Exponents can be fractions too. When a number or term is raised to a fractional power, it is called taking the root of that number or term. This expression can be converted into a more convenient form: $x^{\left(\frac{a}{b}\right)} = \sqrt[b]{x^a}$

The $\sqrt{\ }$ symbol is known as the *radical sign*, and anything under the radical is called the radicand. But first let's look at an example with real numbers:

$8^{\left(\frac{2}{3}\right)} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$, because $4 \times 4 \times 4 = 64$. Here we treated the 2 as an ordinary exponent and wrote the 3 outside the radical.

▪ Roots and Radicals

Usually the test makers will ask you to simplify roots and radicals.

As with exponents, though, you'll also need to know when such expressions *can't* be simplified.

Square roots require you to find the number that, when multiplied by itself, equals the number under the radical sign. A few examples: $\sqrt{25}$ **= 5, because 5 × 5 = 25**

$\sqrt{100}$ **= 10, because 10 × 10 = 100**

$\sqrt{1}$ **= 1, because 1 × 1 = 1**

$\sqrt{\frac{1}{4}}$ **=** $\frac{1}{2}$**, because** $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Here's another way to think about square roots: **if $x^n = y$, then** $\sqrt[n]{y}$ **= x**

If you take the square root of a variable, however, the answer could be positive or negative.
For example, if you solve $x^2 = 100$ by taking the square root of both sides, x could be 10 or –10. Both values work because 10 × 10 = 100 and –10 × –10 = 100 (recall that a negative times a negative is a positive).

These are similar to square roots, but the number of times the final answer must be multiplied by itself will be three or more. You'll always be able to determine the number of multiplications required from the little number outside the radical, as in this example: $\sqrt[3]{8}$ **= 2, because 2 × 2 × 2 = 8**

Here the little 3 indicates that the correct answer must be multiplied by itself a total of three times to equal 8.

A few more examples: $\sqrt[3]{27}$ **= 3, because 3 × 3 × 3 = 27**

$\sqrt[4]{625}$ **= 5, because 5 × 5 × 5 × 5 = 625**

$\sqrt[4]{1}$ **= 1, because 1 × 1 × 1 × 1 = 1**

- **Simplifying Roots**

Roots can only be simplified when you're multiplying or dividing them. Equations that add or subtract roots cannot be simplified. That is, you can't add or subtract roots. You have to work out each root separately and then perform the operation. For example, to solve $\sqrt{9}+\sqrt{4}=?$, do not add the 9 and 4 together to get $\sqrt{13}$. Instead, $\sqrt{9}+\sqrt{4}=3+2=5$.

You can multiply or divide the numbers under the radical sign as long as the roots are of the same degree—that is, both square roots, both cube roots, etc. You cannot multiply, for example, a square root by a cube root. Here's the rule in general form: $\sqrt[n]{x}\times\sqrt[n]{y}=\sqrt[n]{x\times y}$

Here are some examples with actual numbers. We can simplify the expressions below because every term in them is a square root. To simplify multiplication or division of square roots, combine everything under a single radical sign.

$$\sqrt{12}\times\sqrt{3}=\sqrt{12\times 3}=\sqrt{36}=6$$
$$\sqrt{50}\times\sqrt{2}=\sqrt{50\times 2}=\sqrt{100}=10$$
$$\frac{\sqrt{32}}{\sqrt{2}}=\sqrt{\frac{32}{2}}=\sqrt{16}=4$$

You can also use this rule in reverse. That is, a single number under a radical sign can be split into two numbers whose product is the original number. For example: $\sqrt{200}=\sqrt{100\times 2}=\sqrt{100}\times\sqrt{2}=10\sqrt{2}$

The reason we chose to split 200 into 100 × 2 is because it's easy to take the square root of 100, since the result is an integer, 10. The goal in simplifying radicals is to get as much as possible out from under the radical sign. When splitting up square roots this way, try to think of the largest perfect square that divides evenly into the original number. Here's another example: $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$

It's important to remember that as you've seen earlier, you can't add or subtract roots. You have to work out each root separately and then add (or subtract). For example, to solve $\sqrt{25} + \sqrt{9}$, you cannot add 25 + 9 and put 34 under a radical sign. Instead, it is: 5 + 3 = 8.

- **Absolute Value**

The absolute value of a number is the distance that number is from zero, and it's indicated with vertical bars, like this: |8|. Absolute values are always positive or zero—never negative. So, the absolute value of a positive number is that number: |8| = 8. The absolute value of a negative number is the number without the negative sign: |–12| = 12. Here are some other examples: |5| = 5; | –4.234 | = 4.234; $\left|\frac{3}{7}\right| = \frac{3}{7}$; $\left|\frac{-1}{4}\right| = \frac{1}{4}$; |0| = 0

It is also possible to have expressions within absolute value bars: 3 – 2 + |3 – 7|
Think of absolute value bars as parentheses. Do what's inside them first, then tackle the rest of the problem. You can't just make that –7 positive because it's sitting between absolute value bars. You have to work out the math first: 3 – 2 + | –4 |. Now you can get rid of the bars and the negative sign from that 4. Thus, we get: 3 – 2 + 4 = 5

Some important conversions

1 cm = 10 mm **1 inch = 2.5 cms** **1 foot = 12 inches = 30 cms**
1 yard = 3 feet = 36 inches = 90 cms (approximately) **1 mile = 1.6 kms = 1760 yards = 5280 feet**
16 ounces = 1 pound (lb) **1 pound (lb) = 0.454 kg = 454 gms** **1 kg = 2.2 pound (lbs)**
1 dollar = 100 cents **1 nickel = 5 cents** **1 dime = 10 cents** **1 quarter= 25 cents**

EXERCISE

1. In many numbers between 100 & 200 does the digit 3 appear?
a) 10 b) 11 c) 19 d) 20 e) 21

2. The value of $32^{0.16} \times 32^{0.04}$ is
a) $2\sqrt{2}$ b) $\sqrt{2}$ c) 0.2 d) 2 e) $4\sqrt{2}$

3. The square root of 0.9 can be
a) 0.03 b) 0.3 c) 0.003 d) 0.081 e) 0.94

4. The difference between the largest and the smallest of the fractions 6/11, 3/8 and 11/16 is
a) 3 /8 b) 5/16 c) 15 / 88 d) 25/ 176 e) 5/8

5. $\sqrt{3} + \sqrt{2} = 3.146$ then $1/(\sqrt{3} - \sqrt{2})$ is
a) 3.614 b) 3.641 c) 3.146 d) 0.316 e) 0.3146

6. The greatest number that will divide 728 and 901 leaving remainders of 8 and 5 is
a) 19 b) 18 c) 17 d) 16 e) 15

7. $10^9 - 10$ is not divisible by
a) 11 b) 10 c) 9 d) 4 e) None of these

8. If seven numbers, each divisible by 4 are added, the sum will be divided by divisible by
a) 7 b) 4 c) 28 d) 5 e) All of these

9. How many numbers between 100 and 300 are divisible by both 7 and 11?
a) 3 b) 4 c) 5 d) 2 e) 1

10. Three bells begin tolling at the same time and continue to do so at intervals of 21, 28 and 30 seconds respectively. After how many seconds will the bells toll together for the first time after they begin tolling?
a) 7 b) 30 c) 420 d) 630 e) 1764

11. If x is greater than 0 but less than 1, which of the following is the largest?
a) $1/x^2$ b) $1/x$ c) x d) x^2 e) x^3

12. If x= -1, which of the following is the largest?
a) $2x$ b) x c) $x/2$ d) x^2 e) x^3

13. If n is an integer, which of the following must be even?
a) n – 1 b) n+1 c) 3n+1 d) 2n+2 e) 3(n+1)

14. If m, n, o, p and q are integers then m(n+o)(p-q) must be even, when which of the following is even?
a) m+n b) n+p c) m d) o e) p

15. If 9^{32} is divided by 5, then the remainder would be
a) 1 b) 2 c) 3 d) 4 e) None of these

16. The sum of two numbers is 27 and the difference of their squares is 243. What is the difference between the numbers?
a) 10 b) 9 c) 12 d) 15 e) None of these

17. Find the least number which must be added to 15463 so that the resulting number is exactly divisible by 107
a) 50 b) 51 c) 52 d) 30 e) 45

18. A number when divided by 221 leaves a remainder 64. What is the remainder when the same number is divided by 13?
a) 0 b) 1 c) 7 d) 11 e) 12

19. Find the greatest number which will divide 38, 45 and 52 and leave remainders 2,3 and 4 respectively.

a) 7 b) 8 c) 5 d) 12 e) 6

20. If the sum of two positive integers is 24 and the difference of their squares is 48, what is the product of the two integers?

a) 108 b) 119 c) 128 d) 135 e) 143

21. There are a few cards in the collection. If they are counted out 3 at a time, there are 2 cards left out, but if they are counted out 4 at a time, there is 1 card left out. How many cards are there in the collection?

a) 101 b) 103 c) 106 d) 107 e) 109

22. Mary Lou has 969 dimes in a piggy bank. What is the least number of dimes that she can remove from the bank so that she could divide the remaining dimes equally among 7 people?

a) 2 b) 3 c) 4 d) 5 e) 6

23. For how many of the integers from 10 to 99 is at least one of the two digits a 4?

a) 9 b) 10 c) 18 d)19 e) 20

24. The sum of three consecutive odd integers, x, y and z, in ascending order, is 39. What is the sum of the three consecutive odd integers that immediately follow z?

a) 78 b) 57 c) 48 d) 45 e) 42

25. There are 125 chips on a table. If as many of the chips as possible are to be arranged into an equal number of 3-chip and 4-chip stacks and the remaining chips are to be removed, how many of the chips are to be removed?

a) One b) Two c) Five d) Six e) Seven

2. ALGEBRA

Algebra Terms

1. **Constant.** A numerical quantity that does not change.

2. **Variable.** An unknown quantity written as a letter. A variable can be represented by any letter in the English alphabet; *x* or *y* are common on the GRE, but you'll see others as well. Variables may be associated with specific things, like *x* number of apples or *y* dollars. Other times, variables have no specific association, but you'll need to manipulate them to show that you understand basic algebraic principles.

3. **Coefficient.** A coefficient is a number that appears next to a variable and tells how many of that variable there are. For example, in the term $4x$, 4 is the coefficient and tells us there are four *x*s. In the term $3x^3$, 3 is the coefficient and tells us there are three x^3s.

4. **Term.** The product of a constant and a variable. Or, a quantity separated from other quantities by addition or subtraction. For example, in the equation $3x^3 + 2x^2 – 7x + 4 = x – 1$, the side to the left of the equal sign contains four terms ($3x^3$, $2x^2$, $–7x$, 4), while the right side contains two terms (x, $–1$). The constants, 4 and –1, are considered terms because they are coefficients of variables raised to the zero power: $4 = 4x^0$. So technically, every algebraic term is the product of a constant and a variable raised to some power.

5. **Expression.** Any combination of terms. An expression can be as simple as a single constant term, like 5, or as complicated as the sum or difference of many terms, each of which is a combination of constants and variables, such as $\frac{\{(x^2+2)^3-6x\}}{7x^5}$. Expressions don't include an equal sign—this is what differentiates expressions from equations. If you're given the value of every variable in the expression, then you can calculate the numerical value of the expression. Lacking those values, expressions can't be solved, although they can often be simplified.

6. **Equation.** Two expressions linked by an equal sign. Much of the algebra on the GRE consists of solving equations.

Inputs and Outputs: Simple Substitutions

One of the simplest kinds of algebraic problems on the GRE operates like this, as in the following example:

If $x = 2$, what is the value of $\frac{3x^2+6}{3}$?

There's nothing to do but simply input 2 into the expression in place of *x* and do the math:

$\frac{3(2)^2+6}{3}=\frac{(3)(4)+6}{3}=\frac{12+6}{3}=\frac{18}{3}=6$ So here, an input of 2 yields an output of 6.

You may see something like this in the beginning of the Math section, but things will most likely get more complicated after that. For example, the inputs themselves may be a bit more complex than a single number; in fact, an input may itself contain variables:

If $2y + 8x = 11$, what is the value of $3(2y + 8x)$?

You might see this equation bubbling over with variables and panic. Don't. Since the expression $2y + 8x$ appears in both parts of the question, and we're told this expression equals 11, we can simply substitute that figure in place of the expression on the right to get $3(11) = 33$.

A question may also involve multiple substitutions. For instance:

$z = \frac{4y}{x^2}$, $y = 3x$, and $x = 2$, then what is the value of z?

To approach this problem, you just have to input 2 for x to find y, and then input those values into the equation for z. Substituting 2 for x into $y = 3x$ gives $y = 3(2) = 6$. Inputting $x= 2$ and $y = 6$ into the equation for z gives:

$$z = \frac{4y}{x^2} = \frac{4(6)}{2^2} = \frac{24}{4} = 6$$

- **Simplifying Algebraic Expressions**

Before we move on to solving more complicated equations, we need to cover a few simplification tools that allow us to change algebraic expressions into simpler but equivalent forms.

- **Distributing**

The rule of distribution states: $a(b+c) = (a \times b) + (a \times c)$

The a in this expression can be any kind of term, meaning it could be a variable, a constant, or a combination of the two. When you distribute a factor into an expression within parentheses, multiply each term inside the parentheses by the factor outside the parentheses. $4(x + 2)$, for example, would become $4x + (4)(2)$, or $4x + 8$. Let's try a harder one: $3y(y^2 – 6)$. Distributing the $3y$ term across the terms in the parentheses yields:

$$3y(y^2 - 6) = 3y^3 - 18y$$

But the true value of distributing becomes clear when you see a distributable expression in an equation.

- **Factoring**

Factoring an expression is the opposite of distributing. $4x^3 – 8x^2$ is one mean-looking expression, right? Or so it seems, until you realize that both terms share the greatest common factor $4x^2$, which you can factor out:

$$4x^3 - 8x^2 = 4x^2(x - 2)$$

By distributing and factoring, you can group or ungroup quantities in an equation to make your calculations simpler, depending on what the other terms in the equation look like. Sometimes distributing will help; other times, factoring will be the way to go. Here are a few more examples of both techniques:

$3(x+y+4) = 3x+3y+12$	3 is distributed.
$2x+4x+6x+8x = 2x(1+2+3+4)$	$2x$ is factored out.
$x^2(x-1) = x^3 - x^2$	x^2 is distributed.
$xy^2(xy^2 + x^2y) = x^2y^4 + x^3y^3$	xy^2 is distributed.
$14xy^2 - 4xy + 22y = 2y(7xy - 2x + 11)$	$2y$ is factored out.

- **Combining Like Terms**

After factoring and distributing, you can take additional steps to simplify expressions or equations. Combining like terms is one of the simplest techniques you can use. It involves adding or subtracting the coefficients of variables that are raised to the same power. For example, by combining like terms, the expression $x^2 - x^3 + 4x^2 + 3x^3$ can be simplified by adding the coefficients of the variable x^3 (–1 and 3) together and the coefficients of x^2 (1 and 4) together to get: $2x^3 + 5x^2$

Variables that have different exponential values are not like terms and can't be combined. Two terms that do not share a variable are also not like terms and cannot be combined regardless of their exponential value. For example, you can't combine:

$$x^4 + x^2$$
or
$$y^2 + x^2$$

You can, however, factor the first expression to get $x^2(x^2 + 1)$, which you should do if it helps you answer the question.

- **Linear Equations with One Variable**

To solve an equation, you have to isolate the variable you're solving for. That is, you have to "manipulate" the equation until you get the variable alone on one side of the equal sign. By definition, the variable is then equal to everything on the other side of the equal sign. You can't manipulate an equation the way you used to manipulate your little brother or sister. When manipulating equations, there are rules. Here's the first and most fundamental.

- **Whatever you do to one side of an equation, you must do to the other side.**

If you divide one side of an equation by 3, divide the other side by 3. If you take the square root of one side of an equation, take the square root of the other.

By treating the two sides of the equation in the same way, you don't change what the equation means. You change the *form* of the equation into something easier to work with—that's the point of manipulating it—but the equation remains true since both sides stay equal.

Take, for instance, the equation $3x + 2 = 5$. You can do anything you want to it as long as you do the same thing to both sides. Here, since we're trying to get the variable x alone on the left, the thing to do is subtract 2 from that side of the equation. But we can only do that if we subtract 2 from the other side as well:

$$3x + 2 - 2 = 5 - 2$$
$$3x = 3$$

Now we can just divide both sides by 3 to get $x = 1$, and we're done.

You should use the simplification techniques you learned above (distributing, factoring, and combining like terms) to help you solve equations. For example: $3y(y^2 + 6) = 3y^3 + 36$

That seems fairly nasty, since there aren't any like terms to combine. But wait a sec . . . what if you distribute that $3y$ on the left side of the equation? That would give: $3y^3 + 18y = 3y^3 + 36$

Shiver our timbers! Now we can subtract $3y^3$ from both sides to get $18y = 36$

and then simply divide both sides by 18 to get $y = 2$.

- **Reverse PEMDAS**

$$2 + \frac{3(2\sqrt{x} + 3)}{2} = 17$$

In this equation, poor little *x* is being square rooted, multiplied by 2, added to 3, and encased in parentheses—all in the numerator of a fraction. That's hardly what we'd call "alone time." You've got to get him out of there! Undo all of these operations to liberate *x* and solve the equation.

Let SADMEP be your guide: First, subtract 2 from both sides of the equation: $\frac{3(2\sqrt{x}+3)}{2}=15$

There's no addition or division possible at this point, but we can multiply both sides by 2 to get rid of the fraction: $3(2\sqrt{x}+3)=30$

Now divide both sides by 3 (see you later, parentheses!): $2\sqrt{x}+3=10$

Now we're in position to subtract 3 from each side: $2\sqrt{x}=7$

Divide both sides by 2: $\sqrt{x}=\frac{7}{2}$

Finally, square each side to get rid of the square root: $x=\frac{49}{4}$

- **Variables in the Denominator**

Remember, the key to solving equations is to isolate the variable, but how to do this depends on where the variable is located. A variable in the numerator of a fraction is usually pretty easy to isolate. But if the variable is in the denominator, things get more complicated. See what you can make of this one: $\frac{1}{x+2}+3=7$

Following SADMEP, start by subtracting the 3: $\frac{1}{x+2}=4$

But now you have to get the *x* out of the denominator, and the only way to do that is to multiply both sides of the equation by that denominator, $x + 2$: $1=4(x+2)$

Divide both sides by 4: $\frac{1}{4}=x+2$

Subtract 2 from each side: $-\frac{7}{4}=x$

- **Equations with Absolute Value**

To solve an equation in which the variable is within absolute value bars, you have to follow a two-step process:

1. Isolate the expression within the absolute value bars.
2. Divide the equation in two.

If $|x + 3| = 5$, then $x =$

Since both 5 and –5 within absolute value bars equal 5, the expression inside the bars can equal 5 or –5 and the equation will work out. That's why we have to work through both scenarios. So we're actually dealing with two equations:

$x + 3 = 5$

$x + 3 = –5$

For a complete solution, we need to solve both. In the first equation, $x = 2$. In the second equation, $x = -8$. So the solutions to the equation $|x + 3| = 5$ are $x = \{-8, 2\}$. Both work. Substitute them back into the equation if you have any doubts and to reinforce why we need to solve two equations to get a full answer to the question.

- **Equations with Exponents and Radicals**

Absolute value equations aren't the only ones with more than one possible answer. Exponents and radicals can also have devilish effects on algebraic equations that are similar to those caused by absolute value. Consider the equation $x^2 = 25$. Seems pretty simple, right? Just take the square root of both sides, and you end up with $x = 5$. But remember the rule of multiplying negative numbers? When two negative numbers are multiplied together the result is positive. In other words, –5 squared *also* results in 25: $-5 \times -5 = 25$. This means that whenever you have to take the square root to simplify a variable brought to the second power, the result will be two solutions, one positive and one negative: $\sqrt{x^2} = \pm x$. (The only exception is if $x = 0$.) You'll see what we mean by working through this question:

If $2x^2 = 72$, what is the value of x?

To solve this problem, we first divide both sides by 2 to get $x^2 = 36$. Now we need to take the square root of both sides: $x = \pm 6$.

Recognizing when algebraic equations can have more than one possible answer is especially important in Quantitative Comparison questions. Sometimes it may appear that the relationship between columns A and B leans in a definite direction, until you notice that one of the columns can actually have more than one possible value. Keep your eyes peeled for this common nuance.

- **Linear Equations with Two or More Variables**

Some GRE questions contain *two* variables. Those questions also contain two equations, and you can use the two equations in conjunction to solve for the variables. These two equations together are called a ***system of equations* or *simultaneous equations*.**

There are two ways to solve simultaneous equations. The first method involves substitution, and the second involves adding or subtracting one equation from the other. Let's look at both techniques.

- **Solving by Substitution**

You've already seen examples of substitution in the Input/Output discussion earlier in the chapter, so you should be familiar with the mechanics of plugging values or variables from one place into another place to find what you're looking for. In these next examples, both pieces of the puzzle are equations. Our method will be to find the value of one variable in one equation and then plug that into a second equation to solve for a different variable. Here's an example:

If $x - 4 = y - 3$ and $2y = 6$, what is x?

You've got two equations, and you have to find x. The first equation contains both x and y. The second equation contains only y. To solve for x, you first have to solve for y in the second equation and substitute that value into the first equation. If $2y = 6$, then dividing both sides by 2, $y = 3$. Now, substitute 3 for y in the equation $x - 4 = y - 3$:

$$\begin{aligned} x - 4 &= 3 - 3 \\ x - 4 &= 0 \\ x &= 4 \end{aligned}$$

That's all there is to it. But here's one that's more likely to give you trouble:

If $3x = y + 5$ and $2y - 2 = 12k$, what is x in terms of k?

Notice anything interesting? There are *three* variables in this one. To solve for x in terms of k, we have to first get x and k into the same equation. To make this happen, we can solve for y in terms of k in the second equation and then substitute that value into the first equation to solve for x:

$$\begin{aligned} 2y-2&=12k\\ 2y&=12k+2\\ y&=6k+1 \end{aligned}$$

Then substitute $6k + 1$ for y in the equation $3x = y + 5$:

$$\begin{aligned} 3x&=y+5\\ 3x&=(6k+1)+5\\ 3x&=6k+6\\ x&=2k+2 \end{aligned}$$

This is our answer, since x is now expressed in terms of k. Note that you could also solve this problem by solving for y in terms of x in the first equation and substituting that expression in for y in the second equation. Either way works.

- **Solving by Adding or Subtracting**

The amazing thing about simultaneous equations is that you can actually add or subtract the entire equations from each other. Here's an example:

If $6x + 2y = 11$ and $5x + y = 10$, what is $x + y$?

Look what happens if we subtract the second equation from the first:

$$\begin{array}{r} 6x+2y=11\\ \underline{-(5x+y=10)}\\ x+y=1 \end{array}$$

To add or subtract simultaneous equations, you need to know what variable or expression you want to solve for, and then add or subtract accordingly. We made the example above purposely easy to show how the method works. But you won't always be given two equations that you can immediately add or subtract from each other to isolate the exact variable or expression you seek, as evidenced by this next example:

If $2x + 3y = -6$ and $-4x + 16y = 13$, what is the value of y?

We're asked to solve for y, which means we've got to get rid of x. But one equation has $2x$ and the other has $-4x$, which means the x terms won't disappear by simply adding or subtracting them.

Golden Rule of Algebra comes to the rescue: If you do the same thing to both sides of an equation, you don't change the meaning of the equation. That means that in this case, we could multiply both sides of $2x + 3y = -6$ by 2, which would give us $2(2x + 3y) = 2(-6)$. Using the trusty distributive law, and multiplying out the second part, gives us $4x + 6y = -12$. Now we're in a position to get rid of those pesky x terms by adding this new (but equivalent) form of equation 1 to equation 2:

$$\begin{array}{r} 4x+6y=-12\\ \underline{+(-4x+16y=13)}\\ 22y=1\\ y=\dfrac{1}{22} \end{array}$$

As you practice with these types of problems, you'll get a sense for which method works best for you.

- <u>**Binomial and Quadratic Equations**</u>

A binomial is an expression containing two terms. The terms $(x + 5)$ and $(x - 6)$ are both binomials. A quadratic expression takes the form $ax^2 + bx + c$, where $a \neq 0$. Quadratics closely resemble the products formed when binomials are multiplied. That's why we treat these topics together. We'll start off with binomials, and work our way to quadratics.

❖ Multiplying Binomials

The acronym will help you remember how to multiply binomials. This acronym is **<u>FOIL</u>**, and it stands for **First + Outer + Inner + Last.** The acronym describes the order in which we multiply the terms of two binomials to get the correct product. $(x+1)(x+3)$

First, multiply the first (F) terms of each binomial: $\boldsymbol{x \times x = x^2}$

Next, multiply the outer (O) terms of the binomials: $x \times 3 = 3x$

Then, multiply the inner (I) terms: $1 \times x = x$

And then multiply the last (L) terms: $1 \times 3 = 3$

Add all these terms together: $x^2 + 3x + x + 3$

Finally, combine like terms, and you get: $x^2 + 4x + 3$

Here are a few more examples of multiplied binomials to test your FOILing faculties:

$$(y+3)(y-7) = y^2 - 7y + 3y - 21 = y^2 - 4y - 21$$
$$(-x+2)(4x+6) = -4x^2 - 6x + 8x + 12 = -4x^2 + 2x + 12$$
$$(3a+2b)(6c-d) = 18ac - 3ad + 12bc - 2bd$$

Note that the last one doesn't form a quadratic equation, and that none of the terms can be combined. That's okay. When presented with binomials, follow FOIL wherever it leads.

❖ <u>Working with Quadratic Equations</u>

A quadratic equation will always have a variable raised to the power of 2, like this:

$$\boldsymbol{x^2 = 10x - 25}$$

Your job will be to solve for the given variable, as we've done with other algebraic equations throughout this section. The basic approach, however, is significantly different from what you've done so far. Instead of isolating the variable on the left, you'll want to get everything on the left side of the quadratic equation, leaving 0 on the right. In the example above, that means moving the $10x$ and -25 to the left side of the equation:

$$\boldsymbol{x^2 - 10x + 25 = 0}$$

Now it's looking like most of the products of binomials we saw in the previous section, except instead of being just a quadratic expression, it's a quadratic equation because it's set equal to 0.

To solve the equation, we need to factor it. Factoring a quadratic equation means rewriting it as a product of two terms in parentheses, like this: $\boldsymbol{x^2 - 10x + 25 = (x - 5)(x - 5)}$

How did we know to factor the equation into these binomials? Here's the secret: Factoring quadratic equations on the GRE always fits the following pattern: $\boldsymbol{(x \pm m)(x \pm n)}$

Essentially, we perform FOIL in reverse. When we approach a quadratic like $x^2 - 10x + 25$, the two numbers we're looking for as our m and n terms need to *multiply to give the last number* in the equation. The last number in the equation is 25, so we need to find two numbers whose product is 25. Some pairs of numbers that work are:

1 and 25
–1 and –25
5 and 5
–5 and –5

Further, the sum of the two numbers needs *to give the middle number* in the equation. Be very careful that you don't ignore the sign of the middle number: $\boldsymbol{x^2 - 10x + 25}$

Since you're subtracting $10x$, the middle number is –10. That means the m and n numbers we seek not only need to multiply to 25 but also need to add to –10. Going back to our factor list above, –5 and –5 is the only pair that works. Substituting this into the pattern gives: $\boldsymbol{(x - 5)(x - 5)}$

And since we originally set the equation equal to 0, we now have: $\boldsymbol{(x - 5)(x - 5) = 0}$

For the product of two terms to equal 0, that means that either one could be 0. Here both terms are $(x - 5)$, so $x - 5$ must equal 0.

$$\boldsymbol{x - 5 = 0}$$
$$\boldsymbol{x = 5}$$

In this example, m and n are equal, which is why we end up with only one answer. But that's usually not the case. Let's look at another example, using different numbers: $\boldsymbol{x^2 = -10x - 21}$

To solve for x, first move everything to the left to set the equation equal to 0: $\boldsymbol{x^2 + 10x + 21 = 0}$

Now we need to figure out what numbers fit our pattern: $\boldsymbol{(x \pm m)(x \pm n)}$

We know from our equation that $m \times n$ needs to equal 21, and $m + n$ needs to equal 10. So, which numbers work? Let's look at $m \times n = 21$ first: **1 and 21; –1 and –21; 3 and 7; –3 and –7.**

We can eliminate (1 and 21) and (–1 and –21), since neither of these pairs add up to 10. The third pair, 3 and 7, adds to 10, so we can stop right there and plug these numbers into our pattern:

$$(x + 3)(x + 7) = 0$$

If you need to double-check your factoring, just FOIL the resulting binomials, which should bring you right back to the original quadratic. Since we now have the product of two different binomials sets equal to 0, one of the two terms needs to be 0. So, either $(x + 3) = 0$ or $(x + 7) = 0$, which means x could be equal to –3 or –7. There's no way to determine for sure, since both values work.

❖ QUADRATIC FACTORING PATTERNS

There are three patterns of quadratics that commonly appear on the GRE. Learn them now, and you'll work faster on test day.

Pattern 1: $x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2$ Example: $x^2 + 6xy + 9 = (x + 3)^2$

Pattern 2: $x^2 - 2xy + y^2 = (x - y)(x - y) = (x - y)^2$ Example: $x^2 - 10xy + 25 = (x - 5)^2$

Pattern 3: $(x + y)(x - y) = x^2 - y^2$ Example: $(x + 4)(x - 4) = x^2 - 16$

You may be wondering why we wrote the last pattern with the factored form first. We wrote it this way because this is the way it often appears on the GRE: You'll be given an expression that fits the pattern on the left of the equal sign $[(x + y)(x - y)]$, and you'll need to recognize its equivalent form of $(x^2 - y^2)$.

Here's an example of how you can use this pattern on test day:

What is the value of $(\sqrt{85} + \sqrt{72})(\sqrt{85} - \sqrt{72})$

Luckily, quadratic pattern 3 helps us avoid all that work. That pattern states that whenever we have the sum of two values multiplied by the difference of those same values, the whole messy expression is equal to $x^2 - y^2$. Here, if we let $x = \sqrt{85}$ and $y = \sqrt{72}$, then $x^2 - y^2 = \sqrt{85}^2 - \sqrt{72}^2$. If this doesn't look any better to you, then you're not realizing that the squared symbol (the little 2) and the square root symbol (the $\sqrt{\ }$) cancel each other out. This gives the much simpler expression 85 – 72, which equals 13.

This is an excellent example of how changing your math mindset will help you on the GRE. In the difficult-looking problem we just tackled, your first instinct may have been to hack your way through the numbers. However, if you instead suspected that the GRE test makers probably wouldn't present a problem like this if there wasn't a more elegant solution, then you might have searched for an easy way in. Quadratic factoring pattern 3 does the trick.

➢ Inequalities

An inequality is like an equation, but instead of relating equal quantities, it specifies exactly how two quantities are *not* equal. There are four types of inequalities:

1. $x > y$ x is greater than y.
2. $x < y$ x is less than y.
3. $x \geq y$ x is greater than or equal to y.
4. $x \leq y$ x is less than or equal to y.

So, for example, $x + 3 \leq 2x$ may be read as "$x + 3$ is less than or equal to $2x$." Similarly, $y > 0$ is another way of saying "y is greater than 0." Inequalities may also be written in compound form, such as $4 < y - 7 < 3y - 10$. This is really just two separate inequalities: $4 < y - 7$ and $y - 7 < 3y - 10$. Another way to think about this compound inequality is that $y - 7$, the expression stuck in the middle, is between 4 and $3y - 10$.

❖ Solving Inequalities

Solving inequalities is a lot like solving equations: Get the variable on one side of the inequality and all the numbers on the other, using the algebraic rules you've already learned. There is, however, one exception to this, and it's a *crucial* exception. This exception is crucial, so we'll repeat it:

- **<u>The Inequality Exception</u>**

Multiplying or dividing both sides of an inequality by a negative number requires you to flip the direction of the inequality.

Let's try some examples. Solve for *x* in the inequality $\frac{x}{2} - 3 < 2y$.

First we knock that 3 away from the *x* by adding 3 to both sides: $\frac{x}{2} < 2y + 3$

That 2 in the denominator is quite annoying, but by now you should know the fix for that—just multiply both sides by 2 and it will disappear from the left side of the inequality: $x < 2(2y + 3)$

We can simplify the right side with our handy distributive law, multiplying the 2 by both terms in the parentheses to get a final answer of: $x < 4y + 6$

We're done. The *x* stands alone, and we know that it's less than the expression 4*y* + 6. Now try this one:

Solve for *x* in the inequality $\frac{4}{x} \geq -2$.

Here are the steps, all at once:

$$\frac{4}{x} \geq -2$$
$$4 \geq -2x$$
$$-2 \leq x$$

Notice that in this example the inequality had to be flipped, since both sides had to be divided by –2 to isolate the variable in the end.

To help remember that multiplication or division by a negative number reverses the direction of the inequality, **remember that if *x* > *y*, then –*x* < –*y***. Just as 5 is greater than 4, –5 is less than –4. The larger the number, the smaller it becomes when you make it negative. That's why multiplying or dividing inequalities by negatives requires switching the direction of the inequality sign.

- **Inequalities with Two Variables**

Another type of inequality problem involves two variables. For these, you'll be given a range of values for each of the two variables. For example:

$$-8 \leq a \leq 0$$
$$5 \leq b \leq 25$$

This is just another way of saying that *a* is between –8 and 0, inclusive, and that *b* is between 5 and 25, inclusive. *Inclusive* means that we include the values at each end, which is what's meant by the *greater than or equal to* and *less than or equal to* signs. If the test makers didn't want –8 and 0 to be possible values for *a*, for example, they would have to use simple greater-than and less-than signs (> and <). That scenario corresponds to the word *exclusive*, which means you should exclude the values at each end.

Once the range of the two variables has been established, the problem will then ask you to determine the range of values for some expression involving the two variables. For example, you could be asked for the range of values of *a* – *b.* This is really just asking for the smallest and largest possible values of *a* – *b.*

One good way to tackle these problems is to whip up a handy Inequality Table, a table with columns for each of the two variables and one column for the expression whose range you're trying to determine. First write in the largest and smallest values for *a* and *b* from the original inequalities. In this example, the extreme values for *a* are –8 and 0, and for *b* are 5 and 25. Write these values in the table so that each combination of *a* and *b* is represented. There will be four combinations total: the smallest value of *a* with the smallest value of *b*; the smallest value of *a* with the largest value of *b*; the largest value of *a* with the smallest value of *b*; and the largest value of *a* with the largest value of *b*. Now simply evaluate the expression you're asked about for each of the four combinations in the table—in this case, $a - b$:

a	***b***	***a* – *b***
8	5	–8 – 5 = –13
–8	25	–8 – 25 = –33
0	5	0 – 5 = –5
0	25	0 – 25 = –25

The Inequality Table shows us that the smallest possible value of $a - b$ is –33 and the largest is –5. Writing this as a compound inequality gives our final answer:
$\mathbf{-33 \le a - b \le -5}$

❖ Inequality Ranges

The previous question demonstrated how inequalities can be used to express the range of values that a variable can take. There are a few ways that inequality problems may involve ranges. We consider three scenarios below.

• OPERATIONS ON RANGES

Ranges can be added, subtracted, or multiplied. Consider the following:

If $4 < x < 7$, what is the range of $2x + 3$?

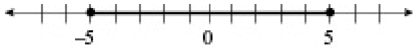
To solve this problem, manipulate the range like an inequality until you have a solution. Begin with the original range: $4 < x < 7$

Since the range we're ultimately looking for contains $2x$, we need to turn the x in the inequality above into that. We can do this by multiplying the whole inequality by 2: $8 < 2x < 14$

Since we're doing the same thing to all parts of the inequality, this manipulation doesn't change its value or meaning. In other words, we're simply invoking the Golden Rule: **<u>Do unto one part what we do unto the other part</u>**. But we're not there yet, because the range we seek is $2x + 3$, not plain old $2x$. No problem: Just add 3 to the inequality across the board, and you have the final answer: $11 < 2x + 3 < 17$

Always remember the crucial rule about multiplying inequalities: If you multiply or divide a range by a negative number, you must flip the greater-than or less-than signs. For example, if you multiply the range $2 < x < 8$ by –1, the new range will be $-2 > -x > -8$.

• ABSOLUTE VALUE AND SINGLE RANGES

Absolute values do the same thing to inequalities that they do to equations. You have to split the inequality into two parts, one reflecting the positive value of the inequality and one reflecting the negative value. You'll see an example just below. If the absolute value is less than a given quantity, then the solution will be a single range with a lower and an upper bound. An example of a single range would be the numbers between –5 and 5, as seen in the following number line: 

A single range question will look something like this:

Solve for x in the inequality $|2x - 4| \le 6$.

First, split the inequality into two. In keeping with the rule for negative numbers, you'll have to flip around the inequality sign when you write out the inequality for the negative scenario:

$2x - 4 \leq 6$
$2x - 4 \geq -6$

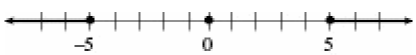
Solve the first: $2x - 4 \leq 6$; $2x \leq 10$; $x \leq 5$

Then solve the second: $2x - 4 \geq -6$; $2x \geq -2$; $x \geq -1$

So x is greater than or equal to –1 and less than or equal to 5. In other words, x lies between those two values. So you can write out the value of x in a single range, $-1 \leq x \leq 5$.

- **ABSOLUTE VALUE AND DISJOINTED RANGES**

You won't always find that the value of the variable lies between two numbers. Instead, you may find that the solution is actually two separate ranges: one whose lower bound is negative infinity and whose upper bound is a real number, and one whose lower bound is a real number and whose upper bound is infinity. Yeah, words make it sound confusing. A number line will make it clearer. An example of a disjointed range would be all the numbers smaller than –5 and larger than 5, as shown below:



On the GRE, disjointed ranges come up in problems in which the absolute value is greater than a given quantity, such as the following:

Solve for x in the inequality $|3x + 4| > 16$.

You know the drill. Split 'er up, then solve each inequality:

$3x + 4 > 16$
$3x + 4 < -16$

Again, notice that we have to switch the inequality sign in the second case because of the negative-number rule.

Solving the first:
$3x + 4 > 16$. Thus, $3x > 12$. Therefore, $x > 4$.

And the second: $3x + 4 < -16$. Thus, $3x < -20$ and therefore, $x < -\frac{20}{3}$

Notice that x is greater than the positive number and smaller than the negative number. In other words, the possible values of x don't lie *between* the two numbers; they lie outside the two numbers. So you need two separate ranges to show the possible values of x: $-\infty < x < -\frac{20}{3}$ and $4 < x < \infty$. There are two distinct ranges for the possible values of x in this case, which is why the ranges are called *disjointed*. It doesn't mean they can bend their fingers back all the way—that's *double-jointed*.

➢ Made-up Symbols

You may see a made-up symbol problem on the GRE involving little graphics you've never seen before in a math context. Sure, they look weird, but they often involve variables and equations, which is why we thought we'd cover them here in the algebra section. Anyway, there's a silver lining to this weirdness: Made-up symbol problems *always* give you exact instructions of what to do. Follow the instructions precisely, and you'll do just fine. Let's see how this works with an example:

If $x\ \Omega\ y = 5x + 2y - 10$, what is $3\ \Omega\ 4$?

Look at the first part of the problem again: $x\ \Omega\ y = 5x + 2y - 10$. All that's really saying is that whenever we have two numbers separated by a Ω, we need to take 5 times the first number and 2 times the second number, add them together, and subtract 10. That's it—these are the only instructions we need to follow.

For the expression $3\ \Omega\ 4$, then, x is 3, which we multiply by 5 to get 15. Similarly, y is 4, which we're told to multiply by 2, which gives us 8. Adding these together gives us 23 and subtracting 10 brings us to 13 and a quick and easy point. Quick and easy, that is, if you don't panic and just follow the directions to a T.

Try one more example to get the hang of this symbol business:

If $j@k = \frac{k}{2} - 13j$, what is $5@-1$?

Forget the actual variables for a moment and focus on the instructions given: Whenever we have two numbers separated by an @, we need to divide the second number by 2, multiply the first number by 13 and subtract the two numbers. So, substituting 5 for j and –1 for k: $5@-1 = \frac{-1}{2} - (13 \times 5) = \frac{-1}{2} - 65 = -65\frac{1}{2}$ or 65.5

- **Groups**

The basic idea is that some people or things belong to one group, others belong to another group, and still others belong to *both* groups or *neither* group. For example, at a certain country club, some members play golf, some play tennis, others play both sports, and still others prefer reading to playing. You'll be given some of the specific numbers in such a problem and then asked to determine the missing values.

The approach you should use depends on whether the problem concerns two or three groups. We'll cover the most effective techniques for both cases. You won't see problems with more than three groups.

- **Problems with Two Groups**

All you need for two-group problems is this formula:

group 1 + group 2 – both + neither = total

where

group 1 = the number of entities in one of the two groups

group 2 = the number of entities in the other group

both = the number of entities in both groups

neither = the number of entities in neither group

total = the total number of entities

You'll most likely be given values for all of the parts of this formula except for one. You'll then have to determine the value of the missing part. Let's see how this works in the following example.

At a certain animal refuge, 180 animals have four legs, 240 are warm-blooded, and 85 both have four legs and are warm-blooded. If the animal refuge has 500 animals in total, how many animals at the refuge have neither four legs nor are warm-blooded?

We'll let *group 1* be the 180 animals that have four legs and *group 2* be the 240 animals that are warm-blooded. Since we're also given the number of animals that belong to *both* groups and the *total* number of animals, the missing value is the number of animals that belong to neither group. Not surprisingly, that's what the question is after. Plugging the values into the formula gives:

$$180 + 240 - 85 + \textit{neither} = 500$$

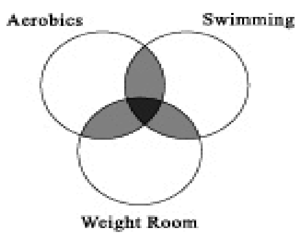
Solving for *neither* is a simple matter of solving this linear equation with one variable, something we discussed way back in the algebra section. Here goes:

$$180 + 240 - 85 + n = 500$$
$$335 + n = 500$$
$$n = 165$$

But things get a bit more difficult if they throw in an extra group.

❖ Problems with Three Groups

The easy of solving sums with 3 groups is Venn diagrams which consist of intersecting circles, in which each circle represents the number of entities in a particular group. For example:

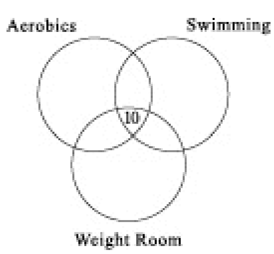


You'll notice that the circles overlap. The upside-down triangular section in the middle, with the darker shading, represents the number of entities that belongs to all three groups. Sections in which only two circles overlap, indicated with the lighter shading, represent the number of entities that belongs to two overlapping groups. The outermost section of each circle, the part that doesn't overlap with any of the other circles, represents the number of entities that belongs to each group alone. For example, in the swimming circle, the outermost section represents the number of swimmers who neither lift weights nor do aerobics.

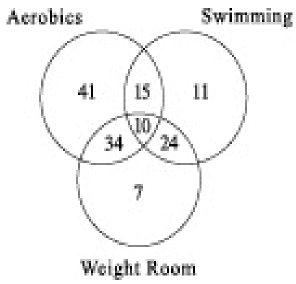
The key to three-group problems is to work *from the inside out*. Begin with the entities that belong to all three groups, then address the entities that belong to two groups, and finally deal with the entities that belong to only one group.

Example: At the Get Fit Athletic Club, every member swims, lifts weights, does aerobics, or participates in some combination of these three activities. Sixty members swim, 75 lift weights, and 100 do aerobics. If 34 members both lift weights and swim, 25 members both do aerobics and swim, 44 members both use the weight room and do aerobics and 10 participate in all three activities, how many members belong to the Get Fit Athletic Club?

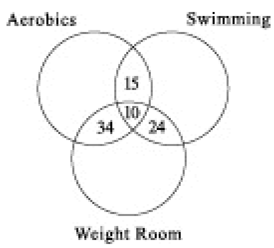
We'll start by filling in values, *working from the inside out*. Since 10 people belong to all three groups, we'll write 10 in the middle section:



Next, we'll fill in the values for people who belong to two groups. We're told that 25 members participate in aerobics and in swimming, and it would be very tempting to write 25 in the section above the 10. Keep in mind, however, that 10 of these 25 people *have already been accounted for*: The 10 people who participate in all three activities are included among those who participate in aerobics and swimming. That leaves 25 – 10 = 15 people for the aerobics/swimming overlap section above the 10 in the middle. Similarly, 44 people do aerobics and use the weight room. Since 10 of these people have already been accounted for in the middle section, that leaves 44 – 10 = 34 people in the section that overlaps aerobics and weight room. Also, 34 people swim and use the weight room, so we'll write 34 – 10 = 24 in the section overlapping those categories. That brings us to here:



Next we need to fill in values for people who belong to only one group. As with the previous step, we have to be very careful not to count anyone more than once. For example, we're told that 60 members swim. That means that the total of *all* the numbers in the swimming circle must be 60. We already have 10 + 15 + 24 = 49 members in the swimming circle. That leaves 60 – 49 = 11 members for the outermost section of the swimming circle. Similarly, since 75 members use the weight room, that leaves 75 – 10 – 34 – 24 = 7 members for the outermost section of the weight room circle. Finally, since the total of the aerobics circle must be 100, the number in the remaining section must be 100 – 10 – 34 – 15 = 41. Filling these numbers in the appropriate sections of our Venn diagram yields the following:



The problem asks for the total number of club members, and we're told specifically that every member participates in at least one of these activities. That means that no member exists outside of our three circles. We can therefore add all of the values in our diagram to arrive at the total number of club members. This gives us a final answer of 41 + 15 + 11 + 34 + 10 + 24 + 7 = 142. Venn diagrams are the only way to go on three-group problems.

EXERCISE

1. What is the value of $x^2 +12x+36$ when $x = 994$?

a) 11928 b) 98836 c) 100000 d) 988036 e) 1000000

2. If $x^2 – y^2 = 28$ and $x-y = 8$, what is the average of x and y?

a) 1.75 b) 3.5 c) 7 d) 8 e) 10

3. Vidya and Vandana solved a quadratic equation. In solving it, Vidya made a mistake in the constant term and got the roots as 6 and 2, while Vandana made a mistake in the co-efficient of x only and obtained the roots as -7 and -1. The correct roots of the equation are

a) 6,-1 b) -7, 2 c) -6, -2 d) 7, 1 e) 16,7

4. If $4y-3x = 5$, what is the smallest integer value of x for which $y > 100$?

a) 130 b) 131 c) 132 d) 395 e) 396

5. A jar contains only red, white and blue marbles. The number of red marbles is 4/5th the number of white ones, and the number of white ones is 3/4th the number of blue ones. If there are 470 marbles in all, how many of them are blue?

a) 120 b) 135 c) 150 d) 184 e) 200

6. What is the greater of two numbers whose product is 900 and whose sum exceeds their difference by 30?

a) 15 b) 60 c) 75 d) 90 e) 100

7. A shop owner charges $150 per T.V. and $90 per refrigerator. Last week she sold 5 more T.V.s than refrigerators. If her total sales for these two items were $2910, what was the total number of T.V.s and refrigerators that she sold?

a) 20 b) 22 c) 21 d) 24 e) 23

8. In a bag containing black and white balls, half the number of white ones equals to one-third the number of black balls and twice the whole number of balls, exceeds 3 times the number of black balls by four. How many balls does the bag contain?
a) 15 b) 18 c) 19 d) 20 e) 22

9. Rs. 38 is divided among A, B and C such that B has Rs. 5 more than A and C has Rs 10 more than B. How much did C get?
a) 15 b) 17 c) 19 d) 20 e) 21

10. Find the number such that if 5, 15 and 35 are added to it, the product of the first and the third result is equal to the square of the second?
a) 5 b) 7 c) 8 d) 9 e) 10

11. A man is 5 times as old as his son. The sum of the squares of their ages is 2106. Find the age of the man.
a) 35 b) 45 c) 50 d) 55 e) 60

12. The sum of reciprocals of two positive consecutive numbers is 15/56. Find the smaller number.
a) 5 b) 6 c) 7 d) 8 e) 9

13. A fraction becomes ½ when 1 is subtracted from the numerator and 2 is added to the denominator, and it becomes 1/3 when 7 is subtracted from the numerator and 2 from the denominator. What is the fraction?
a) 8/15 b) 14/27 c) 15/31 d) 15/26 e)18/27

14. In a class when 4 students sit on each bench, 3 benches are left vacant. If 3 students sit on each bench, 3 students are left standing. Find the number of students in the class.
a) 60 b) 42 c) 48 d) 50 e) 54

15. A number consists of two digits whose product is 30. If the digit be interchanged the new number is more than the original number by 9, the number is
a) 65 b) 56 c) 46 d) 84 e) 54

16. If the roots of the equation $9x^2 + 3ax + 4 = 0$ are equal, then the value of a can be
a) 2 b) -4 c) -2 d) 6 e) Can't be determined

17. Dick travelled four seventh of a journey by train, 5/6th of the balance by bus and he walked the remaining distance of 3 kms. What is the total distance travelled by Dick?
a) 28 b) 42 c) 48 d) 56 e) 70

18. An instructor scored a student's test of 50 questions by subtracting 2 times the number of incorrect answers from the number of correct answers. If the student answered all of the questions and received a score of 38, how many questions did the student answer correctly?

a) 19 b) 38 c) 41 d) 44 e) 46

19. If a total of x identical disks can be arranged in 8 stacks of equal height or in 12 stacks of equal height, the least possible value of x is

a) 96 b) 48 c) 36 d) 24 e) 12

20. On a legislative committee, the number of males is 3 fewer than twice the number of females. If a female replaced one male, there would be an equal number of males and females on the committee. How many members are on the committee?

a) 14 b) 12 c) 10 d) 9 e) 7

21. In a certain brick wall, each row of bricks above the bottom row contains one less brick than the row just below it. If there are 5 rows in all and a total of 75 bricks in the wall, how many bricks does the bottommost row contain?

a) 14 b) 15 c) 16 d) 17 e) 18

22. A kennel sold 3 puppies of breed X and 2 puppies of breed Y for a total of $690. If each breed Y puppy was sold for 20 percent less than each breed X puppy, how much did each breed X puppy sell for?

a) 120 b) 127.70 c) 138 d) 150 e) 156.70

23. Rs. 429 is made up of one rupee, 50 paise and 25 paise coins. The number of these coins are in the proportion of 5:6:7. Total number of coins is

a)796 b) 1120 c) 792 d) 884 e) 894

24. Kim bought a total of $2.65 worth of postage stamps in four denominations. If she bought an equal number of 5-cent and 25-cent stamps and twice as many 10-cent stamps as 5-cent stamps, what is the least number of 1-cent stamps she could have bought?

a) 5 b) 10 c) 15 d) 20 e) 25

25. A certain truck travelling at 55 mph gets 4.5 miles per gallon of diesel fuel consumed. Travelling at 60 mph, the truck gets 3.5 miles per gallon. On a 500-mile trip, if the truck used a total of 120 gallons of diesel fuel and travelled part of the trip at 55 miles per hour and the rest at 60 mph, how many miles did the it travel at 55 mph?

a) 140 b) 200 c) 250 d) 300 e) 360

3. Average

Average = $\frac{Sum\ of\ the\ values}{Total\ number\ of\ values}$

Average speed = $\frac{Total\ Distance}{Total\ Time}$

Average speed when the same distance is travelled twice at the speed of x kmph and y kmph: $\frac{2xy}{x+y}$ kmph

Average of averages or combined average of two sets:

$$\frac{Sum\ of\ the\ first\ set\ of\ values + Sum\ of\ the\ other\ set\ of\ values}{Total\ number\ of\ 1st\ set\ of\ values + Total\ number\ of\ 2nd\ set\ of\ values}$$

NOTE: If there are more sets of numbers, then the sums of all the sets have to be added and divided by the total number of entities in all the sets.

EXERCISE

1. The average of seven numbers is 5. If the average of the first six of these numbers is 4, what is the seventh number?

a) 14 b) 12 c) 11 d) 15 e) 13

2. The average of nine numbers is 12. If each number is increased by 2, then the average of the new set of numbers is

a) 15 b) 13 c) 18 d) 16 e) 14

3. The average age of 5 boys is 12 years, and the average age of 3 other boys is 16 years. The average age of all the 8 boys is

a) 14.5 b) 15.5 c) 13.5 d) 16.5 e) 17.5

4. The average of 11 numbers is 53. The average of the first six numbers is 50 and that of the last six numbers is 55. The sixth number is

a) 53 b) 52.5 c) 48.5 d) 47 e) 46.5

5. The average age of 30 boys in a class is 16 years. The average of the ages of the boys and the teacher is 17 years. The age of the teacher is

a) 37 b) 39 c) 33 d) 47 e) 46

6. The average age of 10 students in a group increased by 0.4 years when a girl of age 6 years is replaced by another girl. The age of the new girl is

a) 16.4 b) 10 c) 12 d) 16 e) 17

7. The average height of 25 students in a class is 140 cms. Five newly admitted students increase the average height by 1 cm. The average height of the set of 5 new students is

a) 146.2 b) 147 c) 145 d) 146 e) 195

8. The average of three consecutive numbers is n. If the next two consecutive numbers are also included, the average of the five numbers will

a) Remains unchanged b) Increases by 0.5 c) Increases by 1 d) Increases by 1.5 e) Decreases by 1

9. The batting average for 40 innings of a cricket player is 50 runs. His best score exceeds his lowest score by 172 runs. If these two innings are excluded, the average of the remaining 38 innings becomes 48 runs. His highest score is

a) 172 b) 173 c) 174 d) 175 e) 176

10. Nine men went to a hotel. Eight of them spent Rs 30 each over their meals and the ninth spent Rs 20 more than the average expenditure of all the nine people. The total money spent by all was

a) 260 b) 262.5 c) 290 d) 292.5 e) 293

11. In an examination, the average marks obtained by Shailesh in English, Hindi and Drawing were 50. His average marks in Math, Science, Social Science and Craft were 70. If the average marks in all the seven subjects were 58, his score in Math was

a) 68 b) 70 c) 72 d) 74 e) Can't be determined

12. The average of the first six odd primes is

a) 6.67 b) 9.33 c) 4.83 d) 6.83 e) None of these

13. The average of five consecutive even numbers is 38. The largest of these is

a) 40 b) 38 c) 44 d) 42 e) 55

14. The average age of A, B, C and D is 58 years. If the average age of B, C, D and E is 60 years and the ages of A and E are in the ratio of 7:8, the age of E is

a) 64 b) 62 c) 66 d) 63 e) 61

15. The average of 11 numbers is 50. The average of the first six numbers is 49 and that of the last six numbers is 52. The sixth number is

a) 50 b) 52 c) 54 d) 56 e) 58

16. A batsman increases his average by 2 runs when he makes 63 runs in his 12^{th} inning. What is his average after 12 innings?

a) 39 b) 40 c) 41 d) 42 e) 43

17. On a journey across Delhi, a taxi averages 30 kmph for 60% of the distance, 20 kmph for 20% of it and 10 kmph for the remainder. The average speed for the whole journey is

a) 20 b) 22.5 c) 25 d) 24.625 e) 26

18. The average of marks obtained by 120 candidates was 35. If the average of passed candidates was 39 and that of the failed candidates was 15, the number of candidates who passed the examination was

a) 100 b) 110 c) 120 d) 150 e) 170

19. A class of 30 students obtained on an average 45 marks. On rechecking it was found that marks had been wrongly entered in two cases. After correction these marks were increased by 24 and 6 respectively. The correct average marks per student are

a) 47 b) 46 c) 48 d) 54 e) 59

20. If $1/a + 1/b = 1/c$ and $ab = c$, what is the average of a and b?

a) 0 b) ½ c) 1 d) $c/2$ e) $2c/3$

21. A man travels a certain distance at the rate of 20 kmph and returns at the rate of 30 kmph. The average speed for the whole journey in kmph is

a) 22 b) 27 c) 25 d) 26 e) 24

22. Average monthly income of a family of four earning members was Rs. 2940. One of the earning members died and therefore the average income came down to 2600. The income of the deceased was

a) 3280 b) 3960 c) 2770 d) 5540 e) 5830

23. The average of 8 readings is 24.3 out of which the average of first two is 18.5 and that of the next three is 21.2. If the sixth reading is 3 less than the seventh and 8 less than the 8^{th}, what is the sixth reading?

a) 24.8 b) 26.5 c) 27.6 d) 29.4 e) 30.02

24. The average salary per head of all the employees of an institution is $60. The average salary per head of 12 officers is $400 and the average salary per head of the rest is $56. Find the total number of employees in the institution.

a) 1000 b)900 c) 1030 d) 1032 e) 950

25. The average weight of a class of 35 students is 47.5 kg. If the weight of the teacher is included, the average rises by 500 gm. What is the weight of the teacher?

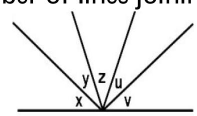
a)60 b) 61 c) 65.5 d) 62 e) 50

4. BASIC GEOMETRY

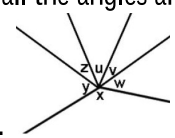
ANGELS AND LINES

An angle of $90^\circ$is a right angle; an angle less than $90^\circ$is acute angle; an angle between $90^\circ$and $180^\circ$is an obtuse angle; $180^\circ$and 360°angle between and is a reflex angle.

The sum of all angles made on one side of a straight line AB at a point O by any number of lines joining the line AB at O is 180°. In the fig. below, the sum of the angles u, v, x, y and z is equal to 180°.



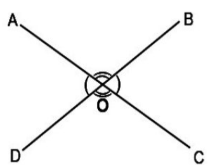
When any number of straight lines join at a point, the sum of all the angles around that point is 360°. In the figure below, the sum of the angles u, v, w, x, y, and z is equal 360°.



Two angles whose sum is $90^\circ$are said to be complementary angles and two angles whose sum $180^\circ$is are said to be supplementary angles.

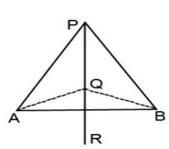
When two straight lines intersect, vertically opposite angles are equal. In Fig.1.03 given below, $\angle AOB$and $\angle COD$ are vertically opposite angles and $\angle BOC$and $\angle AOD$are vertically opposite angles. So, we have

$\angle AOB = \angle COD$and $\angle BOC = \angle AOD$



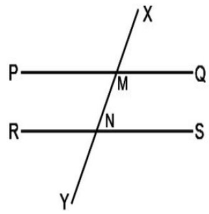
- Two lines which make an angle of $90^\circ$with each other are said to be PERPENDICULAR to each other.
- If a line $l_1$passes through the mid-point of another line $l_2$then the line $l_1$is said to be the BISECTOR of the line l_2i.e., the line $l_2$is divided into two equal parts.
- If a line $l_1$is drawn at the vertex of an angle dividing the angle two equal parts, then the line $l_1$is said to be the ANGULAR BISECTOR of the angle. Any point on the angular bisector of an angle is EQUIDISTANT from the two arms of the angle.
- If a line $l_1$is perpendicular to line $l_2$as well as passes through the mid-point of line l_2, then the line $l_1$is said to be the PERPENDICULAR BISECTOR of the line l_2.

Any point on the perpendicular bisector of a line is EQUIDISTANT from both ends of the line.



In the given figure, line PQ is the perpendicular bisector of line AB. A point P on the perpendicular bisector of AB will be equidistant from A and B, i.e., PA = PB Similarity. For any point R on the perpendicular bisector PQ, RA = RB.

PARALLEL LINES



When a straight line cuts/ intersects two or more parallel lines, the intersecting line is called the TRANSVERSAL. When a straight line XY cuts two parallel lines PQ and RS [as shown in the figure], the following are the relationships between various angles that are formed. [M and N are the points of intersection of XY with PQ and RS respectively]

(a) Alternate angles are equal, i.e.

$\angle PMN = \angle MNS$and $\angle QMN = \angle MNR$

(b) Corresponding angles are equal, i.e.

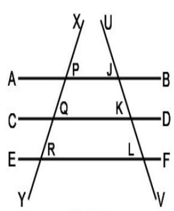
$$\angle XMQ = \angle MNS; \angle QMN = \angle SNY$$
$$\angle XMP = \angle MNR; \angle PMN = \angle RNY$$

(c) Sum of interior angles on the same side of the cutting line is equal to 180°, i.e

$\angle QMN + \angle MNS = 180^\circ$and $\angle PMN + \angle MNR = 180^\circ$

(d) Sum of exterior angles on the same side of the transversal is equal to 180°, i.e

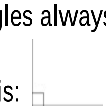
$\angle XMQ + \angle SNY = 180^\circ$and $\angle XMP + \angle RNY = 180^\circ$



If three or more parallel lines make intercepts on a transversal in a certain proportion, then they make intercepts in the same proportion on any other transversal as well. In the figure below, the lines AB, CD and EF are parallel and the transversal XY cuts them at the points P, Q and R. If we now take a second transversal, UV, cutting the three parallel lines at the points J, K and L, then we have **PQ/QR = JK/KL.**

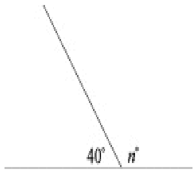
If three or more parallel lines make equal intercepts on one transversal, they make equal intercepts on any other transversal as well.

- **Acute, Obtuse, and Right Angles**

Angles that measure less than 90° are called ***acute angles***. Angles that measure more than 90° are called ***obtuse angles***. A special angle that appears quite often on GRE is called **a *right angle*.** Right angles always measure exactly 90° and are indicated by a little square where the angle measure would normally be, like this: 

- **Straight Angles**

Multiple angles that meet at a single point on a line are called straight angles. The sum of the angles meeting at a single point on a straight line is always equal to 180°. You may see something like this, asking you to solve for *n*:



The two angles (one marked by 40° and the one marked by *n*°) meet at the same point on the line. Since the sum of the angles on the line must be 180°, you can plug 40° and *n*° into a formula like this:

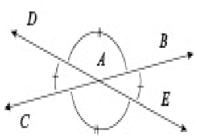
$$40° + n° = 180°$$

Subtracting 40° from both sides gives $n° = 140°$, or $n = 140$.

<u>Note: Angles measuring more than 180° but less than 360° are called reflex angles.</u>

- **Vertical Angles**

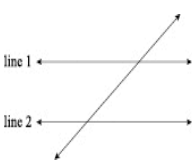
When two lines intersect, the angles that lie opposite each other, called ***vertical angles***, are always equal.



Angles $\angle DAC$ and $\angle BAE$ are vertical angles and are therefore equal. Angles $\angle DAB$ and $\angle CAE$ are also vertical, equal angles.

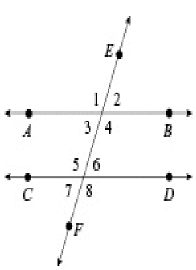
- **Parallel Lines**

A more complicated version of this figure that you may see on the GRE involves two parallel lines intersected by a third line, called a ***transversal***. *Parallel* means that the lines run in exactly the same direction and never intersect. Here's an example:



line 1 is parallel to line 2

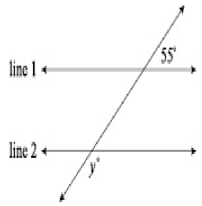
Don't assume lines are parallel just because they look like they are—the question will always tell you if two lines are meant to be parallel. Even though this figure contains many angles, it turns out that it only has two *kinds* of angles: big angles (obtuse) and little angles (acute). All the big angles are equal to each other, and all the little angles are equal to each other. Furthermore, ***any big angle + any little angle* = 180°**. This is true for any figure with two parallel lines intersected by a third line.



The eight angles created by these two intersections have special relationships to each other:

- Angles 1, 4, 5, and 8 are equal to one another. Angle 1 is vertical to angle 4, and angle 5 is vertical to angle 8.
- Angles 2, 3, 6, and 7 are equal to one another. Angle 2 is vertical to angle 3, and angle 6 is vertical to angle 7.
- The sum of any two adjacent angles, such as 1 and 2 or 7 and 8, equals 180º because they form a straight angle lying on a line.
- The sum of *any big angle* + *any little angle* = 180°, since the big and little angles in this figure combine into straight lines and all the big angles are equal and all the little angles are equal. So, a big and little angle don't need to be next to each other to add to 180°; *any* big plus *any* little will add to 180°. For example, since angles 1 and 2 sum to 180º, and since angles 2 and 7 are equal, the sum of angles 1 and 7 also equals 180º.

By using these rules, you can figure out the degrees of angles that may seem unrelated. For example:



If line 1 is parallel to line 2, what is the value of *y* in the given figure?

Again, this figure has only two kinds of angles: big angles (obtuse) and little angles (acute). We know that 55° is a little angle, so *y* must be a big angle. Since *big angle* + *little angle* = 180°, you can write: $y° + 55° = 180°$

Solving for *y*: $y° = 180° – 55°$; $y = 125$

Perpendicular Lines
Two lines that meet at a right angle are called ***perpendicular lines.*** If you're told two lines are perpendicular, just think 90°.

Polygon Basics
A polygon is a two-dimensional figure with three or more straight sides. Polygons are named according to the number of sides they have.

Number of Sides	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon
12	dodecagon
n	*n*-gon

All polygons, no matter how many sides they possess, share certain characteristics:

- The sum of the interior angles of a polygon with *n* sides is **(n – 2)180°.** For instance, the sum of the interior angles of an octagon is (8 – 2)180°= 6(180°) = 1080°.
- A polygon with equal sides and equal interior angles is a **regular polygon**.
- The **sum of the exterior angles** of any polygon is **360°**.
- The perimeter of a polygon is the sum of the lengths of its sides.
- The area of a polygon is the measure of the area of the region enclosed by the polygon. Each polygon tested on the GRE has its own unique area formula, which we'll cover below.

For the most part, the polygons tested in GRE math include triangles and quadrilaterals.

- **Triangles**

Of all the geometric shapes, triangles are among the most commonly tested on the GRE. But since the GRE tends to test the same triangles over and over, you just need to master a few rules and a few diagrams. We'll look at some special triangles shortly, but first we'll explain four very special rules.

TRIANGLES

➢ **Sum of the three angles of a triangle is** 180°

➢ The exterior angle or triangle (at each vertex) is equal to the sum of the two opposite interior angles. (Exterior angle is the angle formed at any vertex, by one side and the extended portion of the second side at that vertex).

➢ A line perpendicular to a side and passing through the midpoint of the side is said to be the perpendicular bisector of the side. It is not necessary that the perpendicular bisector of a side should pass through the opposite vertex in a triangle in general.

➢ The perpendicular drawn to a side from the opposite vertex is called the altitude to that side.

➢ The line joining the midpoint of a side with the opposite vertex is called the median drawn to that side. A median divides the triangle into two equal halves as far as the area is concerned.

➢ An **equilateral** triangle is one in which all the sides are equal (and hence, all angles are equal, i.e., each of the angles is equal to 60°). An **isosceles** triangle is one in which two sides are equal (and hence, the angles opposite to them are equal). A **scalene** triangle is one in which no two sides are equal.

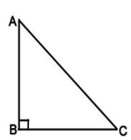
➢ In an isosceles triangle, the unequal side is called the BASE. The angle where the two equal sides meet is called the VERTICAL ANGLE. In an isosceles triangle, the perpendicular drawn to the base from the vertex opposite the base (i.e., the altitude drawn to the base) bisects the base as well as the vertical angle. That is, the altitude drawn to the base will also be the perpendicular bisector of the base as well as the angular bisector of the vertical angle. It will also be the median drawn to the base.

➢ In an equilateral triangle, the perpendicular bisector, the medium and the altitude drawn to a particular side coincide and that will also be the angular bisector of the opposite vertex. If a is the side of an equilateral triangle, then its altitude is equal to $\sqrt{3}a/2$

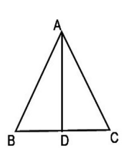
➢ Sum of any two sides of a triangle is greater than the third side; difference of any two sides of a triangle is less than the third side.

➢ If the sides are arranged in the ascending order of their measurement, then the angles opposite the sides (in the same order) will also be in ascending order (i.e. greater angle has greater side opposite to it); if the sides are arranged in descending order of their measurement, the angles opposite the sides in the same order will also be in descending order (i.e., smaller angle has smaller side opposite to it).

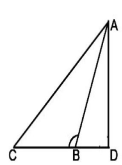
➢ There can be only one right angle or only one obtuse angle in any triangle. There can also not be one right angle and an obtuse angle both present at the same time in a triangle. Hypotenuse is the side opposite the right angle in a right-angled triangle. In a right-angled triangle, hypotenuse is the largest side. In an obtuse angled triangle, the side opposite the obtuse angle is the largest side.



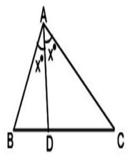
In a right-angled triangle, the square on the hypotenuse (the side opposite the right angle) is equal to the sum of the squares on the other two sides. In the given fig, $AC^2 = AB^2 + BC^2$



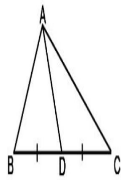
In an acute angled triangle, the square of the side opposite the acute angle is less than the sum of the squares of the other two sides by a quantity equal to twice the product of one of these two sides and the projection of the second side on the first side. In the given fig., $AC^2 = AB^2 + BC^2 - 2BC.BD$



In an obtuse angled triangle, the square of the side opposite the obtuse angle is greater than the sum of the squares of the other two sides by a quantity equal to twice the product of one of the sides containing the obtuse angle and the projection of the second side on the first side. In the given fig. $AC^2 = AB^2 + BC^2 = 2BC.BD$

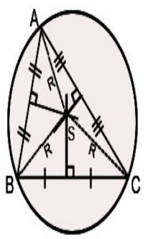


In triangle, the internal bisector of an angle bisects the opposite side in the ratio of the other two sides. In triangle ABC, if AD is the angular bisector of angle A, then BD/DC = AB/AC. This is called the angular Bisector Theorem (refer to the above fig.)



In $\triangle ABC$ if AD is the median from A to side BC (meeting BC at its mid point D), then $2(AD^2 + BD^2) = AB^2 + AC^2$. This is called the **Apollonius Theorem**. This will be helpful in calculating the lengths of the three medians given the lengths of the three sides of the triangle.

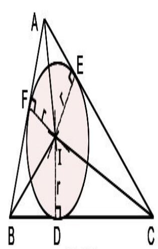
GEOMETRIC CENTRES OF A TRIANGLE



CIRCUMCENTRE

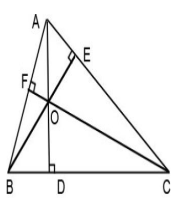
The three perpendicular bisectors of a triangle meet at a point called circumcentre of the triangle and it is represented by S. The circumcentre of a triangle is equidistant from its vertices is called circumradius (represented by R) of the triangle. The circle drawn with the circumcentre as centre and circumradius as radius is called the circumcircle of the triangle and passes through all there vertices of the triangle.

INCENTRE



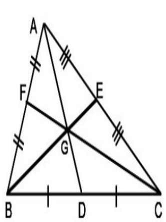
The internal bisectors of the three angles of a triangle meet at a point called incentre of the triangle and it is represented by I. incentre is equidistant from the three sides of the triangle i.e., the perpendicular's drawn from the in centre to the three sides are equal in length and this length is called the inradius (represented by r) of the triangle. The circle drawn with incentre as centre and inradius as radius is called the incircle of the triangle and it touches all three sides on the inside. In the figure above $\angle BIC = 80^\circ + \frac{1}{2}A$ Where I is the incentre $\angle CIA = 90^\circ + \frac{1}{2}B$ and $\angle AIB = 90^\circ + \frac{1}{2}C$

ORTHOCENTRE



The three altitudes meet at a point called Orthocenter and it is represented by O (refer to the figure above) $\angle = 180^\circ - A, \angle COA = 180^\circ - B, \angle AOB = 180^\circ - C$

CENTROID

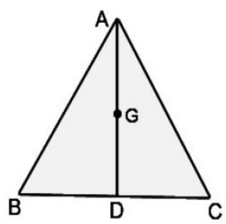


The three medians of a triangle meet at a point called the centroid and it is represented by G.

Important points about geometric centers of a triangle

- Please note the following important points pertaining to the geometric centers of a triangle ABC. In an acute angled triangle, the circumcenter lies inside the triangle. In a right-angled triangle, the circumcenter lies on the hypotenuse of the triangle (it is the midpoint of the hypotenuse). In an obtuse angled triangle, the circumcenter lies outside the triangle.

- In an acute angled triangle, the orthocenter lies inside the triangle. In a right-angled triangle, the vertex where the right angle is formed (i.e., the vertex opposite the hypotenuse) is the orthocenter. In an obtuse angled triangle, the orthocenter lies outside the triangle.
- In a right-angled triangle the length of the median drawn to the hypotenuse is equal to half the hypotenuse. This median is also the circumradius of the right-angled triangle.
- Centroid divides each of the medians in the ratio 2:1, the part of the median towards the vertex being twice in length to the part towards the side.
- Inradius is less than half of any of the three altitudes of the triangle.
- In an isosceles triangle, the centroid, the orthocenter, the circumcentre and incentre, all lie on the median to the base.



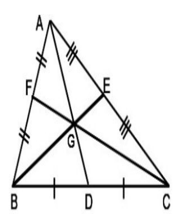
- In an equilateral triangle, the centroid, the orthocenter, the circumcentre and the incentre, all coincide.

Hence, in equilateral triangle ABC shown in the figure, AD is the median, altitude, angular bisector and perpendicular bisector. G is the centroid which divides the median in the ratio of 2:1. Hence, AG =2/3 and GD = 1/3 AD.

But since AD is also the perpendicular bisector and angular bisector and since G is the circumcentre as well as the incentre, AG will b the circumradius and GD will be the inradius of the equilateral triangle ABC Since AD is also the side of the equilateral triangle ABC. Since AD is also the altitude, its length is equal to $\sqrt{3}a/2$where a is the side of the equilateral triangle. Hence, the circum radius of the equilateral $\frac{2}{3} * \frac{\sqrt{3}}{2}.a = a/\sqrt{3}$and the inradious $\frac{1}{3} * \frac{\sqrt{3}}{2}.a = a/2\sqrt{3}$.

Since the radii of the circumcircle and the incircle of an equilateral triangle are in the ratio 2 : 1, the areas of the circumcircle and the incircle of an equilateral triangle will be in the ratio of 4:1.

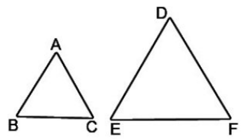
When the three medians of a triangle (i.e., the medians to the three sides of a triangle from the corresponding opposite vertices) are drawn, the resulting six triangles are equal in area and the area of each of these triangles in turn is equal to one-sixth of the area of the original triangle.



In the figure AD, BE and CF are the medians drawn to the three sides. The three medians meet at the centroid G. The six resulting triangles AGF, BGF, BGD, CGD, CGE and AGE are equal in area and each of them is equal to $\frac{1}{6}th$ of the area of triangle ABC.

SIMILARITY OF TRIANGLES

Two triangles are said to be similar if the three angles of one triangle are equal to the three angles of the second triangle. Similar triangles are alike in shape only. The corresponding angles of two similar triangles are equal but the corresponding sides are only proportional and not equal.



For example, if $\triangle ABC$ is similar to $\triangle DEF$ where $\angle A = \angle D, \angle B = \angle E \, and \, \angle C = \angle F$ then we have ratios of the corresponding sides equal:

$$\frac{AB}{DE} = \frac{BC}{EE} = \frac{CA}{FD}$$

By "corresponding sides", we mean that if we take a side opposite to a particular angle in one triangle, we should consider the side opposite to the equal angle in the second triangle. In this case, since AB is the side opposite to $\angle C \, in \, \angle ABC$, and since $\angle C = \angle F$, we have DE which is the side opposide to $\angle F \, in \, \triangle DEF$.

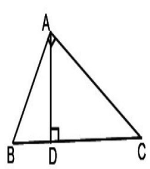
Two triangles are similar if,

- the three angles of one triangle are respectively equal to the three angles of the second triangle, or
- two sides of one triangle are proportional to two sides of the other and the included angles are equal, or
- if the three sides are of one are proportional to the three sides of another triangle.

In two similar triangles,

(a) Ratio of corresponding sides = Ratio of heights (altitudes) = Ratio of the lengths of the medians = Ratio of the lengths of the angular bisectors = Ratio of inradii = Ratio of circumradii = Ratio of perimeters.

(b) Ratio of areas = Ratio of squares of corresponding sides.



In a right-angled triangle, the altitude drawn to the hypotenuse divides the given triangle into two similar triangles, each of which is in turn similar to the original triangle. In triangle ABC in the figure, ABC is a right- angled triangle where $\angle$is a right angle. AD is triangle the perpendicular drawn to the hypotenuse BC. The triangles ABD, CAD and CBA are similar because of the equal angles given below.

In triangle ABC,

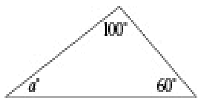
$\angle A = 90°$, if $\angle B = \theta$, then $\angle C = 90° - \theta$ In triangle ABD ,$\angle ADB = 90°$ we already know that $\angle B = \theta$, hence $\angle BAD = 90° - \theta$
Triangle ABD, $\angle ABD = 90°$we already know that $\angle C = 90° - \theta$, Hence $\angle CAD = \theta$
We can write down the relationship between the sides in these three triangles. The important relationships that emerge out of this exercise are:

1 $AD^2 = BD.DC$ **2** $AB^2 = BC.BD$ **3** $AC^2 = CB.CD$

▪ The Four Rules of Triangles

Commit these four rules to memory.

1. The Rule of Interior Angles. ***The sum of the interior angles of a triangle always equals 180°.*** Interior angles are those on the inside. Whenever you're given two angles of a triangle, you can use this formula to calculate the third angle. For example:



What is the value of *a* in the figure above?

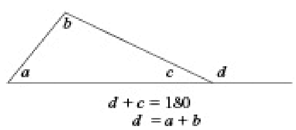
Since the sum of the angles of a triangle equals 180°, you can set up an equation:

$$a° + 100° + 60° = 180°$$

Isolating the variable and solving for a gives $a = 20$.

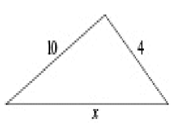
2. The Rule of Exterior Angles. An exterior angle of a triangle is the angle formed by extending one of the sides of the triangle past a vertex (or the intersection of two sides of a figure). In the given figure, d is the exterior angle.

Since, together, d and c form a straight angle, they add up to 180°: $d + c = 180°$. According to the first rule of triangles, the three angles of a triangle always add up to 180°, so $a + b + c = 180°$. Since $d + c = 180°$ and $a + b + c = 180°$, d must be equal to $a + b$ (the remote interior angles). This generalizes to all triangles as the following rules: ***The exterior angle of a triangle plus the interior angle with which it shares a vertex is always 180°. The exterior angle is also equal to the sum of the measures of the remote interior angles.***



3. The Rule of the Sides. The length of any side of a triangle must be greater than the difference and less than the sum of the other two sides. In other words: ***difference of other two sides < one side < sum of other two sides.***

Although this rule won't allow you to determine a precise length of the missing side, it will allow you to determine a range of values for the missing side, which is exactly what the test makers would ask for in such a problem. Here's an example:



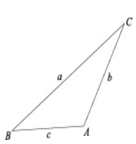
What is the range of values for x in the triangle above?

Since the difference of 10 and 4 is 6, and the sum of 10 and 4 is 14, we can determine the range of values of x:

$$6 < x < 14$$

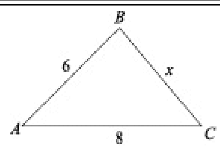
Keep in mind that x must be *inside* this range. That is, x could *not* be 6 or 14.

4. The Rule of Proportion. ***In every triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle***. Take a look at the following figure and try to guess which angle is largest.



In this figure, side a is clearly the longest side and $\angle A$ is the largest angle. Meanwhile, side c is the shortest side and $\angle C$ is the smallest angle. So $\boldsymbol{c < b < a}$ and $\angle C < \angle B < \angle A$. This proportionality of side lengths and angles holds true for all triangles.

Use this rule to solve the question below:

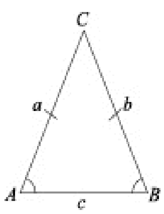


What is one possible value of x if $\angle C < \angle A < \angle B$?

A. 4
B. 5
C. 7
D. 10
E. 15

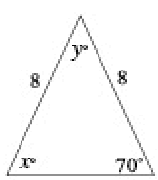
According to the rule of proportion, the longest side of a triangle is opposite the largest angle, and the shortest side of a triangle is opposite the smallest angle. The question tells us that angle C < angle A < angle B. So, the largest angle in triangle ABC is angle B, which is opposite the side of length 8. We know too that the smallest angle is angle C, since angle C < angle A. This means that the third side, with a length of x, measures between 6 and 8 units in length. The only choice that fits this criterion is 7, choice **C**.

- **Isosceles Triangles**



An isosceles triangle has two equal sides and two equal angles, like shown in the figure:

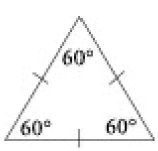
The tick marks indicate that sides a and b are equal, and the curved lines inside the triangle indicate that angle A equals angle B. Notice that the two equal angles are the ones opposite the two equal sides. Let's see how we might put this knowledge to use on the test. Check out this next triangle:



With two equal sides, this is an isosceles triangle. Even though you're not explicitly told that it has two equal angles, any triangle with two equal sides must also have two equal angles. This means that x must be 70°, because angles opposite equal sides are equal. Knowing that the sum of the angles of any triangle is 180°, we can also calculate y: $y + 70 + 70 = 180$, or $y = 40$.

- **Equilateral Triangles**

An equilateral triangle has three equal sides and three equal angles, like this:



The tick marks tip us off that the three sides are equal. We can precisely calculate the angles because the 180° of an equilateral triangle broken into three equal angles yields 60° for each. As soon as you're given one side of an equilateral triangle, you'll immediately know the other two sides, because all three have the same measure. If you ever see a triangle with three equal sides, you'll immediately know its angles measure 60°. Conversely, if you see that a triangle's three interior angles all measure 60°, then you'll know its sides must all be equal.

- **Right Triangles**

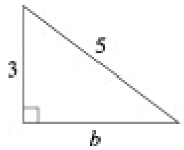
A right triangle is any triangle that contains a right angle. The side opposite the right angle is called the ***hypotenuse*.** The other two sides are called ***legs*.** The angles opposite the legs of a right triangle add up to 90°. That makes sense, because the right angle itself is 90° and every triangle contains 180° total, so the other two angles must combine for the other 90°. Right triangles are so special that they even have their own rule, known as the Pythagorean theorem.

- **THE PYTHAGOREAN THEOREM**

It's one of the most famous theorems in all of math, and it is tested with regularity on the GRE, to boot. Here it is:

In a right triangle, $a^2 + b^2 = c^2$, where c is the length of the hypotenuse and a and b are the lengths of the two legs.

And here's a simple application:



What is the value of b in the triangle above?

The little square in the lower left corner lets you know that this is a right triangle, so you're clear to use the Pythagorean theorem. Substituting the known lengths into the formula gives:

$$3^2 + b^2 = 5^2$$
$$9 + b^2 = 25$$
$$b^2 = 16$$
$$b = 4$$

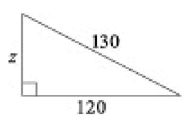
This is therefore the world-famous 3-4-5 right triangle.

- ***Pythagorean Triples.***

Because right triangles obey the Pythagorean theorem, only a specific few have side lengths that are all integers. For example, a right triangle with legs of length 3 and 5 has a hypotenuse of length $\sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34} = 5.83$. Positive integers that obey the Pythagorean theorem are called **Pythagorean triples**, and these are the ones you're likely to see on your test as the lengths of the sides of right triangles. Here are some common ones:

{3, 4, 5}   {5, 12, 13}   {7, 24, 25}   {8, 15, 17}   {9,40,41}   {12,35,37}

In addition to these Pythagorean triples, you should also watch out for their multiples. For example, {6, 8, 10} is a Pythagorean triple, since it is a multiple of {3, 4, 5}, derived from simply doubling each value. This knowledge can significantly shorten your work in a problem like this:



What is the value of z in the triangle above?

Sure, you could calculate it out using the Pythagorean theorem, but who wants to square 120 *and* 130 and then work the results into the formula?

But armed with our Pythagorean triples, it's no problem for us. The hypotenuse is 130, and one of the legs is 120. The ratio between these sides is 130:120, or 13:12. This exactly matches the {5, 12, 13} Pythagorean triple. So we're missing the 5 part of the triple for the other leg. However, since the sides of the triangle in the question are 10 times longer than those in the {5, 12, 13} triple, the missing side must be 5 × 10 = 50.

- **30-60-90 RIGHT TRIANGLES**

Right-angle triangles include two extra-special ones that appear with astounding frequency on the GRE. They are 30-60-90 right triangles and 45-45-90 right triangles. When you see one of these, instead of working out the Pythagorean theorem, you'll be able to apply standard ratios that exist between the length of the sides of these triangles.

The name derives from the fact that these triangles have angles of 30°, 60°, and 90°. So, what's so special about that? This: The side lengths of 30-60-90 triangles always follow a specific pattern. If the short leg opposite the 30° angle has length *x*, then the hypotenuse has length 2*x*, and the long leg, opposite the 60° angle, has length $x\sqrt{3}$. Therefore:

The sides of every 30-60-90 triangle will follow the ratio 1: $\sqrt{3}$:2

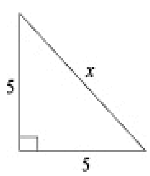
Thanks to this constant ratio, if you know the length of just one side of the triangle, you'll immediately be able to calculate the lengths of the other two. If, for example, you know that the side opposite the 30º angle is 2 meters, then by using the ratio you can determine that the hypotenuse is 4 meters, and the leg opposite the 60º angle is $2\sqrt{3}$ meters.

- **45-45-90 RIGHT TRIANGLES**

A 45-45-90 right triangle is a triangle with two angles of 45° and one right angle. It's sometimes called an ***isosceles right triangle***, since it's both isosceles and right. Like the 30-60-90 triangle, the lengths of the sides of a 45-45-90 triangle also follow a specific pattern. If the legs are of length *x* (the legs will always be equal), then the hypotenuse has length $x\sqrt{2}$.

The sides of every 45-45-90 triangle will follow the ration of $1:1:\sqrt{2}$

This ratio will help you when faced with triangles like this:



This right triangle has two equal sides, which means the two angles other than the right angle must be 45° each. So we have a 45-45-90 right triangle, which means we can employ the $1:1:\sqrt{2}$ ratio. But instead of being 1 and 1, the lengths of the legs are 5 and 5. Since the lengths of the sides in the triangle above are five times the lengths in the $\{1,1,\sqrt{2}\}$ right triangle, the hypotenuse must be $5\times\sqrt{2}$, or $5\sqrt{2}$.

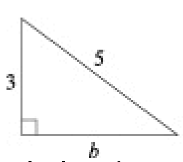
- **Area of a Triangle**

The formula for the area of a triangle is:

area = $\frac{1}{2}\times \text{base} \times \text{height}$ or where s =

Keep in mind that the base and height of a triangle are *not* just any two sides of a triangle. The base and height must be perpendicular, which means they must meet at a right angle.

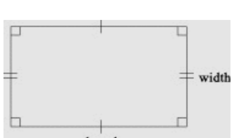
Let's try an example. What's the area of this triangle?



Note that the area is not $\frac{1}{2}\times 3\times 5$, because those two sides do not meet at a right angle. To calculate the area, you must first determine *b.* You'll probably notice that this is the 3-4-5 right triangle you saw earlier, so *b* = 4. Now you have two perpendicular sides, so you can correctly calculate the area as follows: ½ * 3 *4 = 6.

We move now to quadrilaterals, which are four-sided figures.

- **Rectangles**



A rectangle is a quadrilateral in which the opposite sides are parallel and the interior angles are all right angles. The opposite sides of a rectangle are equal, as indicated in the figure below:

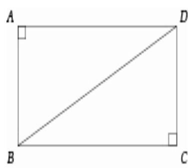
- **Area of a Rectangle**

The formula for the area of a rectangle is: **<u>area = base × height or simply *a = bh*</u>**

Since the base is the length of the rectangle and the height is the width, just multiply the length by the width to get the area of a rectangle i.e. **<u>area = length x width</u>**

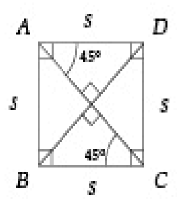
- **Diagonals of a Rectangle**

The two diagonals of a rectangle are always equal to each other, and either diagonal through the rectangle cuts the rectangle into two equal right triangles. In the figure below, the diagonal BD cuts rectangle *ABC*D into congruent right triangles BAD and BCD. ***Congruent*** means that those triangles are exactly identical.



Since the diagonal of the rectangle forms right triangles that include the diagonal and two sides of the rectangle, if you know two of these values, you can always calculate the third with the Pythagorean theorem. For example, if you know the side lengths of the rectangle, you can calculate the length of the diagonal. If you know the diagonal and one side length, you can calculate the other side length.

- **<u>Squares</u>**



They, along with circles, are perhaps the most symmetrical shapes in the universe—nothing to sneeze at. A square is so symmetrical because its angles are all 90º and all four of its sides are equal in length. Like a rectangle, a square's opposite sides are parallel and it contains four right angles. But squares one-up rectangles by virtue of their equal sides.

- **Area of a Square**

The formula for the area of a square is: **area = <u>s^2 or $d^2/2$</u>**

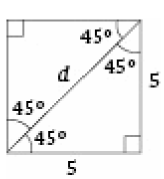
In this formula, s is the length of a side. Since the sides of a square are all equal, all you need is one side to figure out a square's area.

- **Diagonals of a Square**

The square has two more special qualities:

- Diagonals bisect each other at right angles and are equal in length.
- Diagonals bisect the verte*x* angles to create 45º angles. (This means that one diagonal will cut the square into two 45-45-90 triangles, while two diagonals break the square into four 45-45-90 triangles.)

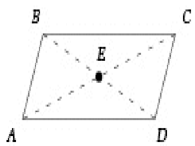
Because a diagonal drawn into the square forms two congruent 45-45-90 triangles, if you know the length of one side of the square, you can always calculate the length of the diagonal:



Since d is the hypotenuse of the 45-45-90 triangle that has legs of length 5, according to the ratio $1:1:\sqrt{2}$, you know that $d = 5\sqrt{2}$. Similarly, if you know only the length of the diagonal, you can use the same ratio to work backward to calculate the length of the sides.

- **<u>Parallelograms</u>**

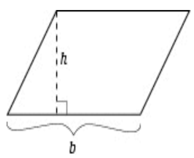
A parallelogram is a quadrilateral whose opposite sides are parallel. That means that rectangles and squares qualify as parallelograms, but so do four-sided figures that don't contain right angles.



In a parallelogram, opposite sides are equal in length. That means that in the figure above, BC = AD and AB = DC. Opposite angles are equal: angle*ABC* = angleADC and angleBAD = angleBCD. Adjacent angles are supplementary, which means they add up to 180°. Here, an example is angle*ABC* + angleBCD = 180°.

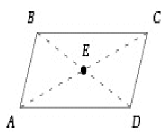
- **Area of a Parallelogram**

The area of a parallelogram is given by the formula: **area = *bh***



In this formula, *b* is the length of the base, and h is the height. As shown in the figure, the height of a parallelogram is represented by a perpendicular line dropped from one side of the figure to the side designated as the base.

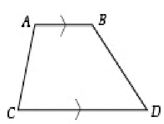
- **Diagonals of a Parallelogram**



- The diagonals of a parallelogram bisect (split) each other: BE = ED and AE = EC
- One diagonal splits a parallelogram into two congruent triangles: ΔABD = ΔBCD
- Two diagonals split a parallelogram into two pairs of congruent triangles: ΔAEB = ΔDEC and ΔBEC = ΔAED

Trapezoids

A trapezoid is a quadrilateral with one pair of parallel sides and one pair of nonparallel sides. Here's an example:



In this trapezoid, AB is parallel to CD (shown by the arrow marks), whereas AC and BD are not parallel.

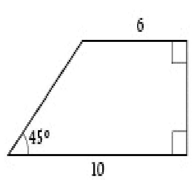
- **Area of a Trapezoid**

$$\text{area} = \frac{s_1 + s_2}{2} h$$

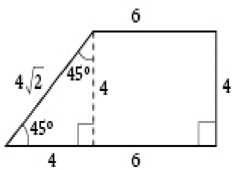
In this formula, s1 and s2 are the lengths of the parallel sides (also called the bases of the trapezoid), and h is the height. In a trapezoid, the height is the perpendicular distance from one base to the other.

If you come across a trapezoid question on the GRE, you may need to use your knowledge of triangles to solve it. Here's an example of what we mean:

Find the area of the given figure.

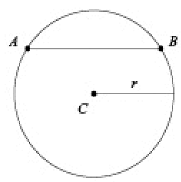


First of all, we're not told that the figure is a trapezoid, but we can infer as much from the information given. Since both the line labeled 6 and the line labeled 10 form right angles with the line connecting them, the 6 and 10 lines must be parallel. Meanwhile, the other two lines (the left and right sides of the figure) cannot be parallel because one connects to the bottom line at a right angle, while the other connects with that line at a 45° angle. So we can deduce that the figure is a trapezoid. The bases of the trapezoid are the parallel sides, and we're told their lengths are 6 and 10. But to find the area, we also need to find the height, which isn't given. To do that, split the trapezoid into a rectangle and a 45-45-90 triangle by drawing in the height.



Once you've drawn in the height, you can split the base that's equal to 10 into two parts: The base of the rectangle is 6, and the leg of the triangle is 4. Since the triangle is 45-45-90, the two legs must be equal. This leg, though, is also the height of the trapezoid. So the height of the trapezoid is 4. Now you can plug the numbers into the formula: $\text{area} = \frac{6+10}{2}(4) = 8(4) = 32$

Another way to find the area of the trapezoid is to find the areas of the triangle and the rectangle, then add them together: **area = $\frac{1}{2}$(4 × 4) + (6 × 4). Thus, $\frac{1}{2}$(16) + 24. Thus, 8 + 24 = 32.**



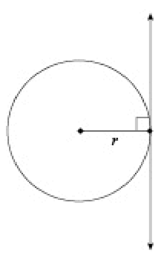
Circles

A circle is the collection of points equidistant from a given point, called the center. Circles are named after their center points. All circles contain 360°. The distance from the center to any point on the circle is called the ***radius*** (r). Radius is the most important measurement in a circle, because if you know a circle's radius, you can figure out all its other characteristics, such as its area, diameter, and circumference. The diameter (d) of a circle stretches between endpoints on the circle, passing through the center. A chord also extends from endpoint to endpoint on the circle, but it does not necessarily pass through the center. In the following figure, point C is the center of the circle, *r* is the radius, and AB is a chord.

- **Tangent Lines**

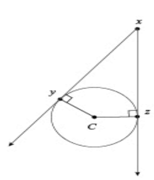
Tangents are lines that intersect a circle at only one point. Just like everything else in geometry, tangent lines are defined by certain fixed rules.

Here's the first: A radius whose endpoint is the intersection point of the tangent line and the circle is always

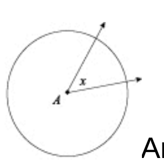


perpendicular to the tangent line, as shown in the following figure:

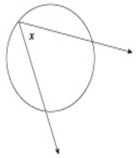
And the second rule: Every point in space outside the circle can extend exactly two tangent lines to the circle. The distances from the origin of the two tangents to the points of tangency are always equal. In the figure below, XY = XZ.



- **Central Angles and Inscribed Angles**



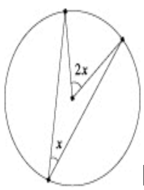
An angle whose vertex is the center of the circle is called a central angle.



The degree of the circle (the slice of pie) cut by a central angle is equal to the measure of the angle. If a central angle is 25º, then it cuts a 25º arc in the circle.

An ***inscribed angle*** is an angle formed by two chords originating from a single point.

An inscribed angle will always cut out an arc in the circle that is twice the size of the degree of the inscribed angle. For example, if an inscribed angle is 40º, it will cut an arc of 80º in the circle.

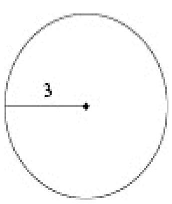


If an inscribed angle and a central angle cut out the same arc in a circle, the central angle will be twice as large as the inscribed angle.

Circumference of a Circle

The circumference is the perimeter of the circle—that is, the total distance around the circle. The formula for circumference of a circle is: **circumference = $2\pi r$ or πd**

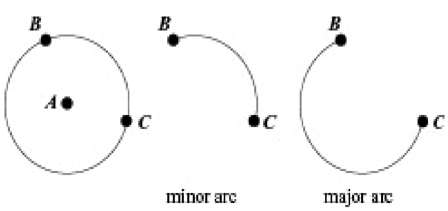
In this formula, *r* is the radius. Since a circle's diameter is always twice its radius, the formula can also be written c = πd, where d is the diameter. Let's find the circumference of the circle below:



Plugging the radius into the formula, $c = 2\pi r = 2\pi\ (3) = 6\pi$.

- **Arc Length**

An arc is a part of a circle's circumference. An arc contains two endpoints and all the points on the circle between the endpoints. By picking any two points on a circle, two arcs are created: a major arc, which is by definition the longer arc, and a minor arc, the shorter one.



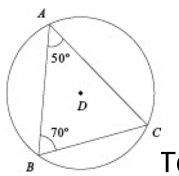
Since the degree of an arc is defined by the central or inscribed angle that intercepts the arc's endpoints, you can calculate the arc length as long as you know the circle's radius and the measure of either the central or inscribed angle.

The arc length formula is: **arc length** = $\frac{n}{360} \times 2\pi r$

In this formula, *n* is the measure of the degree of the arc, and *r* is the radius. This makes sense, if you think about it: There are 360° in a circle, so the degree of an arc divided by 360 gives us the fraction of the total circumference that arc represents. Multiplying that by the total circumference (2p*r*) gives us the length of the arc.

Here's the sort of arc length question you might see on the test:

Circle *D* has radius 9. What is the length of arc *AB*?



To figure out the length of arc *AB,* we need to know the radius of the circle and the measure of angle*C,* the inscribed angle that intercepts the endpoints of arc *AB.* The question provides the radius of the circle, 9, but it throws us a little curveball by not providing the measure of angle*C.* Instead, the question puts angleC in a triangle and tells us the measures of the other two angles in the triangle. Like we said, only a little curveball: You can easily figure out the measure of angle*C* because, as you know by now, the three angles of a triangle add up to 180º:

angle C = 180° – (50° + 70°) **Thus, angle *C*= 180° – 120°** **Thus, angle *C* = 60°**

Since angle *C* is an inscribed angle, arc *AB* must be twice its measure, or 120º. Now we can plug these values into the formula for arc length:

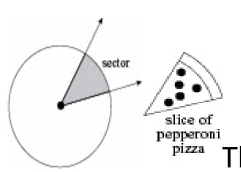
$$\text{arc ab} = \frac{120}{360} \times 2\pi(9) = \frac{1}{3} \times 18\pi = 6\pi$$

- **Area of a Circle**

If you know the radius of a circle, you can figure out its area. The formula for area is: **$\underline{\text{area} = \pi r^2 \text{ or } \pi d^2 / 4}$**
In this formula, *r* is the radius. So when you need to find the area of a circle, your real goal is to figure out the radius. In easier questions the radius will be given. In harder questions, they'll give you the diameter or circumference and you'll have to use the formulas for those to calculate the radius, which you'll then plug into the area formula.

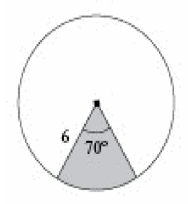
- **Area of a Sector**

A sector of a circle is the area enclosed by a central angle and the circle itself. It's shaped like a slice of pizza. The shaded region in the figure on the left below is a sector. The figure on the right is a slice of pepperoni pizza. See the resemblance?



The area of a sector is related to the area of a circle just as the length of an arc is related to the circumference. To find the area of a sector, find what fraction of 360° the sector makes up and multiply this fraction by the total area of the circle. In formula form: **area of a sector =** $\frac{n}{360} \times \pi r^2$

In this formula, *n* is the measure of the central angle that forms the boundary of the sector, and *r* is the radius. An example will help. Find the area of the sector in the figure below:



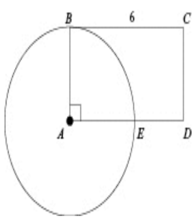
The sector is bounded by a 70° central angle in a circle whose radius is 6. Using the formula, the area of the sector is: $\frac{70}{360} \times \pi(6)^2 = \frac{7}{36} \times 36\pi = 7\pi$

o Mish-Mashes: Figures with Multiple Shapes

The trick in these problems is to understand and be able to manipulate the rules of each figure individually, while also recognizing which elements of the mish-mashes overlap. For example, the diameter of a circle may also be the side of a square, so if you use the rules of circles to calculate that length, you can then use that answer to determine something about the square, such as its area or perimeter.

Mish-mash problems often combine circles with other figures. Here's an example:

What is the length of minor arc *BE* in circle *A* if the area of rectangle *ABCD* is 18?

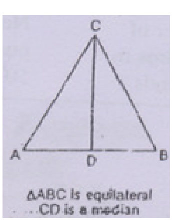


To find the length of minor arc *BE,* you have to know two things: the radius of the circle and the measure of the central angle that intersects the circle at points *B* and *E.* Because *ABCD* is a rectangle, and rectangles only have right angles, angle *BAD* is 90°. In this question, they tell you as much by including the right-angle sign. But in a harder question, they'd leave the right-angle sign out and expect you to deduce that angle *BAD* is 90° on your own. And since that angle also happens to be the central angle of circle *A* intercepting the arc in question, we can determine that arc *BE* measures 90°.

The key is to realize that the radius of the circle is equal to the width of the rectangle. So let's work backward from the rectangle to give us what we need to know about the circle. The area of the rectangle is 18, and its length is 6. Since the area of a rectangle is simply its length multiplied by its width, we can divide 18 by 6 to get a width of 3. As we've seen, this rectangle width doubles as the circle's radius, so we're in business: radius = 3. All we have to do is plug in the values we found into the arc length formula, and we're done.

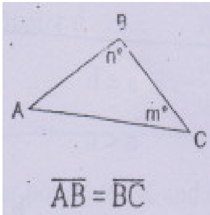
$$\begin{aligned}\text{length of minor arc BE} &= \frac{90}{360} \times 2\pi(3) \\ &= \frac{1}{4} \times 6\pi \\ &= \frac{6\pi}{4} \\ &= \frac{3\pi}{2}\end{aligned}$$

EXERCISE

1. 

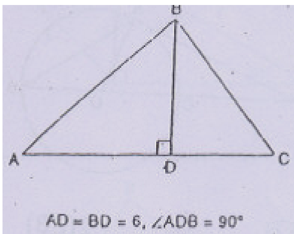
If CD = 3, then the area of triangle ACB is

a) 6 b) 3 c) 3/2 d) $3\sqrt{3}$ e) $(3\sqrt{3}) / 2$

2. 

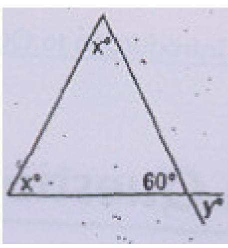
Which of the following CAN be true? SELECT ALL THAT APPLY

i) n>m   ii) m>n   iii) m=n

3. 

If triangle DBC is an isosceles right-angle triangle, what is the area of triangle ABC?

a) 36   b) $30\sqrt{2}$ c) 24   d) 18   e) It can't be determined

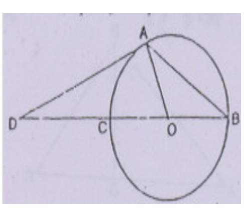
4. 

If the length of one of the sides of the triangle is 6, what is the perimeter of the given triangle?

a) 36   b) 18   c) 12   d) $9\sqrt{3}$ e) It can't be determined

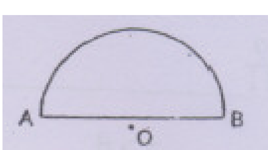
5. Circle O has radius 1 unit. What is the difference between its circumference and area?

a) 2   b) 2π c) π d) $\pi/2$ e) $\pi/4$

6. 

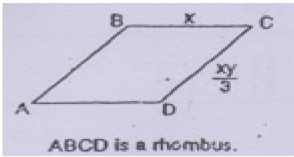
Measure of angle AOB = 120^0. Length of OC = 3 cm. What is the perimeter of sector AOC?

a) $6+2\pi$ b) $6+\pi$ c) $3+\pi$ d) $2+\pi$ e) π

7. 

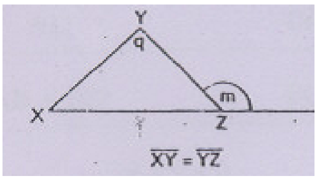
If the length of arc AB = 6π, what is the area of the given circle?

a) It can't be determined   b) 36π c) 18π d) 9π e) 6π

8. 

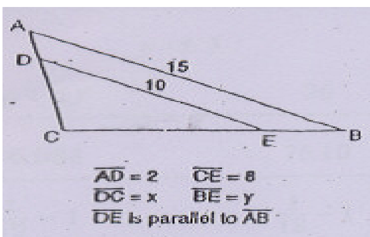
What is the relation between x and y?

a) x= y=3 b) x<y c) y<x d) x =2y e) It can’t be determined

9. 

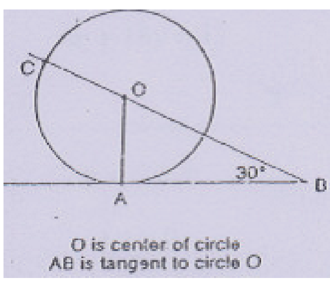
q can be expressed as

a) 2m – 180 b) m – 180 c) m + 60 d) m/2 e) m/3

10. 

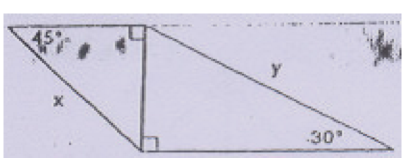
What is the value of x and y?

a) 1 b) 2 c) 3 d) 4 e) 5

11. 

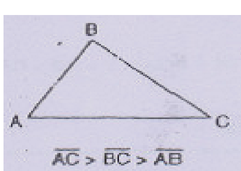
If OA = 5, What is the difference between area of the sector AOC and area of triangle AOB?

a) 25 b) 5 c) (5 - 25√3) d) (25π)/3 - (25√3)/2 e) None of these

12. 

The ratio between x and y (x:y) is

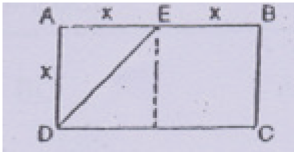
a) 2:1 b) 1: √2 c) 1: √3 d) 1:1 e) It can’t be determined

13. 

Which of the following MUST be true?

i) C is the smallest angle ii) Measure of angle C < 60^0

iii) Measure of angle C = 60^0 iv) Measure angle A = Measure angle B

14. 

What is $\frac{\text{Area of DEBC}}{\text{Area of rectangle ABCD}}$?

a) – b) – c) – d) – e) –

15. In a square WXYZ (not shown), point V is the mid-point of side YZ and the area of triangle XYV is 4/5. What is the area of square WXYZ?

a) 2 b) 8/5 c) 4 d) 16/5 e) 18/5

16. Which of the following could be the lengths of the sides of a triangle?

a) (1,2,3) b) (3,6,9) c) (4,8,16) d) (5,10,20) e) (7,8,13)

17. A, B, C, D and E are all distinct points that lie in the same plane. If seg AB is parallel to seg CD and seg AC is parallel to set BD, which of the following is a set of points all of which could lie on the same line?

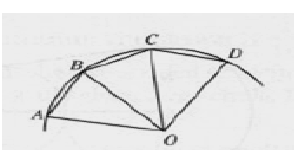
a) {A,B,C,E} b) {B,C,D,E} c) {C,D,E} d) {A, C,D} e) {A,B,D}

18. If the length of a rectangle is one-third the perimeter of the rectangle, then the width of the rectangle is what fraction of the perimeter?

a) 1/3 b) 2/3 c) 1/6 d) 1/12 e) None of these

19. When the base and height of an isosceles right triangle are each decreased by 4, the area decreases by 72. What is the height of the original triangle?

a) 4 b) 8 c) 16 d) 20 e) 32

20. 

The given figure represents part of a regular polygon with n sides, inscribed in a circle with center O. In terms of *n*, what is the measure of $\angle$OBC?

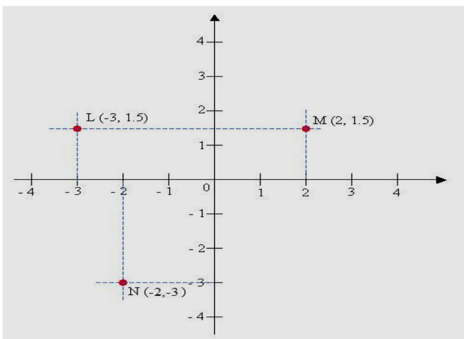
a)— b) 180 – *n* c) 180*n* - 360 d) 180 - — e) 90 - —

5. CO-ORDINATE GEOMETRY

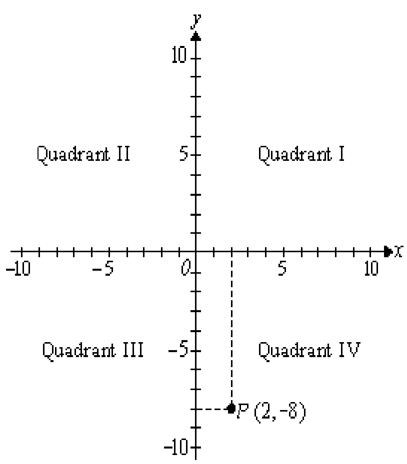
Coordinate Plane

The coordinate plane is a basic concept for coordinate geometry. It describes a two-dimensional plane in terms of two perpendicular axes: x and y. The x-axis indicates the horizontal direction while the y-axis indicates the vertical direction of the plane. In the coordinate plane, points are indicated by their positions along the x and y-axes.

For example: In the coordinate plane below, point L is represented by the coordinates (–3, 1.5) because it is positioned on –3 along the x-axis and on 1.5 along the y-axis. Similarly, you can figure out why the points M = (2, 1.5) and N = (–2, –3).



The rectangular coordinate plane, or *xy*-plane, is shown below:



- The *x*-axis and *y*-axis intersect at the origin *O* (0,0).
- They partition the plane into four quadrants, as shown.
- Each point in the plane has coordinates (*x, y*) that give its location with respect to the axes.

For example, the point *P*(2, -8) is located 2 units to the right of the *y*-axis and 8 units below the *x*-axis.

Slopes

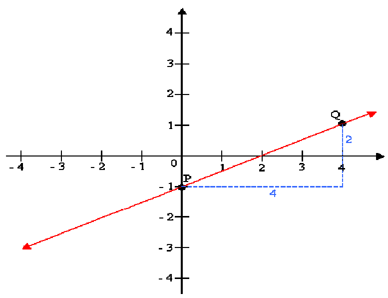
On the coordinate plane, the slant of a line is called the slope. Slope is the ratio of the change in the y-value over the change in the x-value.

Given any two points on a line, you can calculate the slope of the line by using this formula:

$$slope = \frac{\text{change in } y \text{ value}}{\text{change in } x \text{ value}} \quad \text{OR} \quad \boxed{\frac{\text{Difference between the y- co-ordinates}}{\text{Difference between the x- co-ordinates}}} \quad \text{OR} \quad \boxed{\frac{\text{RISE}}{\text{RUN}}}$$

For example: Given two points, P = (0, –1) and Q = (4,1), on the line we can calculate the slope of the line.

$$slope = \frac{change\ in\ y\ value}{change\ in\ x\ value} = \frac{1-(-1)}{4-0} = \frac{2}{4} = \frac{1}{2}$$

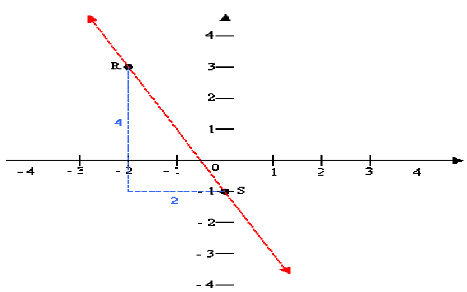


Negative Slope

Let's look at a line that has a negative slope.

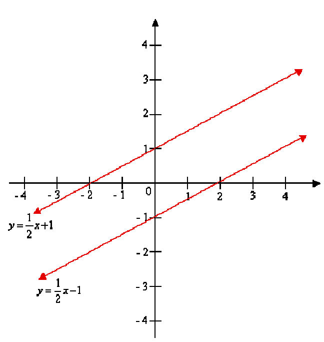
For example: Consider the two points, R(–2, 3) and S(0, –1) on the line. What would be the slope of the line?

$$slope = \frac{change\ in\ y\ value}{change\ in\ x\ value} = \frac{-1-3}{0-(-2)} = \frac{-4}{2} = -\frac{2}{1}$$



Slopes Of Parallel Lines

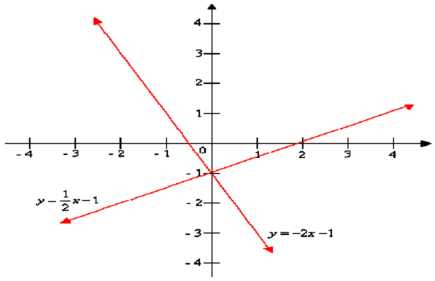
In coordinate geometry, two lines are parallel if their slopes (*m*) are equal.



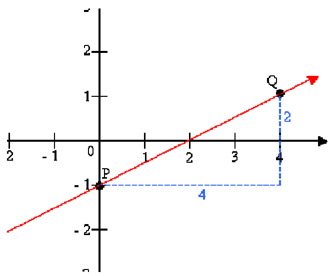
For example: The line $y = \frac{1}{2}x - 1$ is parallel to the line $y = \frac{1}{2}x + 1$. Their slopes are both the same.

Slopes Of Perpendicular Lines

In the coordinate plane, two lines are perpendicular if the product of their slopes (*m*) is –1.



For example: The line $y=\frac{1}{2}x-1$ is perpendicular to the line $y = -2x - 1$. The product of the two slopes is $\frac{1}{2}\times(-2)=-1$



Intercepts

The y-intercept is where the line intercepts (meets) the y-axis and the x-intercept is where the line intercepts (meets) the x-axis.

For example: In the above diagram, the line intercepts the y-axis at (0,–1). Its y-intercept is equal to –1.
The line intercepts the x-axis at (2,0). Its x-intercept is equal to 2.

Equation Of A Line

In coordinate geometry, the equation of a line can be written in the form, $y = mx + b$, where *m* is the slope and *b* is the y-intercept.

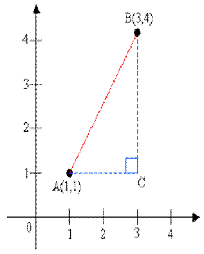
$$y = \overset{\text{slope}}{\overset{\downarrow}{m}}x + \underset{\text{y-intercept}}{\underset{\uparrow}{b}}$$

For example: The equation of the line in the above diagram is: $y=\frac{1}{2}x-1$

Distance Formula

In the coordinate plane, you can use the Pythagorean Theorem to find the distance between any two points.

The distance between the two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



For example: To find the distance between A(1,1) and B(3,4), we form a right angled triangle with $\overline{AB}$ as the hypotenuse. The length of $\overline{AC}$ = 3 – 1 = 2. The length of $\overline{BC}$ = 4 – 1 = 3. Applying Pythagorean Theorem:

$\overline{AB}^2 = 2^2 + 3^2$. $\overline{AB}^2 = 13$. $\overline{AB} = \sqrt{13}$

Midpoint Formula

To find a point that is halfway between two given points, get the average of the x-values and the average of the y-values.

The midpoint between the two points (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

For example: The midpoint of the points A(1,4) and B(5,6) is $\left(\frac{1+5}{2}, \frac{4+6}{2}\right) = \left(\frac{6}{2}, \frac{10}{2}\right) = (3,5)$

REMEMBER THESE POINTS

⇒y = mx + c where m is the slope.

⇒The slopes of two parallel lines, m1 and m2 are equal if the lines are parallel. If the two lines are perpendicular, m1*m2 = -1.

⇒Finding the y-intercept:- Put x=0 in the given equation, c is the y-intercept.

⇒Finding the x-intercept:- Put y=0. in the given equation, -c/m is the x-intercept.

⇒Equation of a straight line parallel to the y-axis at a distance 'a' from it is **x=a.**

⇒Equation of a straight line parallel to the x-axis at a distance 'b' from it is **y=b.**

⇒Equation of a line parallel to the x-axis and passing throught the point (a,b) is **y=b**.

⇒Equation of a line perpendicular to x-axis and passing through (a,b) is **x=a.**

⇒Equation of a line parallel to the y-axis and passing through (a,b) is **x=a**.

⇒Equation of a line perpendicular to the y-axis and passing through (a,b) is **y=b.**

⇒Equation of x-axis is y=0 and equation of y-axis is **x=0**.

⇒The equation of a straight line which cuts off intercepts a and b on the x-axis and y-axis is **x/a+y/b=1**.

⇒The equation of a straight line passing through the origin (0,0) is **y=mx**.

⇒The equation of a straight line passing through the origin and making equal angle with both the axes is **y=±x**

EXERCISE

1. Find the distance between the two points A (3,2) and B (7,5)

a) 4   b) 5   c) 6   d) 7   e) 7

2. Find the co-ordinates of point of intersection of two lines given by the equation: 3x+4y=7 and 5x-4y=9.
a) (2,4) b) (1/4, 2) c) (4,2) d) (3,2) e) (2, 1/4)

3. Find the equation of a line whose slope is 0.5 and which passes through the point (3,2).
a) 2y=x+1 b) y=2+x c) x=y+2 d) 2x=y+1 e) xy=2

4. A line with slope -1/2 intersects x-axis. A point Q (3,-5) lies on the same line. Find the point of intersection between the given line and the x-axis.
a) (-7,0) b) (7,0) c) (0,7) d) (0,-7) e) None of these

5. Line intersects x-axis at A and y-axis at B. If A is (10,0) and B is (0,10), find the equation of the line
a) x+y = 10 b) 10y=x c) 10x=y d) y=10+x e) x=10+y

6. Find the slope of the line passing through (-3,7) and having y-intercept -2.
a) -3 b) 3 c) 4 d) -4 e) 5

7. What kind of a quadrilateral is formed by the vertices (0,0), (4,3), (3,5), (-1,2)
a) Trapezium b) Parallelogram c) Square d) Rectangle e) Rhombus

8. A (a,0) and B (3a,0) are the vertices of an equilateral triangle ABC. What are the co-ordinates of C?
a) (2a,+/-a√3) b) (a,+/-a√3) c) (2a,+/-√a) d) (2a,+/-3√a) e) (a,+/-a√3)

9. The co-ordinates of the vertices of A and B are (6,0) and (0,-8) respectively. What is the area of the square ABCD?
a) 100 b) 99 c) 100.5 d) 99.4 e) 101

10. If points A (2,5), B (-7,2) and C(a,3) are collinear. Find the co-ordinates of C
a) 4 b) -3 c) 5 d) -4 e) 0.4

11. If P (2,1) and Q (8,1) are two of the vertices of a rectangle, which of the following could not be another of the vertices?
a) (2,8) b) (8,2) c) (2,-8) d) (-2,8) e) (8,8)

12. What are the co-ordinates of the point on the graph of y=3x+4 that has equal co-ordinates?
a) 6,6 b) 4,4 c) 2,2 d) -2,-2 e) -4,-4

13. Find the equation of the line having a slope of 4 passing through the point (5,-2)
a) y=4x+20 b) y=4x-22 c) y=5x-2 d) y=5x+15 e) $y=\frac{1}{4}x+2$

14. The vertices of triangle ABC are (3, 4), (3, 8) and (7,4). The area of the triangle ABC is
a) 6 b) 8 c) 20 d) 12 e) 16

15. Find the equation of a straight line passing through (3,4) and parallel to the straight line y=4x+3.
a) y=4x-8 b) y=4x-10 c) y=4x-12 d) y=4x-20 e) y=4x-18

16. A triangle has three vertices A (0,6), B (8,0) and C(8,6). If it is a right angle triangle, find the area of the triangle
a) 10 b) 6 c) 8 d) 12 e) 24

17. Find the equation of the line passing through the points (1,3) and (5,7).
a) y=x+2 b) y=x+1 c) y=x+3 d) y=x-2 e) y=x-4

18. Find the equation of a straight line which passes through the origin and whose angle with x-axis is 45^0.
a) x+y=1 b) x=y c) x-y=2 d) y=4x e) x=10+y

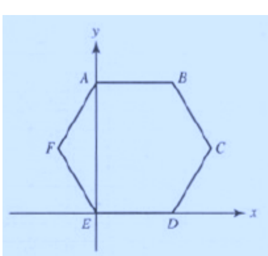
19. The co-ordinates of the vertices of quadrilateral ABCD are A (0,0), B (8,0), C (10,4) and D (2,4) respectively. Find the area of quadrilateral ABCD.
a) 30 b) 38 c) 28 d) 25 e) 32

20. What is the product of the slope of the sides of a rectangle, if none of the sides is parallel to any of the axes?
a) Can't be determined b) 1 c) -1 d) ½ e) -1/2

21. If line *l* has equation: y=5x-10 and line k is perpendicular to it which of the following is always true of line *k*?
a) *k* has a positive slope b) *k* has a negative slope c) *k* has a positive x-intercept
d) *k* has a positive y-intercept e) *k* passes through the origin

22. Line q is given by the equation y= -x+8, and line r is given by the equation 4y=3x-24. If line r intersects the y-axis at point A, line q intersects the y-axis at point B, and both lines intersect the x-axis at point C, what is the area of triangle ABC?
a) 14 b) 24 c) 48 d) 56 e) 112

23. The graphs of y=3x+2 and y=3x-4 in the xy-plane can intersect in at most how many points?
a) 0 b) 1 c) 2 d) 3 e) Infinite

24. 

In the given figure, ABCDEF is a regular hexagon. What is the slop of $\overleftrightarrow{FE}$?
a) – ½ b) -√3 c) - √2 d) √3 e) ½

6. RATIO, PROPORTION, VARIATION AND ALLIGATION

- **Ratios**

Ratios look like fractions and are related to fractions, but they don't quack like fractions. Whereas a fraction describes a part of a whole, a ratio compares one part to another part.

A ratio can be written in a variety of ways. Mathematically, it can appear as $\frac{3}{1}$ or as 3:1. In words, it would be written out as "the ratio of 3 to 1." Each of these three forms of the ratio 3:1 means the same thing: that there are three of one thing for every one of another. For example, if you have three red alligators and one blue alligator, then your ratio of red alligators to blue alligators would be 3:1. For the GRE, you must remember that ratios compare parts to parts rather than parts to a whole. Why do you have to remember that? Because of questions like this:

> For every 40 games a baseball team plays, it loses 12 games. What is the ratio of the team's losses to wins?
>
> (A) 3:10
>
> (B) 7:10
>
> (C) 3:7
>
> (D) 7:3
>
> (E) 10:3

The question says that the team loses 12 of every 40 games, but it asks you for the ratio of losses to *wins*, not losses to *games*. So the first thing you have to do is find out how many games the team wins per 40 games played: 40 – 12 = 28. So for every 12 losses, the team wins 28 games, for a ratio of 12:28. You can reduce this ratio by dividing both sides by 4 to get 3 losses for every 7 wins, or 3:7. Choice **C** is therefore correct. If you instead calculated the ratio of losses to games played (part to whole), you might have just reduced the ratio 12:40 to 3:10, and then selected choice **A**. For good measure, the test makers include 10:3 to entice anyone who went with 40:12 before reducing. There's little doubt that on ratio problems, you'll see an incorrect *part: whole* choice and possibly these other kinds of traps that try to trip you up.

- **Proportions**

Just because you have a ratio of three red alligators to one blue alligator doesn't mean that you can only have three red alligators and one blue one. It could also mean that you have six red and two blue alligators or that you have 240 red and 80 blue alligators. **Ratios compare only relative magnitude**. To know how many of each color alligator you actually have, in addition to knowing the ratio, you also need to know how many *total* alligators there are. This concept forms the basis of another kind of ratio problem you may see on the GRE, a problem that provides you with the ratio among items and the total number of items, and then asks you to determine the number of one particular item in the group. Sounds confusing, but as always, an example should clear things up:

Egbert has red, blue, and green marbles in the ratio of 5:4:3 and he has a total of 36 marbles. How many blue marbles does Egbert have?

First let's clarify what this means. For each *group* of 5 red marbles, Egbert has a *group* of 4 blue marbles and a *group* of 3 green marbles. If he has one group of each, then he'd simply have 5 red, 4 blue, and 3 green marbles for a total of 12. But he doesn't have 12—we're told he has 36. The key to this kind of problem is determining how many groups of each item must be included to reach the total. We have to multiply the total we'd get from having one group of each item by a certain factor that would give us the total given in the problem. Here, as we just saw, having one group of each color marble would give Egbert 12 marbles total, but since he has 36 marbles, we have to multiply by a factor of 3 (since 36 ÷ 12 = 3). That means Egbert has 3 groups of red marbles with 5 marbles in each group, for a total of 3 × 5 = 15 red marbles. Multiplying the other marbles by our factor of 3 gives us 3 × 4 = 12 blue marbles, and 3 × 3 = 9 green marbles. Notice that the numbers work out, because 15 + 12 + 9 does add up to 36 marbles total. The answer to the question is therefore 12 blue marbles.

So here's the general approach: Add up the numbers given in the ratio. Divide the total items given by this number to get the factor by which you need to multiply each group. Then find the item type you're looking for and multiply its ratio number by the factor you determined. In the example above, that would look like this:

5 (red) + 4 (blue) + 3 (green) = 12
36 ÷ 12 = 3 (factor)
4 (blue ratio) × 3 (factor) = 12 (answer)
$5x + 4x + 3x = 36$ Thus, $12x = 36$ Therefore, $x = 3$ Thus, blue = (4)(3) = 12

- **<u>Variation</u>**

Direct Variation / Direct Proportion	**Inverse Variation / Inverse Proportion**
($y \propto x$) When two variable quantities have a constant (unchanged) ratio, their relationship is called a direct variation. It is said that one variable "varies directly" as the other. A relationship between two variables in which one is a constant multiple of the other. In particular, when one variable changes the other changes in proportion to the first. The constant ratio is called the constant of variation. The formula for direct variation is $\boldsymbol{y = kx}$**,** where $\boldsymbol{k}$ is the constant of variation. "y varies directly as x" Solving for k: **k =** $\frac{y}{x}$ In a direct variation, the two variables change in the same sense. If one increases, so does the other and vice versa.	($y \propto \frac{1}{x}$) A relationship between two variables in which the product is a constant. When one variable increases the other decreases in proportion so that the product is unchanged. The statement " y varies inversely as x means that when x increases, y decreases by the same factor. In other words, the expression xy is constant: $\boldsymbol{y = \frac{k}{x}}$ where k is the constant of variation. We can also express the relationship between x and y as: **k= xy** In inverse variation, the two variables change in the opposite sense. If one increases, the other decreases.

- **<u>Alligation</u>**

When two or more quantities of different products are mixed together to produce a mixture of a mean value, the ratios of their amounts are inversely proportional to the differences of their values from the mean value. In other words, if the ingredients are mixed in a ratio, then:

$$\frac{Quantity\ of\ cheaper\ or\ lesser\ quantity}{Quantity\ of\ dearer\ or\ more\ quantity} = \frac{Dearer\ or\ more\ quantity - Mean\ or\ required\ quantity}{Mean\ or\ required\ quantity - Cheaper\ or\ lesser\ quantity}$$

OR

Quantity of cheaper or lesser quantity

Quantity of dearer or more quantity

Mean quantity or required quantity

Dearer or more quantity – Mean or required quantity

Mean or required quantity – Cheaper or lesser quantity

EXERCISE

1. The ratio of Tom's and Harry's ages is 3:5 and the sum of their ages is 80 years. What is the ratio of their ages after 10 years?
   a) 1:3 b) 2:3 c) 3:4 d) 5:7 e) 13:15

2. In a mixture of 35 litres, the ratio of milk and water is 4:1. If 1 litre of water is added to the mixture, what will be the new ratio of milk and water?
   a) 3:2 b) 5:2 c) 7:3 d) 7:2 e) 4:2

3. Two brothers, P and Q have their annual income in the ratio of 8:5 and their annual expenditure in the ratio of 5:3. If they save $1200 and $1000 per annum respectively, find their total annual income.
   a) 17400 b) 18000 c) 18200 d) 18500 e) 19000

4. The sum of two numbers is 40 and their difference is 4. Find the ratio of numbers.
   a) 5:4 b) 6:5 c) 7:3 d) 9:4 e) 11:9

5. A mixture of milk and honey has a ratio 4:5 in the first vessel and a ratio of 5:1 in the second vessel. In what ratio should the mixture be extracted from each vessel and poured into the third vessel, so that the ratio of milk and honey becomes 5:4 in the third vessel?
   a) 5:3 b) 6:5 c) 5:2 d) 4:3 e) 3:2

6. The amounts of time that three workers worked on a special job are in the ratio of 1:2:5. If they worked a combined total of 104 hours, how many hours did the worker who worked the longest spend on the project?
   a) 50 b) 55 c) 60 d) 65 e) 70

7. In a certain game, 3 nurbs are equal to 2 zimps and 6 clabs are equal to 1 zimp. 4 clabs are equal to how many nurbs?
   a) 1 b) 2 c) 3 d) 4 e) 5

8. In a certain company, the ratio of officers to workers is 5:72. If 8 additional workers were to be hired, the ratio of officers to workers would be 5:74. How many officers does the company have?
a) 15 b) 20 c) 25 d) 30 e) 35

9. If the ratio of boys to girls in the class is 3:5 and the class contains 32 students, how many additional boys would have to enroll to make the ratio of boys to girls 1:1?
a) 4 b) 6 c) 8 d) 10 e) 12

10. Three men A, B and C agreed to share the expenses of a trip in the ratio 2:3:4. A paid $250 to porters, B paid $700 for the stuff and C paid $ 490 for travelling expenses. How much should A and C together pay to B to settle their agreed share?
a) 200 b) 210 c) 220 d) 240 e) 300

11. A grey hound follows a deer and takes 6 leaps for every 7 leaps of the deer deer but 5 leaps of the grey hound are equal to 6 leaps of the deer. Compare the ratio of the hound and the deer.
a) 18: 35 b) 36:35 c) 1:1 d) 39:35 e) 54:35

12. A man wishes to divide his monthly savings of $846 among his two sons and a daughter in the ratio 1/4:1/5:1/3 respectively. How much should he give to his daughter?
a) 216 b) 270 c) 300 d) 360 e) 400

13. Three numbers bear a ratio of 2:3:4 to one another. If the sum of the reciprocals of the first and the third number exceeds the reciprocal of the second number by 5/12, find the second number
a) 1 b) 2 c) 3 d) 5 e) 6

14. $1087 are divided among A, B and C so that if $10, $12 and $15 are diminished from the shares of A, B and C respectively, the remainders will be in the ratio of 5:7:9. What is the share of B?
a) 162 b) 262 c) 362 d) 462 e) 562

15. What quantity of sugar costing Rs. 6.10 per kg must be mixed with 126 kgs of sugar priced at Rs. 2.85 per kg so that 20% may be gained by selling the mixture at Rs 4.80/kg
a)60 b) 63 c) 65 d) 67 e) 69

16. A mixture of 70 liters of alcohol and water contains 10% of water. How much water must be added to make the water 37% in the resulting mixture?
a) 15 b) 20 c) 25 d) 30 e) 35

17. The force of attraction between two bodies of mass m1 and m2 respectively varies directly as the product of their masses and inversely as the square of the distance between them. If the masses of the bodies and the distance between them are doubled, the force of attraction will become
a) 4 times b) 2 times c) half d) 3 times e) unchanged

18. What is the least whole number which when subtracted from the terms which are in the ratio of 6:7 gives a ratio less than 16:21?

    a) 1 b) 2 c) 3 d) 5 e) 6

19. A certain test consists of 8 sections with 25 questions numbered from 1 to 25, in each section. If a student answered all of the even-numbered questions correctly and ¾ of the odd-numbered questions correctly, what was the total number of questions he answered correctly?

    a) 150 b) 172 c) 174 d) 175 e) 176

20. Lou and Selma were hired to paint a room for a total of $72. They completed the job with Lou working 3 hours and 20 minutes and Selma working 2 hours and 40 minutes. If they decided to split the $72 in proportion to the amount of time each spent on the job, how much did Lou receive?

    a) 32 b) 36 c) 40 d)41.14 e) 43.20

21. Of the 40 schools invited to participate in a research study, 7/8 agreed to participate. If 60 questionnaires were sent to each of the participating schools and 4/5 of these questionnaires were completed and returned, what was the total number of questionnaires completed and returned?

    a)28 b) 1344 c) 1680 d) 1920 e) 2100

22. Which three of the following fractions are equivalent?

    v=5/80, w=0.05/0.08, x = 0.5/8.0, y=0.05/0.8 and z = 0.05/0.008

    a) v, w and x b) v, x and y c) w, x and y

    d) w, y and z e) x, y and z

23. Of the people who responded to a market survey, 120 preferred Brand X and the rest preferred Brand Y. If the respondents indicated a preference for Brand X over Brand Y by a ratio of 3:1, how many people responded to the survey?

    a) 80 b) 160 c) 240 d) 360 e) 480

    s

24. The current ratio of men to women on a certain board of trustees is 2 to 5. If 4 men were added to the board, the ratio of men to women would be 2 to 3. How many men are currently on the board?

    a) 2 b) 4 c) 5 d) 6 e) 8

25. Tom and Carlos begin to play a series of 4 games with 400 chips each. At the end of each game, there is a loser who must surrender half of his chips to the winner. If Tom wins only the first and the third games, how many chips does he have after the pay off at the end of the fourth game?

    a) 200 b) 275 c) 400 d) 525 e) 550

7. Percentage, Profit and Loss

The basic concept behind percents is pretty simple: Percent means divide by 100. This is true whether you see the word percent or you see the percentage symbol, %. For example, 45% is the same as $\frac{45}{100}$ or .45.

Here's one way percent may be tested: **4 is what percent of 20?**

The first thing you have to know how to do is translate the question into an equation. It's actually pretty straightforward as long as you see that "is" is the same as "equals," and "what" is the same as "*x*." So we can rewrite the problem as 4 equals *x* percent of 20, or: $4 = x\%(20)$

Since a percent is actually a number out of 100, this means: $4 = \frac{x}{100}(20)$

Now just work out the math: $$\begin{aligned} 4 &= \frac{20x}{100} \\ 400 &= 20x \\ x &= 20 \end{aligned}$$ Therefore, 4 is 20% of 20.

Percent problems can get tricky, because some seem to be phrased as if the person who wrote them doesn't speak English. The GRE test makers do this purposefully because they think that verbal tricks are a good way to test your math skills. And who knows—they may even be right. Here's an example of the kind of linguistic trickery we're talking about:

What percent of 2 is 5?

Because the 2 is the smaller number and because it appears first in the question, your first instinct may be to calculate what percent 2 is of 5. But as long as you remember that "is" means "equals" and "what" means "*x*" you'll be able to correctly translate the word problem into math:

$$\begin{aligned} x\%(2) &= 5 \\ \frac{x}{100}(2) &= 5 \\ \frac{2x}{100} &= 5 \\ 2x &= 500 \\ x &= 250 \end{aligned}$$

So 5 is 250% of 2.

You may also be asked to figure out a percentage based on a specific occurrence. For example, if there are 200 cars at a car dealership, and 40 of those are used cars, then we can divide 40 by 200 to find the percentage of used cars at the dealership: $\frac{40}{200} = \frac{4}{20} = \frac{1}{5} = 20\%$. The general formula for this kind of calculation is:

Percent of a specific occurrence = $\frac{\text{the number of specific occurences}}{\text{the total number}} \times 100\%$

- **Converting Percents into Fractions or Decimals**

Converting percents into fractions or decimals is an important GRE skill that may come into play in a variety of situations.

- To convert from a percent to a fraction, take the percentage number and place it as a numerator over the denominator 100. If you have 88 percent of something, then you can quickly convert it into the fraction $\frac{88}{100}$.
- To convert from a percent to a decimal, you must take a decimal point and insert it into the percent number two spaces from the right: 79% equals .79, while 350% equals 3.5.

- **<u>Percent Increase and Decrease</u>**

One of the most common ways the GRE tests percent is through the concept of percent increase and decrease. There are two main varieties: problems that give you one value and ask you to calculate another, and problems that give you two values and ask you to calculate the percent increase or decrease between them. Let's have a look at both.

- **<u>ONE VALUE GIVEN</u>**

In this kind of problem, they give you a single number to start, throw some percentage increases or decreases at you, and then ask you to come up with a new number that reflects these changes. For example, if the price of a \$10 shirt increases 10%, the new price is the original \$10 plus 10% of the \$10 original. If the price of a \$10 shirt decreases 10%, the new price is the original \$10 minus 10% of the \$10 original.

One of the classic blunders test takers make on this type of question is to forget to carry out the necessary addition or subtraction after figuring out the percent increase or decrease. Perhaps their joy or relief at accomplishing the first part distracts them from finishing the problem. In the problem above, since 10% of \$10 is \$1, some might be tempted to choose \$1 as the final answer, when in fact the answer to the percent increase question is \$11, and the answer to the percent decrease question is \$9.

Try the following example on your own. Beware of the kind of distracter just discussed above.

A vintage bowling league shirt that cost \$20 in 1990 cost 15% less in 1970. What was the price of the shirt in 1970?

(A) \$3

(B) \$17

(C) \$23

(D) \$35

(E) \$280

First find the price decrease (remember that 15% = .15): \$20 × .15 = \$3
Since the price of the shirt was less back in 1970, subtract \$3 from the \$20 1990 price to get the actual amount this classic would have set you back way back in 1970 (presumably before it achieved "vintage" status): \$20 – \$3 = \$17
If you finished only the first part of the question and looked at the choices, you might have seen \$3 in choice **A** and forgotten to finish the problem. **B** is the choice that gets the point. Try the next problem.

The original price of a banana in a store is \$2.00. During a sale, the store reduces the price by 25% and Joe buys the banana. Joe then raises the price of the banana 10% from the price at which he bought it and sells it to Sam. How much does Sam pay for the banana?

This question asks you to determine the cumulative effect of two successive percent changes. The key to solving it is realizing that each percentage change is dependent on the last. You have to work out the effect of the first percentage change, come up with a value, and then use that value to determine the effect of the second percentage change.

We begin by finding 25% of the original price: $\frac{25}{100} \times \$2 = \frac{\$50}{100} = \$.50$

Now subtract that \$.50 from the original price: \$2 – \$.50 = \$1.50

That's Joe's cost. Then increase \$1.50 by 10%: $\frac{10}{100} \times \$1.50 = \frac{\$15}{100} = \$.15$

Sam buys the banana for \$1.50 + \$.15 = \$1.65. A total rip-off, but still 35 cents less than the original price.

Some test takers, sensing a shortcut, are tempted to just combine the two percentage changes on double-percent problems. This is not a real shortcut. Here, if we reasoned that the first percentage change lowered the price 25%, and the second raised the price 10%, meaning that the total change was a reduction of 15%, then we'd get:

$$\frac{15}{100} \times \$2 = \frac{\$30}{100} = \$.30$$

Subtract that \$.30 from the original price: \$2 – \$.30 = \$1.70 = WRONG!

If you see a double-percent problem on the GRE, it will include this sort of wrong answer as a trap.

❖ TWO VALUES GIVEN

In the other kind of percent increase/decrease problem, they give you both a first value and a second value, and then ask for the percent by which the value changed from one to the other. If the value goes up, that's a percent increase problem. If it goes down, then it's a percent decrease problem. Luckily, we have a handy formula for both:

percent increase = $\frac{\text{difference between the two numbers}}{\text{smaller of the two numbers}} \times 100\%$ OR $\boxed{\frac{\text{Increase} \times 100}{\text{Original}}}$

percent decrease = $\frac{\text{difference between the two numbers}}{\text{greater of the two numbers}} \times 100\%$ OR $\boxed{\frac{\text{Decrease} \times 100}{\text{Original}}}$

To borrow some numbers from the banana example, Sam pays \$1.65 for a banana that was originally priced at \$2.00. The percent decrease in the banana's price would look like this:

$$\text{Percent decrease} = \frac{\$2.00 - \$1.65}{\$2.00} \times 100\% = \frac{\$.35}{\$2.00} \times 100\% = .175 \times 100\% = 17.5\%$$

So Sam comes out with a 17.5% discount from the original price, despite lining Joe's pockets in the process.

A basic question of this type would simply provide the two numbers for you to plug into the percent decrease formula. A more difficult question might start with the original banana question above, first requiring you to calculate Sam's price of \$1.65 and then asking you to calculate the percent decrease from the original price on top of that.

- **Common Fractions, Decimals, and Percents**

Some fractions, decimals, and percents appear frequently on the GRE. Being able to quickly convert these into each other will save time on the exam, so it pays to memorize the following table.

Fraction	Decimal	Percent
$\frac{1}{8}$	0.125	12.5%
$\frac{1}{6}$	$0.1\overline{6}$ (the little line above the 6 means that 6 repeats indefinitely, so 0.166 = .166666...)	$16\frac{2}{3}\%$
$\frac{1}{5}$	0.2	20%
$\frac{1}{4}$	0.25	25%
$\frac{1}{3}$	$0.33\overline{3}$	$33\frac{1}{3}\%$
$\frac{3}{8}$	0.375	37.5%
$\frac{2}{5}$	0.4	40%
$\frac{1}{2}$	0.5	50%
$\frac{5}{8}$	0.625	62.5%
$\frac{2}{3}$	$0.66\overline{6}$	$66\frac{2}{3}\%$
$\frac{3}{4}$	0.75	75%
$\frac{4}{5}$	0.8	80%
$\frac{7}{8}$	0.875	87.5%

EXERCISE

1. In an examination 36% are the pass marks. If a student gets 17 marks and fails by 10 marks, what are the maximum marks?
   a) 100 b) 60 c) 75 d) 90 e) 72

2. A earns 25% more than B. By how much percent is B's income less than that of A?
   a) 25 b) 24 c) 30 d) 20 e) 28

3. In an election, one of the two candidates gets 42% of the total votes and still loses by 368 votes. What is the total number of votes?
   a) 2000 b) 2300 c) 2500 d) 3680 e) 4200

4. A man gave 30% of his money to his wife, 40% of the remainder to his son and the remaining money equally to his three daughters. If each daughter gets Rs 224, what does the wife get?
   a) 520 b) 1600 c) 1500 d) 480 e) 300

5. The price of sugar is decreased by 10%. To get back to its original value, the new price must be increased by what percent?

a) 9.09 b) 10 c) 11 d) 11.11 e) 10.5

6. An article is sold at Rs 45 at a loss of 10%. If it is sold at Rs 65, the gain percent is
a) 10 b) 15 c) 20 d) 15 e) 30

7. By selling 150 mangoes, a fruit seller gains the S.P. of 30 mangoes. His gain is what percent?
a) 30 b) 25 c) 20 d) 15 e) 28

8. A soap company sells a soap at Rs 15 and gives a spoon worth Rs 1.80 free with it, making a profit of 10%. The cost price of soap is
a) 12 b) 13.25 c) 13.75 d) 11.75 e) 14.10

9. The entry ticket to a trade fair was increased by 20%. Owing to that, the number of visitors was reduced by 10%. The daily money receipts, however, increased by
a) 8 b) 10 c) 12 d) 50 e) 55

10. The cost of 1 kg of sweets increases every year by 10%. Therefore, after two years, the cost of 1 kg of sweets will be increased by what percent?
a) 10 b) 20 c) 21 d) 25 e) None of these

11. The marks obtained by two candidates A and B in an exam are 625 and 575 respectively. Marks of A exceed that of B by what percent?
a) 7.8 b) 8 c) 8.7 d) 8.8 e) 8.9

12. A reduction of 20% in the price of sugar enables a purchaser to obtain 5 kg more sugar for Rs 100. The price of sugar per kg before reduction is
a) 4.50 b) 5 c) 5.50 d) 8 e) 8.50

13. A man spends 25% of his income on house rent, 45% of his income on food and 40% of the balance on conveyance. If he is left with Rs 540, his income is
a) 4500 b) 3000 c) 1350 d) 900 e) 2800

14. In a competitive exam, all the candidates securing 60% or more marks were selected. Mohan secured 240 marks, which were 60 less than the last selected candidate. The total marks were
a) 400 b) 450 c) 500 d) 550 e) 600

15. If 18% of x is the same as 90% of y, then 60% of x is
a) 120% of y b) 20% of y c) 30% of y d) None of the above e) Data insufficient

16. The price of kerosene increases by 25%. By how much percentage a family must reduce its consumption of kerosene, so as not to increase the monthly expenditure on kerosene?

a) 15 b) 20 c) 25 d) 30 e) 35

17. A volleyball team has won 30 games out of the 40 played. If it has 12 more games to play, how many of these must the team win to make its record 75% of the season?
a) 7 b) 9 c) 10 d) 5 e) 12

18. Of the 4800 employees of the company, 1/3 are officers. If the officers' staff were to be reduced by ½, what percent of the remaining employees would then be officers?
a) 12 b) 15 c) 16 d) 20 e) 11

19. On a certain day, a vendor began his business with some apples. Between the opening and noon, he sold 40% of the apples and between noon and closing, he sold 60% of the apples which remained. What percent of the original apples did he sell?
a) 75 b) 70 c) 65 d) 50 e) 76

20. 8% of x subtracted from x is equal to multiplying x by which number?
a) 0.91 b) 0.92 c) 0.93 d) 0.95 e) 0.96

21. A man sold two cameras for $924 each. On one he gained 20% and on the other he lost 20%. How much does he lose in the whole transaction? (Loss Percentage)
a) 4% b) 10% c) 7% d) 8% e) 6%

22. A shop owner buys egg for P cents per dozen and sells them for P/4 cents per egg. At this rate, what is the profit percentage?
a)100 b) 107 c) 200 d) 175 e) 160

23. Sam sells a shirt at a profit of 25%. Had he bought it at 25 percent less and sold it for Rs 25 less, he would have gained 25%. The cost price of the shirt is
a) 60 b) 25 c) 80 d) 100 e) 120

24. Out of a number of electronic items, a person purchases 60% refrigerator. 5% of these were found to be defective. The percentage of defective refrigerators in all is
a) 3 b) 6 c) 9 d) 10 e) 11

25. Since 1950, when Tom was discharged from the army, he has gained two pounds every year. In 1980, he was 40% heavier than he was in 1950. What % of his 1995's weight was his 1980's weight?
a) 80 b) 85 c) 87.5 d) 90 e) 95

8. COUNTING AND PROBABILITY

Counting

Suppose that a job has two different parts. There are *m* different ways of doing the first part, and there are *n* different ways of doing the second part. The problem is to find the number of ways of doing the entire job. For each way of doing the first part of the job, there are *n* ways of doing the second part. Since there are *m* ways of doing the first part, the total number of ways of doing the entire job is *m* x *n*. The formula that can be used is

Number of ways = m x n

For any problem that involves two actions or two objects, each with a number of CHOICES and asks for the number of combinations, the above formula can be used.

Example: William wants a sandwich and a drink for lunch. If a restaurant has 4 choices of sandwiches and 3 choices of drinks, how many different ways can he order his lunch?

Solution

Since there are 4 choices of sandwiches and 3 choices of drinks, using the formula: Number of ways = 4(3) = 12

William can order lunch in 12 different ways.

Factorials

You may see some problems on your exam in which a number is followed by an exclamation point, like this: 5!. An exclamation point used in math symbolizes a *factorial*. A factorial stands for the product of all the numbers up to and including the given number. So $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Some more examples: $3! = 3 \times 2 \times 1 = 6$
$55! = 55 \times 54 \times 53 \times \ldots \times 3 \times 2 \times 1$ = a really huge number you would never be expected to solve for GRE

0! = 1

The proof of this last example is beyond the scope of what you need to know for the GRE. Just remember that 0! = 1 by definition.

The factorial of *n* also signifies the number of ways that the *n* elements of a group can be ordered. So, if you decide to ditch grad school and become a wedding planner instead, and need to figure out how many different ways six people can sit at a table with six chairs, 6! is the way to go: $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ possible seating arrangements. You'll astound the other wedding planners with this quick calculation, never revealing the true source of your knowledge.

As you might guess from the name, factorials have many factors. Recall that a factor is a number that divides into another number with no remainder. Whenever you take the factorial of a number, the result will be divisible by all of the integers up to and including the original number. For example, 6! is divisible by 6, 5, 4, 3, 2, and 1, and all of those numbers are factors of 6!. This is all inherent in the definition of factorial, but it's good to understand it in these terms too.

Simplifying Factorials

The test makers may ask you to work out a problem that involves factorials in fractions, and as you'll soon see, this becomes downright necessary in permutation and combination problems. The trick is to cancel before calculating. As you've seen in earlier examples, canceling with fractions means dividing the numerator and denominator by the same number. A little cancellation makes complicated-looking factorial problems much easier to solve. Check it out:

What is $\frac{100!}{98!}$?

This expression looks like it might be a huge number. And, in fact, trying to calculate 98! or 100! would be near impossible without a computer or ultra-fancy calculator. Fortunately, we can simplify this equation significantly:

$$\frac{100!}{98!} = \frac{100 \times 99 \times 98 \times 97 \times \ldots}{98 \times 97 \times 96 \times \ldots} = 100 \times 99 = 9{,}900$$

This works, because everything after and including the 98 cancels out in both the numerator and the denominator, leaving 100 × 99 in the numerator and 1 in the denominator. Here's another way to think about this:

$$\frac{100!}{98!} = \frac{100 \times 99 \times 98!}{98!}$$

The 98! in the numerator cancels out with the 98! in the denominator, leaving only 100 × 99 = 9,900.

Before you get stuck into multiplying out the factorials in the top and bottom parts of the fraction and then dividing the results, first cancel out what you can.

Permutations

Permutations of Unlike Objects of Number *n* taken *r* at a time.

Formula: $^{n}P_{r}$ =

In permutations there is no choosing, you are just arranging. The order of the objects is important.

When arranging numerals, 123 is different from 321 and 213. The rule here is that the number of ways to "arrange" n DIFFERENT objects is n! UNLESS you want to arrange them in smaller groups, then you follow the rule of n!/(n-r)!

Permutations Including like Objects. Formula: n! / r^1! r^2! .....
where r^1 is the number of one object which are alike and r^2 is the number of a second object alike, and so on.

There is no choosing. The order of the objects is important. You have some objects which are the same.

Example:
You want to arrange 10 books on a shelf but you have 2 copies of one book and 3 copiesof another. The number of ways you can arrange them is 10! / 2! x 3! which equals 3,628,800 ÷ 2 x 6 = 302,400.

Combinations

Combinations are where small groups of size r are made from a larger group of size n. Order is not important in combinations. **Formula: $^{n}C_{r}$ =**

Example: There are five meal options in the cafeteria of a certain school. Assuming that a different meal must be eaten each day, and each different type of meal must be eaten once before any type of meal can be eaten a second time, how many different orders of meals can a student eat in the first five days?

Solution: The answer is 120. There are five types of meals, so the total number of possibilities is 5!. (the "!" stands for factorial), or 5 x 4 x 3 x 2 x 1 = 120. Since a different meal is assigned to every day, you must reduce the amount that you multiply by on a daily basis from 5 to 4 to 3 to 2 to 1.

What if two or more samples are chosen at a time? If we have objects a, b, c, d and want to arrange them two at a time--that is, *ab*, *bc*, *cd*, etc. (We have four combinations taken two at a time). If you have four different combinations taken two at a time, you can write this as ${}^4C_2$, which can be written as

$${}^4C_2 = \frac{4 \times 3}{2 \times 1}$$

Examples

$${}^8C_3, = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \qquad {}^{10}C_4, = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$$

Example: This time we have 7 different books and we want to give three to a friend for Christmas. How many different gifts of three books could we give?

We have a large group of seven and want to see how many smaller groups of three we could make. You can see that order is not important, since the gift is the same no matter which of the 3 books is on the top or the bottom. The answer is 7! / 3!(7-3)! or 5040 ÷ 6 x 24 = 35.

Simple Probability

In general, the probability of an event is the number of favorable outcomes divided by the total number of possible outcomes.

Probability= (No. of favorable outcomes) / (No. of possible outcomes)

Example: What is the probability that a card drawn at random from a deck of cards will be an ace?

Solution

In this case there are four favorable outcomes:

(1) the ace of spades

(2) the ace of hearts

(3) the ace of diamonds

(4) the ace of clubs.

Since each of the 52 cards in the deck represents a possible outcome, there are 52 possible outcomes. Therefore, the probability is 4/52 or 1/13.

The same principle can be applied to the problem of determining the probability of obtaining different totals from a pair of dice.

Example: What is the probability that when a pair of six-sided dice are thrown, the sum of the numbers equals 5?

Solution: There are 36 possible outcomes when a pair of dice is thrown. Consider that if one of the dice rolled is a 1, there are six possibilities for the other die. If one of the dice rolled a 2, the same is still true. And the same is true if one of the dice is a 3,4,5, or 6. If this is still confusing, look at the following (abbreviated) list of outcomes: [(1,1),(1,2),(1,3),(1,4),(1,5),(1,6); (2,1),(2,2),(2,3)… (3,1),(3,2),3,3)… (4,1)…(5,1)…(6,1)….

The total number of outcomes is 6 × 6 = 36. Since four of the outcomes have a total of 5 [(1,4),(4,1),(2,3),(3,2)], the probability of the two dice adding up to 5 is 4/36 = 1/9.

Example: What is the probability that when a pair of six-sided dice are thrown, the sum of the number equals 12?

Solution: We already know the total number of possible outcomes is 36, and since there is only one outcome that sums to 12, (6,6--you need to roll double sixes), the probability is simply 1/36.

The Range of Probability

The probability, *P*, of any event occurring will always be **$0 \le P \le 1$**. A probability of 0 for an event means that the event will *never* happen. A probability of 1 means the event will always occur. For example, drawing a green card from a standard deck of cards has a probability of 0; getting a number less than seven on a single roll of one die has a probability of 1.

Thus, you can automatically eliminate any answer choices that are less than 0 or greater than 1.

Probability of Multiple Events

For questions involving single events, the formula for simple probability is sufficient. **For questions involving multiple events, the answer combines the probabilities for each event in ways that may seem counter-intuitive.** The following strategy is excellent for acquiring a better feel for probability questions involving multiple events or for **making a quick guess** if time is short. We will focus on questions involving two events.

- If two events have to occur together, generally an **"and"** is used. Take a look at **Statement 1:** "I will only be happy today if I get email and win the lottery." The "and" means that **both events are expected to happen together**.
- If both events do **not necessarily have to occur together**, an **"or"** may be used as in **Statement 2,** "I will be happy today if I win the lottery or have email."

Consider **Statement 1.** Your chances of getting email may be relatively high compared to your chances of winning the lottery, but if you expect both to happen, your chances of being happy are slim. Like placing all your bets at a race on one horse, you've decreased your options, and therefore you've decreased your chances. **The odds are better if you have more options**, say if you choose horse 1 or horse 2 or horse 3 to win. In **Statement 2**, we have more options; in order to be happy we can either win the lottery or get email.

1. A **and** B: the probability **<** than the **individual** probabilities of **either** A **or** B.
2. A **or** B must occur: the probability **>** the **individual** probabilities of either A or B. This is an excellent strategy for eliminating certain answer choices.

These two types of probability are formulated as follows:

Probability of A and B: **P(A and B) = P(A) × P(B).**

In other words, the probability of A and B both occurring is the product of the probability of A and the probability of B.

Probability of A or B: **P(A or B) = P(A) + P(B).**

In other words, the probability of A or B occurring is the sum of the probability of A and the probability of B.

Look at the following examples.

Example : If a coin is tossed twice, what is the probability that on the first toss the coin lands heads and on the second toss the coin lands tails?

a) 1/6 b) 1/3 c) ¼ d) ½ e) 1

Solution: First note the "and" in between event A (heads) and event B (tails). That means we expect both events to occur together, and that means fewer options, a less likely occurrence, and a lower probability. Expect the answer to be less than the individual probabilities of either event A or event B, so less than ½. Therefore, eliminate d and e. Next we follow the rule P(A and B) = P(A) × P(B). If event A and event B have to happen together, we multiply individual probabilities. ½ × ½ = ¼. Answer c is correct.

NOTE: Multiplying probabilities that are less than 1 (or fractions) always gives an answer that is smaller than the probabilities themselves.

Example: If a coin is tossed twice what is the probability that it will land either heads both times or tails both times?
a)1/8 b)1/6 c)1/4 d)1/2 e)1
Solution: Note the "or" in between event A (heads both times) and event B (tails both times). That means more options, more choices, and a higher probability than either event A or event B individually. To figure out the probability for event A or B, consider all the possible outcomes of tossing a coin twice: heads, heads; tails, tails; heads, tails; tails, heads. **Since only one coin is being tossed, the order of heads and tails matters. Heads, tails and tails, heads are sequentially different and therefore distinguishable and countable events.**
We can see that the probability for event **A is ¼ and that the probability for event B is ¼. We expect a greater probability given more options, and therefore we can eliminate choices a, b and c, since these are all less than or equal to ¼.** Now we use the rule to get the exact answer. P(A or B) = P(A) + P(B). If either event 1 or event 2 can occur, the individual probabilities are added: ¼ + ¼ = 2/4 = ½. Answer d is correct.

NOTE: We could have used simple probability to answer this question. The total number of outcomes is 4: heads, heads; tails, tails; heads, tails; tails; heads, while the desired outcomes are 2. The probability is therefore 2/4 = ½.

Independent and Dependent Events

By independent we mean that the first event does not affect the probability of the second event. Coin tosses are independent. They cannot affect each other's probabilities; the probability of each toss is independent of a previous toss and will always be 1/2. Separate drawings from a deck of cards are independent events if you put the cards back.

An example of a dependent event, one in which the probability of the second event is affected by the first, is drawing a card from a deck but not returning it. By not returning the card, you've decreased the number of cards in the deck by 1, and you've decreased the number of whatever kind of card you drew. I

If you draw an ace of spades, there are 1 fewer aces and 1 fewer spades. This affects our simple probability: (number of favorable outcomes)/ (total number of outcomes. This type of probability is formulated as follows:

If A and B are not independent, then the probability of A and B is P(A and B) = P(A) × P(B|A) where P(B|A) is the conditional probability of B given A.

Example: If someone draws a card at random from a deck and then, without replacing the first card, draws a second card, what is the probability that both cards will be aces?
Solution: Event A is that the first card is an ace. Since 4 of the 52 cards are aces, P(A) = 4/52 = 1/13. Given that the first card is an ace, what is the probability that the second card will be an ace as well? Of the 51 remaining cards, 3 are aces. Therefore, p(B|A) = 3/51 = 1/17, and the probability of A and B is 1/13 × 1/17 = 1/221.

Example: If there are 30 red and blue marbles in a jar, and the ratio of red to blue marbles is 2:3, what is the

probability that, drawing twice, you will select two red marbles if you return the marbles after each draw?
Solution: First, let's determine the number of red and blue marbles respectively. The ratio 2:3 tells us that the total of 30 marbles must be broken into 5 groups of 6 marbles, each with 2 groups of red marbles and 3 groups of blue marbles. Setting up the equation $2x + 3x = 5x = 30$ employs the same reasoning. Solving, we find that there are 12 red marbles and 18 blue marbles. We are asked to draw twice and return the marble after each draw. Therefore, the first draw does not affect the probability of the second draw. We return the marble after the draw, and therefore, we return the situation to the initial conditions before the second draw. Nothing is altered in between draws, and therefore, the events are independent.

Drawing a red marble would be 12/30 = 2/5. The same is true for the second draw. Since we want two red marbles in a row, the question is really saying that we want a red marble on the first draw and a red marble on the second draw. The "and" means we should expect a lower probability than 2/5. Understanding that the "and" is implicit can help you eliminate choices d and e which are both too big. Therefore, our total probability is P(A and B) = P(A) ×. P(B) = 2/5 × 2/5 = 4/25.

Now consider the same question with the condition that you do not return the marbles after each draw. The probability of drawing a red marble on the first draw remains the same, 12/30 = 2/5. The second draw, however, is different. The initial conditions have been altered by the first draw. We now have only 29 marbles in the jar and only 11 red. We simply use those numbers to figure our new probability of drawing a red marble the second time, 11/29. The events are dependent and the total probability is P(A and B) = P(A) ×. P(B) = 2/5 × 11/29 = 22/145.

If you return every marble you select, the probability of drawing another marble is unaffected; the events are independent. If you do not return the marbles, the number of marbles is affected and therefore dependent.

Mutually Exclusive Events

Another type of probability deals with mutually exclusive events. What do we mean by mutually exclusive events? And what does it mean for two events not to be mutually exclusive? Consider the following example of drawing cards:
Example: What is the probability that a card selected from a deck will be either an ace or a spade?
a)2/52 b)2/13 c)7/26 d)4/13 e)17/52
Solution: First, identify events A and B and notice the "or" in between them. That means a greater probability than either A or B individually. We expect the answer to be greater than 4/52(ace)=1/13 or 13/52(spade)= 1/4. Eliminate a and b. The tricky part of this question lies in the fact that when we figure probability, we are really just counting, and sometimes, we count twice. In this case we have counted the ace of spades twice. If you don't see this, consider what the 4 in 4/52 stands for: ace of hearts, ace of diamonds, ace of clubs, ace of spades. The 13 in 13/52 stands for all the spades: 1,2,3…King, Ace(of spades). Therefore if we just combined the probabilities by the rule for P(A or B) = P(A) + P(B) we would be over counting. We have to subtract 1/52, the ace of spades that was counted twice. Our answer becomes 4/52 + 13/52 - 1/52 = 16/52 = 4/13.

Another way to think about the question is to just count aces and spades; that is, use simple probability. There are 13 spades in a deck and 3 aces other than the ace of spades already included in the 13 spades. Therefore, there are 16 desired outcomes out of a total of 52 possible outcomes, or 16/52 = 4/13.

In the above example, events A and B are not mutually exclusive. Figuring the probability for event A includes part of the probability of event B, and we must therefore subtract out this "over-counted" probability to get the correct answer. The following example illustrates mutually exclusive events:

Example: What is the probability that a card from a deck will be either an ace or a king?

a)1/169 b)1/26 c)2/13 d)4/13 e)8/13

Solution: The question asks for either an ace or a king. Since there are four kings and four aces in a deck, the probabilities for event A and event B are the same, 4/52 = 1/13. Our answer must be more than this, so eliminate a and b. Do kings and aces have anything to do with each other? Is there such a thing as an ace of kings or a king of aces? No, so we don't have to worry about having over-counted; the events are mutually exclusive. The probability is straightforward: P(A or B) = P(A) + P(B) = 1/13 + 1/13 = 2/13. C is correct.

Again we could have used simple probability. Count the total number of kings and aces (4+4) and divide by the total number of cards in a deck: 8/52 = 2/13.

Conditional Probabilities

A conditional probability is the probability of an event given that another event has occurred.

Example: What is the probability that the total of two dice will be greater than 8 given that the first die is a 6?

Solution: This can be computed by considering only outcomes for which the first die is a 6. Then, determine the proportion of these outcomes that total more than 8. All the possible outcomes for two dice are shown in the section on simple probability. There are 6 outcomes for which the first die is a 6: (6,1),(6,2),(6,3),(6,4),(6,5),(6,6), and of these, there are four that total more than 8. The probability of a total greater than 8 given that the first die is 6 is therefore 4/6 = 2/3.

1. Probability of A and B

If A and B are independent, then the probability that events A and B both occur is p(A and B) = p(A) × p(B). In other words, the probability of A and B both occurring is A * B.

What is the probability that a coin will come up with heads twice in a row?

Problem: Two events must occur: a heads on the first toss and a heads on the second toss. Since the probability of each event is 1/2, the probability of both events is: 1/2 × 1/2 = 1/4. Now consider a similar problem:

Problem: Someone draws a card at random out of a deck, replaces it, and then draws another card at random. What is the probability that the first card is the ace of clubs and the second card is a club (any club)?

Since there is only one ace of clubs in the deck, the probability of the first event is 1/52. Since 13/52 = 1/4 of the deck is composed of clubs, the probability of the second event is 1/4. Therefore, the probability of both events is 1/52 × 1/4 = 1/208.

What's the probability of A and B (2 of 2) if A and B are not independent?

If A and B are not independent, then the probability of A and B is p(A and B) = p(A) × p(B|A) where p(B|A) is the conditional probability of B given A. If someone draws a card at random from a deck and then, without replacing the first card, draws a second card, what is the probability that both cards will be aces? Event A is that the first card is an ace. Since 4 of the 52 cards are aces, p(A) = 4/52 = 1/13. Given that the first card is an ace, what is the probability that the second card will be an ace as well? Of the 51 remaining cards, 3 are aces. Therefore, p(B|A) = 3/51 = 1/17, and the probability of A and B is 1/13 × 1/17 = 1/221.

2. Probability of A or B

If events A and B are mutually exclusive, then the probability of A or B is simply:

p(A or B) = p(A) + p(B).

What is the probability of rolling a die and getting either a 1 or a 6? Since it is impossible to get both a 1 and a 6, = mutually exclusive. Therefore,
p(1 or 6) = p(1) + p(6) = 1/6 + 1/6 = 1/3

If the events A and B are NOT mutually exclusive , subtract shared.
p(A or B) = A+B – p(A and B).

The logic behind this formula is that when p(A) and p(B) are added, the occasions on which A and B both occur are counted twice. To adjust for this, p(A and B) is subtracted.

Example: What is the probability that a card selected from a deck will be either an ace or a spade? (Possible to get both? If yes, NOT mutually exclusive)

Solution: The relevant probabilities are: p(ace) = 4/52 and p(spade) = 13/52.
The only way an ace and a spade can both be drawn is to draw the ace of spades. There is only one ace of spades, so p(ace and spade) = 1/52. The probability of an ace or a spade can be computed as p(ace)+p(spade)-p(ace and spade) = 4/52 + 13/52 - 1/52 = 16/52 = 4/13.

⇒**Multiple Trials**

Frequently, probability questions on the GRE won't be limited to a single draw, or trial, but will instead involve repeated draws. When a question involves drawing multiple times from the same group of entities, you need to distinguish between draws *with replacement* and draws *without replacement*. Let's illustrate the difference using our marble example:

1. 1 You select a marble, note its color, and put it back in the bag. You then select a marble again. This is called ***drawing with replacement***.
2. You select a marble and put it aside. Then you draw another marble from those remaining. This is called ***drawing without replacement***.

The GRE will always make it clear which method is being used either by including the actual phrase *with replacement* or *without replacement* or by explicitly describing the method of selection in a way that makes it obvious which mechanism is in play. Let's look at an example of each type.

- **DRAWING WITH REPLACEMENT**

A bag contains 12 red, 13 white, and 15 black marbles. What is the probability of selecting two black marbles in a row if the selection is made with replacement?

The number of black marbles, or favorable outcomes, is 15. The total number of marbles is 40. First, use the probability formula to find the probability of selecting a black marble on the first draw: $\frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}} = \frac{15}{40} = \frac{3}{8}$

Since this problem involves drawing with replacement, we'll need to put the black marble selected on the first draw back into the bag before selecting again. So the bag will still contain 15 black marbles out of 40 total for the second draw. The probability of drawing a black marble on the second draw is thus the same $\frac{3}{8}$.

Now, even though the marbles are coming from the same bag, these two events—a black marble on the first draw and a black marble on the second—are independent; that is, what happens on one draw doesn't affect what happens on the other. To get the probability of two black marbles in a row, we can therefore multiply the individual probabilities: $\frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

- **DRAWING WITHOUT REPLACEMENT**

Now let's see what happens when we *don't* put the first marble back into the bag after selecting it:

A bag contains 12 red, 13 white, and 15 black marbles. What is the probability of selecting two black marbles in a row if the selection is made without replacement?

The probability of the first marble being black is $\frac{3}{8}$, just as before. For the second draw, however, only 14 black marbles remain out of 39 total. (Remember, we took a black marble out of the bag and *did not* put it back.) This means that the probability of the second marble being black is $\frac{14}{39}$. By assuming the first draw was favorable (a black marble selected), we adjusted the figures for our second probability. Since these figures are already adjusted to account for the first favorable outcome, the second drawing is independent from the first, so we can still multiply the individual probabilities to get the chances of selecting two black marbles in a row:

$$\frac{3}{8} \times \frac{14}{39} = \frac{1}{4} \times \frac{7}{13} = \frac{7}{52}$$

This can't be reduced any further, so it's the final answer.

- **The Probability of Something NOT Happening**

If you're told that the chance of snow tomorrow is 25%, it's likely you recognize without much thought that the chance that it will *not* snow is 75%. Here's the formula you used, whether you were aware of it or not:

The probability of an event NOT happening = 1 – The probability of that event

This formula can turn very hard probability questions into easier ones. Consider this next one:

Example: A bag contains 12 red, 13 white, and 15 black marbles. What is the probability of selecting at least one red or one white marble in two draws if the selection is made with replacement?

The first draw might be black, and you still could have a favorable outcome if the second draw is red or white. Similarly, the second draw could be black, and you'd still have a favorable outcome if the first draw is red or white. And of course, a first *and* a second draw of red or white would also count as a favorable outcome. So how do we deal with this ambiguity?

Simple: Use the formula for "NOT happening." It's far easier in this case to calculate the probability of *not* getting at least one red or white marble in two draws because this is actually the same thing as drawing two black marbles, with replacement. We already calculated this earlier as $\frac{9}{64}$. The probability of drawing at least one red or white in two draws is 1 minus the probability of that NOT happening, which is simply the probability of drawing two black marbles. The answer is therefore: $1-\frac{9}{64}=\frac{55}{64}$

EXERCISE

1. How many 3-digit positive integers are odd and do not contain the digit 5?
   a) 90 b) 180 c) 188 d) 200 e) 288

2. How many menu options of 2 soups, 2 appetizers, 3 main courses, a dessert and tea or coffee are possible from 6 soups, 5 appetizers, 6 main courses and 4 desserts?
   a) 51 b) 120 c) 2000 d) 12000 e) 24000

3. Jim and Tom are among 8 members of a club. If at least one of the two is to be selected for a 4-member team, how many teams are possible?
   a) 70 b) 55 c) 48 d) 4 e) 3

4. The letters of the word LOGARITHM are arranged at random. Find the probability that the arrangements start and end with vowels.
   a) 1/12 b) 2/12 c) 5/12 d) 7/12 e) 3/12

5. Seven papers are to be set for an examination of which three papers are of languages. Find the probability that in the timetable the three language papers are not consecutive.
   a) 1/7 b) 2/7 c) 4/7 d) 3/7 e) 6/7

6. 8 Indians and 3 Americans are to stand in a row at random. Find the probability that no two Americans are together.
   a) 12/55 b) 20/55 c) 28/55 d) 31/55 e) 26/55

7. In a game A throws a total of 16 with three dice. If B throws next the same three dice simultaneously, what are B's chances of winning the game?
   a) 2/54 b) 3/54 c) 4/9 d) 1/54 e) 3/22

8. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is
   a) 21/54 b) 16/81 c) 41/92 d) 18/45 e) 18/32

9. The class has 12 men and 4 women. Suppose 3 students are selected at random from the class, what is the probability that they all are men?

a) 10/ 22 b) 31/54 c) 45/94 d) 16/54 e) 11/28

10. A and B play a game where each is asked to select a number from 1 to 25. If the two numbers match, both of them win a prize. What is the probability that they will not win a prize in the single trial?
a) 1/625 b) 1/25 c) 2/25 d) 4/5 e) 24/25

11. Find the chances of throwing more than 15 in one throw of 3 dice.
a) 1/62 b) 5/108 c) 20/26 d) 41/56 e) 4/25

12. A bag contains 5 white, 7 black and 4 red balls. Find the probability that three balls drawn at random are all white?
a) 1/56 b) 52/108 c) 2/7 d) 4/5 e) 14/25

13. What are the chances of throwing equal faces greater than one in 2 successive throws of an ordinary die?
a) 21/54 b) 32/45 c) 5/14 d) 5/36 e) 5/12

14. Out of all the integers 1 to 100, a number is selected at random. What is the probability that the selected number is not divisible by 7?
a) 12/45 b) 28/51 c) 14/55 d) 43/50 e) 7/50

15. Four letters are written to different persons and they had to be put in the four addressed envelopes without being the addresses looked at. What is the probability that the letters would go into the right envelopes?
a) 1/16 b) 1/24 c) 1/48 d) 1/32 e) 1/128

16. In a hostel having 100 students, 40 drink tea, 30 drink coffee and 10 drink milk. Of these, 15 take tea and coffee both, 5 take coffee and milk both and none takes tea and milk both. If two students are picked up at random from the hostel, find the probability that they drink only coffee.
a) 2/110 b) 2/100 c) 4/110 d) 1/110 e) 1/50

17. A room has three electric lamps. From a collection of 12 electric bulbs of which 6 are good, 3 bulbs are selected at random and put in the lamps. Find the probability that room is lighted by at least one of the bulbs.
a) 11/12 b) 10/11 c) 6/11 d) 4/11 e) 3/11

18. A five digit number is made by using digits 8, 7, 4, 5, 3 without repetition. What is the probability that the number is divisible by 9?
a) 1/32 b) 1/9 c) 1/50 d) 1/40 e) 1

19. A bag contains 3 red and 2 black balls and another bag contains 2 red and 3 black balls. What is the probability of choosing a black ball from a randomly chosen bag?
    a) ½ b) 6/10 c) 4/10 d) 2/10 e) 3/10

20. Out of all the integers from 1 to 100, a number is selected at random. What is the probability that the number is divisible by 9?
    a) 11/100 b) 1/11 c) 1/89 d) 89/100 e) 11/89

21. From a pack of 52 cards, 2 cards are picked up at random. What is the probability that one is a king and the other is a queen?
    a) 8/663 b) 1/115 c) 10/89 d) 9/26 e) 25/87

22. An organization consists of 25 members including 4 doctors. A committee of 4 is to be formed at random. Find the probability that the committee contains exactly 2 doctors.
    a) 126/1265 b) 63/105 c) 55/67 d) 35/227 e) 36/107

23. The letters of the word EQUATION are arranged at random. Find the probability that the arrangement starts with a vowel and ends with a consonant.
    a) ½ b) 65/89 c) 25/39 d) 15/56 e) 41/56

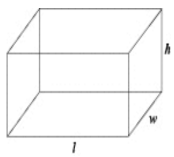
24. A ticket is drawn from a set of 20 tickets numbered 1 to 20 and kept aside. Then another ticket is drawn. Find the probability that both show an even number.
    a) 11/25 b) 27/50 c) 9/38 d) 33/45 e) 1/20

25. The probability that a student A can solve a problem is 1/3, that B can solve it is ½ and that C can solve it is ¼. If all of them try it independently, what is the probability that the problem is solved?
    a) ¾ b) 6/10 c) 13/20 d) 35/67 e) 3/10

9. SOLID GEOMETRY

Rectangular Solids / Cuboids

A rectangular solid is a prism with a rectangular base and edges that are perpendicular to its base. In English, it looks a lot like a cardboard box.



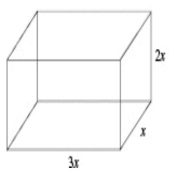
A rectangular solid has three important dimensions: length (*l*), width (*w*), and height (*h*). If you know these three measurements, you can find the solid's volume, surface area, and diagonal length.

- **Volume of a Rectangular Solid**

The formula for the volume of a rectangular solid builds on the formula for the area of a rectangle. As discussed earlier, the area of a rectangle is equal to its length times its width. The formula for the volume of a rectangular solid adds the third dimension, height, to get: **volume = *lwh***

Here's a good old-fashioned example:

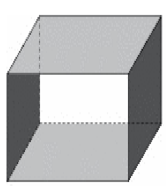
What is the volume of the figure presented below?



The length is $3x$, the width is x, and the height is $2x$. Just plug the values into the volume formula and you're good to go: $v = (3x)(x)(2x) = 6x^3$.

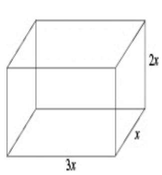
- **Surface Area of a Rectangular Solid**

The surface area of a solid is the area of its outermost skin. In the case of rectangular solids, imagine a cardboard box all closed up. The surface of that closed box is made of six rectangles: The sum of the areas of the six rectangles is the surface area of the box. To make things even easier, the six rectangles come in three congruent pairs. We've marked the congruent pairs by shades of gray in the image below:



Two faces have areas of $l \times w$, two faces have areas of $l \times h$, and two faces have areas of $w \times h$. The surface area of the entire solid is the sum of the areas of the congruent pairs:

Surface area = $2lw + 2lh + 2wh$.

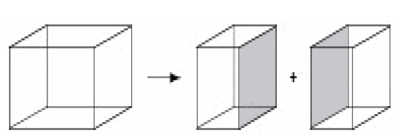


Let's try the formula out on the same solid we saw above. Find the surface area of this:

Again, the length is $3x$, the width is x, and the height is $2x$. Plugging into the formula, we get:

surface area $= 2(3x)(x) + 2(3x)(2x) + 2(x)(2x) = 6x^2 + 12x^2 + 4x^2 = 22x^2$

- **DIVIDING RECTANGULAR SOLIDS**

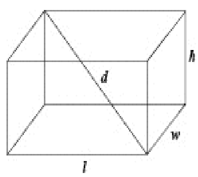


A number of possible questions could be created from this scenario. For example, you may be asked to find the combined surface area of the two new boxes. Or maybe a Quantitative Comparison question would ask you to compare the volume of the original solid with that of the two new ones. Actually, the volume remains unchanged, but the surface area increases because two new sides (shaded in the diagram) emerge when the box is cut in half. You may need to employ a bit of reasoning along with the formulas you're learning to answer a difficult question like this, but it helps to know this general rule: **Whenever a solid is cut into smaller pieces, its surface area increases, but its volume is unchanged.**

- **Diagonal Length of a Rectangular Solid**

The diagonal of a rectangular solid, *d,* is the line segment whose endpoints are opposite corners of the solid. Every rectangular solid has four diagonals, each with the same length, that connect each pair of opposite vertices. Here's one diagonal drawn in:

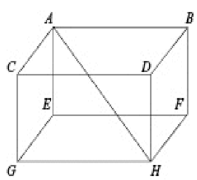
It's possible that a question will test to see if you can find the length of a diagonal. Here's the formula:



$$d = \sqrt{l^2 + w^2 + h^2}$$

Again, *l* is the length, *w* is the width, and *h* is the height. The formula is like a pumped-up version of the Pythagorean theorem. Check it out in action:

What is the length of diagonal *AH* in the rectangular solid below if *AC* = 5, *GH* = 6, and *CG* = 3?

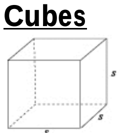


The question gives the length, width, and height of the rectangular solid, so you can just plug those numbers into the formula:

$$AH = \sqrt{5^2 + 6^2 + 3^2} = \sqrt{25 + 36 + 9} = \sqrt{70}$$

The problem could be made more difficult if it forced you to first calculate some of the dimensions before plugging them into the formula.

Cubes



A cube is a three-dimensional square. The length, width, and height of a cube are equal, and each of its six faces is a square. Here's what it looks like—pretty basic:

- **Volume of a Cube**

The formula for finding the volume of a cube is essentially the same as the formula for the volume of a rectangular solid: We just need to multiply the length, width, and height. However, since a cube's length, width, and height are all equal, the formula for the volume of a cube is even easier: **volume = s^3**

In this formula, *s* is the length of one edge of the cube.

- **Surface Area of a Cube**

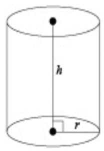
Since a cube is just a rectangular solid whose sides are all equal, the formula for finding the surface area of a cube is the same as the formula for finding the surface area of a rectangular solid, except with *s* substituted in for *l, w,* and *h.* This boils down to: **area = $6s^2$**

- **Diagonal Length of a Cube**

The formula for the diagonal of a cube is also adapted from the formula for the diagonal length of a rectangular solid, with s substituted for l, w, and h. This yields $\sqrt{3s^2}$, which simplifies to: **diagonal = $s\sqrt{3}$**

Right Circular Cylinders

A right circular cylinder looks like one of those cardboard things that toilet paper comes on, except it isn't hollow. It has two congruent circular bases and looks like this:



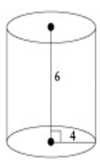
The height, h, is the length of the line segment whose endpoints are the centers of the circular bases. The radius, *r,* is the radius of its base.

- **Volume of a Right Circular Cylinder**

The volume of this kind of solid is the product of the area of its base and its height. Because a right circular cylinder has a circular base, its volume is equal to the area of the circular base times the height or:

$$\text{volume} = \pi r^2 h$$

Find the volume of the cylinder below:



This cylinder has a radius of 4 and a height of 6. Using the volume formula, its volume = $\pi(4)^2(6) = 96\pi$.

- **Area of a Right Circular Cylinder**

Total area of a right circular cylinder = Area of the base +Area of the lid + Curved surface area

Thus, total area of a right circular cylinder = $\pi r^2 + \pi r^2 + 2\pi rh = 2\pi r(r+h)$

EXERCISE

1. The area of a playground is 5600 sq meters. What will be the cost of covering it with grass sheet 1 cm deep, if cost of the grass sheet is $2.80 per cubic meter?

a) 144 b) 150.50 c) 156.80 d) 158.60 e) 160.70

2. Cost of painting four walls of a room 45m x 20 m at $5 per sq. m. is $6500. Find the height of the room.

a) 8m b) 10m c) 12m d) 15m e) 20m

3. The perimeter of a circle is equal to the perimeter of a square. Find the ratio of their areas.

a) 4:1 b) 11:7 c) 14:11 d) 22:7 e) Can't be determined

4. The areas of three adjacent faces of a rectangular box are p, q and r sq. cm. The volume of the box is given by

a) p+q+r b) $\sqrt{pqr}$ c) (pqr)/3 d) pqr e) pqr/2

5. A piece of metal in the form of rectangular solid weighs 3 kg. What is the weight in kgs of a similar piece of same metal whose dimensions are twice those of the first piece?

a) 6 b) 18 c) 24 d) 12 e) 9

6. Perimeter of square A is 2/3rd the perimeter of square B, and the perimeter of square B is 2/3rd perimeter of square C. If the area of square A is 16 squints, what is the area of square C?
a) 24 b) 36 c) 72 d) 64 e) 81

7. Three metal cubes whose edges are 3, 4 and 5 cm respectively are melted and molded without any loss of metal into a single cube. Find the edge of the new cube.
a) 4 b) 5 c) 6 d) 6.5 e) 7

8. A cone of height 7 cm and base of diameter 6 cm is carved from a rectangular block of iron 10 by 5 by 2 cm. Find the percentage of iron wasted.
a) 14 b) 24 c) 34 d) 44 e) 54

9. How many bricks each measuring 25 cm 12.5 cm and 7.5 cm will be required to build a wall 5 m long, 3 m high and 20 cm thick?
a) 1200 b) 1220 c) 1240 d) 1260 e) 1280

10. A rectangular wooden box is 10 cm wide, 10 cm long and 5 cm high. What is the greatest possible distance between any two points on the box?
a) 10 b) 12 c) 14 d) 15 e) 17

11. How many spherical show buttons of diameter 5 mm can be made from a spherical ball of radius 5 cm?
a) 800 b) 8000 c) 80000 d) 800000 e) 8000000

12. The dimensions of a brick are 24 cm, 12 cm, and 8 cm. How many bricks will be required to build a wall 24 m, 8 m and 6 m if 20% of the wall is filled with mortar?
a) 400000 b) 450000 c) 500000 d) 550000 e) 600000

13. The radius of a right circular cylinder is increased by 50%. What is the percentage increase in the volume?
a) 75 b) 100 c) 125 d) 150 e) 250

14. The lower part of a tent is a right circular cylinder and the upper part is a right cone. The diameter of the base is 70 m and the total height is 15m and the height of the cylindrical part is 3 m. Find the cost of the material at the rate of $10 per sq. meter.
a) 45600 b) 46500 c) 47100 d) 47300 e) 48200

15. A hemisphere of silver of radius 7 cm is cast into a right circular cone of height 49 cm. What is the radius of the base?

a) 7 b) $\sqrt{12}$ c) $\sqrt{14}$ d) 10 e) 10.5

16. Half- cubic meter of gold sheet is extended by hammering so as to cover an area of one hectare. The thickness of the sheet in meters is

a) 10.5 b) 0.05 c) 0.005 d) 0.0005 e) 0.00005

17. If each side of the cube is doubled, then its volume

a) is doubled b) becomes 4 times c) becomes 6 times d) becomes 8 times e) None of these

18. A tank 3m long, 2m wide and 1.5m deep is dug in a field 22m long and 14m wide. If the earth dug out is evenly spread over the field, the rise in the level of the field, in cms, will be

a) 299 b) 29 c)2.92 d) 4.15 e) 5.12

19. The perimeter of a rectangular field is 140 m and its breadth is 30 m. The length of its diagonal in meters is

a) 40 b) 50 c) 60 d) 28 e) 56

20. A metallic sphere of radius 2 cm is dropped into a cylindrical vessel filled with water up to a height of 12 cm. If the height of the vessel is 20 cm and its diameter is 6 cm, find the rise in the level of water when the sphere is completely submerged.

a) 32/27 b) 31/27 c) 30/27 d) 32/26 e) 32/25

21. The ice compartment of a refrigerator is 10 inch long, 4 inch wide and 5 inch deep. How many ice cubes will it hold if edge of each cube is 2 inches?

a) 24 b) 30 c) 20 d) 28 e) 25

22. If the radius of the base of a right circular cylinder is halved, keeping the height same, what is the ratio of the volume of the reduced cylinder to that of the original one?

a) 4:1 b) 2:1 c) 1:1 d) 1:4 e) 3:1

23. The length, breadth and height of a cuboid are in the ratio of 5:4:2. If the total surface area is 1216 cm^2, find its volume.

a) 2476 b) 2560 c) 2660 d) 2700 e) 2760

24. If the dimensions of a rectangular crate, in feet, are 5 by 6 by 7, which of the following CANNOT be the sum of the surface areas of two of the faces?

a) 60 b) 70 c) 72 d) 90 e) All of these

25. The curved surface of the cylinder is three times the area of its base, while the height exceeds the radius by 7 cm. Find the volume of the cylinder.

a) 4116π b) 4216π c) 4316π d) 4416π e) 4516π

10. TIME, DISTANCE AND WORK

Distance = Time * Speed   Time = Distance / Speed   Speed = Distance / Time

Note: Time and rate/speed are in inverse proportion

If you bike at a rate of 10 miles per hour, and you bike for 2 hours, you're going to cover 10 × 2 = 20 miles. What gets confusing is the various ways they state these problems, but rest assured they always give you enough information to set up an equation and solve for the variable you seek. Here's an example: **Jim roller-skates 6 miles per hour. One morning, Jim roller-skates continuously for 60 miles. How many hours did Jim spend roller-skating?**

rate × time = distance

Then fill in the values you know: **6 miles per hour × time = 60 miles.** Time taken = 10 hours.

Try the next one: **A traveler begins driving from California and heads east across the United States. If she drives at a rate of 528,000 feet per hour, and drives 4.8 hours without stopping, how many miles has she traveled? (1 mile = 5,280 feet)**

The reason you need to go the extra mile (so to speak) is because the rate is given in feet per hour but the distance traveled is given in miles. The units must jibe for the problem to work, so a conversion is necessary. Luckily, the numbers are easy to work with: The driver drives 528,000 feet per hour, and 5,280 feet equal one mile. So to change her rate into miles per hour, simply divide: $\frac{528{,}000 \text{ feet per hour}}{5{,}280 \text{ feet per mile}}$ **= 100 miles per hour**

Now plug the values from the problem, including this new one, into the formula:
100 miles per hour × 4.8 hours = distance. Thus, the distance traveled is 480 miles.

- 1 km / hr = 5/18 m/sec and 1 m/sec = 18/5 km / hr.

- If two bodies move in the opposite directions at the speed of x km/hr and y km/hr respectively, they approach each other at the relative speed of (x+y) km/hr i.e. the speeds get added if the bodies move in opposite direction.

- If two bodies move in the parallel or same directions at the speed of x km/hr and y km/hr respectively, they approach each other at the relative speed of (x-y) km/hr(If x>y) or (y-x) km/hr(If y>x) i.e. the speeds get subtracted if the bodies move in the same direction.

- If two trains of length *a* meters and *b* meters are moving in the **opposite** direction at *u* m/s and *v* m/s, then the time taken by the trains to cross each other is: ——
- If two trains of length *a* meters and *b* meters are moving in the **same** direction at *u* m/s and *v* m/s, then the time taken by the trains to cross each other is: ——

- If the speed of a boat in still water is *u* m/s and the speed of the stream is *v* m/s, then:
Speed downstream = **u + v m/s**   Speed upstream = **u – v m/s**

- If the speed downstream is **a** m/s and the speed upstream is **b** m/s then:
Speed in still water= **½ (a+b) m/s**   Speed of the stream = **½ (a-b) m/s**

- If A can do a work in a days, then in 1 day, A can do 1/ath part of work. Similarly, if B can do the same work in b days, then in 1 day, B can do 1/b th part of work. If both of them work together, then in 1 day, they can work : (1/a + 1/b)th part of work. Thus, **in 1 day they could do a+b/ab part of work** and they would finish the work in **ab/ a+ b days**.

- If a tap fills a cistern in x minutes, another tap fills the cistern in y minutes and a third tap empties or exhausts it in z minutes, then in 1 minute, (1/x+1/y-1/z) part of the cistern is filled.

- $\frac{M1 * R1 * T1}{W1} = \frac{M2 * R2 * T2}{W2}$ where M = Number of men or labour employed; R = Rate of the work T = Time taken to do the work and W = Work done

The formula could also be manipulated as:

$M_1 * T_1 = M_2 * T_2$ or $M_1 * R_1 * T_1 = M_2 * R_2 * T_2$ or $\frac{M1 * T1}{W1} = \frac{M2 * T2}{W2}$ or $M_1 * R_1 = M_2 * R_2$.

Four workers can dig a 40-foot well in 4 days. How long would it take for 8 workers working at the same rate to dig a 60-foot well?

Solution: Using $\frac{M1 * R1 * T1}{W1} = \frac{M2 * R2 * T2}{W2}$ **we get,** $\frac{4*4}{40} = \frac{8*?}{60}$ **.** Thus, it would take **3 days** of work for the 8 workers to dig the 60-foot well.

EXERCISE

1. A car finishes a journey in 10 hours at a speed of 48 kmph. To finish the journey in 7.5 hours, the speed of the car should be increased by
a) 10 kmph b) 16 kmph c) 18 kmph d) 60 kmph e) 64 kmph

2. A train running between two stations arrives at its destination 10 minutes late when it travels at 40 kmph and 16 minutes early when it travels at 30 kmph. The distance between the two stations in km is
a) 10 b) 12 late c) 15 d) 20 e) 24

3. A clerk walks from his house at 4 km/hr and reaches his office 5 minutes late. If his speed is 5 km/hr, he will reach 10 minutes early. How far in km is his office?
a) 5/12 b) 5/2 c) 5 d) 10 e) 15

4. A bus travelling at 50 kmph leaves Burbank at 9 am. A plane, travelling at 300 kmph in the same direction leaves Burbank at 1 p.m. How many minutes does the plane take to overtake the bus?
a) 24 b) 30 c) 36 d) 40 e) 48

5. An aircraft flying 800 km/hr covers the first 200 km at 100 km/hr, the second 200 km at 200km/hr, the third 200 km at 300 km/hr and the last 200 km at 400 km/hr. The average speed of the aircraft in kmph is
a) 180 b) 192 c) 200 d) 224 e) 250

6. One hour after Pam started walking from A to B, a distance of 58 miles, Sue started along the same road from B to A. If Pam's walking rate was 4 mph and Sue's speed was 5 mph, how many miles had Sue walked when they met?

a) 18 b) 20 c) 25 d) 30 e) 32

7. Flowers in a basket are doubled every minute. In an hour the basket is full. The basket would be half full after how many minutes?

a) 45 b) 50 c) 55 d) 58 e) 59

8. It takes an hour for a saree to dry in the sun. If 25 such sarees are all dried separately and simultaneously, they all would dry in how many hours?

a) 1 b) 2 c) 5 d) 25 e) Can't be determined

9. A monkey ascends a greased pole 21 m high. In the first minute it ascends 5 m and in the next minute it descends 3m. If he continues this process, in how many minutes will it reach the top?

a) 10.5 b) 17 c) 21 d) 40 e) None of these

10. Mary can type 500 words in 10 minutes and Sam can type 400 words in 10 minutes. They can together type 3600 words in

a) 100 b) 80 c) 50 d) 40 e) 20

11. Three persons A, B and C can do a piece of work in 15 days, 6 days and 10 days respectively. In how many days will all the three finish three times the similar work?

a) 3 b) 4 c) 6 d) 8 e) 9

12. Two mail sorters r and s work at similar rates. If r can sort x letters in 60 minutes, how long will it take for both the sorters working together but independently to sort x letters?

a) 60 minutes b) 45 minutes c) 30 minutes d) 25 minutes e) 20 minutes

13. H, R and K working alone can do a piece of work in 9, 8 and 6 days respectively. They jointly finish the work and earn Rs 522. The earnings should be divided in the ratio

a) 6:9:8 b) 9:8:6 c) 8:9:12 d) 6:8:9 e) 7:8:9

14. John can do certain work in 30 hours. If he and his son work together, the time taken is 20 hours. The son working in the same capacity as when he was working with his father, can finish the work in

a) 20 hours b) 30 hours c) 50 hours d) 60 hours e) 90 hours

15. Mohan can do a piece of work in 20 days and Harish in 25 days. They work together for 5 days and then Harish leaves. Mohan will finish the remaining work in how many days?

a) 15 b) 12 c) 11 d) 10 e) 9

16. A and B can paint a home in 10 and 15 days respectively. They started painting but unfortunately A had to leave after some days and B finished the remaining task in 5 days. After how many days did A leave?
a) 4 b) 6 c) 8 d) None of these e) Information inadequate

17. Ron can do a piece of work in 80 days. He works at it for 10 days and then Juan alone finishes the remaining work in 42 days. Had the two worked together from the beginning, they would have completed the work in
a) 20 b) 24 c) 25 d) 27 e) 30

18. A certain number of men promise to do a job in 10 days. But 10 of them do not come. If the rest of the men can do the job in 12 days, the original number of men was
a) 20 b) 40 c) 50 d) 60 e) 80

19. A contractor undertakes to do a job in 300 days. He employs 200 men and after 100 days finds that only a quarter of the work is done. The number of additional men needed to finish the work in time is
a) 100 b) 150 c) 200 d) 250 e) 300

20. Three men or six women can do a piece of work in 20 days. In how many days can 12 men and 8 women do the same piece of work?
a) 3 ½ days b) 3 ¾ days c) 4 days d) 4 4/5 days e) 5 days

21. A cistern normally filled in 8 hours, takes 2 hours longer due to a leak. If the cistern is full, the leak shall empty it in
a) 20 hours b) 24 hours c) 30 hours d) 40 hours e) 48 hours

22. There are two taps which can fill a tank in 60 minutes and 75 minutes respectively. There is another exhausting tap in it. If all the three taps are opened simultaneously, the tank gets filled up in 50 minutes. When the tank is full, the third tap can empty it in how many minutes?
a) 100 b) 125 c) 150 d) 200 e) 225

23. A pipe can fill a tank in 15 hours. The tank develops a hole and 10% of the water leaks out. The pipe will now fill the tank in
a) 16 hours 40 mins b) 17 hours 30 mins c) 18 hours d) 18 hours 40 mins c) 19 hours 20 mins

24. Two pipes A and B fill a tank separately in 24 minutes and 40 minutes respectively and a waste pipe C releases 30 litres per minute. If all the pipes are opened, the tank is filled in an hour. The capacity of the tank in liters is
a) 600 b) 750 c) 800 d) 900 e) 1200

25. Two pipes A and B fill a cistern in 24 and 32 minutes respectively. Assuming that both pipes are opened simultaneously, when must the first pipe be closed so that the cistern in filled in 16 minutes?
a) After 6 mins b) After 8 mins c) After 10 mins d) After 12 mins e) None of these

11. STATISTICS

RANGE

1. Range is the difference between the largest number and smallest number in a set.
2. If Range of a list is 0, then the list will contain all identical elements. Or if a set contains all identical elements, then range = 0.
3. If a set contains only one item, then range = 0.

MODE

Mode is the most frequently recurring number/numbers among the given set of numbers.
A given set of data values may have more than one mode.

MEAN AND MEDIAN

Mean

Arithmetic **Geometric** **Harmonic**

- **Arithmetic mean**: $(\bar{x}) = \frac{\Sigma x}{N}$ or $\frac{\Sigma fx}{\Sigma f}$ where Σ is the sum of all the values

  Arithmetic Mean (Average) = total of quantities / number of quantities

- **Geometric mean** : $(x_1 * x_2 * x_3 * x_4 * \ldots * x_n)^{1/n}$ or $\sqrt[n]{x_1 * x_2 * x_3 * x_4 * \ldots * x_n}$

- **Harmonic mean**: $n / (1/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + \ldots + 1/x_n)$

Harmonic mean ≤ Geometric mean ≤ Arithmetic mean

Median:

The median is the "middle" number in a group (when arranged in ascending or descending order) consisting of an odd number of numbers, and the average of the two middle numbers if there are an even number of numbers.

For a given set of consecutive even/odd (or evenly spread) numbers, mean = median.

For a given set of consecutive integers median = mean

For a set of consecutive integers, the median is the average of the first and the last integers.

So, if you have the following numbers
112, 113, 114, 115, 116, 117
You don't need to do: sum/6 to get the average
You can simply do 112+117/6 which gives the mean and the median!!!

For odd number of consecutive integers median = mean OR
For any arithmetic progression, the median = mean.

Combined mean: If the mean of two series and their sizes are given, then the combined mean for the resultant series is:

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

where: $\bar{x}$ = Combined mean of the two series

n_1 = Size of the first series

n_2 = Size of the second series

$\bar{x}_1$ = Mean of the first series

$\bar{x}_2$ = Mean of the second series

- **Corrected mean:** $\dfrac{\textbf{Incorrect } \Sigma - \textbf{Incorrect value(s)} + \textbf{Correct value(s)}}{\textbf{Number of values (N)}}$

STANDARD DEVIATION

Calculation of Standard Deviation (SD):

(i) Find the mean of the set of numbers
ii) Find the difference between each of the numbers and the mean
iii) Square the differences and take the mean of the differences
iv) Take the positive square root of this value

IN SHORT: S.D. = Sq. root (mean of the squares-square of the mean)

Steps in calculation the standard deviation:

1. Find the mean, $\bar{x}$ or M ,of the values.
2. For each value x_i calculate its deviation ($x - \bar{x}$) or ($\overline{x -}$ M) from the mean.
3. Calculate the squares of these deviations.
4. Find the mean of the squared deviations i.e. divide the sum of the squared deviations by n. This quantity is the **variance** σ.
5. Take the square root of the variance. This calculation is described by the following formula:

$$s = \sqrt{\frac{\Sigma(X-M)^2}{n}}$$

Variance : $\sigma = s^2$ M = x i.e. the arithmetic mean of the values x, defined as: $\frac{\Sigma x}{N}$

NOTE: 1. If Range or SD of a list is 0, then the list will contain all identical elements.
2. A set of numbers with range of zero means that all of the numbers are the same, hence the dispersion of the numbers from its mean is zero. In other words, If the range is 0, then the SD must also be 0, because there is no variance in the values.
3. SD does not change when the same constant is added to or subtracted from all the members of the set.
4. You only need to know the difference between values and total number of values to compute SD.
5. If we know all the numbers of the list, there is a definite SD, regardless of what it is, we can compute it and get an answer – this is helpful for DS questions.
6. The SD of any list is not dependent on the average, but on the deviation of the numbers from the average. So just by knowing that two lists having different averages doesn't say anything about their standard deviation - different averages can have the same SD.
7. The sum of the deviations of the elements from the mean must be 0.
8. Closer the more values to the MEAN, lower the SD.
9. If you multiply all terms by x then **SD =x times old SD** and the **mean = x times old mean.**
10. For comparing the SD for two sets any information about mean, median, mode and range are insufficient unless you can determine the individual terms from the given data.
11. If mean = maximum value it means that all values are equal and SD is 0.
12. Variance is the square of the standard deviation.
13. SD ranks the dispersion (deviation) of the numbers in a list. The more alike the numbers are, the less the dispersion, so the less the standard deviation.
14. The more uneven members are dispersed around their arithmetic average, the more their SD.
15. For data with approximately the same mean, the greater the range, the greater the SD.

EXERCISE

6” 2” 10” 2” 5”

Given are the measures of rainfall for 5 consecutive days during the winter. For the measure of those 5 days, which of the following is true?

I. The median equals the mode. II. The median equals the arithmetic mean. III. The range equals the median

a) I only b) II only c) III only d) I and II only e) I and III only

2. The only test scores for the students in a certain class are 15, 30, 40 30, x, 50 and 30. If x equals one of the other scores and is a multiple of 5, what is the mode for the class?

a) 15 b) 30 c) 40 d) 50 e) Any of these

3. If half the range of the increasing series{11, A, 23, B, C, 68, 73}is equal to its median, what is the median of the series?

a) 23 b) 31 c) 33 d) 41 e) 62

4. The average (mean) GRE score for a group of M students in Montana is 1400, while the mean GRE score for a group of V students in Virginia is 1050. When the scores of both groups are combined, the mean is 1300. What is the value of V/M? $\boxed{M / V ?}$

a) ¼ b) ½ c) 5/2 d) 7/2 e) Can’t be determined

5. The average (mean) of eight numbers is 8. If 2 is subtracted from each of four of the numbers, what is the new average (or mean)?
a) 5.5 b) 6.0 c) 6.5 d) 7.0 e) 7.5

6. The arithmetic mean of scores of a group of students in a test was 52. The brightest 20% of them secured a mean score of 80 and the dullest 25% a mean score of 31. Find the mean score of the remaining 55% of the group.
a) 51.36 b) 52 c) 53.11 d) 54 e) Can't be determined

7. If the sum of a and b is c, what is the arithmetic mean of a and b in terms of a, b and c?
a) 2c b) c–(a+b) c) c/2 d) (a+b) / c e) c/ (a+b)

8. The median of the observations 8,11,13,15, x+1, x+3, 30, 35, 40, 43 arranged in an ascending order is 22. Find x.
a) 15 b) 20 c) 25 d) 28 e) 30

9. Five coins were simultaneously tossed 1000 times and at each toss the number of heads were observed. The number of tosses during which 0, 1, 2, 3, 4 and 5 heads were obtained are shown in the table below. Find the mean number of heads per toss.

No. of heads per toss	0	1	2	3	4	5
No. of tosses	38	144	342	287	164	25

a) 2.47 b) 146.6 c) 411.66 d) 495 d) None of these

10. Which of the following is true for the following the data?

Marks	52	58	60	65	68	70	75
No. of students	7	5	4	6	3	3	2

i) Mean < Mode ii) Median > Mode iii) Range< Median<Mean iv) Mean< Median=Mode

a) Only i and ii b) Only i and iii c) Only ii and iv d) Only i and iv e) None

11. The mean annual salary of all employees in a company is Rs 25000. The mean salary of male and female employees is Rs 27000 and Rs 17000 respectively. Find the percentage of males employed in the company.
a) 20 b) 25 c) 40 d) 60 e) 80

12. What is the difference between the range and the median of the following data?

Marks	20	9	50	40	25	80
No. of students	6	4	7	8	16	2

a) 31 b) 40 c) 46 d) 56 e) 71

13. What is the range of people with neither washing machine nor dishwasher if 14% use washing machine and 27% use dishwasher?

a) 13 b) 14 c) 59 d) 73 e) Can't be determined

14. In a set of positive, distinct integers {a, b, c, d, e}, median is 16. What is the minimum value of a +b +c +d +e?

a) 48 b) 50 c) 54 d) 80 e) None of these

15. A student obtained a mean of 100 observations as 40. It was later discovered that he had wrongly copied down an observation 50 instead of 40. Calculate the correct mean.

a) 39.9 b) 40 c) 40.1 d) 41.1 e) Can't be determined

16. What is the difference between the standard deviation of 3, 4, 5, 6 and that of 81, 82, 83, 84?

a) -1 b) 0 c) 1 d) 1.12 e) $\sqrt{2}$

17. In a list of four positive even numbers, the mean, median and mode are all equal. Which of the following CANNOT be done to the list if the mean, median, and mode are to remain equal?

a) Add one number to the list b) Add one number to the list that is greater than the mean
c) Add two distinct numbers to the list d) Add 2 to each number in the list
e) Remove the first and last numbers from the list

18. The average of five positive even integers is 60. If p is the greatest of these integers, what is the greatest possible value of p?

a) 296 b) 292 c) 64 d) 60 e) None of these

19. If the median of *n* consecutive odd integers is 6, which of the following must be the average of the *n* integers?

a) n / 2 b) n c) 6/ n d) 6 e) 12

20. Adam delivered *n* pizzas on Monday, 5 times as many pizzas on Tuesday as on Monday, 3 fewer pizzas on Wednesday than on Tuesday, and 7 more pizzas on Thursday than on Tuesday. What is the average number of pizzas he delivered per day over the 4 days?

a) $3n+1$ b) $3n+3$ c) $4n+1$ d) $\frac{3n+5}{2}$ e) $\frac{4n+19}{4}$

21. What is the mean of the squares of the first 10 natural numbers

a) 5.5 b) 35.8 c) 38.5 d) 53.8 e) None of these

12. SEQUENCE, SERIES AND PROGRESSION

A sequence is a list of numbers that follows a particular pattern. If you get a sequence problem, you'll probably be given at least one of the terms in the sequence, along with the rule that defines the pattern. You probably won't have to figure out the pattern on your own.

However, what could make sequence problems tough is the notation. Each term in a sequence has the same variable, but each has a different subscript. This subscript indicates a particular term. For example:
a_1 = the first term
a_2 = the second term
a_{10} = the tenth term
a_n = the *n*th term
a_{n+1} = the term immediately after the *n*th term

For example, to indicate that the second term of a sequence is 5 and that the third term is 7, the test makers might write: $a_2 = 5$ $a_3 = 7$.
This subscript notation can also be used to indicate how each term relates to the others. For example: $a_{n+1} = a_n - 3$.
This just means that each successive term is three less than the previous term. Here's an example, using the notation we just discussed:
If a_n = the *n*th term in a sequence, and $a_1 = 3$ and $a_{n+1} = a_n + 2$, what is the value of a_{10}?

Let's use 1 as *n* to keep things simple: $a_1 = a_n = 3$

So a_{n+1} is the same as saying a_{1+1} or a_2. This second term we're told is equal to the first term, a_n, plus 2, which means that the second term will be 3 + 2, or 5. So the notation, which looks intimidating, is really a shorthand way of saying that each successive term is two more than the previous term. Writing out this sequence from the first to the tenth terms gives 3, 5, 7, 9, 11, 13, 15, 17, 19, 21. The tenth term is 21, and so $a_{10} = 21$.

Arithmetic Sequences / Arithmetic Progression

In an arithmetic sequence, the difference between each term and the next is constant. This is the kind of sequence we saw in the previous example. In addition to understanding the notation and concepts for sequences, you should know the formula for arithmetic sequences: $\boldsymbol{a_n = a_1 + (n-1)d}$
where
a_n = the *n*th term
a_1 = the first term
d = the difference between consecutive terms
This formula is useful if you need to determine the value of some very high term and don't want to write down a long sequence of numbers. In our previous example, the first term (a_1) is 3 and the difference between consecutive terms is 2. If you plug these numbers into the equation above looking for a_{10}, you'll get the same answer, 21, that we got earlier. In this example, it's just as easy to write out the terms. But to determine the *100th* term in that sequence, we'll need to plug the numbers into the formula: $a_{100} = 3 + (100 - 1)2 = 3 + 99 \times 2 = 201$.

Common difference (d): $\mathbf{T_n - T_{n-1} = T_{n-1} - T_{n-2}}$ For e.g. $\mathbf{T_3 - T_2 = T_2 - T_1}$.
Thus, $\mathbf{T_3 + T_1 = 2T_2}$ Thus, $\mathbf{T_2 = (T_3 + T_1)/2}$
Note: d could be negative or positive and less than 1 or greater than 1.

NOTE: Take : 3 terms in an A.P. :a-d, a, a+d
4 terms in an A.P. :a-3d, a-d, a+d, a+3d
5 terms in an A.P. :a-2d, a-d, a, a+d,a+2d

Sum of the first n terms in an A.P.

S_n = **n/2 (a + *l*)** where a is the 1st term and *l* is the last term or S_n = **n/2 (2a + (n-1) d)**

Geometric Sequences and Exponential Growth / Geometric Progression

In a geometric sequence, the *ratio* between one term and the next is constant, not the actual difference between the terms. For example, in the sequence 3, 9, 27, 81, each successive term is three times greater than the preceding one, but the actual difference between the terms changes: 9 – 3 = 6, 27 – 9 = 18, and so on. Geometric sequences exhibit *exponential growth*, as opposed to the constant growth of arithmetic sequences. Here's an example of the kind of geometric sequence that the test makers might toss at you: $g_1 = 4$ $g_n = 2g_{n-1}$.

Trying out some terms, this means that $g_2 = (2)g_1$, $g3 = (2)g_2$, and so on. In other words, the first term is 4, and each successive term is twice the value of the preceding term. Writing out the first few terms lets us see that the ratio between terms is constant and thus confirms that this is a geometric sequence: 4, 8, 16, 32, . . .
The ratio between consecutive terms is always 2, even though the differences between the terms increase as you move to the right.

As with arithmetic sequences, you should learn the special formula for geometric sequences, just in case it's not convenient to list out all of the terms up to the one you're looking for: $\boldsymbol{g_n = g_1 r^{n-1}}$
where
g_n = the *n*th term
g_1 = the first term
r = the ratio between consecutive terms

Let's use the formula to calculate the value of the tenth term in the geometric sequence defined by $g_1 = 4$ and $g_n = 2g_{n-1}$. We already know $r = 2$:

$g_{10} = 4 \times (2)^{10-1} = 4 \times 2^9 = 4 \times 512 = 2{,}048$

Common ratio (r): $\mathbf{T_n / T_{n-1} = T_{n-1} / T_{n-2}}$ For e.g. $\mathbf{T_3 / T_2 = T_2 / T_1}$. **Thus,** $\mathbf{(T_2)^2 = T_3 * T_1}$
Note: r could be less than 1 or greater than 1 and positive or negative.

NOTE: Consider: 3 terms in a G.P. :a/r, a, ar
4 terms in a G.P. :a/r^3, a/r, ar, ar^3
5 terms in a G.P. : a/r^2, a/r, a, ar, ar^2

Sum of the first n terms in a G.P. (S_n)

$\mathbf{S_n = (a(r^n - 1) / (r - 1))}$ **if r > 1**
$\mathbf{S_n = (a(1 - r^n) / (1 - r))}$ **if r < 1 and n is finite**
$\mathbf{S_n = (a / (1 - r))}$ **if r < 1 and n is infinite**
$\mathbf{S_n = (na)}$ **if r = 1**

EXERCISE

1. What is the least number that must be added to 3^{78} so that the resulting number is divisible by 10?
a) 1 b) 3 c) 7 d) 9 e) None of these

2. What is the least number that must be subtracted from 7^{29} to get a multiple of 5?
a) 1 b) 2 c) 3 d) 4 e) None of these

3. What is the remainder when 6^{58} is divided by 9?
a) 0 b) 3 c) 6 d) 7 e) 8

4. The first two terms of a sequence are 6 and -8. The successive terms are formed by adding 2 to the previous term and multiplying the resulting sum by -1. What is the absolute value of the difference between the 70th and the 101st terms?
a) It can't be determined b) 14 c) 8 d) 6 e) 2

5. In the sequence 12, 24, 72, 264,…. where 12 is the first term, which of the following is an expression for the *n*th term?
a) $12xn$ b) n^{12} c) 12^n d) $4^{(n+1)} - n$ e) $4^n + 8$

6. The consecutive multiples of 7 from -84 to 7k, k>0, are added together. If the total is 189, what is k?
a) 52 b) 51 c) 50 d) 27 e) 14

7. Find the sum of the first 9 terms of the series ¾. 2/3, 7/12.
a) 30/4 b) 15/4 c) 4/15 d) 16/4 e) 18/4

8. Find the sum of the first 15 terms of the series whose nth term is (4n+1).
a) 494 b) 499 c) 495 d) 496 e) 497

9. Sum of three numbers in an A.P. is 27 and their product is 504. Find the first number.
a) 3 b) 5 c) 6 d) 4 e) 8

10. Sum of three numbers in a G.P. is 38 and their product is 1728. Find the first number.
a) 7 b) 8 c) 9 d) 11 e) 4

11. In an A.P., the first term is 2, the last term is 29, and sum of the terms is 155. Find the common difference.
a) 3 b) 4 c) 5 d) 6 e) 7

12. The sum of 15 terms of an A.P. is 600 and the common difference is 5. Find the first term.
a) 4 b) 5 c) 6 d) 7 e) 8

13. An A.P. has 23 terms. Sum of the middle three terms is 144 and the sum of the last three terms is 264. What is the 16th term?

a) 64 b) 65 c) 66 d) 67 e) 68

14. Find the sum of the series 5/2, 4, 11/2, 7, …. upto 21 terms.

a) 366 b) 367 c) 368 d) 368.5 e) 367.5

15. Find the sum of the first six terms of the series 3, 9, 27, 81,…

a) 1010 b) 1092 c) 1091 d) 1093 e) 2019

16. Find the sum of the infinite series 1, 0.5, 0.25,…

a) 1 b) 3 c) 2 d) 5 e) Can't be determined

17. Find the 10th term of the series 4, 16, 64, 256, 1024,…

a) 4^{10} b) 4^{8} c) 4^{7} d) 4^{6} e) 4^{5}

18. The sum of the three numbers in an A.P. is 24. If the numbers be decreased by 2,4, 4 respectively, they form a G.P.. Find the first term of the A.P.

a) 8 b) 9 c) 10 d) 11 e) 12

19. The first term of a G.P. is 1. The sum of the third and the fifth term is 90. What is the common ratio of the G.P.?

a) +/-1 b) +/-2 c) +/-3 d) +/-4 e) +/-5

20. How many terms of the A.P. 1, 4, 7, … are needed to give the sum 715?

a) 20 b) 21 c) 22 d) 23 e) 24

21. For A.P. a1, a2, a3,… if a4/a7 = 2/3, find a6/a8.

a) 5/4 b) 6/4 c) 4/15 d) 4/5 e) 18/4

22. Find the value of 1.3+2.4+3.5+…. up to the 6th term.

a) 20.3 b) 21.5 c) 22.8 d) 23.9 e) 24.3

23. Find the 4th term of the G.P. whose fifth term is 81 and the eighth term is 2187.

a) 20 b) 27 c) 22 d) 23 e) 24

13. CLOCKS AND CALENDARS AND FUNCTIONS

CLOCKS AND CALENDARS

CLOCK: The face of a clock or a watch is a circle which is divided into 60 minute spaces. The minutes hand passes over 60-minute spaces while the hours hand goes over 5-minute spaces. That is, in 60-minutes, the minutes hand gains 55 minutes on the hour hand.

In every hour:

- The minutes hand revolves around the clock and travels the distance equivalent to the circumference of the clock and traces 360 degrees and the hour hand traces 30 degrees and travels the distance equivalent to $1/12^{th}$ the circumference of the clock every hour.
- The hands coincide once
- They are twice at right angles when the hands are 15 minutes spaces apart.
- They point in the opposite directions once when they are 30 minutes spaces apart. The hands are in the same straight line when they are coincident or opposite to each other.

In one minute: The minutes hand makes an angle of 6 degrees and the hour hand makes an angle of ½ degrees.

Clock too fast, too slow: If a clock indicates 7:10 when the correct time is 7:00, it is said to be 10 min too fast. If it indicates 6:50, when the correct time is 7:00, it is said to be 10 min slow.

CALENDAR: The following facts should be remembered about a calendar.

- In an ordinary year, there are 365 days, i.e. 52 weeks + 1 day. Therefore, an ordinary year contains 1 odd day.
- A leap year contains 366 days, i.e. 52 weeks + 2 days. Therefore, a leap year contains two odd days. February has 29 days in a leap year.
- A leap year is divisible by 4. The turn of century is a leap year if it is divisible by 400. For instance, 1900 was not a leap year but 2000 was a leap year.
- The Gregorian calendar repeats itself after 400 years.

EXERCISE

1. Find the measures in degrees of the smaller angle formed by the hour hand and the minute hand of a clock at:

a) 11:20 p.m. b) 20 mins past 2 c) 10 mins to 6 a.m.
d) Quarter to 11 e) 30 mins past mid noon

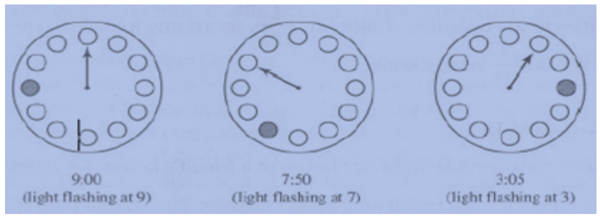
2. The minute hand is twice as long as the hour hand. What is the ratio of the distance travelled by the minute hand in 3 hours and the distance by the hour hand in 9 hours?

a) 2:1 b) 3:1 c) 8:1 d) 9:1 e) None of these

3. What is the difference in the degree measures of the angles formed by the hour hand and the minute hand of a clock at 12:35 and 12:36 p.m.?

a) 1 b) 5 c) 5.5 d) 6 e) 30

4. An ultramodern clock in a school has 12 lights in place of numerals from 1 to 12. The clock has no hour hand; instead a flashing light signals the hour. The clock does, however, have a minute hand, which starts in a vertical position pointing up at 00, the start of the hour, and rotates clockwise through 360 degrees in 60 minutes. Here is how the clock shows various times:



At 9:30 in the morning, while the teacher is out of the classroom, some mischievous students rotate the clock through 90 degrees counterclockwise, without touching the hand. In the next instant, before the clock display changes, the teacher reenters the room. She glances at the clock. What time does she see?

a) 6:15 b) 3:00 c) 3:45 d) 12:15 e) 12:45

5. How many pairs of days are feasible for the two extra days of the leap year?

a) 2 b) 7 c) 21 d) 42 e) None of these

FUNCTIONS

A function relates an input to an output. It is like a machine that has an input and an output. And the output is related somehow to the input. Functions have only one output for a given input.

Input ⇒FUNCTION⇒Output For instance: $\underbrace{f}_{\text{function name}}(\underbrace{x}_{\text{input}}) = \underbrace{x^2}_{\text{what to output}}$ We say *"f of x equals x squared"*

What goes **into** the function is put inside parentheses () after the name of the function:
So ***f(x)*** shows us the function is called "***f***", and "***x***" goes **in**
And we usually see what a function does with the input: $\boldsymbol{f(x) = x^2}$ shows us that function "***f***" takes "***x***" and squares it.

Example: with $\mathbf{f(x) = x^2}$: an input of 4 becomes an output of 16. In fact we can write **f(4) = 16**.

The "x" is Just a Place-Holder! Don't get too concerned about "x", it is just there to show us where the input goes and what happens to it. It could be anything!
So this function: $f(x) = 1 - x + x^2$ is the same function as: $f(q) = 1 - q + q^2$ or $h(A) = 1 - A + A^2$ or $w(\theta) = 1 - \theta + \theta^2$
The variable (x, q, A, etc) is just there so we know where to put the values. Thus, $f(\mathbf{2}) = 1 - \mathbf{2} + \mathbf{2}^2 = 3$

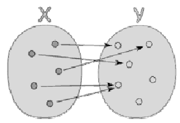
Sometimes There is No Function Name: Sometimes a function has no name, and we see something like: $y = x^2$

But there is still: an input (x) a relationship (squaring) and an output (y)

A Function is Special: A function has **special rules**:

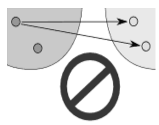
It must work for **every** possible input value And it has only **one relationship** for each input value

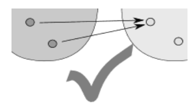
A function is a **special** type of relation where: **every element** in the domain is included, and any input produces **only one output** (not this **or** that).



Formal Definition of a Function

A function relates **each element** of a set with **exactly one** element of another set (possibly the same set).



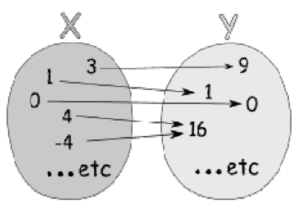


(one-to-many) **(many-to-one)**

This is **NOT** OK in a function But this **is** OK in a function

When a relationship does **not** follow those two rules then it is **not a function**. It is still a **relationship.**

Example: The relationship x → x^2 could also be written as a table:

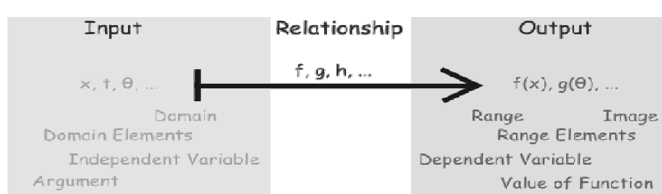


X: x	Y: x^2
3	9
1	1
0	0
4	16
-4	16

It is a function, because: Every element in X is related to Y and No element in X has two or more relationships. So it follows the rules. (Notice how both **4** and **-4** relate to **16**, which is allowed.)

In the example above: the set "X" is called the **Domain**, and the set "Y" is called the **Codomain**, and the set of elements that get pointed to in Y (the actual values produced by the function) is called the **Range**.
A function takes elements from a set (the **domain**) and relates them to elements in a set (the **codomain**). All the outputs (the actual values related to) are together called the **range.**

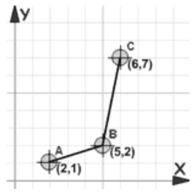
There could be a lot of different names and ways of writing functions. For instance:



<u>Ordered Pairs:</u> Another way to think about functions is to write the input and output of a function as an "ordered pair", such as (4,16). An input and its matching output are together called an **ordered pair.**
They are called **ordered** pairs because the input always comes first, and the output second: (input, output).
So it looks like this: (**x**, **f(x)**). Example: **(4,16)** means that the function takes in "4" and gives out "16".

<u>Set of Ordered Pairs:</u> A function can then be defined as **a set of ordered pairs**: Example: **{(2,4), (3,5), (7,3)}** is a function that says "2 is related to 4", "3 is related to 5" and "7 is related 3".
Also, notice that: the domain is **{2,3,7}** (the input values) and the range is **{4,5,3}** (the output values).

<u>A Benefit of Ordered Pairs:</u>



We can graph them...... because they are also coordinates! So a set of coordinates is also a function (if they follow the rules above, that is)

<u>EXERCISE</u>

1. Let the function g be defined by g(x) = 2x-3. If g(k) = g (2k-3), what is the value of g(4k)?

a) 3 b) 12 c) 21 d) It can't be determined e) None of these

2. The table below shows selected values for function f. If f(x) = cr^x , where c and r are constants. What is f (–)?

x	$f(x)$
0	16
1	4
2	1
3	¼

a) 1/4 b) 2 c) 4 d) 16 e) Can't be determined

3. According to the table, for what value of x does g(f(x)) = -1?

x	f(x)	g(x)
-1	-2	4
0	0	3
1	2	2
2	4	1
3	6	0
4	8	-1

a) 4 b) 3 c) 2 d) 0 e) None of these

4. Let $f(x)$ be defined for any positive integer x greater than 2 as he sum of all prime numbers less than x. For example $f(4)$ = 2+3 = 5 and f (8) = 7+5+3+2=17. What is the value of $f(81) – f(78)$?

a) 79 b) 73 c) 71 d) Can't be determined e) None of these

5. What is the domain of f(x) = ?

a) $x< 1$ b) $x\leq 1$ c) $x > 1$ d) $x\leq 0$ e) $x<0$

14. NUMERIC ENTRY

1. Let A= {all 3-digit positive integers with the digit 1 in the ones place}, and let B= {all 3-digit positive integers with the digit 2 in the tens place}. How many elements are there in AU B?

2. Yan needs $2.37 in postage to mail a letter. If he has 60-cent, 37-cent, 23-cent, 5-cent and 1-cent stamps, at least 10 of each, what is the smallest number of stamps he can use to get the exact postage he needs?

3. A population of bacteria doubles every 2 hours. What is the percent increase after 4 hours?

4. Six chairs are placed in a row to seat six people. How many different seating arrangements are possible if two of the people insist on sitting next to each other?

5. Let *x, y and z* be consecutive even integers. If the product of 3 and *y* is 32 more than the sum of *x* and *z*, what is the median of the numbers in set S = {*x, y, z, 2x, 2y, 2z*}?

6. If $f(x) = x^2 + 3$ and $g(x) = x-5$, evaluate $f(g(9))$.

7. A line intersects two parallel lines, forming eight angles. If one of the angles has measure a^0 , how many of the other seven angles are supplementary to it?

8. A rectangular box with length 22 inches, width 5 inches, and height 5 inches is to be packed with steel balls of radius 2 inches. What is the maximum number of balls that can fit into the box, provided that no balls should protrude from the box?

9. Mary has d dollars to spend and goes on a shopping spree. First she spends 2/5 of her money on shoes. Then she spends ¾ of what's left on a few books. Finally she buys a raffle ticket that costs 1/3 of her remaining dollars. What fraction of d is left?

10. Ten pounds of mixed nuts contain 50 percent peanuts. How many pounds of peanuts must be added so that the final mixture has 60 percent peanuts?

11. At John Adams High School, 120 students take programming, and 200 students take statistics. Of these, 50 students take both programming and statistics. An additional 80 students take neither programming nor statistics. If a student at this school is picked at random, what is the probability that he or she takes programming but not statistics?

12. A hat contains the integers 1 to 100, inclusive. If a number is drawn at random from the hat, what is the probability that a multiple of 5 or a multiple of 8 is drawn?

13. A typing class in elementary school is divided into three groups. The Red Robins, with 6 students, has an average typing speed of 60 words per minute; the Blue Wax Bills, with 10 students, has an average typing speed of 45 words per minute; and the Gold Finches, with 16 students, has an average typing speed of 30 words per minute. What is the average (arithmetic mean) of the typing speeds, in words per minute, for the class?

14. A charter company will provide a plane for a fare of $300 per person if there are between 50 and 100 passengers. If there are more than 100 passengers, then, for each additional passenger over 100, the fare will be reduced by $2 for every passenger. How much revenue will the company make if 120 passengers take the trip?

15. A solid white cube with an edge of 8 inches is painted. The cube is then sliced into 512 1- inch cubes. How many of these cubes have exactly 2 red faces?

16. When an elastic object, such as a coil spring or rubber band, is subjected to a force *f*, an increase in length, called a strain, occurs. Hooke's law states that force *f* is directly proportional to strain *s*. Suppose that a coil spring has a natural length of 4 feet and that a force of 60 pounds stretches the length to 6 feet. What magnitude of force, in pounds, would stretch the spring to a length of 7 feet?

17. Rachel is hanging posters in her new apartment, which includes a bedroom, a living room, and a den. She has 7 different posters. Assuming that she plans to place exactly one poster in each of the three rooms, how many choices does she have?

18. A can contains ¼ pound of cashews. The can is then filled with a mixture that has equal weights of cashews, pecans and walnuts. If the final weight is 1 pound, what faction of the final nut mixture is cashew?

19. The weight of an object on or beneath the surface of the Moon varies directly as the distance of the object from the center of the Moon. The radius of the Moon is approximately 1,080 miles. If an object weighs 60 pounds on the surface of the Moon, how far beneath the surface, in miles, would it have to be to weigh 50 pounds?

20. For all values of x and y, let x*y be defined as x*y= $\frac{4xy}{3}$. If 6*a = 2, then a=

21. A hardware store owner finds that she can expect to sell *n* sets of wrenches per month if the price per set, in dollars, is: *p(n)* = $\frac{3000}{a+n}$ where *a* is a constant. If, according to this function, 25 sets of wrenches are sold in a month at $100 per set, how many sets can the owner expect to sell in a month if she raises the price to $200 per set?

22. If the *n*th term of a sequence is given by the expression 2 x 4^{n-1} , what is the units digit of the 131st term in the sequence?

23. If $f(x)=3x^2$, at what x-co-ordinate do the graphs of $f(x)$ and $f(x-1)$ intersect?

24. The faces of a cube are numbered with integers from 1 to 6 so that the sum of the numbers on opposite faces is 7. Thus, 1 is opposite 6, 2 is opposite 5 and 3 is opposite 4. If the cube is thrown on a flat surface so that 4 shows on the top face, what is the probability that 6 is on the bottom face of the cube?

25. The average (arithmetic mean) of five positive even integers is 60. If p is the greatest of these integers, what is the greatest possible value of p?

26.

Ingredients in each pizza

	A	B	C
Dough	15 ounces	8 ounces	18 ounces
Sauce	25 ounces	6 ounces	20 ounces
Toppings	16 ounces	18 ounces	28 ounces

Number of pizzas sold

	January	February	March
A	30	80	70
B	70	165	110
C	100	140	120

Based on the information given in the table, what was the total number of ounces of toppings sold by the pizza parlor in February?

27. $(a \times 2) + (a \times 2^2) + (b \times 2^3) + (b \times 2^4) = 42$. If a and b are positive integers, what is the value of ab?

28. For even integers x greater than 2, let $\backslash x/$ be defined as the product of the even integers from 2 to x, inclusive. For example, $\backslash 6/ = 2*4*6 = 48$. If $\frac{\backslash 8/}{\backslash 6/} = \backslash a/$, what is the value of a?

29.

FAVORITE RIDE	NUMBER OF PEOPLE
Roller coaster	93
Swings	69
Merry-go-round	18
Bumper cars	45
Tilt-a- whirl	x
Log ride	y

The table above shows the results of a survey of 300 people at an amusement part. Each person chose exactly one ride as his or her favorite. If 10 people were undecided, and x and y are both positive integers, what is the greatest possible value of y?

30. If f is a positive integer, $fg>0$, and $6f+2g=25$, what is the sum of all possible values of g?

31. A bag of dry concrete covers an area of 9 square feet. If only whole bags of dry concrete can be purchased, how many bags must be purchased to pave a sidewalk that is 3.5 feet wide and 225 feet long?

32. If t is a positive integer, and $18t$ is the cube of an integer, then what is the least possible value of t?

33. The average (arithmetic mean) of 6 distinct numbers is 71. One of these numbers is -24, and the rest of the numbers are positive. If all of the numbers are even integers with at least two digits, what is the greatest possible value of any of the 6 numbers?

34. On a number line, point D is 2/5 of the way from point C to point E and is located at -2. If C is at -10, what is the co-ordinate of point E?

35. Stephanie, Damon and Karissa have been contracted to paint an office building that contains 72 rooms. If Stephanie paints half as many rooms as Karissa and 12 more than Damon, how many rooms does Karissa paint?

36. If x and y are real numbers, and $2\sqrt{x-4} - 2 = 1$, and $|y - 5| < 2$, what is the smallest possible integer value of $x + y$?

37. One-third of the air in a tank is removed with each stroke of a pump. What percent of the original amount of air remains in the tank after five strokes?

38. Carlos paid $154.0 for two tickets to a concert. This price included a 25 percent handling fee for each ticket and a $2 transaction fee for the total sale. What was the price for a single ticket before the additional fee?

39. On January 1, 1993, Geraldine purchased a rare stamp for $ 300. The value of the rare stamp increased by 15 percent each year. If Geraldine decided to sell the stamp on January 1 of the first year in which its value had at least doubled since she purchased it, then in which year did Geraldine sell the stamp?

40. Pierre receives a weekly allowance of $8, plus $3for each chore he completes during the week. Armand receives a weekly allowance of $6, plus $8 for each chore he completes during the week. Neither of them receives any other money. In a certain week, if they both complete the same number of chores, but Armand receives twice as much money as Pierre, then what is the total dollar amount that Armand receives in that week? (Disregard the $ sign)

41. Three identical cubes, each with edges of length 8, are to be cut into a total of 384 identical rectangular solids of length 4. If the width and height of each solid are integers, what is the surface area of each solid?

42. Segment PR is tangent to a circle with center O (not shown) at point Q. If triangle POR is a right triangle, QR =3, OR =5, and OP = 20/3, then PQ =

43. Points J, K,L, M and N all lie on the same line. L is the mid-point of seg JK and the length of seg JL is 3. If K is the midpoint of seg JM, and M is the mid point of Seg JN, then seg JN =?

44. The first three terms of a geometric sequence are k, $6k$ and $36k$. For how many values of k between 1 and 10 inclusive, does the sequence contain only even integers?

45. If a and b are distinct integers such that $ab<1$ and $b \neq 0$,what is the greatest possible value of $\frac{a}{b}$?

46. A local bank offers its customers two checking account plans:

> Plan A: An unlimited number of checks can be written each month for a monthly account maintenance fee of \$ 7.50.
>
> Plan B: A monthly account maintenance fee of \$2.50 and a transaction fee of \$0.50 for each check written during the month.

If Plan A costs a certain customer <u>less</u> than Plan B, what is the <u>least</u> number of checks that this customer writes per month?

47. For how many pairs of positive integers (a, b) is $5a + 7b < 20$?

48. If x and y are positive integers and $(x^{1/2} y^{1/4})^8 = 144$, what is the smallest possible value of $y - x$?

49. In a certain flower shop, only 3 vases of flowers and 1 wreath can be displayed, in this order, at one time. If there are 10 vases of flowers and 4 wreathes to choose from, how many different arrangements of vases and wreaths are possible?

50. It takes 10 people working at the same rate 5 hours to pick 300 apples. How many hours would it take 15 people to pick 900 apples at twice the rate?