Projective Monomial Curve and their Cohen Macaulayness

Pranjal Srivastava

IIT Gandhinagar

January 15, 2020

Proposal Defense

Contents

Projective Monomial Curve

$\mathcal{S} = \{a_1, a_2, ..., a_p\}$ with $a_i \in \mathbb{N}$, $0 < a_1 < a_2 < ... < a_p = d$ and $gcd(a_1, a_2, ..., a_p) = 1$. Let numerical semigroup Γ generated by $\mathcal{S}$ and $\tilde{S}$ is generated by $\alpha_0 = (d, 0), \alpha_1 = (d - a_1, a_1), ..., \alpha_{p-1} = (d - a_{p-1}, a_{p-1}), \alpha_p = (0, d)$. Let K be a field and u, v indeterminates over K. The ring
$K[\tilde{S}] = K[u^d, u^{d-a_1}v^{a_1}, ..., u^{d-a_{p-1}}v^{a_{p-1}}, v^d] \subseteq K[u, v]$.
The **Projective Monomial curve** $\mathfrak{C}$ associated to $\mathcal{S}$ is the 1-dimensional irreducible projective variety whose homogeneous coordinate ring is $K[\tilde{S}]$. Let $R = K[X_0, X_1, ..., X_p]$, a polynomial ring over K.
The $K - algebra$ homomorphism $\phi : R \to K[u, v]$ defined by $\phi(X_i) = u^{d-a_i}v^{a_i}$, for $i = 0, ..., p$
We will find the ideal $Ker\phi =: \mathfrak{p}$**, the basis of** S**.** The ideal $\mathfrak{p}$ is a homogeneous prime ideal in R, called the defining ideal of $\mathfrak{C}$,.

Example(a)
Let $S = \{3, 5, 7\}$ be a sequence.
Let $\phi : K[X_0, X_1, X_2, X_3] \rightarrow K[u, v]$
given by according to definition, then projective monomial curve is $\mathfrak{C} = Z(Ker\phi)$ and coordinate ring is $K[u^7, u^4v^3, u^2v^5, v^7]$.
Question 1: Is $K[u^7, u^4v^3, u^2v^5, v^7]$ Cohen Macaulay?
Answer: Yes
Example(b): Let $S = \{1, 3, 4\}$ be a sequence. Then coordinate ring of projective monomial curve is $K[u^4, u^3v^1, u^3v^1, v^4]$.
Question 2: Is $K[u^4, u^3v^1, u^3v^1, v^4]$ Cohen Macaulay?
Answer: No(Proved by Macaulay)

Definition: Let (A, m, k) be a noetherian local ring and M a finite $A-$ module. M is called **Cohen – Macaulay** if $depthM = dimM$. If A is itself a Cohen-Macaulay module, then A is Cohen-Macaulay Ring.

A noetherian ring A is said to be CM ring if $A_{\mathfrak{m}}$ is a CM local ring for every maximal ideal $\mathfrak{m}$ of A.

Example:

(a) The ring $K[[X_1, ..., X_n]]$ is CM.

(b) $K[U^4, U^3V, UV^3, V^4]$ of $K[U, V]$ is not CM.

Condition to Check for CM by Basis

Definition Let $\tilde{T}$ be the semigroup of S generated by $\{\alpha_0 = (d, 0), \alpha_p = (0, d)\}$.
The set $\mathcal{B} = \{\alpha \in \tilde{S} | \alpha - \alpha_0 \notin \tilde{S}$, and $\alpha - \alpha_p \notin \tilde{S}\}$ is called the basis of $\tilde{S}$ over $\tilde{T}$.

Theorem 1.1

The following are equivalent:

(i) The semi group ring $K[\tilde{S}]$ is Cohen-Macaulay.

(ii) $K[\tilde{S}]$ a free $K[\tilde{T}]$ -module with basis $t^{\mathcal{B}}$.

(iii) $|\mathcal{B}| = d$.

- For $\alpha = (\alpha_1, \alpha_2) \in \tilde{S}$, let $t^\alpha = s^{\alpha_1} t^{\alpha_2}$.
  Then $t^{\mathcal{B}} = \{t^\alpha | \alpha \in \mathcal{B}\}$ is spanning set of $K[\tilde{S}]$ as a module over $K[\tilde{T}] = K[s^d, t^d]$.
  Proof: The canonical image of $t^{\mathcal{B}}$ forms a K-basis of $K[\tilde{S}]/(s^d, t^d)K[\tilde{S}]$.
  Let $\Gamma = \{f + (s^d, t^d)K[\tilde{S}] | f \in t^B\}$
  Γ is minimal set of generator of $K[\tilde{S}]/(s^d, t^d)K[\tilde{S}]$ over K.
  Consider $N = \sum_{f \in t^B}$, N is a $K[\tilde{T}]$-module which is spanned by $t^{\mathcal{B}}$ over B. Then $R = N + (s^d, t^d)R$,
  by N.A.K. $R = N$.
- Since $\{s^d, t^d\}$ is a system of parameter in $K[\tilde{S}]$ and $K[\tilde{S}]/(s^d, t^d)K[\tilde{S}]$ is a finite dimensional vector space, so $\mathcal{B}$ is finite. $K[\tilde{S}]$ is Cohen- Macaulay if and only if s^d, t^d is a regular sequence in $K[\tilde{S}]$.

- **Affine Semigroup:** An affine semigroup is a semigroup which is finitely generated and can be embedded in $\mathbb{Z}^n$, for some $n \in \mathbb{N}$.
- Let $G(S)$ denote the group generated by an affine semigroup S. Then $G(S) \cong \mathbb{Z}^r$, with r $\in \mathbb{N}$, which we call rank(S).
- For a semigroup S, the smallest cone containing S, $C(S) = \{\sum_{i=0}^{n} \lambda_i \alpha_i | \lambda_i \in \mathbb{R}^+, \alpha_i \in S\}$is called cone generated by S.
- An affine semigroup $S \subseteq \mathbb{N}$ is called simplicial if the cone C(S) is spanned by m linearly independent vectors $\alpha_1, ..., \alpha_m$ of S, where m=rank(S).

Theorem 1.2 (general form)

Let $\mathcal{S} \subset \mathbb{N}^n$ be a simplicial affine semigroup with $G(S) \cap C(S) \subset S$. Let the spanning vectors of the cone $C(S)$ be $\alpha_1, ..., \alpha_n$. Suppose that the index of the group $G(S)$ in $\mathbb{Z}^n$ is h. Then $R = K[S]$ is a Cohen Macaulay ring if and only if the cardinality of the spanning set X^B of $K[S]$ is $|det(A)|/h$.

Lemma 1.3

Let G be a subgroup of$\mathbb{Z}^n$, and the index of G in $\mathbb{Z}^n$ be h. Let M, N denote the group rings $K[\mathbb{Z}^n], K[G]$ respectively. Then the group ring M is a free $N-$module of rank h.

Proof: Since $(\mathbb{Z}^n : G) = h, \mathbb{Z}^n$ is a disjoint union of h coset, i.e $\mathbb{Z}^n = \cup_{i=0}^{h-1}(z_i + G), z_0 = 0, z_i \in \mathbb{Z}^n$ for $i = 1, ..., (h-1)$. So for any $z \in \mathbb{Z}^n$, z can be uniquely written as $z_i + g$ with $0 \leq i \leq (h-1)$. Therefore every monomial x^z in the $k-$ basis of the group ring $M = k[\mathbb{Z}^n]$ can be expressed as $x^z = x^{z_i}x^g$ uniquely, where $x^{z_i} \in M, x^g \in N = k[G]$. So $x^{z_0}, ..., x^{z_{h-1}}$ generate the group ring M as an N-module.

Therefore every monomial x^z in the $k-$ basis of the group ring $M = k[\mathbb{Z}^n]$ can be expressed as $x^z = x^{z_i}x^g$ uniquely, where $x^{z_i} \in M, x^g \in N = k[G]$. So $x^{z_0}, ..., x^{z_{h-1}}$ generate the group ring M as an N-module.
Suppose there are element $w_0, ..., w_{h-1} \in N$ such that $w_0x^{z_0} + ... + w_{h-1}x^{z_{h-1}} = 0$ Each term $w_ix^{z_i}$ contains monomials in the set $\{x^z \in M | z \in z_i + G\}$, the monomial x^z are linearly independent over K. therefore every term $w_ix^{z_i}$ is equal to zero. Since M is domain, so $w_i = 0$ for all i.

Lemma 1.4

Let R and B be finitely generated domains with $B \subseteq R$. Let K, L be fraction fields of R and B respectively. Then R is a free B-module if and only if R can be generated as a B-module by $[K : L]$ elements.

Lemma 1.5

Let $\mathcal{S} \subset \mathbb{Z}^n$ be a simplicial affine semigroup with rank n. Let $\alpha_1, ..., \alpha_n$. denote the spanning vectors of the cone $C(S)$. $R = K[S], B = K[W]$, where W denotes the semigroup generated by $\alpha_1, ..., \alpha_n$. Let M and N be the fraction fields of R and B respectively. Then $[M : N] = |det(A)|/h$. where $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_n \end{pmatrix}$

Proof: We only prove for special case.
S is generated by $\alpha_0 = (d, 0), \alpha_1 = (d - a_1, a_1), ..., \alpha_{p-1} = (d - a_{p-1}, a_{p-1}), \alpha_p = (0, d)$ and W is generated by $\alpha_0 = (d, 0),), \alpha_p = (0, d)$.

$$A = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$

for a given matrix A, we have a map

$$\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$$

defined by $\alpha \to \alpha A$.

This map induces a ring homomorphism

$$\hat{\phi} : K[u, u^{-1}, v, v^{-1}] \to K[u, u^{-1}, v, v^{-1}]$$

which is given by $\hat{\phi}(U^\alpha) = U^{\phi(\alpha)} = U^{\alpha A}$,
for any monomials $U^\alpha \in K[u, u^{-1}, v, v^{-1}]$.
Then $im\hat{\phi} = K[u^d, u^{-d}, v^d, v^{-d}]$ and $im\hat{\phi}$ maps N to $K(u^d, v^d)$.
let $L = K(u, v)$ and we know that $N \subseteq M \subseteq L = K(u, v)$. So $[L : M][M : N] = [L : N] = d^2$.
Now $[L : M] = [fr(K[\mathbb{Z}^2]) : fr(K[G(S)])]$
$= [\mathbb{Z}^2 : G(S)]$(by lemma 1.3 and 1.4).
$= d$

Theorem 1.6(H.Derksen and G. Kemper)

Let R be a Noetherian $\mathbb{N}$-graded algebra over a field K with $K = R_0$ the homogeneous part of degree 0. then the following are equivalent:

(1) R is Cohen-Macaulay.

(2) Every homogeneous system of parameter is R-regular.

(3) If $f_1, ..., f_n$ is homogeneous system of parameters, then R is a free module over $k[f_1, ..., f_n]$.

(4) There exist a homogeneous system of parameters $f_1, ..., f_n$ such that R is a free module over $K[f_1, ..., f_n]$.

Known Condition for Curve to be CM

Proof of Theorem 1.1:

Suppose $K[\tilde{S}]$ is Cohen Macaulay and since $\{s^d, t^d\}$ is a system of parameter, so by theorem 1.6 $K[\tilde{S}]$ is free over $K[s^d, t^d]$.
$(i) \Leftrightarrow (ii)$ By Lemma 1.4 and 1.5, $|B| = d$

Condition for Curve to be CM

(1) If $\mathcal{S} = \{a_1, a_2, ..., a_p\}$ is in arithmetic progression, then $K[\tilde{S}]$ is Cohen Macaulay.

(2) If $\mathcal{S} = \{a_1, a_2, ..., a_p\}$ is in almost arithmetic progression, then $K[\tilde{S}]$ is Cohen Macaulay.

(3) **What are the other conditions we can give on S such that $K[S]$ will become Cohen Macaulay ?**

Problem Description

Gluing of two semigroup: Let S_1 and S_2 be two numerical semigroups minimally generated by$\{n_1, ..., n_r\}$ and $\{n_{r+1}, ..., n_e\}$, respectively. Let $\lambda \in S_1 \setminus \{n_1, ..., n_r\}$ and $\mu \in S_2 \setminus \{n_{r+1}, ..., n_e\}$ be such that $gcd(\lambda, \mu) = 1$. We say that $S =< \mu n_1, ..., \mu n_r, \lambda n_{r+1}, ..., \lambda n_e >$ is a gluing of S_1 and S_2.

Example $S_1 = \{3, 5, 7\}$ and $S_2 = \{9, 11\}$, take $\lambda = 8$ and $\mu = 20$ then $S = S_1 \# S_2 = \{60, 100, 140, 72, 99\}$.

Main Problem:

Let S_1 and S_2 be two numerical semigroups minimally generated by$\{n_1, ..., n_r\}$ and $\{n_{r+1}, ..., n_e\}$, respectively.$K[\tilde{S_1}]$, $K[\tilde{S_2}]$ are corresponding coordinate rings and $K[\tilde{S}]$ is coordinate ring of curve associated to gluing S then

(1) What condition we give on S_1 and S_2 such that $K[\tilde{S}]$ will become Cohen Macaulay?

(2) What is the smallest Cohen Macaulay Ring containing $K[\tilde{S}]$ in a "good" sense.

References

1. Ping Li. Seminormality and the Cohen-Macaulay Property. PhD thesis, Queen's University, 2005.
2. Winfried Bruns; Jürgen Herzog, Cohen–Macaulay rings. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.
3. Li, Ping; Patil, D. P.; Roberts, Leslie G. Bases and ideal generators for projective monomial curves. Comm. Algebra 40 (2012), no. 1, 173–191.
4. Les Reid and Leslie G. Roberts. Non-Cohen-Macaulay projective monomial curves. Journal of Algebra, 291:171–186, 2005.
5. Harm Derksen and Gregor Kemper, Computational Invariant Theory - Encyclopaedia of Mathematical Sciences, Springer, 2002.

THANK YOU