Introduction to Hidden Markov Models

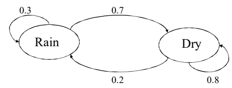
Markov Models

- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- To define Markov model, the following probabilities have to be specified: transition probabilities $a_{ij} = P(s_i \mid s_j)$ and initial probabilities $\pi_i = P(s_i)$

Example of Markov Model



- Two states : 'Rain' and 'Dry'.
- Transition probabilities: P('Rain'|'Rain')=0.3 , P('Dry'|'Rain')=0.7 , P('Rain'|'Dry')=0.2, P('Dry'|'Dry')=0.8
- Initial probabilities: say P('Rain')=0.4 , P('Dry')=0.6 .

Calculation of sequence probability

- By Markov chain property, probability of state sequence can be found by the formula:

$$P(s_{i1}, s_{i2}, \ldots, s_{ik}) = P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1})$$
$$= P(s_{ik} \mid s_{ik-1}) P(s_{i1}, s_{i2}, \ldots, s_{ik-1}) = \ldots$$
$$= P(s_{ik} \mid s_{ik-1}) P(s_{ik-1} \mid s_{ik-2}) \ldots P(s_{i2} \mid s_{i1}) P(s_{i1})$$

- Suppose we want to calculate a probability of a sequence of states in our example, {'Dry','Dry','Rain',Rain'}.

P({'Dry','Dry','Rain',Rain'}) =
P('Rain'|'Rain') P('Rain'|'Dry') P('Dry'|'Dry') P('Dry')=
= 0.3*0.2*0.8*0.6

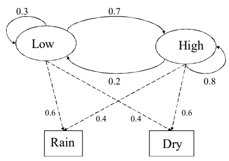
Hidden Markov models.

- Set of states: $\{s_1, s_2, \ldots, s_N\}$
- Process moves from one state to another generating a sequence of states : $s_{i1}, s_{i2}, \ldots, s_{ik}, \ldots$
- Markov chain property: probability of each subsequent state depends only on what was the previous state:

$$P(s_{ik} \mid s_{i1}, s_{i2}, \ldots, s_{ik-1}) = P(s_{ik} \mid s_{ik-1})$$

- States are not visible, but each state randomly generates one of M observations (or visible states) $\{v_1, v_2, \ldots, v_M\}$
- To define hidden Markov model, the following probabilities have to be specified: matrix of transition probabilities A=(a_{ij}), a_{ij}= P(s_i| s_j) , matrix of observation probabilities B=($b_i(v_m)$), $b_i(v_m)$= P(v_m| s_i) and a vector of initial probabilities π=(π_i), π_i = P(s_i) . Model is represented by M=(A, B, π).

Example of Hidden Markov Model



Example of Hidden Markov Model

- Two states : 'Low' and 'High' atmospheric pressure.
- Two observations : 'Rain' and 'Dry'.
- Transition probabilities: P('Low'|'Low')=0.3 , P('High'|'Low')=0.7 , P('Low'|'High')=0.2, P('High'|'High')=0.8
- Observation probabilities : P('Rain'|'Low')=0.6 , P('Dry'|'Low')=0.4 , P('Rain'|'High')=0.4 , P('Dry'|'High')=0.3 .
- Initial probabilities: say P('Low')=0.4 , P('High')=0.6 .

Calculation of observation sequence probability

- Suppose we want to calculate a probability of a sequence of observations in our example, {'Dry','Rain'}.
- Consider all possible hidden state sequences:

P({'Dry','Rain'}) = P({'Dry','Rain'} , {'Low','Low'}) + P({'Dry','Rain'} , {'Low','High'}) + P({'Dry','Rain'} , {'High','Low'}) + P({'Dry','Rain'} , {'High','High'})

where first term is :
P({'Dry','Rain'} , {'Low','Low'})=
P({'Dry','Rain'} | {'Low','Low'}) P({'Low','Low'}) =
P('Dry'|'Low')P('Rain'|'Low') P('Low')P('Low'|'Low)
= 0.4*0.4*0.6*0.4*0.3

Main issues using HMMs :

Evaluation problem. Given the HMM M=(A, B, π) and the observation sequence O=$o_1 o_2 \ldots o_K$, calculate the probability that model M has generated sequence O .

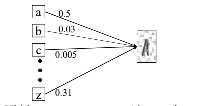
- **Decoding problem.** Given the HMM M=(A, B, π) and the observation sequence O=$o_1 o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence O.
- **Learning problem.** Given some training observation sequences O=$o_1 o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters M=(A, B, π) that best fit training data.

O=$o_1 \ldots o_K$ denotes a sequence of observations $o_k \in \{v_1, \ldots, v_M\}$.

Word recognition example(1).

• Typed word recognition, assume all characters are separated.

• Character recognizer outputs probability of the image being particular character, P(image|character).



Word recognition example(2).

• Hidden states of HMM = characters.

• Observations = typed images of characters segmented from the image v_α . Note that there is an infinite number of observations

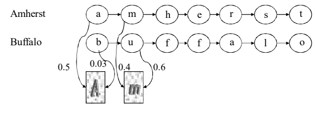
• Observation probabilities = character recognizer scores.

$$B = \big(b_i(v_\alpha)\big) = \big(P(v_\alpha \mid s_i)\big)$$

•Transition probabilities will be defined differently in two subsequent models.

Word recognition example(3).

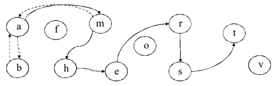
• If lexicon is given, we can construct separate HMM models for each lexicon word.



• Here recognition of word image is equivalent to the problem of evaluating few HMM models.
•This is an application of **Evaluation problem.**

Word recognition example(4).

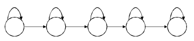
• We can construct a single HMM for all words.
• Hidden states = all characters in the alphabet.
• Transition probabilities and initial probabilities are calculated from language model.
• Observations and observation probabilities are as before.



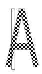
• Here we have to determine the best sequence of hidden states, the one that most likely produced word image.
• This is an application of **Decoding problem.**

Character recognition with HMM example.

• The structure of hidden states is chosen.



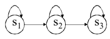
• Observations are feature vectors extracted from vertical slices.



• Probabilistic mapping from hidden state to feature vectors:
  1. use mixture of Gaussian models
  2. Quantize feature vector space.

Exercise: character recognition with HMM(1)

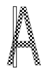
• The structure of hidden states:



• Observation = number of islands in the vertical slice.
•HMM for character 'A' :

Transition probabilities: $\{a_{ij}\} = \begin{pmatrix} .8 & .2 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{pmatrix}$

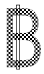
Observation probabilities: $\{b_{jk}\} = \begin{pmatrix} .9 & .1 & 0 \\ .1 & .8 & .1 \\ .9 & .1 & 0 \end{pmatrix}$



•HMM for character 'B' :

Transition probabilities: $\{a_{ij}\} = \begin{pmatrix} .8 & .2 & 0 \\ 0 & .8 & .2 \\ 0 & 0 & 1 \end{pmatrix}$

Observation probabilities: $\{b_{jk}\} = \begin{pmatrix} .9 & .1 & 0 \\ 0 & .2 & .8 \\ .6 & .4 & 0 \end{pmatrix}$



Exercise: character recognition with HMM(2)

• Suppose that after character image segmentation the following sequence of island numbers in 4 slices was observed:
  { 1, 3, 2, 1}

• What HMM is more likely to generate this observation sequence , HMM for 'A' or HMM for 'B' ?

Exercise: character recognition with HMM(3)

Consider likelihood of generating given observation for each possible sequence of hidden states:

• HMM for character 'A':

Hidden state sequence	Transition probabilities		Observation probabilities
$s_1 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3$	$.8 * .2 * .2$	$*$	$.9 * 0 * .8 * .9 = 0$
$s_1 \rightarrow s_2 \rightarrow s_2 \rightarrow s_3$	$.2 * .8 * .2$	$*$	$.9 * .1 * .8 * .9 = 0.0020736$
$s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_3$	$.2 * .2 * 1$	$*$	$.9 * .1 * .1 * .9 = 0.000324$
			Total = 0.0023976

• HMM for character 'B':

Hidden state sequence	Transition probabilities		Observation probabilities
$s_1 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3$	$.8 * .2 * .2$	$*$	$.9 * 0 * .2 * .6 = 0$
$s_1 \rightarrow s_2 \rightarrow s_2 \rightarrow s_3$	$.2 * .8 * .2$	$*$	$.9 * .8 * .2 * .6 = 0.0027648$
$s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_3$	$.2 * .2 * 1$	$*$	$.9 * .8 * .4 * .6 = 0.006912$
			Total = 0.0096768

Evaluation Problem.

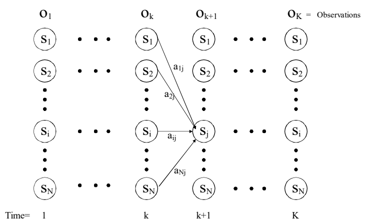
•**Evaluation problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the probability that model M has generated sequence O .

• Trying to find probability of observations $O=o_1 o_2 \ldots o_K$ by means of considering all hidden state sequences (as was done in example) is impractical:
  N^K hidden state sequences - exponential complexity.

• Use **Forward-Backward HMM algorithms** for efficient calculations.

• Define the forward variable $\alpha_k(i)$ as the joint probability of the partial observation sequence $o_1 o_2 \ldots o_k$ and that the hidden state at time k is s_i : $\alpha_k(i) = P(o_1 o_2 \ldots o_k, q_k = s_i)$

Trellis representation of an HMM



Forward recursion for HMM

- Initialization:

$\alpha_1(i)= P(o_1, q_1=s_i) = \pi_i\, b_i(o_1)$, $1<=i<=N$.

- Forward recursion:

$\alpha_{k+1}(i)= P(o_1 o_2 \ldots o_{k+1}, q_{k+1}=s_j) =$

$\Sigma_i P(o_1 o_2 \ldots o_{k+1}, q_k=s_i, q_{k+1}=s_j) =$

$\Sigma_i P(o_1 o_2 \ldots o_k, q_k=s_i)\, a_{ij}\, b_j(o_{k+1}) =$

$[\Sigma_i \alpha_k(i)\, a_{ij}]\, b_j(o_{k+1})$, $1<=j<=N, 1<=k<=K-1$.

- Termination:

$P(o_1 o_2 \ldots o_K) = \Sigma_i P(o_1 o_2 \ldots o_K, q_K=s_i) = \Sigma_i \alpha_K(i)$

- Complexity :

N^2K operations.

Backward recursion for HMM

- Define the forward variable $\beta_k(i)$ as the joint probability of the partial observation sequence $o_{k+1} o_{k+2} \ldots o_K$ given that the hidden state at time k is s_i : $\beta_k(i)= P(o_{k+1} o_{k+2} \ldots o_K | q_k=s_i)$
- Initialization:

$\beta_K(i)= 1$, $1<=i<=N$.

- Backward recursion:

$\beta_k(j)= P(o_{k+1} o_{k+2} \ldots o_K | q_k=s_j) =$

$\Sigma_i P(o_{k+1} o_{k+2} \ldots o_K, q_{k+1}=s_i | q_k=s_j) =$

$\Sigma_i P(o_{k+2} o_{k+3} \ldots o_K | q_{k+1}=s_i)\, a_{ji}\, b_i(o_{k+1}) =$

$\Sigma_i \beta_{k+1}(i)\, a_{ji}\, b_i(o_{k+1})$, $1<=j<=N, 1<=k<=K-1$.

- Termination:

$P(o_1 o_2 \ldots o_K) = \Sigma_i P(o_1 o_2 \ldots o_K, q_1=s_i) =$

$\Sigma_i P(o_1 o_2 \ldots o_K | q_1=s_i)\, P(q_1=s_i) = \Sigma_i \beta_1(i)\, b_i(o_1)\, \pi_i$

Decoding problem

- **Decoding problem.** Given the HMM $M=(A, B, \pi)$ and the observation sequence $O=o_1 o_2 \ldots o_K$, calculate the most likely sequence of hidden states s_i that produced this observation sequence.
- We want to find the state sequence $Q= q_1 \ldots q_K$ which maximizes $P(Q | o_1 o_2 \ldots o_K)$, or equivalently $P(Q , o_1 o_2 \ldots o_K)$.
- Brute force consideration of all paths takes exponential time. Use efficient **Viterbi algorithm** instead.
- Define variable $\delta_k(i)$ as the maximum probability of producing observation sequence $o_1 o_2 \ldots o_k$ when moving along any hidden state sequence $q_1 \ldots q_{k-1}$ and getting into $q_k= s_i$.

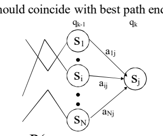
$\delta_k(i) = \max P(q_1 \ldots q_{k-1}, q_k= s_i , o_1 o_2 \ldots o_k)$

where max is taken over all possible paths $q_1 \ldots q_{k-1}$.

Viterbi algorithm (1)

- General idea:

if best path ending in $q_k= s_j$ goes through $q_{k-1}= s_i$ then it should coincide with best path ending in $q_{k-1}= s_i$.



- $\delta_k(i) = \max P(q_1 \ldots q_{k-1}, q_k= s_j , o_1 o_2 \ldots o_k) =$

$\max_i [\, a_{ij}\, b_j(o_k)\ \max P(q_1 \ldots q_{k-1}= s_i , o_1 o_2 \ldots o_{k-1})\,]$

- To backtrack best path keep info that predecessor of s_j was s_i.

Viterbi algorithm (2)

- Initialization:

$\delta_1(i) = \max P(q_1= s_i , o_1) = \pi_i\, b_i(o_1)$, $1<=i<=N$.

- Forward recursion:

$\delta_k(j) = \max P(q_1 \ldots q_{k-1}, q_k= s_j , o_1 o_2 \ldots o_k) =$

$\max_i [\, a_{ij}\, b_j(o_k) \max P(q_1 \ldots q_{k-1}= s_i , o_1 o_2 \ldots o_{k-1})\,] =$

$\max_i [\, a_{ij}\, b_j(o_k)\, \delta_{k-1}(i)\,]$, $1<=j<=N, 2<=k<=K$.

- Termination: choose best path ending at time K

$\max_i [\, \delta_K(i)\,]$

- Backtrack best path.

This algorithm is similar to the forward recursion of evaluation problem, with Σ replaced by max and additional backtracking.

Learning problem (1)

- **Learning problem.** Given some training observation sequences $O=o_1 o_2 \ldots o_K$ and general structure of HMM (numbers of hidden and visible states), determine HMM parameters $M=(A, B, \pi)$ that best fit training data, that is maximizes $P(O | M)$.
- There is no algorithm producing optimal parameter values.
- Use iterative expectation-maximization algorithm to find local maximum of $P(O | M)$ - **Baum-Welch algorithm.**

Learning problem (2)

- If training data has information about sequence of hidden states (as in word recognition example), then use maximum likelihood estimation of parameters:

$$a_{ij}= P(s_i | s_j) = \frac{\text{Number of transitions from state } s_j \text{ to state } s_i}{\text{Number of transitions out of state } s_j}$$

$$b_i(v_m)= P(v_m | s_i)= \frac{\text{Number of times observation } v_m \text{ occurs in state } s_i}{\text{Number of times in state } s_i}$$

Baum-Welch algorithm

General idea:

$$a_{ij}= P(s_i | s_j) = \frac{\text{Expected number of transitions from state } s_j \text{ to state } s_i}{\text{Expected number of transitions out of state } s_j}$$

$$b_i(v_m)= P(v_m | s_i)= \frac{\text{Expected number of times observation } v_m \text{ occurs in state } s_i}{\text{Expected number of times in state } s_i}$$

$\pi_i= P(s_i) =$ Expected frequency in state s_i at time k=1.

Baum-Welch algorithm: expectation step(1)

- Define variable $\xi_k(i,j)$ as the probability of being in state s_i at time k and in state s_j at time k+1, given the observation sequence $o_1\, o_2 \ldots o_K$.

$$\xi_k(i,j) = P(q_k = s_i\,, q_{k+1} = s_j \mid o_1\, o_2 \ldots o_K)$$

$$\xi_k(i,j) = \frac{P(q_k = s_i\,, q_{k+1} = s_j\,, o_1\, o_2 \ldots o_k)}{P(o_1\, o_2 \ldots o_k)} =$$

$$\frac{P(q_k = s_i\,, o_1\, o_2 \ldots o_k)\, a_{ij}\, b_j(o_{k+1})\, P(o_{k+2} \ldots o_K \mid q_{k+1} = s_j)}{P(o_1\, o_2 \ldots o_k)} =$$

$$\frac{\alpha_k(i)\, a_{ij}\, b_j(o_{k+1})\, \beta_{k+1}(j)}{\Sigma_i \Sigma_j\, \alpha_k(i)\, a_{ij}\, b_j(o_{k+1})\, \beta_{k+1}(j)}$$

Baum-Welch algorithm: expectation step(2)

- Define variable $\gamma_k(i)$ as the probability of being in state s_i at time k, given the observation sequence $o_1\, o_2 \ldots o_K$.

$$\gamma_k(i) = P(q_k = s_i \mid o_1\, o_2 \ldots o_K)$$

$$\gamma_k(i) = \frac{P(q_k = s_i\,, o_1\, o_2 \ldots o_k)}{P(o_1\, o_2 \ldots o_k)} = \frac{\alpha_k(i)\, \beta_k(i)}{\Sigma_i\, \alpha_k(i)\, \beta_k(i)}$$

Baum-Welch algorithm: expectation step(3)

- We calculated $\xi_k(i,j) = P(q_k = s_i\,, q_{k+1} = s_j \mid o_1\, o_2 \ldots o_K)$
  and $\gamma_k(i) = P(q_k = s_i \mid o_1\, o_2 \ldots o_K)$
- Expected number of transitions from state s_i to state s_j =
  $= \Sigma_k\, \xi_k(i,j)$
- Expected number of transitions out of state s_i $= \Sigma_k\, \gamma_k(i)$
- Expected number of times observation v_m occurs in state s_i =
  $= \Sigma_k\, \gamma_k(i)$, k is such that $o_k = v_m$
- Expected frequency in state s_i at time k=1 : $\gamma_1(i)$.

Baum-Welch algorithm: maximization step

$$a_{ij} = \frac{\text{Expected number of transitions from state } s_j \text{ to state } s_i}{\text{Expected number of transitions out of state } s_j} = \frac{\Sigma_k\, \xi_k(i,j)}{\Sigma_k\, \gamma_k(i)}$$

$$b_i(v_m) = \frac{\text{Expected number of times observation } v_m \text{ occurs in state } s_i}{\text{Expected number of times in state } s_i} = \frac{\Sigma_k\, \xi_k(i,j)}{\Sigma_{k,o_k = v_m}\, \gamma_k(i)}$$

$$\pi_i = (\text{Expected frequency in state } s_i \text{ at time } k=1) = \gamma_1(i).$$