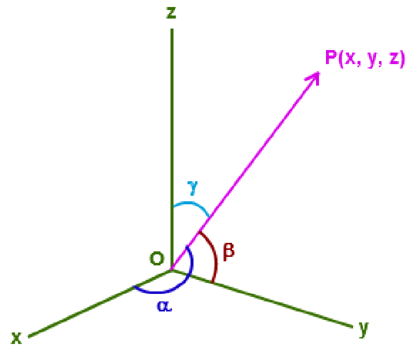
THREE-DIMENSIONAL GEOMETRY QUICK NOTES

Prepared by Neha Agrawal MATHEMATICALLY INCLINED

- **DIRECTION ANGLES OF A VECTOR**



Let α : angle $\vec{OP}$ makes with the positive directions of x axis.
β : angle $\vec{OP}$ makes with the positive directions of y axis.
γ : angle $\vec{OP}$ makes with the positive directions of z axis.

are called the *DIRECTION ANGLES*

- **DIRECTION COSINES OF A VECTOR**

Cosines of these Direction angles are called the *DIRECTION COSINES* of $\vec{OP}$.

They are denoted by l, m and n respectively.

$0 \le \alpha, \beta, \gamma \le \pi$

$l = \cos\alpha$; $m = \cos\beta$; $n = \cos\gamma$

$$\boxed{l^2 + m^2 + n^2 = 1}$$

Also $\vec{PO}$ makes angles $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$ with OX,OY,OZ axes.
So, the direction cosines of $\vec{PO}$ are: $-l, -m, -n$

DIRECTION RATIOS OF A VECTOR

Let l, m and n be the direction cosines of a vector $\vec{r}$ and a, b and c be three numbers such that

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

(i.e if a,b,c are three numbers proportional to the d.c's of a line then a,b,c are called the direction ratios of vector $\vec{r}$)

- **DCs are always UNIQUE and DRs are NOT UNIQUE.**

- If a, b ,c are the direction ratios of a vector, then its direction cosines are given by

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

(signs should be taken all +ve or all -ve)

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LINES

CONCEPT	VECTOR EQUATION	CARTESIAN EQUATION
	EQUATION OF LINES	
POINT - PARALLEL FORM	Line passing through a point whose p.v is $\vec{a}$ and is parallel to a given vector $\vec{b}$ $\vec{r}=\vec{a}+\lambda\vec{b}$	Line passing through a point (x_1,y_1,z_1) and having DR's a,b,c $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ (a,b,c can be replaced by l,m,n)
TWO-POINT FORM	Line passing through two points whose p.v are $\vec{a}$ and $\vec{b}$ $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$	Line passing through two points (x_1,y_1,z_1) and (x_2,y_2,z_2) $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$
ANGLE BETWEEN TWO LINES	Angle between two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ $\cos\theta=\frac{\lvert\vec{b}_1.\vec{b}_2\rvert}{\lvert\vec{b}_1\rvert\lvert\vec{b}_2\rvert}$	Angle between $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ $\cos\theta=\frac{\lvert a_1a_2+b_1b_2+c_1c_2\rvert}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$
CONDITION FOR TWO LINES TO BE PARALLEL	$\vec{b}_1=\lambda\vec{b}_2$	$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
CONDITION FOR TWO LINES TO BE PERPENDICULAR	$\vec{b}_1.\vec{b}_2=0$	$a_1a_2+b_1b_2+c_1c_2=0$
Skew lines: Two lines in space which are neither parallel nor intersecting are called Skew lines. They lie in different planes.		

	SHORTEST DISTANCE BETWEEN TWO SKEW LINES	SHORTEST DISTANCE BETWEEN TWO PARALLEL LINES
	If $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ are two lines then $\dfrac{\left|\left(\vec{b}_1 \times \vec{b}_2\right)\cdot\left(\vec{a}_2 - \vec{a}_1\right)\right|}{\left|\vec{b}_1 \times \vec{b}_2\right|}$	$\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}$ is $\dfrac{\left|\vec{b} \times (\vec{a}_2 - \vec{a}_1)\right|}{\left|\vec{b}\right|}$

PLANES

CONCEPT	VECTOR EQUATION	CARTESIAN EQUATION
EQUATION OF PLANES		
NORMAL FORM	A plane passing having $\hat{n}$ as a unit vector normal to it and at a distance d from the origin $\vec{r}.\hat{n} = d$	$lx + my + nz = d$
POINT-NORMAL FORM	Plane passing through a point whose p.v is $\vec{a}$ and $\perp$ to the vector $\vec{n}$ $(\vec{r} - \vec{a}).\vec{n} = \vec{0}$ $\vec{r}.\vec{n} = d$	Plane passing through a point (x_1,y_1,z_1) and direction ratios of the normal to the plane is a,b,c $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
PLANE THROUGH THREE NON-COLLINEAR POINTS	$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) X (\vec{c} - \vec{a})] = 0$	$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
INTERCEPT FORM		Plane cutting off intercepts a,b,c from x,y,z axes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
PLANE THROUGH INTERSECTION OF TWO PLANES	$\left(\vec{r}.\vec{n}_1 - d_1\right) + \lambda\left(\vec{r}.\vec{n}_2 - d_2\right) = 0$	$\left(A_1x + B_1y + C_1z - D_1\right) + \lambda\left(A_2x + B_2y + C_2z - D_2\right) = 0$

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ANGLE BETWEEN TWO PLANES	Angle between two planes $\vec{r}.\vec{n}_1 = d_1, \vec{r}.\vec{n}_2 = d_2$ is $\cos\theta = \frac{\lvert\vec{n}_1.\vec{n}_2\rvert}{\lvert\vec{n}_1\rvert\lvert\vec{n}_2\rvert}$ (Angle between their normal's)	$\cos\theta = \frac{\lvert a_1a_2 + b_1b_2 + c_1c_2\rvert}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$
CONDITION FOR TWO PLANES TO BE PARALLEL	$\vec{n}_1 \times \vec{n}_2 = \vec{0}$ **OR** $\vec{n}_1 = \lambda\vec{n}_2$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
CONDITION FOR TWO PLANES TO BE PERPENDICULAR	$\vec{n}_1.\vec{n}_2 = 0$	$a_1a_2 + b_1b_2 + c_1c_2 = 0$
DISTANCE BETWEEN A POINT AND A PLANE	$\frac{\lvert\vec{a}.\vec{n} - d\rvert}{\lvert\vec{n}\rvert}$ ($\vec{r}.\vec{n} = d$, where p.v of P is $\vec{a}$)	The length of the $\perp$ from P(x_1,y_1,z_1) to the plane ax+by+cz+d=0 is $\frac{\lvert ax_1 + by_1 + cz_1 + d\rvert}{\sqrt{a^2 + b^2 + c^2}}$
DISTANCE BETWEEN TWO PARALLEL PLANES	$\frac{\lvert d_1 - d_2\rvert}{\lvert\vec{n}\rvert}$ if $\vec{r}.\vec{n} = d_1$ and $\vec{r}.\vec{n} = d_2$	The distance between two parallel planes ax+by+cz+d_1=0 and ax+by+cz+d_2=0 is $\frac{\lvert d_1 - d_2\rvert}{\sqrt{a^2 + b^2 + c^2}}$
CONDITION FOR TWO LINES TO BE CO-PLANAR	Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = \vec{0}$	$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
EQUATION OF A PLANE CONTAINING TWO LINES	$(\vec{r} - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = \vec{0}$ **OR** $(\vec{r} - \vec{a}_2).(\vec{b}_1 \times \vec{b}_2) = \vec{0}$	$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$ **OR**

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ANGLE BETWEEN A LINE AND A PLANE 	Angle between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}.\vec{n} = d$ is $\sin\theta = \left\lvert \frac{\vec{b}.\vec{n}}{\lvert\vec{b}\rvert\lvert\vec{n}\rvert} \right\rvert$	Angle between the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and the plane $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ is $\sin\theta = \left\lvert \frac{Aa+Bb+Cc}{\sqrt{A^2+B^2+C^2}\sqrt{a^2+b^2+c^2}} \right\rvert$
CONDITION FOR A LINE AND A PLANE TO BE PARALLEL	$\vec{n}_1.\vec{n}_2 = 0$	$a_1a_2 + b_1b_2 + c_1c_2 = 0$
CONDITION FOR A LINE AND A PLANE TO BE PERPENDICULAR	$\vec{n}_1 \times \vec{n}_2 = \vec{0}$ **OR** $\vec{n}_1 = \lambda\vec{n}_2$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$