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CAREER INSTITUTE
JEE/NEET EXPERT

MATHEMATICS
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VECTORS (MSP-17)

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Vector Algebra

CHAPTER

JEE Syllabus

Vectors and Scalars, addition of vectors, components of a vector in two dimensional and three dimensional space, scalar and vector products, scalar and vector triple product. Application of vectors to plane geometry.

THIS CHAPTER INCLUDES :

- Vectors and Scalars
- Representation of vectors
- Magnitude of a vectors
- Types of vectors
- Angle between vectors
- Addition of vectors
- Subtraction of vectors
- Section formulae
- Resolution of vectors
- Linearly dependent & independent system of vectors
- Multiplication of a vectors by a scalar
- Scalar (dot) product of two vectors
- Vectors (cross) product of two vectors
- Scalar triple product
- Vector triple product
- Application of vectors to geometry
- Solved examples

Vector algebra deals with addition / subtraction / product of vector quantities. Application of vectors to many geometrical problems cuts short the procedure. Hence geometrical significance of vectors should be well understood.

VECTORS AND SCALARS

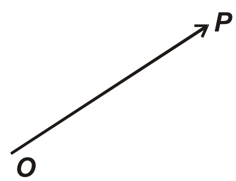
The physical quantities (we deal with) are generally of two types:

Scalar Quantity: A quantity which has magnitude but no sense of direction is called Scalar quantity (or scalar), *e.g.,* mass, volume, density, speed etc.

Vector Quantity: A quantity which has magnitude as well as a sense of direction in space is called a vector quantity, *e.g.,* velocity, force, displacement etc.

NOTATION AND REPRESENTATION OF VECTORS

Vectors are represented by $\vec{a}, \vec{b}, \vec{c}$ and their magnitude (modulus) is represented by a, b, c, or $|\vec{a}|, |\vec{b}|, |\vec{c}|$,..... The vectors are represented by directed line segments.



For example, line segment $\overrightarrow{OP}$ represents a vector with magnitude OP (length of line segment), arrow denotes its direction. O is initial point and P is terminal point.

SOME SPECIAL VECTORS

1. **Null vectors:** A vector with zero magnitude and indeterminate direction, denoted by $\vec{O}$.

2. **Unit vector:** A vector with unit magnitude (one unit), denoted by $\hat{a}$ where $|\hat{a}|$ = 1 unit.

3. **Equal vectors:** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have same sense of direction and $|\vec{a}| = |\vec{b}|$, denoted by $\vec{a} = \vec{b}$.

4. **Like and unlike vectors:** Vectors having same sense of directions are called Like vector and opposite sense of directions are called Unlike vector.

5. **Negative of a vector:** Negative of a vector $\vec{a}$, denoted by $-\vec{a}$, is a vector whose magnitude is $|\vec{a}|$ and direction is opposite of $\vec{a}$.

6. **Collinear vectors:** Vectors having same line of action.

7. **Parallel vectors:** Vectors having same line of action or are parallel to a fixed straight lines.

8. **Coplanar vectors:** The vectors which lie in the same plane. At least three coplanar unequal vectors are required to make the sum zero and at least four if non-coplanar.

9. **Free vectors:** A vector not restricted to pass through a fixed point.

10. **Localized vectors:** A vector restricted to pass through a fixed point.

11. **Co-initial vectors:** Vectors having same initial point.

12. **Position vectors:** Let O be fixed point in space, then vector $\overrightarrow{OP}$ (P is any point in space) is called position vector of P w.r.t. O. If A and B are any two point in space then

$$\overrightarrow{AB} = p.v. \text{ of } B - p.v. \text{ of } A = \overrightarrow{OB} - \overrightarrow{OA}.$$

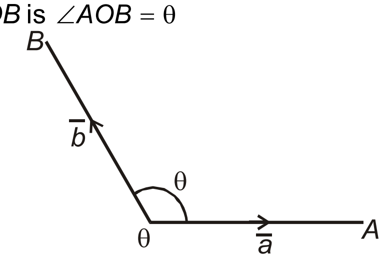
ANGLE BETWEEN TWO VECTORS

The angle between two vectors $\bar{a}$ and $\bar{b}$ represented by OA and OB is $\angle AOB = \theta$

$0 \le \theta \le \pi$

If $\theta = \frac{\pi}{2}$, then vectors are called orthogonal or perpendicular vectors

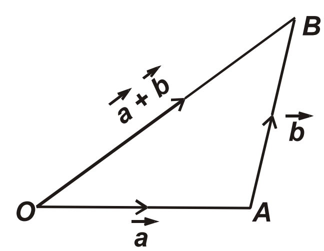
if $\theta = 0$ or π then vectors are called parallel or coincident vectors.



ADDITION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. Draw $\overrightarrow{OA}$ representing vector $\vec{a}$.

Taking terminal point of $\vec{a}$ as initial point of vector $\vec{b}$, draw $\overrightarrow{AB}$ representing vector $\vec{b}$. Then vector $\overrightarrow{OB}$ is called sum of vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} + \vec{b}$ (triangle law of addition)



Triangle Law of Addition

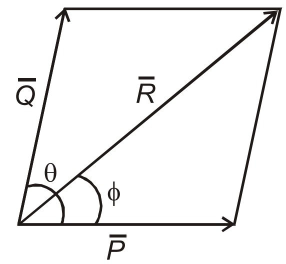
Three vectors are in equilibrium if represented by, three sides of a closed triangle taken in order, (in magnitude and direction).

Converse of triangle law is also true.

Law of polygons : If several vectors, when added, form a closed polygon, their resultant is zero.

Parallelogram law of addition – If two vectors are represented in magnitude and direction, by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction, by the co-initial diagonal of that parallelogram.

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta} \text{ and } \tan\phi = \frac{Q\sin\theta}{P + Q\cos\theta}$$



If $\theta = 0^\circ$ then $R = P + Q$ (maximum)

$\theta = \pi$ then $R = P \sim Q$ (minimum)

If $\theta = \frac{\pi}{2}$ then $R = \sqrt{P^2 + Q^2}$ and $\tan\phi = \frac{Q}{P}$

Properties of Vector Addition

1. Vector addition is commutative.

   $\overline{a} + \overline{b} = \overline{b} + \overline{a}$

2. Vector addition is associative

   $(\overline{a} + \overline{b}) + \overline{c} = \overline{a} + (\overline{b} + \overline{c})$

3. $\overline{O}$ or null vector is additive identity.

   $\overline{A} + \overline{O} = \overline{A} = \overline{O} + \overline{A}$

4. Additive inverse of $\overline{A}$ is $(-\overline{A})$, Since

   $\overline{A} + (-\overline{A}) = \overline{O}$

Illustration 1 :

At what angle (θ) shall two vectors $\overline{P}$ & $\overline{Q}$ $(|\overline{P}| = |\overline{Q}|)$ act, so as to have a resultant n times in magnitude of each. What shall be the range of n?

Solution :

Let $|\overline{P}| = |\overline{Q}| = F$ then magnitude of resultant = nF. By law of parallelogram of addition

$(nF)^2 = F^2 + F^2 + 2F^2\cos\theta$

or $n^2 = 2 + 2\cos\theta$

$\therefore \cos\theta = \frac{n^2 - 2}{2}$

$\theta = \cos^{-1}\left(\frac{n^2 - 2}{2}\right)$

Since $-1 \le \cos\theta \le 1$, So $-1 \le \frac{n^2 - 2}{2} \le 1$

$\Rightarrow -2 \le n^2 - 2 \le 2$

$\Rightarrow 0 \le n^2 \le 4$ or $0 \le n \le 2$

SUBTRACTION OF TWO VECTORS

The subtraction of two vectors $\overline{a}$ and $\overline{b}$ is defined as $\overline{a} - \overline{b} = \overline{a} + (-\overline{b})$

i.e. the vector to be subtracted is reversed and added to other vector.

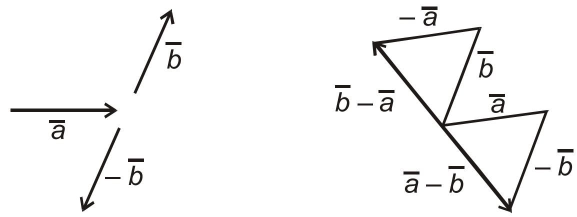
$\overline{a} - \overline{b} \ne \overline{b} - \overline{a}$ not commutative

$\overline{a} - \overline{b} = -(\overline{b} - \overline{a})$ *i.e.* directions are opposite but magnitudes are same

For non - zero vectors $\bar{a}$ & $\bar{b}$

$\bar{a}+\bar{b} \neq \bar{a}-\bar{b}$

$|\bar{a}+\bar{b}|=|\bar{a}-\bar{b}|$ mean $\bar{a} \perp \bar{b}$



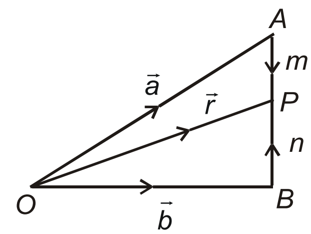
SECTION FORMULAE

Let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}$ then p.v. of point P which divides AB internally in the ratio $m:n$ given by

$$\overrightarrow{OP}=\vec{r}=\frac{m\vec{b}+n\vec{a}}{m+n} \quad (m \neq -n)$$

If P divides AB externally in the ratio $m:n$ then

$$\overrightarrow{OP}=\frac{n\overrightarrow{a}-m\overrightarrow{b}}{n-m}$$



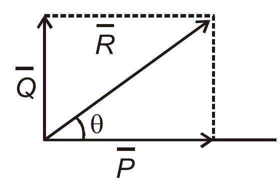
VECTOR RESOLUTION OF A VECTOR (COMPONENTS OF A VECTOR)

A vector $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$ has its x, y and z components as a, b & c respectively.

The vector resolution can be orthogonal or non-orthogonal. Orthogonal resolution means the components of the vector are mutually perpendicular otherwise non-orthogonal. Orthogonal components of vector $\bar{R}$ are as shown.

$P = R\cos\theta$ $\qquad \vec{R}=\bar{P}+\bar{Q}$

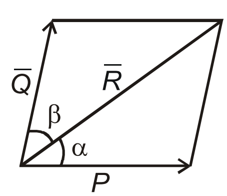
$Q = R\sin\theta$



Non-orthogonal components of $\bar{R}$ are as shown

$$P=\frac{R\sin\beta}{\sin(\alpha+\beta)}$$

$$Q=\frac{R\sin\alpha}{\sin(\alpha+\beta)}$$



LINEAR COMBINATION OF VECTORS

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors, then any vector $\vec{r}$ in the plane of $\vec{a}$ and $\vec{b}$ is uniquely expressed as $\vec{r}=x\vec{a}+y\vec{b}$, where x and y are scalars. (Similar linear combination (unique) exists for three non-coplanar vectors, $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}$).

LINEARLY DEPENDENT AND INDEPENDENT VECTORS

Vectors $\vec{a}_1\vec{a}_2,....,\vec{a}_n$ are said to be linearly dependent if for scalars x_1, x_2,....x_n (not all zero), $x_1\vec{a}_1 + x_2\vec{a}_2 + ... + x_n\vec{a}_n = \vec{0}$. If $x_1\vec{a}_1 + ... + x_n\vec{a}_n = \vec{0} \Rightarrow x_1 = x_2 = ... x_n = 0$ then vectors are called linearly independent.

Note : *1. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then the vectors are linearly independent.*

2. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are linearly dependent then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

3. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanars, then $x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$, $x_2\bar{a} + y_2\bar{b} + z_2\bar{c}$ and $x_3\bar{a} + y_3\bar{b} + z_3\bar{c}$ are coplanar iff.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

4. Two non-zero, non-collinear vectors, or any three non-coplanar vectors are linearly independent.

5. Two collinear or any three coplanar or any four vectors in 3-D are linearly dependent.

6. Three points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear iff there exist scalars x, y and z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ where x + y + z = 0.

7. Four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are said to be coplanar iff there exist scalars x, y, z and w (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$ where x + y + z + w = 0

Illustration 2 :

If the vectors $x\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + y\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + z\hat{k}$ are coplanar, when $x \neq 1$, $y \neq 1$ and $z \neq 1$, then show that

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = 1$$

Solution :

For the given vectors to be coplanar.

$$\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & 1-y & 0 \\ 0 & y-1 & 1-z \\ 1 & 1 & z \end{vmatrix} = 0 \text{ (Operature } R_1 - R_2 \ \& \ R_2 - R_3)$$

$$\Rightarrow (x-1)(1-y)(1-z)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \dfrac{1}{x-1} & \dfrac{1}{1-y} & \dfrac{z}{1-z} \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(1-y)(1-z)\left[\left(\frac{-z}{1-z}-\frac{1}{1-y}\right)+\frac{1}{x-1}\right]=0$$

$$\Rightarrow \frac{1}{1-x}+\frac{1}{1-y}=\frac{-z}{1-z}\ (\because x\neq 1, y\neq 1, z\neq 1)$$

$$\Rightarrow \frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=\frac{1}{1-z}-\frac{z}{1-z}=\frac{1-z}{1-z}=1$$

Multiplication of a Vector by a Scalar

If m is a scalar and $\vec{a}$ is a vector, then $m\vec{a}$ (scalar multiple) is a vector whose magnitude is $|m||\vec{a}|$ and direction is same as of $\vec{a}$ (if m +ve) and opposite that of $\vec{a}$ if m is –ve.

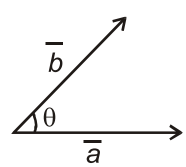
Properties

1. $m(\vec{a}+\vec{b}) = m\vec{a}+m\vec{b}$
2. $(mn)\vec{a} = m(n\vec{a}) = n(m\vec{a})$
3. $(m+n)\vec{a} = m\vec{a}+n\vec{a}$

Scalar (dot) Product of Two Vectors

The scalar product of vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, θ is angle between two vectors.

Properties :



1. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$
2. $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$
3. $(m\vec{a}).\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.(m\vec{b})$
4. If $\theta = 0 \Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|$ (like vectors)
5. If $\theta = \pi \Rightarrow \vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$ (unlike vectors)
6. If $\hat{a}$ and $\hat{b}$ are unit vectors then $\hat{a}.\hat{b} = \cos\theta$ (where θ is angle between them).
7. $\vec{a}.\vec{a} = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a}.\vec{a}}$
8. If $\vec{a} \perp \vec{b} \Rightarrow \vec{a}.\vec{b} = 0$ but $\vec{a}.\vec{b} = 0 \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$.
9. If $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the rectangular coordinate are OX, OY and OZ then $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$, $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$.

10. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

11. Components of a vector $\vec{r}$ in the direction of vector $\vec{a}$ is $\left(\dfrac{\vec{r}.\vec{a}}{|\vec{a}|^2}\right)\vec{a}$ and $\perp$ to $\vec{a}$ is $\vec{r} - \left(\dfrac{\vec{r}.\vec{a}}{|\vec{a}|^2}\right)\vec{a}$.

12. Work done by a force $\vec{F}$ in displacing a particle from A to $B\,(\overrightarrow{AB} = \vec{d})$

$$W = \vec{F}.\overrightarrow{AB} = \vec{F}.\vec{d}$$

Illustration 3 :

Determine the values of c such that for all x (real) the vectors $cx\hat{i} - 6\hat{j} + 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle with each other.

Solution :

If θ is the angle between the given vectors then $\cos\theta = \dfrac{cx^2 - 12 + 6cx}{\sqrt{c^2x^2 + 45}\sqrt{x^2 + 4c^2x^2 + 4}}$

If θ is obtuse then $\cos\theta < 0 \Rightarrow cx^2 + 6cx - 12 < 0 \quad \forall x \in R$

Which is possible if $c < 0$ and $36c^2 + 48c < 0$

$\Rightarrow \quad c < 0$ and $12c\,(3c + 4) < 0$

$\Rightarrow \quad c < 0$ and $c > -4/3$

$\Rightarrow \quad -\dfrac{4}{3} < c < 0$ (but for $c = 0$, $cx^2 + 6cx - 12 < 0 \;\; \forall x$)

Hence $-\dfrac{4}{3} < c \le 0$.

VECTOR (CROSS) PRODUCT OF TWO VECTORS

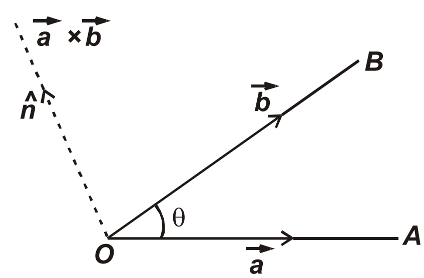
Vector product of two vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ whose direction is that of unit vector $\hat{n}$ which is $\perp$ to both $\vec{a}$ and $\vec{b}$ in such away that $\vec{a}, \vec{b}, \hat{n}$ form a right handed triad (right handed screw system).

Properties

1. In general, $\vec{a} \times \vec{b} \ne \vec{b} \times \vec{a}$. In fact $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

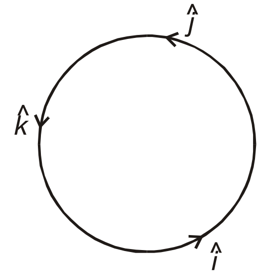
2. For scalar m, $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$



4. If $\vec{a} \parallel \vec{b}$ then $\theta = 0$ or π $\Rightarrow$ $\vec{a} \times \vec{b} = \vec{0}$ (but $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ $\vec{a} \parallel \vec{b}$). In particular $\vec{a} \times \vec{a} = 0$.

5. If $\vec{a} \perp \vec{b}$ then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$ (or $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$)



6. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}$ and

   $\hat{k} \times \hat{i} = \hat{j}$ (use cyclic system)

7. Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by $\pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

8. If θ is angle between $\vec{a}$ and $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

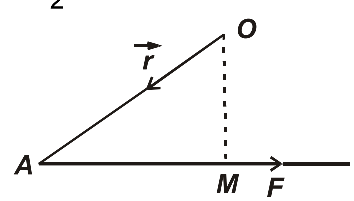
10. (i) If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, then

   Area of parallelogram = $|\vec{a} \times \vec{b}|$

   (ii) If diagonals of parallelogram are $\vec{a}$ and $\vec{b}$, then area of parallelogram = $\dfrac{1}{2}|\vec{a} \times \vec{b}|$

11. Vector moment of a force about a point:

   The vector moment or torque $\vec{M}$ of a force $\vec{F}$ acting at *A* about the point *O* is given by $\vec{M} = \vec{r} \times \vec{F}$



SCALAR TRIPLE PRODUCT

Scalar triple product: If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors then $\vec{a}.(\vec{b} \times \vec{c})$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ denoted $[\vec{a}\ \vec{b}\ \vec{c}]$.

Properties

1. If $\vec{a}, \vec{b}$ and $\vec{c}$ are adjacent sides of a parallelopiped then volume = $[\vec{a}\ \vec{b}\ \vec{c}]$

2. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$ (cyclic permutations of $\vec{a}, \vec{b}$ and $\vec{c}$ makes no change in value) but $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$ etc.

   i.e., Dot and cross can be interchanged, keeping the same cyclic order i.e. $\vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$

3. For scalar *m*, $[m\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ m\vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ m\vec{c}] = m[\vec{a}\ \vec{b}\ \vec{c}]$

4. The value of scalar triple product is zero if any two vectors are equal or parallel

5. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

6. Condition of coplanarity: Three non-colinear, non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are called coplanar iff $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

Vector Triple Product

Vector triple product: If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors their vector triple product is defined by

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

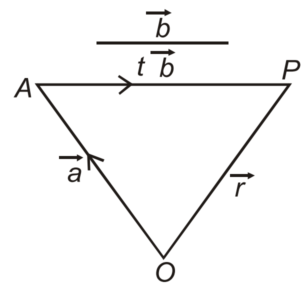
Note : $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

APPLICATION OF VECTORS TO GEOMETRY

1. Vector Equation of Straight Line

Case I : Vector equation of a straight line passing through a point $\vec{a}$ and parallel to vector $\vec{b}$. Let O be the origin and $\overrightarrow{OA} = \vec{a}$. Let P be any point on the line with $\overrightarrow{OP} = \vec{r}$.

Since $\overrightarrow{AP} \parallel \vec{b}$, for some scalar *t*,



$$\overrightarrow{AP} = t\vec{b}$$

$\Rightarrow \quad \vec{r} - \vec{a} = t\vec{b}$

$\Rightarrow \quad \boxed{\vec{r} = \vec{a} + t\vec{b}}$ (required equation)

Case II : Vector equation of a straight line passing through two points $\vec{a}$ and $\vec{b}$.

$$\boxed{\vec{r} = \vec{a} + t(\vec{b} - \vec{a})}$$

2. Vector Equation of a Plane

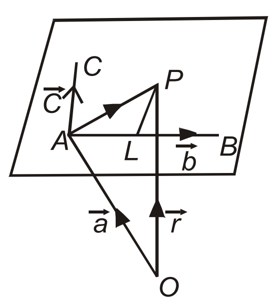
Case I : Vector equation of a plane passing through the point $\vec{a}$ and parallel to two given vectors $\vec{b}$ and $\vec{c}$.

Let O be the origin, $\overrightarrow{OA} = \vec{a}$. Let AB and AC be the lines on the plane | | to vectors $\vec{b}$ and $\vec{c}$. Let $P(\vec{r})$ be any point on the plane, draw $PL \parallel AC$, $(AL \parallel AB)$.

Now $\overrightarrow{AL} = s\vec{b}$ and $\overrightarrow{LP} = t\vec{c}$ (*s, t* are scalars)

Now $\overrightarrow{AP} = \overrightarrow{AL} + \overrightarrow{LP} = s\vec{b} + t\vec{c}$

and $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + s\vec{b} + t\vec{c}$



Required equation $\boxed{\vec{r} = \vec{a} + s\vec{b} + t\vec{c}}$

Another form : $\boxed{[\vec{r}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]}$

Case II : Vector equation of a plane passing through the point $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

$$\boxed{\vec{r} = (1-s-t)\vec{a} + s\vec{b} + t\vec{c}}$$ (where s, t are scalars)

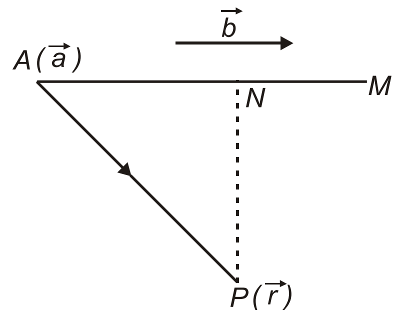
Another form : $\boxed{\vec{r}.(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}) = [\vec{a}\ \vec{b}\ \vec{c}]}$

Case III: Vector equation of plane passing through a point $\vec{a}$ and perpendicular to vector $\vec{n}$:

$$\boxed{(\vec{r} - \vec{a}).\vec{n} = 0}$$

3. Perpendicular distance of a point from a line

Let straight line passes through $A(\vec{a})$ and is || to $\vec{b}$. Then perpendicular distance of point $(\vec{r})$ from the line = $\boxed{\dfrac{|(\vec{r}-\vec{a})\times\vec{b}|}{|\vec{b}|}}$



Another form : $\boxed{\left[(\vec{r}-\vec{a})^2 - \left\{\dfrac{(\vec{r}-\vec{a}).\vec{b}}{|\vec{b}|}\right\}^2\right]^{1/2}}$

4. Perpendicular distance of a point from a plane

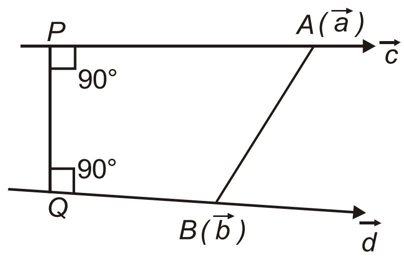
Case I : When plane passes through a point $\vec{a}$ and is || to $\vec{b}$ and $\vec{c}$. Distance of point $P(\vec{r})$ from the plane $\boxed{\dfrac{(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}}$

Case II : When plane passes through the points $\vec{a}, \vec{b}$ and $\vec{c}$.

$$\boxed{\dfrac{(\vec{r}-\vec{a}).(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b})}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}|}}$$

5. Shortest distance between two non-intersecting lines :

Let there be two non-intersecting lines passing through $\vec{a}$ and $\vec{b}$ and are || to $\vec{c}$ and $\vec{d}$. If PQ be the shortest distance between them, then



PQ = projection of $\overrightarrow{AB}$ on the vector $\vec{n}$ (where $\vec{n} = \vec{c}\times\vec{d}$)

Then $\boxed{PQ = \dfrac{(\vec{b}-\vec{a}).(\vec{c}\times\vec{d})}{|\vec{c}\times\vec{d}|}}$

6. Vector equation of the bisector of the angle between two straight lines :

Let the two lines *AB* and *AC* meet at *A* whose p.v. is $\vec{a}$. Let $\vec{b}$ and $\vec{c}$ be vectors || to *AB* and *AC* respectively. Also let *P* be a point on internal bisector of $\angle BAC$ where $\overrightarrow{OP} = \vec{r}$.

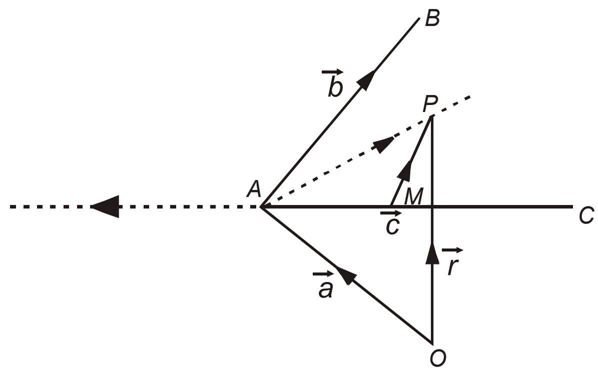
From *P* draw line || *AB* which meets *AC* at *M*.

Now $\angle PAM = \angle PAB = \angle APM$

$\Rightarrow$ $AM = PM = t$ (say)

$\overrightarrow{AM}$ is collinear with $\vec{c}$ and $\overrightarrow{MP}$ is collinear with $\vec{b}$.

Then $\overrightarrow{AM} = t\dfrac{\vec{c}}{|\vec{c}|}, \overrightarrow{MP} = t\dfrac{\vec{b}}{|\vec{b}|}$



Now in ΔAPM, $\overrightarrow{AP} = \overrightarrow{AM} + \overrightarrow{MP} = t\left(\dfrac{\vec{c}}{|\vec{c}|} + \dfrac{\vec{b}}{|\vec{b}|}\right)$,

In ΔOAP, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\Rightarrow$ $\vec{r} = \vec{a} + t\left(\dfrac{\vec{b}}{|\vec{b}|} + \dfrac{\vec{c}}{|\vec{c}|}\right)$, required equation of internal bisector

Equation of external bisector is given by $\vec{r} = \vec{a} + t\left(\dfrac{\vec{b}}{|\vec{b}|} - \dfrac{\vec{c}}{|\vec{c}|}\right)$

Illustration 4 :

Show that the perpendicular distance of the point *A* whose position vector is $\vec{a}$ from the line $\vec{r} = \vec{b} + t\vec{c}$ is

$$\left|\vec{b} - \vec{a} + \frac{\{(\vec{a} - \vec{b}).\vec{c}\}\vec{c}}{|\vec{c}|^2}\right|$$

Solution :

We have

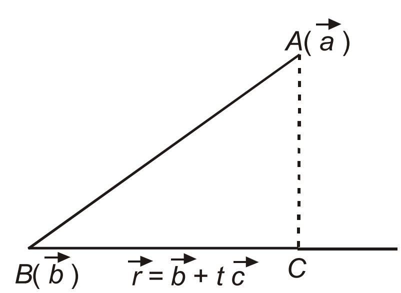
$\overrightarrow{BC}$ = projection of vector $\overrightarrow{BA}$ along the given line

$$= \frac{(\vec{a} - \vec{b}).\vec{c}}{|\vec{c}|^2}\vec{c}$$

Now in ΔABC we have

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$= (\vec{b} - \vec{a}) + \frac{(\vec{a} - \vec{b}).\vec{c}}{|\vec{c}|^2}\vec{c}$$



Therefore perpendicular distance $= |\overrightarrow{AC}| = \left|\vec{b} - \vec{a} + \dfrac{(\vec{a} - \vec{b}).\vec{c}}{|\vec{c}|^2}\vec{c}\right|$

SOLVED EXAMPLES

Example 1 :

If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$ then find $|\vec{a}+\vec{b}+\vec{c}|$ when the angle between $\vec{b}$ and $\vec{c}$ is $\pi/3$.

Solution :

We have $|\vec{a}|=|\vec{b}|=|\vec{c}|=1, \vec{a}.\vec{b}=\vec{a}.\vec{c}=0$

and $\vec{b}.\vec{c}=\cos\frac{\pi}{3}=\frac{1}{2}$

Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{b}.\vec{c} \quad (\because \vec{a}.\vec{c}=\vec{a}.\vec{b}=0)$$

$$=1+1+1+2.\frac{1}{2}=4$$

$\Rightarrow \ |\vec{a}+\vec{b}+\vec{c}|=2$

Example 2 :

Find the distance of the point $B(\hat{i}+2\hat{j}+3\hat{k})$ from the line which passes through $A(4\hat{i}+2\hat{j}+2\hat{k})$ and which is parallel to the vector $C=2\hat{i}+3\hat{j}+6\hat{k}$.

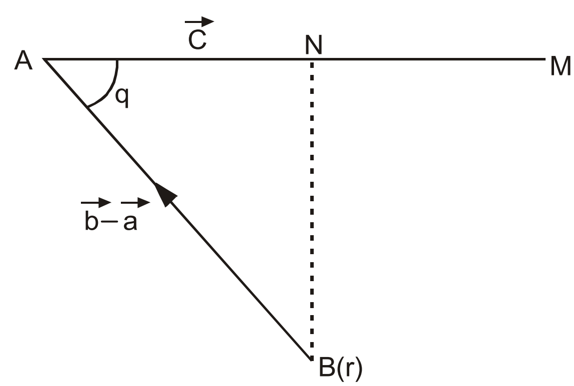
Solution :

Given,

p.v. of $A=\vec{a}=4\hat{i}+2\hat{j}+2\hat{k}$

p.v. of $B=\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

and p.v. of $C=\vec{c}=2\hat{i}+3\hat{j}+6\hat{k}$



Now $\overrightarrow{AB}=\vec{b}-\vec{a}=-3\hat{i}+\hat{k}$.

Then $\overrightarrow{AB}\times\vec{c}=-3\hat{i}+20\hat{j}-9\hat{k}$.

Now $|\overrightarrow{AB}\times\vec{c}|=|\overrightarrow{AB}||\vec{c}|\sin\theta=|\overrightarrow{AB}||\vec{c}|\frac{BN}{|\overrightarrow{AB}|}=|\vec{c}|BN$

$$\Rightarrow BN=\frac{\left|-3\hat{i}+20\hat{j}-9\hat{k}\right|}{|2\hat{i}+3\hat{j}+6\hat{k}|}=\frac{\sqrt{490}}{\sqrt{49}}=\sqrt{10}$$

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