SPECTRUM
CAREER INSTITUTE
JEE/NEET EXPERT

MATHEMATICS

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STRAIGHT LINE -2 D (MSP-XI)

Student's Name: ______________________

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The Straight Line

11

CHAPTER

JEE Syllabus

Recall of Cartesian system of rectangular co-ordinates in a plane, distance formula, area of a triangle, condition for collinearity of 3 points and section formula. Centroid & incentre of a triangle, locus and its equation, translation of axes, slope of a line, parallel and perpendicular lines, intercepts of a line on the co-ordinate axes.

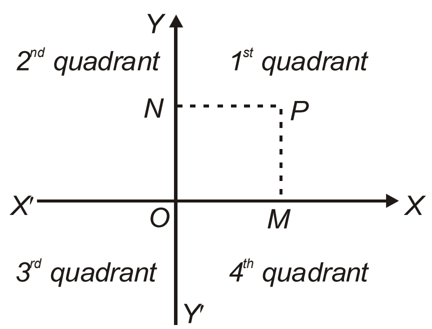
Coordinate geometry is the study of geometry using algebra. The plane curves which would be considered are : straight line, circle, parabola, ellipse and hyperbola.

The system of coordinates used here is cartesian system of coordinates, introduced by philosopher Descartes. It is by far the most important system.

Cartesian System of Rectangular Co-ordinates

To identify a point P in the two dimensional plane, we associate with it, an ordered pair of numbers, in the following way:

Let OX and OY be two fixed perpendicular straight lines in the plane of the paper.



The line OX is called, the axis of x ; the line OY is called, the axis of y ; the two together are called the axes of coordinates. The point O is called the origin (of reference) and OXY is referred to as the Cartesian system. We shall restrict ourselves to rectangular Cartesian system (where OX and OY are perpendicular).

From any point P in the plane, draw lines PM and PN parallel to Y and X axes respectively. The distance $OM = x$ is called as abscissa of P ; the distance $ON = y$ is called as ordinate of P ; the two together, i.e. (x, y), are referred to as the coordinates of P.

Distances measured along OX are positive and those measured opposite to it are negative. Distances measured in the direction of OY are positive and those measured opposite to it are negative.

OXY	=	1st quadrant	Abscissa & ordinate both +ve
$OX'Y$	=	2nd quadrant	Abscissa –ve & ordinate +ve
$OX'Y'$	=	3rd quadrant	Abscissa –ve & ordinate –ve
OXY'	=	4th quadrant	Abscissa +ve & ordinate –ve

THIS CHAPTER INCLUDES :

- Cartesian system of rectangular co-ordinates
- Distance formula
- Area of triangle and condition for collinearity of 3-points
- Area of a plane polygon
- Section formula (mid point)
- Centroid, incentre & ex-centres of a triangle
- Locus & its equation
- Shifting of origin (Translation of axes)
- Rotation of axes
- Slope of a line
- Intercepts of a line
- General equation of a straight line
- Various forms of equation of straight line
- Conditions for two lines to be intersecting, parallel or perpendicular
- Point of Intersection of two lines
- Angle between two lines
- Concurrency of lines
- Family of lines
- Angle bisectors between two lines
- Area of parallelogram or rhombus
- Tricks for brief solutions
- Solved examples

Illustration 1 :

In a triangle *ABC*, the maximum of the three abscissa is –2 and minimum of three ordinates is +1. The triangle shall lie in the quadrant :

(1) *OXY* (2) *OX′ Y* (3) *OX′ Y′* (4) *OXY′*

Solution :

Answer (2)

Since maximum of abscissa is –2, it is clear that x_1, x_2, x_3 all are less than zero *i.e.* on the left of *Y*-axis, so either *OX′Y* or *OX′Y′* quadrants are possible. Since minimum of ordinates is +1, hence all ordinates are positive *i.e.* above *X*-axis, so *OX′Y* is the natural choice.

DISTANCE FORMULA

Distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ *i.e.* length of line *PQ* is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

> **Note :** *This formula is true for all points i.e. the points lying in any quadrant.*

Distance of point $P(x_1, y_1)$ from origin is given by $OP = \sqrt{x_1^2 + y_1^2}$

Illustration 2 :

Three points *A*(2, 3), *B*(2, 6), *C*(5, 3) form a triangle. The triangle *ABC* is

(1) Equilateral (2) Scalene

(3) Right angled & non-isosceles (4) Right angled isosceles

Solution :

Answer (4)

$$AB = \sqrt{(2-2)^2 + (6-3)^2} = 3$$

$$AC = \sqrt{(2-5)^2 + (3-3)^2} = 3$$

$$BC = \sqrt{(2-5)^2 + (6-3)^2} = 3\sqrt{2}$$

$AB^2 + AC^2 = BC^2$ hence triangle is right angled isosceles.

AREA OF TRIANGLE

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ form a triangle then area of ΔABC is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ or } \Delta = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right| = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \text{ 'Stair-method'}$$

Area of ΔOAB, where *O* is origin of co-ordinate system is given by $\Delta = \frac{1}{2}(x_1y_2 - x_2y_1)$

> **Note :** *When point A, B, C are taken in anticlockwise order, the area comes out to be positive otherwise negative. As all areas are necessarily positive quantities so care should be taken. Whenever area of a triangle is given take ± signs.*

Cor. : If $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are the sides of a triangle, then area of triangle

$$\Delta = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where C_1, C_2, C_3 are cofactors of c_1, c_2, c_3 in the determinant above.

Condition for Collinearity of 3-points

If A, B & C are collinear, the area of triangle *ABC* has to be zero *i.e.*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

AREA OF A PLANE POLYGON

Let A_1, A_2 ....A_n are the vertices of a n sided plane polygon, its area is given by Stair method

$$\text{Area of Polygon} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x & y_2 \\ x & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

i.e. $$= \frac{1}{2}\left|[x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \text{..........} + (x_ny_1 - x_1y_n)]\right|$$

Important results

To prove that a given four sided figure is :

1. **Square** : Prove that four sides are equal & diagonals are equal
2. **Rhombus** : Four sides equal, diagonals unequal.
3. **Rectangleyes** : Opposite sides equal & diagonals equal.
4. **Parallelogram** : Opposite sides equal & diagonals unequal.

Note : *Diagonals bisect each other in all the above.*

Illustration 3 :

The area of pentagon *ABCDE*, where *A*(–2, 3), *B*(3, 3), *C*(5, 4), *D*(2, 6) and *E*(–2, 5) is

(1) 29 units (2) $\frac{47}{2}$ units (3) $\frac{29}{2}$ units (4) 15 units

Solution :

Answer (4). By Stair method

$$\text{Area of given pentagon} = \frac{1}{2}\left|\begin{vmatrix} -2 & 3 \\ 3 & 3 \\ 5 & 4 \\ 2 & 6 \\ -2 & 5 \\ -2 & 3 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|(-6 - 9 + 12 - 15 + 30 - 8 + 10 + 12 - 6 + 10)\right|$$

$$= \frac{1}{2}\left|(30)\right|$$

= 15 units

Illustration 4 :

The area of polygonal figure $A(-2, 3)$, $B(4, 3)$, $C(2, 5)$, $D(2, 3)$, $E(0, 3)$ and $F(0, 1)$ is

(1) Zero (2) 4 (3) 8 (4) None of these

Solution :

Answer (2)

By Stair method area comes out to be zero, whereas area comes out to be 4 units. Here ABCD movement is anticlockwise where as *EFAE* is clockwise, Hence +ve and –ve area have cancelled. Care should be taken to roughly plot the figure on co-ordinate axes.

Illustration 5 :

Given five points $O(0, 0)$, $A(0, -2)$, $B(3, 0)$, $C(5, 1)$ and $D(6, 2)$ and a line $2x - 3y = 6$. Choose the wrong statement.

(1) Area ΔOAB = Area of triangle formed by given line with co-ordinate axes.

(2) Area ΔABC < Area ΔOAB

(3) Area ΔABD is zero

(4) Area ΔABC > Area ΔBDC

Solution :

Answer (4)

(a) Area $\Delta OAB = \frac{1}{2}\left|(x_1y_2 - x_2y_1\right| = \frac{1}{2}\left|0 + 6\right|$ = 3 units

(b) Area bound by $x = 0$, $y = 0$ and $2x - 3y - 6 = 0$ is $\frac{1}{2}\left|\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & -6 \end{vmatrix}\right| = \frac{1}{2}\left|-6\right|$ = 3 units

(c) $\Delta ABC = \frac{1}{2}\left|\begin{vmatrix} 0 & -2 & 1 \\ 3 & 0 & 1 \\ 5 & 1 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|2(-2) + 1(3)\right| = \frac{1}{2}$ units

(d) $\Delta ABD = \frac{1}{2}\begin{vmatrix} 0 & -2 & 1 \\ 3 & 0 & 1 \\ 6 & 2 & 1 \end{vmatrix} = \frac{1}{2}\left|[+2(3-6) + 1(6)]\right| = \frac{1}{2}\left|0\right| = 0$

(e) $\Delta BDC = \frac{1}{2}\begin{vmatrix} 3 & 0 & 1 \\ 6 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix} = \frac{1}{2}\left|[3(2-1) + 1(6-10)]\right| = \frac{1}{2}\left|-1\right| = \frac{1}{2}$ units

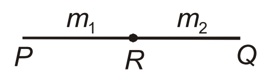
Hence answer (4) is wrong

SECTION FORMULA

Coordinates of the point $R\,(\bar{x},\ \bar{y})$ which divides the join of the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

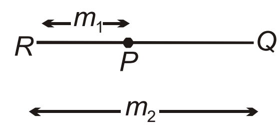
(i) internally, in the ratio $m_1 : m_2$, are

$$\bar{x} = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}\ ;\ \bar{y} = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$



(ii) externally, in the ratio $m_1 : m_2$, are

$$\bar{x} = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}\ ;\ \bar{y} = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$



where $m_1 \neq m_2$

Cor 1 : Mid point ($m_1 = m_2$) of PQ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Cor 2 : If $R\,(\bar{x}, \bar{y})$ divides $P\,(x_1, y_1)$ and $Q\,(x_2, y_2)$ in the ratio

$\lambda : 1$ ($\lambda > 0$) then

$$\bar{x} = \frac{\lambda x_2 \pm x_1}{\lambda \pm 1}\ ;\ \bar{y} = \frac{\lambda y_2 \pm y_1}{\lambda \pm 1}$$

Here '+' sign is taken for internal division & '–' sign for external division.

Cor 3 : For finding the ratio of division, use $\lambda : 1$. If λ comes out to be positive, it indicates internal division otherwise external if λ is negative.

Cor 4 : Line $Ax + By + C = 0$ divides join of $P\,(x_1, y_1)$ and $Q\,(x_2, y_2)$ in the ratio $\left[-\frac{(Ax_1 + By_1 + C)}{(Ax_2 + By_2 + C)}\right]$

Illustration 6 :

Diagonal AC & BD of quadrilateral $ABCD$ are divided in the ratio of $\lambda : 1$ and $t : 1$ respectively by their point of intersection, where $A(0, 2)$, $B(3, 3)$, $C\left(\frac{6}{5}, \frac{46}{5}\right)$ and $D\left(\frac{1}{2}, \frac{37}{4}\right)$ are the vertices. Then λ and t respectively are

(1) 4, 5 (2) 5, 4 (3) 3, 2 (4) 6, 5

Solution :

Answer (2)

Equating the x coordinate of intersection point, we get

$$\frac{\frac{6\lambda}{5}}{\lambda + 1} = \frac{\frac{t}{2} + 3}{t + 1} \quad \ldots\text{(i)}$$

equating y coordinate of intersection point, we get

$$\frac{\frac{46\lambda}{5} + 2}{\lambda + 1} = \frac{\frac{37t}{4} + 3}{t + 1} \quad \ldots\text{(ii)}$$

solving equations (i) & (ii), we get

$\lambda = 5$

$t = 4$

CENTROID, INCENTRE & EX-CENTRES OF A TRIANGLE

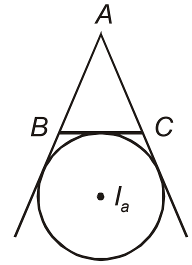
If vertices of ΔABC have coordinates $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

(i) Coordinates of its *centroid* are $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

(ii) Coordinates of its *incentre* are $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$ where $a = BC$, $b = AC$ and $c = AB$

(iii) Coordinates of *excentre* I_a are

$$\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$



If *G, C* and *H* denote the centroid, circumcentre and orthocentre respectively of ΔABC, then, *G, C* and *H* are collinear and *G* divides *CH* internally in the ratio 1 : 2.

Note : *Incentre is the point of intersection of internal bisectors of angles of triangle. Its distance from all three sides is same and called inradius (r) of circle.*

Circumcentre of Triangle

It is the point of intersection of perpendicular bisectors of sides, so its distance from all three vertices is same. If $O(x, y)$ be circumcentre of ΔABC, $[A(x_1, y_1), B(x_2, y_2)$, and $C(x_3, y_3)]$ then $OA^2 = OB^2 = OC^2$

i.e. $(x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2=(x-x_3)^2+(y-y_3)^2$ when solved shall give (x, y).

Alternative method I

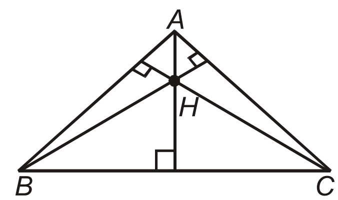
If D, E, F are mid points of BC, AC and AB respectively, then

$$\left.\begin{aligned}&\text{Slope of } BC \times \text{Slope of } OD = -1\\&\text{Slope of } CD \times \text{Slope of } OE = -1\\&\text{Slope of } AB \times \text{Slope of } OF = -1\end{aligned}\right\}\quad \text{Solving any two we get } x \,\&\, y$$

Alternative method II

Circumcentre is given by

$$\left[\frac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \frac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right]$$



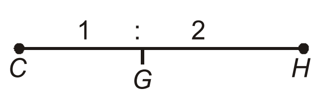
Orthocentre

It is the point of intersection of the perpendicular drawn from the vertices of the triangle on the opposite sides. When vertices and the angles of the triangle are given, then orthocentre is given by,

$$\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$$

Note :

1. *In case of equilateral triangle, centroid, incentre, circumcentre and orthocentre of the triangle lie at the same point.*
2. *Orthocentre H, centroid G and circumcentre C of a triangle are collinear and centroid divides the line joining orthocentre and circumcentre in the ratio of 1 : 2*



3. The orthocentre of the triangle with vertices (0, 0) (x_1, y_1) and (x_2, y_2) is

$$\left\{(y_1 - y_2)\left(\frac{x_1x_2 - y_1y_2}{x_2y_1 - x_1y_2}\right), (x_1 - x_2)\left(\frac{x_1x_2 + y_1y_2}{x_1y_2 - x_2y_1}\right)\right\}$$

Illustration 7 :

Triangle *ABC* has its vertices as $(5\sqrt{2}, 5)$, $(3\sqrt{2}, 9)$ and $(-\sqrt{2}, 2)$ respectively. The centroid is at a distance from origin.

(1) $\sqrt{\frac{108}{5}}$ (2) $\sqrt{\frac{118}{3}}$ (3) $\frac{\sqrt{118}}{3}$ (4) $\frac{\sqrt{108}}{5}$

Solution :

Answer (2)

G, Centroid = $\left(\frac{7\sqrt{2}}{3}, \frac{16}{3}\right)$

OG = $\sqrt{\left(\frac{7\sqrt{2}}{3}\right)^2 + \left(\frac{16}{3}\right)^2}$

$= \frac{1}{3}\sqrt{98 + 256} = \frac{\sqrt{354}}{3} = \sqrt{\frac{118}{3}}$

LOCUS AND ITS EQUATION

When a point moves, so as always to satisfy a given condition or conditions, the path it traces out is called as its locus under these conditions.

Equation to the locus (or curve) is the relation between coordinates of an arbitrarily chosen point on the curve and which relation holds for no other points except those lying on the curve.

Standard method for writing equation of a locus

1. Assume the point whose locus (path) is to be found is (h, k).
2. Make the equation involving (h, k) as per the conditions given.
3. Simplify this equation.
4. Substitute h with x and k with y in the simplified form of equation & you get the equation of locus.

Illustration 8 :

Find the equation of the locus of a point so that sum of its distance from two given point $P(3, 2)$ and $Q(4, 3)$ is 4.

Solution :

1. Let the required variable point be $R(h, k)$
2. $PR + QR = 4$, Hence $\sqrt{(h-3)^2 + (k-2)^2} + \sqrt{(h-4)^2 + (k-3)^2} = 4$
3. $h^2 + k^2 + 13 - 6h - 4k = 16 + h^2 + k^2 + 25 - 8h - 6k - 8\sqrt{(h-4)^2 + (k-3)^2}$

$\Rightarrow 2h + 2k - 28 = -8\sqrt{(h-4)^2 + (k-3)^2}$

$\Rightarrow h + k - 14 = -4\sqrt{(h-4)^2 + (k-3)^2}$

$\Rightarrow h^2 + k^2 + 196 - 28h - 28k + 2hk = 16h^2 + 16k^2 + 400 - 32h - 24k$

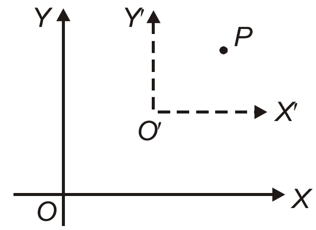
$\Rightarrow 15h^2 + 15k^2 - 2hk - 4h + 4k + 204 = 0$

4. Hence locus is

$$\boxed{15x^2 + 15y^2 - 2xy - 4x + 4y + 204 = 0}$$

Shifting of Origin

Change of axes, by changing origin, the direction of axes remaining the same.

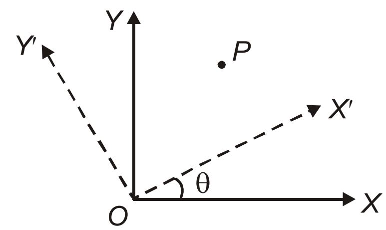


Let OXY and $O'X'Y'$ be two rectangular Cartesian system of axes.

Let P be any point in the plane of the axes and let P and O' have coordinates (x, y) and (h, k) respectively with respect to OXY system. Then the coordinates (x', y') of P with respect to the system $OX'Y'$ are given by $x = x' + h$; $y = y' + k$

Rotation of axes

Change of axes (without changing the origin), by changing the direction of axes, both systems of coordinates being rectangular.



If a point P in the plane of OXY has coordinates (x, y) and (x', y') with respect to the system OXY and $OX'Y'$ respectively, then

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

SLOPE OF A LINE

One of the simplest loci we come across in geometry is a straight line ; the geometrical property which is true in case of this locus is that slope between any two points on a line, is a constant m. This constant is known as slope of the line.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are any two points then slope between A and B is defined as :

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta \quad : \quad 0 \le \theta < \pi \,; \quad \theta \ne \frac{\pi}{2}$$

where θ is the angle of inclination of the line joining A and B with the positive direction of x-axis. If $x_1 = x_2$ then slope is not defined.

Points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are **collinear** (i.e. lie on a straight line) if slope between A and B is equal to slope between B and C.

Slope of line $ax + by + c = 0$ is $\left(-\frac{a}{b}\right)$

Two lines with slopes m_1 and m_2 are

(i) Parallel if $m_1 = m_2$

(ii) Perpendicular if $m_1 m_2 = -1$

Intercepts of a Line

Let a line $L \equiv ax + by + c = 0$, intersects OX - axes at A and OY - axes at B, then OA and OB are called x-intercept and y-intercept of line respectively. For x - intercept, substitute $y = 0$ in the equation

i.e. $ax + c = 0$

$\therefore$ $x = -c/a$ is the x - intercept

Similarly for y - intercept put $x = 0$

$by + c = 0$

$\therefore$ $y = -c/b$ is the y - intercept

Note : *The equation of line having its x and y-intercepts as 'a' and 'b' respectively is* $\frac{x}{a}+\frac{y}{b}=1$ *(called intercept form of line)*

GENERAL EQUATION OF A STRAIGHT LINE

An equation of the form $ax + by + c = 0$ where a & b both are not zero simultaneously, represents a straight line.

Equation of straight line is of first degree in x and / or y.

Slope of a line is denoted by $m = \tan\theta$; where θ is the inclination of line to the positive direction of x-axis. (θ measured anticlockwise from +ve x-axis is taken as positive).

$$\boxed{\text{Slope} = \text{gradient} = m = \tan\theta}$$

Slope of x-axis – zero

Slope of y-axis – Infinite (undefined)

Slope of line equally inclined with both axes is either 1 or –1 ($\theta = 45°$ or $135°$).

Slope of a line joining two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Slope of the straight line is constant not variable.

Slope of line $ax + by + c = 0$ is

$$\boxed{m = -a/b}$$

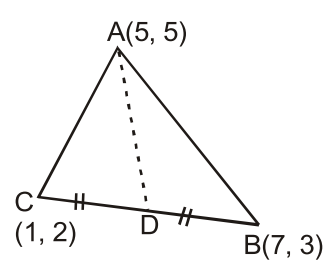
Collinearity of three points A, B & C can be checked by

If slope of AB = slope of BC then A, B, C are collinear

Illustration 9 :

The three vertices of a triangle are $A(5, 5)$, $B(7, 3)$, $C(1, 2)$. Find the slope of median from A.

Solution :



$$D = \left(\frac{1+7}{2}, \frac{2+3}{2}\right)$$

$$= \left(4, \frac{5}{2}\right)$$

$A = (5, 5)$

Hence slope of $AD = \dfrac{5 - 5/2}{5 - 4}$

$$= \frac{5/2}{1}$$

$= 5/2$

The intercepts of a line $ax + by + c = 0$ on x and y-axes can be found by substituting $y = 0$ and $x = 0$ respectively in the equation i.e.,

x intercept $\Rightarrow$ $ax + c = 0$ i.e., $x = -c/a$

y intercept $\Rightarrow$ $by + c = 0$ i.e., $y = -c/b$

Hence length of x intercept = $|-c/a|$ and length of y intercept is $|-c/b|$

Equation (i) x-axis is $y = 0$

(ii) y-axis is $x = 0$

Equation of line (a is a non-zero constant)

(i) parallel to x-axis is $y = a$

(ii) parallel to y-axis is $x = a$

VARIOUS FORMS OF EQUATION OF STRAIGHT LINE

(1) Line with slope m and a given point (x_1, y_1) on it

$y - y_1 = m(x - x_1)$ **(Slope point form)**

(2) Line with two given points (x_1, y_1) and (x_2, y_2) on it

$\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$ **(Two point form)**

(3) Line with given slope m and intercept c on y-axis

$y = mx + c$ **(Slope intercept form)**

(4) Line with given intercepts a and b on x and y axes respectively

$\dfrac{x}{a} + \dfrac{y}{b} = 1$ **(Double intercept form)**

(5) Line at perpendicular distance p from the origin and where, the perpendicular makes angle α with OX

$x \cos\alpha + y \sin\alpha = p$ **(Normal or perpendicular form)**

(6) Line making an angle α with OX and passing through (x_1, y_1) and r as the directed distance of any point $P(x, y)$ on the line

$x = x_1 + r\cos\alpha$; $y = y_1 + r\sin\alpha$ **(Symmetric or parametric form)**

Illustration 10 :

Find the co-ordinates of two points which are $3\sqrt{2}$ distance from the point (1, 3) and lie on a straight line passing from this point and inclined at $\dfrac{2\pi}{3}$ angle to x-axis.

Solution :

Clearly symmetric or parametric form shall be most helpful in such questions

$x = x_1 + r\cos\alpha$; $y = y_1 + r\sin\alpha$

$\alpha = 2\pi/3$ (given)

$x_1 = 1$ (given)

$y_1 = 3$ (given)

$r = \pm 3\sqrt{2}$ (given)

So points are

$$x = 1 \pm 3\sqrt{2}\cos 2\pi/3 = 1 \pm 3\sqrt{2}\left(-\frac{1}{2}\right) = 1 \mp \frac{3}{\sqrt{2}}$$

$$y = 3 \pm 3\sqrt{2}\sin 2\pi/3 = 3 \pm 3\sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = 3 \pm \frac{3\sqrt{3}}{\sqrt{2}}$$

$\left(1+\frac{3}{\sqrt{2}}, 3-\frac{3\sqrt{3}}{\sqrt{2}}\right)$ and $\left(1+\frac{3}{\sqrt{2}}, 3-\frac{3\sqrt{3}}{\sqrt{2}}\right)$ are the required points.

Reduction of General Equation to Different Standard Forms

$ax + by + c = 0$

$\Rightarrow$ $y = \frac{-ax}{b} - \frac{c}{b}$ Slope - intercept form ($y = mx + c$)

$m = \frac{-a}{b}$ & y intercept = $\frac{-c}{b}$

Hence, slope = $-\frac{\text{coefficient of } x}{\text{coefficient of } y}$

$ax + by + c = 0$

$\Rightarrow$ $ax + by = -c$

$\Rightarrow$ $\frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1$ intercept form $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$

Here x intercept = $-c/a$ and intercept on y-axis is $-c/b$.

$ax + by + c = 0$

$ax + by = -c$

$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = -\frac{c}{\sqrt{a^2+b^2}}$ Normal or perpendicular form ($x\cos\alpha + y\sin\alpha = p$)

(i) If $c < 0$ then $-c > 0$

Hence, $p = \frac{-c}{\sqrt{a^2+b^2}}$ and $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

(ii) If $c > 0$ then $\frac{-a}{\sqrt{a^2+b^2}}x - \frac{by}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}$

$p = \frac{c}{\sqrt{a^2+b^2}}, \cos\alpha \frac{-a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{-b}{\sqrt{a^2+b^2}}$

Length of perpendicular from origin is $\frac{\lvert c \rvert}{\sqrt{a^2+b^2}}$

Illustration 11 :

Write the equation $3x - 4y + 5 = 0$ in normal form.

Solution :

$3x - 4y + 5 = 0$

$\Rightarrow$ $-3x + 4y = 5$

$$\therefore \quad \frac{-3}{\sqrt{(-3)^2+4^2}}x + \frac{4}{\sqrt{(-3)^2+4^2}}y = 1$$

$\therefore$ $x \cos \alpha + y \sin \alpha = 1$

where $\cos\alpha = -\frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$

Conditions for Two Lines to be Intersecting, Parallel or Perpendicular

Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

1. Coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

2. Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

3. Perpendicular if $a_1a_2 + b_1b_2 = 0$

4. Intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Line parallel to ax + by + c = 0 is ax + by + k = 0 only constant term has to be changed

Line perpendicular to ax + by + c = 0 is bx – ay + λ = 0

Interchange the coefficients of x and y and change the sign of one of them and change the constant term

Illustration 12 :

Give the equation of lines parallel and perpendicular to line $2x + 3y + 5 = 0$, which pass through origin.

Solution :

Line parallel to given line is $2x + 3y + k = 0$. $k = 0$ if it passed through origin, hence

$2x + 3y = 0$ is the required parallel line.

Any line perpendicular to $2x + 3y + 5 = 0$ is

$3x - 2y + \lambda = 0$ ($\lambda = 0$, since it passes through origin)

$\therefore$ $3x = 2y$ is the required perpendicular line

POINT OF INTERSECTION OF TWO LINES

By solving the two equations of straight lines simultaneously, the point of intersection can be found

for lines $l_1 = a_1x + b_1y + c_1 = 0$

& $l_2 = a_2x + b_2y + c_2 = 0$

Point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}; \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right); \ a_1b_2 \neq a_2b_1$$

distance between two parallel lines

$$d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

Angle between Two Lines

$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$ where ϕ is acute angle and m_1 & m_2 are slopes of two lines

Otherwise $\tan\phi = \pm\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$; one sign gives acute angle and the other gives obtuse angle

Here m_1 or m_2 or both should not be infinite.

If both m_1 & m_2 are undefined, angle = 0

If m_1 *is undefined &* $m_2 = \tan\theta$, *then angle between lines is* $\left(\frac{\pi}{2} - \theta\right)$

For parallel lines $m_1 = m_2$ and for perpendicular lines $m_1m_2 = -1$

1. The two lines which make angles ϕ with the given line $y = mx + c$ and passing through the point (x_1, y_1) are given by $y - y_1 = m_1(x - x_1)$ and $y - y_1 = m_2(x - x_1)$

   where $m_1 = \frac{m - \tan\phi}{1 + m\tan\phi}$; $m_2 = \frac{m + \tan\phi}{1 - m\tan\phi}$

   i.e. $m_1 = \tan(\theta - \phi)$ and $m_2 = \tan(\theta + \phi)$ if $m = \tan\theta$.

2. The length r of the line segment drawn through a given point (x_1, y_1) and making an angle θ with x-axis, to meet the line $ax + by + c = 0$ is given by $r = -\frac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}$

Points in Relation to a Line $ax + by + c = 0$

1. The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side or on the opposite side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have same sign or opposite sign.
2. The ratio in which the line $ax + by + c = 0$ divides the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $-\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$.
3. The length of the perpendicular from $P(x_1, y_1)$ to $ax + by + c = 0$ is $p_1 = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$
4. Coordinates of the foot of the perpendicular drawn from $P(x_1, y_1)$ to the line $ax + by + c = 0$ are given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$ **Do not use square root in 4 and 5 pts.**
5. Coordinates of the image of $P(x_1, y_1)$ in the line $ax + by + c = 0$ are given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{\sqrt{a^2 + b^2}}$$

CONCURRENCY OF LINES

The lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if they pass through the same point ; the condition for concurrency is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

> ***Note :*** *To prove three line to be concurrent show that the point of intersection of any two lines, satisfies the equation of third line.*

Illustration 13 :

Show that the line $3x + 4y - 6 = 0$ does not divide the triangle ABC whose vertices are (0.0), (1,–1) & (–3, 2).

Solution :

$3x + 4y - 6 = 0$

$$\left.\begin{array}{l} x = 0, y = 0 \ \Rightarrow \text{LHS} = -6 \\ x = 1, y = -1 \Rightarrow \text{LHS} = -7 \\ x = -3, y = 2 \Rightarrow \text{LHS} = -7 \end{array}\right]$$

Hence all the three vertices fall on the same side of line, hence the line does not divide the triangle.

Illustration 14 :

The three line $2x + 3y - 5 = 0$; $2x + 4y - 6 = 0$ and $x + 7y - 8 = 0$ are

(1) Parallel
(2) Concurrent
(3) Form a triangle
(4) Neither parallel nor intersecting but skew lines

Solution :

$$\begin{vmatrix} 2 & 3 & -5 \\ 2 & 4 & -6 \\ 1 & 7 & -8 \end{vmatrix}$$

= 2(–32 + 42) – 3(–16 + 6) – 5(14 – 4)

= 20 + 30 – 50 = 0

Hence lines are concurrent

FAMILY OF LINES

Lines through the point of intersection of two given lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by : $(a_1x + b_1y + c_1) + \lambda\,(a_2x + b_2y + c_2) = 0$ where λ is a parameter.

It represents a family of lines. Any particular line (member of the family) can be found from the additional condition stated about the required line.

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle ABC then

(a) Equation of **median** through A is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Equation of **internal bisector** of angle A is $b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

where $b = AC$ and $c = AB$.

ANGLE BISECTORS BETWEEN TWO LINES

The equations of the bisectors of the angles between two intersecting lines

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

Let ϕ be the angle between one of the bisectors and one of the lines $a_1x + b_1y + c_1 = 0$. If $|\tan\phi| < 1$ i.e. $\phi < 45°$, then that bisector is the acute angle bisector of the two given lines. The other equation represents the obtuse angle bisector.

Rule for writing a particular bisector

Write the equations of the lines so that constant terms are positive

If $a_1a_2 + b_1b_2 > 0$ then $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ ...(1)

gives the obtuse angle bisector. If $a_1a_2 + b_1b_2 < 0$ then (1) gives the acute angle bisector.

Note : *If the equations are written so that the constant terms have same sign, then (1) gives bisector of the angle in which the origin lies (and not necessarily the acute angle bisector)*

Illustration 15 :

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the bisector of the angle which contains the origin.

Solution :

Re-writing the given equations so that constant terms are positive, we have

$-4x - 3y + 6 = 0$ ...(i)

and $5x + 12y + 9 = 0$ ...(ii)

Now $a_1a_2 + b_1b_2 = (-4)(5) + (-3)(12)$

$= -20 - 36 = -56 < 0$

So, the origin lies in acute angle.

$\therefore$ The equation of the bisector of the acute angle between the lines (i) and (ii) is

$$\frac{-4x - 3y + 6}{\sqrt{(-4)^2 + (-3)^2}} = + \frac{5x + 12y + 9}{\sqrt{5^2 + (12)^2}}$$

$\Rightarrow$ $-52x - 39y + 78 = 25x + 60y + 45$

$\Rightarrow$ $7x + 9y - 3 = 0$

AREA OF PARALLELOGRAM OR RHOMBUS

Area of a parallelogram or a rhombus, equations of whose sides are given, can be obtained by using the following formula

$$Area = \frac{p_1p_2}{\sin\theta}$$

Where $p_1 = DL$ = distance between lines AB and CD,

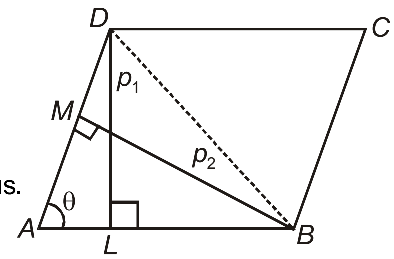
$p_2 = BM$ = distance between lines AD and BC,

θ = angle between adjacent sides AB and AD.

In the case of a rhombus, $p_1 = p_2$. Thus, Area of rhombus = $\dfrac{p_1^2}{\sin\theta}$

Also, area of rhombus = $\dfrac{1}{2}d_1d_2$

where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.



TRICKS FOR BRIEF SOLUTIONS

1. If (x_1, y_1) and (x_2, y_2) are the two vertices of an equilateral triangle, then the third vertex is given by,

$$\left[\frac{(x_1+x_2)\pm\sqrt{3}(y_1-y_2)}{2}, \frac{y_1+y_2\mp\sqrt{3}(x_1-x_2)}{2}\right]$$

2. If (x_1, y_1) and (x_2, y_2) are the vertices of the hypotenuse of a right angle triangle, then the third vertex is given by

$$\left[\frac{(x_1+x_2)\pm(y_1-y_2)}{2}, \frac{(y_1+y_2)\pm(x_1-x_2)}{2}\right]$$

3. Area of the triangle, whose sides are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ is given by

$$A = \frac{1}{2}\left|\sum\frac{(c_1-c_2)^2}{m_1-m_2}\right|$$

4. Area of rhombus, formed by $ax \pm by \pm c = 0$, is given by,

$$A = \left|\frac{2c^2}{ab}\right|$$

5. Area of the parallelogram formed by the lines

$a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$ is given by

$$A = \left|\frac{(d_1-c_1)(d_2-c_2)}{a_1b_2-a_2b_1}\right|$$

SOLVED EXAMPLES

Example 1 :

If O be the origin and if points Q_1 and Q_2 have coordinates (x_1, y_1) and (x_2, y_2) respectively show that $OQ_1 \cdot OQ_2 \cos\angle Q_1OQ_2 = x_1x_2 + y_1y_2$.

Solution :

The cosine formula applied to triangle Q_1OQ_2 gives $\cos\angle Q_2OQ_1 = \dfrac{OQ_1^2 + OQ_2^2 - Q_1Q_2^2}{2\cdot OQ_1\cdot OQ_2}$

$$= \frac{(x_1-0)^2+(y_1-0)^2+(x_2-0)^2+(y_2-0)^2-[(x_1-x_2)^2+(y_1-y_2)^2]}{2\cdot OQ_1\cdot OQ_2} = \frac{2(x_1x_2)+2(y_1y_2)}{2\cdot OQ_1\cdot OQ_2}$$

$$\therefore \quad OQ_1\cdot OQ_2 \cos\angle Q_1OQ_2 = x_1x_2 + y_1y_2$$

Example 2 :

Find the points which divide the line joining the points (–3, –4) and (–8, –7)

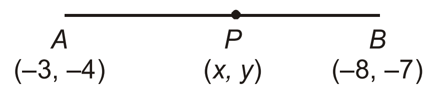
(*i*) *internally in the ratio 7 : 5*

(*ii*) *externally in the ratio 7 : 5*

Solution :

Let $A(-3, -4)$; $B(-8, -7)$

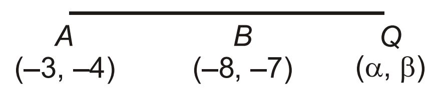
(i) If $P(x, y)$ divides AB internally in the ratio 7 : 5 then



$$x = \frac{7\cdot(-8)+5\,(-3)}{7+5} = -\frac{71}{12}$$

$$y = \frac{7\cdot(-7)+5\,(-4)}{7+5} = -\frac{69}{12}$$

(ii) Let $Q(\alpha, \beta)$ divide AB externally in the ratio of 7 : 5.



$$\alpha = \frac{7\cdot(-8)-5\,(-3)}{7-5} = -\frac{41}{2}$$

$$\beta = \frac{7\cdot(-7)-5\,(-4)}{7-5} = -\frac{29}{2}$$

Example 3 :

Coordinates of the points A, B, C and P are (6, 3), (–3, 5), (4, –2) and (x, y) respectively. Show that $\dfrac{\text{Area of } \Delta\, PBC}{\text{Area of } \Delta\, ABC} = \dfrac{|x+y-2|}{7}$

Solution :

Area of ΔABC = $\dfrac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 4 & -2 & 1 \\ 6 & 3 & 1 \end{vmatrix} = \dfrac{49}{2}$

Area of ΔPBC is modulus of the determinant

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix} = \frac{7}{2}(x+y-2)$$

$$\therefore \quad \frac{\text{Area of } \Delta\, PBC}{\text{Area of } \Delta\, ABC} = \frac{|x+y-2|}{7}$$

Example 4 :

For what value of k are the points (k, 2 – 2k), (1 – k, 2k) (–4 –k, 6 – 2k) collinear ?

Solution :

If points are collinear then $\dfrac{1}{2}\begin{vmatrix} k & 2-2k & 1 \\ 1-k & 2k & 1 \\ -1-k & 6-2k & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ 1-2k & 4k-2 & 0 \\ -4-2k & 4 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ $4(1 - 2k) - (4k - 2)(-4 - 2k) = 0$

$\Rightarrow$ $4 - 8k + 16k - 8 + 8k^2 - 4k = 0$

$\Rightarrow$ $8k^2 + 4k - 4 = 0$ $\quad\Rightarrow\quad$ $2k^2 + k - 1 = 0$

$\Rightarrow$ $k = \dfrac{-1 \pm \sqrt{1+8}}{4} = \dfrac{-1 \pm 3}{4}$ $\quad\Rightarrow\quad$ $k = \dfrac{1}{2}, -1$

Example 5 :

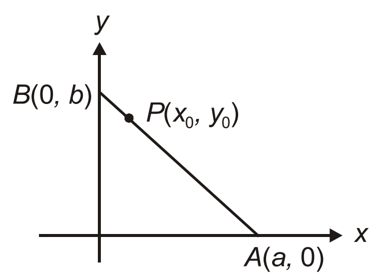
The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.

Solution :

Let the two mutually perpendicular lines be the x and y axes and let $P(x_0, y_0)$ be any point on the locus. Let AB denote the corresponding position of the rod such that $\dfrac{AP}{BP} = 2$ and where $A(a, 0)$ and $B(0, b)$. Then,

$$l^2 = AB^2 = a^2 + b^2$$

$$y_0 = \frac{2b+0}{3} \;;\quad x_0 = \frac{2 \cdot 0 + 1 \cdot a}{3}$$



$$\therefore \left(\frac{3y_0}{2}\right)^2 + (3x_0)^2 = l^2$$

$$y_0^2 + 4x_0^2 = \frac{4l^2}{9}$$

Equation to locus is $4x^2 + y^2 = \dfrac{4l^2}{9}$.

Note : *If* $\dfrac{AP}{BP} = \dfrac{1}{2}$ *then equation to locus would be* $4y^2 + x^2 = \dfrac{4l^2}{9}$

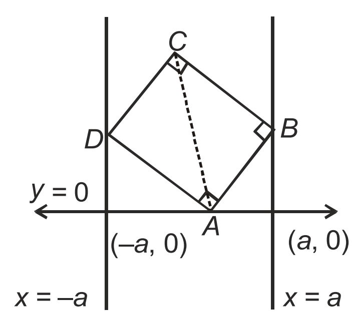
Example 6 :

ABCD is a variable rectangle having its sides parallel to fixed directions. The vertices B and D lie on x = a and x = –a and A lies on the line y = 0. Find the locus of C.

Solution :

Let A be $(x_1, 0)$, B be (a, y_2) and D be $(-a, y_3)$. We are given that AB and AD have fixed directions and hence their slopes are constant, say, m_1 and m_2.

$$\therefore \frac{y_2}{a - x_1} = m_1 \text{ and } \frac{y_3}{-a - x_1} = m_2$$



Further, $m_1 m_2 = -1$, since $ABCD$ is rectangle

$$\frac{y_2}{a - x_1} = m_1 \text{ and } \frac{y_3}{a + x_1} = \frac{1}{m_1} \qquad \ldots(1)$$

Let the coordinates of C be (α, β).

Now mid point of $BD \equiv$ mid point of AC

$\Rightarrow \quad \dfrac{x_1 + \alpha}{2} = 0$ and $\dfrac{y_2 + y_3}{2} = \dfrac{\beta}{2}$

$\therefore \quad \alpha = -x_1$ and $\beta = y_2 + y_3$ …(2)

By (1) and (2) we have : $-(m_1^2 - 1)\,\alpha + m_1\beta = (m_1^2 + 1)\,a$

Locus of C is given by $m_1 y = (m_1^2 + 1)\,a + (m_1^2 - 1)\,x$

Example 7 :

Vertices of a triangle are (2 cosθ, sinθ), (sinθ, cosθ) (–cosθ, –2 sinθ). Find the locus of its centroid, as θ varies.

Solution :

Let centroid of the Δ be (h, k), then

$$h = \frac{2\cos\theta - \cos\theta + \sin\theta}{3};\ k = \frac{\sin\theta - 2\sin\theta + \cos\theta}{3}$$

$\Rightarrow \quad 3h = \cos\theta + \sin\theta$; $3k = \cos\theta - \sin\theta$

squaring and adding,

$\Rightarrow \quad 9h^2 + 9k^2 = 2$

$\therefore \quad$ Required locus is, $9(x^2 + y^2) = 2$.

Example 8 :

The extremities of a diagonal of a square are (a, 0) and (0, b), where a > 0, b > 0. Find the coordinates of the extremities of the other diagonal.

Solution :

Let $ACBD$ be the given square with $A(a, 0)$ and $B(0, b)$

Coordinates of the mid-point M of $AB \equiv \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

$$AB = \sqrt{a^2 + b^2} \Rightarrow AM = BM = CM = DM = \frac{\sqrt{a^2 + b^2}}{2}$$

Slope of $AB = -\dfrac{b}{a} \Rightarrow$ slope of $CD = \dfrac{a}{b} = \tan\theta$ (say)

$$\Rightarrow \quad \cos\theta = \frac{b}{\sqrt{a^2 + b^2}},\ \sin\theta = \frac{a}{\sqrt{a^2 + b^2}}$$

Equation of CD in parametric form is $\dfrac{x - \frac{a}{2}}{\frac{b}{\sqrt{a^2 + b^2}}} = \dfrac{y - \frac{b}{2}}{\frac{a}{\sqrt{a^2 + b^2}}} = r$

To find coordinates of C & D put $r = \pm\dfrac{\sqrt{a^2 + b^2}}{2}$

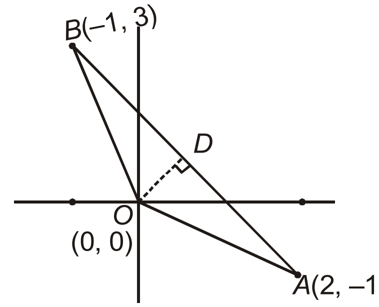
$$\Rightarrow \quad C \equiv \left(\frac{a + b}{2}, \frac{a + b}{2}\right) \text{ and } D = \left(\frac{a - b}{2}, \frac{b - a}{2}\right)$$

Example 9 :

Find the coordinates of the orthocentre of the triangle whose vertices are (0, 0), (2, –1) and (–1, 3).

Solution :

Let OAB be the triangle as shown in the figure.



Slope $AB = \dfrac{3-(-1)}{-1-2} = -\dfrac{4}{3}$

Slope of altitude $OD = \dfrac{3}{4}$ and its equation is $y = \dfrac{3}{4}x$ …(1)

Slope $OB = -3$

$\therefore$ Slope of altitude through $A = \dfrac{1}{3}$

Eq. of altitude from A is $y + 1 = \dfrac{1}{3}(x-2)$

$3y - x + 5 = 0$ …(2)

Solving (1) and (2), the coordinates of orthocentre are $x = -4$, $y = -3$

Example 10 :

If p and p_1 be the lengths of the perpendiculars drawn from the origin upon the straight lines $x \sin\theta + y\cos\theta = \dfrac{1}{2} a \sin 2\theta$ and $x\cos\theta - y\cos\theta = a\cos 2\theta$, prove that $4p^2 + p_1^2 = a^2$.

Solution :

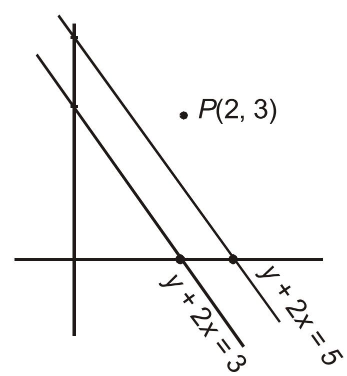
We have $p = \left|\dfrac{-\frac{1}{2}a\sin 2\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right| \Rightarrow p^2 = \dfrac{1}{4}a^2\sin^2 2\theta$ *and* $p_1 = \left|\dfrac{-a\cos 2\theta}{\sqrt{\sin^2\theta+\cos^2\theta}}\right| \Rightarrow p_1^2 = a^2\cos^2 2\theta$

Now $4p^2 + p_1^2 = a^2\sin^2 2\theta + a^2\cos^2 2\theta = a^2$

Example 11:

Find the equation of the line passing through (2, 3) and making intercepts of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$.

Solution :



$y + 2x = 3$ …(1)

$y + 2x = 5$ …(2)

Equation of the line passing through $P(2, 3)$ is

$\dfrac{x-2}{\cos\theta} = \dfrac{y-3}{\sin\theta} = r$ …(3)

The points of intersection of (3) with (2) and (1) are

$A(2 + r\cos\theta, 3 + r\sin\theta)$

and $B(2 + (r-2)\cos\theta, 3 + (r-2)\sin\theta)$ respectively for some r.

$\therefore$ $(3 + r\sin\theta) + 2(2 + r\cos\theta) = 3$ …(4)

$(3 + (r-2)\sin\theta) + 2(2 + (r-2)\cos\theta) = 5$ …(5)

By (4) and (5)

$\sin\theta + 2\cos\theta = -1$

$\sin\theta = -1 - 2\cos\theta$

$\sin^2\theta = (-1 - 2\cos\theta)^2$

$\cos\theta\,(5\cos\theta + 4) = 0$

$\therefore$ $\cos\theta = 0$ or $\cos\theta = -\frac{4}{5}$ and accordingly $\sin\theta = 1$ or $\frac{3}{5}$.

(3) now gives equations of required lines.

Example 12 :

Find the coordinates of those points on the line 3x + 2y = 5 which are equidistant from the lines 4x + 3y = 7 and 2y = 5.

Solution :

It is obvious that the desired points lie on $3x + 2y = 5$ as well as on the bisectors of angles formed by the lines

$4x + 3y = 7$ …(1)

$2y - 5 = 0$ …(2)

Equations to the angle bisectors are $\frac{4x+3y-7}{5} = \pm\frac{2y-5}{2}$

$8x + 6y - 14 = \pm(10y - 25)$

$8x + 16y - 39 = 0$; $8x - 4y + 11 = 0$

Solving $3x + 2y = 5$ and $8x + 16y = 39$ we get $x = \frac{1}{16},\ y = \frac{77}{32}$

Solving $3x + 2y = 5$ and $8x - 4y + 11 = 0$ we get $x = -\frac{1}{14},\ y = \frac{73}{28}$

Example 13 :

The line L has intercepts a and b on the coordinate axes. While keeping the origin fixed, if the coordinate axes are rotated through a fixed angle, the same line has intercepts p and q on the rotated axes. Show that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$.

Solution :

Suppose that the axes are rotated in the anticlockwise direction through an angle α. The equation of the line L with respect to the old axes is given by $\frac{x}{a} + \frac{y}{b} = 1$. To find the equation of L with respect to the new rotated axes, we replace x by $x\cos\alpha - y\sin\alpha$ and y by $x\sin\alpha + y\cos\alpha$ so that the equation of L with respect to the new axes is

$$\frac{1}{a}(x\cos\alpha - y\sin\alpha) + \frac{1}{b}(x\sin\alpha + y\cos\alpha) = 1$$

$$\Rightarrow \left(\frac{1}{a}\cos\alpha + \frac{1}{b}\sin\alpha\right)x + \left(\frac{1}{b}\cos\alpha - \frac{1}{a}\sin\alpha\right)y = 1 \quad .....(1)$$

Since p and q are the intercepts made by this line on coordinate axes we have

$$\frac{1}{p} = \frac{1}{a}\cos\alpha + \frac{1}{b}\sin\alpha \text{ and } \frac{1}{q} = \frac{1}{a}\sin\alpha + \frac{1}{b}\cos\alpha$$

Squaring and adding we get $\frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{a^2} + \frac{1}{b^2}$

Example 14 :

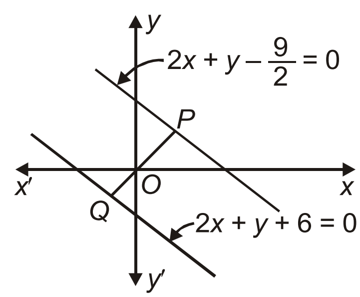
A straight line through the origin 'O' meets the parallel lines 4x + 2y = 9 and 2x + y + 6 = 0 at the points P and Q respectively. Then find the ratio in which the point 'O' divides the segment PQ.

Solution :

Taking PQ in any direction, ratio of OP and OQ will be same, so taking $PQ \perp$ to the given parallel lines

$OP = \perp$ from (0, 0) on $2x + y - \frac{9}{2} = 0$.

$$= \left|\frac{-9/2}{\sqrt{4+1}}\right| = \frac{9}{2\sqrt{5}}$$



$OQ = \perp$ from (0, 0) on $2x + y + 6 = 0$

$$= \left|\frac{6}{\sqrt{4+1}}\right| = \frac{6}{\sqrt{5}}$$

$$\therefore \quad \frac{OP}{OQ} = \frac{9/2\sqrt{5}}{6/\sqrt{5}} = \frac{9}{12} = \frac{3}{4}$$

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