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MATHEMATICS
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Limits, Continuity and Differentiability

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CHAPTER

Syllabus

Limits of functions, Continuity, Differentiability, Differentiation of the sum, difference, product and quotient of two functions, differentiation of trigonometric, Inverse trigonometric, Logarithmic, Exponential, Composite and Implicit functions, Derivatives of order upto two.

THIS CHAPTER INCLUDES:

Let y = f(x) be a real valued function which is defin... ood of the point at x = a. A real number l is called limit ... hen $x \to a$ *i.e.* $\lim\limits_{x\to a} f(x) = l$.

The notion of continuity occurs in may application of calculus. We may intuitively thi... ose functions whose graphs we can draw without lifting the pencil off the paper.

EXISTENCE OF LIMIT

$\lim\limits_{x\to a} f(x) = l$ exists if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \text{finite}$$

or $\lim\limits_{h\to 0} f(a+h) = \lim\limits_{h\to 0} f(a-h) = \text{finite}$

$\longrightarrow$ $\qquad$ $\longleftarrow$

$x = (a-h)$ $\qquad$ $x = a$ $\qquad$ $x = (a+h), (h > 0)$

$x \to a^-$ $\qquad$ $x \to a^+$

where $\lim\limits_{x\to a^+} f(x)$ is called right hand limit and $\lim\limits_{x\to a^-} f(x)$ is called left hand limit.

Conclusion :

$\lim\limits_{x\to a} f(x)$ exist if its left hand limit (LHL) at $x = a$, and right hand limit (RHL) at x...

i.e. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = \text{finite}$, otherwise limit does not exist.

Illustration 1 :

Find $\lim\limits_{x\to 0} f(x)$ if $f(x) = \dfrac{|x|}{x}$.

Solution :

L.H.L. = $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{|0-h|}{0-h} = \lim_{h\to 0}\frac{|-h|}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

R.H.L $= \lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$= \lim_{h \to 0} \frac{|0+h|}{0+h} = \lim_{h \to 0} \frac{|h|}{h} = 1$

Here Left hand limit ≠ right hand limit, so limit does not exist.

Illustration 2 :

Find $\lim_{x \to 2} f(x)$ if $f(x) = \frac{[x]}{x^2+1}$, where $[x]$ denote the greatest integer less than or equal to x.

Solution :

R. H. L. $\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h)$

$= \lim_{h \to 0} \frac{[2+h]}{(2+h)^2+1}$

$\lim_{h \to 0} \frac{2}{(2+h)^2+1} = \frac{2}{3}$

L.H.L. $\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h)$

$= \lim_{h \to 0} \frac{[2-h]}{(2-h)^2+1}$ $\quad$ $[2 - h] = 1$ (since $2 - h < 2$)

$= \lim_{h \to 0} \frac{1}{(2-h)^2+1} = \frac{1}{3}$

$\therefore$ LHL ≠ RHL, so $\lim_{x \to 2} f(x)$ does not exist.

Note : *Generally limits for greatest integer function do not exist when $x \to$ integer.*

Illustration 3 :

Find the value of $\lim_{x \to 2^-} \left[\frac{x}{2}\right]^3 + \frac{x}{2}^3$ where [.] denote the step function.

Solution:

$\lim_{x \to 2^-} \left(\left[\frac{x}{2}\right]^3 + \frac{x}{2}^3\right)$

$= \lim_{h \to 0} \left(\left[\frac{2-h}{2}\right]^3 + \frac{(2-h)^3}{2}\right)$

$\therefore$ $2 - h < 2$

So $\frac{2-h}{2} < 1$

$\left[\frac{2-h}{2}\right] = 0$

$\Rightarrow \lim_{h \to 0} \frac{(2-h)^3}{2} = 4$

Indeterminate Forms

A function $f(x)$, takes the form $f(x) = \frac{0}{0}$, at $x = a$ then we say that is indeterminate at $x = a$

Some indeterminate forms are $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^\circ, \infty^\circ, 1^\infty$ etc.

Algebra of Limits

If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exists then

(i) $\lim\limits_{x\to a}[f(x)\pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(ii) $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ $\left(\lim\limits_{x\to a} g(x) \neq 0\right)$

(iii) $\lim\limits_{x\to a}[K\ f(x)] = K \lim\limits_{x\to a} f(x)$

(iv) $\lim\limits_{x\to a}[f(x)]^{g(x)} = \left[\lim\limits_{x\to a} f(x)\right]^{\lim\limits_{x\to a} g(x)}$

(v) $\lim\limits_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)}$

(vi) $\lim\limits_{x\to a} \log f(x) = \log\left[\lim\limits_{x\to a} f(x)\right]$

(vii) $\lim\limits_{x\to a} f(x)\times g(x) = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} g(x)$

(viii) $\lim\limits_{x\to a} fog(x) = f\left[\lim\limits_{x\to a} g(x)\right]$ (Provided f(x) is contineous at x = $\lim\limits_{x\to a} g(x)$)

(ix) Sandwich theorem – If there is a function $h(x)$ such that $f(x) \leq h(x) \leq g(x)$ $\forall$ x and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$

Then $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x)$ *or* $\lim\limits_{x\to a} f(x)$

L. Hospital's Rule

If $f(x)$ and $g(x)$ are function of x such that $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Here $f'(x)$ and $g'(x)$ both are differentiation of $f(x)$ and $g(x)$ w.r.t. x.

Note : *1. In applying this rule f(x) and g(x) are to be differentiated separately.*

2. L. Hospital's Rule is true even if f(x) = g(x) = 0 when $x \to \infty$

i.e. $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$

3. Forms $0 \times \infty$ and $\infty - \infty$, reduced either to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ for using L. Hospital's rule

4. 0^0, 1^∞, ∞^0 such types of form can be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking log of the given expression.

Some Particular Limits

1. $\lim\limits_{x\to0} \dfrac{\sin x}{x} = \lim\limits_{x\to0} \dfrac{x}{\sin x} = \lim\limits_{x\to0} \dfrac{\tan x}{x} = \lim\limits_{x\to0} \dfrac{x}{\tan x} = 1$

2. $\lim\limits_{x\to0} \cos x = \lim\limits_{x\to0} \sec x = 1$ [x is measured in radian]

3. $\lim\limits_{x\to0} \dfrac{\sin^{-1}x}{x} = \dfrac{\tan^{-1}x}{x} = 1$

4. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$

5. $\lim\limits_{x\to\infty} \dfrac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

6. $\lim\limits_{x\to0} \left[\dfrac{\sin x}{x}\right] = 0$, where [.] represent step function

7. $\lim\limits_{x\to0} \left[\dfrac{x}{\sin x}\right] = 1$

8. $\lim\limits_{x\to0} [\cos x] = 0$ [$x \to 0$ then $\cos x \to 1$, not exact equal to 1]

9. $\lim\limits_{x\to0} \left[\dfrac{\tan x}{x}\right] = 1$

Note : *$\sin x < x < \tan x$ for $0 < |x| < \frac{\pi}{2}$*

10. $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$, if $a > 0$, $n \in$ R and if $a < 0$, $n \in$ Z.

11. $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}=\dfrac{m}{n}a^{m-n}$

12. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a$

13. $\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x}=1$ and $\lim\limits_{x\to 0}\dfrac{\log_e(1-x)}{x}=-1$

14. $\lim\limits_{n\to\infty}x^n=\begin{cases}0, & 0<x<1\\ \infty, & x>1\end{cases}$

15. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$

Some Important Expansions

1. $a^x=1+x\log a+\dfrac{x^2}{2!}(\log a)^2+\dfrac{x^3}{3!}(\log a)^3+....$

2. $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+....$

3. $\log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+....$ (if |x| < 1)

   Note : *For getting exp. of a^{-x}, e^{-x}, log(1 – x) replace x with –x in exp. of e^x, a^x and log(1 – x)*

4. $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+....$

5. $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+....$

6. $\tan x=x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+....$

7. $\sin hx=x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+....$

8. $\cos hx=1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+....$

9. $(1+x)^{1/x}=e\left[1-\dfrac{x}{2}+\dfrac{11}{24}x^2-.....\right]$

EVALUATION OF LIMITS

By Factorization

In this method factorise (if possible) both numerator and denominator and cancel those terms which give $\frac{0}{0}$ form.

Illustration 4 :

$$\lim_{x\to 3}\frac{x^3-4x-15}{x^3+x^2-6x-18}$$

Solution:

$$= \lim_{x\to 3}\frac{(x-3)(x^2+3x+5)}{(x-3)(x^2+4x+6)}$$

$$= \lim_{x\to 3}\frac{(x-3)(x^2+3x+5)}{(x-3)(x^2+4x+6)}$$

$$= \lim_{x\to 3}\frac{x^2+3x+5}{x^2+4x+6}=\frac{23}{27}$$

Illustration 5 :

$$\lim_{x\to 0}\frac{\cos^2 x+\cos x-2}{\sin^2 x}$$

Solution:

$$= \lim_{x\to 0}\frac{\cos^2 x+2\cos x-\cos x-2}{1-\cos^2 x}$$

$$= \lim_{x\to 0}\frac{(\cos x+2)(\cos x-1)}{(1+\cos x)(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{-(\cos x+2)}{1+\cos x}=-\frac{3}{2}$$

By Substitution

Illustration 6 :

$$\lim_{x\to \frac{\pi}{2}}\frac{2x-\pi}{\cos x}$$

$\lim_{x\to \frac{\pi}{2}}\frac{2x-\pi}{\cos x}$, Put $x = \frac{\pi}{2}+h$ when $x \to \frac{\pi}{2}$, $h \to 0$

So $\lim_{h\to 0} \dfrac{2\left(\dfrac{\pi}{2}+h\right)-\pi}{\cos\left(\dfrac{\pi}{2}+h\right)}$

$= \lim_{h\to 0} \dfrac{2h}{\sin h} = 2$

Illustration 7 :

Find the value $\lim_{x\to 0} \dfrac{(\cos x)^{1/3}+(\cos x)^{1/2}-2}{\sin^2 x}$

Solution:

$$\lim_{x\to 0} \frac{(\cos x)^{1/3}+(\cos x)^{1/2}-2}{\sin^2 x}$$

$$= \lim_{x\to 0} \frac{(\cos x)^{1/3}+(\cos x)^{1/2}-2}{1-\cos^2 x}$$

Put $\cos x = t^6$

when $x \to 0$, $\cos x \to 1$, then $t \to 1$

So, $\lim_{t\to 1} \dfrac{t^2+t^3-2}{1-t^{12}}$

Using L. Hospital's rule

$$\lim_{t\to 1} \frac{2t+3t^2}{0-12t^{11}} = -\frac{5}{12}$$

Illustration 8 :

Find $\lim_{x\to \pi} \dfrac{2^{-\sin x}-1}{x(x-\pi)}$

Solution:

$$\lim_{x\to \pi} \frac{2^{-\sin x}-1}{x(x-\pi)}$$

Put $x = \pi + t$, if $x \to \pi$, then $t \to 0$

$$\therefore \quad \lim_{t\to 0} \frac{2^{-\sin(\pi+t)}-1}{(\pi+t)(t)} = \lim_{t\to 0} \frac{2^{\sin t}-1}{(\pi+t)t}$$

$$= \lim_{t\to 0} \frac{2^{\sin t}-1}{\sin t} \times \frac{\sin t}{t} \times \frac{1}{\pi + t}$$

$$= \log 2 \times 1 \times \frac{1}{\pi} = \frac{1}{\pi}\log 2$$

Evaluation of limit when $x \to \infty$:

In case of limits, when $x \to \infty$, put $x = \frac{1}{t}$ so that t $\to$ 0 when $x \to \infty$ or

If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are algebraic function, then divide both N^r and D^r by terms which have highest power of x, then apply the limit.

Illustration 9 :

$$\lim_{x\to\infty} \frac{2x^3+4x^2+5}{9x^3+4x^2+7}$$

Solution :

$$\lim_{x\to\infty} \frac{2x^3+4x^2+5}{9x^3+4x^2+7}$$

$$\lim_{x\to\infty} \frac{2+\frac{4}{x}+\frac{5}{x^3}}{9+\frac{4}{x}+\frac{7}{x^3}} = \frac{2}{9} \qquad \text{(divided } N^r \text{ and } D^r \text{ by } x^3\text{)}\left[\because \frac{1}{\infty}=0\right]$$

Illustration 10 :

Find $\lim_{x\to\infty} \frac{\sqrt{x+2}}{\sqrt{x+\sqrt{x}}+\sqrt{x+5}}$

Solution :

$$\lim_{x\to\infty} \frac{\sqrt{x+2}}{\sqrt{x+\sqrt{x}}+\sqrt{x+5}}$$

$$\lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x}}}{\sqrt{1+\sqrt{\frac{1}{x}}}+\sqrt{1+\frac{5}{x}}} \qquad \text{(divide by } \sqrt{x} \text{ both } N^r \text{ and } D^r\text{)}$$

$$= \frac{\sqrt{1+0}}{\sqrt{1+\sqrt{0}}+\sqrt{1+0}} = \frac{1}{2}$$

Illustration 11:

Find $\lim_{x\to\infty} \frac{x^4 \sin\left(\frac{1}{x}\right) - x^3}{1-|x|^3}$

Solution:

$$\lim_{x\to\infty}\frac{x^4\sin\left(\frac{1}{x}\right)-x^3}{1-|x|^3}$$

Put $x=\frac{1}{t}$, if $x\to\infty$, then $t\to 0$

$$\text{So }\lim_{t\to 0}\frac{\frac{1}{t^4}\sin t-\frac{1}{t^3}}{1-\left|\frac{1}{t}\right|^3}$$

$$=\quad \lim_{t\to 0}\frac{\frac{\sin t}{t}-1}{|t|^3-1}\qquad (t>0)$$

$$=\quad \lim_{t\to 0}\frac{\frac{\sin t}{t}-1}{t^3-1}=\frac{1-1}{-1}=0$$

By Expansion

In this method use expansion formulae and solve

Illustration 12 :

$$\lim_{x\to 0}\frac{\tan x-x-\frac{x^3}{3}}{x^5}$$

Solution :

$$\lim_{x\to 0}\frac{\tan x-x-\frac{x^3}{3}}{x^5}$$

$$\lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\text{.....terms containing higher power of }x\right)-x-\frac{x^3}{3}}{x^5}\quad \text{(Using exp. formula)}$$

$$=\lim_{x\to 0}\frac{\frac{2}{15}x^5(1+\text{terms containing higher power of }x)}{x^5}$$

$$=\frac{2}{15}$$

Illustration 13 :

Find the value of $\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e+\frac{ex}{2}}{x^2}$

Solution :

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e+\frac{ex}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{e[1-\frac{x}{2}+\frac{11}{24}x^2-....]-e+\frac{ex}{2}}{x^2} \qquad (\because (1+x)^{\frac{1}{x}}=e[1-\frac{x}{2}+\frac{11}{24}x^2-....])$$

$$=\lim_{x\to 0}\frac{[e-\frac{ex}{2}+\frac{11e}{24}x^2+...\text{terms containing higher power of } x]-e+\frac{ex}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{\frac{11e}{24}x^2[1+\text{terms containing higher power of } x]}{x^2}$$

$$=\frac{11e}{24}$$

Illustration 14 :

Find the value of $\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$

Solution :

$$\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$$

$$\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}...\right)-(1+\sin x+\frac{1}{2!}\sin^2 x+...)}{x-\sin x}$$

$$=\lim_{x\to 0}\frac{(x-\sin x)+\frac{1}{2}(x^2-\sin^2 x)+\frac{1}{3!}(x^3-\sin^3 x)+..}{x-\sin x}$$

$$=\lim_{x\to 0}\frac{(x-\sin x)[1+\frac{1}{2}(x^2-\sin x)+...]}{(x-\sin x)} \qquad = 1$$

Using Some Standard Limits

In this method, we some standard limits given below

If $\lim_{x\to a} f(x)=0$ then

(i) $\lim_{x\to a}\frac{\sin f(x)}{f(x)}=1$ (ii) $\lim_{x\to a}\cos f(x)=1$

(iii) $\lim_{x\to a}\frac{a^{f(x)}-1}{f(x)}=\log a$ (iv) $\lim_{x\to a}\frac{\log(1+f(x))}{f(x)}=1$

Illustration 15 :

Find the following limits

(i) $\lim_{x\to 0} \frac{3^x - 1}{\tan x}$ (ii) $\lim_{x\to 3} \frac{\sin(e^{x-3}-1)}{\log(x-2)}$

(iii) $\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$

Solution :

(i) $\lim_{x\to 0} \frac{3^x - 1}{\tan x}$

$= \lim_{x\to 0} \frac{3^x - 1}{x} \times \frac{x}{\tan x}$

$=$ log 3 × 1 = log 3

(ii) $\lim_{x\to 3} \frac{\sin(e^{x-3}-1)}{\log(x-2)}$

$= \lim_{x\to 3} \frac{\sin(e^{x-3}-1)}{e^{x-3}-1} \times \frac{e^{x-3}-1}{\log(x-2)}$

$= \lim_{x\to 3} \frac{\sin(e^{x-3}-1)}{e^{x-3}-1} \times \frac{e^{x-3}-1}{x-3} \times \frac{x-3}{\log[1+(x-3)]}$

$=$ 1 × 1 × 1 = 1

(iii) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3 \cos x}$

$= \lim_{x\to 0} \frac{\sin x \times 2\sin^2 \frac{x}{2}}{x^3 \cos x} = \lim_{x\to 0} \frac{2}{4} \times \frac{\sin x}{x} . \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \frac{1}{\cos x}$

$= \frac{2}{4} \times 1 \times 1 \times 1 = \frac{1}{2}$

Illustration 16 :

If α and β are the roots of the equation $ax^2 + bx + c = 0$ then find the value of $\lim_{x\to\frac{1}{\alpha}} \frac{1-\cos(cx^2+bx+a)}{\left(x-\frac{1}{\alpha}\right)^2}$

Solution:

If α and β is a root of the equation $ax^2 + bx + c$, then equation whose root are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is given by $cx^2 + bx + a = 0$, so $cx^2 + bx + a = \left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)$

Now $\lim_{x\to\frac{1}{\alpha}} \dfrac{1-\cos(cx^2+bx+a)}{\left(x-\frac{1}{\alpha}\right)^2}$

$$= \lim_{x\to\frac{1}{\alpha}} \frac{2\sin^2\frac{(cx^2+bx+a)}{2}}{\left(x-\frac{1}{\alpha}\right)^2}$$

$$= \lim_{x\to\frac{1}{\alpha}} \frac{2\sin^2\frac{1}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{\frac{1}{4}\left(x-\frac{1}{\alpha}\right)^2\left(x-\frac{1}{\beta}\right)^2}\times\frac{\frac{1}{4}\left(x-\frac{1}{\beta}\right)^2}{1}$$

$$= \lim_{x\to\frac{1}{\alpha}} \frac{1}{2}\left[\frac{\sin\frac{\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{2}}{\frac{1}{2}\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}\right]^2\times\left(x-\frac{1}{\beta}\right)^2$$

$$= \frac{1}{2}\times 1\times\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2 \quad = \frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$$

Evaluation of Exponential Limit

(a) Evaluation of limit form 1^∞:

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$ then

$$\lim_{x\to a}\left[1+f(x)\right]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

(ii) If $\lim_{x\to a} f(x) = 1, \lim_{x\to a} g(x) = \infty$

then $\lim_{x\to a}\left[f(x)\right]^{g(x)} = e^{\lim_{x\to a}[f(x)-1]\times g(x)}$

(b) Evaluation of the limit form 0^o, 0^∞, or ∞^o:

If $\lim_{x\to a}\left[f(x)\right]^{g(x)}$ gives any one form (0^o, 0^∞ or ∞^o)

then $\lim_{x\to a}\left[f(x)\right]^{g(x)} = e^{\lim_{x\to a} g(x)\log f(x)}$

Note : $\lim_{x\to a}(1+f(x))^{\frac{1}{f(x)}} = e\left(\text{where} \lim_{x\to a} f(x) = 0\right)$

Illustration 17 :

Find the value of following limits

(i) $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{5x}}$ (ii) $\lim\limits_{x\to 0}\left(\frac{1+3x^2}{1+5x^2}\right)^{\frac{1}{x^2}}$

(iii) $\lim\limits_{x\to 0}\left(\frac{1^x+2^x+3^x+...n^x}{n}\right)^{\frac{1}{x}}$ (iv) $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{\cot x}$

Solution :

(i) $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{5x}} = \lim\limits_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^{\frac{3}{5}}$

$= e^{\frac{3}{5}}\left(\because \lim\limits_{x\to 0}(1+f(x))^{\frac{1}{f(x)}} = e\right)$

(ii) $\lim\limits_{x\to 0}\left(\frac{1+3x^2}{1+5x^2}\right)^{\frac{1}{x^2}}$

This is a 1^∞ form

$\lim\limits_{x\to 0}\left(\frac{1+3x^2}{1+5x^2}\right)^{\frac{1}{x^2}} = e^{\lim\limits_{x\to 0}\left(\frac{1+3x^2}{1+5x^2}-1\right)\times\frac{1}{x^2}}$

$= e^{\lim\limits_{x\to 0}\frac{-2x^2}{1+5x^2}\times\frac{1}{x^2}} = e^{-2}$

(iii) $\lim\limits_{x\to 0}\left(\frac{1^x+2^x+3^x+.....+n^x}{n}\right)^{\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\left(\frac{1^x+2^x+3^x+...+n^x}{n}-1\right)\times\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\left(\frac{1^x+2^x+...+n^x-n}{n}-1\right)\times\frac{1}{x}}$

$= e^{\lim\limits_{x\to 0}\frac{1}{n}\left[\frac{(1^x-1)+(2^x-1)+....+(n^x-1)}{x}\right]}$

$= e^{\lim\limits_{x\to 0}\frac{1}{n}\left[\frac{1^x-1}{x}+\frac{2^x-1}{x}+\frac{3^x-1}{3}+......+\frac{n^x-1}{x}\right]}$

$= e^{\frac{1}{n}[\log 1+\log 2+\log 3+....+\log n]}$

$= e^{\frac{1}{n}\log 1..2..3.....n}$

$= e^{\log(1..2..3.....n)^{\frac{1}{n}}}$

$= (1...2...3......n)^{\frac{1}{n}} = (n!)^{\frac{1}{n}}$

(iv) $\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cot x}$ $\qquad (\infty^{0})$

$$\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cot x} = e^{\lim_{x \to \frac{\pi}{2}} \cot x \log \tan x} \qquad (0 \times \infty)$$

$$= e^{\lim_{x \to \frac{\pi}{2}} \frac{\log \tan x}{\tan x}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= e^{\lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x}\sec^2 x}{\sec^2 x}} \qquad \text{(using L. Hospital's rule)}$$

$$= e^{\frac{1/\infty}{1}} = e^{0} = 1$$

Evaluation of limits using Newton-Leibnitz's Theorem

Let $I(x) = \int_{\phi(x)}^{\psi(x)} f(t)dt$

Newton-Leibnitz's theorem states that

$$I'(x) = f(\psi(x)).\psi'(x) - f(\phi(x)).\phi'(x)$$

Illustration 18 :

If $\lim_{x \to 0} \int_0^x \frac{t^2 dt}{(x - \sin x)\sqrt{a + t}} = 1$, then value of a is ....

Solution :

Given $\lim_{x \to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{x - \sin x} = 1$

$\therefore \lim_{x \to 0} \frac{\frac{x^2}{\sqrt{a+x}} \cdot 1 - 0}{1 - \cos x} = 1$ [Using $N - L$ theorem]

$\Rightarrow \lim_{x \to 0} \frac{x^2}{\sqrt{a+x}\, 2\left(\frac{\sin x/2}{x/2}\right)^2 \cdot \frac{x^2}{4}} = 1$

$\Rightarrow \sqrt{a} = 2 \quad \therefore \quad a = 4$

CONTINUITY & DIFFERENTIABILITY

Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ if and only if $f(a^-) = f(a^+) = f(a)$ = finite number, *i.e.,* $\lim_{x\to a} f(x)$ exists finitely, $f(a)$ is a finite number and $\lim_{x\to a} f(x) = f(a)$.

More precisely, for given $\varepsilon > 0$ $\delta > 0$ such that $0 \le |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

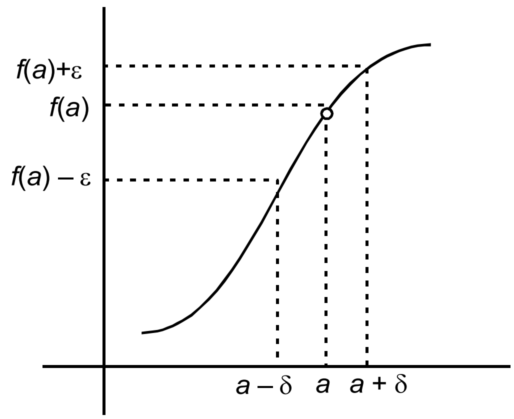


Illustration 19 :

Consider the function

$$f(x) = \begin{cases} 5x - 4, & \text{if } 0 < x \le 1 \\ 4x^3 - 3x, & \text{if } 1 < x < 2 \end{cases}$$

Solution :

At $x = 1$

L.H.L. $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5x - 4)$

Put $x = 1 - h$. As $x \to 1^-$, $h \to 0$

$\therefore$ L.H.L. $= \lim_{h\to 0}[5(1-h)-4] = 5(1-0)-4 = 1$

R.H.L. $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x^3 - 3x)$

Put $x = 1 + h$. As $x \to 1^+$, $h \to 0$

$\therefore$ R.H.L. $= \lim_{h\to 0}[4(1+h)^3 - 3(1+h)]$

$= [4(1+0)^3 - 3(1 + 0)] = 1$

$f(1) = 5(1) - 4 = 1$

Since *L.H.L.= R.H.L.* $= f(1)$,

$\therefore$ $f(x)$ is continuous at $x = 1$.

Continuity of a Function in Open Interval

A function is said to be continuous in an open interval (a, b), if it is continuous at each point of (a, b).

Continuity in Closed Interval

A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if

1. $f(x)$ is continuous from right at $x = a$, i.e.,

$$\lim_{h \to 0} f(a+h) = f(a)$$

2. $f(x)$ is continuous from left at $x = b$, i.e.,

$$\lim_{h \to 0} f(b-h) = f(b)$$

3. f(x) is continuous at each point of the open interval (a, b)

Illustration 20 :

The graph of a function $f(x)$ is given.

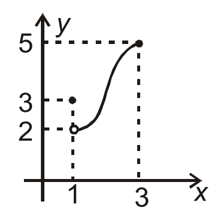
Which of the following statements is not correct?

(a) $f(x)$ is continuous on $(1, 3)$

(b) $f(x)$ is continuous on $(1, 3]$

(c) $f(x)$ is continuous on $[1, 3]$

(d) None of these



Solution :

At $x = 1$

$f(1) = 3, \lim_{x \to 1^+} f(x) = 2$

$\therefore$ $f(x)$ is not continuous from right

At $x = 3$

$f(3) = 5, \lim_{x \to 3^-} f(x) = 5$ $\therefore$ $f(x)$ is continuous from left

At all the points $c \in (1,3)$ $f(x)$ is continuous because there is no jump in the graph.

Hence statement of option (3) is incorrect.

The nest illustration offers an example of piece - wise continuous function.

Illustration 21 :

If $f(x) = \begin{cases} 1-2x & ; x < 0 \\ 2 & ; x = 0 \\ x^2+2 & ; x > 0 \end{cases}$, then at $x = 0$

(1) f is continuous

(2) f is continuous from left

(3) f is continuous from right

(4) f has removable discontinuity

Solution :

At $x = 0$

$$\text{LHL} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}(1-2(-h)) = \lim_{h\to 0}(1+2h) = 1$$

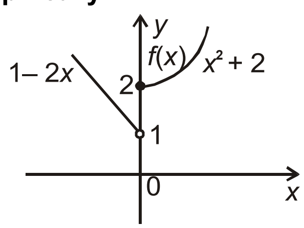
$$\text{RHL} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}(0+h)^2 + 2 = 2$$

$f(0) = 2$

$\therefore$ LHL $\neq$ RHL = $f(0)$

$\Rightarrow$ f is continuous from right but discontinuous from left. $\therefore$ only (3) is correct

Graphically:



Properties of Continuous Function

1. If f and g are continuous at $x = a$, then
   (a) $f + g$ is continuous at $x = a$
   (b) $f - g$ is continuous at $x = a$
   (c) fg is continuous at $x = a$
   (d) f/g is continuous at $x = a$, provided $g(a) \neq 0$
   (e) kf is continuous at $x = a$, where k is any real constant
   (f) $[f(x)]^{m/n}$ is continuous at $x = a$, provided $[f(x)]^{m/n}$ is defined on an interval containing a, and m, n are integers
2. If f is continuous at a and g is continuous at $f(a)$ then gof is continuous at a
3. If f is continuous at x = a and g is discontinuous at $x = a$, then $f + g$ and $f - g$ are discontinuous at $x = a$, whereas fg may be continuous at $x = a$.
4. If f is continuous at $x = a$ and $f(a) \neq 0$, then there exists an open interval $(a - \delta, a + \delta)$ such that $\forall x \in (a-\delta, a+\delta), f(x)$ has the same sign as $f(a)$
5. If f is a continuous function defined on $[a, b]$ such that $f(a).f(b) < 0$, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).
6. If f is a continuous function defined on $[a, b]$ and k is any real number between $f(a)$ and $f(b)$, then there exists atleast one solution of the equation $f(x) = k$ in the open interval (a, b).
7. If a function f is continuous on a closed interval $[a, b]$, then it is bounded on $[a, b]$ i.e., there exists real number k and K such that

   $k \leq f(x)$ for all $x \in [a, b]$
8. Every polynomial is continuous at every point of the real line.
9. Every rational function is continuous at every point where its denominator is different from zero.
10. Logarithmic functions, Exponential functions, Trigonometric functions, Inverse circular function and Absolute value functions are continuous in their domain of definition.

Illustration 22 :

$f + g$ may be a continuous function if

(1) f is continuous and g is discontinuous

(2) f is discontinuous and g is continuous

(3) f and g both are discontinuous

(4) None of these

Solution :

Consider $h(x) = f(x) + g(x)$

If $f(x)$ is continuous & $g(x)$ is discontinuous, then let us assume that $h(x)$ is continuous.

Now, $g(x) = \underbrace{h(x)}_{\text{cont.fn}} - \underbrace{f(x)}_{\text{cont.fn}} \Rightarrow g(x)$ is a continuous function (by property 1(b) above)

Which is contradictory to the given fact that $g(x)$ is a discontinuous function.

Hence our assumption that $h(x)$ is continuous is wrong.

i.e., if f is continuous and g is discontinuous then $f + g$ can't be a continuous function.

i.e., (1) is wrong.

Similarly (2) is wrong.

For (3), we take $f(x) = x - [x]$ (Discontinuous function)

$g(x) = x + [x]$ (Discontinuous function)

$(f + g)(x) = (x - [x]) + (x + [x])$

$= 2x$ (Continuous function) $\therefore$ (3) is correct.

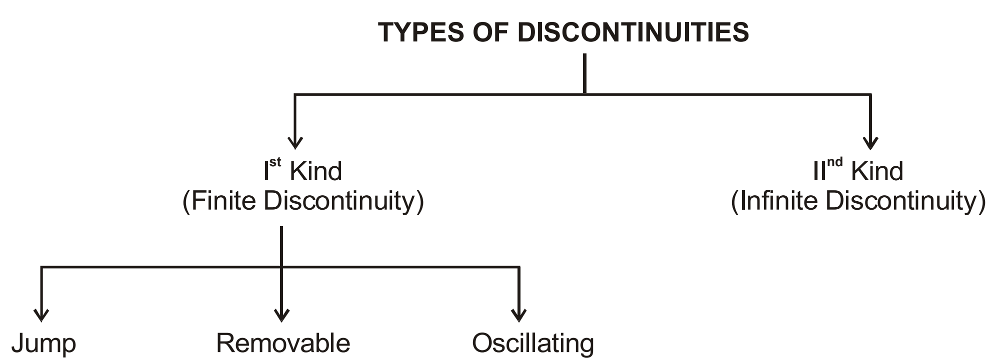
DISCONTINUITY OF FUNCTIONS

A function $f(x)$, which is not continuous at a point $x = a$, is said to be discontinuous at that point.

The discontinuity may arise due to any of the following reasons

1. $\lim\limits_{h\to 0} f(a-h) \neq \lim\limits_{h\to 0} f(a+h)$, i.e., LHL and RHL exist, but are not equal
2. $\lim\limits_{h\to 0} f(a-h) = \lim\limits_{h\to 0} f(a+h) \neq f(a)$, LHL and RHL exist and are equal, but are different from $f(a)$.
3. $f(a)$ is not defined.
4. At least one of the limits $\lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$ does not exist or at least one of these limits is ∞ or $-\infty$.

Types of Discontinuities



(when a function keeps on oscillating as $x \to$ some number, $f(x)$ doesn't come closer & closer to any number)

Removable Discontinuity

If $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ exist and are equal, but are not equal to $f(a)$, then the function $f(x)$ is said to have a removable discontinuity at $x = a$. However, by suitably defining the function at $x = a$, $f(x)$ can be made continuous at $x = a$.

Discontinuity of the First Kind

If $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ exist but are not equal, then the function $f(x)$ is said to have a discontinuity of the first kind at $x = a$. Also called **jump discontinuity**.

If $\lim_{h\to 0} f(a-h)$ exists but not equal to $f(a)$, then the function $f(x)$ is said to have a discontinuity of the first kind from the left at $x = a$.

Similarly, if $\lim_{h\to 0} f(a+h)$ exists but not equal to $f(a)$, then the function $f(x)$ is said to have a discontinuity of the first kind from the right at $x = a$.

Discontinuity of the Second Kind

If at least one of the limits $\lim_{h\to 0} f(a-h)$ or $\lim_{h\to 0} f(a+h)$ does not exist or at least one of these limits is ∞ or $-\infty$, then the function $f(x)$ is said to have a discontinuity of the second kind at $x = a$.

If $\lim_{h\to 0} f(a-h)$ does not exist or is equal to ∞ or $-\infty$, then the function $f(x)$ is said to have a discontinuity of the second kind from the left at $x = a$. Discontinuity of the second kind from the right is similarly defined.

Illustration 23 :

If $f(x) = \dfrac{1}{x^2 - 17x + 66}$ then $f\left(\dfrac{2}{x-2}\right)$ is discontinuous at x =

(1) $2, \dfrac{7}{3}, \dfrac{25}{11}$ (2) $2, \dfrac{8}{3}, \dfrac{24}{11}$

(3) $2, \dfrac{7}{3}, \dfrac{24}{11}$ (4) None of these

Solution :

Let $u = \dfrac{2}{x-2}$

$\because f(u) = \dfrac{1}{(u-6)(u-11)}$

$\therefore$ $f(u)$ is undefined when u is undefined, $u = 6$, $u = 11$

i.e., at $x = 2, \dfrac{2}{x-2} = 6, \dfrac{2}{x-2} = 11$

$x = 2$, $x = 7/3$, $x = 24/11$

$\therefore$ (3) is correct

Illustration 24 :

The function $f(x) = |2\ sgn\ 2x| + 2$ has

(1) Jump discontinuity

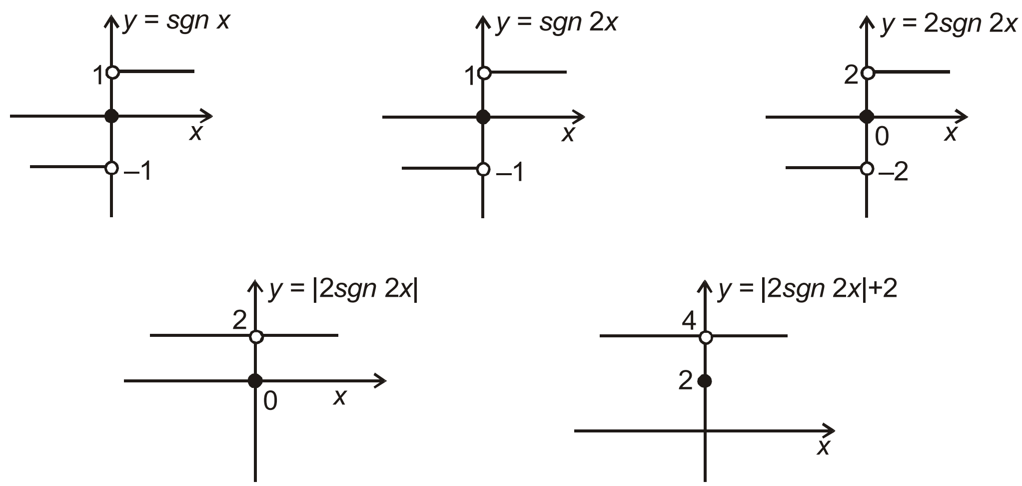
(2) Removable discontinuity

(3) Infinite discontinuity

(4) No discontinuity at $x = 0$

Solution :

Graphical Method



At $x = 0$

LHL = RHL = $4 \neq f(0) = 2$

$\therefore$ (2) is correct

Analytical Method :

To find RHL at $x = 0$, we put $x = 0 + h$ (where h is small positive no.) and let h approach towards 0.

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\left|2\,sgn\,\underbrace{2h}_{+ve}\right| + 2 = |\,2\times 1\,| + 2 = 4$$

Similarly, $\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\left|2\,sgn(2(-h))\right| + 2 = \lim_{h\to 0}|\,2\times(-1)\,| + 2 = 4$

$f(0) = |2\ sgn\ 2 \times 0| + 2 = |2 \times 0\ | + 2 = 2$

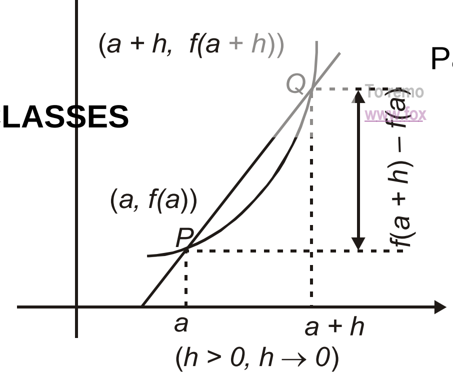
$LHL = RHL = 4 \neq f(0) = 2$

$\therefore$ $f(x)$ has removable discontinuity at $x = 0$

DIFFERENTIABILITY OF FUNCTIONS

Differentiability at a Point

A function $y = f(x)$ is said to be differentiable at a point a if at $x = a$ left hand derivative $f'(a^-)$ and right hand derivative $f'(a+)$ both exist finitely and are equal. There common value is called derivative of $f(x)$ at $x = a$. Right hand derivative at $x = a$ is defined as:



$$f'(a+) \equiv \lim_{h\to 0} \frac{f(a-h)-f(a)}{h} (h > 0)$$

Left hand derivative at $x = a$:

$$f'(a-) \equiv \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} (h > 0)$$

Thus $f(x)$ is differentiable at $x = a$ if $f'(a^-) = f'(a^+)$ some fixed finite quantity.

Differentiability on an Interval

A function $f(x)$ is said to be differentiable on an open interval (a, b) if $f(x)$ is differentiable at every point of this interval (a, b).

It is differentiable on a closed interval [a, b] if it is differentiable on the open interval (a, b) and the limits

$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a}$$

and $Lf'(b) = \lim_{h\to 0} \frac{f(a-h)-f(b)}{(b-h)-b}$ exist.

Properties of Differentiable Functions

1. Every polynomial function, exponential function and constant function is differentiable at each point of the real line.
2. Logarithmic functions, Trigonometric functions and Inverse Trigonometric functions are differentiable in their domain of definition.
3. The sum, difference, product and quotient of two differentiable functions is differentiable.
4. The composition of differentiable functions is a differentiable function.
5. If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
6. If $f(x)$ and $g(x)$ both are not differentiable at a point, then the sum function $f(x) + g(x)$ and the product function $f(x).g(x)$ can still be differentiable at that point.

Illustration 25 :

At the point $x = 1$, the function $f(x) = \begin{cases} x^3 - 1 & ; \quad 1 < x < \infty \\ x - 1 & ; \quad -\infty < x \le 1 \end{cases}$ is

(1) Continuous and differentiable

(2) Continuous and non differentiable

(3) Discontinuous and differentiable

(4) Discontinuous and non differentiable

Solution :

$$\text{RHL} = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}(1+h)^3 - 1$$

$$= \lim_{h\to 0}(1+3h+3h^2+h^3) - 1 = 0$$

$\text{LHL} = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}((1-h)-1) = 0$, $f(1) = 1 - 1 = 0$

$\because$ LHL = RHL = $f(1)$ $\therefore$ $f(x)$ is continuous at $x = 1$.

Now, $f(x)$ may or may not be differentiable at $x = 1$

$$\text{RHD} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\left((1+h)^3-1\right)-(1-1)}{h}$$

$$= \lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h} = \lim_{h\to 0}(3+3h+h^2) = 3$$

$$\text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{\left((1-h)-1\right)-0}{-h}$$

$$= \lim_{h\to 0}\frac{(-h)}{(-h)} = 1$$

$\because$ $LHD \neq RHD$ $\therefore$ $f(x)$ is not differentiable at $x = 1$

$\because$ (2) is correct.

Relation between Continuity and Differentiability

1. If a function $f(x)$ is differentiable at a point $x = a$ then it is continuous at $x = a$.
2. If $f(x)$ is only continuous at a point $x = a$, there is no guarantee that $f(x)$ is differentiable there.
3. If $f(x)$ is not differentiable at $x = a$ then it may or may not be continuous at $x = a$.
4. If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.
5. If left hand derivative and right hand derivative of $f(x)$ at $x = a$ are finite (they may or may not be equal) then $f(x)$ is continuous at $x = a$.

Differentiation of Basic Elementary Functions

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(\log_e x) = \frac{1}{x}(x > 0)$
9. $\frac{d}{dx}(a^x) = a^x(\ln a)(a > 0)$
10. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
11. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
12. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$

13. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, -\infty < x < \infty$

14. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, x \in R - [-1, 1]$

15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, x \in R - [-1, 1]$

Differentiation of Explicit Functions

1. Scalar product rule: $\frac{d}{dx}(cf(x)) = cf'(x)$ (where c is a constant)

2. Sum and difference rule: $\frac{d}{dx}\{f(x) \pm g(x)\} = f'(x) \pm g'(x)$

3. Product rule: $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$

4. Quotient rule: $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} (g(x) \neq 0)$

Illustration 26 :

Find $\frac{dy}{dx}$ if

(a) $y = 5x^{2/3} - 3x^{5/2} + 2x^{-3}$

(b) $y = x^3 \tan^{-1} x$

(c) $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

Solution:

(a) $\frac{dy}{dx} = \frac{d}{dx}\{5x^{2/3} - 3x^{5/2} + 2x^{-3}\}$

$$= 5\frac{d}{dx}\left(x^{2/3}\right) - 3\frac{d}{dx}\left(x^{5/2}\right) + 2\frac{d}{dx}(x^{-3})$$

$$= 5.\frac{2}{3}x^{\frac{2}{3}-1} - 3.\frac{5}{2}x^{\frac{5}{2}-1} + 2(-3)x^{-3-1} = \frac{10}{3\sqrt[3]{x}} - \frac{15}{2}x^{\frac{3}{2}} - \frac{6}{x^4}$$

(b) $\frac{dy}{dx} = \frac{d}{dx}(x^3 \tan^{-1} x) = \left(\frac{d}{dx}x^3\right)\tan^{-1} x + x^3 \frac{d}{dx}(\tan^{-1} x)$

$$= 3x^2 \tan^{-1} x + x^3 \frac{1}{1+x^2} = 3x^2 \tan^{-1} x + \frac{x^3}{1+x^2}$$

(c) $\dfrac{dy}{dx} = \dfrac{(\sin x + \cos x)'(\sin x - \cos x) - (\sin x + \cos x)(\sin x - \cos x)'}{(\sin x - \cos x)^2}$ (′ denote differentiate w.r.t. x)

$$= \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \frac{-(1 - 2\sin x \cos x) - (1 + 2\sin x \cos x)}{(\sin x - \cos x)^2} = \frac{-2}{(\sin x - \cos x)^2}$$

Differentiation of composite functions: (Chain Rule)

If function $h(x)$ is composition of two functions $f(x)$ and $g(x)$ as $h(x) = f(g(x))$. Now if $h(x) = f(t)$ and $t = g(x)$

then, $\dfrac{d}{dx}(h(x)) = \dfrac{dh(x)}{dt} \times \dfrac{dt}{dx} = \dfrac{df(t)}{dt} \times \dfrac{dg(x)}{dx} = f'(t)g'(x)$

$\Rightarrow \dfrac{d}{dx}(h(x)) = f'(g(x))g'(x)$.

This rule is called Chain Rule.

This rule can be extended to the compositions of any finite number of functions, for example, if $y = f(g(h(x)))$

Then, $\dfrac{dy}{dx} = f'(g(h(x)))g'(h(x))h'(x)$

Illustration 27 :

Let $y = \log_e(\cos^3 x^4)$ find $\dfrac{dy}{dx}$.

Solution :

Put $\cos^3 x^4 = t$, $\cos x^4 = z$, $x^4 = w$

then $\dfrac{dy}{dx} = \dfrac{d}{dt}\log t \times \dfrac{dz^3}{dz} \times \dfrac{d\cos w}{dw} \times \dfrac{dx^4}{dx}$

$$= \frac{1}{t} \times 3z^2 \times (-\sin w) \times 4x^3 = \frac{12x^3 \cos^2(x^4)\sin(x^4)}{\cos^3(x^4)}$$

Differentiation of Function represented Parametrically :

If x and y are represented as : $x = \phi(t)$ and $y = \psi(t)$ $(\alpha < t < \beta)$, where $\phi(t)$ and $\psi(t)$ are differentiable functions and $\phi'(t) \neq 0$, then y defined as a single valued function (continuous) of x is differentiable and its derivative is given by:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\psi'(t)}{\phi'(t)}$$ (t is a parameter)

Remark : $\dfrac{d^2y}{dx^2} \neq \dfrac{\frac{d^2y}{dt^2}}{\frac{dx^2}{dt^2}}$

Illustration 28 :

Find $\frac{dy}{dx}$ if $y = e^t \cos t$, $x = e^t \sin t$ (t being a parameter)

Solution:

We have $\frac{dy}{dx} = \frac{d}{dt}(e^t \cos t)$

$= e^t \cos t - e^t \sin t = e^t (\cos t - \sin t)$

and $\frac{dx}{dt} = \frac{d}{dt}(e^t \sin t)$

$= e^t \sin t + e^t \cos t = e^t (\cos t + \sin t)$

Then, $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{e^t(\cos t - \sin t)}{e^t(\cos t + \sin t)} = \frac{\cos t - \sin t}{\cos t + \sin t}$

LOGARITHMIC DIFFERENTIATION

If the given functions are of the following types:

1. Consisting of product and quotient of a number of functions or
2. Consisting of function of x raised to a power which is also a function of x i.e., $\{f(x)\}^{g(x)}$ where $f(x)$ and $g(x)$ are function of x

   Then first we take logarithm of whole expression and then differentiate w.r.t. the suitable variable.

Illustration 29 :

Let $y = \{f(x)\}^{g(x)}$, find $\frac{dy}{dx}$.

Solution :

Taking logarithm of both sides, we get $\log_e |y| = g(x)\log_e |(f(x))|$

Now differentiate w.r.t. x, we get $\frac{1}{y}\frac{dy}{dx} = g'(x)\log_e |f(x)| + g(x)\frac{1}{f(x)}f'(x)$

$= g'(x)\log_e |(f(x)| + \frac{g(x)}{f(x)}f'(x)$

$\Rightarrow \frac{dy}{dx} = (f(x))^{g(x)}\{g'(x)\log|f(x)| + \frac{f'(x)}{f(x)}g(x)\}$

Remember : (Direct rule)

$$\boxed{\frac{dy}{dx} \equiv \text{Derivative of } (f(x)^{g(x)} \text{ treating } f(x) \text{ constant} + \text{Derivative of} (f(x)^{g(x)} \text{ treating } g(x) \text{ constant.}}$$

Illustration 30 :

Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2\sqrt{x+3)^2}}}$

Solution :

Taking log of both sides, we get $\log|y| = \log(\sqrt{x-1}) - \log\left\{\sqrt[3]{(x+2)^2}\,\sqrt{(x+3)^2}\right\}$

$$= \frac{1}{2}\log|(x-1)| - \frac{2}{3}\log|x+2| - \log|x+3|$$

differentiating w.r.t. x, we have $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(x-1)} - \frac{2}{3(x+2)} - \frac{1}{(x+3)}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{(x-1)}}{\sqrt[3]{(x+2)^2}\sqrt{(x+3)^2}}\left\{\frac{3(x+2)(x+3) - 4(x-1)(x+3) - 6(x-1)(x+2)}{6(x-1)(x+2)(x+3)}\right\}$$

Differentiation by using Trigonometric Transformations

With the help of trigonometric transformations, the labour involved in compiting derivation can be reduced. Note some standard trigonometric substitutions

1. $\sqrt{a^2 - x^2}$ $\Rightarrow$ put $x = a \sin\theta$ or $a \cos\theta$
2. $\sqrt{a^2 + x^2}$ $\Rightarrow$ put $x = a \tan\theta$ or $a \cot\theta$
3. $\sqrt{x^2 - a^2}$ $\Rightarrow$ put $x = a \sec\theta$ or $a\,\mathrm{cosec}\,\theta$
4. $\sqrt{(a-x)/(a+x)}$ $\Rightarrow$ put $x = a \cos\theta$ or $a \cos 2\theta$
5. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ $\Rightarrow$ put $x = a \sin 2\theta$
6. $\sqrt{2ax - x^2}$ $x = 2a\sin^2\theta$
7. $\sqrt{(x-\alpha)(\beta-x)}$ or $\sqrt{\dfrac{(x-\alpha)}{(\beta-x)}}$ $x = \alpha\cos^2\theta + \beta\sin^2\theta$ (Important)

   (Where $\alpha < x < \beta$)

Illustration 31 :

If $y = \tan^{-1}\left(\dfrac{3a^2x - x^3}{a(a^2 - 3x^2)}\right)$, find $\dfrac{dy}{dx}$.

Solution :

We have, $y = \tan^{-1}\left(\dfrac{3(x/a) - (x/a)^3}{1 - 3(x/a)^2}\right)$, put $\dfrac{x}{a} = \tan\theta$, we have

$$\Rightarrow y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

Now differentiate w.r.t. x,

$$\frac{dy}{dx}=\frac{3}{\left(1+\left(\frac{x}{a}\right)^2\right)}\frac{1}{a}=\frac{3a}{(x^2+a^2)}$$

Differentiation of Inverse Function

If $y = f(x)$ and $x = g(y)$ are inverse functions ($g(f(x)) = x$ or $f(g(y)) = y$), then their derivatives are reciprocal $g'(y)=\frac{1}{f'(x)}$.

Illustration 32 :

If $y=\ln\sqrt{1+x^2}$, using differentiation of inverse function, find $\frac{dy}{dx}$.

Solution :

At $x > 0$, the inverse function $x=\sqrt{e^{2y}-1}$ has derivative $\frac{dx}{dy}=\frac{e^{2y}}{\sqrt{e^{2y}-1}}$.

Hence $\frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{\sqrt{e^{2y}-1}}{e^{2y}}=\frac{x}{x^2+2}$

Differentiation of a Function with respect to a Function

This type of differentiation is similar to parametric form type differentiation. Let $y = f(x)$ and $z = f(x)$ be two functions of x and we have to differentiate $f(x)$ w.r.t. $f(x)$ i.e. we have to find $\frac{dy}{dz}$. Then first compute $\frac{dy}{dx}=f'(x)$ and $\frac{dz}{dx}=\phi'(x)$.

Now $\frac{dy}{dz}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)}=\frac{f'(x)}{\phi'(x)}$.

Illustration 33 :

Find the differential coefficient of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution :

Let $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

We have to find $\frac{dy}{dz}$.

Now, put $x = \tan\theta$, then

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x$$

and $z = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

We have $\frac{dy}{dx} = \frac{2}{1+x^2}$ and $\frac{dz}{dx} = \frac{2}{1+x^2}$

$\Rightarrow \frac{dy}{dz} = 1$. (In fact, $y = z \Rightarrow \frac{dy}{dz} = 1$)

Differentiation of implicit function

If a differentiable function $y = y(x)$ satisfies the equation $f(x,y) = 0$, then we differentiate it w.r.t x considering y as a function of x and solve the equation $\frac{d}{dx}f(x,y) = 0$ for $\frac{dy}{dx}$

Note : *You will find this formula useful in case of implicit function*

$$\frac{dy}{dx} = -\frac{\text{differentiate } f \text{ w.r.t. } x \text{ keeping } y \text{ constant}}{\text{differentiate } f \text{ w.r.t. } y \text{ keeping } x \text{ constant}}.$$

Illustration 34 :

Find $\frac{dy}{dx}$ if $x^3 + x^2y + y^2 = 0$.

Solution :

Differentiating w.r.t. x considering y as a function of x, we get

$$3x^2 + 2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \quad \Rightarrow (x^2+2y)\frac{dy}{dx} = -(3x^2+2xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2+2xy}{x^2+2y}$$

Differentiation of Determinant whose Elements are Functions of Variable *x*

Let $\Delta(x) = \begin{vmatrix} f_{11}(x) & f_{12}(x)\ldots f_{1n}(x) \\ f_{21}(x) & f_{22}(x)\ldots f_{2n}(x) \\ \vdots & \\ f_{n1}(x) & f_{n2}(x)\ldots f_{nn}(x) \end{vmatrix}$, then

$$\frac{d}{dx}\Delta(x)=\begin{vmatrix} f'_{11}(x) & f'_{12}(x)\ldots f'_{1n}(x) \\ f_{21}(x) & f_{22}(x)\ldots f_{2n}(x) \\ \vdots & \\ f_{n1}(x) & f_{n2}(x)\ldots f_{nn}(x)\end{vmatrix}+\begin{vmatrix} f_{11}(x) & f_{12}(x)\ldots f_{1n}(x) \\ f'_{21}(x) & f'_{22}(x)\ldots f'_{2n}(x) \\ \vdots & \\ f_{n1}(x) & f_{n2}(x)\ldots f_{nn}(x)\end{vmatrix}+\ldots+\begin{vmatrix} f_{11}(x) & f_{12}(x)\ldots f_{1n}(x) \\ f_{21}(x) & f_{22}(x)\ldots f_{2n}(x) \\ \vdots & \\ f'_{n1}(x) & f'_{n2}(x)\ldots f'_{nn}(x)\end{vmatrix}$$

i.e. $$\frac{d}{dx}\Delta(x)=\begin{vmatrix}\text{differentiate 1st row} \\ \text{keeping other rows as} \\ \text{they are.}\end{vmatrix}+\begin{vmatrix}\text{differentiate 2nd row} \\ \text{keeping remaining rows} \\ \text{as they are.}\end{vmatrix}+\ldots+\begin{vmatrix}\text{differentiate } n\text{th row} \\ \text{keeping remaining rows} \\ \text{as they are.}\end{vmatrix}$$

Illustration 35 :

If $f(x)=\begin{vmatrix}\sin x & e^{2x}+e^{-x} & \log x \\ \cos x & 2e^{2x}-e^{-x} & \frac{1}{x} \\ -\sin x & 4e^{2x}+e^{-x} & -\frac{1}{x^2}\end{vmatrix}$ find $f'(x)$.

Solution :

Using rule for differentiation of determinant, we get

$$f'(x)=\begin{vmatrix}\cos x & 2e^{2x}-e^{-x} & \frac{1}{x} \\ \cos x & 2e^{2x}-e^{-x} & \frac{1}{x} \\ -\sin x & 4e^{2x}+e^{-x} & -\frac{1}{x^2}\end{vmatrix}+\begin{vmatrix}\sin x & e^{2x}+e^{-x} & \log x \\ -\sin x & 4e^{2x}+e^{-x} & -\frac{1}{x^2} \\ -\sin x & 4e^{2x}+e^{-x} & -\frac{1}{x^2}\end{vmatrix}+\begin{vmatrix}\sin x & e^{2x}+e^{-x} & \log x \\ \cos x & 2e^{2x}-e^{-x} & \frac{1}{x} \\ -\cos x & 8e^{2x}-e^{-x} & \frac{2}{x^3}\end{vmatrix}$$

$$=0+0+\begin{vmatrix}\sin x & e^{2x}+e^{-x} & \log x \\ \cos x & 2e^{2x}-e^{-x} & \frac{1}{x} \\ -\cos x & 8e^{2x}-e^{-x} & \frac{2}{x^3}\end{vmatrix}=\begin{vmatrix}\sin x & e^{2x}+e^{-x} & \log x \\ \cos x & 2e^{2x}-e^{-x} & \frac{1}{x} \\ -\cos x & 8e^{2x}-e^{-x} & \frac{2}{x^3}\end{vmatrix}$$

HIGHER ORDER DERIVATIVES

When $y = f(x)$ is differentiable w.r.t. x, the differential coefficient $\frac{dy}{dx}$ or $f'(x)$ so obtained, being a function of x, can be differentiated again (if it is differentiable), to obtain a quantity, which is termed as 2nd differential coefficient of $f(x)$ and denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$. Similarly we can obtain third, fourth and higher order differential coefficient of $f(x)$, denoted by $\frac{d^3y}{dx^3}(=f'''(x)),\frac{d^4y}{dx^4}(\equiv f^{(iv)}(x))$ and so on.

If u and v are functions differentiable n times, then

$(c_1u + c_2v)^{(n)} = c_1u^{(n)} + c_2v^{(n)}$ (where c_1 and c_2 are constants and $u^{(n)}$ denote derivative of n^{th} order)

Leibintz formula **:** (n^{th} order derivative of product of two functions)

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$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{1.2}u^{(n-2)}v^{(2)} + \ldots + uv^{(n)} = \sum_{r=0}^{n} C(n,r)u^{(n-r)}v^{(r)}$$

where $u^{(0)} = u$, $v^{(0)} = v$ and $C(n,r) = \frac{n!}{(n-r)!r!}$ (binomial coefficients)

Some basic formulae : (n) denote nth derivative.

(1) $(x^m)^{(n)} = m(m-1) \ldots\ldots (m-n+1)\, x^{m-n} = \frac{\lfloor m\, x^{m-n}}{\lfloor m-n}$

(2) $(a^x)^{(n)} = a^x(\ln a)^n$ $(a > 0)$

(3) $(e^x)^{(n)} = e^x$

(4) $(\ln x)^{(n)} = (-1)^{n-1}\, \frac{(n-1)!}{x^n}$

(5) $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$

(6) $(\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$

SOLVED EXAMPLES

Example 1 :

Evaluate the following limits

(i) $\lim_{n\to\infty}\left[\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ...n \text{ terms}\right]$

(ii) $\lim_{x\to\infty}\left[\frac{x^2+3x+2}{x^2+5x+1}\right]^x$

(iii) $\lim_{y\to 0}\frac{b^{\sqrt{y}} - b^{\frac{1}{\sqrt{y}}}}{b^{\sqrt{y}} + b^{\frac{1}{\sqrt{y}}}}, b > 1$

(iv) $\lim_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3} = 1$, find a and b

Solution :

(i) $\lim_{n\to\infty}\left[\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ...\frac{1}{(3n-2)(3n+1)}\right]$

$$= \lim_{n\to\infty}\frac{1}{3}\left[\frac{3}{1.4} + \frac{3}{4.7} + \frac{3}{7.10} + ...\frac{3}{(3n-2)(3n+1)}\right]$$

$$= \lim_{n\to\infty}\frac{1}{3}\left[\left(1-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{10}\right) + ...\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{3}\left[1 - \frac{1}{3n+1}\right] = \frac{1}{3}$$

(ii) $\lim_{x\to\infty}\left[\dfrac{x^2+3x+2}{x^2+5x+1}\right]^x$ (1^∞ form)

$= e^{\lim_{x\to\infty}\left[\frac{x^2+3x+2}{x^2+5x+1}-1\right]x}$

$= e^{\lim_{x\to\infty}\frac{(-2x+1)x}{x^2+5x+1}} = e^{\lim_{x\to\infty}\frac{-2x^2+x}{x^2+5x+1}}$

$= e^{\lim_{x\to\infty}\frac{-2+\frac{1}{x}}{1+\frac{5}{x}+\frac{1}{x^2}}} = e^{-2}$

(iii) $\lim_{y\to 0}\dfrac{b^{\sqrt{y}}-b^{\frac{1}{\sqrt{y}}}}{b^{\sqrt{y}}+b^{\frac{1}{\sqrt{y}}}}$

Let $y = t^2$, $y \to 0 \Rightarrow t \to 0$

$\lim_{t\to 0}\dfrac{b^t-b^{\frac{1}{t}}}{b^t+b^{\frac{1}{t}}} = \lim_{t\to 0}\dfrac{b^{t-\frac{1}{t}}-1}{b^{t-\frac{1}{t}}+1}$ $\left(\text{Divide both } N^r \text{ and } D^r \text{ by } b^{\frac{1}{t}}\right)$

$= \dfrac{b^{-\infty}-1}{b^{-\infty}+1} = -1$

(iv) $\lim_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{x^3} = 1$, then find a and b

$$= \lim_{x\to 0}\frac{x\left[1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-.....\right)\right]-b\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....\right]}{x^3} = 1$$

$$= \lim_{x\to 0}\frac{x(1+a-b)-x^3\left(\frac{a}{2!}-\frac{b}{3!}\right)+x^5\left(\frac{a}{4!}-\frac{b}{5!}\right)+.....}{x^3} = 1$$

But this value is equal to 1.

For this $1 + a - b = 0$ and $-\dfrac{a}{2!}+\dfrac{b}{3!} = 1$

From these equation a = $-\dfrac{5}{2}, b = \dfrac{-3}{2}$

Example 2 :

$\lim_{x\to 0}\dfrac{x^a\sin^b x}{\sin x^c}$ *is a finite and non-zero then find the relation between a, b and c, where a, b and c* $\in N$

Solution :

$$\lim_{x\to 0}\frac{x^a \sin^b x}{\sin x^c}$$

$$= \lim_{x\to 0} x^a \frac{\sin^b x}{x^b}\frac{x^c}{\sin x^c}\times\frac{x^b}{x^c}$$

$$= \lim_{x\to 0} x^{a+b-c}\left(\frac{\sin x}{x}\right)^b\left(\frac{x}{\sin x}\right)^c$$

$$= \lim_{x\to 0} x^{a+b-c}\times 1\times 1$$

This term is finite and non zero if $a + b - c = 0$

So relation is $a + b = c$

Example 3 :

$$\lim_{x\to 0}\frac{2\log\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{\tan 5x}$$

Solution :

$$\lim_{x\to 0}\frac{2\log\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{\tan 5x}$$

$$= \lim_{x\to 0}\frac{\log\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\tan 5x}$$

$$= \lim_{x\to 0}\frac{\log\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{\tan 5x}$$

$$= \lim_{x\to 0}\frac{\log(1+\sin x)}{\tan 5x}$$

$$= \lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x}\frac{\sin x}{\tan 5x}$$

$$= \lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x}\frac{\sin x}{x}\frac{5x}{\tan 5x}\frac{1}{5}$$

$$= 1 \times 1 \times 1 \times \frac{1}{5}$$

$$= \frac{1}{5}$$

Example 4 :

$$\lim_{n\to\infty}\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)+.....\left(1+\frac{1}{3^{2^n}}\right)$$

Solution :

$$\lim_{n\to\infty}\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)+.....\left(1+\frac{1}{3^{2^n}}\right)$$

$$= \lim_{n\to\infty}\frac{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)+.....\left(1+\frac{1}{3^{2^n}}\right)}{1-\frac{1}{3}} \quad \left[\because \text{multiply and divide by}\left(1-\frac{1}{3}\right)\text{both } N^r \text{ and } D^r\right]$$

$$= \lim_{n\to\infty}\frac{\left(1-\frac{1}{3^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)+.....\left(1+\frac{1}{3^{2^n}}\right)}{\frac{2}{3}}$$

$$= \lim_{n\to\infty}\frac{\left(1-\frac{1}{3^4}\right)\left(1+\frac{1}{3^4}\right)+.....+\left(1+\frac{1}{3^{2^n}}\right)}{\frac{2}{3}}$$

$$= \lim_{n\to\infty}\frac{1-\frac{1}{3^{2^{n+1}}}}{\frac{2}{3}} = \frac{1-0}{\frac{2}{3}} = \frac{3}{2}$$

Example 5 :

Find $\lim_{x\to\infty}\frac{x^2+x+1}{e^{[x]}}$

Solution :

We know that

$$x-1<[x]\le x$$

$$\Rightarrow \quad e^{x-1}<e^{[x]}\le e^x$$

$$\Rightarrow \quad \frac{1}{e^{x-1}}>\frac{1}{e^{[x]}}\ge\frac{1}{e^x}$$

$$\Rightarrow \quad \frac{x^2+x+1}{e^{x-1}}>\frac{x^2+x+1}{e^{[x]}}\ge\frac{x^2+x+1}{e^x}$$

Now, $\lim_{x\to\infty}\frac{x^2+x+1}{e^x}$ and $\lim_{x\to\infty}\frac{x^2+x+1}{e^{x-1}}=\lim_{x\to\infty}\frac{2x+1}{e^{x-1}}=\frac{1}{\infty}=0$

$$= \lim_{x\to\infty}\frac{2x+1}{e^x} = \lim_{x\to\infty}\frac{2}{e^x}$$

[So using Sandwich theorem.]

$$= \frac{2}{\infty}=0$$

$$\lim_{x\to\infty}\frac{x^2+x+1}{e^{[x]}}=0$$

Example 6 :

If f(2) = 9 and f'(2) = 4 then find the value of $\lim_{x\to 2}\frac{\sqrt{f(x)}-3}{f(x)-9}$

Solution :

$$\lim_{x\to 2}\frac{\sqrt{f(x)}-3}{f(x)-9} \qquad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 2}\frac{\frac{f'(x)}{2\sqrt{f(x)}}-0}{f'(x)-0} \qquad \lim_{x\to 2}\frac{1}{2\sqrt{f(x)}}=\frac{1}{6}$$ [Using L. Hospital's rule]

Example 7 :

Find the value of $\lim_{h\to 0}\frac{\log(1+2h)-2\log(1+h)}{h^2}$

Solution :

$$\lim_{h\to 0}\frac{\log(1+2h)-2\log(1+h)}{h^2} \qquad \left(\frac{0}{0}\right)$$

$$\lim_{h\to 0}\frac{\frac{2}{1+2h}-\frac{2}{1+h}}{2h}$$ [Using L. Hospital's rule]

$$\lim_{h\to 0}\frac{\frac{-4}{(1+2h)^2}+\frac{2}{(1+h)^2}}{2}$$ [Using L. Hospital's rule]

$$=\frac{\frac{-4}{1+0}+\frac{2}{1+0}}{2}=-1$$

Example 8 :

If f : R $\rightarrow$ *R be a differentiable function such that f(3) = 2 and f'(3) =* $\frac{1}{2}$ *then find the value of* $\lim_{x\to 3}\frac{\int_2^{f(x)} 3t^2}{x-3}.dt$

Solution :

$$\lim_{x\to 3}\frac{\int_2^{f(x)} 3t^2}{x-3}.dt \qquad \left(\frac{0}{0}\right)$$

$$=\lim_{x\to 3}\frac{3[f(x)]^2\times f'(x)}{1-0}$$ [Using L. Hospital's rule]

$$=\frac{3[f(3)]^2\times f'(3)}{1-0}=3\times 4\times\frac{1}{2}=6$$

Example 9 :

$$If\ f(x)=\begin{cases}\dfrac{8^x-4^x-2^x+1^x}{x^2} & \text{where } x>0\\ e^x\sin x+[px+k\log 4 & \text{whe re } x\le 0\end{cases}$$ *is continuous at x= 0,find k.*

Solution :

We have, $f(0-) = \lim\limits_{h\to 0}$ $(e^{-h}\sin(-h) - \pi h + k\log 4)$

$\lim\limits_{h\to 0}$ $(-e^{-h}\sin(h) - \pi h + k\log 4) = k\log 4$

and $f(0+) = \lim\limits_{h\to 0}\dfrac{8^h-4^h+2^h+1}{h^2}$ $(h>0)$

$\lim\limits_{h\to 0}\left(\dfrac{4^h-1}{h}\right)\left(\dfrac{2^h-1}{h}\right)$ = (log 4) (log 2) = 2(log 2)^2

It is given that $f(x)$ is continuous at $x = 0$, then $f(0-) = f(0+)$

$\Rightarrow$ $k\log 4 = 2(\log 2)^2$ $\quad\Rightarrow$ $2k\log 2 = 2(\log 2)^2 \Rightarrow \boxed{k=\log 2}$

Example 10 :

$$If\quad f(x)=\begin{cases}\dfrac{(5^x-1)^3}{(1-\cos x)In(1+x)} & ,\ x\ne 0\\ a & ,\ x=0\end{cases}.$$

Find value of a if f(x) is continuous at x = 0.

Solution :

It is given that $f(x)$ is continuous at $x = 0$. Then $\lim\limits_{x\to 0} f(x)=f(0)$.

Now, $\underset{x\to 0}{Lim} f(x) = \underset{x\to 0}{Lim}\dfrac{(5^x-1)^3}{(1-\cos x)In(1+x)}$

$$= \underset{x\to 0}{Lim}\frac{(5^x-1)^3}{2\sin^2\left(\frac{x}{2}\right)In(1+x)} = \lim_{x\to 0}\frac{\left(\frac{5^x-1}{x}\right)^3}{2.\frac{1}{4}\left(\frac{\sin(x/2)}{x/2}\right)\frac{\ln(1+x)}{x}}=\frac{(\log 5)^3}{1/2}=2(\log 5)^3$$

Therefore, $a = (\log 5)^3$.

Example 11 :

Discuss the continuity of the function

$$F(x) = \begin{cases} \dfrac{a^{[x]+x}-1}{[x]+x} & x \neq 0 \\ \ln a & x = 0 \end{cases}$$

At x = 0

where [x] denotes greatest integer $\leq x$ and {x} denotes fractional part of x and a(>e) is any positive arbitrary constant.

Solution :

We know that $[x] + \{x\} = x$ for any $x \in R$ where $0 \leq \{x\} < 1$.

The given function can be expressed as follows :

$$f(x) = \begin{cases} \dfrac{a^{[x]+x}-1}{[x]+x} & x \neq 0 \\ \ln a & x = 0 \end{cases}$$

Now, $f(0-) = \lim\limits_{h\to 0} \dfrac{a^{[-h]-h}-1}{[-h]-h}$ $(0 < h < 1 \Rightarrow -1 < -h < 0 \Rightarrow [-h] = -1)$

$$= \lim_{h\to 0} \frac{a^{-1-h}-1}{(-1)-h} = \frac{a^{-1}-1}{-1} = 1 - \frac{1}{a}$$

and $f(0+) = \lim\limits_{h\to 0} \dfrac{a^{-1-h}-1}{[h]+h} = \lim\limits_{h\to 0} \dfrac{a^{h}-1}{h} = \ln a$ $(0 < h < 1 \Rightarrow [h] = 0)$

Hence $f(x)$ is not continuous at $x = 0$

Example 12 :

Let f(x) be a continuous function defined for $1 \leq x \leq 3$. If f(x) takes rational values for all x and f(2) = 10, then find f(1.5).

Solution :

We show that $f(x)$ must be a constant function. Suppose to the contrary $f(x)$ is not a constant function. Then by the property of continuous function on closed interval [1,3], it attains max and min. at some points in [1,3]. Let M is maximum value and m is minimum value ($m \neq M$) then $f(x)$ takes every value in the interval $[m, M]$ which is a contradiction because $f(x)$ can take only rational values but there are irrational numbers as well between m and M. Therefore $f(x)$ is a constant function. Then $f(1.5) = 10$.

Example 13 :

Suppose the function f satisfies the conditions:

(a) $f(x + y) = f(x) f(y)$ for all x and y and

(b) $f(x) = 1 + x\, g(x)$ where $\lim\limits_{x\to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists and $f'(x) = f(x)$ for all x. Find f(x).

Solution :

Putting $x = y = 0$ in relation (a) we get, $f(0) = f(0)\, f(0) \Rightarrow f(0) = 0$ or 1,using (b) we have $f(0) = 1$.

Now $f'(x) = \lim_{x\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h}$$

$$= f(x)\lim_{h\to 0} \frac{1+hg(h)-1}{h}$$

$= f(x) \lim_{h\to 0} g(h) = f(x) \Rightarrow f'(x) = f(x)$ for all x.

Now, $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = x + c \Rightarrow \log_e f(x) = x + c$

For $x = 0$, $f(0) = 1$, this gives $c = 0$. Hence $f(x) = e^x$.

Example 14 :

Does the function $f(x) = \dfrac{x^3}{4} - \sin \pi x + 3$ *take on the value* $2\dfrac{1}{3}$ *within the interval* $[-2, 2]$

Solution :

$f(x)$ is continuous in the interval $[-2, 2]$ and $f(-2) = 1$ and $f(2) = 5$, therefore $f(x)$ takes every value in $[1, 5]$ since $1 < 2\dfrac{1}{3} < 5$ there is a point c in the interval $[-2, 2]$ where $f(x)$ takes value $2\dfrac{1}{3}$.

Example 15 :

If $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$, *find* $\dfrac{d^2y}{dx^2}$.

Solution :

We have, $\dfrac{dy}{dx} = \left(\dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta}\right)$ differentiate again w.r.t. x we get

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{\dfrac{dx}{d\theta}\cdot\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\cdot\dfrac{d^2x}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^2}\cdot\frac{1}{\left(\dfrac{dx}{d\theta}\right)} = \frac{\dfrac{dx}{d\theta}\cdot\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\cdot\dfrac{d^2x}{d\theta^2}}{\left(\dfrac{dx}{d\theta}\right)^3}$$

Now $\frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta = -3\sin\theta(1-\cos^2\theta) = -3\sin^3\theta$

$$\frac{d^2x}{d\theta^2} = 9\sin^2\theta\cos\theta$$

and $\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta(1-\sin^2\theta) = 3\cos^3\theta$

$$\frac{d^2y}{d\theta^2} = -9\cos^2\theta\sin\theta$$

Therefore, $\frac{d^2y}{dx^2} = \frac{(-3\sin^3\theta)(-9\cos^2\theta\sin\theta) - 3\cos^3\theta(-9\sin^2\theta\cos\theta)}{(-3\sin^3\theta)^3}$

$$\frac{27\sin^4\theta\cos^2\theta + 27\cos^4\theta\sin^2\theta}{-27\sin^9\theta}$$

$$= \frac{\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)}{-\sin^9\theta} = -\frac{\cos^2\theta}{\sin^7\theta} = -\cot^2\theta\,\mathrm{cosec}^5\theta$$

or, $\frac{dy}{dx} = -\cot^3\theta \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\cot^3\theta).\frac{d\theta}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2} = 3\cot^2\theta\,\mathrm{cosec}^2\theta\,\frac{1}{(-3\sin^3\theta)} = -\cot^2\theta\,\mathrm{cosec}^5\theta$$

Example 16 :

If $y = \tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$, *find* $\frac{dy}{dx}$

Solution :

Given $y = \tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{2^x}{1+2^x\cdot 2^{x+1}}\right) = \tan^{-1}\left(\frac{2^{x+1} - 2x^2}{1+2^{x+1}2^x}\right) = \tan^{-1}(2^{x+1}) - \tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get, $\frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(2^{x+1}) - \frac{d}{dx}\tan^{-1}(2^x)$

$$= \frac{1}{1+(2^{x+1})^2}.2^{x+1}(\ln 2) - \frac{1}{1+(2^x)^2}.2^x(\ln 2)$$

$$= \left\{\frac{2^{x+1}}{1+2^{2(x+1)}} - \frac{2^x}{1+2^{2x}}\right\}(\ln 2)$$

Example 17 :

If y = sin (m sin⁻¹x), show that $(1-x^2)\dfrac{d^2y}{dx^2}-\dfrac{xdy}{dx}+m^2y=0$

Solution :

Given, $y = \sin(m\sin^{-1}x)$

Differentiating w.r.t. *x,* we get $\dfrac{dy}{dx}=\cos(m\sin^{-1}x)\dfrac{m}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\dfrac{dy}{dx}=m\cos(m\sin^{-1}x)$ ...(1)

differentiate (1) w.r.t. *x,* we get

$$\frac{(-2x)}{2\sqrt{1-x^2}}\frac{dy}{dx}+\sqrt{1-x^2}\frac{d^2y}{dx^2}=m^2(-\sin(m\sin^{-1}x))\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow -x\frac{dy}{dx}+(1-x^2)\frac{d^2y}{dx^2}=-m^2y$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

Example 18 :

Prove that $\dfrac{d^2x}{dy^2}\left(\dfrac{dy}{dx}\right)^3+\dfrac{d^2y}{dx^2}=0$

Solution:

$$\frac{dy}{dx}=\left(\frac{dx}{dy}\right)^{-1}$$

Differentiating w.r.t. *x,*

$$\frac{d^2y}{dx^2}=-\left(\frac{dx}{dy}\right)^{-2}\frac{d}{dx}\left(\frac{dx}{dy}\right)=-\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\cdot\frac{dy}{dx}=-\left(\frac{d^2x}{dy^2}\right)\left(\frac{dy}{dx}\right)^3$$

$$\Rightarrow \frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}=0$$

Example 19 :

Find $\dfrac{dy}{dx}$ *if* $y=e^{x^{e^{x^{e^{x\ldots\infty}}}}}$

Solution :

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Given function y can be expressed as $y = e^{x^y}$

Taking logarithm, we get $\log y = x^y$

Again taking log, gives, $\log(|\log y|) = y \log|x|$

differentiate w.r.t x, $\frac{1}{(\log y)} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log|x| + \frac{1}{x}y$

$$\Rightarrow \left(\frac{1}{y\log y} - \log|x|\right)\frac{dy}{dx} = \frac{y}{x} \quad \Rightarrow \frac{1 - y\log|x|\log y}{y\log y}\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2\log y}{x(1 - y\log y\log|x|)}$$

Example 20 :

If g is inverse of f and $f'(x) = \sin x$, prove that $g'(x) = \text{cosec}\,(g(x))$.

Solution :

It is given that g is inverse of f, then $g(x) = f^{-1}(x) \Rightarrow fg(x) = x$

Differentiate w.r.t. x, using chain rule we get $f'(g(x))\, g'(x) = 1$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sin(g(x))}$$

$$\Rightarrow g'(x) = \text{cosec}(g(x))$$

Example 21 :

If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ be polynomials of degree 3,4 and 5 respectively, then show that

$$\Delta(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$ *is divisible by $f(x)$, where ' ′ ' (dash) denotes the derivative.*

Solution :

Here α is a repeated root of equation $f(x) = 0$

$\therefore$ $f(\alpha) = 0$, $f'(\alpha) = 0$ and therefore $f(x)$ will be of the form: $f(x) = a_0\,(x - \alpha)^2$ where a_0 is constant.

Now $\Delta'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ (other two det are zero because 2nd and 3rd rows contain constant)

$$\therefore \Delta'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} = 0$$

Also $\Delta(\alpha) = 0$ and since $\Delta(x)$ is a polynomial in x, α is repeated root of $\Delta(x) = 0$. Therefore $\Delta(x) \equiv (x-\alpha)^2\,\phi(x)$

Therefore, $\Delta(x) = \frac{\phi(x)f(x)}{a_0} = f(x)\psi(x) \quad \left(\text{where } \psi(x) = \frac{\phi(x)}{a_0}\right)$

Hence, $\Delta(x)$ is divisible by $f(x)$.

Example 22 :

Find the derivative of the function $f(x)=(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *at* $x=\frac{\pi}{4}$.

Solution :

$(\log_{\sin x}\cos x)^{-1}=\log_{\cos x}\sin x$ and $\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x$, therefore $f(x)$ is expressed as :

$$f(x)=(\log_{\cos x}\sin x)^2+2\tan^{-1}x=\left(\frac{\log_e\sin x}{\log_e\cos x}\right)^2+2\tan^{-1}x \quad 0<x<\frac{\pi}{2}$$

Differentiate w.r.t. x, $f'(x)=2\left(\frac{\log_e\sin x}{\log_e\cos x}\right)\left\{\frac{\cot x\log_e\cos x+\tan x\log_e\sin x}{(\log_e\cos x)^2}\right\}+\frac{2}{1+x^2}$ at $x=\frac{\pi}{4}$,

$$f'\left(\frac{\pi}{4}\right)=2\left(\frac{\log_e\left(\frac{1}{\sqrt{2}}\right)}{\log_e\left(\frac{1}{\sqrt{2}}\right)}\right)\left\{\frac{\log_e\left(\frac{1}{\sqrt{2}}\right)+\log_e\left(\frac{1}{\sqrt{2}}\right)}{\left(\log_e\left(\frac{1}{\sqrt{2}}\right)\right)^2}\right\}+\frac{2}{1+\left(\frac{\pi}{4}\right)^2}$$

$$=\frac{4}{\left(\log_e\frac{1}{\sqrt{2}}\right)}+\frac{32}{16+\pi^2}=-\frac{8}{(\log_e 2)}+\frac{32}{16+\pi^2}=8\left(\frac{4}{16+\pi^2}-\frac{1}{\log_e 2}\right)$$

Example 23 :

If $y=\frac{1}{1+x^{n-m}+x^{p-m}}+\frac{1}{1+x^{m-n}+x^{p-n}}+\frac{1}{1+x^{m-p}+x^{n-p}}$, *prove that* $\frac{dy}{dx}=0$

Solution :

$$y=\frac{1}{1+\frac{x^n}{x^m}+\frac{x^p}{x^m}}+\frac{1}{1+\frac{x^m}{x^n}+\frac{x^p}{x^n}}+\frac{1}{1+\frac{x^m}{x^p}+\frac{x^n}{x^p}}$$

$$\Rightarrow y=\frac{x^m}{x^m+x^n+x^p}+\frac{x^n}{x^n+x^m+x^p}+\frac{x^p}{x^p+x^m+x^n} \quad \Rightarrow y=\frac{x^m+x^n+x^p}{x^m+x^n+x^p}=1$$

Now, differentiating w.r.t. x, $\frac{dy}{dx}=0$

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