Transform Calculus

(MA 20202)

Assignment-7

1. Solve the PDE using Fourier transform technique

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0$$

s.t. $u(x, 0) = f(x), -\infty < x < \infty$;
both u and $\frac{\partial u}{\partial x}$ vanish as $|x| \to \infty$;
and u is bounded as $y \to \infty$.

2. Solve the PDE using Fourier transform technique

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0$$

s.t. $\frac{\partial u}{\partial y}(x, 0) = g(x), -\infty < x < \infty$;
u is bounded $y \to \infty$;
both u and $\frac{\partial u}{\partial x}$ are bounded as $|x| \to \infty$.

3. Solve the PDE using Fourier transform technique

$$\nabla^2 \phi = 0, \quad y > 0$$

s.t. both $\phi(x, y)$ and $\frac{\partial \phi}{\partial x}(x, y) \to 0$ as $\sqrt{x^2 + y^2} \to \infty$;

$$\phi(x, y) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

4. Solve the following heat conduction problem using the Laplace transform technique. $u(x, t)$ denotes the temperature at the location x at any time t.

   (a) $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$

   subject to

   i. $u(x, 0) = 0, \ \forall x$
   ii. $u(0, t) = u_0, \ \forall t$

iii. u is finite $\forall x$ and $\forall t$.

(b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$

subject to

i. $u(x, 0) = 0,\ \forall x$

ii. $u(0, t) = 1,\ \forall t$

iii. $\lim_{x \to \infty} u(x, t) = 0, \forall t$.

5. Solve 1-D heat conduction problem given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

subject to

(a) $u(0, t) = 0,\ \forall t;$

(b) $u(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases};$

(c) $u(x, t)$ is bounded $\forall x$ and $\forall t$

using the Fourier sine transformation technique.

6. Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

subject to

(a) $\frac{\partial u}{\partial t}(0, t) = u_0, \forall t;$

(b) $u(x, 0) = 0, \forall x;$

(c) $u(x, t)$ is bounded $\forall x$ and $\forall t$

using the Fourier cosine transform technique.

7. Solve the 1-D heat conduction problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to

(a) $u(x, 0) = f(x), \forall x;$

(b) $u(x,t) \to 0$ as $|x| \to \infty$;

(c) $u(x,t)$ is bounded $\forall x$ and $\forall t$

using the Fourier transform technique.

Take $f(x) = \begin{cases} 0, & x < 1 \\ a, & x > 1 \end{cases}$ and obtain the particular solution.

8. Solve the 1-D wave propagation equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to

(a) $u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 0,\ \forall x$;

(b) $u(0,t) = f(t),\ \forall t$;

(c) $u(x,t)$ is bounded $\forall x$ and $\forall t$

using the Laplace transform technique.

9. Solve the 1-D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

subject to

(a) $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x), \forall x$;

(b) both $u(x,t)$ and $\frac{\partial u}{\partial x}(x,t) \to 0$ as $|x| \to \infty$

using the Fourier transform technique.