ENPM667 Fall 2020, Final Project

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1 First Component

For this project, we consider a crane modeled as a frictionless cart, moving in one-dimension along a track. There are two loads attached to the cart, $m1$ and $m2$, as seen in 1, at lengths of $l1$ and $l2$ respectively. An external force F is applied to the cart of mass M and creates a displacement x.

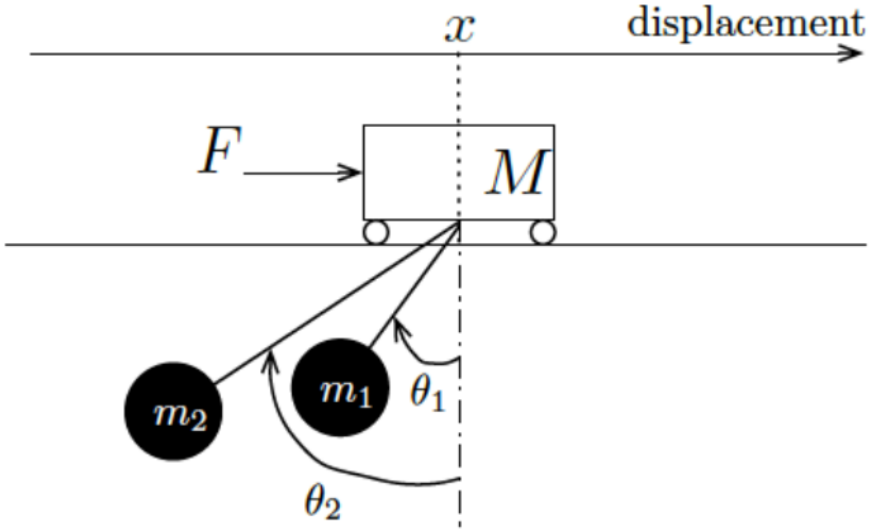

Figure 1: Given problem setup of crane (frictionless cart) with two loads attached.

1.1 Dynamics of the system

The equations are determined using the Euler Lagrange method.
The position vectors of individual elements are

$$\text{Position of M} = P_M = \begin{bmatrix} x & 0 \end{bmatrix}^T \tag{1.1}$$

$$\text{Position of } m_1 = P_{m_1} = \begin{bmatrix} x - l_1 sin\theta_1 & -l_1 cos\theta_1 \end{bmatrix}^T \tag{1.2}$$

$$\text{Position of } m_2 = P_{m_2} = \begin{bmatrix} x - l_2 sin\theta_2 & -l_2 cos\theta_2 \end{bmatrix}^T \tag{1.3}$$

The velocities are obtained by differentiating the above vectors with respect to time.

$$\text{Velocity of M} = V_M = \begin{bmatrix} \dot{x} & 0 \end{bmatrix}^T \tag{1.4}$$

$$\text{Velocity of } m_1 = V_{m_1} = \begin{bmatrix} \dot{x} - l_1 \dot{\theta}_1 cos\theta_1 & l_1 \dot{\theta}_1 sin\theta_1 \end{bmatrix}^T \tag{1.5}$$

$$\text{Velocity of } m_2 = V_{m_2} = \begin{bmatrix} \dot{x} - l_2 \dot{\theta}_2 cos\theta_2 & l_2 \dot{\theta}_2 sin\theta_2 \end{bmatrix}^T \tag{1.6}$$

The kinetic and potential energies of the individual elements are given by

$$\text{Kinetic Energy of M} = K.E_1 = \frac{1}{2}M||V_M||^2 = \frac{1}{2}M\dot{x}^2 \tag{1.7}$$

$$\text{Kinetic Energy of } m_1 = K.E_2 = \frac{1}{2}m_1||V_{m_1}||^2 = \frac{1}{2}m_1(\dot{x}^2 + l_1^2\dot{\theta}_1{}^2 - 2l_1\dot{x}\dot{\theta}_1 cos\theta_1) \tag{1.8}$$

$$\text{Kinetic Energy of } m_2 = K.E_3 = \frac{1}{2}m_2||V_{m_2}||^2 = \frac{1}{2}m_2(\dot{x}^2 + l_2^2\dot{\theta}_2{}^2 - 2l_2\dot{x}\dot{\theta}_2 cos\theta_2) \tag{1.9}$$

$$\text{Potential Energy of M} = P.E_1 = MgP_{My} = 0 \tag{1.10}$$

$$\text{Potential Energy of } m_1 = P.E_2 = m_1 g P_{m_1 y} = -m_1 g l_1 cos\theta_1 \tag{1.11}$$

$$\text{Potential Energy of } m_2 = P.E_3 = m_2 g P_{m_2 y} = -m_2 g l_2 cos\theta_2 \tag{1.12}$$

The Lagrangian is calculated from the total energies:

$$\mathcal{L} = K.E_1 + K.E_2 + K.E_3 - (P.E_1 + P.E_2 + P.E_3)$$

By substituting the kinetic and potential energies in the above formula and simplifying, we get

$$\mathcal{L}(x, \theta_1, \theta_2) = \frac{1}{2}[\dot{x}^2(M + m_1 + m_2) + m_1 l_1^2 \dot{\theta}_1{}^2 + m_1 l_2^2 \dot{\theta}_2{}^2] - m_1 l_1 \dot{x}\dot{\theta}_1 cos\theta_1 - m_2 l_2 \dot{x}\dot{\theta}_2 cos\theta_2 \tag{1.13}$$

$$+ m_1 g l_1 cos\theta_1 + m_2 g l_2 cos\theta_2 \tag{1.14}$$

Since the generalized coordinates are chosen to be x, θ_1, θ_2, we first get the equation of motion for the cart based on x.

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x}(M + m_1 + m_2) - m_1 l_1 \dot{\theta}_1 cos\theta_1 - m_2 l_2 \dot{\theta}_2 cos\theta_2 \tag{1.15}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \ddot{x}(M + m_1 + m_2) - m_1 l_1(\ddot{\theta}_1 cos\theta_1 - sin\theta_1) \quad [1.16]$$

$$\frac{\partial \mathcal{L}}{\partial x1} = 0 \quad [1.17]$$

$$F_1 = (M + m_1 + m_2)\ddot{x} - m_1 l_1(\ddot{\theta}_1 cos\theta_1 - \dot{\theta}_1^{\ 2} sin\theta_1) - m_2 l_2(\ddot{\theta}_2 cos\theta_2 - \dot{\theta}_2^{\ 2} sin\theta_2) \quad [1.18]$$

First we calculate the equation of motion based on θ_1:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 \dot{x} cos\theta_1 \quad [1.19]$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1(\ddot{x} cos\theta_1 - \dot{x}\dot{\theta}_1 sin\theta_1) \quad [1.20]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = m_1 l_1 \dot{x}\dot{\theta}_1 sin\theta_1 - m_1 g l_1 sin\theta_1 \quad [1.21]$$

$$F_2 = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} cos\theta_1 + m_1 g l_1 sin\theta_1 \quad [1.22]$$

Finally, calculating equation of motion based on θ_2:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_2 \dot{x} cos\theta_2 \quad [1.23]$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2(\ddot{x} cos\theta_2 - \dot{x}\dot{\theta}_2 sin\theta_2) \quad [1.24]$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_2 \dot{x}\dot{\theta}_2 sin\theta_2 - m_2 g l_2 sin\theta_2 \quad [1.25]$$

$$F_3 = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} cos\theta_2 + m_2 g l_2 sin\theta_2 \quad [1.26]$$

The forces acting on the system are $F_1 = F, F_2 = F_3 = 0$. Thus, the nonlinear equations of motion are given by

$$F = (M + m_1 + m_2)\ddot{x} - m_1 l_1(\ddot{\theta}_1 cos\theta_1 - \dot{\theta}_1^{\ 2} sin\theta_1) - m_2 l_2(\ddot{\theta}_2 cos\theta_2 - \dot{\theta}_2^{\ 2} sin\theta_2) \quad [1.27]$$

$$0 = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} cos\theta_1 + m_1 g l_1 sin\theta_1 \quad [1.28]$$

$$0 = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \ddot{x} cos\theta_2 + m_2 g l_2 sin\theta_2 \quad [1.29]$$

By rearranging, the equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are given by

$$\ddot{\theta}_1 = \frac{1}{l_1}(\ddot{x}cos\theta_1 - gsin\theta_1) \tag{1.30}$$

$$\ddot{\theta}_2 = \frac{1}{l_2}(\ddot{x}cos\theta_2 - gsin\theta_2) \tag{1.31}$$

By substituting in the above equations, $\ddot{x}$ is given by

$$\ddot{x} = \frac{1}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2} N \tag{1.32}$$

where

$$N = F - g(m_1 sin\theta_1 cos\theta_1 + m_2 sin\theta_2 cos\theta_2) - m_1 l_1 sin\theta_1 \dot{\theta_1}^2 - m_2 l_2 sin\theta_2 \dot{\theta_2}^2$$

By using the above equation of $\ddot{x}$, θ_1 and θ_2 can be written as

$$\ddot{\theta}_1 = \frac{1}{l_1}[-gsin\theta_1 + \frac{Ncos\theta_1}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2}] \tag{1.33}$$

$$\ddot{\theta}_2 = \frac{1}{l_2}[-gsin\theta_2 + \frac{Ncos\theta_2}{M + m_1 sin^2\theta_1 + m_2 sin^2\theta_2}] \tag{1.34}$$

The values of $\ddot{x}$, $\ddot{\theta}_1$, $\ddot{\theta}_2$ from equations 1.1, 1.33 and 1.34 respectively constitute the nonlinear state-space system.

1.2 Linearization of the system

The linearization method used for this system is small angle approximation. We assume that the angles θ_1 and θ_2 are very small (approaching zero), and thus higher powers and orders of differentiation of θ_1 and θ_2 are neglected. This results in the following:

$$sin\theta_1 = \theta_1, sin\theta_2 = \theta_2$$
$$cos\theta_1 = cos\theta_2 = 1$$

By applying these results to 1.1,1.33 and 1.34, we get the linearized equations of motion.

$$\ddot{x} = \frac{1}{M}[F - g(m_1\theta_1 + m_2\theta_2)] \tag{1.35}$$

$$\ddot{\theta}_1 = \frac{1}{l_1}[\frac{F - g(m_1\theta_1 + m_2\theta_2)}{M} - g\theta_1] \tag{1.36}$$

$$\ddot{\theta}_2 = \frac{1}{l_2}[\frac{F - g(m_1\theta_1 + m_2\theta_2)}{M} - g\theta_2] \tag{1.37}$$

Using Eq 1.35,1.36 and 1.37, as well as the given equilibrium point, $x = 0, \theta_1 = 0, \theta_2 = 0$, the linearized state space representation of the system can be written as

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g\frac{m_1}{M} & 0 & -g\frac{m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{g}{l_1}(1+\frac{m_1}{M}) & 0 & -\frac{g}{l_1}\frac{m_2}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g}{l_1}\frac{m_1}{M} & 0 & -\frac{g}{l_2}(1+\frac{m_2}{M}) & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{l_1 M} \\ 0 \\ \frac{1}{l_2 M} \end{bmatrix} u(t) \qquad [1.38]$$

where $u(t)$ is the input to the system, i.e. the external force F.

The resulting linearized system is then

$$\dot{X}(t) = AX(t) + Bu(t) \qquad [1.39]$$

where the state is

$$X(t) = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \qquad [1.40]$$

1.3 Controllability of the linearized system

For analyzing the controllability of the system, we will use Popov Belevitch Hautus (PBH) test. We examine the matrix

$$[(\lambda I - A)|B] = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & g\frac{m_1}{M} & 0 & g\frac{m_2}{M} & 0 & \frac{1}{M} \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{g}{l_1}(1+\frac{m_1}{M}) & \lambda & \frac{g}{l_1}\frac{m_2}{M} & 0 & \frac{1}{l_1 M} \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & \frac{g}{l_1}\frac{m_1}{M} & 0 & \frac{g}{l_2}(1+\frac{m_2}{M}) & \lambda & \frac{1}{l_2 M} \end{bmatrix}$$

This matrix is full rank $\forall \lambda \in C$, so the system is controllable. Also, we know that $M, m_1, m_2, l_1, l_2 > 0$, $M >> m_1, m_2$, and $g \approx 10$.

We can also examine the controllability matrix

$$co = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \qquad [1.41]$$

```
co =

[        0,       1/M,                                                 0,           -(10*(l1*m2 + 
[      1/M,         0,            -(10*(l1*m2 + l2*m1))/(M^2*l1*l2),
[        0, 1/(M*l1),                                                 0, -(10*(M*l2 + l1*m2 + l2
[1/(M*l1),          0, -(10*(M*l2 + l1*m2 + l2*m1))/(M^2*l1^2*l2),
[        0, 1/(M*l2),                                                 0, -(10*(M*l1 + l1*m2 + l2
[1/(M*l2),          0, -(10*(M*l1 + l1*m2 + l2*m1))/(M^2*l1*l2^2),
```

We know that $|co| \neq 0$ for a full rank matrix and correspondingly a completely controllable system. Thus:

$$-\frac{(g^6 \cdot (l_1 - l_2)^2)}{(M^6 \cdot l_1^6 \cdot l_2^6)} \neq 0 \tag{1.42}$$

From solving the previous equation, we result in one further condition for complete controllability:

$$l_1 \neq l_2 \tag{1.43}$$

1.4 LQR Controller

Next, we select parameter values for the system: $M = 1000Kg$, $m_1 = m_2 = 100Kg$, $l_1 = 20m$, $l_2 = 10m$. After substituting the given values, as well as $g \approx 10$, the linear system can be written as

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{11}{20} & 0 & -\frac{1}{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{10} & 0 & -\frac{11}{l0} & 0 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20000} \\ 0 \\ \frac{1}{10000} \end{bmatrix} u(t) \tag{1.44}$$

We can check the system stability via the eigenvalues:

$$eig(A) = \begin{bmatrix} 0.0000 + 0.0000i \\ 0.0000 + 0.0000i \\ 0.0000 + 0.7356i \\ 0.0000 - 0.7356i \\ 0.0000 + 1.0531i \\ 0.0000 - 1.0531i \end{bmatrix} \tag{1.45}$$

Every eigenvalue has 0 real part, so the system is only locally stable about the equilibrium point.

To examine the controllability of this system, we first calculate the controllability matrix:

$$\begin{aligned} co &= \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \\ &= 1e-03 \cdot \begin{bmatrix} 0 & 1 & 0 & -0.15 & 0 & 0.1475 \\ 1 & 0 & -0.15 & 0 & 0.1475 & 0 \\ 0 & 0.05 & 0 & -0.0325 & 0 & 0.0236 \\ 0.05 & 0 & -0.0325 & 0 & 0.0236 & 0 \\ 0 & 0.1 & 0 & -0.115 & 0 & 0.1298 \\ 0.1 & 0 & -0.115 & 0 & 0.1298 & 0 \end{bmatrix} \end{aligned} \tag{1.46}$$

The controllability matrix appears to be full rank, however the values are very small so the result may not be reliable. We then apply the PBH test again:

$$[(\lambda I - A)|B] = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 1 & 0 & \frac{1}{1000} \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{11}{20} & \lambda & \frac{1}{20} & 0 & \frac{1}{20000} \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{11}{l0} & \lambda & \frac{1}{10000} \end{bmatrix}$$

This matrix is full rank $\forall\lambda \in C$, so we can conclude that the system is indeed controllable.

We next simulate the nonlinear system and choose parameters for LQR Control of the linearized system. We must find parameters to satisfy our cost of form

$$J = \int_0^\infty (X^T Q X + u^T R u) dt \qquad [1.47]$$

To solve this, we use the algebraic Riccati equation:

$$P \cdot A + A^T P - PBR^{-1}B^T P = -Q \qquad [1.48]$$

and solve for P. Then, we can formulate the state feedback equation

$$u(t) = -KX(t) = -(R^{-1}B^T P)X(t) \qquad [1.49]$$

We start with $Q = C^T C$, where C comes from our output equation:

$$Y(t) = CX(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} X(t) = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} \qquad [1.50]$$

and $R = 1$. This yields

$$K = \begin{bmatrix} 1.0000 & 48.9849 & -1.5587 & -78.9411 & -0.7885 & -39.7166 \end{bmatrix} \qquad [1.51]$$

However, when we examine the stability of the system using Lyapunov's indirect method, we obtain:

$$eig(A - BK) = \begin{bmatrix} -0.0001 + 1.0430i \\ -0.0001 - 1.0430i \\ -0.0204 + 0.0204i \\ -0.0204 - 0.0204i \\ -0.0001 + 0.7285i \\ -0.0001 - 0.7285i \end{bmatrix} \qquad [1.52]$$

which do all have negative real parts, so the system is stable. However, several of the eigenvalues have real parts very close to zero, which may cause regions of marginal stability.

We can see this in the visualization in Figure 2, where the simulated system runs for 30 seconds with a constant external force $F = 5N$. The cart position x is shown in green before applying LQR and blue after, and similarly the load angles are in cyan before LQR and red after.

We see that the load angles and thus the loads are oscillating, which is undesired behavior; however, the preliminary LQR parameters have very slightly affected this issue. We can increase the gain on Q to attempt to stabilize the system.

In Figure 3, we have tuned the parameters and expanded the time window to $t = 300s = 5min$. A force of 5N is applied as before, but the cost factor on the input is changed to $R = 0.1$ to minimize the oscillations in angle. As before, we start with $Q = C^T C$ and then adjust the parameters relating to x and θ:

$$Q = \begin{bmatrix} 5000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad [1.53]$$

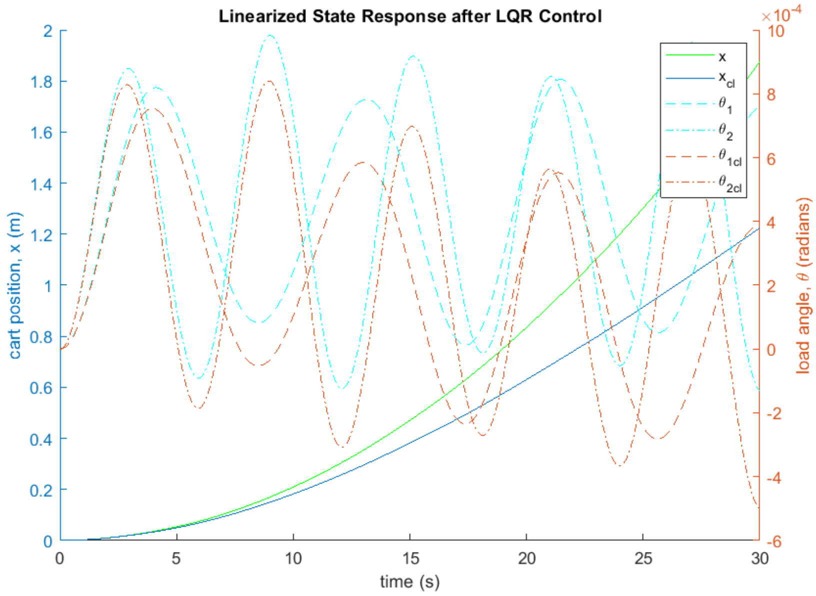

Figure 2: Simulation of original system vs the system after first attempt at LQR control, where the states after applying the feedback are noted with $_{cl}$.

We can see that the oscillations in θ_1, θ_2 decrease to an asymptote with this adjustment of the LQR parameters. The resulting gain matrix is

$$K = \begin{bmatrix} 223.6068 & 723.9204 & -169.9189 & -789.2091 & -106.3756 & -449.7062 \end{bmatrix} \quad [1.54]$$

Finally, we check the stability of the feedback system again:

$$eig(A - BK) = \begin{bmatrix} -0.3000 + 0.3138i \\ -0.3000 - 0.3138i \\ -0.0111 + 1.0409i \\ -0.0111 - 1.0409i \\ -0.0087 + 0.7259i \\ -0.0087 - 0.7259i \end{bmatrix} \quad [1.55]$$

These eigenvalues are also all in the negative left half plane, so the feedback system is stable. Additionally, the real parts are farther from 0, which removes the question of marginal stability.

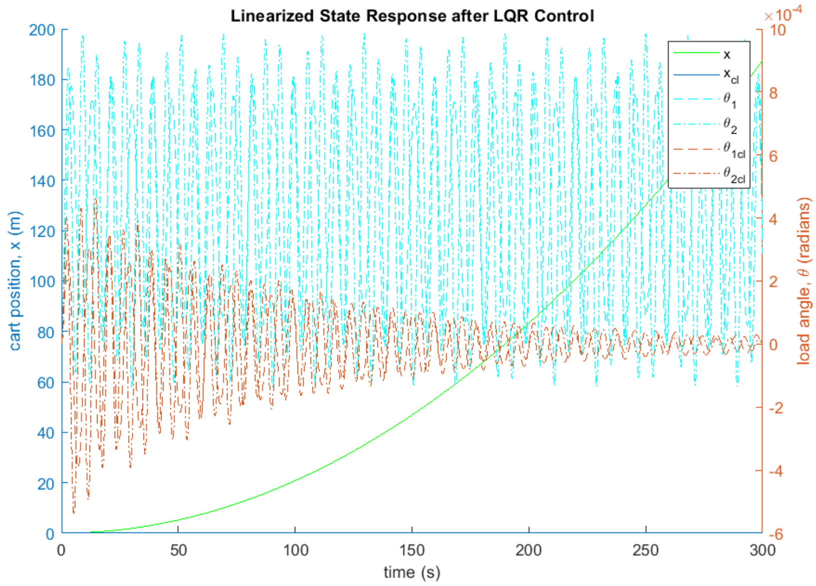

Figure 3: Simulation of feedback system after LQR control with tuned parameters.

2 Second Component

Using the previously given parameters, $M = 1000Kg$, $m_1 = m_2 = 100Kg$, $l_1 = 20m$, $l_2 = 10m$, we now examine the observability of the system and construct an observer for output feedback control. Note that parameters from Part D are used; although the assignment references Part C, no specific parameters were generated there, only conditions on the variables.

2.1 Observability of the system

Our original system used the complete output x, θ_1, θ_2, but for analysis we examine the observability of other combinations of output states as well. These can be found in Table 1.

Output Vector	C matrix	Rank of Observability matrix
x	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	**6**
θ_1, θ_2	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	4
x, θ_2	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	**6**
x, θ_1, θ_2	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	**6**

Table 1: Observability analysis for given output vectors.

In order to obtain these results, we calculated the observability matrix for each C and observed whether it was full rank or not for complete observability. In this case, the observability matrix is given by

$$ob = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix} \quad [2.1]$$

```
%%  Part 2 - Observability
% Check observability of other output vectors
% y=x
C1 = [1 0 0 0 0 0;
      0 0 0 0 0 0;
      0 0 0 0 0 0];
ob = obsv(A,C1);
observability = rank(ob)
 % y=theta1,theta2
C2 = [0 0 0 0 0 0;
      0 0 1 0 0 0;
      0 0 0 0 1 0];
ob = obsv(A,C2);
observability = rank(ob)
 % y=x,theta2
C3 = [1 0 0 0 0 0;
      0 0 0 0 0 0;
      0 0 0 0 1 0];
ob = obsv(A,C3);
observability = rank(ob)

% Check observability of original system
% y=x,theta1,theta2
ob = obsv(sys_ss);
observability = rank(ob)
```

2.2 Luenberger Observer

We now construct a state observer to estimate the state externally. To do this, we construct the state of the observer as

$$\dot{\hat{X}} = A\hat{X} + Bu + L(Y - \hat{Y}) \quad [2.2]$$

Then, the error $e = X - \hat{X}$ in the observed state is

$$\begin{aligned} \dot{e} = \dot{X} - \dot{\hat{X}} &= (AX + Bu) - (A\hat{X} + Bu + L(Y - \hat{Y})) \\ &= AX - A\hat{X} - L(CX - C\hat{X}) \\ &= (A - LC)e \end{aligned} \quad [2.3]$$

Now, we can examine the estimator state $A_e = A - LC$. We want the observer's estimate to converge on the state faster than the state changes, which means the poles of A_e must be faster than the original system's by some factor. We first check the eigenvalues of the original system from Eq 1.45, however, these have all zero real parts. Therefore, to have the observer converge

5-10 times faster than the state, we can arbitrarily choose poles with negative real parts. Here, we use pole placement of $[-1, -2, -3, -4, -5, -6]$ to construct L.

We can now simulate the closed-loop observer system.

$$\begin{bmatrix} \dot{X} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & (A - LC) \end{bmatrix} \begin{bmatrix} X \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \tag{2.4}$$

$$Y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} X \\ e \end{bmatrix} \tag{2.5}$$

2.2.1 Output x

We use the C matrix shown on line 1 of Table 1 and the pole placement as above. Then, our observer gain is

$$L = 1.0e + 03 \cdot \begin{bmatrix} 0.0210 & 0 & 0 \\ 0.1733 & 0 & 0 \\ -2.8024 & 0 & 0 \\ 0.1054 & 0 & 0 \\ 2.1021 & 0 & 0 \\ -1.4428 & 0 & 0 \end{bmatrix} \tag{2.6}$$

After simulation, this yields the error response shown in Figure 4 which was given initial conditions of $[1, 1, \pi, \pi, \pi, \pi]$.

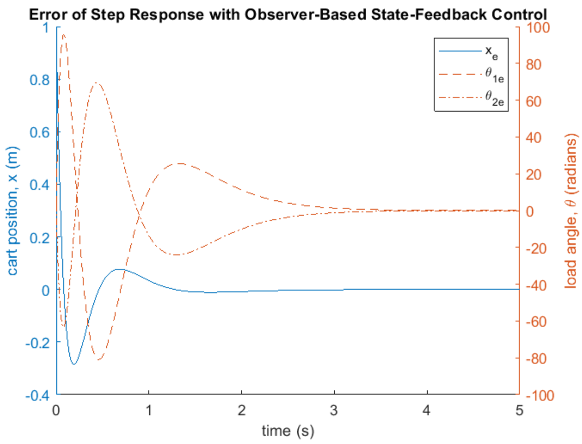

Figure 4: Error Response for Output x.

2.2.2 Output x, θ_2

We use the C matrix shown on line 3 of Table 1 and the pole placement as above. Then, our observer gain is

$$L = \begin{bmatrix} 13.0749 & 0 & -0.8281 \\ 56.2522 & 0 & -8.5387 \\ -87.2678 & 0 & 19.5049 \\ -19.0634 & 0 & 10.7358 \\ 0.3521 & 0 & 7.9251 \\ 3.4788 & 0 & 13.1865 \end{bmatrix} \tag{2.7}$$

After simulation, this yields the error response shown in Figure 5 which was given initial conditions of $[1, 1, \pi, \pi, \pi, \pi]$.

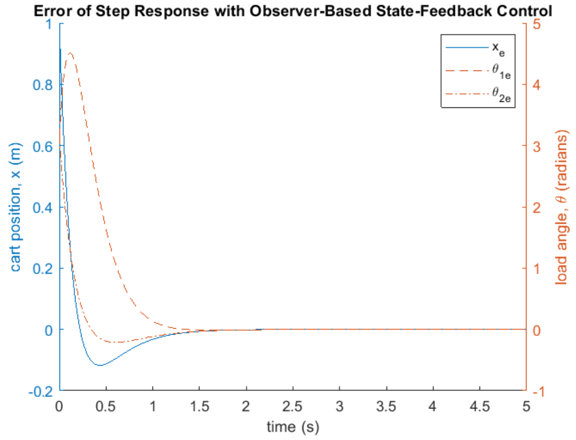

Figure 5: Error Response for Output x, θ_2.

2.2.3 Output x, θ_1, θ_2

We use the C matrix shown on line 4 of Table 1 and the pole placement as above. Then, our observer gain is

$$L = \begin{bmatrix} 8.5631 & -0.8851 & 0.0000 \\ 17.5219 & -4.9674 & -1.0000 \\ -0.9140 & 9.4369 & -0.0000 \\ -4.1173 & 20.9280 & -0.0501 \\ 0.0000 & -0.0000 & 3.0000 \\ 0.0000 & -0.1000 & 0.9000 \end{bmatrix} \qquad [2.8]$$

After simulation, this yields the error response shown in Figure 6 which was given initial conditions of $[1, 1, \pi, \pi, \pi, \pi]$.

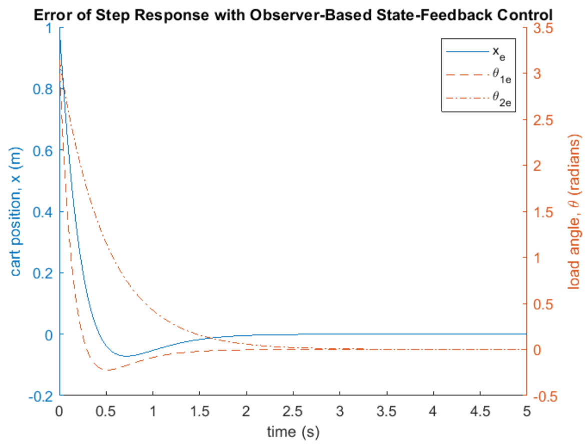

Figure 6: Error Response for Output x, θ_1, θ_2.

2.3 Output feedback controller

We use output of x for this section as our smallest observable output vector. The cost for the LQG controller is given by

$$J = E\{\lim_{t \to \infty} \frac{1}{\tau} \int_0^{\tau} \begin{bmatrix} x^T & u^T \end{bmatrix} Q_{xu} \begin{bmatrix} x \\ u \end{bmatrix} dt\} \qquad [2.9]$$

where there are noise elements, w, v added to our system equations:

$$\dot{X} = AX + Bu + w \qquad [2.10]$$

$$\dot{Y} = CX + v \qquad [2.11]$$

The second gain matrix is then

$$Q_{wv} = E(\begin{bmatrix} w \\ v \end{bmatrix} \begin{bmatrix} w' & v' \end{bmatrix}) \qquad [2.12]$$

With a naive selection of identity matrix for Q and randomly generated Gaussian white noise, we can obtain the feedback system shown in Figure 7.

```
KLQG =

  A =
                        x_e       x_dot_e      theta1_e  theta1_dot_e      theta2_e  theta2_dot_e
   x_e               -3.211             1             0             0             0             0
   x_dot_e            -8.58        -0.049       -0.9985       0.07829       -0.9992        0.0392
   theta1_e           10.31             0             0             1             0             0
   theta1_dot_e       1.237      -0.00245       -0.5499      0.003914      -0.04996       0.00196
   theta2_e           1.379             0             0             0             0             1
   theta2_dot_e       5.211       -0.0049      -0.09985      0.007829          -1.1       0.00392

  B =
                   x(1)     x(2)     x(3)
   x_e            3.211  -0.5768   -1.326
   x_dot_e        8.579   -1.237    0.393
   theta1_e      -10.31   0.5593    4.929
   theta1_dot_e  -1.237   -0.321   -2.667
   theta2_e      -1.379    1.011    2.351
   theta2_dot_e  -5.211   0.5226    2.839

  C =
             x_e       x_dot_e      theta1_e  theta1_dot_e      theta2_e  theta2_dot_e
   F          -1           -49         1.533         78.29        0.7674          39.2

  D =
       x(1)  x(2)  x(3)
   F      0     0     0
```

Figure 7: Feedback system parameters derived from LQG Controller.

We can now observe the output from the LQG system and the corresponding response from the original system. These are shown in Figure 8.

In future work, the gain matrices should be iterated on to obtain the desired response.

For the problem of tracking a constant reference, we assume $x^{\cdot}$ and $u^{\cdot}$, which are asymptotic variables which reach the given reference value for time $t \longrightarrow \infty$. So,

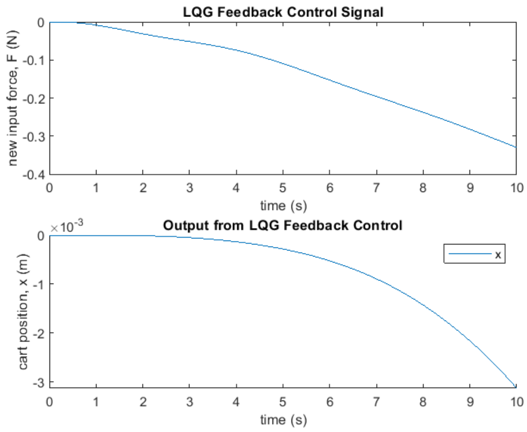

Figure 8: Top: Output force obtained from LQG controller; Bottom: output position of cart, x.

$$x^{\cdot} = \lim t \longrightarrow \infty x \quad [2.13]$$

$$u^{\cdot} = \lim t \longrightarrow \infty u \quad [2.14]$$

These asymptotic variables should satisfy the equation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x^{\cdot} \\ u^{\cdot} \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} y_d \quad [2.15]$$

where y_d is the desired output of the system.