1. *Solution.* Set $g(x) = f(x) - 5$. Since a, b, c, d are all roots of $g(x)$, we must have

$$g(x) = (x-a)(x-b)(x-c)(x-d)\,h(x)$$

for some $h(x) \in \mathbb{Z}[x]$. Let k be an integer such that $f(k) = 8$, giving $g(k) = f(k) - 5 = 3$. Using the factorization above, we find that

$$3 = (k-a)(k-b)(k-c)(k-d)\,h(x).$$

By the Fundamental Theorem of Arithmetic, we can only express 3 as the product of at most three distinct integers $(-3, 1, -1)$. Since $k-a, k-b, k-c, k-d$ are all distinct integers, we have too many terms in the product, leading to a contradiction. $\square$

2. *Solution.* Assume for the sake of contradiction that three such distinct primes exist, and let these primes be $\sqrt[3]{p_1}, \sqrt[3]{p_2}, \sqrt[3]{p_3}$. By definition of an arithmetic sequence, set

$$\sqrt[3]{p_1} = a,\ \sqrt[3]{p_2} = a + kd,\ \sqrt[3]{p_3} = a + md, m > k.$$

Subtracting gives:

$$\sqrt[3]{p_2} - \sqrt[3]{p_1} = kd$$
$$\sqrt[3]{p_3} - \sqrt[3]{p_1} = md.$$

Multiply the first equation by m and the second by k in order to equate the two

$$m\sqrt[3]{p_2} - m\sqrt[3]{p_1} = k\sqrt[3]{p_3} - k\sqrt[3]{p_1} = mkd.$$

Rearraging this equation, we get

$$m\sqrt[3]{p_2} - k\sqrt[3]{p_3} = (m-k)\sqrt[3]{p_1}. \tag{1.1}$$

Now, cubing this gives Using this equation and some rearrangement, we get:

$$m^3 p_2 - 3\left(m^2 p_2^{\frac{2}{3}}\right)\left(k p_3^{\frac{1}{3}}\right) + 3\left(m p_2^{\frac{1}{3}}\right)\left(k^2 p_3^{\frac{2}{3}}\right) - k^3 p_3 = (m-k)^3 p_1.$$

Moving the integer terms over to the RHS and factoring out $3\left(m p_2^{\frac{1}{3}}\right)\left(k p_3^{\frac{1}{3}}\right)$ from the LHS gives

$$\left[3\left(m p_2^{\frac{1}{3}}\right)\left(k p_3^{\frac{1}{3}}\right)\right]\left(k p_3^{\frac{1}{3}} - m p_2^{\frac{1}{3}}\right) = (m-k)^3 p_1 - m^3 p_2 + k^3 p_3.$$

From Equation 1.1, we know that $kp_3^{\frac{1}{3}} - mp_2^{\frac{1}{3}} = (k-m)\sqrt[3]{p_1}$. Therefore, substituting this into the above equation gives

$$3\left(mp_2^{\frac{1}{3}}\right)\left(kp_3^{\frac{1}{3}}\right)\left((k-m)\sqrt[3]{p_1}\right) = (m-k)^3 p_1 - m^3 p_2 + k^3 p_3.$$

Leaving only the cube roots on the left hand side gives

$$\sqrt[3]{p_1 p_2 p_3} = \frac{(m-k)^3 p_1 - m^3 p_2 + k^3 p_3}{3mk(k-m)}. \tag{1.2}$$

Lemma. If $\sqrt[3]{a}$ is rational, then $\sqrt[3]{a}$ is an integer.

Proof. Set $\sqrt[3]{a} = \frac{x}{y}$ where the fraction is in lowest terms, therefore $\gcd(x, y) = 1$. We desire to show that we must have the denominator, y, equal to 1. Cubing this equation and rearranging gives $ay^3 = x^3$. Assume for the sake of contradiction that y has a prime divisor, say p. If $p \mid y$ then we must also have $p \mid x$ from $ay^3 = x^3$. However, this contradicts $\gcd(x, y) = 1$. Therefore, it is impossible for y to have any prime divisors, and we must have $y = 1$, implying that $\sqrt[3]{a}$ is an integer. □

Since the RHS of Equation 1.2 is a rational number, the Lemma above implies that $\sqrt[3]{p_1 p_2 p_3}$ must be an integer. From the Fundamental Theorem of Arithmetic, this means that we must have $p_1 = p_2 = p_3$, contradiction.

□

3. p, p^2+2 are primes

$\Rightarrow p \neq 2.$

For $p = 3$, $3^2+2 = 11$ which is a prime.

Let $p \equiv \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \bmod 3$

$\Rightarrow p^2+2 \equiv \begin{Bmatrix} 1^2+2 \\ 2^2+2 \end{Bmatrix} \equiv \begin{Bmatrix} 3 \\ 6 \end{Bmatrix} \equiv \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \bmod 3$

$\Rightarrow$ For any prime $p \neq 3$, p^2+2 isn't a prime ∎

4. Someone post it in the channel, wasn't that tough!

5. Induction!

6. $M_{17} = 2^{17}-1 = 131071$. There's no elegant reason why M_{17} IS A PRIME. $\sqrt{131071} \approx 362$. Then you'll have to divide it by all the primes below 362 which is just sad :(Don't worry if you didn't do the problem.

7. Trivial!