ASSIGNMENT 2

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1.

2. Let the primes be a,b and c. Let x = $a^{1/3}$, y = $b^{1/3}$ and z = $c^{1/3}$. For any p,q $\in \mathbb{Z}$, x - y =pd & y - z = qd, where d is difference of the AP.
   Dividing both the equations we get, q(x - y) = p(y - z)
   $\implies$ qx + pz = y(p + q)
   $\implies q^3x^3 + p^3z^3 + 3pqzx(pq + zx) = y^3(p + q)^3$
   $\implies q^3a + p^3c + 3pqzx(pq + zx) = b(p + q)^3$
   $\implies 3pqzx(pq + zx) = b(p + q)^3 - q^3a + p^3c$
   Clearly, LHS is irrational as cube root of a prime number is irrational and RHS is rational as all the numbers are rational, therefore they cannot be different terms of an AP.

3. We know that each number is either one of these forms, 3q, 3q + 1 or 3q + 2, where q $\in \mathbb{Z}$. If p =3q and p is prime then the only possibility is q = 1.
   When p = 3, $p^2 + 2 = 11$ which is a prime.
   Now, let p = 3q + 1, so $p^2 + 2 = 9q^2 + 6q + 1 + 2 = 3(3q^2 + 2q + 1)$ and hence $p^2 + 2$ is not a prime.
   Now, let p = 3q + 2, so $p^2 + 2 = 9q^2 + 12q + 4 + 2 = 3(3q^2 + 4q + 2)$ and hence $p^2 + 2$ is not a prime in this case as well.
   So, the only solution is p = 3.

4. Let p be not a prime, then there exists an integer q such that 1 < q < p and q divides p.
   As p divides (p-1)! + 1, so q also must divide it. But q divides (p-1)! as q $\in \{2, 3, ..., p - 1\}$ so q must divide 1, which is impossible.
   Therefore, p is a prime.

5. We know that $F_n = 2^{2^n} + 1$,
   $F_0 = 3 = 5 - 2 = F_1 - 2$, so the given relation is true for n = 1. Let it be true for some n $\in \mathbb{N}$.
   $F_0F_1..F_{n-1} = F_n - 2$
   $\implies F_0F_1...F_{n-1}F_n = F_n^2 - 2F_n$
   $= (2^{2^n} + 1)(2^{2^n} - 1)$
   $= 2^{2^{n+1}} - 1$
   $= F_{n+1} - 2$. So it is true for n + 1 also.
   Therefore, by induction it is true for each such n.

6.

7.