Numbers Made Dumber

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Assignment 2

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1. Given the polynomial

   $$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

   with integer coefficients $a_1, a_2, \ldots, a_n$ and given also that there exist four distinct integers a, b, c, d such that

   $$f(a) = f(b) = f(c) = f(d) = 5$$

   show that there is no integer k such that $f(k) = 8$.

2. Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression.

3. For which prime p is $p^2 + 2$ also prime?

4. Show that if $p > 1$ and p divides $(p-1)! + 1$, then p is prime.

5. Show that $F_0 F_1 \ldots F_{n-1} = F_n - 2$ for all $n \geq 1$, where F_i is the i-th fermat number.

6. Evaluate the Mersenne number M_{17}, and determine whether it is prime.

7. Are the following statements true or false, where a and b are positive integers and p is prime? In each case, give a proof or counterexample.

   - If $gcd(a, p^2) = p$ then $gcd(a^2, p^2) = p^2$.
   - If $gcd(a, p^2) = p$ and $gcd(b, p^2) = p^2$ then $gcd(ab, p^4) = p^3$.
   - If $gcd(a, p^2) = p$ and $gcd(b, p^2) = p$ then $gcd(ab, p^4) = p^2$.
   - If $gcd(a, p^2) = p$ then $gcd(a + p, p^2) = p$.