Assignment 1

05-04-2021

1. Prove that $\frac{21n+4}{14n+3}$ is irreducible for every natural n.
2. Find all integers n such that $n^2 + 2n + 2$ divides $n^3 + 4n^2 + 4n - 14$.
3. For natural numbers a, n, m, prove that $\gcd(a^m - 1, a^n - 1) = a^{\gcd(m,n)} - 1$.
4. Let the integers a_n and b_n be defined by the relationship
$$a_n + b_n\sqrt{2} = (1 + \sqrt{2})^n$$
for all integers $n \geq 1$. Prove that $\gcd(a_n, b_n) = 1$ for all integers $n \geq 1$.
5. If p is an odd prime, and a, b are relatively prime positive integers, prove that
$$\gcd(a + b, \frac{a^p + b^p}{a + b}) = 1 \textit{ or } p.$$
6. If $a|bc$ and $\gcd(a, b) = 1$, prove that $a|c$.
7. Prove that the expression
$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$
is an integer for all pairs of integer $n \geq m \geq 1$.
8. Let $n, p > 1$ be positive integers and p be a prime. Given that $n|p - 1$ and $p|n^3 - 1$, prove that $4p - 3$ is a perfect square.
9. Find all pairs of positive integers a, b such that
$$\frac{a^2 + b}{b^2 - a} \text{ and } \frac{b^2 + a}{a^2 - b}$$
are both integers.
10. For $m > 1$, it can be proven that the integer sequence $f_m(n) = \gcd(n + m, mn + 1)$ has a fundamental period T_m. In other words,
$$\forall n \in \mathbb{N}, f_m(n + T_m) = f_m(n).$$
Find an expression for T_m in terms of m.