ASSIGNMENT 1

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1. Let a = 21n+4 and b = 14n+3. We know that gcd(a,b)=1 when there exist integers x and y such that ax + by = 1. Now, (21n+4)(-2) + (14n+3)(3) = 1, so for x = -2 and y = 3, ax + by = 1 and therefore gcd(a,b)=1.

   Therefore, $\frac{21n+4}{14n+3}$ is irreducible.

2. $\frac{n^3+4n^2+4n-14}{n^2+2n+2}$ = n + 2 - $\frac{2n+18}{n^2+2n+2}$

   The quantity $\frac{2n+18}{n^2+2n+2}$ must be an integer. So its absolute value can either be 0 or $\geq$ 1.

   So n = -9, or (2n + 18) $\geq$ (n^2 + 2n + 2) ;
   $n^2 \leq$ 16, So n $\in$ {-4, -3, -2, -1, 0, 1, 2, 3, 4}.
   But, when n $\in$ {-3, 2, 3} then $\frac{2n+18}{n^2+2n+2}$ is not an integer. Hence, final answer is n $\in$ {-9, -4, -2, -1, 0, 1, 4}.

3.

4. $a_n + b_n\sqrt{2} = (1+\sqrt{2})^n$. Observe that $a_2 = 3$ and $b_2 = 2$ and gcd(a_2, b_2) = gcd(3, 2) = 1.
   Now, $(1+\sqrt{2})^{n+1} = a_{n+1} + b_{n+1}\sqrt{2}$
   $= (a_n + b_n\sqrt{2})(1+\sqrt{2})$
   $= (a_n + 2b_n) + (a_n + b_n)\sqrt{2}$
   By comparing, $a_{n+1} = a_n + 2b_n$; $b_{n+1} = a_n + b_n$.
   gcd(a_{n+1}, b_{n+1}) = gcd(a_{n+1} - b_{n+1}, b_{n+1})
   = gcd((a_n + 2b_n) - (a_n + b_n) , (a_n + b_n))
   = gcd(b_n, a_n + b_n) = gcd(b_n, a_n) = gcd(a_n, b_n).
   Therefore, gcd(a_{n+1}, b_{n+1}) = gcd(a_n, b_n) = ... = gcd(a_2, b_2) = gcd(3,2) = 1.

5. $a^p = (a+b-b)^p = C_0(a+b)^p - C_1(a+b)^{p-1}b + ... + C_{p-1}(a+b)b^{p-1} - C_pb^p$
   $= (a+b)^2 \times k + pb^{p-1}$ - b^p [Where k is an integer]
   $(a^p + b^p) = (a+b)^2 \times k + pb^{p-1}$
   gcd(a + b, $\frac{a^p+b^p}{a+b}$) = gcd(a + b, pb^{p-1}).
   Now, as gcd(a + b, b) = gcd(a, b) = 1, So (a + b) does not divide b and therefore, gcd(a + b, b^{p-1}) = 1. So, gcd(a + b, pb^{p-1}) = gcd(a + b, p). As p is prime, gcd can be either 1 or p. If (a + b) is a multiple of p, then gcd = p or else the gcd will be 1.

6. As gcd(a, b) = 1, w.l.o.g. let b = aq + r, where r < a.
   Now, a|bc implies bc = ka for some integer k,
   $\implies (aq+r)c = ka$
   Dividing by a, —qc + $\frac{rc}{a}$ = k.
   $\implies \frac{rc}{a} = k - qc$ As RHS is an integer, LHS must also be an integer, and as r < a, hence a $\nmid$ r, $\therefore$ a | c.

7. gcd(m,n) = am + bn for some integers a and b, The given equation becomes $\frac{am}{n}$ $\binom{n}{m}$ + b $\binom{n}{m}$ = a $\binom{n-1}{m-1}$ + b$\binom{n}{m}$, which is an integer as n $\geq$ m $\geq$ 1.

8. As n | p - 1, p - 1 = an, and p > n.
p | (n - 1)($n^2 + n + 1$) and as p > n, so p $\nmid$ (n - 1), so p | $n^2 + n + 1$
$n^2 + n + 1$ = pb = b(1 + an) $\implies n^2 + (1 - ab)n + 1 - b = 0$
As n is an integer the discriminant of the quadratic must be a whole square, set b = 1 and observe that D = $(1 - a)^2$, 4p - 3 = $4n^2 + 4n + 1 =$ $(2n + 1)^2$.

9. By symmetry of given equations, let us assume a = b for a case of solutions, then $\frac{a+1}{a-1}$ is an integer for a = b = 2 and a = b = 3.

10. Let t = fundamental period of the function.
gcd(n + t + m, mn + mt + 1) = gcd(n + m, mn + 1)
$\implies$ gcd(n + t + m, mn + mt + 1 - m(n + t + m)) = gcd(n + m, mn + 1 - m(n + m))
$\implies$ gcd(n + t + m, 1 - m^2) = gcd(n + m, 1 - m^2)
$\implies$ t = k(1 - m^2), As t is fundamental period so k should be minimum and k$\neq$0, so k = 1, and hence t = 1 - m^2.