ASSIGNMENT 3 - SOLUTIONS

Q1

Solution. Notice that we may write x in the form $5k+3$ and $11m+4$.

$$x = 5k + 3 = 11m + 4$$

Taking this equation mod 5 we arrive at $11m + 4 \equiv 3 \pmod 5 \implies m \equiv -1 \pmod 5$. We substitute $m = 5m_1 - 1$ to give us $x = 11(5m_1 - 1) + 4 = 55m_1 - 7$.

Therefore $x \equiv 48 \pmod{55}$ which means $\boxed{x = 55k + 48}$ for some integer k. □

Q2

Solution. We notice that if x is 7-safe, 11-safe, and 13-safe then we must have

$$\begin{cases} x \equiv 3, 4 \pmod 7 \\ x \equiv 3, 4, 5, 6, 7, 8 \pmod{11} \\ x \equiv 3, 4, 5, 6, 7, 8, 9, 10 \pmod{13} \end{cases}$$

By Chinese Remainder Solution this renders solutions mod 1001. We have 2 choices for the value of x mod 7, 6 choices for the value of x mod 11 and 8 choices for the value of x mod 13. Therefore, we have $2 \cdot 6 \cdot 8 = 96$ total solutions mod 1001.

We consider the number of solutions in the set

$$\{1, 2, \cdots, 1001\}, \{1002, \cdots, 2002\}, \{2003, \cdots, 3003\}, \cdots, \{9009, \cdots, 10010\}.$$

From above there are $96 \cdot 10 = 960$ total solutions. However we must subtract the solutions in the set $\{10,001; 10,002; \cdots; 10,010\}$.

We notice that only $x = 10,006$ and $x = 10,007$ satisfy $x \equiv 3, 4 \pmod 7$.

	$10,006$	$10,007$
$\pmod 7$	3	4
$\pmod{11}$	7	8
$\pmod{13}$	9	10

These values are arrived from noting that $10,006 \equiv -4 \pmod{7 \cdot 11 \cdot 13}$ and $10,007 \equiv -3 \pmod{7 \cdot 11 \cdot 13}$. Therefore $x = 10,006$ and $x = 10,007$ are the two values we must subtract off.

In conclusion we have $960 - 2 = \boxed{958}$ solutions. □

Q3 *Solution.* Note that

$$\binom{n}{7} = \frac{n!}{(n-7)!7!} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2^4 \cdot 3^2 \cdot 5 \cdot 7}.$$

In order for this to be divisible by $12 = 2^2 \cdot 3$, the numerator must be divisible by $2^6 \cdot 3^3$. (We don't care about the 5 or the 7; by the Pigeonhole Principle these will be canceled out by factors in the numerator anyway.) Therefore we wish to find all values of n such that

$$2^6 \cdot 3^3 \mid n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6).$$

We start by focusing on the factors of 3, as these are easiest to deal with. By the Pigeonhole Principle, the expression must be divisible by $3^2 = 9$. Now, if $n \equiv 0, 1, 2, 3, 4, 5, \text{ or } 6 \pmod 9$, one of these seven integers will be a multiple of 9 as well as a multiple of 3, and so in this case the expression is divisible by 27. (Another possibility is if the numbers n, $n-3$, and $n-6$ are all divisible by 3, but it is easy to see that this case has already been accounted for.)

Now, we have to determine when the product is divisible by 2^6. If n is even, then each of $n, n-2, n-4, n-6$ is divisible by 2, and in addition exactly two of those numbers must be divisible by 4. Therefore the divisibility is sure. Otherwise, n is odd, and $n-1, n-3, n-5$ are divisible by 2.

- If $n-3$ is the only number divisible by 4, then in order for the product to be divisible by 2^6 it must also be divisible by 16. Therefore $n \equiv 3 \pmod{16}$ in this case.

- If $n-1$ and $n-5$ are both divisible by 4, then in order for the product to be divisible by 2^6 one of these numbers must also be divisible by 8. Therefore $n \equiv 1, 5 \pmod 8 \implies n \equiv 1, 5, 9, 13 \pmod{16}$.

Pooling all our information together, we see that $\binom{n}{7}$ is divisible by 12 iff n is such that

$$\begin{cases} n \equiv 0, 1, 2, 3, 4, 5, 6 & \pmod 9, \\ n \equiv 0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14 & \pmod{16}. \end{cases}$$

There are 7 possibilities modulo 9 and 13 possibilities modulo 16, so by CRT there exist $7 \times 13 = 91$ solutions modulo $9 \times 16 = 144$. Therefore, as more and more numbers n are checked, the probability that $\binom{n}{7}$ is divisible by 12 approaches $\frac{91}{144}$. The requested answer is $91 + 144 = \boxed{235}$.

□

Q4

Solution. Let the points on the grid be of the form

$$(x, y) = (a + m, b + n), \qquad 99 \geq m, n \geq 0.$$

We are going to use the Chinese Remainder Theorem to have every single term have a common divisor among the two coordinates. For the remainder of the problem assume that the sequence $\{p_j\}$ is a sequence of distinct prime numbers. Let $a \equiv 0 \left(\bmod \prod_{i=1}^{100} p_i\right)$. Then let

$$\begin{cases} b & \equiv 0 \pmod{p_1} \\ b+1 & \equiv 0 \pmod{p_2} \\ \cdots \\ b+99 & \equiv 0 \pmod{p_{100}}. \end{cases}$$

We find that repeating this process with letting $a + 1 \equiv 0 \left(\bmod \prod_{i=101}^{200} p_i\right)$ and defining similarly

$$\begin{cases} b \equiv 0 \pmod{p_{101}} \\ b+1 \equiv 0 \pmod{p_{102}} \\ \cdots \\ b+99 \equiv 0 \pmod{p_{200}} \end{cases}$$

gives us the following:

$$\begin{cases} a & \equiv 0 \left(\bmod \prod_{i=1}^{100} p_i\right) \\ a+1 & \equiv 0 \left(\bmod \prod_{i=101}^{200} p_i\right) \\ \cdots \\ a+99 & \equiv 0 \left(\bmod \prod_{i=9901}^{10000} p_i\right) \end{cases} \quad \text{and} \quad \begin{cases} b & \equiv 0 \left(\bmod \prod_{i=0}^{99} p_{100i+1}\right) \\ b+1 & \equiv 0 \left(\bmod \prod_{i=0}^{99} p_{100i+2}\right) \\ \cdots \\ b+99 & \equiv 0 \left(\bmod \prod_{i=1}^{100} p_{100i}\right). \end{cases}$$

This notation looks quite intimidating; take a moment to realize what it is saying. It is letting each $a+k$ be divisible by 100 distinct primes, then letting b be divisible by the first of these primes, $b+1$ be divisible by the second of these primes and so forth. This is precisely what we did in our first two examples above. By CRT we know that a solution exists, therefore we have proven the existence of a 100×100 grid.

$\square$

Motivation. This problem requires a great deal of insight. When I solved this problem, my first step was to think about completing a 1×100 square as we did above. Then you have to think of how to extend this method to a 2×100 square and then generalizing the method all the way up to a 100×100 square. Notice that our construction is not special for 100, we can generalize this method to a $x \times x$ square! $\square$