A report on

APPLICATION OF GROUP THEORY IN MUSIC THEORY

submitted in the partial fulfilment of the requirement for the award of degree of

BACHELOR OF TECHNOLOGY

(Electrical Engineering)

by

Pranshu Kukreti	Pulkit Kapoor
(2K18/EE/141)	(2K18/EE/148)

under supervision of

Dr. Dinesh Udar

DEPARTMENT OF APPLIED MATHEMATICS

DELHI TECHNOLOGICAL UNIVERSITY

(2020-21)

Certificate

This is to certify that Pranshu Kukreti and Pulkit Kapoor, B. Tech. students in the Department of Electrical Engineering have submitted a project report on "APPLICATION OF GROUP THEORY IN MUSIC THEORY" in partial fulfillment of the requirement for award of degree of Bachelor/Master of Technology in Electrical Engineering, during the academic year 2020-21.

It is a record of the students' research work prepared under my supervision and guidance.

Dr. Dinesh Udar

Professor

Department of Applied Mathematics, Delhi Technological University

Abstract.

This paper critically analysis the behavior and the relationship that exist between musical notes and group theory. The musical notes form additive abelian group modulo 12. Musical Actions of the Dihedral Groups. . Finally, the work came up with some propositions due to the musical notes behavior and their proofs one of which was name Dido's Theorem.

Content

Introduction

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra. Other well-known algebraic structures, such as rings, fields, and vectors space can all be seen as groups endowed with additional operations and axioms. Various physical systems, such as crystals and the hydrogen atom, can be modeled by symmetry groups. Thus, group theory has many important applications in Physics, chemistry, and materials science. Group theory is also central to public key cryptography. The modern concept of abstract group developed out of several fields of mathematics (Wussing, 2007). The idea of group theory although developed from the concept of abstract algebra, yet can be applied in many other areas of mathematical areas and other field in sciences and as well as in music.

❖ *Music Theory*

Music theory is a tool and framework with which we explain our listening experience. However, both the tool and the term “listening experience” are loosely defined. They are dependent on the music. During the mid-15th century, composers began constructing their pieces around a particular pitch, called the tonic. This pitch was quickly established at the start of the piece and all other pitches were heard relative to it. Intervals and chords were labeled as consonant or dissonant. A feeling of tension occurred in various ways, such as when resolution was delayed, or when the music leapt to distant keys (more than two accidentals removed from the tonic). Resolution to the tonic was crucial to ending the piece. After over two centuries of tonal music, listeners have begun to expect music to resolve in particular ways. Along with the development of tonal music was the development of tonal theory. Its structure and notation allowed theorists to describe the listener’s expectation. Thus, it provided an explanation for our reaction to particular harmonies. It explained our feeling of surprise at a particular chord and our feeling of finality at the end of a piece. Around the turn of the 19th century, composers pushed the boundaries of tonal music. They began using dissonant chords with unprecedented freedom and resolved them in new ways. Eventually, their pieces no longer fit the framework of tonal music. Tonal theory no longer provided an adequate explanation for our listening experience. Thus, a new framework was constructed called atonal music theory. Discussions in music require a certain vocabulary. The following terms are defined in the appendix:

- interval
- half step (semitone) & whole step (whole tone)
- flat, sharp, natural & accidental
- enharmonic equivalence
- major, minor, mode
- parallel & relative
- scale degree
- triad

❖ *Group Theory*

Group theory is the branch of pure mathematics which is emanated from abstract algebra. Due to its abstract nature, it was seeming to be an arts subject rather than a science subject. In fact, was considered pure abstract and not practical. Even students of group theory after being introduced to the course seems not to believe as to whether the subject has any practical application in real life, because of its abstract nature (Tosk, 2013). The problem prompts the researchers to study the different ways in which group can be express concretely both from theoretical and practical point of view, with intention of bringing its real-life application in musical notes. This paper aim at taking some concepts of group theory to study and understand musical notes in relation to the group's axioms. The main objective is to see these musical notes interpretation algebraically as regard to their behaviour.

Definition of Terms

We present here some few definitions that will help us to be familiar with concepts in music and abstract algebra. Musical Notes Musical notes are the following notes $C\ C\#\ D\ D\#\ E\ F\ F\#\ G\ G\#\ A\ A\#\ B$ when logically combined, give out pleasant sound to the ear. The first note which is C, is called the root note.

C# is called the 2nd note

D is called the 3rd note

D# is called the 4th note

E is called the 5th note

F is called the 5th note

F# is called the 6th note

G is called the 7th note

G# is called the 8th note

A is called the 10th note

A# is called the 11th note

B is called the 12th note

- ***Musical Flat b***

Musical flats can be defined as the movement of sound from one pitch to the one lower, and it is donated to b. For example, movement from F to any other note to the left.

- ***Musical Sharp #***

This can be considered as the movement of sound from a pitch (note) to another pitch higher, and it bis denoted by #. For example, movement from F to any other note to the right on the musical notes.

- ***Tone***

This simply meant any movement from a musical note to the next note two steps forward or backward on the musical notes. For example, movement from F to G or to D#.

- ***Semitone***

This can be defined as any movement from a musical note to the next note a step forward or backward on the musical notes (Scales). For example, movement from F to F# or F to E.

- ***Chord***

A chord is produced when two, three or more notes are sounded together. Transposition Transposition involves playing or writing a given melody at a different pitch higher or lower other than the original

Pitch Classes and Integers Modulo 12

As the ancient Greeks noticed, any two pitches that differ by a whole number of octaves6 sound alike. Thus we identify any two such pitches, and speak of pitch classes arising from this equivalence relation. Like most modern music theorists, we use equal tempered tuning, so that the octave is divided into twelve pitch classes as follows.

C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
0	1	2	3	4	5	6	7	8	9	10	11	0

A	*A*♯ *B*♭	*B*	*C*	*C*♯ *D*♭	*D*	*D*♯ *E*♭	*E*	*F*	*F*♯ *G*♭	*G*	*G*♯ *A*♭	*A*

The interval between two consecutive pitch classes is called a half-step or semitone. The notation ♯ means to move up a semitone, while the notation ♭ means to move down a semitone. Note that some pitches have two letter names. This is an instance of enharmonic equivalence. Music theorists have found it useful to translate pitch classes to integers modulo 12 taking 0 to be C as in Figure 1. Mod 12 addition and subtraction can be read off of this clock; for example 2 + 3 = 5 mod 12, 11 + 4 = 3 mod 12, and 1 − 4 = 9 mod 12. We can also determine the musical interval from one pitch class to another; for example, the interval from D to G♯ is 6A pitch y is an octave above a pitch x if the frequency of y is twice that of x. 4 six semitones. This description of pitch classes in terms of Z12 .This translation from pitch classes to integers modulo 12 permits us to easily use abstract algebra for modelling musical events, as we shall see in the next two sections.

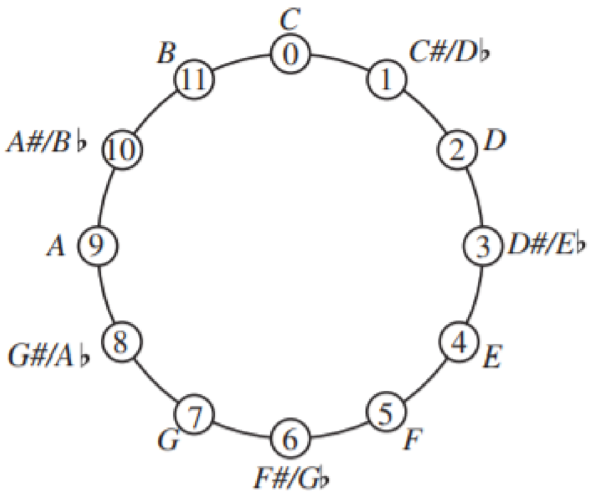


Figure 1: The musical clock.

Numbering of the Musical Notes

C $C\#$ D $D\#$ E F $F\#$ G $G\#$ A $A\#$ B

0 1 2 3 4 5 6 7 8 9 10 11

Note that $B\# = C$

It shows that the musical notes form a group of integers of Modulo 12.

That is $Z12 = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\ \}$. Let the operation be $* = \# = +$ Result of the behavior of the musical note on Groups.

Musical Notes as it related to groups axiom. Without loss of generality

I. **Closureness** $E, F \in Z12$, hence $E * F = A \in Z12$

II. **Associativity** $E, F\ and\ F\# \in Z12$, hence $(E * F) * F\# = E * (F * F\#) = A * F\# = E * B = D\# = D\#$

III. **Identity** $\boldsymbol{F} \in Z12 \ni C \in Z12$, hence $F * C = C * F = F$

IV. **Inverse** $\boldsymbol{F} \in Z12\ \exists G \in Z12$, hence $F * G = G * F = C \in Z12$

Therefore, musical note behavior satisfied all the mathematical group axioms.

v. Furthermore, $\forall F, G \in Z12\ F * G = G * F = C \in Z12$

This shows that it is not just a group, but also an **Abelian group**. With the behavior of the musical notes we have just seen, we personally suggest for the root note of musical scales (notes) to be algebraically named as the identity note.

Table 1: List of Musical Notes and their inverse

Note	Inverse
C	C
C#	B
D	A#
D#	A
E	G#
F	G
F#	F#

The behavior of the musical note as related to table 1. Gave us insight to formulate this Proposition which we intend call it Dido's theorem.

Dido's theorem

If G is cyclic, then there is at least an element which is unique with its inverse.

Proof

Suppose G is cyclic

$\Longrightarrow \forall x \in G$, each $x \in G$ can be written in the form $x = g\ m$ for some $g \in G$ Where $m \in Z \ni$ $some\ y \in G \ni x * y = e \in G$

$\Longrightarrow x = y$ where e is the identity element $y = x - 1 \Longrightarrow x = x - 1$

This completes the proof.

Remark table 1 give better understanding of the proposition above

Theorem 3.1

Let $H \leq G$ be groups and $g \in G$. Then:

(i) $g \in gH$

(ii) Two left cosets of H in G are either identical or disjoint.

(iii) The number of elements in gH is $|H|$

The Result of theorem 3.1

$Z12 = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$. $H = \{C, C\#, B\} \Longrightarrow H \leq Z12$

$DH = \{D * C, D * C\#, D * B\}$ $DH = \{D, D\#, C\#\}$

Clearly, $D \in DH$ And again, $|H| = |DH| = 3$. Furthermore, for some $A, F \in Z12$ $(A * F)H = DH$

$\Longrightarrow$ two left cosets are identical in this case for some $A, G \in Z12$

Theorem 3.2 (Langrange's Theorem)

The order of a subgroup of a finite group is a factor of the order of the group.

The Result of Theorem 3.2

$|Z12| = 12$ Since $|Z12| = 12$ $|H| = 3 \Rightarrow |H| = Z12$ $|Z12|$ $|H| \diagup = 12$ $3 \diagup = 4$ It is true that the order of a subgroup divides the order of a group.

Theorem 3.3

Every subgroup of a cyclic group is cyclic.

The Result of theorem 3.3

From $Z12 = \{C, C\#, D, D\#, E, F, F\#, G, G\#, A, A\#, B\}$ For $C \in Z12$

C^0 $= C$ C^1 $= C\#$ C^2 $= D$ C^3 $= D\#$ C^4 $= E$ C^5 $= F$ C^6 $= F\#$
C^7 $= G$ C^8 $= G\#$ C^9 $= A$ C^10 $= A\#$ C^11 $= B$ C^12 $= C$

Musical notes are cyclic. That is $Z12 = \langle C \rangle$.

Consider the subgroup $H = \{ C, C\#, B\}$ $B \in H \leq Z12$

B^0 $= B$

B^1 $= C$

B^2 $= C\#$

Clearly, H is cyclic.

Using musical notes we are satisfied with the theorem which states that "every cyclic group has a subgroup which is also cyclic.

Musical Actions of the Dihedral Groups

The dihedral group of order 24 is the group of symmetries of a regular 12-sided polygon. Algebraically, the dihedral group of order 24 is generated by two elements s and t , such that

$$s^{12} = 1, \quad t^2 = 1, \quad tst = s^{-1}$$

Let's take some time to make sense of these properties in terms of symmetries. The operation s is rotation by (360/n)degree and the operation t is reflection about an axis.

➢ *First musical action*

The first musical action of the dihedral group of order 24 we consider arises via the familiar compositional techniques of transposition and inversion. A transposition moves a sequence of pitches up or down. When singers decide to sing a song in a higher register, for example, they do this by transposing the melody. An inversion, on the other hand, reflects a melody about a fixed axis, just as the face of a clock can be reflected about the 0-6 axis. Often, musical inversion turns upward melodic motions into downward melodic motions.

➢ *Second musical action*

The second action of the dihedral group of order 24 that we explore has only come to the attention of music theorists in the past two decades. It's not that no one made these music moves before, but rather that they were not formalized. Its origins lie in the P,L, and R operations of the 19th-century music theorist Hugo Riemann. The parallel operation maps a major triad to its parallel minor and vice versa. The leading tone exchange operation takes a major triad to the minor triad obtained by lowering only the root note by a semitone. The operation raises the fifth note of a minor triad by a semitone. The relative operation R maps a major triad to its relative minor, and vice versa. For example,

- $P(C - major) = c - minor$
- $L(C - major) = e - minor$
- $R(C - major) = a - minor$

Transposition and Inversion

The transposition Tn moves a pitch-class or pitch-class set up by n (mod 12). (Note: moving down by n is equivalent to moving up by 12−n.)

$\boldsymbol{Tn : Z12 \rightarrow Z12 \ni Tn\ (X) : X + n}$

Consider the pitch a. Inversion In inverts the pitch about C(0) and then transposes it by n. That is, In = −a + n (mod 12).

$\boldsymbol{In : Z12 \rightarrow Z12 \ni In\ (X) : -x + n}$ **where n is in mod12**

As is well known, these transpositions and inversions have a particularly nice representation in terms of the musical clock . The transposition T1 corresponds to clockwise rotation of the clock by 1/12 of a turn, while I_0 corresponds to a reflection of the clock about the 0-6 axis. Hence T_1 and I_0 generate the dihedral group of symmetries of the 12-gon.

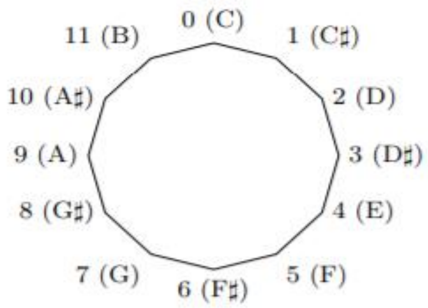
Since $(T_1)^n = T_n$ and $T_n \circ I_0 = I_n$, we see that the 12 transpositions and 12 inversions form the dihedral group of order 24.

The compositions

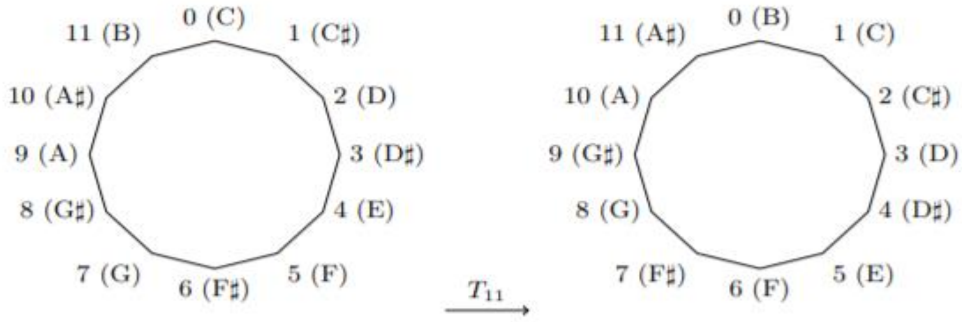
$T_m \circ T_n = T_{m+n \bmod 12}$ $\quad$ $T_m \circ I_n = I_{m+n \bmod 12}$

$I_m \circ T_n = I_{m-n \bmod 12}$ $\quad$ $I_m \circ I_n = T_{m-n \bmod 12}$

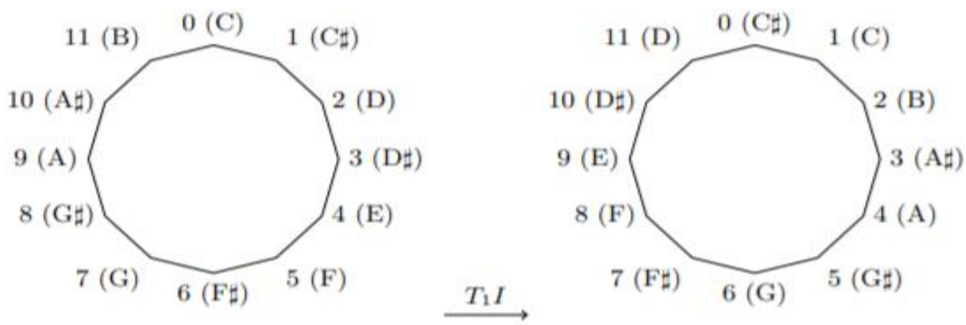
are easy to verify. This group is often called the T/I-group. The first action of the dihedral group of order 24 on the set of major and minor triads that we study is defined via the T/I-group.



Consider the transposition T_{11}. It sends C to B, C♯to C, etc. That is,



Now consider the inversion $T_1 I$. It sends C to D♯, D♯to C, D to B, etc. This gives:



Major and Minor Triads

Triadic harmony has been in use for hundreds of years and is still used every day in popular music. In this section we use the integers modulo 12 to define major and minor triads; in this way we can consider them as objects upon which the dihedral group of order 24 may act.

A triad consists of three simultaneously played notes. A major triad consists of a root note, a second note 4 semitones above the root, and a third note 7 semitones above the root. For example, the C-major triad consists of {0, 4, 7} = {C, E, G} and is represented as a chord polygon in Figure 2. See [18] for beautiful illustrations of the utility of chord polygons. Since any major triad is a subset of the pitch-class space Z12, and transpositions and inversions act on Z12, we can also apply transpositions and inversions to any major triad. Figure 2 shows what happens when we apply I0 to the C-major triad. The resulting triad is not a major triad, but instead a minor triad.

A *minor triad* consists of a root note, a second note 3 semitones above the root, and a third note 7 semitones above the root. For example, the fminor triad consists of {5, 8, 0} = {F, A♭, C} and its chord polygon appears in Figure 2.

Altogether, the major and minor triads form the set S of *consonant* triads, which are called consonant because of their smooth sound. A consonant triad is named after its root. For example, the C-major triad consists of {0, 4, 7} = {C, E, G} and the f-minor triad consists of {5, 8, 0} = {F, A♭, C}. Musicians commonly denote major triads by upper-case letters and minor

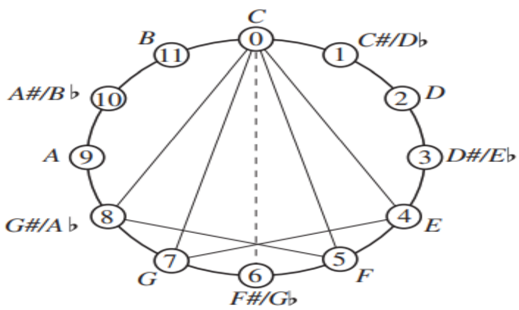


Figure 2: I_0 applied to a C-major triad yields an f-minor triad.

Major Triads	Minor Triads
$C = \langle 0, 4, 7\rangle$	$\langle 0, 8, 5\rangle = f$
$C\sharp = D\flat = \langle 1, 5, 8\rangle$	$\langle 1, 9, 6\rangle = f\sharp = g\flat$
$D = \langle 2, 6, 9\rangle$	$\langle 2, 10, 7\rangle = g$
$D\sharp = E\flat = \langle 3, 7, 10\rangle$	$\langle 3, 11, 8\rangle = g\sharp = a\flat$
$E = \langle 4, 8, 11\rangle$	$\langle 4, 0, 9\rangle = a$
$F = \langle 5, 9, 0\rangle$	$\langle 5, 1, 10\rangle = a\sharp = b\flat$
$F\sharp = G\flat = \langle 6, 10, 1\rangle$	$\langle 6, 2, 11\rangle = b$
$G = \langle 7, 11, 2\rangle$	$\langle 7, 3, 0\rangle = c$
$G\sharp = A\flat = \langle 8, 0, 3\rangle$	$\langle 8, 4, 1\rangle = c\sharp = d\flat$
$A = \langle 9, 1, 4\rangle$	$\langle 9, 5, 2\rangle = d$
$A\sharp = B\flat = \langle 10, 2, 5\rangle$	$\langle 10, 6, 3\rangle = d\sharp = e\flat$
$B = \langle 11, 3, 6\rangle$	$\langle 11, 7, 4\rangle = e$

Figure 3: The set S of consonant triads.

triads by lower-case letters as indicated in the table of all consonant triads in Figure 3.

The table also reflects the component wise action of the T/I-group because of this ordering. In the table, an application of T1 to an entry gives the entry immediately below it, for example

T1(0, 4, 7)= (T1(0), T1(4), T1(7))

= (1, 5, 8)

The PLR-Group

We notice that in the T/I-Group, we can start with any major triad, and using only those two transformations, get all 24 major and minor triads. Another way of navigating the major and minor triads is through the PLR-group, initiated by David Lewin. As we'll find, the PLR-group has a beautiful geometric depiction called the Tonnetz.

Consider the three functions $P, L, R : S \to S$ defined as follows:

$$P\langle y_1, y_2, y_3\rangle = I_{y_1+y_3}\langle y_1, y_2, y_3\rangle$$

$$L\langle y_1, y_2, y_3\rangle = I_{y_2+y_3}\langle y_1, y_2, y_3\rangle$$

$$R\langle y_1, y_2, y_3\rangle = I_{y_1+y_2}\langle y_1, y_2, y_3\rangle$$

These are called parallel, leading tone exchange, and relative. These are contextual inversions because the axis of inversion depends on the aggregate input triad. Notably, the functions P, L, and R are not defined component wise, and this distinguishes them from inversions of the form In, where the axis of inversion is independent of the input triad. For P, L, and R the axis of inversion on the musical clock when applied to hy1, y2, y3i is indicated in the table below

Function	Axis of Inversion Spanned by
P	$\frac{y_1+y_3}{2}, \frac{y_1+y_3}{2}+6$
L	$\frac{y_2+y_3}{2}, \frac{y_2+y_3}{2}+6$
R	$\frac{y_1+y_2}{2}, \frac{y_1+y_2}{2}+6$

SUMMARY AND CONCLUSSION

The results of this findings in our paper, shows that musical notes behaviors satisfied all group axioms and are related to group theory. Since music is food for the soul and mind, we suggest that a good understanding of group theory to musician can help in composing best musical composition that will give satisfaction to Audience and as well bring healings to their minds.

REFERENCES

Ada Zhang, (2009) The framework of music theory as represented with Groups 9-11 Adam Townsend (2011). Mathematics and Music theory, A paper presentation of undergraduate Mathematics Colloquium at the University College London (http://ucl.sneffel.com) 5-8

Alissa, S. Crans, (2009). Musical Actions of Dihedral groups 479-482

Emma Rose, (2011). Mathematics and Music 5-6

H.Wussing (2007). The Genesis of the abstract group concept: A contribution to the History of the origin of Abstract Group theory, New York: Dover Publications http://en.Wikipedia.org/wiki/Group(Mathematics)

Heath, Thomas, (1965). A History of Greek Mathematics, Vol. 1 Oxford University press. Oxford.

Samuel H. Tsok and I.B. Mshelia (2013). Application of Group Theory to a Local Game Called “Tsorry Checkerboard” (A Case of Klein Four- Group). International Organization of Scientific Research Journal of Mathematics (IOSR-JM), Volume 7, Issue 3, 04-06 www.iosrjournals.org

Thomas M. Fiore (1993). Music and Mathematics 7-15 (fioret@umich.edu)

https://arxiv.org/pdf/1407.5757.pdf