

BINOMIAL THEOREM

1. BINOMIAL EXPRESSION :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x - y$, $xy + \frac{1}{x}$, $\frac{1}{z} - 1$, $\frac{1}{(x-y)^{1/3}} + 3$ etc.

2. BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM**.

If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then :

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n y^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r$$

This theorem can be proved by induction.

Observations :

- The number of terms in the expansion is $(n+1)$ i.e. one more than the index.
- The sum of the indices of x & y in each term is n .
- The binomial coefficients of the terms $({}^nC_0, {}^nC_1, \dots)$ equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{n-r}$
- Symbol nC_r can also be denoted by $\binom{n}{r}$, $C(n, r)$ or A_r^n .

Some important expansions :

- $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$.
- $(1 - x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \dots + (-1)^n \cdot {}^nC_n x^n$.

Note : The coefficient of x^r in $(1 + x)^n = {}^nC_r$ & that in $(1 - x)^n = (-1)^r \cdot {}^nC_r$

Illustration 1 : Expand : $(y + 2)^6$.

Solution : ${}^6C_0 y^6 + {}^6C_1 y^5 \cdot 2 + {}^6C_2 y^4 \cdot 2^2 + {}^6C_3 y^3 \cdot 2^3 + {}^6C_4 y^2 \cdot 2^4 + {}^6C_5 y^1 \cdot 2^5 + {}^6C_6 \cdot 2^6$
 $= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Illustration 2 : Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution : ${}^7C_0, {}^7C_1 \left(-\frac{2y^2}{5}\right), {}^7C_2 \left(-\frac{2y^2}{5}\right)^2, {}^7C_3 \left(-\frac{2y^2}{5}\right)^3$

Illustration 3 : If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and -6 respectively then m is -

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- (A) 6 (B) 9 (C) 12 (D) 24

Solution : $(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{(m)(m-1) \cdot x^2}{2} + \dots\right] \left[1 - nx + \frac{n(n-1)}{2} x^2 + \dots\right]$

$$\text{Coefficient of } x = m - n = 3 \quad \text{.....(i)}$$

$$\text{Coefficient of } x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6 \quad \text{.....(ii)}$$

Solving (i) and (ii), we get

$$m = 12 \text{ and } n = 9.$$

Do yourself - 1 :

(i) Expand $\left(3x^2 - \frac{x}{2}\right)^5$

(ii) Expand $(y + x)^n$

Pascal's triangle :

$(x+y)^0$	1	1
$(x+y)^1$	$x + y$	1 1
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$	1 4 6 4 1

Pascal's triangle

- (i) **Pascal's triangle** - A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for $n = 0$, the second for $n = 1$, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.
- (ii) Pascal triangle is formed by binomial coefficient.
- (iii) The number of terms in the expansion of $(x+y)^n$ is $(n + 1)$ i.e. one more than the index.
- (iv) The sum of the indices of x & y in each term is n .
- (v) Power of first variable (x) decreases while of second variable (y) increases.
- (vi) Binomial coefficients are also called **combinatorial coefficients**.
- (vii) Binomial coefficients of the terms equidistant from the beginning and end are equal.
- (viii) r^{th} term from the beginning in the expansion of $(x + y)^n$ is same as r^{th} term from end in the expansion of $(y + x)^n$.
- (ix) r^{th} term from the end in $(x + y)^n$ is $(n - r + 2)^{\text{th}}$ term from the beginning.

3. IMPORTANT TERMS IN THE BINOMIAL EXPANSION :

- (a) **General term:** The general term or the $(r + 1)^{\text{th}}$ term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r x^{n-r} y^r$$

Illustration 4 : Find : (a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

Also, find the relation between a and b , so that these coefficients are equal.

Solution :

- (a) In the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$, the general term is :

$$T_{r+1} = {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{22-3r}$$

putting $22 - 3r = 7$

$$\therefore 3r = 15 \Rightarrow r = 5$$

$$\therefore T_6 = {}^{11}C_5 \frac{a^6}{b^5} \cdot x^7$$

Hence the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5 a^6 b^{-5}$.

Ans.

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

- (b) In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, general term is :

$$T_{r+1} = {}^{11}C_r (ax)^{11-r} \left(\frac{-1}{bx^2}\right)^r = (-1)^r {}^{11}C_r \frac{a^{11-r}}{b^r} \cdot x^{11-3r}$$

putting $11 - 3r = -7$

$$\therefore 3r = 18 \Rightarrow r = 6$$

$$\therefore T_7 = (-1)^6 \cdot {}^{11}C_6 \frac{a^5}{b^6} \cdot x^{-7}$$

Hence the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6 a^5 b^{-6}$.

Ans.

Also given :

$$\text{Coefficient of } x^7 \text{ in } \left(ax^2 + \frac{1}{bx}\right)^{11} = \text{coefficient of } x^{-7} \text{ in } \left(ax - \frac{1}{bx^2}\right)^{11}$$

$$\Rightarrow {}^{11}C_5 a^6 b^{-5} = {}^{11}C_6 a^5 b^{-6}$$

$$\Rightarrow ab = 1 \quad (\because {}^{11}C_5 = {}^{11}C_6)$$

which is the required relation between a and b.

Ans.

Illustration 5 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{1/4}\right)^{1000-r} \left(8^{1/6}\right)^r = {}^{1000}C_r 3^{\frac{1000-r}{2}} 2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is $\{0, 2, 4, \dots, 1000\}$

Hence, number of rational terms is 501

Ans.

(b) Middle term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

- (i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$
 (ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Important Note :

Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$$\Rightarrow {}^nC_r \text{ will be maximum } \begin{cases} \text{When } r = \frac{n}{2} \text{ if } n \text{ is even} \\ \text{When } r = \frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ if } n \text{ is odd} \end{cases}$$

\Rightarrow The term containing greatest binomial coefficient will be middle term in the expansion of $(1+x)^n$

Illustration 6 : Find the middle term in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$

Solution : The number of terms in the expansion of $\left(3x - \frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{\text{th}}$ and $\left(\frac{9+3}{2}\right)^{\text{th}}$ are two middle terms. They are given by T_5 and T_6

$$\therefore T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_4 3^5 x^5 \cdot \frac{x^{12}}{6^4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{3^5}{2^4 \cdot 3^4} x^{17} = \frac{189}{8} x^{17}$$

$$\text{and } T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4 \cdot x^4 \cdot \frac{x^{15}}{6^5} = \frac{-9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{3^4}{2^5 \cdot 3^5} x^{19} = -\frac{21}{16} x^{19} \quad \text{Ans.}$$

(c) Term independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Illustration 7 : The term independent of x in $\left[\sqrt{\frac{x}{3}} + \sqrt{\frac{3}{2x^2}}\right]^{10}$ is -

- (A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

Solution : General term in the expansion is

$${}^{10}C_r \left(\frac{x}{3}\right)^{\frac{r}{2}} \left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10} \cdot \frac{3^{5-r}}{2^{\frac{10-r}{2}}} \quad \text{For constant term, } \frac{3r}{2} = 10 \Rightarrow r = \frac{20}{3}$$

which is not an integer. Therefore, there will be no constant term.

Ans. (D)

Do yourself - 2 :

(i) Find the 7th term of $\left(3x^2 - \frac{1}{3}\right)^{10}$

(ii) Find the term independent of x in the expansion : $\left(2x^2 - \frac{3}{x^3}\right)^{25}$

(iii) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$ (b) $\left(2x^2 - \frac{1}{x}\right)^7$

(d) Numerically greatest term :

Let numerically greatest term in the expansion of $(a + b)^n$ be T_{r+1} .

$$\Rightarrow \begin{cases} |T_{r+1}| \geq |T_r| \\ |T_{r+1}| \geq |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \leq r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case I : When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer equal to m , then T_m and T_{m+1} will be numerically greatest term.

Case II : When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m , then T_{m+1} will be the numerically greatest term.

Illustration 8 : Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \frac{1}{5}$

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \leq r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|} - 1 \leq r \leq \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

solving we get $2 \leq r \leq 3$

$$\therefore r = 2, 3$$

so, the greatest terms are T_{2+1} and T_{3+1} .

\therefore Greatest term (when $r = 2$)

$$T_3 = {}^{11}C_2 \cdot 3^9 \cdot (-5x)^2 = 55 \cdot 3^9 = T_4$$

From above we say that the value of both greatest terms are equal.

Ans.

Illustration 9 : Given T_3 in the expansion of $(1 - 3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \leq r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\frac{1}{-3x}\right|} - 1 \leq 2 \leq \frac{7}{1+\left|\frac{1}{-3x}\right|}$$

Let $|x| = t$

$$\frac{21t}{3t+1} - 1 \leq 2 \leq \frac{21t}{3t+1}$$

$$\begin{cases} \frac{21t}{3t+1} \leq 3 \\ \frac{21t}{3t+1} \geq 2 \end{cases} \Rightarrow \begin{cases} \frac{4t-1}{3t+1} \leq 0 \Rightarrow t \in \left[-\frac{1}{3}, \frac{1}{4}\right] \\ \frac{15t-2}{3t+1} \geq 0 \Rightarrow t \in \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{2}{15}, \infty\right) \end{cases}$$

$$\text{Common solution } t \in \left[\frac{2}{15}, \frac{1}{4}\right] \Rightarrow x \in \left[-\frac{1}{4}, -\frac{2}{15}\right] \cup \left[\frac{2}{15}, \frac{1}{4}\right]$$

Do yourself -3 :

- (i) Find the numerically greatest term in the expansion of $(3 - 2x)^9$, when $x = 1$.
- (ii) In the expansion of $\left(\frac{1}{2} + \frac{2x}{3}\right)^n$ when $x = -\frac{1}{2}$, it is known that 3rd term is the greatest term. Find the possible integral values of n .

4. PROPERTIES OF BINOMIAL COEFFICIENTS :

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n = \sum_{r=0}^n {}^nC_r x^r ; n \in \mathbb{N} \quad \dots(i)$$

where $C_0, C_1, C_2, \dots, C_n$ are called combinatorial (binomial) coefficients.

- (a) The sum of all the binomial coefficients is 2^n .

Put $x = 1$, in (i) we get

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n \Rightarrow \sum_{r=0}^n {}^nC_r = 2^n \quad \dots(ii)$$

- (b) Put $x = -1$ in (i) we get

$$C_0 - C_1 + C_2 - C_3 + \dots + C_n = 0 \Rightarrow \sum_{r=0}^n (-1)^r {}^nC_r = 0 \quad \dots(iii)$$

- (c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1} .

$$\text{From (ii) \& (iii), } C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

- (d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$$(e) \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

$$(f) {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2} = \dots = \frac{n(n-1)(n-2) \dots (n-r+1)}{r(r-1)(r-2) \dots 1}$$

$$(g) {}^nC_r = \frac{r+1}{n+1} \cdot {}^{n+1}C_{r+1}$$

Illustration 10 : Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \dots + {}^{10}C_{10} = {}^{26}C_{11}$

Solution : LHS = ${}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \dots + {}^{25}C_{10}$
 $\Rightarrow {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \dots + {}^{25}C_{10}$
 $\Rightarrow {}^{12}C_{11} + {}^{12}C_{10} + \dots + {}^{25}C_{10}$
 $\Rightarrow {}^{13}C_{11} + {}^{13}C_{10} + \dots + {}^{25}C_{10}$
 and so on. \therefore LHS = ${}^{26}C_{11}$

Aliter :

LHS = coefficient of x^{10} in $\{(1+x)^{10} + (1+x)^{11} + \dots + (1+x)^{25}\}$

$$\Rightarrow \text{coefficient of } x^{10} \text{ in } \left[(1+x)^{10} \frac{\{1+x\}^{16} - 1}{1+x-1} \right]$$

$$\Rightarrow \text{coefficient of } x^{10} \text{ in } \frac{[(1+x)^{26} - (1+x)^{10}]}{x}$$

$$\Rightarrow \text{coefficient of } x^{11} \text{ in } [(1+x)^{26} - (1+x)^{10}] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$$

Illustration 11 : A student is allowed to select at most n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select books is 63, find the value of n .

Solution : Given student selects at most n books from a collection of $(2n+1)$ books. It means that he selects one book or two books or three books or or n books. Hence, by the given condition-

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_n = 63 \quad \dots(i)$$

But we know that

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_{2n+1} = 2^{2n+1} \quad \dots(ii)$$

Since ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_n) + ({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + \dots + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$$

$$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_n) + ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + \dots + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$$

$$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$$

$$\Rightarrow 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_n) = 2^{2n+1} \quad [\text{from (i)}]$$

$$\Rightarrow 2 + 2 \cdot 63 = 2^{2n+1} \quad \Rightarrow 1 + 63 = 2^{2n}$$

$$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n} \quad \therefore 2n = 6$$

Hence, $n = 3$.

Ans.

Illustration 12 : Prove that :

$$(i) \quad C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$$

$$(ii) \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

Solution :

$$(i) \quad \text{L.H.S.} = \sum_{r=1}^n r \cdot {}^nC_r = \sum_{r=1}^n r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}$$

$$= n \sum_{r=1}^n {}^{n-1}C_{r-1} = n \cdot [{}^{n-1}C_0 + {}^{n-1}C_1 + \dots + {}^{n-1}C_{n-1}]$$

$$= n \cdot 2^{n-1}$$

Aliter : (Using method of differentiation)

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n \quad \text{.....(A)}$$

Differentiating (A), we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + n.C_nx^{n-1}.$$

Put $x = 1$,

$$C_1 + 2C_2 + 3C_3 + \dots + n.C_n = n.2^{n-1}$$

$$\begin{aligned} \text{(ii) L.H.S.} &= \sum_{r=0}^n \frac{C_r}{r+1} = \frac{1}{n+1} \sum_{r=0}^n \frac{n+1}{r+1} {}^nC_r \\ &= \frac{1}{n+1} \sum_{r=0}^n {}^{n+1}C_{r+1} = \frac{1}{n+1} [{}^{n+1}C_1 + {}^{n+1}C_2 + \dots + {}^{n+1}C_{n+1}] = \frac{1}{n+1} [2^{n+1} - 1] \end{aligned}$$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1} \quad (\text{where } C \text{ is a constant})$$

$$\text{Put } x = 0, \text{ we get, } C = -\frac{1}{n+1}$$

$$\therefore \frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}$$

$$\text{Put } x = 1, \text{ we get } C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

$$\text{Put } x = -1, \text{ we get } C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots = \frac{1}{n+1}$$

Illustration 13 : If $(1+x)^n = \sum_{r=0}^n {}^nC_r x^r$, then prove that $C_1^2 + 2.C_2^2 + 3.C_3^2 + \dots + n.C_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$

$$\text{Solution : } (1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n \quad \text{.....(i)}$$

Differentiating both the sides, w.r.t. x , we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \dots + n.C_nx^{n-1} \quad \text{.....(ii)}$$

also, we have

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n \quad \text{.....(iii)}$$

Multiplying (ii) & (iii), we get

$$(C_1 + 2C_2x + 3C_3x^2 + \dots + C_nx^{n-1})(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n) = n(1+x)^{2n-1}$$

Equating the coefficients of x^{n-1} , we get

$$C_1^2 + 2C_2^2 + 3C_3^2 + \dots + n.C_n^2 = n. {}^{2n-1}C_{n-1} = \frac{(2n-1)!}{((n-1)!)^2}$$

Ans.

Illustration 14 : Prove that : $C_0 - 3C_1 + 5C_2 - \dots + (-1)^n(2n+1)C_n = 0$

$$\text{Solution : } T_r = (-1)^r(2r+1)C_r = 2(-1)^r \cdot {}^nC_r + (-1)^r {}^nC_r$$

$$\Sigma T_r = 2 \sum_{r=1}^n (-1)^r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^n (-1)^r {}^nC_r = 2 \sum_{r=1}^n (-1)^r \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^n (-1)^r \cdot {}^nC_r$$

$$= 2[{}^{n-1}C_0 - {}^{n-1}C_1 + \dots] + [{}^nC_0 - {}^nC_1 + \dots] = 0$$

Illustration 15 : Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$

Solution : $(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \dots + (-1)^n {}^{2n}C_{2n}x^{2n}$ (i)

and $(x + 1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \dots + {}^{2n}C_{2n}$ (ii)

Multiplying (i) and (ii), we get

$$(x^2 - 1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + \dots + (-1)^n {}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + \dots + {}^{2n}C_{2n}) \quad \text{....(iii)}$$

Now, coefficient of x^{2n} in R.H.S.

$$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + (-1)^n ({}^{2n}C_{2n})^2$$

$$\therefore \text{General term in L.H.S., } T_{r+1} = {}^{2n}C_r (x^2)^{2n-r} (-1)^r$$

$$\text{Putting } 2(2n - r) = 2n$$

$$\therefore r = n$$

$$\therefore T_{n+1} = {}^{2n}C_n x^{2n} (-1)^n$$

$$\text{Hence coefficient of } x^{2n} \text{ in L.H.S.} = (-1)^n \cdot {}^{2n}C_n$$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n \cdot {}^{2n}C_n$$

Illustration 16 : Prove that : ${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{2n-2}C_n + {}^nC_2 \cdot {}^{2n-4}C_n + \dots = 2^n$

Solution : L.H.S. = Coefficient of x^n in $[{}^nC_0(1+x)^{2n} - {}^nC_1(1+x)^{2n-2} + \dots]$

$$= \text{Coefficient of } x^n \text{ in } [(1+x)^2 - 1]^n$$

$$= \text{Coefficient of } x^n \text{ in } x^n(x+2)^n = 2^n$$

Illustration 17 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ then show that the sum of the products of

the C_i 's taken two at a time represented by : $\sum_{0 \leq i < j \leq n} C_i C_j$ is equal to $2^{2n-1} - \frac{2n!}{2 \cdot n! \cdot n!}$

Solution : Since $(C_0 + C_1 + C_2 + \dots + C_{n-1} + C_n)^2$

$$= C_0^2 + C_1^2 + C_2^2 + \dots + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + \dots + C_0C_n + C_1C_2 + C_1C_3 + \dots + C_1C_n + C_2C_3 + C_2C_4 + \dots + C_2C_n + \dots + C_{n-1}C_n)$$

$$(2^n)^2 = {}^{2n}C_n + 2 \sum_{0 \leq i < j \leq n} C_i C_j$$

Hence $\sum_{0 \leq i < j \leq n} C_i C_j = 2^{2n-1} - \frac{2n!}{2 \cdot n! \cdot n!}$ **Ans.**

Illustration 18 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ then prove that $\sum_{0 \leq i < j \leq n} (C_i + C_j)^2 = (n-1) {}^{2n}C_n + 2^{2n}$

Solution : L.H.S. $\sum_{0 \leq i < j \leq n} (C_i + C_j)^2$

$$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \dots + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \dots + (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \dots + (C_2 + C_n)^2 + \dots + (C_{n-1} + C_n)^2$$

$$= n(C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2) + 2 \sum_{0 \leq i < j \leq n} C_i C_j$$

$$= n \cdot {}^{2n}C_n + 2 \cdot \left\{ 2^{2n-1} - \frac{2n!}{2 \cdot n! \cdot n!} \right\} \quad \text{\{from Illustration 17\}}$$

$$= n \cdot {}^{2n}C_n + 2^{2n} - 2^n C_n = (n-1) \cdot {}^{2n}C_n + 2^{2n} = \text{R.H.S.}$$

Do yourself - 4 :

- (i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} =$
 (A) 2^{n-1} (B) $2^n C_n$ (C) 2^n (D) 2^{n+1}
- (ii) If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, $n \in \mathbb{N}$. Prove that
- (a) $3C_0 - 8C_1 + 13C_2 - 18C_3 + \dots$ upto $(n+1)$ terms $= 0$, if $n \geq 2$.
- (b) $2C_0 + 2^2 \frac{C_1}{2} + 2^3 \frac{C_2}{3} + 2^4 \frac{C_3}{4} + \dots + 2^{n+1} \frac{C_n}{n+1} = \frac{3^{n+1} - 1}{n+1}$
- (c) $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + \dots + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{((n+1)!)^2}$

5. MULTINOMIAL THEOREM :

Using binomial theorem, we have $(x+a)^n = \sum_{r=0}^n {}^n C_r x^{n-r} a^r$, $n \in \mathbb{N}$

$$= \sum_{r=0}^n \frac{n!}{(n-r)!r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r, \text{ where } s+r=n$$

This result can be generalized in the following form.

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{r_1+r_2+\dots+r_k=n} \frac{n!}{r_1!r_2!\dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$$

$$\text{The general term in the above expansion } \frac{n!}{r_1!r_2!r_3!\dots r_k!} x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_k^{r_k}$$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + \dots + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases :

$$(i) \quad (x+y+z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

$$(ii) \quad (x+y+z+u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Illustration 19 : Find the coefficient of $x^2 y^3 z^4 w$ in the expansion of $(x-y-z+w)^{10}$

Solution : $(x-y-z+w)^{10} = \sum_{p+q+r+s=10} \frac{10!}{p!q!r!s!} (x)^p (-y)^q (-z)^r (w)^s$

We want to get $x^2 y^3 z^4 w$ this implies that $p=2, q=3, r=4, s=1$

$$\therefore \text{Coefficient of } x^2 y^3 z^4 w \text{ is } \frac{10!}{2!3!4!1!} (-1)^3 (-1)^4 = -12600$$

Ans.

Illustration 20 : Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3 .

Solution : Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

$$\text{Coefficient of } x^2y^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$$

Ans.

Illustration 21 : Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution : The general term in the expansion of $(2 - x + 3x^2)^6 = \frac{6!}{r!s!t!} 2^r (-x)^s (3x^2)^t$,

where $r + s + t = 6$.

$$= \frac{6!}{r!s!t!} 2^r \times (-1)^s \times (3)^t \times x^{s+2t}$$

For the coefficient of x^5 , we must have $s + 2t = 5$.

But, $r + s + t = 6$,

$\therefore s = 5 - 2t$ and $r = 1 + t$, where $0 \leq r, s, t \leq 6$.

Now $t = 0 \Rightarrow r = 1, s = 5$.

$t = 1 \Rightarrow r = 2, s = 3$.

$t = 2 \Rightarrow r = 3, s = 1$.

Thus, there are three terms containing x^5 and coefficient of x^5

$$= \frac{6!}{1! 5! 0!} \times 2^1 \times (-1)^5 \times 3^0 + \frac{6!}{2! 3! 1!} \times 2^2 \times (-1)^3 \times 3^1 + \frac{6!}{3! 1! 2!} \times 2^3 \times (-1)^1 \times 3^2$$

$$= -12 - 720 - 4320 = -5052.$$

Ans.

Illustration 22 : If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that (a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \frac{1}{2}(3^n - a_n)$

Solution : (a) We have

$$(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r \quad \dots(A)$$

Replace x by $\frac{1}{x}$

$$\therefore \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r \left(\frac{1}{x}\right)^r$$

$$\Rightarrow (x^2 + x + 1)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$$

$$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \quad \{\text{Using (A)}\}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$$a_{2n-r} = a_r \text{ for } 0 \leq r \leq 2n.$$

Hence $a_r = a_{2n-r}$.

(b) Putting $x=1$ in given series, then

$$\begin{aligned} a_0 + a_1 + a_2 + \dots + a_{2n} &= (1+1+1)^n \\ a_0 + a_1 + a_2 + \dots + a_{2n} &= 3^n \quad \dots(1) \end{aligned}$$

But $a_r = a_{2n-r}$ for $0 \leq r \leq 2n$

\therefore series (1) reduces to

$$2(a_0 + a_1 + a_2 + \dots + a_{n-1}) + a_n = 3^n.$$

$$\therefore a_0 + a_1 + a_2 + \dots + a_{n-1} = \frac{1}{2}(3^n - a_n)$$

Do yourself - 5 :

(i) Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

6. APPLICATION OF BINOMIAL THEOREM :

Illustration 23 : If $(6\sqrt{6} + 14)^{2n+1} = [N] + F$ and $F = N - [N]$; where $[.]$ denotes greatest integer function, then NF is equal to

- (A) 20^{2n+1} (B) an even integer (C) odd integer (D) 40^{2n+1}

Solution : Since $(6\sqrt{6} + 14)^{2n+1} = [N] + F$

Let us assume that $f = (6\sqrt{6} - 14)^{2n+1}$; where $0 \leq f < 1$.

$$\begin{aligned} \text{Now, } [N] + F - f &= (6\sqrt{6} + 14)^{2n+1} - (6\sqrt{6} - 14)^{2n+1} \\ &= 2 \left[{}^{2n+1}C_1 (6\sqrt{6})^{2n} (14) + {}^{2n+1}C_3 (6\sqrt{6})^{2n-2} (14)^3 + \dots \right] \end{aligned}$$

$$\Rightarrow [N] + F - f = \text{even integer.}$$

Now $0 < F < 1$ and $0 < f < 1$

so $-1 < F - f < 1$ and $F - f$ is an integer so it can only be zero

$$\text{Thus } NF = (6\sqrt{6} + 14)^{2n+1} (6\sqrt{6} - 14)^{2n+1} = 20^{2n+1}.$$

Ans. (A,B)

Illustration 24 : Find the last three digits in 11^{50} .

Solution : Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \dots + {}^{50}C_{48} 10^2 + {}^{50}C_{49} 10 + {}^{50}C_{50}$
 $= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \dots + {}^{50}C_{47} 10^3}_{1000K} + 49 \times 25 \times 100 + 500 + 1$

$$\Rightarrow 1000K + 123001$$

$$\Rightarrow \text{Last 3 digits are 001.}$$

Illustration 25 : Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution : When 2222 is divided by 7 it leaves a remainder 3.

So adding & subtracting 3^{5555} , we get :

$$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$$

For E_1 : Now since $2222-3 = 2219$ is divisible by 7, therefore E_1 is divisible by 7

($\because x^n - a^n$ is divisible by $x - a$)

For E_2 : 5555 when divided by 7 leaves remainder 4.

So adding and subtracting 4^{2222} , we get :

$$\begin{aligned} E_2 &= 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222} \\ &= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222} \end{aligned}$$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7

($\because x^n + a^n$ is divisible by $x + a$ when n is odd)

Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

Do yourself - 6 :

- (i) Prove that $5^{25} - 3^{25}$ is divisible by 2.
- (ii) Find the remainder when the number 9^{100} is divided by 8.
- (iii) Find last three digits in 19^{100} .
- (iv) Let $R = (8 + 3\sqrt{7})^{20}$ and $[.]$ denotes greatest integer function, then prove that :
 - (a) $[R]$ is odd
 - (b) $R - [R] = 1 - \frac{1}{(8 + 3\sqrt{7})^{20}}$
- (v) Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$.

7. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in \mathbb{Q}$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \infty$ provided $|x| < 1$.

Note :

- (i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. $(n+1)$ & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$
- (ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol nC_r cannot be used to denote the coefficient of the general term.
- (iii) Following expansion should be remembered ($|x| < 1$).
 - (a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots \infty$
 - (b) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots \infty$
 - (c) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots \infty$
 - (d) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \infty$
 - (e) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + \frac{(-1)^r(r+1)(r+2)}{2!}x^r + \dots$
 - (f) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{2!}x^r + \dots$
- (iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

8. APPROXIMATIONS :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1+x)^n = 1 + nx$, approximately.

This is an approximate value of $(1+x)^n$

Illustration 26 : If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}}$

Solution :

$$\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1 - \frac{3}{2}x + 1 - \frac{5x}{3}}{2\left(1 + \frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 + \frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 - \frac{x}{8}\right)$$

$$= \frac{1}{2}\left(2 - \frac{x}{4} - \frac{19}{6}x\right) = 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$$

Ans.

Illustration 27 : The value of cube root of 1001 upto five decimal places is –

- (A) 10.03333 (B) 10.00333 (C) 10.00033 (D) none of these

Solution :

$$(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1 + \frac{1}{1000}\right)^{1/3} = 10\left\{1 + \frac{1}{3} \cdot \frac{1}{1000} + \frac{1/3(1/3-1)}{2!} \frac{1}{1000^2} + \dots\right\}$$

$$= 10\{1 + 0.0003333 - 0.00000011 + \dots\} = 10.00333$$

Ans. (B)

Illustration 28 : The sum of $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \dots$ is –

- (A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution : Comparing with $1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

$$nx = 1/4 \quad \dots\dots(i)$$

and $\frac{n(n-1)x^2}{2!} = \frac{1.3}{4.8}$

or $\frac{nx(nx-x)}{2!} = \frac{3}{32} \Rightarrow \frac{1}{4}\left(\frac{1}{4} - x\right) = \frac{3}{16} \quad \text{(by (i))}$

$$\Rightarrow \left(\frac{1}{4} - x\right) = \frac{3}{4} \Rightarrow x = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2} \quad \dots\dots(ii)$$

putting the value of x in (i)

$$n(-1/2) = 1/4 \Rightarrow n = -1/2$$

\therefore sum of series $= (1+x)^n = (1-1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$

Ans. (A)

9. EXPONENTIAL SERIES :

- (a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.
- (b) Logarithms to the base ' e ' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm**.
- (c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$; where x may be any real or complex number & $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$
- (d) $a^x = 1 + \frac{x}{1!} \ln a + \frac{x^2}{2!} \ln^2 a + \frac{x^3}{3!} \ln^3 a + \dots \infty$, where $a > 0$
- (e) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \infty$

10. LOGARITHMIC SERIES :

- (a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$, where $-1 < x \leq 1$
- (b) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$, where $-1 \leq x < 1$

- Remember :**
- (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty = \ln 2$ (ii) $e^{\ln x} = x$; for all $x > 0$
- (iii) $\ln 2 = 0.693$ (iv) $\ln 10 = 2.303$

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. If the coefficients of x^7 & x^8 in the expansion of $\left[2 + \frac{x}{3}\right]^n$ are equal, then the value of n is :

(A) 15 (B) 45 (C) 55 (D) 56

BT0001

2. Set of value of r for which, ${}^{18}C_{r-2} + 2 \cdot {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains :

(A) 4 element (B) 5 elements (C) 7 elements (D) 10 elements

BT0002

3. If the constant term of the binomial expansion $\left(2x - \frac{1}{x}\right)^n$ is -160 , then n is equal to -

(A) 4 (B) 6 (C) 8 (D) 10

BT0003

4. The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)\dots\left(x-\frac{1}{2^{49}}\right)$ is equal to -

(A) $-2\left(1 - \frac{1}{2^{50}}\right)$

(B) +ve coefficient of x

(C) -ve coefficient of x

(D) $-2\left(1 - \frac{1}{2^{49}}\right)$

BT0004

5. The largest real value for x such that $\sum_{k=0}^4 \left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right) = \frac{8}{3}$ is -

(A) $2\sqrt{2} - 5$ (B) $2\sqrt{2} + 5$ (C) $-2\sqrt{2} - 5$ (D) $-2\sqrt{2} + 5$

BT0005

6. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

BT0006

7. Number of rational terms in the expansion of $(\sqrt{2} + \sqrt[4]{3})^{100}$ is :

(A) 25 (B) 26 (C) 27 (D) 28

BT0007

8. Given $(1 - 2x + 5x^2 - 10x^3)(1+x)^n = 1 + a_1x + a_2x^2 + \dots$ and that $a_1^2 = 2a_2$ then the value of n is-

(A) 6 (B) 2 (C) 5 (D) 3

BT0008

9. The sum of the co-efficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is -
 (A) 2.6^{10} (B) 3.6^{10} (C) 6^{11} (D) none

BT0009

10. Co-efficient of α^t in the expansion of,
 $(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3}(\alpha + q)^2 + \dots + (\alpha + q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is :
 (A) $\frac{{}^m C_t (p^t - q^t)}{p - q}$ (B) $\frac{{}^m C_t (p^{m-t} - q^{m-t})}{p - q}$ (C) $\frac{{}^m C_t (p^t + q^t)}{p - q}$ (D) $\frac{{}^m C_t (p^{m-t} + q^{m-t})}{p - q}$

BT0010

11. Let $\binom{n}{k}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum
 $\binom{99}{97} + \binom{98}{96} + \binom{97}{95} + \dots + \binom{3}{1} + \binom{2}{0}$ equals -
 (A) $\binom{99}{97}$ (B) $\binom{100}{98}$ (C) $\binom{99}{98}$ (D) $\binom{100}{97}$

BT0011

[COMPREHENSION TYPE]

Paragraph for question nos. 12 to 14

If $n \in \mathbb{N}$ and if $(1 + 4x + 4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where $a_0, a_1, a_2, \dots, a_{2n}$ are real numbers.

12. The value of $2 \sum_{r=0}^n a_{2r}$, is
 (A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$
13. The value of $2 \sum_{r=1}^n a_{2r-1}$, is-
 (A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$
14. The value of a_{2n-1} is -
 (A) 2^{2n} (B) $n \cdot 2^{2n}$ (C) $(n-1)2^{2n}$ (D) $(n+1)2^{2n}$
15. If $n \in \mathbb{N}$ & n is even, then $\frac{1}{1 \cdot (n-1)!} + \frac{1}{3! \cdot (n-3)!} + \frac{1}{5! \cdot (n-5)!} + \dots + \frac{1}{(n-1)! \cdot 1!} =$
 (A) 2^n (B) $\frac{2^{n-1}}{n!}$ (C) $2^n n!$ (D) none of these

BT0012

BT0012

BT0012

BT0013

EXERCISE (O-2)

[ONE OR MORE THAN ONE CORRECT CHOICE TYPE]

1. If it is known that the third term of the binomial expansion $(x + x^{\log_{10} x})^5$ is 10^6 then x is equal to-
 (A) 10 (B) $10^{-5/2}$ (C) 100 (D) 5 BT0014
2. In the expansion of $(x^3 + 3 \cdot 2^{-\log_{\sqrt{2}} \sqrt{x^3}})^{11}$
 (A) there appears a term with the power x^2 (B) there does not appear a term with the power x^2
 (C) there appears a term with the power x^{-3} (D) the ratio of the co-efficient of x^3 to that of x^{-3} is $1/3$ BT0015
3. In the expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right)^{10}$, the term which does not contain x is-
 (A) ${}^{11}C_4 - {}^{10}C_3$ (B) ${}^{10}C_7$ (C) ${}^{10}C_4$ (D) ${}^{11}C_5 - {}^{10}C_5$ BT0016
4. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1 x + A_2 x^2 + \dots$. If A_0, A_1, A_2 are in A.P. then the value of n is-
 (A) 2 (B) 3 (C) 5 (D) 7 BT0017
5. Consider $E = \left(\sqrt[8]{x} + \sqrt[5]{y} \right)^z = I + f, 0 \leq f < 1$
 (A) If $x = 5, y = 2, z = 100$, then number of irrational terms in expansion of E is 98
 (B) If $x = 5, y = 2, z = 100$, then number of rational terms in expansion of E is 4
 (C) If $x = 16, y = 1$ & $z = 6$, then $I = 197$
 (D) If $x = 16, y = 1$ & $z = 6$, then $f = (\sqrt{2} - 1)^6$ BT0018
6. Greatest term in the binomial expansion of $(a + 2x)^9$ when $a = 1$ & $x = \frac{1}{3}$ is :
 (A) 3^{rd} & 4^{th} (B) 4^{th} & 5^{th} (C) only 4^{th} (D) only 5^{th} BT0019
7. Let $(5 + 2\sqrt{6})^n = p + f$ where $n, p \in \mathbb{N}$ and $0 < f < 1$ then the value of $f^2 - f + pf - p$ is -
 (A) a natural number (B) a negative integer (C) a prime number (D) are irrational number BT0020
8. If $(9 + \sqrt{80})^n = I + f$ where I, n are integers and $0 < f < 1$, then -
 (A) I is an odd integer (B) I is an even integer
 (C) $(I + f)(1 - f) = 1$ (D) $1 - f = (9 - \sqrt{80})^n$ BT0021

9. If $\sum_{r=1}^{10} r(r-1) {}^{10}C_r = k \cdot 2^9$, then k is equal to-
- (A) 10 (B) 45 (C) 90 (D) 100

BT0022

10. The sum $\frac{{}^{11}C_0}{1} + \frac{{}^{11}C_1}{2} + \frac{{}^{11}C_2}{3} + \dots + \frac{{}^{11}C_{11}}{12}$ equals $\left(\text{where } \binom{n}{r} \text{ denotes } {}^nC_r \right)$
- (A) $\frac{2^{11}}{12}$ (B) $\frac{2^{12}}{12}$
(C) $\frac{2^{11}-1}{12}$ (D) $\frac{2^{12}-1}{12}$

BT0023

11. **Statement-1 :** The sum of the series ${}^nC_0 \cdot {}^mC_r + {}^nC_1 \cdot {}^mC_{r-1} + {}^nC_2 \cdot {}^mC_{r-2} + \dots + {}^nC_r \cdot {}^mC_0$ is equal to ${}^{n+m}C_r$, where nC_r 's and mC_r 's denotes the combinatorial coefficients in the expansion of $(1+x)^n$ and $(1+x)^m$ respectively.
- Statement-2 :** Number of ways in which r children can be selected out of (n+m) children consisting of n boys and m girls if each selection may consist of any number of boys and girls is equal to ${}^{n+m}C_r$.
- (A) Statement-1 is true, statement-2 is true ; statement-2 is a correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true ; statement-2 is NOT a correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

BT0024

12. Which of the following statement(s) is/are correct ?

- (A) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \dots + \infty = 4$
(B) Integral part of $(9+4\sqrt{5})^n$, $n \in \mathbb{N}$ is even.
(C) $({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n)^2 = 1 + {}^{2n}C_1 + {}^{2n}C_2 + \dots + {}^{2n}C_{2n}$
(D) $\frac{1}{(3+2x)^2}$ can be expanded as infinite series in ascending powers of x only if $|x| < \frac{2}{3}$.

BT0025

13. If for $n \in \mathbb{I}$, $n > 10$; $1 + (1+x) + (1+x)^2 + \dots + (1+x)^n = \sum_{k=0}^n a_k \cdot x^k$, $x \neq 0$ then

- (A) $\sum_{k=0}^n a_k = 2^{n+1}$ (B) $a_{n-2} = \frac{n(n+1)}{2}$
(C) $a_p > a_{p-1}$ for $p < \frac{n}{2}$, $p \in \mathbb{N}$ (D) $(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10} ({}^{n+1}C_{10} - {}^{n+1}C_9)$

BT0026

14. Let $P(n) = \sum_{r=0}^n \frac{(-1)^r}{r+1} {}^nC_r$. Now which of the following holds good ?

(A) $|P_{10}|$ is harmonic mean of $|P_9|$ & $|P_{11}|$ (B) $\sum_{r=5}^{10} P(r)P(r-1) = -\frac{6}{55}$
 (C) $|P_{10}|$ is arithmetic mean of $|P_9|$ & $|P_{11}|$ (D) $\sum_{r=5}^{10} P(r)P(r-1) = \frac{6}{55}$

BT0027

15. Let $(1+x)^m = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_mx^m$, where $C_r = {}^mC_r$ and $A = C_1C_3 + C_2C_4 + C_3C_5 + C_4C_6 + \dots + C_{m-2}C_m$, then -

(A) $A \geq {}^{2m}C_{m-2}$ (B) $A < {}^{2m}C_{m-2}$
 (C) $A > C_0^2 + C_1^2 + C_2^2 + \dots + C_m^2$ (D) $A < C_0^2 + C_1^2 + C_2^2 + \dots + C_m^2$

BT0028

EXERCISE (S-1)

1. (a) If the coefficients of $(2r+4)^{\text{th}}$, $(r-2)^{\text{th}}$ terms in the expansion of $(1+x)^{18}$ are equal, find r .
 (b) If the coefficients of the r^{th} , $(r+1)^{\text{th}}$ & $(r+2)^{\text{th}}$ terms in the expansion of $(1+x)^{14}$ are in AP, find r .
 (c) If the coefficients of 2^{nd} , 3^{rd} & 4^{th} terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

BT0031

2. Find the term independent of x in the expansion of (i) $\left[\sqrt{\frac{x}{3}} + \frac{\sqrt{3}}{2x^2}\right]^{10}$ (ii) $\left[\frac{1}{2}x^{1/3} + x^{-1/5}\right]^8$

BT0032

3. Prove that the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ & the term independent of x in $\left(x - \frac{2}{x}\right)^{10}$ is $1 : 32$.

BT0033

4. Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$.

BT0034

5. Let $(1+x^2)^2 \cdot (1+x)^n = \sum_{k=0}^{n+4} a_k \cdot x^k$. If a_1, a_2 & a_3 are in AP, find n .

BT0035

6. Let $f(x) = 1 - x + x^2 - x^3 + \dots + x^{16} - x^{17} = a_0 + a_1(1+x) + a_2(1+x)^2 + \dots + a_{17}(1+x)^{17}$, find the value of a_2 .

BT0036

7. Find the coefficient of x^r in the expression :

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \dots + (x+2)^{n-1}$$

BT0037

8. Find numerically greatest term in the expansion of :

(i) $(2+3x)^9$ when $x = \frac{3}{2}$

BT0038

(ii) $(3-5x)^{15}$ when $x = \frac{1}{5}$

BT0039

9. (a) Show that the integral part in each of the following is odd. $n \in \mathbb{N}$

(A) $(5+2\sqrt{6})^n$ (B) $(8+3\sqrt{7})^n$

BT0040

- (b) Show that the integral part in each of the following is even. $n \in \mathbb{N}$

(A) $(3\sqrt{3}+5)^{2n+1}$ (B) $(5\sqrt{5}+11)^{2n+1}$

BT0041

10. Let $N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + \dots + 2000 \cdot {}^{2000}C_{2000}$. Prove that N is divisible by 2^{2003} .

BT0042

11. Prove the following identities using the theory of permutation where $C_0, C_1, C_2, \dots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in \mathbb{N}$:

(a) $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n!n!}$

BT0043

(b) $C_0 C_1 + C_1 C_2 + C_2 C_3 + \dots + C_{n-1} C_n = \frac{(2n)!}{(n+1)!(n-1)!}$

BT0044

(c) $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n = \frac{2n!}{(n-r)!(n+r)!}$

BT0045

(d) $\sum_{r=0}^{n-2} ({}^n C_r \cdot {}^n C_{r+2}) = \frac{(2n)!}{(n-2)!(n+2)!}$

BT0046

(e) ${}^{100}C_{10} + 5 \cdot {}^{100}C_{11} + 10 \cdot {}^{100}C_{12} + 10 \cdot {}^{100}C_{13} + 5 \cdot {}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$

BT0047

12. If $C_0, C_1, C_2, \dots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in \mathbb{N}$, then prove the following :

(a) $C_1 + 2C_2 + 3C_3 + \dots + n \cdot C_n = n \cdot 2^{n-1}$

BT0048

(b) $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = (n+2)2^{n-1}$

BT0049

(c) $C_0 + 3C_1 + 5C_2 + \dots + (2n+1)C_n = (n+1)2^n$

BT0050

(d) $(C_0+C_1)(C_1+C_2)(C_2+C_3) \dots (C_{n-1}+C_n) = \frac{C_0 \cdot C_1 \cdot C_2 \dots C_{n-1} (n+1)^n}{n!}$

BT0051

(e) $1 \cdot C_0^2 + 3 \cdot C_1^2 + 5 \cdot C_2^2 + \dots + (2n+1) C_n^2 = \frac{(n+1)(2n)!}{n!n!}$

BT0052

13. Prove that

(a) $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \dots + \frac{n \cdot C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

BT0053

$$(b) \quad C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$$

BT0054

$$(c) \quad 2 \cdot C_0 + \frac{2^2 \cdot C_1}{2} + \frac{2^3 \cdot C_2}{3} + \frac{2^4 \cdot C_3}{4} + \dots + \frac{2^{n+1} \cdot C_n}{n+1} = \frac{3^{n+1} - 1}{n+1}$$

BT0055

$$(d) \quad C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$$

BT0056

14. Given that $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, find the values of :

$$(i) \quad a_0 + a_1 + a_2 + \dots + a_{2n} ;$$

$$(ii) \quad a_0 - a_1 + a_2 - a_3 + \dots + a_{2n} ;$$

$$(iii) \quad a_0^2 - a_1^2 + a_2^2 - a_3^2 + \dots + a_{2n}^2$$

BT0057

15. Find the sum of the series $\sum_{r=0}^n (-1)^r \cdot {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \dots \text{up to } m \text{ terms} \right]$

BT0058

16. Find the coefficient of

$$(a) \quad x^4 \text{ in the expansion of } (1 + x + x^2 + x^3)^{11}$$

BT0059

$$(b) \quad x^4 \text{ in the expansion of } (2 - x + 3x^2)^6$$

BT0060

17. Find the coefficient of

$$(a) \quad x^2 y^3 z^4 \text{ in the expansion of } (ax - by + cz)^9.$$

BT0061

$$(b) \quad a^2 b^3 c^4 d \text{ in the expansion of } (a - b - c + d)^{10}.$$

BT0062

EXERCISE (S-2)

1. Let a and b be the coefficient of x^3 in $(1 + x + 2x^2 + 3x^3)^4$ and $(1 + x + 2x^2 + 3x^3 + 4x^4)^4$ respectively. Find the value of $(a - b)$.

BT0063

2. Find the index n of the binomial $\left(\frac{x}{5} + \frac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient ($n \in \mathbb{N}$).

BT0064

3. Find the sum of the roots (real or complex) of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$.

BT0065

4. Let $a = (4^{1/401} - 1)$ and let $b_n = {}^nC_1 + {}^nC_2 \cdot a + {}^nC_3 \cdot a^2 + \dots + {}^nC_n \cdot a^{n-1}$. Find the value of $(b_{2006} - b_{2005})$

BT0066

5. For which positive values of x , fourth term in the expansion of $(5 + 3x)^{10}$, is greatest.

BT0067

6. Let $P = (2 + \sqrt{3})^5$ and $f = P - [P]$, where $[P]$ denotes the greatest integer function.

Find the value of $\left(\frac{f^2}{1-f}\right)$.

BT0068

7. If $(7 + 4\sqrt{3})^n = p + \beta$ where n & p are positive integers and β is a proper fraction show that $(1 - \beta)(p + \beta) = 1$.

BT0069

8. Find the coefficient of x^{49} in the polynomial

$$\left(x - \frac{C_1}{C_0}\right) \left(x - 2^2 \cdot \frac{C_2}{C_1}\right) \left(x - 3^2 \cdot \frac{C_3}{C_2}\right) \dots \dots \dots \left(x - 50^2 \cdot \frac{C_{50}}{C_{49}}\right), \text{ where } C_r = {}^{50}C_r.$$

BT0070

9. Prove that $\sum_{K=0}^n {}^nC_K \sin Kx \cdot \cos(n-K)x = 2^{n-1} \sin nx$.

BT0071

10. If $\binom{n}{r}$ denotes nC_r , then

(a) Evaluate : $2^{15} \binom{30}{0} \binom{30}{15} - 2^{14} \binom{30}{1} \binom{29}{14} + 2^{13} \binom{30}{2} \binom{28}{13} \dots - \binom{30}{15} \binom{15}{0}$

BT0072

(b) Prove that : $\sum_{r=1}^n \binom{n-1}{n-r} \binom{n}{r} = \binom{2n-1}{n-1}$

BT0073

(c) Prove that : $\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$

BT0074

EXERCISE (JM)

1. Let $S_1 = \sum_{j=1}^{10} j(j-1)^{10} C_j$, $S_2 = \sum_{j=1}^{10} j^{10} C_j$ and $S_3 = \sum_{j=1}^{10} j^{210} C_j$.

[AIEEE-2010]

Statement-1 : $S_3 = 55 \times 2^9$.

Statement-2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

- (1) Statement-1 is true, Statement-2 is true ; Statement-2 is a correct explanation for Statement-1.
- (2) Statement-1 is true, Statement-2 is true ; Statement-2 is not a correct explanation for Statement-1.
- (3) Statement-1 is true, Statement-2 is false.
- (4) Statement-1 is false, Statement-2 is true.

BT0075

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- [AIEEE 2011]
 (1) -144 (2) 132 (3) 144 (4) -132
BT0076
3. If n is a positive integer, then $(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$ is : [AIEEE 2012]
 (1) a rational number other than positive integers (2) an irrational number
 (3) an odd positive integer (4) an even positive integer
BT0077
4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is : [JEE-Main 2013]
 (1) 4 (2) 120 (3) 210 (4) 310
BT0078
5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to :- [JEE(Main)-2014]
 (1) $\left(16, \frac{251}{3}\right)$ (2) $\left(14, \frac{251}{3}\right)$ (3) $\left(14, \frac{272}{3}\right)$ (4) $\left(16, \frac{272}{3}\right)$
BT0079
6. The sum of coefficients of integral powers of x in the binomial expansion of $(1 - 2\sqrt{x})^{50}$ is : [JEE(Main)-2015]
 (1) $\frac{1}{2}(3^{50} - 1)$ (2) $\frac{1}{2}(2^{50} + 1)$ (3) $\frac{1}{2}(3^{50} + 1)$ (4) $\frac{1}{2}(3^{50})$
BT0080
7. If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is :- [JEE(Main)-2016]
 (1) 729 (2) 64 (3) 2187 (4) 243
BT0081
8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \dots + ({}^{21}C_{10} - {}^{10}C_{10})$ is :- [JEE(Main)-2017]
 (1) $2^{20} - 2^{10}$ (2) $2^{21} - 2^{11}$ (3) $2^{21} - 2^{10}$ (4) $2^{20} - 2^9$
BT0082
9. The sum of the co-efficients of all odd degree terms in the expansion of $(x + \sqrt{x^3 - 1})^5 + (x - \sqrt{x^3 - 1})^5$, $(x > 1)$ is - [JEE(Main)-2018]
 (1) 0 (2) 1 (3) 2 (4) -1
BT0083
10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : [JEE(Main)- 2019]
 (1) 14 (2) 6 (3) 4 (4) 8
BT0084

11. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is [JEE(Main)- 2019]
 (1) 12 (2) 15 (3) 10 (4) 14
BT0085
12. If $\sum_{r=0}^{25} \left\{ {}^{50}C_r \cdot {}^{50-r}C_{25-r} \right\} = K \left({}^{50}C_{25} \right)$, then K is equal to : [JEE(Main)- 2019]
 (1) $2^{25} - 1$ (2) $(25)^2$ (3) 2^{25} (4) 2^{24}
BT0086
13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3} + \frac{3}{x}\right)^8$ equals 5670 is : [JEE(Main)- 2019]
 (1) 6 (2) 8 (3) 0 (4) 4
BT0087
14. The value of r for which ${}^{20}C_r \cdot {}^{20}C_0 + {}^{20}C_{r-1} \cdot {}^{20}C_1 + {}^{20}C_{r-2} \cdot {}^{20}C_2 + \dots + {}^{20}C_0 \cdot {}^{20}C_r$ is maximum, is [JEE(Main)- 2019]
 (1) 20 (2) 15 (3) 11 (4) 10
BT0088
15. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \dots + a_{50}x^{50}$, for all $x \in \mathbb{R}$, then $\frac{a_2}{a_0}$ is equal to:- [JEE(Main)- 2019]
 (1) 12.50 (2) 12.00 (3) 12.75 (4) 12.25
BT0089
16. Let $S_n = 1 + q + q^2 + \dots + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \dots + \left(\frac{q+1}{2}\right)^n$, where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 \cdot S_1 + \dots + {}^{101}C_{101} \cdot S_{100} = \alpha T_{100}$, then α is equal to :- [JEE(Main)- 2019]
 (1) 2^{100} (2) 200 (3) 2^{99} (4) 202
BT0090
17. The total number of irrational terms in the binomial expansion of $\left(7^{1/5} - 3^{1/10}\right)^{60}$ is : [JEE(Main)- 2019]
 (1) 55 (2) 49 (3) 48 (4) 54
BT0091
18. If some three consecutive in the binomial expansion of $(x + 1)^n$ is powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficient is :- [JEE(Main)- 2019]
 (1) 964 (2) 625 (3) 227 (4) 232
BT0092
19. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1+x+x^2)^9$ is : [JEE(Main)- 2019]
 (1) -84 (2) 84 (3) 126 (4) -126
BT0093

20. If ${}^{20}C_1 + (2^2) {}^{20}C_2 + (3^2) {}^{20}C_3 + \dots + (20^2) {}^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to:
[JEE(Main)- 2019]
 (1) (420, 18) (2) (380, 19) (3) (380, 18) (4) (420, 19)

BT0094

21. The term independent of x in the expansion of $\left(\frac{1}{60} - \frac{x^8}{81}\right) \cdot \left(2x^2 - \frac{3}{x^2}\right)^6$ is equal to :
[JEE(Main)- 2019]
 (1) 36 (2) - 108 (3) - 72 (4) - 36

BT0095

22. The coefficient of x^7 in the expression $(1+x)^{10} + x(1+x)^9 + x^2(1+x)^8 + \dots + x^{10}$ is :
 (1) 120 (2) 330 (3) 210 (4) 420
[JEE(Main)- 2020]
 BT0096

23. If the sum of the coefficients of all even powers of x in the product $(1+x+x^2+\dots+x^{2n})(1-x+x^2-x^3+\dots+x^{2n})$ is 61, then n is equal to _____. **[JEE(Main)- 2020]**
 BT0097

24. If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$, then
[JEE(Main)- 2020]
 (1) $\alpha + \beta = 60$ (2) $\alpha + \beta = -30$ (3) $\alpha - \beta = -132$ (4) $\alpha - \beta = 60$

BT0098

25. In the expansion of $\left(\frac{x}{\cos \theta} + \frac{1}{x \sin \theta}\right)^{16}$, if ℓ_1 is the least value of the term independent of x when $\frac{\pi}{8} \leq \theta \leq \frac{\pi}{4}$ and ℓ_2 is the least value of the term independent of x when $\frac{\pi}{16} \leq \theta \leq \frac{\pi}{8}$, then the ratio $\ell_2 : \ell_1$ is equal to :
[JEE(Main)- 2020]
 (1) 1 : 8 (2) 1 : 16 (3) 8 : 1 (4) 16 : 1

BT0099

26. If $C_r \equiv {}^{25}C_r$ and $C_0 + 5.C_1 + 9.C_2 + \dots + (101).C_{25} = 2^{25}.k$, then k is equal to _____. **[JEE(Main)- 2020]**
 BT0100

EXERCISE (JA)

1. For $r = 0, 1, \dots, 10$, let A_r , B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then $\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$ is equal to -
 (A) $B_{10} - C_{10}$ (B) $A_{10} (B_{10}^2 - C_{10} A_{10})$ (C) 0 (D) $C_{10} - B_{10}$

[JEE 2010, 5]

BT0101

2. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5 : 10 : 14. Then $n =$
[JEE (Advanced) 2013, 4M, -1M]

BT0102

3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is -
 (A) 1051 (B) 1106 (C) 1113 (D) 1120
[JEE(Advanced)-2014, 3(-1)]
BT0103
4. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3) \dots (1 + x^{100})$ is **[JEE 2015, 4M, -0M]**
BT0104
5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \dots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)^{51}C_3$ for some positive integer n . Then the value of n is **[JEE(Advanced)-2016, 3(0)]**
BT0105
6. Let $X = \binom{10}{1}C_1^2 + 2\binom{10}{2}C_2^2 + 3\binom{10}{3}C_3^2 + \dots + 10\binom{10}{10}C_{10}^2$, where ${}^{10}C_r$, $r \in \{1, 2, \dots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____. **[JEE(Advanced)-2018, 3(0)]**
BT0106

7. Suppose $\det \begin{bmatrix} \sum_{k=0}^n k & \sum_{k=0}^n {}^nC_k k^2 \\ \sum_{k=0}^n {}^nC_k k & \sum_{k=0}^n {}^nC_k 3^k \end{bmatrix} = 0$, holds for some positive integer n . Then $\sum_{k=0}^n \frac{{}^nC_k}{k+1}$ equals **[JEE(Advanced)-2019, 3(0)]**
BT0107

ANSWER KEY

Do yourself-1

$$(i) {}^5C_0 x(3x^2)^5 + {}^5C_1 (3x^2)^4 \left(-\frac{x}{2}\right) + {}^5C_2 (3x^2)^3 \left(-\frac{x}{2}\right)^2 + {}^5C_3 (3x^2)^2 \left(-\frac{x}{2}\right)^3 + {}^5C_4 (3x^2)^1 \left(-\frac{x}{2}\right)^4 + {}^5C_5 \left(-\frac{x}{2}\right)^5$$

$$(ii) {}^nC_0 y^n + {}^nC_1 y^{n-1} \cdot x + {}^nC_2 y^{n-2} \cdot x^2 + \dots + {}^nC_n \cdot x^n$$

Do yourself-2

$$(i) \frac{70}{3} x^8; (ii) \frac{25!}{10! 5!} 2^{15} 3^{10}; (iii) (a) -20; (b) -560x^5, 280x^2$$

Do yourself-3

$$(i) 4^{\text{th}} \& 5^{\text{th}} \text{ i.e. } 489888 \quad (ii) n = 4, 5, 6$$

Do yourself-4

(i) C

Do yourself-5

$$(i) -272160 \text{ or } -{}^{10}C_5 \times {}^5C_2 \times 108$$

Do yourself-6

$$(ii) 1 \quad (iii) 001 \quad (v) 1$$

EXERCISE (O-1)

1. C 2. C 3. B 4. A 5. A 6. C 7. B
8. A 9. B 10. B 11. D 12. B 13. A 14. B 15. B

EXERCISE (O-2)

1. A,B 2. B,C,D 3. A,C,D 4. A,B 5. A,C 6. B 7. B
8. A,C,D 9. B 10. D 11. A 12. A,C 13. B,C,D 14. A,D
15. B,D

EXERCISE (S-1)

1. (a) $r = 6$ (b) $r = 5$ or 9 2. (i) $\frac{5}{12}$ (ii) $T_6 = 7$ 4. $\frac{17}{54}$ 5. $n = 2$ or 3 or 4
6. 816 7. ${}^nC_r (3^{n-r} - 2^{n-r})$ 8. (i) $T_7 = \frac{7 \cdot 3^{13}}{2}$ (ii) 455×3^{12}
14. (i) 3^n (ii) 1, (iii) a_n 15. $\frac{(2^{mn} - 1)}{(2^n - 1)(2^{mn})}$ 16. (a) 990 (b) 3660
17. (a) $-1260 \cdot a^2 b^3 c^4$; (b) -12600

EXERCISE (S-2)

1. 0
2. $n = 12$
3. 500
4. 2^{10}
5. $\frac{5}{8} \leq x \leq \frac{20}{21}$
6. 722
8. -22100
10. (a) $\binom{30}{15}$

EXERCISE (JM)

1. 3
2. 1
3. 2
4. 3
5. 4
6. 3

7. Bonus

Note : In the problem 'number of terms should be 13 instead of 28', then (1) will be the answer

8. 1
9. 3
10. 4
11. 2
12. 3
13. 3
14. 1
15. 4
16. 1
17. 4
18. 4
19. 2
20. 1
21. 4
22. 2
23. 30
24. 3
25. 4
26. 51

EXERCISE (JA)

1. D
2. 6
3. C
4. 8
5. 5
6. 646
7. 6.20