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S. No.	CHAPTER NAME	Pg.No.
1	COMPLEX NUMBERS	01-50
2	PROBABILITY	51-112
3	JEE-MAIN (STATISTICS & REASONING)	113-154

E MATHEMATICS

Chapter Contents 01

COMPLEX NUMBERS

JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.

JEE (Advanced) Syllabus :

Complex Numbers : Complex numbers as ordered pairs of reals, Representation of complex numbers in the form a + ib and their representation in a plane, Argand diagram, algebra of complex numbers, modulus and argument (or amplitude) of a complex number, square root of a complex number, triangle inequality.

Important Notes

COMPLEX NUMBERS

1. DEFINITION :

Complex numbers are defined as expressions of the form $a + ib$ where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $z = a + ib$. 'a' is called real part of z (Re z) and 'b' is called imaginary part of z (Im z).

Every Complex Number Can Be Regarded As

- Purely real if $b = 0$
- Imaginary if $b \neq 0$
  - Purely imaginary if $a = 0$

Note :

(i) The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complex Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

(ii) Zero is both purely real as well as purely imaginary but not imaginary.

(iii) $i = \sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$; $i^3 = -i$; $i^4 = 1$ etc.

In general $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, where $n \in I$

(iv) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

***Illustration* 1 :** The value of $i^{57} + 1/i^{125}$ is :-

(A) 0 (B) $-2i$ (C) $2i$ (D) 2

Solution :

$$i^{57} + 1/i^{125} = i^{56}.\ i + \frac{1}{i^{124}.i}$$

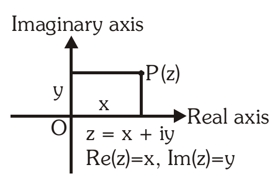
$$= \left(i^4\right)^{14} i + \frac{1}{\left(i^4\right)^{31} i}$$

$$= i + \frac{1}{i} = i + \frac{i}{i^2} = i - i = 0$$

Ans. (A)

2. ARGAND DIAGRAM :

Master Argand had done a systematic study on complex numbers and represented every complex number $z = x + iy$ as a set of ordered pair (x, y) on a plane called complex plane (Argand Diagram) containing two perpendicular axes. Horizontal axis is known as Real axis & vertical axis is known as Imaginary axis.



All complex numbers lying on the real axis are called as purely real and those lying on imaginary axis as purely imaginary.

3. ALGEBRAIC OPERATIONS :

Fundamental operations with complex numbers :

(a) Addition $(a + bi) + (c + di) = (a + c) + (b + d)i$

(b) Subtraction $(a + bi) - (c + di) = (a - c) + (b - d)i$

(c) Multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(d) Division $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di}\cdot\dfrac{c-di}{c-di} = \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

Note :

(i) The algebraic operations on complex numbers are similar to those on real numbers treating i as a polynomial.

(ii) Inequalities in complex numbers (non-real) are not defined. There is no validity if we say that complex number (non-real) is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless.

(iii) In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$ but in complex numbers, $z_1^{\ 2} + z_2^{\ 2} = 0$ does not imply $z_1 = z_2 = 0$.

Illustration 2 : $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if $\theta =$

(A) $2n\pi \pm \dfrac{\pi}{3}, n \in I$ (B) $n\pi + \dfrac{\pi}{3}, n \in I$ (C) $n\pi \pm \dfrac{\pi}{3}, n \in I$ (D) none of these

Solution : $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e.,

$$\frac{(3+2i\sin\theta)}{(1-2i\sin\theta)} \times \frac{(1+2i\sin\theta)}{(1+2i\sin\theta)} = \frac{\left(3-4\sin^2\theta\right)+i\left(8\sin\theta\right)}{\left(1+4\sin^2\theta\right)}$$

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0 \quad \Rightarrow \quad 3 - 4\sin^2\theta = 0 \text{ (only if } \theta \text{ be real)}$$

$$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\sin\frac{\pi}{3}\right)^2$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$$

Ans. (C)

Do yourself - 1 :

(i) Determine least positive value of n for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$

(ii) Find the value of the sum $\sum_{n=1}^{5}(i^n + i^{n+2})$, where $i = \sqrt{-1}$.

4. EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts are respectively equal.

Illustration 3 : The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ are

(A) x = –1, y =3 (B) x = 3, y = –1 (C) x = 0, y = 1 (D) x = 1, y = 0

Solution : $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i \Rightarrow (4 + 2i)\, x + (9 - 7i)\, y - 3i - 3 = 10i$

Equating real and imaginary parts, we get $2x - 7y = 13$ and $4x + 9y = 3$.

Hence x = 3 and y = –1. **Ans.(B)**

Illustration 4 : Find the square root of 7 + 24 i.

Solution : Let $\sqrt{7+24i} = a + ib$

Squaring $a^2 - b^2 + 2iab = 7 + 24i$

Compare real & imaginary parts $a^2 - b^2 = 7$ & $2ab = 24$

By solving these two equations

We get $a = \pm 4$, $b = \pm 3$

$\sqrt{7+24i} = \pm(4 + 3i)$

Illustration 5 : If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution : We have , $x = -5 + 2\sqrt{-4}$

$\Rightarrow x + 5 = 4i \Rightarrow (x + 5)^2 = 16i^2$

$\Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now,

$x^4 + 9x^3 + 35x^2 - x + 4$

$\Rightarrow x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$\Rightarrow x^2(0) - x(0) + 4(0) - 160 \Rightarrow -160$ **Ans.**

Do yourself - 2 :

(i) Find the value of $x^3 + 7x^2 - x + 16$, where $x = 1 + 2i$.

(ii) If $a + ib = \frac{c+i}{c-i}$, where c is a real number, then prove that : $a^2 + b^2 = 1$ and $\frac{b}{a}=\frac{2c}{c^2-1}$.

(iii) Find square root of –15 – 8i

5. THREE IMPORTANT TERMS : CONJUGATE/MODULUS/ARGUMENT :

(a) CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\,\mathrm{Re}(z)$

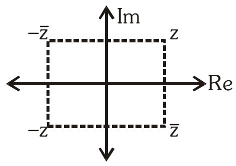
(ii) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(iii) $z\bar{z} = a^2 + b^2$, which is purely real

(iv) If z is purely real, then $z - \bar{z} = 0$

(v) If z is purely imaginary, then $z + \bar{z} = 0$

(vi) If z lies in the 1st quadrant, then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.



(b) Modulus :

If P denotes complex number z = x + iy, then the length OP is called modulus of complex number z. It is denoted by |z|.

$$OP = |z| = \sqrt{x^2 + y^2}$$

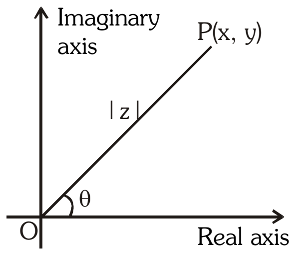
Geometrically $|z|$ represents the distance of point P from origin. ($|z| \geq 0$)

Note : Unlike real numbers, $|z| = \begin{bmatrix} z & \text{if} & z > 0 \\ -z & \text{if} & z < 0 \end{bmatrix}$

is not correct.

(c) Argument or Amplitude :

If P denotes complex number z = x + iy and if OP makes an angle θ with real axis, then θ is called one of the arguments of z.



$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive real axis)

Note :

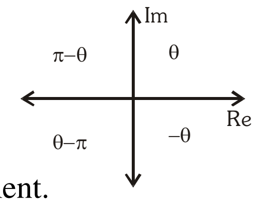
(i) Argument of a complex number is a many valued function. If θ is the argument of a complex number, then $2n\pi + \theta$; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

(ii) The unique value of θ such that $-\pi < \theta \leq \pi$ is called **Amplitude (principal value of the argument).**

(iii) Principal argument of a complex number z = x + iy can be found out using method given below :

(a) Find $\theta = \tan^{-1}\left|\dfrac{y}{x}\right|$ such that $\theta \in \left(0, \dfrac{\pi}{2}\right)$.

(b) Use given figure to find out the principal argument according as the point lies in respective quadrant.



(iv) Unless otherwise stated, amp z implies principal value of the argument.

(v) The unique value of $\theta = \tan^{-1}\dfrac{y}{x}$ such that $0 < \theta \leq 2\pi$ is called **least positive argument.**

(vi) If z = 0, arg(z) is not defined

(vii) If z is real & negative, arg(z) = π.

(viii) If z is real & positive, arg(z) = 0

(ix) If $\theta = \dfrac{\pi}{2}$, z lies on the positive side of imaginary axis.

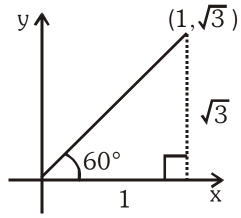
(x) If $\theta = -\dfrac{\pi}{2}$, z lies on the negative side of imaginary axis.

By specifying the modulus & argument a complex number is defined completely. Argument impart direction & modulus impart distance from origin.

For the complex number 0 + 0i the argument is not defined and this is the only complex number which is given by its modulus only.

Illustration 6 : Find the modulus, argument, principal value of argument, least positive argument of complex numbers (a) $1 + i\sqrt{3}$ (b) $-1 + i\sqrt{3}$ (c) $1 - i\sqrt{3}$ (d) $-1 - i\sqrt{3}$

Solution : (a) For $z = 1 + i\sqrt{3}$



$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

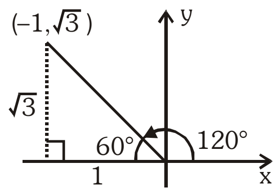
$\arg(z) = 2n\pi + \dfrac{\pi}{3}, \quad n \in I$

Least positive argument is $\dfrac{\pi}{3}$

If the point is lying in first or second quadrant then amp(z) is taken in anticlockwise direction.

In this case $\text{amp}(z) = \dfrac{\pi}{3}$

(b) For $z = -1 + i\sqrt{3}$



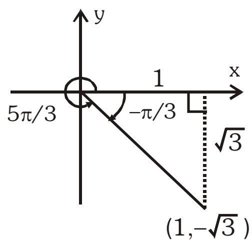
$|z| = 2$

$\arg(z) = 2n\pi + \dfrac{2\pi}{3}, \ n \in I$

Least positive argument $= \dfrac{2\pi}{3}$

$\text{amp}(z) = \dfrac{2\pi}{3}$

(c) For $z = 1 - i\sqrt{3}$



$|z| = 2$

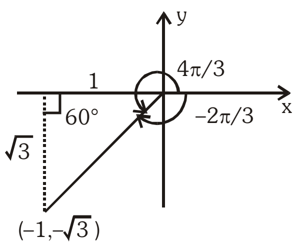
$\arg(z) = 2n\pi - \dfrac{\pi}{3}, n \in I$

Least positive argument $= \dfrac{5\pi}{3}$

If the point lies in third or fourth quadrant then consider amp(z) in clockwise direction.

In this case $\text{amp}(z) = -\dfrac{\pi}{3}$

(d) For $z = -1 - i\sqrt{3}$



$|z| = 2$

$\arg(z) = 2n\pi - \dfrac{2\pi}{3}, \ n \in I$

Least positive argument $= \dfrac{4\pi}{3}$

$\text{amp}(z) = -\dfrac{2\pi}{3}$

Illustration 7 : Find modulus and argument for z = 1 – sin α + i cos α, α ∈ (0,2π)

Solution : $|z| = \sqrt{(1-\sin\alpha)^2 + (\cos\alpha)^2} = \sqrt{2-2\sin\alpha} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$

Case (i) For $\alpha \in \left(0, \frac{\pi}{2}\right)$, z will lie in I quadrant.

$$\text{amp}(z) = \tan^{-1}\frac{\cos\alpha}{1-\sin\alpha} \Rightarrow \text{amp}(z) = \tan^{-1}\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \tan^{-1}\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$$

$$\Rightarrow \arg z = \tan^{-1}\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Since $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\therefore$ amp (z) $= \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$, $|z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

Case (ii) at $\alpha = \frac{\pi}{2}$: $z = 0 + 0i$

$|z| = 0$

amp (z) is not defined.

Case (iii) For $\alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, z will lie in IV quadrant

so amp (z) $= -\tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\frac{\alpha}{2} + \frac{\pi}{4} \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore$ amp (z) $= -\left(\frac{\alpha}{2} + \frac{\pi}{4} - \pi\right) = \frac{3\pi}{4} - \frac{\alpha}{2}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$

Case (iv) at $\alpha = \frac{3\pi}{2}$: $z = 2 + 0i$

$|z| = 2$

amp (z) = 0

Case (v) For $\alpha \in \left(\frac{3\pi}{2}, 2\pi\right)$

z will lie in I quadrant

arg (z) $= \tan^{-1}\tan\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)$

Since $\frac{\alpha}{2} + \frac{\pi}{4} \in \left(\pi, \frac{5\pi}{4}\right)$

$\therefore \arg z = \frac{\alpha}{2} + \frac{\pi}{4} - \pi = \frac{\alpha}{2} - \frac{3\pi}{4}$, $|z| = \sqrt{2}\left(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}\right)$

Do yourself - 3 :

Find the modulus and amplitude of following complex numbers :

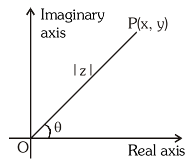
(i) $-2+2\sqrt{3}i$ **(ii)** $-\sqrt{3}-i$ **(iii)** $-2i$ **(iv)** $\dfrac{1+2i}{1-3i}$ **(v)** $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

6. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometrical Representation) :

Every complex number $z = x + i y$ can be represented by a point on the cartesian plane known as complex plane by the ordered pair (x, y). There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

For $z = x + iy$; $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$ and $\theta = \tan^{-1}\dfrac{y}{x}$



Note :

(i) Distance between the two complex numbers z_1 & z_2 is given by $|z_1 - z_2|$.

(ii) $|z - z_0| = r$, represents a circle, whose centre is z_0 and radius is r.

Illustration 8 : Find the locus of :

(a) $|z-1|^2 + |z+1|^2 = 4$ (b) $\text{Re}(z^2) = 0$

Solution : (a) Let $z = x + iy$

$\Rightarrow$ $(|x + iy - 1|)^2 + (|x + iy + 1|)^2 = 4$

$\Rightarrow$ $(x-1)^2 + y^2 + (x+1)^2 + y^2 = 4$

$\Rightarrow$ $x^2 - 2x + 1 + y^2 + x^2 + 2x + 1 + y^2 = 4 \Rightarrow x^2 + y^2 = 1$

Above represents a circle on complex plane with center at origin and radius unity.

(b) Let $z = x + iy$

$\Rightarrow$ $z^2 = x^2 - y^2 + 2xyi$

$\therefore$ $\text{Re}(z^2) = 0$

$\Rightarrow$ $x^2 - y^2 = 0 \Rightarrow y = \pm x$

Thus $\text{Re}(z^2) = 0$ represents a pair of straight lines passing through origin.

Illustration 9 : If z is a complex number such that $z^2 = (\bar{z})^2$, then

(A) z is purely real (B) z is purely imaginary

(C) either z is purely real or purely imaginary (D) none of these

Solution : Let $z = x + iy$, then its conjugate $\bar{z} = x - iy$

Given that $z^2 = (\bar{z})^2 \Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy \Rightarrow 4ixy = 0$

If $x \neq 0$ then $y = 0$ and if $y \neq 0$ then $x = 0$. **Ans. (C)**

Illustration 10 : Among the complex number z which satisfies $|z - 25i| \leq 15$, find the complex numbers z having

(a) least positive argument (b) maximum positive argument

(c) least modulus (d) maximum modulus

Solution : The complex numbers z satisfying the condition

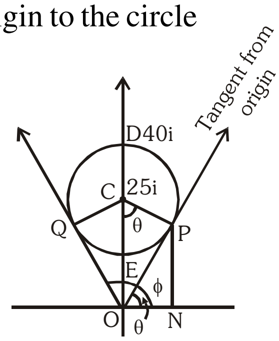
$|z - 25i| \leq 15$

are represented by the points inside and on the circle of radius 15 and centre at the point C(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle

Here θ = least positive argument

and ϕ = maximum positive argument



$\therefore$ In $\Delta OCP, OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$

and $\sin\theta = \dfrac{OP}{OC} = \dfrac{20}{25} = \dfrac{4}{5}$

$\therefore$ $\tan\theta = \dfrac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\therefore$ $z_p = 20(\cos\theta + i\sin\theta) = 20\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\therefore$ $z_p = 12 + 16i$

Similarly $z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

Do yourself - 4 :

(i) Find the distance between two complex numbers $z_1 = 2 + 3i$ & $z_2 = 7 - 9i$ on the complex plane

(ii) Find the locus of $|z - 2 - 3i| = 1$.

(iii) If z is a complex number, then $z^2 + \bar{z}^2 = 2$ represents -

(A) a circle (B) a straight line (C) a hyperbola (D) an ellipse

(c) Trigonometric / Polar Representation :

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$; $\arg z = \theta$; $\bar{z} = r(\cos\theta - i\sin\theta)$

Note : $\cos\theta + i\sin\theta$ is also written as CiS θ.

Euler's formula :

The formula $e^{ix} = \cos x + i\sin x$ is called Euler's formula.

It was introduced by Euler in 1748, and is used as a method of expressing complex numbers.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Euler's identities.

(d) Exponential Representation :

Let z be a complex number such that $|z| = r$ & $\arg z = \theta$, then $z = r.e^{i\theta}$

Illustration 11 : Express the following complex numbers in polar and exponential form :

(i) $\dfrac{1+3i}{1-2i}$ (ii) $\dfrac{i-1}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$

Solution : (i) Let $z=\dfrac{1+3i}{1-2i}=\dfrac{1+3i}{1-2i}\times\dfrac{1+2i}{1+2i}=-1+i$

$|z|=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\tan\alpha=\left|\dfrac{1}{-1}\right|=1=\tan\dfrac{\pi}{4}\Rightarrow\alpha=\dfrac{\pi}{4}$

$\because$ Re(z) < 0 and Im(z) > 0 $\Rightarrow$ z lies in second quadrant.

$\therefore$ $\theta=\arg(z)=\pi-\alpha=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

Hence Polar form is $z=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

and exponential form is $z=\sqrt{2}\,e^{\frac{3\pi}{4}i}$

(ii) Let $z=\dfrac{i-1}{\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}}=\dfrac{i-1}{\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}}=\dfrac{2(i-1)}{(1+i\sqrt{3})}$

$\Rightarrow$ $z=\dfrac{2(i-1)}{(1+i\sqrt{3})}\times\dfrac{(1-i\sqrt{3})}{(1-i\sqrt{3})}\Rightarrow z=\left(\dfrac{\sqrt{3}-1}{2}\right)+i\left(\dfrac{\sqrt{3}+1}{2}\right)$

$\because$ Re(z) > 0 and Im(z) > 0 $\Rightarrow$ z lies in first quadrant.

$\therefore$ $|z|=\sqrt{\left(\dfrac{\sqrt{3}-1}{2}\right)^2+\left(\dfrac{\sqrt{3}+1}{2}\right)^2}=\sqrt{\dfrac{2(3+1)}{4}}=\sqrt{2}$.

$\tan\theta=\left|\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right|=\tan\dfrac{5\pi}{12}\Rightarrow\alpha=\dfrac{5\pi}{12}$

Hence Polar form is $z=\sqrt{2}\left(\cos\dfrac{5\pi}{12}+i\sin\dfrac{5\pi}{12}\right)$

and exponential form is $z=\sqrt{2}e^{\frac{5\pi}{12}i}$

Illustration 12 : If $x_n=\cos\left(\dfrac{\pi}{2^n}\right)+i\sin\left(\dfrac{\pi}{2^n}\right)$, then $x_1x_2x_3.........\infty$ is equal to -

(A) –1 (B) 1 (C) 0 (D) ∞

Solution : $x_n=\cos\left(\dfrac{\pi}{2^n}\right)+i\sin\left(\dfrac{\pi}{2^n}\right)=1\times e^{i\frac{\pi}{2^n}}$

$x_1x_2x_3.........\infty$

$=e^{i\frac{\pi}{2^1}}.e^{i\frac{\pi}{2^2}}---e^{i\frac{\pi}{2^n}}=e^{i\left(\frac{\pi}{2}+\frac{\pi}{2^2}+---+\frac{\pi}{2^n}\right)}$

$=\cos\left(\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+........\right)+i\sin\left(\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+........\right)=-1$

$\left(\text{as }\dfrac{\pi}{2}+\dfrac{\pi}{2^2}+\dfrac{\pi}{2^3}+........=\dfrac{\pi/2}{1-1/2}=\pi\right)$ **Ans. (A)**

Do yourself - 5 :

Express the following complex number in polar form and exponential form :

(i) $-2+2i$ **(ii)** $-1-\sqrt{3}i$ **(iii)** $\dfrac{(1+7i)}{(2-i)^2}$ **(iv)** $(1-\cos\theta+i\sin\theta),\ \theta\in(0,\pi)$

7. IMPORTANT PROPERTIES OF CONJUGATE :

(a) $z+\overline{z}=2\,\text{Re}(z)$ **(b)** $z-\overline{z}=2i\,\text{Im}(z)$ **(c)** $\overline{(\overline{z})}=z$

(d) $\overline{z_1+z_2}=\overline{z}_1+\overline{z}_2$ **(e)** $\overline{z_1-z_2}=\overline{z}_1-\overline{z}_2$

(f) $\overline{z_1\,z_2}=\overline{z}_1\,.\,\overline{z}_2$. In general $\overline{z_1z_2.........z_n}=\overline{z}_1.\overline{z}_2.........\overline{z}_n$

(g) $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z}_1}{\overline{z}_2}$; $z_2\neq 0$ **(h)** If $f(\alpha+i\beta)=x+iy\Rightarrow f(\alpha-i\beta)=x-iy$

8. IMPORTANT PROPERTIES OF MODULUS :

(a) $|z|\geq 0$ **(b)** $|z|\geq \text{Re}(z)$ **(c)** $|z|\geq \text{Im}(z)$

(d) $|z|=|\overline{z}|=|-z|=|-\overline{z}|$ **(e)** $z\overline{z}=|z|^2$

(f) $|z_1\,z_2|=|z_1|.|z_2|$. In general $|z_1z_2.........z_n|=|z_1|.|z_2|.........|z_n|$

(g) $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$, $z_2\neq 0$

(h) $|z^n|=|z|^n,\ n\in I$

(i) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\text{Re}(z_1\overline{z}_2)$

(j) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\alpha-\beta)$, where α,β are $\arg(z_1)$, $\arg(z_2)$ respectively.

(k) $|z_1+z_2|^2+|z_1-z_2|^2=2\left[|z_1|^2+|z_2|^2\right]$

(l) $\big||z_1|-|z_2|\big|\leq|z_1+z_2|\leq|z_1|+|z_2|$ **[Triangle Inequality]**

(m) $\big||z_1|-|z_2|\big|\leq|z_1-z_2|\leq|z_1|+|z_2|$ **[Triangle Inequality]**

9. IMPORTANT PROPERTIES OF AMPLITUDE :

(a) $\text{amp}\,(z_1.\,z_2)=\text{amp}\ z_1+\text{amp}\ z_2+2\,k\pi$; $k\in I$

(b) $\text{amp}\left(\dfrac{z_1}{z_2}\right)=\text{amp}\ z_1-\text{amp}\ z_2+2\,k\pi$; $k\in I$

(c) $\text{amp}(z^n)=n\,\text{amp}(z)+2k\pi$; $n,k\in I$

where proper value of k must be chosen so that RHS lies in $(-\pi,\pi]$.

Illustration 13 : Find amp z and |z| if $z=\left[\dfrac{(3+4i)(1+i)(1+\sqrt{3}i)}{(1-i)(4-3i)(2i)}\right]^2$.

Solution : $\text{amp}\ z=2\left[\text{amp}(3+4i)+\text{amp}(1+i)+\text{amp}(1+\sqrt{3}i)-\text{amp}(1-i)-\text{amp}(4-3i)-\text{amp}(2i)\right]+2k\pi$

where $k\in I$ and k chosen so that amp z lies in $(-\pi,\pi]$.

$$\Rightarrow \quad \text{amp}\, z = 2\left[\tan^{-1}\frac{4}{3}+\frac{\pi}{4}+\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)-\tan^{-1}\left(-\frac{3}{4}\right)-\frac{\pi}{2}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp } z = 2\left[\tan^{-1}\frac{4}{3}+\cot^{-1}\frac{4}{3}+\frac{\pi}{3}\right]+2k\pi \Rightarrow \text{amp}\, z = 2\left[\frac{\pi}{2}+\frac{\pi}{3}\right]+2k\pi$$

$$\Rightarrow \quad \text{amp } z = -\frac{\pi}{3} \quad [\text{at } k=-1]$$ **Ans.**

Also,

$$|z| = \left|\frac{(3+4i)(1+i)\left(1+\sqrt{3}i\right)}{(1-i)(4-3i)(2i)}\right|^2 \Rightarrow |z| = \left(\frac{|3+4i||1+i||1+\sqrt{3}i|}{|1-i||4-3i||2i|}\right)^2$$

$$\Rightarrow \quad |z| = \left(\frac{5\times\sqrt{2}\times 2}{\sqrt{2}\times 5\times 2}\right)^2 = 1$$ **Ans.**

Aliter

$$z = \left[\frac{(3+4i)(1+i)\left(1+\sqrt{3}i\right)}{(1-i)(4-3i)(2i)}\right]^2 \Rightarrow z = \left[-\frac{\sqrt{3}+i}{2}\right]^2 \Rightarrow z = \frac{2-2\sqrt{3}i}{4} = \frac{1}{2}-\frac{\sqrt{3}i}{2}$$

Hence $|z| = 1$, $\text{amp}(z) = -\frac{\pi}{3}$.

Illustration 14 : If $\left|\frac{z-i}{z+i}\right| = 1$, then locus of z is -

(A) x-axis (B) y-axis (C) x = 1 (D) y = 1

Solution : We have, $\left|\frac{z-i}{z+i}\right| = 1 \Rightarrow \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = 1$

$$\Rightarrow \frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = 1 \Rightarrow x^2+(y-1)^2 = x^2+(y+1)^2 \Rightarrow 4y = 0; y = 0$$

which is x-axis **Ans. (A)**

Illustration 15: If $|z_1 + z_2|^2 = |z_1|^2+|z_2|^2$ then $\left(\frac{z_1}{z_2}\right)$ is -

(A) zero or purely imaginary (B) purely imaginary

(C) purely real (D) none of these

Solution : Here let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $|z_1| = r_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $|z_2| = r_2$

$$\therefore \quad |(z_1+z_2)|^2 = \left|(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)\right|^2$$

$$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2) = |z_1|^2 + |z_2|^2 \text{ if } \cos(\theta_1-\theta_2) = 0$$

$\therefore \quad \theta_1 - \theta_2 = \pm\frac{\pi}{2}$

$\Rightarrow$ $\text{amp}(z_1) - \text{amp}(z_2) = \pm\frac{\pi}{2} \Rightarrow \text{amp}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \Rightarrow \frac{z_1}{z_2}$ is purely imaginary **Ans. (B)**

Illustration 16: z_1 and z_2 are two complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}$ is unimodular (whose modulus is one), while z_2 is not unimodular. Find $|z_1|$.

Solution : Here $\left|\frac{z_1 - 2z_2}{2 - z_1\overline{z_2}}\right| = 1 \Rightarrow \frac{|z_1 - 2z_2|}{|2 - z_1\overline{z_2}|} = 1$

$\Rightarrow \quad |z_1 - 2z_2| = |2 - z_1\overline{z_2}| \Rightarrow \quad |z_1 - 2z_2|^2 = |2 - z_1\overline{z_2}|^2$

$\Rightarrow \quad (z_1 - 2z_2)(\overline{z_1 - 2z_2}) = (2 - z_1\overline{z_2})(\overline{2 - z_1\overline{z_2}})$

$\Rightarrow \quad (z_1 - 2z_2)(\overline{z_1} - 2\overline{z_2}) = (2 - z_1\overline{z_2})(2 - \overline{z_1}z_2)$

$\Rightarrow \quad z_1\overline{z_1} - 2z_1\overline{z_2} - 2z_2\overline{z_1} + 4z_2\overline{z_2} = 4 - 2\overline{z_1}z_2 - 2z_1\overline{z_2} + z_1\overline{z_1}z_2\overline{z_2}$

$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2 \Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 4|z_2|^2 - 4 = 0$

$\Rightarrow \quad (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

But $|z_2| \neq 1$ (given)

$\therefore \quad |z_1|^2 = 4$

Hence, $\quad |z_1| = 2$.

Illustration 17: The locus of the complex number z in argand plane satisfying the inequality

$\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 \quad \left(\text{where } |z-1| \neq \frac{2}{3}\right)$ is -

(A) a circle (B) interior of a circle (C) exterior of a circle (D) none of these

Solution : We have, $\log_{1/2}\left(\frac{|z-1|+4}{3|z-1|-2}\right) > 1 = \log_{1/2}\left(\frac{1}{2}\right)$

$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2}$ $\quad [\because \log_a x \text{ is a decreasing function if } a < 1]$

$\Rightarrow \quad 2|z-1| + 8 < 3|z-1| - 2$ as $|z - 1| > 2/3$

$\Rightarrow \quad |z-1| > 10$

which is exterior of a circle. **Ans. (C)**

***Illustration 18*:** If $\left|z - \frac{4}{z}\right| = 2$, then the greatest value of $|z|$ is -

(A) $1 + \sqrt{2}$ (B) $2 + \sqrt{2}$ (C) $\sqrt{3} + 1$ (D) $\sqrt{5} + 1$

Solution : We have $|z| = \left|z - \frac{4}{z} + \frac{4}{z}\right| \le \left|z - \frac{4}{z}\right| + \frac{4}{|z|} = 2 + \frac{4}{|z|}$

$\Rightarrow \quad |z|^2 \le 2|z| + 4 \qquad \Rightarrow \quad (|z|-1)^2 \le 5$

$\Rightarrow \quad |z| - 1 \le \sqrt{5} \qquad \Rightarrow \quad |z| \le \sqrt{5} + 1$

Therefore, the greatest value of $|z|$ is $\sqrt{5}+1$. **Ans. (D)**

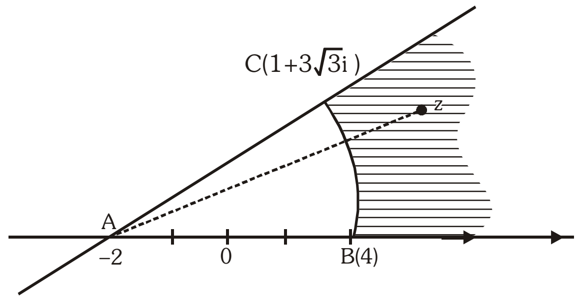
Illustration 19: Shaded region is given by -

(A) $|z + 2| \ge 6,\ 0 \le \arg(z) \le \frac{\pi}{6}$

(B) $|z + 2| \le 6,\ 0 \le \arg(z + 2) \le \frac{\pi}{3}$

(C) $|z + 2| \ge 6,\ 0 \le \arg(z + 2) \le \frac{\pi}{3}$

(D) None of these



Solution : Note that AB = 6 and $1 + 3\sqrt{3}i = -2 + 3 + 3\sqrt{3}i = -2 + 6\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2 + 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore \quad \angle BAC = \frac{\pi}{3}$

Thus, shaded region is given by $|z + 2| \ge 6$ and $0 \le \arg(z + 2) \le \frac{\pi}{3}$ **Ans. (C)**

Do yourself - 6 :

(i) The inequality $|z - 4| < |z - 2|$ represents region given by -

(A) $\mathrm{Re}(z) > 0$ (B) $\mathrm{Re}(z) < 0$ (C) $\mathrm{Re}(z) > 3$ (D) none

(ii) If $z = re^{i\theta}$, then the value of $|e^{iz}|$ is equal to -

(A) $e^{-r\cos\theta}$ (B) $e^{r\cos\theta}$ (C) $e^{r\sin\theta}$ (D) $e^{-r\sin\theta}$

10. SECTION FORMULA AND COORDINATES OF ORTHOCENTRE, CENTROID, CIRCUMCENTRE, INCENTRE OF A TRIANGLE :

If z_1 & z_2 are two complex numbers then the complex number $z = \frac{nz_1 + mz_2}{m + n}$ divides the join of z_1 & z_2 in the ratio m : n.

Note :

(i) If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$; where $a + b + c = 0$ and a,b,c are not all simultaneously zero, then the complex numbers z_1, z_2 & z_3 are collinear.

(ii) If the vertices A, B, C of a triangle represent the complex numbers z_1, z_2, z_3 respectively, then :

- Centroid of the $\Delta ABC = \dfrac{z_1 + z_2 + z_3}{3}$

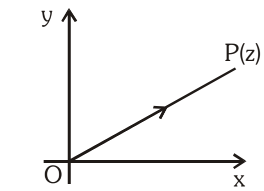
- Orthocentre of the Δ ABC =

$$\frac{(a \sec A)z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C} \quad \text{or} \quad \frac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\tan A + \tan B + \tan C}$$

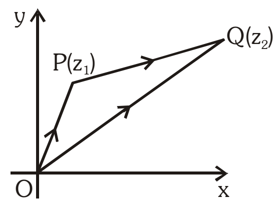
- Incentre of the $\Delta ABC = \dfrac{(az_1 + bz_2 + cz_3)}{(a + b + c)}$

- Circumcentre of the $\Delta ABC = \dfrac{(z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C)}{(\sin 2A + \sin 2B + \sin 2C)}$

11. VECTORIAL REPRESENTATION OF A COMPLEX NUMBER :

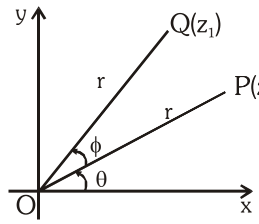


(a) In complex number every point can be represented in terms of position vector. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

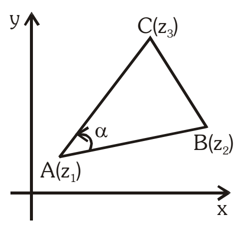


(b) If $P(z_1)$ & $Q(z_2)$ be two complex numbers on argand plane then $\vec{PQ}$ represents complex number $z_2 - z_1$.

Note :



(i) If $\vec{OP} = z = re^{i\theta}$ then $\vec{OQ} = z_1 = re^{i(\theta+\phi)} = z.e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$ i.e. $\dfrac{z_1}{|z_1|} = \dfrac{z}{|z|}e^{i\phi}$

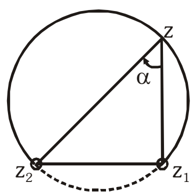


(ii) In general, if z_1, z_2, z_3 be the three vertices of ΔABC then

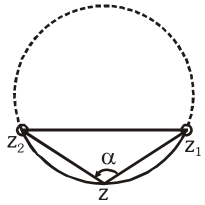
$\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$. Here $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$.

(iii) Note that the locus of z satisfying $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ is:

Case (a) $0 < \alpha < \pi/2$
Locus is major arc of circle as shown excluding z_1 & z_2



Case (b) $\frac{\pi}{2} < \alpha < \pi$



Locus is minor arc of circle as shown excluding z_1 & z_2

(iv) If A, B, C & D are four points representing the complex numbers

z_1, z_2, z_3 & z_4 then AB || CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB $\perp$ CD if $\frac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary.

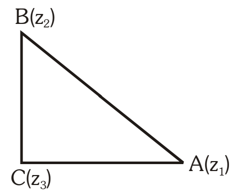
(v) If z_1, z_2, z_3 are the vertices of an equilateral triangle where z_0 is its circumcentre then

(1) $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ (2) $z_1^2 + z_2^2 + z_3^2 = 3 z_0^2$

Illustration 20 : Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

Solution : In the isosceles triangle ABC, AC = BC and BC$\perp$AC. It means that AC is rotated through angle $\pi/2$ to occupy the position BC.

Hence we have, $\frac{z_2 - z_3}{z_1 - z_3} = e^{+i\pi/2} = +i \Rightarrow z_2 - z_3 = +i(z_1 - z_3)$



$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -\left(z_1^2 + z_3^2 - 2z_1z_3\right)$

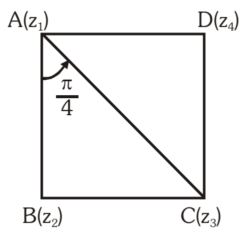
$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2 = 2(z_1 - z_3)(z_3 - z_2)$

$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

Illustration 21: If the vertices of a square ABCD are z_1, z_2, z_3 & z_4 then find z_3 & z_4 in terms of z_1 & z_2.

Solution : Using vector rotation at angle A

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\frac{\pi}{4}}$$



$\because |z_3 - z_1| = AC$ and $|z_2 - z_1| = AB$

Also AC = $\sqrt{2}$ AB

$\therefore |z_3 - z_1| = \sqrt{2}|z_2 - z_1|$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

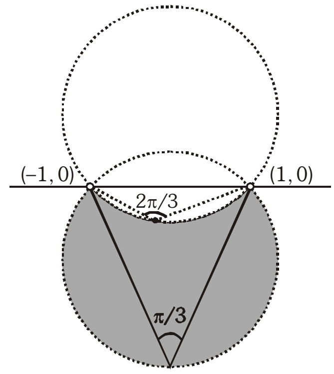
$\Rightarrow z_3 - z_1 = (z_2 - z_1)(1 + i)$

$\Rightarrow z_3 = z_1 + (z_2 - z_1)(1 + i)$

Similarly $z_4 = z_2 + (1 + i)(z_1 - z_2)$

Illustration 22 : Plot the region represented by $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ in the Argand plane.

Solution : Let us take $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$, clearly z lies on the minor arc of the circle passing through (1, 0) and (–1, 0). Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$ means that 'z' is lying on the major arc of the circle passing through (1, 0) and (–1, 0). Now if we take any point in the region included between two arcs say $P_1(z_1)$ we get $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$



Thus $\frac{\pi}{3} \le \arg\left(\frac{z+1}{z-1}\right) \le \frac{2\pi}{3}$ represents the shaded region (excluding points (1, 0) and (–1, 0)).

Do yourself - 7 :

(i) A complex number z = 3 + 4i is rotated about another fixed complex number z_1 = 1 + 2i in anticlockwise direction by 45° angle. Find the complex number represented by new position of z in argand plane.

(ii) If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

(iii) If $A(z_1)$, $B(z_2)$, $C(z_3)$ are vertices of ΔABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(iv) If a & b are real numbers between 0 and 1 such that the points z_1 = a + i, z_2 = 1 + bi and z_3 = 0 form an equilateral triangle then a and b are equal to :-

(A) a = b = 1/2 (B) a = b = $2 - \sqrt{3}$ (C) a = b = $-2 + \sqrt{3}$ (D) a = b = $\sqrt{2} - 1$

(v) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, find locus of z.

12. DE'MOIVRE'S THEOREM :

The value of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$ if 'n' is integer & it is one of the values of $(\cos\theta + i\sin\theta)^n$ if n is a rational number of the form p/q, where p & q are co-prime.

Note : Continued product of the roots of a complex quantity should be determined by using theory of equations.

Illustration 23: If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and also $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

(a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

(b) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

Solution : Let $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$ & $z_3 = \cos\gamma + i\sin\gamma$.

$\therefore$ $z_1 + z_2 + z_3 = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i\,.\,0 = 0$ .... (i)

(a) Also $\frac{1}{z_1} = (\cos\alpha + i\sin\alpha)^{-1} = \cos\alpha - i\sin\alpha$

$\frac{1}{z_2} = \cos\beta - i\sin\beta,\ \frac{1}{z_3} = \cos\gamma - i\sin\gamma$

$\therefore$ $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$ .... (ii)

$= 0 - i\,.\,0 = 0$

Now $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$= 0 - 2z_1z_2z_3\left(\frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2}\right) = 0 - 2z_1z_2z_3\,.\,0 = 0$ {using (i) and (ii)}

or $(\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

or $\cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0 + i.0$

Equating real and imaginary parts on both sides,

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and $\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

(b) If $z_1 + z_2 + z_3 = 0$ then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

$\therefore$ $(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

or $\cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma$

$= 3\{\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\}$

Equating imaginary parts on both sides, $\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

(c) Equating real parts on both sides, $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$

Do yourself - 8 :

(i) If $z_r = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$, $r = 0,1,3,4,.........$, then $z_1z_2z_3z_4z_5$ is equal to -

(A) –1 (B) 0 (C) 1 (D) none of these

(ii) If $(x - 1)^4 - 16 = 0$, then the sum of non-real complex values of x is -

(A) 2 (B) 0 (C) 4 (D) none of these

(iii) If $(\sqrt{3} - i)^n = 2^n$, $n \in Z$, then n is a multiple of -

(A) 6 (B) 10 (C) 9 (D) 12

13. CUBE ROOT OF UNITY :

(a) The cube roots of unity are 1, $\frac{-1 + i\sqrt{3}}{2}(\omega)$, $\frac{-1 - i\sqrt{3}}{2}(\omega^2)$.

(b) If ω is one of the imaginary cube roots of unity then $1 + \omega + \omega^2 = 0$. In general $1 + \omega^r + \omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3 & $1 + \omega^r + \omega^{2r} = 3$ if $r = 3\lambda$; $\lambda \in I$

(c) In polar form the cube roots of unity are :

$1 = \cos 0 + i\sin 0$; $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

(d) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.

(e) The following factorisation should be remembered : ($a, b, c \in R$ & ω is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

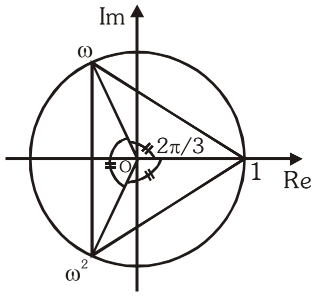


Illustration 24 : If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to (where $n \in I$) -

(A) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

Solution :

$$\alpha = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3};$$

$$\beta = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

$$\alpha^n + \beta^n = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n + \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^n$$

$$= \left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right) + \left(\cos\frac{2n\pi}{3} - i\sin\left(\frac{2n\pi}{3}\right)\right) = 2\cos\left(\frac{2n\pi}{3}\right)$$ **Ans. (A)**

Illustration 25 : If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is imaginary cube root of unity), then find the value of $\frac{\alpha - 1}{\beta - 1} + \frac{\beta - 1}{\gamma - 1} + \frac{\gamma - 1}{\alpha - 1}$.

Solution : We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore$ $(x - 1)^3 + 8 = 0$

$\therefore$ $(x - 1)^3 = (-2)^3$

$\Rightarrow$ $\left(\frac{x-1}{-2}\right)^3 = 1 \Rightarrow \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$ (cube roots of unity)

$\therefore$ $x = -1, 1 - 2\omega, 1 - 2\omega^2$

Here $\alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$

$\therefore$ $\alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$

Then $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right) = \frac{1}{\omega} + \frac{1}{\omega} + \omega^2 = \omega^2 + \omega^2 + \omega^2$

Therefore $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = 3\omega^2$. **Ans.**

Do yourself - 9 :

(i) If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^2$ equals : -

(A) ω (B) -4ω (C) ω^2 (D) 4ω

(ii) If ω is a non real cube root of unity, then the expression $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$ is equal to :-

(A) 0 (B) 3 (C) 1 (D) 2

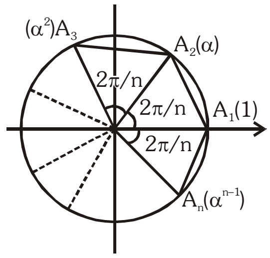
14. n^{th} ROOTS OF UNITY :

If $1, \alpha_1, \alpha_2, \alpha_3 .... \alpha_{n-1}$ are the n, n^{th} root of unity then :

(a) They are in G.P. with common ratio $e^{i(2\pi/n)}$

(b) Their arguments are in A.P. with common difference $\frac{2\pi}{n}$

(c) The points represented by n, n^{th} roots of unity are located at the vertices of a regular polygon of n sides inscribed in a unit circle having center at origin, one vertex being on positive real axis.



(d) $1^p + \alpha_1^p + \alpha_2^p + ... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n

$= n$ if p is an integral multiple of n

(e) $(1-\alpha_1)(1-\alpha_2)......(1-\alpha_{n-1}) = n$

(f) $(1+\alpha_1)(1+\alpha_2).......(1+\alpha_{n-1}) = 0$ if n is even and

$= 1$ if n is odd.

(g) $1.\alpha_1.\alpha_2.\alpha_3.........\alpha_{n-1} = 1$ or -1 according as n is odd or even.

Illustration 26: Find the value $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - \cos\frac{2\pi k}{7}\right)$

Solution :

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}\right) - \sum_{k=1}^{6}\left(\cos\frac{2\pi k}{7}\right) = \sum_{k=1}^{6}\sin\frac{2\pi k}{7} - \sum_{k=0}^{6}\cos\frac{2\pi k}{7} + 1$$

$= \sum_{k=0}^{6}$ (Sum of imaginary part of seven seventh roots of unity)

$-\sum_{k=0}^{6}$ (Sum of real part of seven seventh roots of unity) $+1 = 0 - 0 + 1 = 1$

15. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :

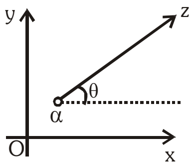
(a) $$\cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{n+1}{2}\right)\theta.$$

(b) $$\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin\left(\frac{n+1}{2}\right)\theta.$$

Note : If $\theta = (2\pi/n)$ then the sum of the above series vanishes.

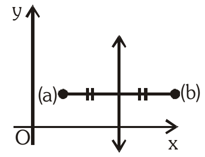
16. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

(a) $\text{amp}(z-\alpha) = \theta$ is a ray emanating from the complex



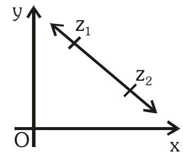
point α and inclined at an angle θ to the x–axis.

(b) $|z-a| = |z-b|$ is the perpendicular bisector of the segment joining a & b.

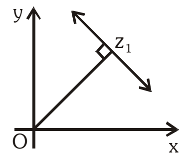


(c) The equation of a line joining z_1 & z_2 is given by ;

$z = z_1 + t\,(z_1 - z_2)$ where t is a parameter.



(d) $z = z_1\,(1 + it)$ where t is a real parameter, is a line through the point z_1 &



perpendicular to z_1.

(e) The equation of a line passing through z_1 & z_2 can be expressed in the determinant form as

$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$. This is also the condition for three complex numbers to be collinear.

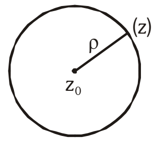
(f) Complex equation of a straight line through two given points z_1 & z_2 can be written as $z\left(\bar{z}_1 - \bar{z}_2\right) - \bar{z}\left(z_1 - z_2\right) + \left(z_1\bar{z}_2 - \bar{z}_1 z_2\right) = 0$, which on manipulating takes the form as $\bar{\alpha} z + \alpha\bar{z} + r = 0$ where r is real and α is a non zero complex constant.

(g) The equation of circle having centre z_0 & radius ρ is :

$|z - z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0 z_0 - \rho^2 = 0$ which is of the form

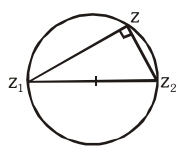
$z\bar{z} + \bar{\alpha} z + \alpha\bar{z} + r = 0$, r is real centre $= -\alpha$ & radius $= \sqrt{\alpha\bar{\alpha} - r}$.

Circle will be real if $\alpha\bar{\alpha} - r \geq 0$.



(h) $\arg\left(\dfrac{z - z_2}{z - z_1}\right) = \pm\dfrac{\pi}{2}$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

this equation represents the circle described on the line segment joining z_1 & z_2 as diameter.



(i) Condition for four given points z_1, z_2, z_3 & z_4 to be concyclic is, the number $\dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z_4 - z_2}{z_4 - z_1}$ is real. Hence the equation of a circle through 3 non collinear points z_1, z_2 & z_3 can be taken as

$\dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$ is real $\Rightarrow \dfrac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)} = \dfrac{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}$

Miscellaneous Illustration :

Illustration 27 : If z is a point on the Argand plane such that $|z - 1| = 1$, then $\frac{z-2}{z}$ is equal to -

(A) tan (arg z) (B) cot (arg z) (C) i tan (arg z) (D) none of these

Solution : Since $|z - 1| = 1$,

$\therefore$ let $z - 1 = \cos\theta + i\sin\theta$

Then, $z - 2 = \cos\theta + i\sin\theta - 1$

$$= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \text{.... (i)}$$

and $z = 1 + \cos\theta + i\sin\theta$

$$= 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \quad \text{.... (ii)}$$

From (i) and (ii), we get $\frac{z-2}{z} = i\tan\frac{\theta}{2} = i\tan(\arg z)\left(\because \arg z = \frac{\theta}{2} \text{ from } (ii)\right)$ **Ans. (C)**

Illustration 28 : Let a be a complex number such that $|a| < 1$ and $z_1, z_2, \ldots, z_n$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \ldots a^k$, then show that vertices of the polygon lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$.

Solution : We have, $z_k = 1 + a + a^2 + \ldots.. + a^k = \frac{1-a^{k+1}}{1-a}$

$$\Rightarrow \quad z_k - \frac{1}{1-a} = \frac{-a^{k+1}}{1-a} \quad \Rightarrow \quad \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|} \qquad (\because |a| < 1)$$

$\therefore$ Vertices of the polygon $z_1, z_2, \ldots.., z_n$ lie within the circle $\left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$

Illustration 29 : If z_1 and z_2 are two complex numbers and $C > 0$, then prove that

$$|z_1 + z_2|^2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$$

Solution : We have to prove that : $|z_1 + z_2|^2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

i.e. $|z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 \le (1 + C)|z_1|^2 + (1 + C^{-1})|z_2|^2$

or $z_1\bar{z}_2 + \bar{z}_1 z_2 \le C|z_1|^2 + C^{-1}|z_2|^2$

or $C|z_1|^2 + \frac{1}{C}|z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2 \ge 0$ (using Re $(z_1\bar{z}_2) \le |z_1\bar{z}_2|$)

or $\left(\sqrt{C}|z_1| - \frac{1}{\sqrt{C}}|z_2|\right)^2 \ge 0$ which is always true.

Illustration 30 : If $\theta \in [\pi/6, \pi/3]$, i = 1, 2, 3, 4, 5 and $z^4\cos\theta_1 + z^3\cos\theta_2 + z^2\cos\theta_3 + z\cos\theta_4 + \cos\theta_5 = 2\sqrt{3}$, then show that $|z| > \frac{3}{4}$

Solution : Given that $\cos\theta_1.z^4+\cos\theta_2.z^3+\cos\theta_3.z^2+\cos\theta_4.z+\cos\theta_5=2\sqrt{3}$

or $\left|\cos\theta_1.z^4+\cos\theta_2.z^3+\cos\theta_3.z^2+\cos\theta_4.z+\cos\theta_5\right|=2\sqrt{3}$

$$2\sqrt{3}\le\left|\cos\theta_1.z^4\right|+\left|\cos\theta_2.z^3\right|+\left|\cos\theta_3.z^2\right|+\left|\cos\theta_4.z\right|+\left|\cos\theta_5\right|$$

$\because$ $\theta_i\in[\pi/6,\ \pi/3]$

$\therefore$ $\dfrac{1}{2}\le\cos\theta_i\le\dfrac{\sqrt{3}}{2}$

$2\sqrt{3}\le\dfrac{\sqrt{3}}{2}|z|^4+\dfrac{\sqrt{3}}{2}|z|^3+\dfrac{\sqrt{3}}{2}|z|^2+\dfrac{\sqrt{3}}{2}|z|+\dfrac{\sqrt{3}}{2}$

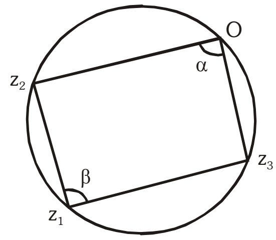
$\Rightarrow 3\le|z|^4+|z|^3+|z|^2+|z|$ $\qquad\Rightarrow 3<|z|+|z|^2+|z|^3+|z|^4+|z|^5+..........\infty$

$\Rightarrow 3<\dfrac{|z|}{1-|z|}$ $\qquad\Rightarrow 3-3|z|<|z|$ $\qquad\Rightarrow 4|z|>3$

$\therefore$ $|z|>\dfrac{3}{4}$

Illustration 31 : If z_1, z_2, z_3 are complex numbers such that $\dfrac{2}{z_1}=\dfrac{1}{z_2}+\dfrac{1}{z_3}$, show that the points represented by z_1, z_2, z_3 lie on a circle passing through the origin.

Solution : We have, $\dfrac{2}{z_1}=\dfrac{1}{z_2}+\dfrac{1}{z_3}\Rightarrow\dfrac{1}{z_1}-\dfrac{1}{z_2}=\dfrac{1}{z_3}-\dfrac{1}{z_1}\Rightarrow\dfrac{z_2-z_1}{z_1z_2}=\dfrac{z_1-z_3}{z_1z_3}$

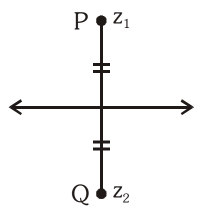


$\Rightarrow\dfrac{z_2-z_1}{z_3-z_1}=\dfrac{-z_2}{z_3}$ $\qquad\Rightarrow\quad \arg\left(\dfrac{z_2-z_1}{z_3-z_1}\right)=\arg\left(\dfrac{-z_2}{z_3}\right)$

$\arg\left(\dfrac{z_2-z_1}{z_3-z_1}\right)=\pi+\arg\left(\dfrac{z_2}{z_3}\right)\Rightarrow$ or $\beta=\pi-\arg\dfrac{z_3}{z_2}=\pi-\alpha\ =\alpha+\beta=\pi$

Thus the sum of a pair of opposite angle of a quadrilateral is 180°. Hence, the points 0, z_1, z_2 and z_3 are the vertices of a cyclic quadrilateral i.e. lie on a circle.

Illustration 32: Two given points P & Q are the reflection points w.r.t. a given straight line if the given line is the right bisector of the segment PQ. Prove that the two points denoted by the complex numbers z_1 & z_2 will be the reflection points for the straight line $\bar{\alpha}z+\alpha\bar{z}+r=0$ if and only if ; $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$, where r is real and α is non zero complex constant.



Solution : Let $P(z_1)$ is the reflection point of $Q(z_2)$ then the perpendicular bisector of z_1 & z_2 must be the line

$$\bar{\alpha}z+\alpha\bar{z}+r=0 \qquad .... \text{(i)}$$

Now perpendicular bisector of z_1 & z_2 is, $|z-z_1|=|z-z_2|$

or $(z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$-z\bar{z}_1-z_1\bar{z}+z_1\bar{z}_1=-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$ $\qquad$ ($z\bar{z}$ cancels on either side)

or $(\bar{z}_2-\bar{z}_1)z+(z_2-z_1)\bar{z}+z_1\bar{z}_1-z_2\bar{z}_2=0$ $\qquad .... \text{(ii)}$

Comparing (i) & (ii) $\dfrac{\bar{\alpha}}{\bar{z}_2-\bar{z}_1}=\dfrac{\alpha}{z_2-z_1}=\dfrac{r}{z_1\bar{z}_1-z_2\bar{z}_2}=\lambda$

$\therefore \quad \bar{\alpha}=\lambda\left(\bar{z}_2-\bar{z}_1\right)$ .... (iii) $\quad \alpha=\lambda\left(z_2-z_1\right)$ .... (iv)

$r=\lambda\left(z_1\bar{z}_1-z_2\bar{z}_2\right)$ .... (v)

Multiplying (iii) by z_1; (iv) by $\bar{z}_2$ and adding

$\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$

Note that we could also multiply (iii) by z_2 & (iv) by $\bar{z}_1$ & add to get the same result.

Hence $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$

Again, let $\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$ is true w.r.t. the line $\bar{\alpha}z+\alpha\bar{z}+r=0$.

Subtracting $\bar{\alpha}\left(z-z_1\right)+\alpha(\bar{z}-\bar{z}_2)=0$

or $\left|(z-z_1)\right|\left|\bar{\alpha}\right|=\left|\alpha\right|\left|(\bar{z}-\bar{z}_2)\right|$ or $\left|z-z_1\right|=\left|\overline{z-z_2}\right|=\left|z-z_2\right|$

Hence 'z' lies on the perpendicular bisector of joins of z_1 & z_2.

ANSWERS FOR DO YOURSELF

1 : **(i)** $n = 4$ **(ii)** 0

2 : **(i)** $-17 + 24i$ **(iii)** $\pm(1-4i)$

3 : **(i)** $|z| = 4$; $\text{amp}(z)=\dfrac{2\pi}{3}$ **(ii)** $|z|=2$; $\text{amp}(z)=-\dfrac{5\pi}{6}$ **(iii)** $|z|=2$; $\text{amp}(z)=-\dfrac{\pi}{2}$

(iv) $|z|=\dfrac{1}{\sqrt{2}}$; $\text{amp}(z)=\dfrac{3\pi}{4}$ **(v)** $|z|=2$; $\text{amp}(z)=\dfrac{\pi}{3}$

4 : **(i)** 13 units **(ii)** locus is a circle on complex plane with center at (2,3) and radius 1 unit. **(iii)** C

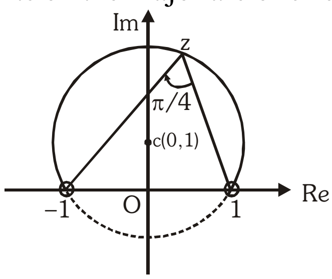
5 : **(i)** $2\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$; $2\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ **(ii)** $2\left(\cos\dfrac{4\pi}{3}+i\sin\dfrac{4\pi}{3}\right)$; $2e^{i\left(\frac{4\pi}{3}\right)}$

(iii) $\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$; $\sqrt{2}e^{i\left(\frac{3\pi}{4}\right)}$ **(iv)** $2\sin\left(\dfrac{\theta}{2}\right)\left(\cos\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)+i\sin\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)\right)$; $2\sin\left(\dfrac{\theta}{2}\right)e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$

6 : **(i)** C **(ii)** D

7 : **(i)** $1+(2+2\sqrt{2})i$ **(ii)** 0 **(iii)** $z_2 = z_3 + i(z_1 - z_3)$ **(iv)** B

(v) Locus is all the points on the major arc of circle as shown excluding points 1 & –1.



8 : **(i)** C **(ii)** A **(iii)** D

9 : **(i)** D **(ii)** B

ELEMENTARY EXERCISE

1. Simplify and express the result in the form of a + bi

(a) $\left(\frac{1+2i}{2+i}\right)^2$ **CN0001**

(b) $-i(9 + 6i)(2 - i)^{-1}$ **CN0002**

(c) $\left(\frac{4i^3-i}{2i+1}\right)^2$ **CN0003**

(d) $\frac{(2+i)^2}{2-i}-\frac{(2-i)^2}{2+i}$ **CN0004**

(e) A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$. **CN0005**

2. Given that x, y $\in$ R, solve :

(a) $(x + 2y) + i\,(2x - 3y) = 5 - 4i$ **CN0006**

(b) $(x + iy) + (7 - 5i) = 9 + 4i$ **CN0007**

(c) $x^2 - y^2 - i\,(2x + y) = 2i$ **CN0008**

3. Find the square root of :

(a) $9 + 40\,i$ **CN0009**

(b) $-11 - 60\,i$ **CN0010**

(c) $50\,i$ **CN0011**

4. (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$ **CN0012**

(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$ **CN0013**

5. Solve the following equations over C and express the result in the form a + ib, a, b $\in$ R.

(a) $ix^2 - 3x - 2i = 0$ **CN0014**

(b) $2(1 + i)x^2 - 4(2 - i)x - 5 - 3i = 0$ **CN0015**

6. Locate the points representing the complex number z on the Argand plane :

(a) $|z + 1 - 2i| = \sqrt{7}$ **CN0016**

(b) $|z - 1|^2 + |z + 1|^2 = 4$ **CN0017**

(c) $\left|\frac{z-3}{z+3}\right| = 3$ **CN0018**

(d) $|z - 3| = |z - 6|$ **CN0019**

7. If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'. **CN0020**

8. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle $z_1z_2z_3$ is equilateral. **CN0021**

9. For what real values of x & y are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex ? **CN0022**

10. If $(x + iy)^{1/3} = a + bi$, then prove that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$. **CN0023**

11. (a) Prove the identity, $|1-z_1\bar{z}_2|^2-|z_1-z_2|^2=\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$ **CN0124**

(b) Prove the identity, $|1+z_1\bar{z}_2|^2+|z_1-z_2|^2=\left(1+|z_1|^2\right)\left(1+|z_2|^2\right)$ **CN0125**

(c) For any two complex numbers, prove that $|z_1+z_2|^2+|z_1-z_2|^2=2\left[|z_1|^2+|z_2|^2\right]$. Also give the geometrical interpretation of this identity. **CN0026**

12. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z-4|+|z+4|=16$. **CN0027**

Paragraph for question nos. 13 to 15

Consider a complex number $w=\dfrac{z-i}{2z+1}$, where $z = x + iy$ and $x, y \in R$.

13. If the complex number w is purely imaginary then locus of z is -

(A) a straight line

(B) a circle with centre $\left(-\dfrac{1}{4},\dfrac{1}{2}\right)$ and radius $\dfrac{\sqrt{5}}{4}$.

(C) a circle with centre $\left(\dfrac{1}{4},-\dfrac{1}{2}\right)$ and passing through origin.

(D) neither a circle nor a straight line. **CN0028**

14. If the complex number w is purely real then locus of z is

(A) a straight line passing through origin

(B) a straight line with gradient 3 and y intercept (–1)

(C) a straight line with gradient 2 and y intercept 1.

(D) none **CN0028**

15. If $|w| = 1$ then the locus of P(z) is

(A) a point circle (B) an imaginary circle (C) a real circle (D) not a circle. **CN0028**

EXERCISE (O-1)

1. If $z+z^3=0$ then which of the following must be true on the complex plane?

(A) $\mathrm{Re}(z)<0$ (B) $\mathrm{Re}(z)=0$ (C) $\mathrm{Im}(z)=0$ (D) $z^4=1$ **CN0029**

2. Number of integral values of n for which the quantity $(n+i)^4$ where $i^2=-1$, is an integer is

(A) 1 (B) 2 (C) 3 (D) 4 **CN0030**

3. Let $i=\sqrt{-1}$. The product of the real part of the roots of $z^2-z=5-5i$ is

(A) – 25 (B) – 6 (C) – 5 (D) 25 **CN0031**

4. There is only one way to choose real numbers M and N such that when the polynomial $5x^4+4x^3+3x^2+Mx+N$ is divided by the polynomial x^2+1, the remainder is 0. If M and N assume these unique values, then M – N is

(A) – 6 (B) – 2 (C) 6 (D) 2 **CN0032**

5. The complex number z satisfying $z+|z|=1+7i$ then the value of $|z|^2$ equals

(A) 625 (B) 169 (C) 49 (D) 25 **CN0033**

6. Number of values of z (real or complex) simultaneously satisfying the system of equations $1 + z + z^2 + z^3 + ... + z^{17} = 0$ and $1 + z + z^2 + z^3 + ... + z^{13} = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4

CN0034

7. If $\dfrac{x-3}{3+i} + \dfrac{y-3}{3-i} = i$ where $x, y \in R$ then

(A) $x = 2$ & $y = -8$ (B) $x = -2$ & $y = 8$ (C) $x = -2$ & $y = -6$ (D) $x = 2$ & $y = 8$

CN0035

8. Number of complex numbers z satisfying $z^3 = \bar{z}$ is

(A) 1 (B) 2 (C) 4 (D) 5

CN0036

9. Let $z = 9 + bi$ where b is non zero real and $i^2 = -1$. If the imaginary part of z^2 and z^3 are equal, then b^2 equals

(A) 261 (B) 225 (C) 125 (D) 361

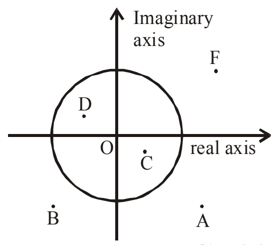
CN0037

10. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

(A) i (B) i – 1 (C) –1 (D) 0

CN0038

11. The diagram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F, which is



(A) A (B) B

(C) C (D) D

CN0039

12. If $z = x + iy$ & $\omega = \dfrac{1 - iz}{z - i}$ then $|\omega| = 1$ implies that, in the complex plane

(A) z lies on the imaginary axis (B) z lies on the real axis

(C) z lies on the unit circle (D) none

CN0040

13. On the complex plane locus of a point z satisfying the inequality

$2 \le |z - 1| < 3$ denotes

(A) region between the concentric circles of radii 3 and 1 centered at (1, 0)

(B) region between the concentric circles of radii 3 and 2 centered at (1, 0) excluding the inner and outer boundaries.

(C) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner and outer boundaries.

(D) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner boundary and excluding the outer boundary.

CN0041

14. The complex number z satisfies $z + |z| = 2 + 8i$. The value of $|z|$ is

(A) 10 (B) 13 (C) 17 (D) 23

CN0042

15. Let $Z_1 = (8 + i)\sin\theta + (7 + 4i)\cos\theta$ and $Z_2 = (1 + 8i)\sin\theta + (4 + 7i)\cos\theta$ are two complex numbers. If $Z_1 \cdot Z_2 = a + ib$ where $a, b \in R$ then the largest value of $(a + b)\ \forall\ \theta \in R$, is

(A) 75 (B) 100 (C) 125 (D) 130

CN0043

16. The locus of z, for arg z = $-\pi/3$ is

(A) same as the locus of z for arg z = $2\pi/3$

(B) same as the locus of z for arg z = $\pi/3$

(C) the part of the straight line $\sqrt{3}x + y = 0$ with $(y < 0, x > 0)$

(D) the part of the straight line $\sqrt{3}x + y = 0$ with $(y > 0, x < 0)$

CN0044

17. If z_1 & $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin & if $\frac{\text{Im}\, z_1}{\text{Re}\, z_1} = \sqrt{2} - 1$ then the value of n is equal to :

(A) 8 (B) 12 (C) 16 (D) 24

CN0045

18. All real numbers x which satisfy the inequality $\left|1 + 4i - 2^{-x}\right| \le 5$ where $i = \sqrt{-1}$, $x \in R$ are

(A) $[-2, \infty)$ (B) $(-\infty, 2]$ (C) $[0, \infty)$ (D) $[-2, 0]$

CN0046

19. For $Z_1 = \sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$; $Z_2 = \sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; $Z_3 = \sqrt[6]{\frac{1+i}{\sqrt{3}-i}}$ which of the following holds good?

(A) $\sum |Z_1|^2 = \frac{3}{2}$

(B) $|Z_1|^4 + |Z_2|^4 = |Z_3|^{-8}$

(C) $\sum |Z_1|^3 + |Z_2|^3 = |Z_3|^{-6}$

(D) $|Z_1|^4 + |Z_2|^4 = |Z_3|^8$

CN0047

20. Number of real or purely imaginary solution of the equation, $z^3 + iz - 1 = 0$ is :

(A) zero (B) one (C) two (D) three

CN0048

21. A point 'z' moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

CN0049

22. If z is a complex number satisfying the equation $|z + i| + |z - i| = 8$, on the complex plane then maximum value of $|z|$ is

(A) 2 (B) 4 (C) 6 (D) 8

CN0050

23. Let z_r $(1 \le r \le 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$

and $|30z_1 + 20z_2 + 15z_3 + 12z_4| = k|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$.

Then the value of k equals

(A) $|z_1z_2z_3|$ (B) $|z_2z_3z_4|$ (C) $|z_3z_4z_1|$ (D) $|z_4z_1z_2|$

CN0051

24. Let Z be a complex number satisfying the equation $(Z^3 + 3)^2 = -16$ then $|Z|$ has the value equal to

(A) $5^{1/2}$ (B) $5^{1/3}$ (C) $5^{2/3}$ (D) 5

CN0052

25. If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2-z_3|}=\frac{4}{|z_3-z_1|}=\frac{5}{|z_1-z_2|}$,

then the value of $\frac{9}{z_2-z_3}+\frac{16}{z_3-z_1}+\frac{25}{z_1-z_2}$ equals

(A) 0 (B) 3 (C) 4 (D) 5

CN0053

26. The area of the triangle whose vertices are the roots $z^3+iz^2+2i=0$ is

(A) 2 (B) $\frac{3}{2}\sqrt{7}$ (C) $\frac{3}{4}\sqrt{7}$ (D) $\sqrt{7}$

CN0054

27. Consider two complex numbers α and β as

$\alpha=\left(\frac{a+bi}{a-bi}\right)^2+\left(\frac{a-bi}{a+bi}\right)^2$, where a, b $\in$ R and $\beta=\frac{z-1}{z+1}$, $z \neq \pm1$ where $|z|=1$, then

(A) Both α and β are purely real
(B) Both α and β are purely imaginary
(C) α is purely real and β is purely imaginary
(D) β is purely real and α is purely imaginary

CN0055

28. Let Z is complex satisfying the equation $z^2-(3+i)z+m+2i=0$, where $m \in R$. Suppose the equation has a real root. The additive inverse of non real root, is

(A) $1-i$ (B) $1+i$ (C) $-1-i$ (D) -2

CN0056

29. The minimum value of $|z-1+2i|+|4i-3-z|$ is

(A) $\sqrt{5}$ (B) 5 (C) $2\sqrt{13}$ (D) $\sqrt{15}$

CN0057

30. If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(A) $1-i\sqrt{3}$ (B) $-1+i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

CN0058

31. Let C_1 and C_2 are concentric circles of radius 1 and 8/3 respectively having centre at (3, 0) on the argand plane. If the complex number z satisfies the inequality, $\log_{1/3}\left(\frac{|z-3|^2+2}{11|z-3|-2}\right)>1$ then :

(A) z lies outside C_1 but inside C_2
(B) z lies inside of both C_1 and C_2
(C) z lies outside both of C_1 and C_2
(D) none of these

CN0059

32. Identify the incorrect statement.

(A) no non zero complex number z satisfies the equation, $\bar{z}=-4z$
(B) $\bar{z}=z$ implies that z is purely real
(C) $\bar{z}=-z$ implies that z is purely imaginary
(D) if z_1, z_2 are the roots of the quadratic equation $az^2+bz+c=0$ such that $\text{Im}(z_1 z_2) \neq 0$ then a, b, c must be real numbers .

CN0060

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33. The equation of the radical axis of the two circles represented by the equations, $|z-2|=3$ and $|z-2-3i|=4$ on the complex plane is :

(A) $3y+1=0$ (B) $3y-1=0$ (C) $2y-1=0$ (D) none

CN0061

34. $z_1=\dfrac{a}{1-i}$; $z_2=\dfrac{b}{2+i}$; $z_3=a-bi$ for $a, b \in R$

if $z_1-z_2=1$ then the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand's plane is given by

(A) $\dfrac{1}{9}(1+7i)$ (B) $\dfrac{1}{3}(1+7i)$ (C) $\dfrac{1}{3}(1-3i)$ (D) $\dfrac{1}{9}(1-3i)$

CN0062

35. Consider the equation $10z^2-3iz-k=0$, where z is a complex variable and $i^2=-1$. Which of the following statements is True?

(A) For all real positive numbers k, both roots are pure imaginary.

(B) For negative real numbers k, both roots are pure imaginary.

(C) For all pure imaginary numbers k, both roots are real and irrational.

(D) For all complex numbers k, neither root is real.

CN0063

36. Number of complex numbers z such that $|z|=1$ and $\left|\dfrac{z}{\bar{z}}+\dfrac{\bar{z}}{z}\right|=1$ is

(A) 4 (B) 6 (C) 8 (D) more than 8

CN0064

37. If z is a complex number satisfying the equation $|z-(1+i)|^2=2$ and $\omega=\dfrac{2}{z}$, then the locus traced by 'ω' in the complex plane is

(A) $x-y-1=0$ (B) $x+y-1=0$ (C) $x-y+1=0$ (D) $x+y+1=0$

CN0065

38. If P and Q are respectively by the complex numbers z_1 and z_2 such that $\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}\right|=\left|\dfrac{1}{z_1}-\dfrac{1}{z_2}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

(A) $\dfrac{z_1-z_2}{2}$ (B) $\dfrac{z_1+z_2}{2}$ (C) $\dfrac{z_1+z_2}{3}$ (D) z_1+z_2

CN0066

39. If z_1 & z_2 are two non-zero complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$, then $\text{Arg } z_1-\text{Arg } z_2$ is equal to

(A) $-\pi$ (B) $-\pi/2$ (C) 0 (D) $\pi/2$

CN0067

40. A particle starts from a point $z_0=1+i$, where $i=\sqrt{-1}$. It moves horizontally away from origin by 2 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 particle moves $\sqrt{5}$ units in the direction of $2\hat{i}+\hat{j}$ and then it moves through an angle of $\text{cosec}^{-1}\sqrt{2}$ in anticlockwise direction of a circle with centre at origin to reach a point z_2. The arg z_2 is given by

(A) $\sec^{-1}2$ (B) $\cot^{-1}0$ (C) $\sin^{-1}\left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)$ (D) $\cos^{-1}\left(\dfrac{-1}{2}\right)$

CN0068

41. Consider $az^2 + bz + c = 0$, where a, b, c $\in$ R and $4ac > b^2$.

(i) If z_1 and z_2 are the roots of the equation given above, then which one of the following complex numbers is purely real?

(A) $z_1\bar{z}_2$ (B) $\bar{z}_1 z_2$ (C) $z_1 - z_2$ (D) $(z_1 - z_2)i$

(ii) In the argand's plane, if A is the point representing z_1, B is the point representing z_2 & $z = \dfrac{\overrightarrow{OA}}{\overrightarrow{OB}}$ then

(A) z is purely real (B) z is purely imaginary

(C) $|z| = 1$ (D) ΔAOB is a scalene triangle.

CN0069

42. If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is equal to :

(A) 5/3 (B) 8/3 (C) 11/3 (D) none of these

CN0070

43. Given $z_p = \cos\left(\frac{\pi}{2^P}\right) + i\sin\left(\frac{\pi}{2^P}\right)$, then $\lim\limits_{n\to\infty} (z_1 z_2 z_3 .... z_n) =$

(A) 1 (B) – 1 (C) i (D) – i

CN0071

44. The maximum & minimum values of $|z + 1|$ when $|z + 3| \leq 3$ are :

(A) (5 , 0) (B) (6 , 0) (C) (7 , 1) (D) (5 , 1)

CN0072

45. If $|z| = 1$ and $|\omega - 1| = 1$ where $z, \omega \in C$, then the largest set of values of $|2z - 1|^2 + |2\omega - 1|^2$ equals

(A) [1, 9] (B) [2, 6] (C) [2, 12] (D) [2, 18]

CN0073

46. If Arg $(z + a) = \frac{\pi}{6}$ and Arg $(z - a) = \frac{2\pi}{3}$; $a \in R^+$, then

(A) z is independent of a (B) $|a| = |z + a|$ (C) $z = a \text{ Cis } \frac{\pi}{6}$ (D) $z = a \text{ Cis } \frac{\pi}{3}$

CN0074

47. If z_1, z_2, z_3 are the vertices of the Δ ABC on the complex plane which are also the roots of the equation, $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the Δ ABC to be equilateral triangle is

(A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$ (C) $\alpha^2 = 3\beta$ (D) $\alpha = 3\beta^2$

CN0075

48. The locus represented by the equation, $|z - 1| + |z + 1| = 2$ is :

(A) an ellipse with focii (1 , 0) ; (– 1 , 0)

(B) one of the family of circles passing through the points of intersection of the circles $|z - 1| = 1$ & $|z + 1| = 1$

(C) the radical axis of the circles $|z - 1| = 1$ and $|z + 1| = 1$

(D) the portion of the real axis between the points (1 , 0) ; (– 1 , 0) including both.

CN0076

Paragraph for question nos. 49 to 51

Let A, B, C be three sets of complex numbers as defined below.

$A = \{z : |z+1| \le 2 + \text{Re}(z)\}$, $B = \{z : |z-1| \ge 1\}$ and $C = \left\{z : \left|\frac{z-1}{z+1}\right| \ge 1\right\}$

49. The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 (B) 5 (C) 6 (D) 10

CN0077

50. The area of region bounded by $A \cap B \cap C$ is

(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

CN0077

51. The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude is

(A) -1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) -2

CN0077

[MATCH THE COLUMN]

52. Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

Column-I (equations in z)	**Column-II** (principal value of arg (z))
(A) $z^2 - z + 1 = 0$	(P) $-2\pi/3$
(B) $z^2 + z + 1 = 0$	(Q) $-\pi/3$
(C) $2z^2 + 1 + i\sqrt{3} = 0$	(R) $\pi/3$
(D) $2z^2 + 1 - i\sqrt{3} = 0$	(S) $2\pi/3$

CN0078

EXERCISE (O-2)

1. Let z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 is :

(A) an isosceles right angled triangle (B) a right angled triangle which is not isosceles

(C) an equilateral triangle (D) an isosceles triangle which is not right angled.

CN0079

2. Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\ \text{CiS}\left(\frac{\pi}{4} + \theta\right)$ where $\theta = \text{amp } z$. If 'O' is the origin, then the Δ OPQ is :

(A) isosceles but not right angled (B) right angled but not isosceles

(C) right isosceles (D) equilateral.

CN0080

3. If $z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{\text{amp } z}\right)$ equals

(A) 1 (B) π (C) 3π (D) 4

CN0081

4. z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, then the value of $z^{2000} + \frac{1}{z^{2000}} + 1$ is equal to

(A) 0 (B) – 1 (C) $\sqrt{3}+1$ (D) $1-\sqrt{3}$

CN0082

5. The complex number ω satisfying the equation $\omega^3 = 8i$ and lying in the second quadrant on the complex plane is

(A) $-\sqrt{3}+i$ (B) $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ (C) $-2\sqrt{3}+i$ (D) $-\sqrt{3}+2i$

CN0083

6. If $z^4 + 1 = \sqrt{3}\,i$

(A) z^3 is purely real (B) z represents the vertices of a square of side $2^{1/4}$

(C) z^9 is purely imaginary (D) z represents the vertices of a square of side $2^{3/4}$.

CN0084

7. Let z is a complex number satisfying the equation $Z^6 + Z^3 + 1 = 0$. If this equation has a root $re^{i\theta}$ with $90° < \theta < 180°$ then the value of 'θ' is

(A) 100° (B) 110° (C) 160° (D) 170°

CN0085

8. If A and B be two complex numbers satisfying $\frac{A}{B}+\frac{B}{A} = 1$. Then the two points represented by A and B and the origin form the vertices of

(A) an equilateral triangle

(B) an isosceles triangle which is not equilateral

(C) an isosceles triangle which is not right angled

(D) a right angled triangle

CN0086

9. If $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{2008}$ are $(2009)^{th}$ roots of unity, then the value of $\sum_{r=1}^{2008} r(\alpha_r + \alpha_{2009-r})$ equals

(A) 2009 (B) 2008 (C) 0 (D) – 2009

CN0087

10. If $x = \frac{1+\sqrt{3}i}{2}$ then the value of the expression, $y = x^4 - x^2 + 6x - 4$, equals

(A) $-1+2\sqrt{3}i$ (B) $2-2\sqrt{3}i$ (C) $2+2\sqrt{3}i$ (D) none

CN0088

11. (a) If $w(\neq 1)$ is a cube root of unity and $(1 + w)^7 = A + Bw$, then A & B are respectively the numbers

(A) 0, 1 (B) 1, 1 (C) 1, 0 (D) –1, 1

(b) If $(w \neq 1)$ is a cube root of unity then $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} =$

(A) 0 (B) 1 (C) i (D) w

CN0089

12. If $z_n = \cos\dfrac{\pi}{(2n+1)(2n+3)} + i\sin\dfrac{\pi}{(2n+1)(2n+3)}$, then $\underset{n\to\infty}{\text{Limit}}\,(z_1 \,.\, z_2 \,.\, z_3 \,.\, ...\, z_n) =$

(A) $\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$ (B) $\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$

(C) $\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}$ (D) $\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}$

CN0090

13. The straight line $(1 + 2i)z + (2i - 1)\bar{z} = 10i$ on the complex plane, has intercept on the imaginary axis equal to

(A) 5 (B) 5/2 (C) – 5/2 (D) – 5

CN0091

14. If $\cos\theta + i\sin\theta$ is a root of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ then the value of $\sum\limits_{r=1}^{n} a_r \cos r\theta$ equals (where all coefficient are real)

(A) 0 (B) 1 (C) –1 (D) none

CN0092

15. Intercept made by the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + r = 0$ on the real axis on complex plane, is

(A) $\sqrt{(\alpha+\bar{\alpha}) - r}$ (B) $\sqrt{(\alpha+\bar{\alpha})^2 - 2r}$ (C) $\sqrt{(\alpha+\bar{\alpha})^2 + r}$ (D) $\sqrt{(\alpha+\bar{\alpha})^2 - 4r}$

CN0093

16. If Z_r ; r = 1, 2, 3,..., 50 are the roots of the equation $\sum\limits_{r=0}^{50} (Z)^r = 0$, then the value of $\sum\limits_{r=1}^{50} \dfrac{1}{Z_r - 1}$ is

(A) – 85 (B) – 25 (C) 25 (D) 75

CN0094

17. All roots of the equation, $(1 + z)^6 + z^6 = 0$:

(A) lie on a unit circle with centre at the origin

(B) lie on a unit circle with centre at (–1, 0)

(C) lie on the vertices of a regular polygon with centre at the origin

(D) are collinear

CN0095

18. It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is 60° then $\left|\dfrac{z_1 + z_2}{z_1 - z_2}\right|$ can be expressed as $\dfrac{\sqrt{N}}{7}$ where N is natural number then N equals

(A) 126 (B) 119 (C) 133 (D) 19

CN0096

One ore more than one is/are correct :

19. If z is a complex number which simultaneously satisfies the equations $3|z - 12| = 5|z - 8i|$ and $|z - 4| = |z - 8|$ then the Im(z) can be

(A) 15 (B) 16 (C) 17 (D) 8

CN0097

20. Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^4 = iz$.
Which of the following statement(s) is/are correct?

(A) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(B) $\sum_{k=1}^{3} \text{Amp}(z_k) = \frac{\pi}{2}$

(C) Centroid of the triangle formed by z_1, z_2 and z_3 is $\left(\frac{1}{\sqrt{3}}, \frac{-1}{3}\right)$

(D) Area of triangle formed by z_1, z_2 and z_3 is $\frac{3\sqrt{3}}{2}$ **CN0098**

21. If $z \in C$, which of the following relation(s) represents a circle on an Argand diagram?

(A) $|z - 1| + |z + 1| = 3$
(B) $(z - 3 + i)(\bar{z} - 3 - i) = 5$
(C) $3|z - 2 + i| = 7$
(D) $|z - 3| = 2$

CN0099

22. The locii of a point P(z) in the complex plane satisfying the $\left|z + \frac{1}{z}\right| = 2$ are two circles C_1 and C_2.
These circles

(A) have centres on real axis.
(B) cut each other orthogonally.
(C) are congruent
(D) have exactly two common tangents.

CN0100

23. Let A and B be two distinct points denoting the complex numbers α and β respectively. A complex number z lies between A and B where $z \neq \alpha$, $z \neq \beta$. Which of the following relation(s) hold good?

(A) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$

(B) $\exists$ a positive real number 't' such that $z = (1 - t)\alpha + t\beta$

(C) $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0$

(D) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ **CN0101**

24. Let z_1, z_2, z_3 are the coordinates of the vertices of the triangle $A_1A_2A_3$. Which of the following statements are equivalent.

(A) $A_1A_2A_3$ is an equilateral triangle.

(B) $(z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$, where ω is the cube root of unity.

(C) $\frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_2}{z_1 - z_3}$

(D) $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} = 0$ **CN0102**

25. Let tangents at $A(z_1)$ and $B(z_2)$ are drawn to the circle $|z| = 2$. Then which of the following is/are **CORRECT** ?

(A) The equation of tangent at A is given by $\frac{z}{z_1}+\frac{\bar{z}}{\bar{z}_1}=2$.

(B) If tangents at $A(z_1)$ and $B(z_2)$ intersect at $P(z_p)$, then $z_p=\frac{2z_1z_2}{z_1+z_2}$.

(C) Slope of tangent at $A(z_1)$ is $\frac{1}{i}\left(\frac{z_1+\bar{z}_1}{z_1-\bar{z}_1}\right)$

(D) If points $A(z_1)$ and $B(z_2)$ on the circle $|z| = 2$ are such that $z_1 + z_2 = 0$, then tangents intersect at $\frac{\pi}{2}$.

CN0103

26. Equation of a straight line on the complex plane passing through a point P denoting the complex number α and perpendicular to the vector $\overrightarrow{OP}$ where 'O' is the origin can be written as

(A) $\text{Im}\left(\frac{z-\alpha}{\alpha}\right)=0$

(B) $\text{Re}\left(\frac{z-\alpha}{\alpha}\right)=0$

(C) $\text{Re}(\bar{\alpha}\,z)=0$

(D) $\bar{\alpha}\,z+\alpha\bar{z}-2|\alpha|^2=0$

CN0104

27. If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the imaginary n^{th} roots of unity then the product $\prod_{r=1}^{n-1}(i-\alpha_r)$ (where $i=\sqrt{-1}$) can take the value equal to

(A) 0 (B) 1 (C) i (D) (1 + i)

CN0105

28. If the expression $(1 + ir)^3$ is of the form of $s(1 + i)$ for some real 's' where 'r' is also real and $i = \sqrt{-1}$, then the value of 'r' can be

(A) $\cot\frac{\pi}{8}$ (B) $\sec\pi$ (C) $\tan\frac{\pi}{12}$ (D) $\tan\frac{5\pi}{12}$

CN0106

29. Let point z moves on $|z - 1| = 1$ such that minimum & maximum value of $|z-2\sqrt{6}i|$ are m & M respectively, then-

(A) $m + M = 10$ (B) $m^2 + M^2 = 52$ (C) $m + M = 8$ (D) $m^2 + M^2 = 16$

CN0107

30. Locus of all points z in argand plane which satisfy $|z^2 + 1| = |z^2 - 1|$ is -

(A) same as $\text{Re}(z) = 0$

(B) same as $\text{Re}(z^2) = 0$

(C) a pair of straight lines

(D) a circle with unit radius

CN0108

EXERCISE (S-1)

1. Find the modulus, argument and the principal argument of the complex numbers.

(i) $6(\cos 310° - i \sin 310°)$ (ii) $-2(\cos 30° + i \sin 30°)$ (iii) $\dfrac{2+i}{4i+(1+i)^2}$

CN0109

2. (a) Let z is complex number satisfying the equation, $z^2 - (3+i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root, then find the value of *m*. **CN0110**

(b) a, b, c are real numbers in the polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$ If two roots of the equation $P(z) = 0$ are 2 and i, then find the value of 'a'.

CN0111

3. Find the real values of x & y for which $z_1 = 9y^2 - 4 - 10ix$ and $z_2 = 8y^2 - 20i$ are conjugate complex of each other.

CN0112

4. (a) Solve the following equation $z^2 - (3-2i)z = (5i-5)$ expressing your answer in the form of $(a+ib)$. **CN0113**

(b) If $(1-i)$ is a root of the equation $z^3 - 2(2-i)z^2 + (4-5i)z - 1 + 3i = 0$, then find the other two roots.

CN0114

5. (a) If $iz^3 + z^2 - z + i = 0$, then find $|z|$. **CN0115**

(b) Let z_1 and z_2 be two complex numbers such that $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, find $|z_1|$.

CN0116

(c) Find the minimum value of the expression $E = |z|^2 + |z-3|^2 + |z-6i|^2$ (where $z = x+iy$, $x,y \in R$)

CN0117

6. Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]\ldots\ldots\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right] \text{ is equal to } \left(1-\frac{1}{2^{2^n}}\right)(1+i) \text{ where } n \geq 2.$$

CN0118

7. Let z be a complex number such that $z \in c \backslash R$ and $\dfrac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$.

CN0119

8. Let $z = (0, 1) \in C$. Express $\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

CN0120

9. Among the complex numbers z satisfying the condition $\left|z + 3 - \sqrt{3}i\right| = \sqrt{3}$, find the number having the least positive argument.

CN0121

10. If A, B and C are the angles of a triangle $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$, where $i = \sqrt{-1}$, then find the value of D.

CN0122

11. Dividing f(z) by $z-i$, we get the remainder i and dividing it by $z+i$, we get the remainder $1+i$. Find the remainder upon the division of f(z) by z^2+1.

CN0123

12. (a) Find all non–zero complex numbers z satisfying $\bar{z} = i z^2$. **CN0124**

(b) If the complex numbers $z_1, z_2, \ldots\ldots z_n$ lie on the unit circle $|z| = 1$ then show that $|z_1 + z_2 + \ldots\ldots + z_n| = |z_1^{-1} + z_2^{-1} + \ldots\ldots + z_n^{-1}|$. **CN0125**

13. Let $|z| = 2$ and $w = \frac{z+1}{z-1}$ where z, w $\in$ C (where C is the set of complex numbers). If M and n respectively be the greatest and least modulus of w, then find the value of (2010m + M).

CN0126

14. If ω is the imaginary cube root of unity, then find the number of pairs of integers (a, b) such the $|a\omega + b| = 1$.

CN0127

EXERCISE (S-2)

1. Find the sum of the series $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) \ldots\ldots (n-1)(n-\omega)(n-\omega^2)$ where ω is one of the imaginary cube root of unity.

CN0128

2. Resolve $z^5 + 1$ into linear & quadratic factors with real coefficients. Deduce that : $4.\sin\frac{\pi}{10}.\cos\frac{\pi}{5} = 1$.

CN0129

3. Prove that , with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p , p', q , q' are all real.

(i) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.

(ii) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$.

State whether these equal roots are real or complex. **CN0130**

4. (a) Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by

$A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1-i)z + (1+i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

CN0131

(b) If z is a non-real complex number, then find the minimum value of $\frac{\operatorname{Im} z^5}{\operatorname{Im}^5 z}$. **CN0132**

5. Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ **CN0133**

(b) $\operatorname{Re}\left(\frac{z+2i}{iz+2}\right) \le 4 \quad (z \ne 2i)$ **CN0134**

(c) $\operatorname{Arg}(z+i) - \operatorname{Arg}(z-i) = \pi/2$ **CN0135**

(d) $\operatorname{Arg}(z-a) = \pi/3$ where $a = 3 + 4i$. **CN0136**

6. If the equation $(z+1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots.\ z_7$, find the value of

(a) $\sum_{r=1}^{7} \operatorname{Re}(z_r)$ and **CN0137**

(b) $\sum_{r=1}^{7} \operatorname{Im}(z_r)$ **CN0138**

7. If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z - 2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

CN0139

8. Let $z_i(i = 1, 2, 3, 4)$ represent the vertices of a square all of which lie on the sides of the triangle with vertices (0, 0), (2, 1) and (3, 0). If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is $\frac{m}{n}$ (where m and n are in their lowest form). Find the value of (m + n).

CN0140

9. Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients satisfying $f(i) = 0$ and $f(1 + i) = 5$. Find the value of $a^2 + b^2 + c^2 + d^2$. (where $i = \sqrt{-1}$) **CN0141**

10. A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

CN0142

11. Evaluate : $\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$. **CN0143**

12. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum 3 + 4i and the other two with product 13 + i. Find the value of 'b'. **CN0144**

EXERCISE (JM)

1. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[AIEEE-2012]**

(1) on the imaginary axis.

(2) either on the real axis or on a circle passing through the origin.

(3) on a circle with centre at the origin.

(4) either on the real axis or on a circle not passing through the origin. **CN0145**

2. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals

[JEE (Main)-2013]

(1) $-\theta$ (2) $\frac{\pi}{2} - \theta$ (3) θ (4) $\pi - \theta$∴

CN0146

3. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$:

[JEE(Main)-2014]

(1) is equal to $\frac{5}{2}$ (2) lies in the interval (1, 2)

(3) is strictly greater than $\frac{5}{2}$ (4) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

CN0147

4. A complex number z is said to be unimodular if |z| = 1. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a : **[JEE(Main)-2015]**

(1) circle of radius 2 (2) circle of radius $\sqrt{2}$
(3) straight line parallel to x-axis (4) straight line parallel to y-axis

CN0148

5. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE(Main)-2016]**

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

CN0149

6. Let ω be a complex number such that 2ω + 1 = z where z = $\sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to :- **[JEE(Main)-2017]**

(1) 1 (2) –z (3) z (4) –1

CN0150

7. If α, β ∈ C are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to- **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

CN0151

8. Let z be a complex number such that |z| + z = 3 + i (where i = $\sqrt{-1}$). Then |z| is equal to :- **[JEE(Main)-2019]**

(1) $\frac{5}{4}$ (2) $\frac{\sqrt{41}}{4}$ (3) $\frac{\sqrt{34}}{3}$ (4) $\frac{5}{3}$

CN0152

9. If $\frac{z-\alpha}{z+\alpha}(\alpha \in R)$ is a purely imaginary number and |z| = 2, then a value of α is : **[JEE(Main)-2019]**

(1) 1 (2) 2 (3) $\sqrt{2}$ (4) $\frac{1}{2}$

CN0153

10. Let Z_1 and Z_2 be two complex numbers satisfying $|Z_1| = 9$ and $|Z_2-3-4i|=4$. Then the minimum value of $|Z_1-Z_2|$ is : **[JEE(Main)-2019]**

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

CN0154

11. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}\left(i = \sqrt{-1}\right)$, then $\left(1 + iz + z^5 + iz^8\right)^9$ is equal to **[JEE(Main)-2019]**

(1) –1 (2) 1 (3) 0 (4) $\left(-1+2i\right)^9$

CN0155

12. Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$, then :- **[JEE(Main)-2019]**

(1) $5\text{Im}(\omega) < 1$ (2) $4\text{Im}(\omega) > 5$ (3) $5\text{Re}(\omega) > 1$ (4) $5\text{Re}(\omega) > 4$

CN0156

13. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then : **[JEE(Main)-2019]**

(1) $\bar{z}w = i$ (2) $\bar{z}w = -i$ (3) $z\bar{w} = \frac{1-i}{\sqrt{2}}$ (4) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

CN0157

14. The equation $|z-i| = |z-1|$, $i = \sqrt{-1}$, represents: **[JEE(Main)-2019]**

(1) the line through the origin with slope -1. (2) a circle of radius 1.

(3) a circle of radius $\frac{1}{2}$. (4) the line through the origin with slope 1.

CN0158

15. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x,y) lies on a : **[JEE(Main)-2020]**

(1) circle whose centre is at $\left(-\frac{1}{2},-\frac{3}{2}\right)$ (2) circle whose diameter is $\frac{\sqrt{5}}{2}$

(3) straight line whose slope is $\frac{3}{2}$ (4) straight line whose slope is $-\frac{2}{3}$

CN0159

16. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$. If $a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b = \sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation: **[JEE(Main)-2020]**

(1) $x^2 - 102x + 101 = 0$ (2) $x^2 + 101x + 100 = 0$

(3) $x^2 - 101x + 100 = 0$ (4) $x^2 + 102x + 101 = 0$ **CN0160**

17. If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then **[JEE(Main)-2020]**

(1) $b^2 - b = 42$ (2) $b^2 + b = 12$ (3) $b^2 + b = 72$ (4) $b^2 - b = 30$

CN0161

18. If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be **[JEE(Main)-2020]**

(1) $\sqrt{\frac{17}{2}}$ (2) $\sqrt{10}$ (3) $\sqrt{8}$ (4) $\sqrt{7}$

CN0162

19. Let z be complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|z| = \frac{5}{2}$. Then the value of $|z + 3i|$ is : **[JEE(Main)-2020]**

(1) $\sqrt{10}$ (2) $2\sqrt{3}$ (3) $\frac{7}{2}$ (4) $\frac{15}{4}$

CN0163

EXERCISE (JA)

1. Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then

[JEE 2010, 3M]

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

CN0164

2. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

[JEE 2010, 3M]

CN0165

3. Match the statements in **Column-I** with those in **Column-II.**

[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I	Column II
(A) The set of points z satisfying $\left\|z - i\|z\|\right\| = \left\|z + i\|z\|\right\|$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\|z + 4\| + \|z - 4\| = 10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $\|w\| = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(r) the set of points z satisfying $\|\text{Im } z\| \le 1$
(D) If $\|w\| = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\|\text{Re } z\| \le 2$
	(t) the set of points z satisfying $\|z\| \le 3$

[JEE 10, 3+3+8]

CN0166

4. **Comprehension (3 questions together)**

Let a,b and c be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \text{...(E)}$$

(i) If the point P(a,b,c), with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(A) 0 (B) 12 (C) 7 (D) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to -

(A) –2 (B) 2 (C) 3 (D) –3

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is - **[JEE 2011, 3+3+3]**

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞ **CN0167**

5. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011, 4M]**
CN0168

6. Let $\omega = e^{i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that
$a + b + c = x$
$a + b\omega + c\omega^2 = y$
$a + b\omega^2 + c\omega = z$.
Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[JEE 2011, 4M]**
CN0169

7. Match the statements given in **Column I** with the values given in **Column II**

Column I	Column II
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for	(s) π
$|z| = 1, z \ne 1$ is given by	(t) $\frac{\pi}{2}$

[JEE 2011, 2+2+2+2M]
CN0170

8. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**

Column I	Column II
(A) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number}, |z|=1, z \ne \pm 1\right\}$ is	(p) $(-\infty,-1) \cup (1,\infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty,0) \cup (0,\infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r) $[2,\infty)$
(D) If $f(x) = x^{3/2}(3x - 10)$, $x \ge 0$, then $f(x)$ is increasing in	(s) $(-\infty,-1] \cup [1,\infty)$
	(t) $(-\infty,0] \cup [2,\infty)$

[JEE 2011, 2+2+2+2M]
CN0171

9. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value - **[JEE 2012, 3M, –1M]**

(A) –1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

CN0172

10. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

[JEE(Advanced) 2013, 2M]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

CN0173

11. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n =

[JEE(Advanced) 2013, 3, (–1)]

(A) 57 (B) 55 (C) 58 (D) 56

CN0174

12. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in C : \operatorname{Re} z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \operatorname{Re} z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1, z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$ **CN0175**

Paragraph for Question 13 and 14

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \operatorname{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \operatorname{Re} z > 0\}$.

13. $\min\limits_{z \in S} |1-3i-z| =$ **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

CN0176

14. Area of S = **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

CN0176

15. The quadratic equation p(x) = 0 with real coefficients has purely imaginary roots. Then the equation p(p(x)) = 0 has **[JEE(Advanced) 2014, 3(–1)]**

(A) only purely imaginary roots (B) all real roots

(C) two real and two purely imaginary roots (D) neither real nor purely imaginary roots.

CN0177

16. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ... 9.

	List-I		**List-II**
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1.	True
Q.	There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2.	False
R.	$\frac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert ... \lvert 1-z_9\rvert}{10}$ equals	3.	1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	4.	2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

[JEE(Advanced) 2014, 3(–1)]

CN0178

17.

	Column-I		**Column-II**
(A)	In $\mathbb{R}^2$, if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$ then possible value(s) of n is (are)	(R)	3
(D)	Let the harmonic mean of two positive real number a and b be 4, If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\lvert q - a\rvert$ is (are)	(S)	4
		(T)	5

[JEE 2015, 8(Each 2M, –1M)]

CN0179

18. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\dfrac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|}$ is **[JEE 2015, 4M, –0M]**

CN0180

19. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s ∈ {1, 2, 3}. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

[JEE(Advanced)-2016, 3(0)]

CN0181

20. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\right\}$, where $i = \sqrt{-1}$. If z = x + iy and z ∈ S, then (x,y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the x-axis for $a \neq 0, b = 0$

(D) the y-axis for $a = 0, b \neq 0$ **[JEE(Advanced)-2016, 4(–2)]**

CN0182

21. Let a, b, x and y be real numbers such that a – b = 1 and y ≠ 0. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ?

[JEE(Advanced)-2017, 4(–2)]

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

CN0183

22. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) **FALSE** ? **[JEE(Advanced)-2018, 4(–2)]**

(A) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line **CN0184**

23. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy\ \left(x, y \in \mathbb{R}, i = \sqrt{-1}\right)$ of the equation $sz + t\overline{z} + r = 0$, where $\overline{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

CN0185

24. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \overline{z}_0}{z_0 - \overline{z}_0 + 2i}$ is

[JEE(Advanced)-2019, 3(–1)]

(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

CN0186

25. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$$

equals ____ **[JEE(Advanced)-2019, 3(0)]**

CN0187

ANSWER KEY

ELEMENTARY EXERCISE

1. **(a)** $\frac{7}{25}+\frac{24}{25}$ i; **(b)** $\frac{21}{5}-\frac{12}{5}$i; **(c)** 3 + 4i; **(d)** $\frac{22}{5}$i; **(e)** 15

2. **(a)** x =1, y = 2; **(b)** (2,9); **(c)** (−2 ,2) or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$ **3.** **(a)** ± (5 + 4i) ; **(b)** ±(5 − 6i); **(c)** ± 5(1 + i)

4. **(a)** −160 ; **(b)** − (77 +108 i)   **5.** **(a)** − i , − 2i **(b)** $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$

6. **(a)** on a circle of radius $\sqrt{7}$ with centre (−1, 2); **(b)** on a unit circle with centre at origin
(c) on a circle with centre (−15/4, 0) & radius 9/4; **(d)** a straight line.

7. a = b = 2 $-\sqrt{3}$ **8.** $z_3=\sqrt{3}(1-i)$ and $z'_3=\sqrt{3}(-1+i)$

9. x = 1, y = −4 or x = −1, y = −4   **12.** $\frac{x^2}{64}+\frac{y^2}{48}=1$ **13.** B   **14.** C   **15.** C

EXERCISE (O-1)

1. B	**2.** C	**3.** B	**4.** C	**5.** A	**6.** A	**7.** B	**8.** D
9. B	**10.** B	**11.** C	**12.** B	**13.** D	**14.** C	**15.** C	**16.** C
17. A	**18.** A	**19.** B	**20.** A	**21.** D	**22.** B	**23.** D	**24.** B
25. A	**26.** A	**27.** C	**28.** C	**29.** C	**30.** C	**31.** A	**32.** D
33. B	**34.** A	**35.** B	**36.** C	**37.** A	**38.** B	**39.** C	**40.** B
41. (i) D; (ii) C		**42.** B	**43.** B	**44.** A	**45.** D	**46.** D	**47.** A

48. D **49.** B **50.** A **51.** B **52.** (A) Q,R; (B) P,S; (C) Q,S; (D) P,R

EXERCISE (O-2)

1. A	**2.** C	**3.** D	**4.** A	**5.** A	**6.** D	**7.** C	**8.** A
9. D	**10.** A	**11.** **(a)** B; **(b)** A		**12.** B	**13.** A	**14.** C	**15.** D
16. B	**17.** D	**18.** C	**19.** C,D	**20.** A,B	**21.** B,C,D	**22.** B,C,D	

23. A,B,C,D **24.** A,B,C,D **25.** A,B,C **26.** B,D **27.** A,B,C,D **28.** B,C,D
29. A,B **30.** B,C

EXERCISE (S-1)

1. **(i)** Modulus = 6, Arg = $2k\pi+\frac{5\pi}{18}$ (k ∈ I), Principal Arg = $\frac{5\pi}{18}$

(ii) Modulus = 2, Arg = $2k\pi+\frac{7\pi}{6}$ (k ∈ I), Principal Arg = $-\frac{5\pi}{6}$

(iii) Modulus = $\frac{\sqrt{5}}{6}$, Arg = $2k\pi-\tan^{-1}2$ (k ∈ I), Principal Arg = $-\tan^{-1}2$

2. **(a)** 2, **(b)** −11/2   **3.** [(−2, 2) ; (−2, −2)]   **4.** **(a)** z = (2 + i) or (1 − 3i); **(b)** z = 1 or 2 − i

5. **(a)** 1, **(b)** 2 **(c)** 30 **8.** $\begin{cases}(1,\ 0) & \text{for} \quad n=4k \\ (1,\ 1) & \text{for} \quad n=4k+1 \\ (0,\ 1) & \text{for} \quad n=4k+2 \\ (0,\ 0) & \text{for} \quad n=4k+3\end{cases}$ **9.** $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$ **10.** – 4

11. $\frac{iz}{2}+\frac{1}{2}+i$ **12.** **(a)** $\frac{\sqrt{3}}{2}-\frac{i}{2}, -\frac{\sqrt{3}}{2}-\frac{i}{2}, i$ **13.** 673 **14.** 6

EXERCISE (S-2)

1. $\left[\frac{n(n+1)}{2}\right]^2-n$ **2.** $(Z+1)(Z^2-2Z\cos 36^\circ+1)(Z^2-2Z\cos 108^\circ+1)$ **4.** **(a)** $\pi-2$, **(b)** – 4

5. **(a)** The region between the concentric circles with centre at (0 , 2) & radii 1 & 3 units

(b) region outside or on the circle with centre $\frac{1}{2}+2i$ and radius $\frac{1}{2}$.

(c) semicircle (in the 1st & 4th quadrant) $x^2+y^2=1$

(d) a ray emanating from the point (3 + 4i) directed away from the origin & having equation $\sqrt{3}x-y+4-3\sqrt{3}=0$

6. (a) $-\frac{7}{2}$, (b) zero **7.** 4 **8.** 41 **9.** 26 **10.** (3+ 7i) **11.** 48(1 – i) **12.** 51

EXERCISE (JM)

1. 2	**2.** 3	**3.** 2	**4.** 1	**5.** 1	**6.** 2	**7.** 2	**8.** 4
9. 2	**10.** 1	**11.** 1	**12.** 3	**13.** 2	**14.** 4	**15.** 2	**16.** 1
17. 4	**18.** 4	**19.** 3					

EXERCISE (JA)

1. A,C,D **2.** 1 **3.** (A) $\rightarrow$ (q,r), (B) $\rightarrow$ (p), (C) $\rightarrow$ (p,s,t), (D) $\rightarrow$ (q,r,s,t) **4.** **(i)** D, **(ii)** A, **(iii)** B

5. 5 **6.** Bonus **7.** (A) $\rightarrow$ (q); (B) $\rightarrow$ (p) or (p,q,r,s,t); (C) $\rightarrow$ (s); (D) $\rightarrow$ (t)

8. (A) $\rightarrow$ (s); (B) $\rightarrow$ (t); (C) $\rightarrow$ (r); (D) $\rightarrow$ (r) **9.** D **10.** C **11.** B,C,D

12. C,D **13.** C **14.** B **15.** D **16.** C

17. (A) $\rightarrow$ (P,Q); (B) $\rightarrow$ (P,Q); (C) $\rightarrow$ (P,Q,S,T); (D) $\rightarrow$ (Q,T) **18.** 4 **19.** 1 **20.** A,C,D

21. A,D **22.** A,B,D **23.** A,C,D **24.** 2 **25.** 3.00

Chapter Contents 02

PROBABILITY

JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

Addition and multiplication rules of probability, conditional probability, Bayes Theorem, independence of events, computation of probability of events using permutations and combinations.

JEE (Advanced) Syllabus :

Addition and multiplication rules of probability, conditional probability, Bayes Theorem, independence of events, computation of probability of events using permutations and combinations.

Important Notes

PROBABILITY

1 INTRODUCTION :

The theory of probability has been originated from the game of chance and gambling. In old days, gamblers used to gamble in a gambling house with a die to win the amount fixed among themselves. They were always desirous to get the prescribed number on the upper face of a die when it was thrown on a board. Shakuni of Mahabharat was perhaps one of them. People started to study the subject of probability from the middle of seventeenth century. The mathematicians Huygens, Pascal Fermat and Bernoulli contributed a lot to this branch of Mathematics. A.N. Kolmogorow proposed the set theoretic model to the theory of probability.

Probability gives us a measure of likelihood that something will happen. However probability can never predict the number of times that an occurrence actually happens. But being able to quantify the likely occurrence of an event is important because most of the decisions that affect our daily lives are based on likelihoods and not on absolute certainties.

2. DEFINITIONS :

(a) Experiment : An action or operation resulting in two or more well defined outcomes. e.g. tossing a coin, throwing a die, drawing a card from a pack of well shuffled playing cards etc.

(b) Sample space : A set S that consists of all possible outcomes of a random experiment is called a sample space and each outcome is called a sample point. Often, there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.

e.g. in an experiment of "throwing a die", following sample spaces are possible :

(i) {even number, odd number}

(ii) {a number less than 3, a number equal to 3, a number greater than 3}

(iii) {1,2,3,4,5,6}

Here 3^{rd} sample space is the one which provides most information.

If a sample space has a finite number of points it is called finite sample space and if it has an infinite number of points, it is called infinite sample space. e.g. (i) "in a toss of coin" either a head (H) or tail (T) comes up, therefore sample space of this experiment is S = {H,T} which is a finite sample space. (ii) "Selecting a number from the set of natural numbers", sample space of this experiment is S = {1,2,3,4,......} which is an infinite sample space.

(c) Event : An event is defined as an occurrence or situation, for example

(i) in a toss of a coin, it shows head,

(ii) scoring a six on the throw of a die,

(iii) winning the first prize in a raffle,

(iv) being dealt a hand of four cards which are all clubs.

In every case it is set of some or all possible outcomes of the experiment. Therefore event (A) is subset of sample space (S). If outcome of an experiment is an element of A we say that event A has occurred.

- An event consisting of a single point of S is called a simple or elementary event.
- ϕ is called impossible event and S (sample space) is called sure event.

Note : Probability of occurrence of an event A is denoted by P(A).

(d) **Compound Event :** If an event has more than one sample points it is called **Compound Event**. If A & B are two given events then $A \cap B$ is called compound event and is denoted by $A \cap B$ or AB or A & B.

(e) **Complement of an event :** The set of all outcomes which are in S but not in A is called the complement of the event A & denoted by $\bar{A}$, A^c, A' or 'not A'.

(f) **Mutually Exclusive Events :** Two events are said to be **Mutually Exclusive** (or disjoint or incompatible) if the occurrence of one precludes (rules out) the simultaneous occurrence of the other. If A & B are two mutually exclusive events then $P(A \cap B) = 0$.

Consider, for example, choosing numbers at random from the set {3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

If, Event A is the selection of a prime number,

Event B is the selection of an odd number,

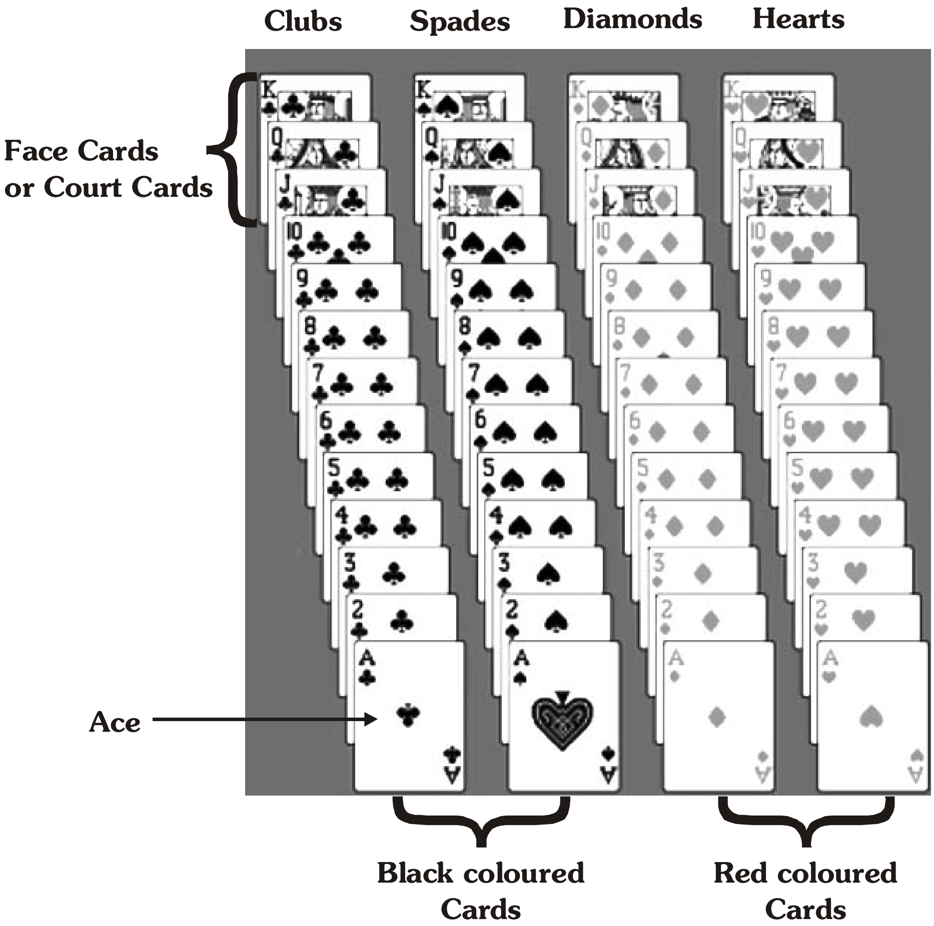
Event C is the selection of an even number,

then A and C are mutually exclusive as none of the numbers in this set is both prime and even. But A and B are not mutually exclusive as some numbers are both prime and odd (viz. 3, 5, 7, 11).

(g) **Equally Likely Events :** Events are said to be **Equally Likely** when each event is as likely to occur as any other event. Note that the term 'at random' or 'randomly' means that all possibilities are equally likely.

(h) **Exhaustive Events :** Events A,B,C........ N are said to be **Exhaustive Events** if no event outside this set can result as an outcome of an experiment. For example, if A & B are two events defined on a sample space S and A & B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.

Note : Playing cards : A pack of playing cards consists of 52 cards of 4 suits, 13 in each, as shown in figure.



Comparative study of Equally likely, Mutually Exclusive and Exhaustive events :

	Experiment	Events	E/L	M/E	Exhaustive
1.	Throwing of a die	A: throwing an odd face {1, 3, 5} B : throwing a composite {4,6}	No	Yes	No
2.	A ball is drawn from an urn containing 2White, 3Red and 4Green balls	E_1 : getting a White ball E_2 : getting a Red ball E_3 : getting a Green ball	No	Yes	Yes
3.	Throwing a pair of dice	A : throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more { 46, 64, 55, 56, 65, 66 }	Yes	No	No
4.	From a well shuffled pack of cards a card is drawn	E_1 : getting a heart E_2: getting a spade E_3 : getting a diamond E_4 : getting a club	Yes	Yes	Yes
5.	From a well shuffled pack of cards a card is drawn	A = getting a heart B = getting a face card	No	No	No

Illustration 1 : A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Solution : Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is

$S = \{HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6\}$.

Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, Ti means tail on the coin and the number i on the die.

Illustration 2 : Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Solution : In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on. Hence, the desired sample space is S = {H, TH, TTH, TTTH, TTTTH,...}

Illustration 3 : A coin is tossed three times, consider the following events.

A : 'no head appears'

B : 'exactly one head appears'

C : 'at least two heads appear'

Do they form a set of mutually exclusive and exhaustive events ?

Solution : The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Events A, B and C are given by

A = {TTT}

B = {HTT, THT, TTH}

C = {HHT, HTH, THH, HHH}

Now,

$A \cup B \cup C$ = {TTT, HTT,THT,TTH,HHT,HTH,THH,HHH} = S

Therefore A,B and C are exhaustive events. Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$. Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive. Hence, A,B and C form a set of mutually exclusive and exhaustive events.

Do yourself - 1 :

(i) Two balls are drawn from a bag containing 2 Red and 3 Black balls, write sample space of this experiment.

(ii) Out of 2 men and 3 women a team of two persons is to be formed such that there is exactly one man and one woman. Write the sample space of this experiment.

(iii) A coin in tossed and if head comes up, a die is thrown. But if tail comes up, the coin is tossed again. Write the sample space of this experiment.

(iv) In a toss of a die, consider following events :

A : An even number turns up. B : A prime number turns up.

These events are -

(A) Equally likely events (B) Mutually exclusive events

(C) Exhaustive events (D) None of these

3. CLASSICAL DEFINITION OF PROBABILITY :

If n represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by P(A) = m/n. There are (n–m) outcomes which are favourable to the event that A does not happen. 'The event A does not happen' is denoted by $\bar{A}$ (and is read as 'not A')

Thus $P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$

i.e. $P(\bar{A}) = 1 - P(A)$

Note :

(i) $0 \le P(A) \le 1$

(ii) $P(A) + P(\bar{A}) = 1$,

(iii) If x cases are favourable to A & y cases are favourable to $\bar{A}$ then P(A) = $\frac{x}{(x+y)}$ and

$P(\bar{A}) = \frac{y}{(x+y)}$. We say that Odds In Favour Of A are x: y & Odds Against A are y : x

OTHER DEFINITIONS OF PROBABILITY :

(a) Axiomatic probability : Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0, 1] satisfying the following axioms :

(i) For any event E, $P(E) \ge 0$ (ii) P(S) = 1

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(E \cap F) = P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots, \omega_n$, i.e., $S = \{\omega_1, \omega_2, \ldots \omega_n\}$

It follows from the axiomatic definition of probability that :

(i) $0 \le P(\omega_i) \le 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \ldots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i)$, $\omega_i \in A$.

(b) Empirical probability : The probability that you would hit the bull's-eye on a dartboard with one throw of a dart would depend on how much you had practised, how much natural talent for playing darts you had, how tired you were, how good a dart you were using etc. all of which are impossible to quantify. A method which can be adopted in the example given above is to throw the dart several times (each throw is a trial) and count the number of times you hit the bull's-eye (a success) and the number of times you miss (a failure). Then an empirical value of the probability that you hit the bull's-eye with any one throw is

$$\frac{\text{number of successes}}{\text{number of successes + number of failures}}.$$

If the number of throws is small, this does not give a particular good estimate but for a large number of throws the result is more reliable.

When the probability of the occurrence of an event A cannot be worked out exactly, an empirical value can be found by adopting the approach described above, that is :

(i) making a large number of trials (i.e. set up an experiment in which the event may, or may not, occur and note the outcome),

(ii) counting the number of times the event does occur, i.e. the number of successes,

(iii) calculating the value of $\frac{\text{number of successes}}{\text{number of trials (i.e. successes + failures)}} = \frac{r}{n}$

The probability of event A occurring is defined as $P(A) = \lim_{n\to\infty}\left(\frac{r}{n}\right)$

$n \to \infty$ means that the number of trials is large (but what should be taken as 'large' depends on the problem).

Illustration 4 : If the letters of INTERMEDIATE are arranged, then the odds in favour of the event that no two 'E's occur together, are-

(A) $\frac{6}{5}$ (B) $\frac{5}{6}$ (C) $\frac{2}{9}$ (D) none of these

Solution : $I \to 2, N \to 1, T \to 2, E \to 3, R \to 1, M \to 1, D \to 1, A \to 1$ (3'E's, Rest 9 letters)

First arrange rest of the letters $= \frac{9!}{2!\ 2!}$,

Now 3'E's can be placed by ${}^{10}C_3$ ways, so favourable cases $= \frac{9!}{2!\ 2!} \times {}^{10}C_3 = 3\times 10!$

Total cases $= \frac{12!}{2!\,2!\,3!} = \frac{11}{2}\times 10!$; Non-favourable cases $= \left(\frac{11}{2} - 3\right)\times 10! = \frac{5}{2}\times 10!$

Odds in favour of the event $= \frac{3}{5/2} = \frac{6}{5}$ **Ans. (A)**

Illustration 5 : From a group of 10 persons consisting of 5 lawyers, 3 doctors and 2 engineers, four persons are selected at random. The probability that the selection contains at least one of each category is-

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) none of these

Solution : $n(S) = {}^{10}C_4 = 210.$

$n(E) = {}^5C_2 \times {}^3C_1 \times {}^2C_1 + {}^5C_1 \times {}^3C_2 \times {}^2C_1 + {}^5C_1 \times {}^3C_1 \times {}^2C_2 = 105$

$\therefore \quad P(E) = \frac{105}{210} = \frac{1}{2}.$ **Ans. (A)**

Illustration 6 : If four cards are drawn at random from a pack of fifty-two playing cards, find the probability that at least one of them is an ace.

Solution : If A is a combination of four cards containing at least one ace (i.e. either one ace, or two aces, or three aces or four aces) then $\bar{A}$ is a combination of four cards containing no aces.

Now $P(\bar{A}) = \dfrac{\text{Number of combinations of four cards with no aces}}{\text{Total number of combinations of four cards}} = {}^{48}C_4 / {}^{52}C_4 = 0.72$

Using $P(A) + P(\bar{A}) = 1$ we have $P(A) = 1 - P(\bar{A}) = 1 - 0.72 = 0.28$

Illustration 7 : A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is $2^n/({}^{2n}C_n)$.

Solution : Let S be the sample space & E be the event that each of the n pairs of balls drawn consists of one white and one red ball.

$\therefore$ $n(S) = ({}^{2n}C_2)\,({}^{2n-2}C_2)\,({}^{2n-4}C_2)...({}^{4}C_2)({}^{2}C_2)$

$$= \left\{\frac{(2n)(2n-1)}{1.2}\right\}\left\{\frac{(2n-2)(2n-3)}{1.2}\right\}\left\{\frac{(2n-4)(2n-5)}{1.2}\right\}......\left\{\frac{4.3}{1.2}\right\}.\left\{\frac{2.1}{1.2}\right\}$$

$$= \frac{1.2.3.4....(2n-1)2n}{2^n} = \frac{2n!}{2^n}$$

and $n(E) = ({}^{n}C_1.{}^{n}C_1)\,({}^{n-1}C_1.{}^{n-1}C_1)\,({}^{n-2}C_1.{}^{n-2}C_1)...({}^{2}C_1.{}^{2}C_1)\,({}^{1}C_1.{}^{1}C_1)$

$= n^2.(n-1)^2.(n-2)^2....2^2.1^2 = [1.2.3.....(n-1)n]^2 = (n!)^2$

$\therefore$ Required Probability, $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{(n!)^2}{(2n)!/2^n} = \dfrac{2^n}{\dfrac{2n!}{(n!)^2}} = \dfrac{2^n}{{}^{2n}C_n}$ **Ans.**

Do yourself - 2 :

(i) A coin is tossed successively three times. Find the probability of getting exactly one head or two heads.

(ii) A bag contains 5 red and 4 green balls. Four balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

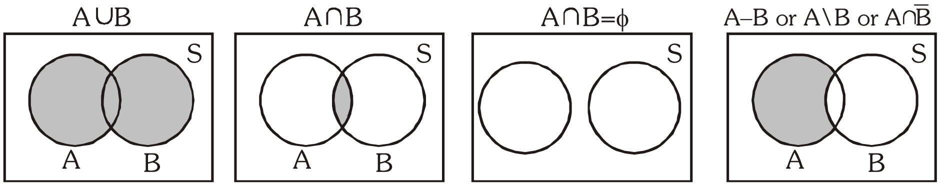
(iii) Two natural numbers are selected at random, find the probability that their sum is divisible by 10.

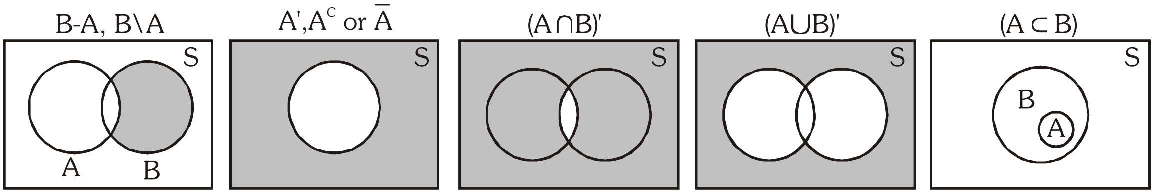
(iv) Five card are drawn successively from a pack of 52 cards with replacement. Find the probability that there is at least one Ace.

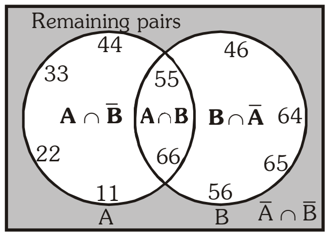
4. VENN DIAGRAMS :

A diagram used to illustrate relationships between sets. Commonly, a rectangle represents the universal set and a circle within it represents a given set (all members of the given set are represented by points within the circle). A subset is represented by a circle within a circle and intersection is indicated by overlapping circles.

Let S is the sample space of an experiment and A, B are two events corresponding to it :







Example : Let us conduct an experiment of tossing a pair of dice.
Two events defined on the experiment are

A : getting a doublet {11, 22, 33, 44, 55, 66}

B : getting total score of 10 or more {64, 46, 55, 56, 65, 66}

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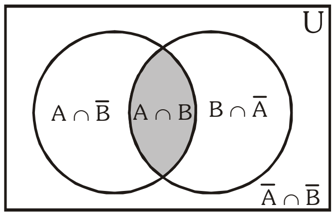
5. ADDITION THEOREM :



$A \cup B = A + B = A$ or B denotes occurrence of at least A or B.

For 2 events A & B :

$\mathbf{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$

(a)

$$\left.\begin{array}{l} P(A\cup B) \\ P(A+B) \\ P(A \text{ or } B) \\ P(\text{occurence of atleast A or B}) \end{array}\right] = \begin{array}{l} P(A)+P(B)-P(A\cap B) \text{ (This is known as generallised addition theorem)} \\ P(A)+P(B\cap \bar{A}) \\ P(B)+P(A\cap \bar{B}) \\ P(A\cap \bar{B})+P(A\cap B)+P(B\cap \bar{A}) \\ 1-P(\bar{A}\cap \bar{B}) \\ 1-P(\overline{A\cup B}) \end{array}$$

Note :

(i) If A & B are mutually exclusive then $P(A\cup B) = P(A) + P(B)$.

(ii) If A & B are mutually exclusive and exhaustive, then $P(A\cup B) = P(A) + P(B) = 1$

(b) $P(\text{only A occurs}) = P(A\backslash B) = P(A-B) = P(A\cap B^C) = P(A) - P(A\cap B)$

(c) $P(\text{either A or B}) = 1 - P(\text{neither A nor B})$

i.e. $P(A\cup B) = 1 - P(\bar{A}\cap\bar{B})$

(d) For any two events A & B

$P(\text{exactly one of A, B occurs}) = P\left(A\cap\bar{B}\right) + P\left(B\cap\bar{A}\right)$

$\Rightarrow$ **P(exactly one of A, B occurs) = $P(A) + P(B) - 2P(A \cap B)$**

$= P(A \cup B) - P(A \cap B) = P(A^c \cup B^c) - P(A^c \cap B^c)$

(e) $P(A \cap B) \le P(A), P(B) \le P(A \cup B) \le P(A) + P(B)$

6. DE MORGAN'S LAW :

If A & B are two subsets of a universal set U, then

(i) $(A \cup B)^c = A^c \cap B^c$ & (ii) $(A \cap B)^c = A^c \cup B^c$

Note :

(a) $(A \cup B \cup C)^C = A^C \cap B^C \cap C^C$ & $(A \cap B \cap C)^C = A^C \cup B^C \cup C^C$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Illustration 8 : Given two events A and B. If odds against A are as 2 : 1 and those in favour of $A \cup B$ are as 3 : 1, then find the range of P(B).

Solution : Clearly $P(A) = 1/3$, $P(A \cup B) = 3/4$.

Now, $P(B) \le P(A \cup B)$

$\Rightarrow P(B) \le 3/4$

Also, $P(B) = P(A \cup B) - P(A) + P(A \cap B)$

$\Rightarrow P(B) \ge P(A \cup B) - P(A)$ $(\because P(A \cap B) \ge 0)$

$\Rightarrow P(B) \ge 3/4 - 1/3$ $\Rightarrow P(B) \ge \dfrac{5}{12}$

$\Rightarrow \dfrac{5}{12} \le P(B) \le \dfrac{3}{4}$ **Ans.**

Illustration 9 : If A and B are two events such that $P(A \cup B) = \dfrac{3}{4}$, $P(A \cap B) = \dfrac{1}{4}$ and $P(A^c) = \dfrac{2}{3}$. Then find -

(i) P(A) (ii) P(B) (iii) $P(A \cap B^c)$ (iv) $P(A^c \cap B)$

Solution : $P(A) = 1 - P(A^c) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$

$P(B) = P(A \cup B) + P(A \cap B) - P(A) = \dfrac{3}{4} + \dfrac{1}{4} - \dfrac{1}{3} = \dfrac{2}{3}$

$P(A \cap B^c) = P(A) - P(A \cap B) = \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{1}{12}$

$P(A^c \cap B) = P(B) - P(A \cap B) = \dfrac{2}{3} - \dfrac{1}{4} = \dfrac{5}{12}$ **Ans.**

Do yourself - 3 :

(i) Draw Venn diagram of (a) $(A^C \cap B^C) \cup (A \cap B)$ (b) $B^C \cup (A^C \cap B)$

(ii) If A and B are two mutually exclusive events, then-

(A) $P(A) \le P(\bar{B})$ (B) $P(\bar{A} \cap \bar{B}) = P(\bar{A}) - P(B)$ (C) $P(\bar{A} \cup \bar{B}) = 0$ (D) $P(\bar{A} \cap B) = P(B)$

(iii) A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

(iv) In a class of 125 students, 70 passed in English, 55 in mathematics and 30 in both. Find the probability that a student selected at random from the class has passed in

(a) at least one subject (b) only one subject.

7. CONDITIONAL PROBABILITY AND MULTIPLICATION THEOREM :

(a) **Conditional Probability :** Let A and B be two events such that P (A) >0. Then P(B|A) denote the conditional probability of B given that A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we led to the definition

$$\mathbf{P(B|A)} = \frac{\mathbf{P(A \cap B)}}{\mathbf{P(A)}} = \text{which is called conditional probability of B given A}$$

(b) **Multiplication Theorem :** $P(A \cap B) = P(A)\ P(B|A)$ which is called compound probability or multiplication theorem. It says the probability that both A and B occur is equal to the probability that A occur times the probability that B occurs given that A has occurred.

Note : For any three events A_1, A_2, A_3 we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\ P(A_2|A_1)\ P(A_3 | (A_1 \cap A_2))$$

Illustration 10 : Two dice are thrown. Find the probability that the numbers appeared have a sum of 8 if it is known that the second die always exhibits 4

Solution : Let A be the event of occurrence of 4 always on the second die

$= \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$; $\therefore$ $n(A) = 6$

and B be the event of occurrence of such numbers on both dice whose sum is 8 = {(6,2), (5,3), (4,4), (3,5), (2,6)}.

Thus, $A \cap B = A \cap \{(4,4)\} = \{(4,4)\}$

$\therefore$ $n(A \cap B) = 1$

$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6} \text{ or } \frac{P(A \cap B)}{P(A)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Illustration 11 : A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag?

Solution : Let A be the event of drawing first ball white and B be the event of drawing second ball blue. Here A and B are dependent events.

$$P(A)=\frac{6}{16},\ P(B\,|\,A)=\frac{7}{15}$$

$$P(AB)=P(A).P(B\,|\,A)=\frac{6}{16}\times\frac{7}{15}=\frac{7}{40}$$

Illustration 12 : A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

Solution : E_1 : Event that first drawn ball is red, second is blue and so on.

E_2 : Event that first drawn ball is blue, second is red and so on.

$$\therefore \quad P(E_1)=\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5} \text{ and } P(E_2)=\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}$$

$$P(E)=P(E_1)+P(E_2)=2\times\frac{4}{8}.\frac{4}{7}.\frac{3}{6}.\frac{3}{5}=\frac{6}{35}$$ **Ans.**

Illustration 13 : If two events A and B are such that $P(\bar{A})=0.3$, P(B) = 0.4 and $P(A\bar{B})=0.5$ then $P(B\,|\,(A\cup\bar{B}))$ equals -

(A) 1/2 (B) 1/3 (C) 1/4 (D) 1/5

Solution : We have $P(B\,|\,(A\cup\bar{B}))=\dfrac{P[B\cap(A\cup\bar{B})]}{P(A\cup\bar{B})}=\dfrac{P[(B\cap A)\cup(B\cap\bar{B})]}{P(A)+P(\bar{B})-P(A\cap\bar{B})}$

$$=\frac{P(AB)}{P(A)+P(\bar{B})-P(A\bar{B})}=\frac{P(A)-P(A\bar{B})}{P(A)+P(\bar{B})-P(A\bar{B})}=\frac{0.7-0.5}{0.7+0.6-0.5}=\frac{0.2}{0.8}=\frac{1}{4}$$ **Ans.(C)**

Illustration 14 : Three coins are tossed. Two of them are fair and one is biased so that a head is three times as likely as a tail. Find the probability of getting two heads and a tail.

Solution : E_1 : Event that head occurs on fair coin

E_2 : Event that, head occurs on biased coin

$$\therefore\ P(E_1)=\frac{1}{2},\ P(E_2)=\frac{3}{4}$$

E : HHT or HTH or THH

$$\Rightarrow P(E)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}+\frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}$$

$$=\frac{7}{16}$$

Illustration 15 : In a multiple choice test of three questions there are five alternative answers given to the first two questions each and four alternative answers given to the last question. If a candidate guesses answers at random, what is the probability that he will get-

(a) Exactly one right answer ? (b) At least one right answer ?

Solution : E_1 : Event that, candidate guesses a correct answer for I question
E_2 : Event that, candidate guesses a correct answer for II question
E_3 : Event that, candidate guesses a correct answer for III question

$$\therefore\ P(E_1)=\frac{1}{5},\ P(E_2)=\frac{1}{5},\ P(E_3)=\frac{1}{4}$$

(a) E : Event that candidate get exactly one correct answer.

$$\therefore\ P(E)=P(E_1)P(\bar{E}_2)P(\bar{E}_3)+P(\bar{E}_1).P(E_2)P(\bar{E}_3)+P(\bar{E}_1)P(\bar{E}_2)P(E_3)$$

$$=\frac{1}{5}.\frac{4}{5}.\frac{3}{4}+\frac{4}{5}.\frac{1}{5}.\frac{3}{4}+\frac{4}{5}.\frac{4}{5}.\frac{1}{4}=\frac{2}{5}$$

(b) E : Event that candidate gets atleast one correct answer

$$\therefore\ P(E)=1-P(\bar{E}_1)P(\bar{E}_2)P(\bar{E}_3)=1-\frac{4}{5}.\frac{4}{5}.\frac{3}{4}=\frac{13}{25}$$

Illustration 16 : A speaks truth in 75% cases and B in 80% cases. What is the probability that they contradict each other in stating the same fact?
(A) 7/20 (B) 13/20 (C) 3/20 (D) 1/5

Solution : There are two mutually exclusive cases in which they contradict each other i.e. $\bar{A}B$ and $A\bar{B}$. Hence required probability = $P(A\bar{B}+\bar{A}B)=P(A\bar{B})+P(\bar{A}B)$

$$=P(A)P(\bar{B})+P(\bar{A})P(B)=\frac{3}{4}.\frac{1}{5}+\frac{1}{4}.\frac{4}{5}=\frac{7}{20}$$ **Ans. (A)**

Do yourself - 4 :

(i) A bag contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the bag and kept aside. From the remaining balls another ball is drawn and kept aside the first. This process is repeated till all the balls are drawn. Then probability that the balls drawn are in sequence of 2 black, 4 white and 3 red is-
(A) $\frac{1}{1260}$ (B) $\frac{1}{7560}$ (C) $\frac{1}{210}$ (D) None of these

(ii) Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that the drawn cards are face cards of same suit ?

8. INDEPENDENT EVENTS :

Two events A & B are said to be independent if occurrence or non occurrence of one does not affect the probability of the occurrence or non occurrence of other.

(a) If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be **Dependent** or **Contingent.** For two independent events A and B : $P(A\cap B) = P(A) . P(B)$. Often this is taken as the definition of independent events.
Note : If A and B are independent events, then
(i) $P(\bar{A}\cap\bar{B})=P(\bar{A}).P(\bar{B})$ (ii) $P(A\cap\bar{B})=P(A).P(\bar{B})$

(b) Three events A, B & C are independent if & only if all the following conditions hold ;
$P(A\cap B) = P(A) . P(B)$; $P(B\cap C) = P(B). P(C)$
$P(C\cap A) = P(C) . P(A)$ and $P(A\cap B\cap C) = P(A) . P(B) . P(C)$
i.e. they must be pairwise as well as mutually independent.

(c) If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive. i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0 \Rightarrow P(A \cap B \cap C) = 0$. However the converse of this is not true.

Note :

Independent events are not in general mutually exclusive & vice versa. Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments.

Illustration 17 : If A & B are independent events such that $P(A \cap \bar{B}) = \frac{1}{3}$ & $P(A \cup B) = \frac{11}{15}$, then $P(A \cap B)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{5}{11}$ (C) $\frac{2}{9}$ (D) $\frac{7}{9}$

Solution : $P(A) - P(A \cap B) = \frac{1}{3}$ & $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{11}{15}$

$$\Rightarrow \quad P(B) = \frac{6}{15} = \frac{2}{5}$$

$$P(A) - P(A)\,P(B) = \frac{1}{3} \Rightarrow P(A) = \frac{5}{9}$$

$$\Rightarrow \quad P(A \cap B) = P(A)\,P(B) = \frac{2}{5} \times \frac{5}{9} = \frac{2}{9}$$

Illustration 18 : A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as : **[IIT 1992]**

A = {The first bulb is defective}

B = {The second bulb is non-defective}

C = {The two bulbs are both defective or both non-defective}

Determine whether

(i) A, B, C are pairwise independent, (ii) A, B, C are independent.

Solution : We have $P(A) = \frac{50}{100}.1 = \frac{1}{2}$; $P(B) = 1.\frac{50}{100} = \frac{1}{2}$; $P(C) = \frac{50}{100}.\frac{50}{100} + \frac{50}{100}.\frac{50}{100} = \frac{1}{2}$

$A \cap B$ is the event that first bulb is defective and second is non-defective.

$$\therefore \quad P(A \cap B) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

$A \cap C$ is the event that both bulbs are defective.

$$\therefore \quad P(A \cap C) = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

Similarly $P(B \cap C) = \frac{1}{4}$

Thus we have $P(A \cap B) = P(A) \cdot P(B)$; $P(A \cap C) = P(A) \cdot P(C)$; $P(B \cap C) = P(B) \cdot P(C)$

$\therefore$ A, B and C are pairwise independent.

There is no element in $A \cap B \cap C$

$\therefore$ $P(A \cap B \cap C) = 0$

$\therefore$ $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

Hence A, B and C are not mutually independent.

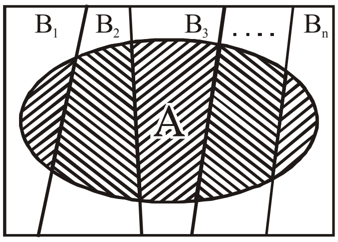
Do yourself - 5 :

(i) For two independent events A and B, the probability that both A & B occur is 1/8 and the probability that neither of them occur is 3/8. The probability of occurrence of A may be -

(A) 1/2 (B) 1/4 (C) 1/8 (D) 3/4

(ii) A die marked with numbers 1,2,2,3,3,3 is rolled three times. Find the probability of occurrence of 1,2 and 3 respectively.

9. TOTAL PROBABILITY THEOREM :



Let an event A of an experiment occurs with its n mutually exclusive & exhaustive events $B_1, B_2, B_3, \ldots\ldots B_n$ then total probability of occurrence of even A is

$$P(A) = P(AB_1) + P(AB_2) + \ldots\ldots + P(AB_n) = \sum_{i=1}^{n} P(AB_i)$$

$$P(A) = P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + \ldots\ldots\ldots\ldots + P(B_n)\,P(A|B_n)$$
$$= \Sigma P(B_i)\,P(A|B_i)$$

Illustration 19 : A purse contains 4 copper and 3 silver coins and another purse contains 6 copper and 2 silver coins. One coin is drawn from any one of these two purses. The probability that it is a copper coin is -

(A) $\frac{4}{7}$ (B) $\frac{3}{4}$ (C) $\frac{2}{7}$ (D) $\frac{37}{56}$

Solution : Let $A \equiv$ event of selecting first purse

$B \equiv$ event of selecting second purse

$C \equiv$ event of drawing a copper coin

Then given event has two disjoint cases: AC and BC

$$\therefore P(C) = P(AC + BC) = P(AC) + P(BC) = P(A)P(C\,|\,A) + P(B)P(C\,|\,B)$$

$$= \frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{6}{8} = \frac{37}{56}$$

Ans. (D)

Illustration 20 : Three groups A, B, C are contesting for positions on the Board of Directors of a Company. The probabilities of their winning are 0.5, 0.3, 0.2 respectively. If the group A wins, the probability of introducing a new product is 0.7 and the corresponding probabilities for group B and C are 0.6 and 0.5 respectively. Find the probability that the new product will be introduced.

Solution : Given P(A) = 0.5, P(B) = 0.3 and P(C) = 0.2

$\therefore$ P(A) + P(B) + P(C) =1

then events A, B, C are exhaustive.

If P(E) = Probability of introducing a new product, then as given

P(E|A) = 0.7, P(E|B) =0.6 and P(E|C) = 0.5

$\therefore$ P(E) = P(A). P(E|A) + P(B). P(E|B) + P(C). P(E|C)

$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5 = 0.35 + 0.18 + 0.10 = 0.63$

Illustration21 : A pair of dice is rolled together till a sum of either 5 or 7 is obtained. Find the probability that 5 comes before 7.

Solution : Let E_1 = the event of getting 5 in a roll of two dice = {(1, 4), (2, 3), (3, 2), (4, 1)}

$$\therefore \quad P(E_1) = \frac{n(E_1)}{n(S)} = \frac{4}{6\times 6} = \frac{1}{9}$$

Let E_2 = the event of getting either 5 or 7

= {(1, 4), (2, 3), (3, 2), (4, 1), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

$$\therefore \quad P(E_2) = \frac{n(E_2)}{n(S)} = \frac{10}{6\times 6} = \frac{5}{18}$$

$\therefore$ the probability of getting neither 5 nor 7 $= P(\bar{E}_2) = 1 - P(E_2) = 1 - \frac{5}{18} = \frac{13}{18}$

The event of getting 5 before 7 = $E_1 \cup (\bar{E}_2 E_1) \cup (\bar{E}_2 \bar{E}_2 E_1) \cup ......$ to ∞

$\therefore$ the probability of getting 5 before 7

$$= P(E_1) + P(\bar{E}_2 E_1) + P(\bar{E}_2\bar{E}_2 E_1) + ... \text{ to } \infty = P(E_1) + P(\bar{E}_2)P(E_1) + P(\bar{E}_2)P(\bar{E}_2)P(E_1) + ... \text{ to } \infty$$

$$= \frac{1}{9} + \frac{13}{18}.\frac{1}{9} + \frac{13}{18}.\frac{13}{18}.\frac{1}{9} + ... \text{ to } \infty$$

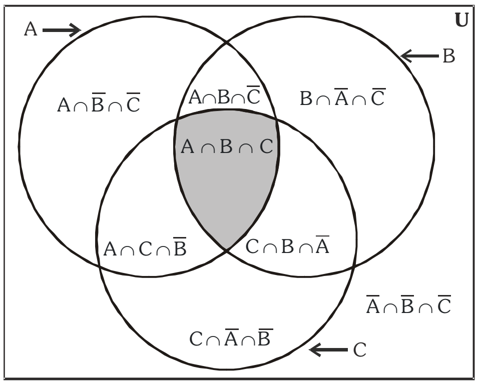
$$= \frac{1}{9}\left[1 + \frac{13}{18} + \left(\frac{13}{18}\right)^2 + ... \text{ to } \infty\right] = \frac{1}{9}.\frac{1}{1-\frac{13}{18}} = \frac{1}{9}.\frac{18}{5} = \frac{2}{5}$$

Do yourself - 6 :

(i) An urn contains 6 white & 4 black balls. A die is rolled and the number of balls equal to the number obtained on the die are drawn from the urn. Find the probability that the balls drawn are all black.

(ii) There are n bags such that i^{th} bag $(1 \le i \le n)$ contains i black and 2 white balls. Two balls are drawn from a randomly selected bag out of given n bags. Find the probability that the both drawn balls are white.

10. PROBABILITY OF THREE EVENTS :



For any three events A,B and C we have

(a) P(atleast one of A, B and C occurs)

$= P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A\cap B)$

$- P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

Note : $P(E_1\cup E_2 \ldots\ldots \cup E_n) = 1 - P(\overline{E}_1\cap\overline{E}_2\ldots\ldots\cap\overline{E}_n)$

(b) P(at least two of A,B,C occur)

$= P(B\cap C) + P(C\cap A) + P(A\cap B) - 2P(A\cap B\cap C)$

(c) P(exactly two of A,B,C occur)

$= P(B\cap C) + P(C\cap A) + P(A\cap B) - 3P(A\cap B\cap C)$

(d) P(exactly one of A,B,C occurs)

$= P(A) + P(B) + P(C) - 2P(B\cap C) - 2P(C\cap A) - 2P(A\cap B) + 3P(A\cap B\cap C)$

Illustration 22 : Let A,B,C be three events. If the probability of occurring exactly one event out of A and B is 1 – a, out of B and C is 1 – 2a, out of C and A is 1 – a and that of occurring three events simultaneously is a^2, then prove that the probability that at least one out of A,B,C will occur is greater than 1/2.

Solution : $P(A) + P(B) - 2P(A\cap B) = 1 - a$ .....(1)

and $P(B) + P(C) - 2P(B\cap C) = 1 - 2a$ .....(2)

and $P(C) + P(A) - 2P(C\cap A) = 1 - a$ .....(3)

and $P(A\cap B\cap C) = a^2$ .....(4)

$\because P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

$$= \frac{1}{2}\{P(A)+P(B)-2P(A\cap B)+P(B)+P(C)-2P(B\cap C)+P(C)+P(A)-2P(C\cap A)\}+P(A\cap B\cap C)$$

$$= \frac{1}{2}\{1-a+1-2a+1-a\}+a^2 \qquad \{\text{from (1), (2), (3) \& (4)}\}$$

$$= \frac{3}{2} - 2a + a^2 = (a-1)^2 + \frac{1}{2} > \frac{1}{2}$$ **Ans.**

Do yourself - 7 :

(i) In a class, there are 100 students out of which 45 study mathematics, 48 study physics, 40 study chemistry, 12 study both mathematics & physics, 11 study both physics & chemistry, 15 study both mathematics & chemistry and 5 study all three subjects. A student is selected at random, then find the probability that the selected student studies

(a) only one subject (b) neither physics nor chemistry

11. BINOMIAL PROBABILITY DISTRIBUTION :

Suppose that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. Each toss or selection is called a trial. In any single trial there will be a probability associated with a particular event such as head on the coin, 4 on the die, or selection of a red marble. In some cases this probability will not change from one trial to the next (as in tossing a coin or die). Such trials are then said to be independent and are often called Bernoulli trials after James Bernoulli who investigated them at the end of the seventeenth century.

Let p be the probability that an event will happen in any single Bernoulli trial (called the probability of success). Then q = 1 – p is the probability that the event will fail to happen in any single trial (called the probability of failure). The probability that the event will happen exactly x times in n trials (i.e., x successes and n – x failures will occur) is given by the probability function.

$$f(x) = P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad \text{......... (i)}$$

where the random variable X denotes the number of successes in n trials and x = 0, 1,........., n.

Example : The probability of getting exactly 2 heads in 6 tosses of a fair coin is

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{6!}{2!4!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{15}{64}$$

The discrete probability function (i) is often called the binomial distribution since for x = 0, 1, 2,...,n, it corresponds to successive terms in the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 + ... + p^n = \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x}$$

Illustration 23 : If a fair coin is tossed 10 times, find the probability of getting

(i) exactly six heads (ii) atleast six heads (iii) atmost six heads

Solution : The repeated tosses of a coin are Bernoulli trials. Let X denotes the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with n = 10 and $p = \frac{1}{2}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x,\ x = 0, 1, 2, ..., n$

Here $n = 10,\ p = \frac{1}{2},\ q = 1 - p = \frac{1}{2}$

Therefore $P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^x = {}^{10}C_x\left(\frac{1}{2}\right)^{10}$

Now (i) $P(X = 6) = {}^{10}C_6\left(\frac{1}{2}\right)^{10} = \frac{10!}{6!\times 4!2^{10}} = \frac{105}{512}$

(ii) P(atleast six heads)

$= P(X \geq 6) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\left(\frac{10!}{6!\times 4!}\right) + \left(\frac{10!}{7!\times 3!}\right) + \left(\frac{10!}{8!\times 2!}\right) + \left(\frac{10!}{9!\times 1!}\right) + \left(\frac{10!}{10!}\right)\right]\frac{1}{2^{10}} = \frac{193}{512}$$

(iii) P(at most six heads) $= P(X \leq 6)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X = 4) + P(X = 5) + P(X=6)$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$

Illustration 24 : India and Pakistan play a 5 match test series of hockey, the probability that India wins at least three matches is -

(A) $\frac{1}{2}$ (B) $\frac{3}{5}$ (C) $\frac{4}{5}$ (D) $\frac{5}{16}$

Solution : India win atleast three matches

$$= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^5(16) = \frac{1}{2}$$

Ans. (A)

Illustration 25 : A coin is tossed 7 times. Each time a man calls head. The probability that he wins the toss on more than three occasions is -

(A) $\frac{1}{4}$ (B) $\frac{5}{8}$ (C) $\frac{1}{2}$ (D) none of these

Solution : The man has to win at least 4 times.

$\therefore$ the required probability

$$= {}^7C_4\left(\frac{1}{2}\right)^4\cdot\left(\frac{1}{2}\right)^3 + {}^7C_5\cdot\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + {}^7C_6\left(\frac{1}{2}\right)^6\cdot\frac{1}{2} + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= ({}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7)\cdot\frac{1}{2^7} = \frac{64}{2^7} = \frac{1}{2}.$$

Ans. (C)

Do yourself - 8 :

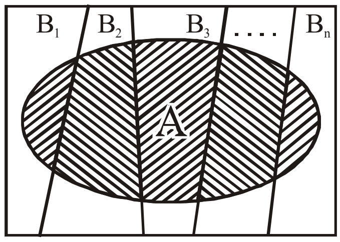
(i) An experiment succeeds twice as often as it fails. Find the probability that in next 6 trials, there will be more than 3 successes.

(ii) Find the probability of getting 4 exactly thrice in 7 throws of a die.

12. BAYE'S THEOREM :

Let an event A of an experiment occurs with its n mutually exclusive & exhaustive events $B_1, B_2, B_3, \ldots\ldots B_n$ & the probabilities $P(A/B_1)$, $P(A/B_2)$....... $P(A/B_n)$ are known, then

$$P(B_i/A) = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$$



Explanation :

$A \equiv$ event what we have ; $B_i \equiv$ event what we want & remaining are alternative events.

Now, $P(AB_i) = P(A).\ P(B_i/A) = P(B_i).\ P(A/B_i)$

$$P(B_i/A) = \frac{P(B_i)\ .\ P(A/B_i)}{P(A)} = \frac{P(B_i)\ .\ P(A/B_i)}{\sum_{i=1}^{n} P(AB_i)}$$

$$P(B_i/A) = \frac{P(B_i)\ .\ P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\ .\ P(A/B_i)}$$

Illustration 26 : Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ?

Solution : Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1)$ = P(a gold coin from box I) = $\frac{2}{2} = 1$

$P(A|E_2)$ = P(a gold coin from box II) = 0

$P(A|E_3)$ = P(a gold coin from box III) = $\frac{1}{2}$

Now, the probability that the other coin in the box is of gold
= the probability that gold coin is drawn from the box I.
= $P(E_1|A)$
By Baye's theorem, we know that

$$P(E_1 | A) = \frac{P(E_1)P(A | E_1)}{P(E_1)P(A | E_1) + P(E_2)P(A | E_2) + P(E_3)P(A | E_3)} = \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

Illustration 27 : A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from the bag B.

Solution : Let E_1 = The event of ball being drawn from bag A

E_2 = The event of ball being drawn from bag B.

E = The event of ball being red.

Since, both the bags are equally likely to be selected, therefore

$$P(E_1) = P(E_2) = \frac{1}{2} \text{ and } P(E|E_1) = \frac{3}{5} \text{ and } P(E|E_2) = \frac{5}{9}$$

$\therefore$ Required probability

$$P(E_2|E) = \frac{P(E_2)P(E|E_2)}{P(E_1).P(E|E_1) + P(E_2)P(E|E_2)} = \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{25}{52}$$

Do yourself - 9 :

(i) A pack of cards was found to contain only 51 cards. If first 13 cards, which are examined, are all red, then find the probability that the missing card is black.

(ii) A man has 3 coins A, B & C. A is fair coin. B is biased such that the probability of occurring head on it is 2/3. C is also biased with the probability of occurring head as 1/3. If one coin is selected and tossed three times, giving two heads and one tail, find the probability that the chosen coin was A.

13. PROBABILITY THROUGH STATISTICAL (STOCHATIC) TREE DIAGRAM :

These tree diagrams are generally drawn by economist and give a simple approach to solve a problem.

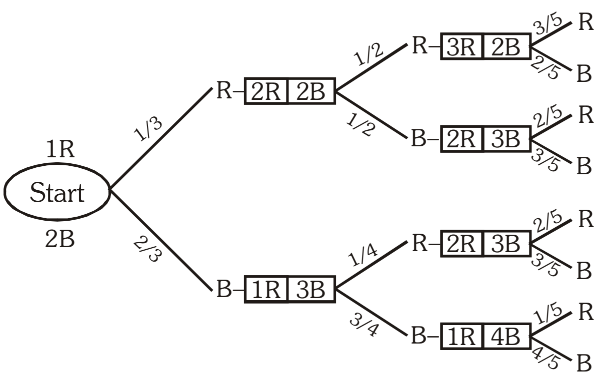
Illustration 28 : A bag initially contains 1 red ball and 2 blue balls. A trial consists of selecting a ball at random, noting its colour and replacing it together with an additional ball of the same colour. Given that three trials are made, draw a tree diagram illustrating the various probabilities. Hence, or otherwise, find the probability that

(a) atleast one blue ball is drawn

(b) exactly one blue ball is drawn

(c) Given that all three balls drawn are of the same colour find the probability that they are all red.

Solution :



Calculations :

$$P(A) = 1 - P(RRR) = 1 - \frac{1}{3}.\frac{1}{2}.\frac{3}{5} = 1 - \frac{1}{10} = \frac{9}{10}$$

$$P(\text{exactly one Blue}) = \frac{1}{3}.\frac{1}{2}.\frac{2}{5} + \frac{1}{3}.\frac{1}{2}.\frac{2}{5} + \frac{2}{3}.\frac{1}{4}.\frac{2}{5} = \frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \frac{3}{15} = \frac{1}{5}$$

$$P(C) = P\left(\frac{RRR}{(RRR \cup BBB)}\right) = \frac{P(RRR)}{P(RRR) + P(BBB)} = \frac{\frac{1}{3}.\frac{1}{2}.\frac{3}{5}}{\frac{1}{3}.\frac{1}{2}.\frac{3}{5} + \frac{2}{3}.\frac{3}{4}.\frac{4}{5}} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{4}{10}} = \frac{1}{5}$$

14. PROBABILITY DISTRIBUTION (Not in JEE) :

(a) A Probability Distribution spells out how a total probability of 1 is distributed over several values of a random variable.

(b) Mean of any probability distribution of a random variable is given by :

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i \quad (\text{ Since } \Sigma p_i = 1)$$

(c) Variance of a random variable is given by, $\sigma^2 = \sum (x_i - \mu)^2 . p_i$

$\sigma^2 = \sum p_i x_i^2 - \mu^2$ (Note that Standard Deviation (SD) = $+\sqrt{\sigma^2}$)

(d) The probability distribution for a binomial variate 'X' is given by ; P (X = r) = ${}^nC_r\, p^r q^{n-r}$ where: p = probability of success in a single trial, q = probability of failure in a single trial and p + q = 1.

(e) Mean of Binomial Probability Distribution (BPD) = np ; variance of BPD = npq.

(f) If p represents a person's chance of success in any venture and 'M' the sum of money which he will receive in case of success, then his expectations or probable value = pM

Illustration 29 : A random variable X has the probability distribution:

X :	1	2	3	4	5	6	7	8
p(X) :	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E = {X is a prime number} and F = {X < 4}, the probability P(E ∪ F) is -

(1) 0.35 (2) 0.77 (3) 0.87 (4) 0.50

Solution : E = x is a prime number

P(E) = P(2) + P(3) + P(5) + P(7) = 0.62

F = (x < 4), P(F) =P(1) + P(2) + P(3) = 0.50

∴ P (E ∪ F) = P(E) + P(F) – P(E ∩ F)

= 0.62 + 0.50 – 0.35 = 0.77

Illustration 30 : The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probability of 2 successes is -

(1) $\frac{128}{256}$ (2) $\frac{219}{256}$ (3) $\frac{37}{256}$ (4) $\frac{28}{256}$

Solution : $\left.\begin{matrix} np = 4 \\ npq = 2 \end{matrix}\right\} \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 8$

$$P(X = 2) = {}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 = 28.\frac{1}{2^8} = \frac{28}{256}$$

Miscellaneous Illustrations :

Illustration 31 : Three persons A, B, C in order cut a pack of cards, replacing them after each cut, on the condition that the first who cuts a spade shall win a prize; find their respective chances.

Solution : Let p be the chance of cutting a spade and q the chance of not cutting a spade from a pack of 52 cards.

Then $p = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{1}{4}$ and $q = 1 - \frac{1}{4} = \frac{3}{4}$

Now A will win a prize if he cuts spade at 1st, 4th, 7th, 10th turns, etc. Note that A will get a second chance if A, B, C all fail to cut a spade once and then A cuts a spade at the 4th turn. Similarly he will cut a spade at the 7th turn when A, B, C fail to cut spade twice, etc.

Hence A's chance of winning the prize $= p + q^3p + q^6p + q^9p + ... = \frac{p}{1-q^3} = \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^3} = \frac{16}{37}$

Similarly B's chance $= (qp + q^4p + q^7p + ...) = q(p + q^3p + q^6p + ...) = \frac{3}{4}.\frac{16}{37} = \frac{12}{37}$

and C's chance $= \frac{3}{4}$ of B's chance $= \frac{3}{4}.\frac{12}{37} = \frac{9}{37}$

Illustration 32 : (a) If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement, determine the probability that the roots of the equation $x^2 + px + q = 0$ are real. **[IIT 1997]**

(b) Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have equal roots.

Solution : (a) If roots of $x^2 + px + q = 0$ are real, then $p^2 - 4q \ge 0$ .... (i)

Both p, q belongs to set S = {1, 2, 3, ... 10} when p = 1, no value of q from S will satisfy (i)

p = 2	q = 1 will satisfy	1 value
p = 3	q = 1, 2	2 value
p = 4	q = 1, 2, 3, 4	4 value
p = 5	q = 1, 2, 3, 4, 5, 6	6 value
p = 6	q = 1, 2, 3, 4, 5, 6, 7, 8, 9	9 value

For p = 7, 8, 9, 10 all the ten values of q will satisfy.

Sum of these selections is 1 + 2 + 4 + 6 + 9 + 10 + 10 + 10 + 10 = 62

But the total number of selections of p and q without any order is $10 \times 10 = 100$

Hence the required probability is $= \frac{62}{100} = 0.62$

(b) Roots equal $\Rightarrow$ $b^2 - 4ac = 0$

$\therefore \quad \left(\frac{b}{2}\right)^2 = ac \qquad \qquad ......(i)$

Each coefficient is an integer, so we consider the following cases :

b = 1 $\quad \therefore \quad \frac{1}{4} = ac$

No integral values of a and c

b = 2 $\quad 1 = ac \quad \therefore \quad (1, 1)$

b = 3 $\quad 9/2 = ac$

No integral values of a and c

b = 4 $\quad 4 = ac \quad \therefore \quad (1, 4), (2, 2), (4, 1)$

b = 5 $\quad 25/2 = ac$

No integral values of a and c

b = 6 $\quad 9 = ac \quad \therefore \quad (3, 3)$

Thus we have 5 favourable ways for b = 2, 4, 6

Total number of equations is 6.6.6 = 216

$\therefore$ Required probability is $\frac{5}{216}$

Illustration 33 : A set A has n elements. A subset P of A is selected at random. Returning the element of P, the set Q is formed again and then a subset Q is selected from it. Find the probability that P and Q have no common elements. **[IIT 1990]**

Solution : Let P be the empty set, or one element set or two elements set .... or n elements set. Then the set Q will be chosen from amongst the remaining n elements or n – 1 elements or n – 2 elements .... or no elements. The probability of P being an empty set is ${}^nC_0/2^n$, the probability of P being one element set is ${}^nC_1/2^n$ and in general, the probability of P being an r element set is ${}^nC_r/2^n$.

When the set P consisting of r elements is chosen from A, then the probability of choosing the set Q from amongst the remaining n – r elements is $2^{n-r}/2^n$. Hence the probability that P and Q have no common elements is given by

$$\sum_{r=0}^{n} \frac{{}^nC_r}{2^n} \cdot \frac{2^{n-r}}{2^n} = \frac{1}{4^n} \sum_{r=0}^{n} {}^nC_r . 2^{n-r} = \left(\frac{1}{4}\right)^n (1+2)^n = \left(\frac{3}{4}\right)^n \qquad \text{[By binomial theorem]}$$

Illustration 34 : The probabilities of three events A, B and C are $P(A) = 0.6$, $P(B) = 0.4$ and $P(C) = 0.5$. If $P(A \cup B) = 0.8$, $P(A \cap C) = 0.3$, $P(A \cap B \cap C) = 0.2$ and $P(A \cup B \cup C) \geq 0.85$, find $P(B \cap C)$.

Solution :

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.8 = 0.6 + 0.4 - P(A \cap B)$

$\therefore \quad P(A \cap B) = 0.2$

Now $P(A \cup B \cup C) = S_1 - S_2 + S_3 = (0.6 + 0.4 + 0.5) - (0.2 + P(B \cap C) + 0.3) + 0.2$

$= 1.5 - 0.3 - P(B \cap C)$

We know $0.85 \leq P(A \cup B \cup C) \leq 1$

or $\quad 0.85 \leq 1.2 - P(B \cap C) \leq 1$

$\therefore \quad 0.2 \leq P(B \cap C) \leq 0.35$

ANSWERS FOR DO YOURSELF

1 : **(i)** $\{B_1R_1, B_2R_1, B_3R_1, B_1R_2, B_2R_2, B_3R_2, B_1B_2, B_2B_3, B_1B_3, R_1R_2\}$

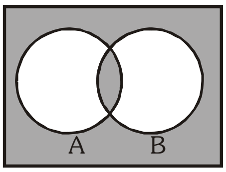
(ii) $\{M_1W_1, M_2W_1, M_1W_2, M_2W_2, M_1W_3, M_2W_3\}$

(iii) {H1, H2, H3, H4, H5, H6, TH,TT}

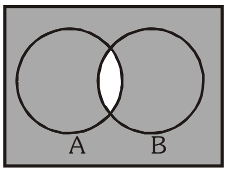
(iv) A

2 : **(i)** $\frac{3}{4}$ **(ii)** $\frac{10}{21}$ **(iii)** 1/10 **(iv)** $\frac{(13)^5 - (12)^5}{(13)^5}$

3 : **(i)** **(a)**



(b)



(ii) A,B,D **(iii)** 11/15

(iv) (a) $\frac{19}{25}$ (b) $\frac{13}{25}$

4 : **(i)** A **(ii)** 1/5525

5 : **(i)** A, B **(ii)** 1/36

6 : **(i)** 2/21, **(ii)** $\frac{1}{n+2}$

7 : **(i)** **(a)** 0.72, **(b)** 0.23

8 : **(i)** $\frac{496}{729}$, **(ii)** ${}^7C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^4$

9 : **(i)** 2/3, **(ii)** 9/25

EXERCISE (O-1)

PART # 1

1. 6 married couples are standing in a room. If 4 people are chosen at random, then the chance that exactly one married couple is among the 4 is-

(A) $\frac{16}{33}$ (B) $\frac{8}{33}$ (C) $\frac{17}{33}$ (D) $\frac{24}{33}$

PB0001

2. The probability that a positive two digit number selected at random has its tens digit at least three more than its unit digit is -

(A) 14/45 (B) 7/45 (C) 36/45 (D) 1/6

PB0002

3. A 5 digit number is formed by using the digits 0,1,2,3,4 & 5 without repetition. The probability that the number is divisible by 6 is :

(A) 8% (B) 17% (C) 18% (D) 36%

PB0003

4. A cube with all six faces coloured is cut into 64 cubical blocks of the same size which are thoroughly mixed. Find the probability that the 2 randomly chosen blocks have 2 coloured faces each -

PB0004

5. A card is drawn at random from a well shuffled deck of cards. Find the probability that the card is a-

(i) king or a red card (ii) club or a diamond (iii) king or a queen (iv) king or an ace

(v) spade or a club (vi) neither a heart nor a king

PB0005

6. A bag contain 5 white, 7 black, and 4 red balls, find the chance that three balls drawn at random are all white.

PB0006

7. If four coins are tossed, Two events A and B are defined as

A : No two consecutive heads occur

B : At least two consecutive heads occur.

Find P(A) and P(B). State whether the events are equally likely, mutually exclusive and exhaustive.

PB0007

8. Thirteen persons take their places at a round table, Find the odds against two particular persons sitting together.

PB0008

9. A has 3 shares in a lottery containing 3 prizes and 9 blanks, B has 2 shares in a lottery containing 2 prizes and 6 blanks. Compare their chances of success.

PB0009

10. Mr. A forgot to write down a very important phone number. All he remembers is that it started with 713 and that the next set of 4 digit involved are 1,7 and 9 with one of these numbers appearing twice. He guesses a phone number and dials randomly. The odds in favour of dialing the correct telephone number, is -

(A) 1 : 35 (B) 1 : 71 (C) 1 : 23 (D) 1 : 36

PB0010

11. Consider a function $f(x)$ that has zeroes 4 and 9. Given that Mr. A randomly selects a number from the set $\{-10, -9, -8, ...8, 9, 10\}$, what is the probability that Mr. A chooses a zero of $f(x^2)$?

PB0011

12. (a) A fair die is tossed. If the number is odd, find the probability that it is prime.

(b) Three fair coins are tossed. If both heads and tails appear, determine the probability that exactly one head appears.

PB0012

13. Mr. A lives at origin on the cartesian plane and has his office at (4,5). His friend lives at (2,3) on the same plane. Mr. A can go to his office travelling one block at a time either in the +y or +x direction. If all possible paths are equally likely then the probability that Mr. A passed his friends house is -

(A) 1/2 (B) 10/21 (C) 1/4 (D) 11/21

PB0013

14. I have 3 normal dice, one red, one blue and one green and I roll all three simultaneously. Let P be the probability that the sum of the numbers on the red and blue dice is equal to the number on the green die. If P is the written in lowest terms as a/b then the value of (a + b) equals -

(A) 79 (B) 77 (C) 61 (D) 57

PB0014

15. There are three passengers on an airport shuttle bus that makes stops at four different hotels. The probability that all three passengers will be staying at different hotels, is -

(A) $\frac{1}{16}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{3}{4}$

PB0015

PART # 2

1. In throwing 3 dice, the probability that atleast 2 of the three numbers obtained are same is -

(A) 1/2 (B) 1/3 (C) 4/9 (D) none

PB0016

2. There are 4 defective items in a lot consisting of 10 items. From this lot we select 5 items at random. The probability that there will be 2 defective items among them is -

(A) $\frac{1}{2}$ (B) $\frac{2}{5}$ (C) $\frac{5}{21}$ (D) $\frac{10}{21}$

PB0017

3. From a pack of 52 playing cards, face cards and tens are removed and kept aside then a card is drawn at random from the remaining cards. If

A: The event that the card drawn is an ace

H: The event that the card drawn is a heart

S : The event that the card drawn is a spade

then which of the following holds ?

(A) $9P(A) = 4P(H)$
(B) $P(S) = 4P(A \cap H)$
(C) $3P(H)= 4P(A \cup S)$
(D) $P(H)= 12P(A \cap S)$

PB0018

4. If two of the 64 squares are chosen at random on a chess board, the probability that they have a side in common is -

(A) 1/9 (B) 1/18 (C) 2/7 (D) none

PB0019

5. Two red counters, three green counters and 4 blue counters are placed in a row in random order. The probability that no two blue counters are adjacent is -

(A) $\frac{7}{99}$ (B) $\frac{7}{198}$ (C) $\frac{5}{42}$ (D) none

PB0020

6. South African cricket captain lost the toss of a coin 13 times out of 14. The chance of this happening was

(A) $\frac{7}{2^{13}}$ (B) $\frac{1}{2^{13}}$ (C) $\frac{13}{2^{14}}$ (D) $\frac{13}{2^{13}}$

PB0021

7. There are ten prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top ten contestants in a mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the 8^{th} contestant goes to select the prize, the probability that the remaining three prizes are one A, one B and one C, is

(A) 1/4 (B) 1/3 (C) 1/12 (D) 1/10

PB0022

8. A coin is tossed and a die is thrown. Find the probability that the outcome will be a head or a number greater than 4.

PB0023

9. A coin is biased so that heads is three times as likely to appear as tails. Find P(H) and P(T). If such a coin is tossed twice find the probability that head occurs at least once.

PB0024

10. Given two independent events A, B such that $P(A) = 0.3$, $P(B) = 0.6$. Determine

(i) P(A and B) (ii) P(A and not B) (iii) P(not A and B) (iv) P(neither A nor B)
(v) P(A or B)

PB0025

11. The probabilities that a student will receive A, B, C or D grade are 0.40, 0.35, 0.15 and 0.10 respectively. Find the probability that a student will receive

(i) not an A grade (ii) B or C grade (iii) at most C grade

PB0026

12. In a single throw of three dice, determine the probability of getting

(i) a total of 5 (ii) a total of atmost 5 (iii) a total of at least 5.

PB0027

13. A natural number x is randomly selected from the set of first 100 natural numbers. Find the probability that it satisfies the inequality. $x + \frac{100}{x} > 50$

PB0028

14. 3 students A, B and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins. Assume no two reach the winning point simultaneously.

PB0029

15. A box contains 7 tickets, numbered from 1 to 7 inclusive. If 3 tickets are drawn from the box without replacement, one at a time, determine the probability that they are alternatively either odd-even-odd or even-odd-even.

PB0030

16. Let a red die, a blue die, a green die and a white die are rolled once, the dice being fair. The outcomes on the red, blue, green and white die denote the numbers, a, b,c and d respectively. Let E denotes the event that absolute value of (a – 1) (b – 2) (c – 3) (d – 6) = 1, then P(E) is -

(A) $\frac{1}{324}$ (B) $\frac{1}{648}$ (C) $\frac{2}{324}$ (D) $\frac{1}{162}$

PB0031

17. 5 different marbles are placed in 5 different boxes randomly. Find the probability that exactly two boxes remain empty. Given each box can hold any number of marbles.

PB0032

18. Let A and B be events such that $P(\overline{A}) = 4/5$, P(B) = 1/3, P(A/B) = 1/6, then :

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) P(B/A)

(d) Are A and B independent?

PB0033

PART # 3

1. Let A & B be two events. Suppose $P(A) = 0.4$, $P(B) = p$ & $P(A \cup B) = 0.7$. The value of p for which A & B are independent is :

(A) 1/3 (B) 1/4 (C) 1/2 (D) 1/5

PB0034

2. A pair of numbers is picked up randomly (without replacement) from the set {1, 2, 3, 5, 7, 11, 12, 13, 17, 19}. The probability that the number 11 was picked given that the sum of the numbers was even, is nearly :

(A) 0.1 (B) 0.125 (C) 0.24 (D) 0.18

PB0035

3. For a biased die the probabilities for the different faces to turn up are given below :

Faces :	1	2	3	4	5	6
Probabilities :	0.10	0.32	0.21	0.15	0.05	0.17

The die is tossed & you are told that either face one or face two has turned up. Then the probability that it is face one is :

(A) 1/6 (B) 1/10 (C) 5/49 (D) 5/21

PB0036

4. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the determinant chosen has the value non negative is :

(A) 3/16 (B) 6/16 (C) 10/16 (D) 13/16

PB0037

5. A card is drawn & replaced in an ordinary pack of 52 playing cards. Minimum number of times must a card be drawn so that there is atleast an even chance of drawing a heart, is

(A) 2 (B) 3 (C) 4 (D) more than four

PB0038

6. A license plate is 3 capital letters (of English alphabets) followed by 3 digits. If all possible license plates are equally likely, the probability that a plate has either a letter palindrome or a digit palindrome (or both), is

(A) $\frac{7}{52}$ (B) $\frac{9}{65}$ (C) $\frac{8}{65}$ (D) none

PB0039

7. Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2 respectively. If they race three times the probability that "the same horse wins all the three races" and the probability that a, b, c each wins one race, are respectively (Assume no dead heat)

(A) $\frac{8}{50};\frac{9}{50}$ (B) $\frac{16}{100},\frac{3}{100}$ (C) $\frac{12}{50};\frac{15}{50}$ (D) $\frac{10}{50};\frac{8}{50}$

PB0040

8. Two cubes have their faces painted either red or blue. The first cube has five red faces and one blue face. When the two cubes are rolled simultaneously, the probability that the two top faces show the same colour is 1/2. Number of red faces on the second cube, is

(A) 1 (B) 2 (C) 3 (D) 4

PB0041

9. A committee of three persons is to be randomly selected from a group of three men and two women and the chair person will be randomly selected from the committee. The probability that the committee will have exactly two women and one man, and that the chair person will be a woman, is/are

(A) 1/5 (B) 8/15 (C) 2/3 (D) 3/10

PB0042

10. An urn contains 3 red balls and n white balls.

Mr. A draws two balls together from the urn. The probability that they have the same colour is 1/2.

Mr. B draws one ball from the urn, notes its colour and replaces it. He then draws a second ball from the urn and finds that both balls have the same colour is, 5/8. The possible value of n is

(A) 9 (B) 6 (C) 5 (D) 1

PB0043

11. The probability that an automobile will be stolen and found within one week is 0.0006. The probability that an automobile will be stolen is 0.0015. The probability that a stolen automobile will be found in one week is

(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6

PB0044

12. A box contains 100 tickets numbered 1, 2, 3,.... ,100. Two tickets are chosen at random. It is given that the maximum number on the two chosen tickets is not more than 10. The minimum number on them is 5, with probability

(A) $\frac{1}{9}$ (B) $\frac{2}{11}$ (C) $\frac{3}{19}$ (D) none

PB0045

13. Two boys A and B find the jumble of n ropes lying on the floor. Each takes hold of one loose end randomly. If the probability that they are both holding the same rope is $\frac{1}{101}$ then the number of ropes is equal to

(A) 101 (B) 100 (C) 51 (D) 50

PB0046

[REASONING TYPE]

14. For children A,B,C and D have 1,3,5 and 7 identical unbiased dice respectively and roll them with the condition that one who obtains an even score, wins. They keep playing till some one or the other wins.

Statement-1: All the four children are equally likely to win provided they roll their dice simultaneously.

Statement-2: The child A is most probable to win the game if they roll their dice in order of A,B,C and D respectively.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0047

15. In one day test match between India and Australia the umpire continues tossing a fair coin until the two consecutive throws either H T or T T are obtained for the first time. If it is H T, India wins and if it is T T, Australia wins.

Statement-1: Both India and Australia have equal probability of winning the toss.

Statement-2: If a coin is tossed twice then the events HT or TT are equiprobable.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0048

PART # 4

1. If E & F are events with $P(E) \leq P(F)$ & $P(E \cap F) > 0$, then :

(A) occurrence of E $\Rightarrow$ occurrence of F

(B) occurrence of F $\Rightarrow$ occurrence of E

(C) non – occurrence of E $\Rightarrow$ non – occurrence of F

(D) none of the above implications holds.

PB0049

2. Events A and C are independent. If the probabilities relating A, B and C are $P(A) = 1/5$; $P(B) = 1/6$; $P(A \cap C) = 1/20$; $P(B \cup C) = 3/8$ then

(A) events B and C are independent

(B) events B and C are mutually exclusive

(C) events B and C are neither independent nor mutually exclusive

(D) events A and C are equiprobable

PB0050

3. An unbiased cubic die marked with 1, 2, 2, 3, 3, 3 is rolled 3 times. The probability of getting a total score of 4 or 6 is

(A) $\frac{16}{216}$ (B) $\frac{50}{216}$ (C) $\frac{60}{216}$ (D) none

PB0051

4. A bag contains 3 R & 3 G balls and a person draws out 3 at random. He then drops 3 blue balls into the bag & again draws out 3 at random. The chance that the 3 later balls being all of different colours is

(A) 15% (B) 20% (C) 27% (D) 40%

PB0052

5. A biased coin with probability P, $0 < P < 1$, of heads, is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, then the value of P is

(A) 1/4 (B) 1/6 (C) 1/3 (D) 1/2

PB0053

6. Two numbers a and b are selected from the set of natural number then the probability that $a^2 + b^2$ is divisible by 5 is

(A) $\frac{9}{25}$ (B) $\frac{7}{18}$ (C) $\frac{11}{36}$ (D) $\frac{17}{81}$

PB0054

7. When a missile is fired from a ship, the probability that it is intercepted is 1/3. The probability that the missile hits the target, given that it is not intercepted is 3/4. If three missiles are fired independently from the ship, the probability that all three hits the target, is

(A) 1/12 (B) 1/8 (C) 3/8 (D) 3/4

PB0055

8. An urn contains 10 balls coloured either black or red. When selecting two balls from the urn at random, the probability that a ball of each colour is selected is $8/15$. Assuming that the urn contains more black balls than red balls, the probability that at least one black ball is selected, when selecting two balls, is

(A) $\frac{18}{45}$ (B) $\frac{30}{45}$ (C) $\frac{39}{45}$ (D) $\frac{41}{45}$

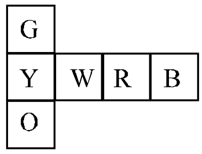
PB0056

9. An unbiased die with numbers 1, 2, 3, 4, 6 and 8 on its six faces is rolled. After this roll if an odd number appears on the top face, all odd numbers on the die are doubled. If an even number appears on the top face, all the even numbers are halved. If the given die changes in this way then the probability that the face 2 will appear on the second roll is -

(A) 2/18 (B) 3/18 (C) 2/9 (D) 5/18

PB0057

10. A butterfly randomly lands on one of the six squares of the T-shaped figure shown and then randomly moves to an adjacent square. The probability that the butterfly ends up on the R square is



(A) 1/4 (B) 1/3
(C) 2/3 (D) 1/6

PB0058

11. A fair coin is tossed a large number of times. Assuming the tosses are independent which one of the following statement, is True?

(A) Once the number of flips is large enough, the number of heads will always be exactly half of the total number of tosses. For example, after 10,000 tosses one should have exactly 5,000 heads.

(B) The proportion of heads will be about 1/2 and this proportion will tend to get closer to 1/2 as the number of tosses increases

(C) As the number of tosses increases, any long run of heads will be balanced by a corresponding run of tails so that the overall proportion of heads is exactly 1/2

(D) All of the above

PB0059

12. The number 'a' is randomly selected from the set {0, 1, 2, 3, ... 98, 99}. The number 'b' is selected from the same set. Probability that the number $3^a + 7^b$ has a digit equal to 8 at the units place, is

(A) $\frac{1}{16}$ (B) $\frac{2}{16}$ (C) $\frac{4}{16}$ (D) $\frac{3}{16}$

PB0060

PART # 5

1. An examination consists of 8 questions in each of which one of the 5 alternatives is the correct one. On the assumption that a candidate who has done no preparatory work chooses for each question any one of the five alternatives with equal probability, the probability that he gets more than one correct answer is equal to

(A) $(0.8)^8$ (B) $3(0.8)^8$ (C) $1 - (0.8)^8$ (D) $1 - 3(0.8)^8$

PB0061

2. An ant is situated at the vertex A of the triangle ABC. Every movement of the ant consists of moving to one of other two adjacent vertices from the vertex where it is situated. The probability of going to any of the other two adjacent vertices of the triangle is equal. The probability that at the end of the fourth movement the ant will be back to the vertex A, is :

(A) 4/16 (B) 6/16 (C) 7/16 (D) 8/16

PB0062

3. A key to room number C_3 is dropped into a jar with five other keys, and the jar is throughly mixed. If keys are randomly drawn from the jar without replacement until the key to room C_3 is chosen, then what are the odds in favour that the key to room C_3 will be obtained on the 2^{nd} try?

(A) 1 : 4 (B) 1 : 5 (C) 1 : 6 (D) 5 : 6

PB0063

4. Lot A consists of 3G and 2D articles. Lot B consists of 4G and 1D article. A new lot C is formed by taking 3 articles from A and 2 from B. The probability that an article chosen at random from C is defective, is

(A) 1/3 (B) 2/5 (C) 8/25 (D) none

PB0064

5. Mr. A and Mr. B each have a bag that contains one ball of each of the colours blue, green, orange, red and violet. 'A' randomly selects one ball from his bag and puts it into B's bag. 'B' then randomly selects one ball from his bag and puts it into A's bag. The probability that after this process the contents of the two bags are the same, is

(A) 1/6 (B) 1/5 (C) 1/3 (D) 1/2

PB0065

6. An instrument consists of two units. Each unit must function for the instrument to operate. The reliability of the first unit is 0.9 & that of the second unit is 0.8. The instrument is tested & fails. The probability that "only the first unit failed & the second unit is sound" is :

(A) 1/7 (B) 2/7 (C) 3/7 (D) 4/7

PB0066

7. A box contains 10 tickets numbered from 1 to 10. Two tickets are drawn one by one without replacement. The probability that the "absolute value of difference between the first drawn ticket number and the second is not less than 4" is

(A) $\frac{7}{30}$ (B) $\frac{7}{15}$ (C) $\frac{11}{30}$ (D) $\frac{10}{30}$

PB0067

8. A purse contains 2 six sided dice. One is a normal fair die, while the other has 2 ones, 2 threes, and 2 fives. A die is picked up and rolled. Because of some secret magnetic attraction of the unfair die, there is 75% chance of picking the unfair die and a 25% chance of picking a fair die. The die is rolled and shows up the face 3. The probability that a fair die was picked up, is

(A) 1/7 (B) 1/4 (C) 1/6 (D) 1/24

PB0068

9. On a Saturday night 20% of all drivers is U.S.A. are under the influence of alcohol. The probability that a driver under the influence of alcohol will have an accident is 0.001. The probability that a sober driver will have an accident is 0.0001. If a car on a Saturday night smashed into a tree, the probability that the driver was under the influence of alcohol, is -

(A) 3/7 (B) 4/7 (C) 5/7 (D) 6/7

PB0069

10. A box has four dice in it. Three of them are fair dice but the fourth one has the number five on all of its faces. A die is chosen at random from the box and is rolled three times and shows up the face five on all the three occasions. The chance that the die chosen was a rigged die, is

(A) $\frac{216}{217}$ (B) $\frac{215}{219}$ (C) $\frac{216}{219}$ (D) none

PB0070

11. On a normal standard die one of the 21 dots from any one of the six faces is removed at random with each dot equally likely to be chosen. The die is then rolled. The probability that the top face has an odd number of dots is

(A) $\frac{5}{11}$ (B) $\frac{5}{12}$ (C) $\frac{11}{21}$ (D) $\frac{6}{11}$

PB0071

Paragraph for question nos. 12 to 14

A JEE aspirant estimates that she will be successful with an 80 percent chance if she studies 10 hours per day, with a 60 percent chance if she studies 7 hours per day and with a 40 percent chance if she studies 4 hours per day. She further believes that she will study 10 hours, 7 hours and 4 hours per day with probabilities 0.1, 0.2 and 0.7, respectively

12. The chance she will be successful, is

(A) 0.28 (B) 0.38 (C) 0.48 (D) 0.58

PB0072

13. Given that she is successful, the chance she studied for 4 hours, is

(A) $\frac{6}{12}$ (B) $\frac{7}{12}$ (C) $\frac{8}{12}$ (D) $\frac{9}{12}$

PB0072

14. Given that she does not achieve success, the chance she studied for 4 hour, is

(A) $\frac{18}{26}$ (B) $\frac{19}{26}$ (C) $\frac{20}{26}$ (D) $\frac{21}{26}$

PB0072

[REASONING TYPE]

15. A fair coin is tossed 3 times consider the events

A : first toss is head

B : second toss is head

C : exactly two consecutive heads or exactly two consecutive tails.

Statement-1: A, B, C are independent events.

Statement-2: A, B, C are pairwise independent.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0073

PART # 6

1. A bowl has 6 red marbles and 3 green marbles. The probability that a blind folded person will draw a red marble on the second draw from the bowl without replacing the marble from the first draw, is

(A) 2/3 (B) 1/4 (C) 5/12 (D) 5/8

PB0074

2. The probability that a radar will detect an object in one cycle is p. The probability that the object will be detected in n cycles is :

(A) $1 - p^n$ (B) $1 - (1 - p)^n$ (C) p^n (D) $p(1 - p)^{n-1}$

PB0075

3. In a certain factory, machines A, B and C produce bolts. Of their production, machines A, B, and C produce 2%, 1% and 3% defective bolts respectively. Machine A produces 35% of the total output of bolts, machine B produces 25% and machine C produces 40%. A bolts is chosen at random from the factory's production and is found to be defective. The probability it was produced on machine C, is

(A) $\frac{6}{11}$ (B) $\frac{23}{45}$ (C) $\frac{24}{43}$ (D) $\frac{3}{11}$

PB0076

4. Three numbers are chosen at random without replacement from {1, 2, 3,...... , 10}. The probability that the minimum of the chosen numbers is 3 or their maximum is 7 is

(A) 1/2 (B) 1/3 (C) 1/4 (D) 11/40

PB0077

5. Two buses A and B are scheduled to arrive at a town central bus station at noon. The probability that bus A will be late is 1/5. The probability that bus B will be late is 7/25. The probability that the bus B is late given that bus A is late is 9/10. Then the probabilities

(i) neither bus will be late on a particular day and

(ii) bus A is late given that bus B is late, are respectively

(A) 2/25 and 12/28 (B) 18/25 and 22/28

(C) 7/10 and 18/28 (D) 12/25 and 2/28

PB0078

6. If at least one child in a family with 3 children is a boy then the probability that exactly 2 of the children are boys, is

(A) 3/7 (B) 4/7 (C) 1/3 (D) 3/8

PB0079

7. From an urn containing six balls, 3 white and 3 black ones, a person selects at random an even number of balls (all the different ways of drawing an even number of balls are considered equally probable, irrespective of their number). Then the probability that there will be the same number of black and white balls among them

(A) 4/5 (B) 11/15 (C) 11/30 (D) 2/5

PB0080

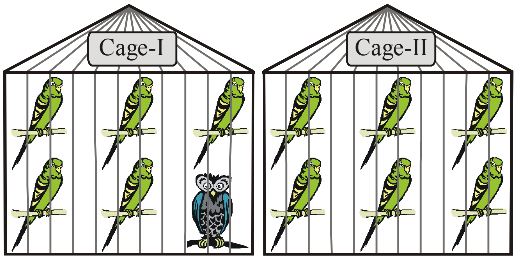
8. There are three main political parties namely 1, 2, 3. If in the adjoining table p_{ij} , (i, j=1, 2, 3) denote the probability that party j wins the general elections contested when party i is in the power. What is the probability that the party 2 will be in power after the next two elections, given that the party 1 is in the power?

$P_{11}=0.7$	$P_{12}=0.2$	$P_{13}=0.1$
$P_{21}=0.5$	$P_{22}=0.3$	$P_{23}=0.2$
$P_{31}=0.3$	$P_{32}=0.4$	$P_{33}=0.3$

(A) 0.27 (B) 0.24 (C) 0.14 (D) 0.06

PB0081

9. Shalu bought two cages of birds : Cage-I contains 5 parrots and 1 owl, and Cage-II contains 6 parrots, as shown



Birds like to fly

One day Shalu forgot to lock both cages and two birds flew from Cage-I to Cage-II. Then two birds flew back from Cage-II to Cage-I. Assume that all birds have equal chance of flying, the probability that the Owl is still in Cage-I, is

(A) 1/6 (B) 1/3 (C) 2/3 (D) 3/4

PB0082

10. Suppose families always have one, two or three children, with probabilities $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{4}$ respectively. Assume everyone eventually gets married and has children, the probability of a couple having exactly four grandchildren is

(A) $\frac{27}{128}$ (B) $\frac{37}{128}$ (C) $\frac{25}{128}$ (D) $\frac{20}{128}$

PB0083

11. Miss C has either Tea or Coffee at morning break. If she has tea one morning, the probability she has tea the next morning is 0.4. If she has coffee one morning, the probability she has coffee next morning is 0.3. Suppose she has coffee on a Monday morning. The probability that she has tea on the following Wednesday morning is

(A) 0.46 (B) 0.49 (C) 0.51 (D) 0.61

PB0084

12. In a maths paper there are 3 sections A, B & C. Section A is compulsory. Out of sections B & C a student has to attempt any one. Passing in the paper means passing in A & passing in B or C. The probability of the student passing in A, B & C are p, q & 1/2 respectively. If the probability that the student is successful is 1/2 then :

(A) $p = q = 1$ (B) $p = q = 1/2$ (C) $p = 1, q = 0$ (D) $p = 1, q = 1/2$

PB0085

[REASONING TYPE]

13. From a well shuffled pack of 52 playing cards a card is drawn at random. Two events A and B are defined as

A : Red card is drawn.

B : Card drawn is either a Diamond or Heart

Statement-1: $P(A + B) = P(AB)$

Statement-2: $A \subseteq B$ and $B \subseteq A$

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

PB0086

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. n different books ($n \geq 3$) are put at random in a shelf. Among these books there is a particular book 'A' and a particular book B. The probability that there are exactly 'r' books between A and B is -

(A) $\frac{2}{n(n-1)}$ (B) $\frac{2(n-r-1)}{n(n-1)}$ (C) $\frac{2(n-r-2)}{n(n-1)}$ (D) $\frac{(n-r)}{n(n-1)}$

PB0087

2. Of all the mappings that can be defined from the set A : $\{1, 2, 3, 4\} \rightarrow B(5, 6, 7, 8, 9\}$, a mapping is randomly selected. The chance that the selected mapping is strictly monotonic, is

(A) $\frac{1}{125}$ (B) $\frac{2}{125}$ (C) $\frac{5}{4096}$ (D) $\frac{5}{2048}$

PB0088

3. A fair die is thrown 3 times. The chance that sum of three numbers appearing on the die is less than 11, is equal to -

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{6}$ (D) $\frac{5}{8}$

PB0089

4. One bag contains 3 white & 2 black balls, and another contains 2 white & 3 black balls. A ball is drawn from the second bag & placed in the first, then a ball is drawn from the first bag & placed in the second. When the pair of the operations is repeated, the probability that the first bag will contain 5 white balls is:

(A) 1/25 (B) 1/125 (C) 1/225 (D) 2/15

PB0090

5. If a, b and c are three numbers (not necessarily different) chosen randomly and with replacement from the set {1, 2, 3, 4, 5}, the probability that (ab + c) is even, is

(A) $\frac{35}{125}$ (B) $\frac{59}{125}$ (C) $\frac{64}{125}$ (D) $\frac{75}{125}$

PB0091

6. A purse contains 100 coins of unknown value, a coin drawn at random is found to be a rupee. The chance that it is the only rupee in the purse, is (Assume all numbers of rupee coins in the purse to be equally likely.)

(A) $\frac{1}{5050}$ (B) $\frac{2}{5151}$ (C) $\frac{1}{4950}$ (D) $\frac{2}{4950}$

PB0092

7. Mr. Dupont is a professional wine taster. When given a French wine, he will identify it with probability 0.9 correctly as French, and will mistake it for a Californian wine with probability 0.1. When given a Californian wine, he will identify it with probability 0.8 correctly as Californian, and will mistake it for a French wine with probability 0.2. Suppose that Mr. Dupont is given ten unlabelled glasses of wine, three with French and seven with Californian wines. He randomly picks a glass, tries the wine, and solemnly says : "French". The probability that the wine he tasted was Californian, is nearly equal to

(A) 0.14 (B) 0.24 (C) 0.34 (D) 0.44

PB0093

8. Sixteen players $s_1, s_2, \ldots\ldots, s_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength. The probability that "exactly one of the two players s_1 & s_2 is among the eight winners" is

(A) $\frac{4}{15}$ (B) $\frac{7}{15}$ (C) $\frac{8}{15}$ (D) $\frac{9}{15}$

PB0094

9. A multiple choice test question has five alternative answers, of which only one is correct. If a student has done his home work, then he is sure to identify the correct answer; otherwise, he chooses an answer at random.

Let E : denotes the event that a student does his home work with P(E) = p and

F : denotes the event that he answer the question correctly.

(a) If p = 0.75 the value of P(E/F) equals

(A) $\frac{8}{16}$ (B) $\frac{10}{16}$ (C) $\frac{12}{16}$ (D) $\frac{15}{16}$

(b) The relation P(E/F) $\geq$ P(E) holds good for

(A) all values of p in [0, 1] (B) all values of p in (0, 1) only

(C) all values of p in [0.5, 1] only (D) no value of p.

(c) Suppose that each question has *n* alternative answers of which only one is correct, and p is fixed but not equal to 0 or 1 then P(E/F)

(A) decreases as *n* increases for all $p \in (0, 1)$

(B) increases as *n* increases for all $p \in (0, 1)$

(C) remains constant for all $p \in (0, 1)$

(D) decreases if $p \in (0, 0.5)$ and increases if $p \in (0.5, 1)$ as *n* increases

PB0095

[MULTIPLE OBJECTIVE TYPE]

10. Which of the following statement(s) is/are correct?

(A) 3 coins are tossed once. Two of them atleast must land the same way. No matter whether they land heads or tails, the third coin is equally likely to land either the same way or oppositely. So, the chance that all the three coins land the same way is 1/2.

(B) Let $0 < P(B) < 1$ and $P(A/B) = P(A/B^c)$ then A and B are independent.

(C) Suppose an urn contains 'w' white and 'b' black balls and a ball is drawn from it and is replaced along with 'd' additional balls of the same colour. Now a second ball is drawn from it. The probability that the second drawn ball is white is independent of the value of 'd'.

(D) A, B, C simultaneously satisfy $P(ABC) = P(A)\cdot P(B)\cdot P(C)$ and $P(AB\overline{C}) = P(A)\cdot P(B)\cdot P(\overline{C})$ and $P(A\overline{B}C) = P(A)\cdot P(\overline{B})\cdot P(C)$ and $P(\overline{A}BC) = P(\overline{A})\cdot P(B)\cdot P(C)$ then A, B, C are independent.

PB0096

11. Identify the correct statement :

(A) If the probability that a computer will fail during the first hour of operation is 0.01, then if we turn on 100 computers, exactly one will fail in the first hour of operation.

(B) A man has ten keys only one of which fits the lock. He tries them in a door one by one discarding the one he has tried. The probability that fifth key fits the lock is 1/10.

(C) Given the events A and B in a sample space. If P(A) = 1, then A and B are independent.

(D) When a fair six sided die is tossed on a table top, the bottom face can not be seen. The probability that the product of the numbers on the five faces that can be seen is divisible by 6 is one.

PB0097

12. Two whole numbers are randomly selected and multiplied. Consider two events E_1 and E_2 defined as

E_1 : Their product is divisible by 5

E_2 : Unit's place in their product is 5.

Which of the following statement(s) is/are correct?

(A) E_1 is twice as likely to occur as E_2.

(B) E_1 and E_2 are disjoint

(C) $P(E_2/E_1) = 1/4$

(D) $P(E_1/E_2) = 1$

PB0098

13. A boy has a collection of blue and green marbles. The number of blue marbles belong to the sets {2, 3, 4, ... 13}. If two marbles are chosen simultaneously and at random from his collection, then the probability that they have different colour is $1/2$. Possible number of blue marbles is :

(A) 2 (B) 3 (C) 6 (D) 10

PB0099

14. If A & B are two events such that $P(B) \neq 1$, B^C denotes the event complementary to B, then

(A) $P\left(A/B^C\right) = \dfrac{P(A) - P(A \cap B)}{1 - P(B)}$

(B) $P(A \cap B) \geq P(A) + P(B) - 1$

(C) $P(A) >< P\left(A/B\right)$ according as $P\left(A/B^C\right) >< P(A)$

(D) $P\left(A/B^C\right) + P\left(A^C/B^C\right) = 1$

PB0100

15. For $P(A) = \dfrac{3}{8}$; $P(B) = \dfrac{1}{2}$; $P(A \cup B) = \dfrac{5}{8}$ which of the following do/does hold good?

(A) $P\left(A^c/B\right) = 2P\left(A/B^c\right)$ (B) $P(B) = P\left(A/B\right)$

(C) $15P\left(A^c/B^c\right) = 8P\left(B/A^c\right)$ (D) $P\left(A/B^c\right) = P\left(A \cap B\right)$

PB0101

16. If E_1 and E_2 are two events such that $P(E_1) = 1/4$, $P(E_2/E_1) = 1/2$ and $P(E_1/E_2) = 1/4$

(A) then E_1 and E_2 are independent

(B) E_1 and E_2 are exhaustive

(C) E_2 is twice as likely to occur as E_1

(D) Probabilities of the events $E_1 \cap E_2$, E_1 and E_2 are in G.P.

PB0102

17. Two events A and B are such that the probability that at least one of them occurs is 5/6 and both of them occurring simultaneously is 1/3. If the probability of not occurrence of B is 1/2 then

(A) A and B are equally likely (B) A and B are independent

(C) $P(A/B) = 2/3$ (D) $3P(A) = 4P(B)$

PB0103

18. The probabilities of events, $A \cap B$, A, B & $A \cup B$ are respectively in A.P. with probability of second term equal to the common difference. Therefore the events A and B are

(A) mutually exclusive (B) independent

(C) such that one of them must occur (D) such that one is twice as likely as the other

PB0104

19. A box contains 11 tickets numbered from 1 to 11. Six tickets are drawn simultaneously at random. Let E_1 denotes the event that the sum of the numbers on the tickets drawn is even and E_2 denotes the event that the sum of the numbers on the tickets drawn is odd. Which of the following hold good?

(A) E_1 and E_2 are equally likely
(B) E_1 and E_2 are exhaustive
(C) $P(E_2) > P(E_1)$
(D) $P(E_1/E_2) = P(E_2/E_1)$

PB0105

20. If $\overline{E}$ & $\overline{F}$ are the complementary events of events E & F respectively & if $0 < P(F) < 1$, then :

(A) $P(E|F) + P(\overline{E}|F) = 1$
(B) $P(E|F) + P(E|\overline{F}) = 1$
(C) $P(\overline{E}|F) + P(E|\overline{F}) = 1$
(D) $P(E|\overline{F}) + P(\overline{E}|\overline{F}) = 1$

PB0106

21. Probability of **n** heads in **2n** tosses of a fair coin can be given by

(A) $\prod_{r=1}^{n}\left(\dfrac{2r-1}{2r}\right)$
(B) $\prod_{r=1}^{n}\left(\dfrac{n+r}{2r}\right)$
(C) $\sum_{r=0}^{n}\left(\dfrac{{}^nC_r}{2^n}\right)^2$
(D) $\dfrac{\sum_{r=0}^{n}\left({}^nC_r\right)^2}{\left(\sum_{r=0}^{n}{}^nC_r\right)^2}$

PB0107

22. Which of the following statements is/are True?

(A) A fair coin is tossed n times where n is a positive integer. The probability that n^{th} toss results in head is 1/2.
(B) The conditional probability that the n^{th} toss results in head given that first $(n-1)$ tosses results in head is $1/2^n$
(C) Let E and F be the events such that F is neither impossible nor sure. If $P(E/F) > P(E)$ then $P(E/F^c) > P(E)$
(D) If A, B and C are independent then the events $(A \cup B)$ and C are independent.

PB0108

[MATRIX MATCH TYPE]

23.

	Column-I	Column-II
(A)	Two different numbers are taken from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The probability that their sum and positive difference, are both multiple of 4, is $x/55$ then x equals	(P) 4
		(Q) 6
(B)	There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is 1/5 then the number of green socks are	(R) 8
(C)	A drawer contains a mixture of red socks and blue socks, atmost 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly $1/2$ that both are red or both are blue. The largest possible number of red socks in the drawer that is consistent with this data, is	(S) 10

PB0109

EXERCISE (S-1)

1. In a box, there are 8 alphabets cards with the letters: S, S, A,A,A,H,H,H. Find the probability that the word 'ASH' will form if :

(i) the three cards are drawn one by one & placed on the table in the same other that they are drawn.

(ii) the three cards are drawn simultaneously.

PB0110

2. There are 2 groups of subjects one of which consists of 5 science subjects & 3 engg. subjects & other consists of 3 science & 5 engg. subjects. An unbiased die is cast . If the number 3 or 5 turns up a subject is selected at random from first group, otherwise the subject is selected from 2^{nd} group. Find the probability that an engg. subject is selected.

PB0111

3. A pair of fair dice is tossed. Find the probability that the maximum of the two numbers is greater than 4.

PB0112

4. In a given race, the odds in favour of four horses A, B, C & D are 1 : 3, 1 : 4, 1 : 5 and 1 : 6 respectively. Assuming that a dead heat is impossible, find the chance that one of them wins the race.

PB0113

5. A covered basket of flowers has some lilies and roses. In search of rose, Sweety and Shweta alternately pick up a flower from the basket but puts it back if it is not a rose. Sweety is 3 times more likely to be the first one to pick a rose. If sweety begin this 'rose hunt' and if there are 60 lilies in the basket, find the number of roses in the basket.

PB0114

6. A certain drug, manufactured by a Company is tested chemically for its toxic nature. Let the event "THE DRUG IS TOXIC" be denoted by H and the event "THE CHEMICAL TEST REVEALS THAT THE DRUG IS TOXIC" be denoted by S. Let P(H) = a, P(S/H) = $P(\overline{S}/\overline{H})$ = 1 – a. Then show that the probability that the drug is not toxic given that the chemical test reveals that it is toxic is free from 'a'.

PB0115

7. Players A and B alternately toss a biased coin, with A going first. A wins if A tosses a Tail before B tosses a Head; otherwise B wins. If the probability of a head is p, find the value of p for which the game is fair to both players.

PB0116

8. The entries in a two-by-two determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ are integers that are chosen randomly and independently, and , for each entry, the probability that the entry is odd is p. If the probability that the value of the determinant is even is 1/2, then find the value of p.

PB0117

9. There are 4 urns. The first urn contains 1 white & 1 black ball, the second urn contains 2 white & 3 black balls, the third urn contains 3 white & 5 black balls & the fourth urn contains 4 white & 7 black balls. The selection of each urn is not equally likely. The probability of selecting i^{th} urn is $\frac{i^2+1}{34}$ (i = 1,2,3,4). If we randomly select one of the urns & draw a ball, then the probability of ball being white is p/q where p and q $\in$ N are in their lowest form. Find (p+q).

PB0118

10. A room has three electric lamps. From a collection of 10 electric bulbs of which 6 are good 3 are selected at random & put in the lamps. Find the probability that the room is lighted.

PB0119

11. Find the minimum number of tosses of a pair of dice so that the probability of getting the sum of the digits on the dice equal to 7 on at least one toss is greater than 0.95.
($\log_{10}2 = 0.3010$; $\log_{10}3 = 0.4771$)

PB0120

12. The probability that a person will get an electric contract is $2/5$ and the probability that he will not get plumbing contract is $4/7$. If the probability of getting at least one contract is $2/3$, what is the probability that he will get both?

PB0121

13. Five horses compete in a race. John picks two horses at random and bets on them. Find the probability that John picked the winner. Assume no dead heat.

PB0122

14. There are 6 red balls and 6 green balls in a bag. Five balls are drawn out at random and placed in a red box. The remaining seven balls are put in a green box. If the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number, is $\frac{p}{q}$ where p and q are relatively prime, then find the value of $(p + q)$

PB0123

15. A lot contains 50 defective & 50 non defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as :
A = { the first bulb is defective}; B = { the second bulb is non defective}
C = { the two bulbs are both defective or both non defective}
Determine whether (i) A,B,C are pair wise independent (ii) A,B,C are independent

PB0124

16. An unbiased normal coin is tossed 'n' times
Let E_1 : event that both Heads and Tails are present in 'n' tosses.
E_2 : event that the coin shows up Heads atmost once.
Find the value of 'n' for which E_1 and E_2 are independent.

PB0125

17. A bomber wants to destroy a bridge . Two bombs are sufficient to destroy it . If four bombs are dropped, what is the probability that it is destroyed, if the chance of a bomb hitting the target is 0.4.

PB0126

18. The chance of one event happening is the square of the chance of a 2^{nd} event, but odds against the first are the cubes of the odds against the 2^{nd} . Find the chances of each (assume that both events are neither sure nor impossible).

PB0127

19. A bag contains N balls, some of which are white, the others are black, white being more in number than back. Two balls are drawn at random from the bag, without replacement. It is found that the probability that the two balls are of the same colour is the same as the probability that they are of different colour. It is given that $180 < N < 220$. If K denotes the number of white balls, find the exact value of $(K + N)$.

PB0128

20. An aircraft gun can take a maximum of four shots at an enemy's plane moving away from it. The probability of hitting the plane at first, second, third & fourth shots are 0.4, 0.3, 0.2 & 0.1 respectively. What is the probability that the gun hits the plane.

PB0129

21. In a batch of 10 articles, 4 articles are defective. 6 articles are taken from the batch for inspection. If more than 2 articles in this batch are defective, the whole batch is rejected Find the probability that the batch will be rejected.

PB0130

22. An author writes a good book with a probability of 1/2. If it is good it is published with a probability of 2/3. If it is not, it is published with a probability of 1/4. Find the probability that he will get atleast one book published if he writes two.

PB0131

23. A uniform unbiased die is constructed in the shape of a regular tetrahedron with faces numbered 2, 2, 3 and 4 and the score is taken from the face on which the die lands. If two such dice are thrown together, find the probability of scoring.

(i) exactly 6 on each of 3 successive throws.

(ii) more than 4 on at least one of the three successive throws.

PB0132

24. A biased coin which comes up heads three times as often as tails is tossed. If it shows heads, a chip is drawn from urn-I which contains 2 white chips and 5 red chips. If the coin comes up tail, a chip is drawn from urn-II which contains 7 white and 4 red chips. Given that a red chip was drawn, what is the probability that the coin came up heads ?

PB0133

25. Each of the 'n' passengers sitting in a bus may get down from it at the next stop with probability p. Moreover, at the next stop either no passenger or exactly one passenger boards the bus. The probability of no passenger boarding the bus at the next stop being $p_{o.}$ Find the probability that when the bus continuous on its way after the stop, there will again be 'n' passengers in the bus.

PB0134

26. A normal coin is continued tossing unless a head is obtained for the first time. Find the probability that
(a) number of tosses needed are atmost 3.
(b) number of tosses are even.

PB0135

27. Before a race the chance of three runners, A, B, C were estimated to be proportional to 5, 3, 2, but during the race A meets with an accident which reduces his chance to 1/3. What are the respective chance of B and C now?

PB0136

28. A is one of the 6 horses entered for a race, and is to be ridden by one of two jockeys B or C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win; if C rides A, his chance is trebled, what are the odds against his winning?

PB0137

29. A real estate man has eight master keys to open several new houses. Only one master key will open a given house. If 40% of these homes are usually left unlocked, find the probability that the real estate man can get into a specific home if he selects three master keys at random.

PB0138

30. A, B are two inaccurate arithmeticians whose chance of solving a given question correctly are (1/8) and (1/12) respectively. They solve a problem and obtained the same result. If it is 1000 to 1 against their making the same mistake, find the chance that the result is correct.

PB0139

31. During a power blackout, 100 persons are arrested on suspect of looting. Each is given a polygraph test. From past experience it is known that the polygraph is 90% reliable when administered to a guilty person and 98% reliable when given to some one who is innocent. Suppose that of the 100 persons taken into custody, only 12 were actually involved in any wrong doing. If the probability that a given suspect is innocent given that the polygraph says he is guilty is a/b where a and b are relatively prime, find the value of (a + b).

PB0140

EXERCISE (S-2)

1. N fair coins are flipped once. The probability that at most 2 of the coins show up as heads is $\frac{1}{2}$. Find the value of N.

PB0141

2. To pass a test a child has to perform successfully in two consecutive tasks, one easy and one hard task. The easy task he can perform successfully with probability 'e' and the hard task he can perform successfully with probability 'h', where $h < e$. He is allowed 3 attempts, either in the order (Easy, Hard, Easy) (option A) or in the order (Hard, Easy, Hard)(option B) whatever may be the order, he must be successful twice in a row. Assuming that his attempts are independent, in what order he choses to take the tasks, in order to maximise his probability of passing the test.

PB0142

3. A box contains three coins two of them are fair and one two – headed. A coin is selected at random and tossed. If the head appears the coin is tossed again, if a tail appears, then another coin is selected from the remaining coins and tossed.

(i) Find the probability that head appears twice.

(ii) If the same coin is tossed twice, find the probability that it is two headed coin.

(iii) Find the probability that tail appears twice.

PB0143

4. Eight players P_1 , P_2 , P_3 ,..........P_8 play a knock out tournament. It is known that whenever the players P_i and P_j play , the player P_i will win if $i < j$. Assuming that the players are paired at random in each round, what is the probability that the players P_4 reaches the final ?

PB0144

5. A doctor is called to see a sick child. The doctor knows (prior to the visit) that 90% of the sick children in that neighbourhood are sick with the flu, denoted by F, while 10% are sick with the measles, denoted by M.

A well known symptom of measles is rash, denoted by R. The probability of having a rash for a child sick with the measles is 0.95. However, occasionally children with the flu also develop a rash, with conditional probability 0.08.

Upon examination the child, the doctor finds a rash. What is the probability that the child has the measles ?

If the probability can be expressed in the form of p/q where $p,q \in N$ and are in their lowest form, find $(p + q)$.

PB0145

6. A permutation of 5 digits from the set {1,2,3,4,5} where each digit is used exactly once, is chosen randomly. Let $\frac{p}{q}$ expressed as rational in lowest form be the probability that the chosen permutation changes from increasing to decreasing, or decreasing to increasing at most once e.g. the strings like 1 2 3 4 5, 5 4 3 2 1, 1 2 5 4 3 and 5 3 2 1 4 are acceptable but strings like 1 3 2 4 5 or 5 3 2 4 1 are not, find $(p + q)$.

PB0146

7. (a) Two natural numbers x and y are chosen at random. Find the probability that $x^2 + y^2$ is divisible by 10.

(b) Two numbers x & y are chosen at random from the set {1,2,3,4,....3n}. Find the probability that $x^2 - y^2$ is divisible by 3.

PB0147

8. A hotel packed breakfast for each of the three guests. Each breakfast should have consisted of three types of rolls, one each of nut, cheese and fruit rolls. The preparer wrapped each of the nine rolls and once wrapped, the rolls were indistinguishable from one another. She then randomly put three rolls in a bag for each of the guests. If the probability that each guest got one roll of each type is m/n where m and n are relatively prime integers, find the value of (m + n).

PB0148

9. A coin has probability 'p' of showing head when tossed. It is tossed 'n' times. Let p_n denote the probability that no two (or more) consecutive heads occur. Prove that,

$$p_1 = 1,\ p_2 = 1 - p^2\ \&\ p_n = (1 - p)p_{n-1} + p(1 - p)p_{n-2},\ \text{for all } n \geq 3.$$

PB0149

10. In a tournament, team X, plays with each of the 6 other teams once. For each match the probabilities of a win, drawn and loss are equal. Find the probability that the team X, finishes with more wins than losses.

PB0150

11. A pair of students is selected at random from a probability class. The probability that the pair selected will consist of one male and one female student is $\frac{10}{19}$. Find the maximum number of students the class can contain.

PB0151

12. 3 students {A, B, C} tackle a puzzle together and offers a solution upon which majority of the 3 agrees. Probability of A solving the puzzle correctly is p. Probability of B solving the puzzle correctly is also p. C is a dumb student who randomly supports the solution of either A or B. There is one more student D, whose probability of solving the puzzle correctly is once again, p. Out of the 3 member team {A, B, C} and one member team {D}, which one is more likely to solve the puzzle correctly.

PB0152

13. In a knockout tournament 2^n equally skilled players; $S_1, S_2, ... S_{2^n}$ are participating. In each round players are divided in pair at random and winner from each pair moves in the next round. If S_2 reaches the semifinal then the probability that S_1 wins the tournament is 1/20. Find the value of 'n'.

PB0153

14. All the face cards from a pack of 52 plying cards are removed. From the remaining pack half of the cards are randomly removed without looking at them and then randomly drawn two cards simultaneously from the remaining. If the probability that, two cards drawn are both aces, is $\frac{p({}^{38}C_{20})}{{}^{40}C_{20}.{}^{20}C_2}$, then find the value of p.

PB0154

EXERCISE (JM)

1. If C and D are two events such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following is :- **[AIEEE-2011]**

(1) $P(C|D) < P(C)$ (2) $P(C|D) = \frac{P(D)}{P(C)}$ (3) $P(C|D) = P(C)$ (4) $P(C|D) \geq P(C)$

PB0155

2. Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of at least one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval :- **[AIEEE-2011]**

(1) $\left[0, \frac{1}{2}\right]$ (2) $\left(\frac{11}{12}, 1\right]$ (3) $\left(\frac{1}{2}, \frac{3}{4}\right]$ (4) $\left(\frac{3}{4}, \frac{11}{12}\right]$

PB0156

3. Let A, B, C be pairwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$. Then $P(A^c \cap B^c | C)$ is equal to: **[AIEEE-2011]**

(1) $P(A^c) - P(B)$ (2) $P(A) - P(B^c)$ (3) $P(A^c) + P(B^c)$ (4) $P(A^c) - P(B^c)$

PB0157

4. Three numbers are chosen at random without replacement from {1, 2, 3, ... , 8}. The probability that their minimum is 3, given that their maximum is 6, is : **[AIEEE-2012]**

(1) $\frac{2}{5}$ (2) $\frac{3}{8}$ (3) $\frac{1}{5}$ (4) $\frac{1}{4}$

PB0158

5. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is : **[JEE-MAIN 2013]**

(1) $\frac{17}{3^5}$ (2) $\frac{13}{3^5}$ (3) $\frac{11}{3^5}$ (4) $\frac{10}{3^5}$

PB0159

6. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\bar{A}) = \frac{1}{4}$, where $\bar{A}$ stands for the complement of the event A. Then the events A and B are : **[JEE(Main)-2014]**

(1) mutually exclusive and independent. (2) equally likely but not independent.
(3) independent but not equally likely. (4) independent and equally likely.

PB0160

7. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is **NOT true** ? **[JEE(Main)-2016]**

(1) E_1, E_2 and E_3 are independent. (2) E_1 and E_2 are independent.

(3) E_2 and E_3 are independent. (4) E_1 and E_3 are independent.

PB0161

8. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one–by–one, with replacement, then the variance of the number of green balls drawn is :- **[JEE(Main)-2017]**

(1) $\frac{6}{25}$ (2) $\frac{12}{5}$ (3) 6 (4) 4

PB0162

9. If two different numbers are taken from the set {0, 1, 2, 3, ..., 10), then the probability that their sum as well as absolute difference are both multiple of 4, is :- **[JEE(Main)-2017]**

(1) $\frac{7}{55}$ (2) $\frac{6}{55}$ (3) $\frac{12}{55}$ (4) $\frac{14}{45}$

PB0163

10. For three events A, B and C, P(Exactly one of A or B occurs) = P(Exactly one of B or C occurs) = P(Exactly one of C or A occurs) = $\frac{1}{4}$ and P(All the three events occur simultaneously) = $\frac{1}{16}$. Then the probability that at least one of the events occurs, is :- **[JEE(Main)-2017]**

(1) $\frac{3}{16}$ (2) $\frac{7}{32}$ (3) $\frac{7}{16}$ (4) $\frac{7}{64}$

PB0164

11. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is: **[JEE(Main)-2018]**

(1) $\frac{2}{5}$ (2) $\frac{1}{5}$ (3) $\frac{3}{4}$ (4) $\frac{3}{10}$

PB0165

12. An urn contains 5 red and 2 green balls. A ball is drawn at random from the urn. If the drawn ball is green, then a red ball is added to the urn and if the drawn ball is red, then a green ball is added to the urn; the original ball is not returned to the urn. Now, a second ball is drawn at random from it. The probability that the second ball is red, is : **[JEE(Main)-2019]**

(1) $\frac{26}{49}$ (2) $\frac{32}{49}$ (3) $\frac{27}{49}$ (4) $\frac{21}{49}$

PB0166

13. An unbiased coin is tossed. If the outcome is a head then a pair of unbiased dice is rolled and the sum of the numbers obtained on them is noted. If the toss of the coin results in tail then a card from a well-shuffled pack of nine cards numbered 1,2,3,...,9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is : **[JEE(Main)-2019]**

(1) $\frac{13}{36}$ (2) $\frac{19}{36}$ (3) $\frac{19}{72}$ (4) $\frac{15}{72}$

PB0167

14. If the probability of hitting a target by a shooter, in any shot, is 1/3, then the minimum number of independent shots at the target required by him so that the probability of hitting the target at least once is greater than $\frac{5}{6}$, is : **[JEE(Main)-2019]**

(1) 6 (2) 5 (3) 4 (4) 3

PB0168

15. Let S = {1, 2, ... , 20}. A subset B of S is said to be "nice", if the sum of the elements of B is 203. Then the probability that a randomly chosen subset of S is "nice" is :-

[JEE(Main)-2019]

(1) $\frac{6}{2^{20}}$ (2) $\frac{5}{2^{20}}$ (3) $\frac{4}{2^{20}}$ (4) $\frac{7}{2^{20}}$

PB0169

16. In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to : **[JEE(Main)-2019]**

(1) $\frac{150}{6^5}$ (2) $\frac{175}{6^5}$ (3) $\frac{200}{6^5}$ (4) $\frac{225}{6^5}$

PB0170

17. In a game, a man wins Rs. 100 if he gets 5 of 6 on a throw of a fair die and loses Rs. 50 for getting any other number on the die. If he decides to throw the die either till he gets a five or a six or to a maximum of three throws, then his expected gain/loss (in rupees) is :

[JEE(Main)-2019]

(1) $\frac{400}{3}$ gain (2) $\frac{400}{3}$ loss (3) 0 (4) $\frac{400}{9}$ loss

PB0171

18. Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls is :

[JEE(Main)-2019]

(1) $\frac{1}{11}$ (2) $\frac{1}{17}$ (3) $\frac{1}{10}$ (4) $\frac{1}{12}$

PB0172

19. If three of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is : **[JEE(Main)-2019]**

(1) $\frac{3}{10}$ (2) $\frac{1}{10}$ (3) $\frac{3}{20}$ (4) $\frac{1}{5}$

PB0173

20. In a workshop, there are five machines and the probability of any one of them to be out of service on a day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is $\left(\frac{3}{4}\right)^3 k$, then k is equal to : **[JEE(Main)-2020]**

(1) $\frac{17}{2}$ (2) 4 (3) $\frac{17}{8}$ (4) $\frac{17}{4}$

PB0174

21. An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for k = 3, 4, 5 otherwise X takes the value –1. Then the expected value of X, is : **[JEE(Main)-2020]**

(1) $\frac{3}{16}$ (2) $-\frac{3}{16}$ (3) $\frac{1}{8}$ (4) $-\frac{1}{8}$

PB0175

22. A random variable X has the following probability distribution :

X :	1	2	3	4	5
P(X) :	K^2	2K	K	2K	$5K^2$

Then $P(X > 2)$ is equal to : **[JEE(Main)-2020]**

(1) $\frac{7}{12}$ (2) $\frac{23}{36}$ (3) $\frac{1}{36}$ (4) $\frac{1}{6}$

PB0176

23. If 10 different balls are to be placed in 4 distinct boxes at random, then the probability that two of these boxes contain exactly 2 and 3 balls is : **[JEE(Main)-2020]**

(1) $\frac{945}{2^{11}}$ (2) $\frac{965}{2^{11}}$ (3) $\frac{945}{2^{10}}$ (4) $\frac{965}{2^{10}}$

PB0177

24. In a box, there are 20 cards, out of which 10 are lebelled as A and the remaining 10 are labelled as B. Cards are drawn at random, one after the other and with replacement, till a second A-card is obtained. The probability that the second A-card appears before the third B-card is :

[JEE(Main)-2020]

(1) $\frac{11}{16}$ (2) $\frac{13}{16}$ (3) $\frac{9}{16}$ (4) $\frac{15}{16}$

PB0178

EXERCISE (JA)

1. (a) Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is -

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$ (C) $\frac{2}{9}$ (D) $\frac{1}{36}$

PB0179

(b) A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is - **[JEE 2010, 3+5]**

(A) $\frac{3}{5}$ (B) $\frac{6}{7}$ (C) $\frac{20}{23}$ (D) $\frac{9}{20}$

PB0180

Paragraph for Question 2 and 3

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

2. The probability of the drawn ball from U_2 being white is -

(A) $\frac{13}{30}$ (B) $\frac{23}{30}$ (C) $\frac{19}{30}$ (D) $\frac{11}{30}$

PB0181

3. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is -

(A) $\frac{17}{23}$ (B) $\frac{11}{23}$ (C) $\frac{15}{23}$ (D) $\frac{12}{23}$

[JEE 2011, 3+3]

PB0181

4. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If P(T) denotes the probability of occurrence of the event T, then -

(A) $P(E) = \frac{4}{5},\ P(F) = \frac{3}{5}$ (B) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(C) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (D) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$

PB0182

[JEE 2011, 4M]

5. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and X_1, X_2, X_3 denotes respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true? **[JEE 2012, 4M]**

(A) $P\left[X_1^c \mid X\right] = \frac{3}{16}$

(B) P[Exactly two engines of ship are functioning $\mid X] = \frac{7}{8}$

(C) $P[X \mid X_2] = \frac{5}{16}$

(D) $P[X \mid X_1] = \frac{7}{16}$

PB0183

6. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is - **[JEE 2012, 4M]**

(A) $\frac{91}{216}$ (B) $\frac{108}{216}$ (C) $\frac{125}{216}$ (D) $\frac{127}{216}$

PB0184

7. Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}, P(Y \mid X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is(are) correct ? **[JEE 2012, 4M]**

(A) $P(X \cup Y) = \frac{2}{3}$ (B) X and Y are independent

(C) X and Y are not independent (D) $P(X^C \cap Y) = \frac{1}{3}$

PB0185

8. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is **[JEE(Advanced) 2013, 2M]**

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$ (C) $\frac{3}{256}$ (D) $\frac{253}{256}$

PB0186

9. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\,p = \alpha\beta$ and $(\beta - 3\gamma)\,p = 2\beta\gamma$. All the given probabilities are assumed of lie in the interval (0,1).

Then $\dfrac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$ **[JEE-Advanced 2013, 4, (–1)]**

PB0187

Paragraph for Question 10 and 11

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

10. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is

(A) $\frac{116}{181}$ (B) $\frac{126}{181}$ (C) $\frac{65}{181}$ (D) $\frac{55}{181}$

[JEE(Advanced) 2013, 3, (–1)]

PB0188

11. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, the probability that all 3 drawn balls are of the same colour is

(A) $\frac{82}{648}$ (B) $\frac{90}{648}$ (C) $\frac{558}{648}$ (D) $\frac{566}{648}$

[JEE(Advanced) 2013, 3, (–1)]

PB0188

12. Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is - **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

PB0189

Paragraph For Questions 13 and 14

Box 1 contains three cards bearing numbers, 1,2,3 ; box 2 contains five cards bearing numbers 1,2,3,4,5; and box 3 contains seven cards bearing numbers 1,2,3,4,5,6,7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the i^{th} box, i = 1,2,3.

13. The probability that $x_1 + x_2 + x_3$ is odd, is -

(A) $\frac{29}{105}$ (B) $\frac{53}{105}$ (C) $\frac{57}{105}$ (D) $\frac{1}{2}$

[JEE(Advanced)-2014, 3(–1)]

PB0190

14. The probability that x_1,x_2,x_3 are in an arithmetic progression, is -

(A) $\frac{9}{105}$ (B) $\frac{10}{105}$ (C) $\frac{11}{105}$ (D) $\frac{7}{105}$

[JEE(Advanced)-2014, 3(–1)]

PB0190

15. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is **[JEE 2015, 4M, –0M]**

PB0191

Paragraph For Questions 16 and 17

Let n_1 and n_2 be the number of red and black balls respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

16. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

[JEE 2015, 4M, –0M]

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$ (B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$ (D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

PB0192

17. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1 and n_2 is(are) **[JEE 2015, 4M, –0M]**

(A) $n_1 = 4$ and $n_2 = 6$ (B) $n_1 = 2$ and $n_2 = 3$

(C) $n_1 = 10$ and $n_2 = 20$ (D) $n_1 = 3$ and $n_2 = 6$

PB0192

18. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is **[JEE(Advanced)-2016, 3(–1)]**

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$ (C) $\frac{78}{93}$ (D) $\frac{75}{83}$

PB0193

Paragraph For Questions 19 and 20

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2, respectively, after two games

19. P(X > Y) is- **[JEE(Advanced)-2016, 3(0)]**

(A) $\frac{1}{4}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{7}{12}$

PB0194

20. P(X = Y) is- **[JEE(Advanced)-2016, 3(0)]**

(A) $\frac{11}{36}$ (B) $\frac{1}{3}$ (C) $\frac{13}{36}$ (D) $\frac{1}{2}$

PB0194

21. Let X and Y be two events such that $P(X)=\frac{1}{3}, P(X|Y)=\frac{1}{2}$ and $P(Y|X)=\frac{2}{5}$. Then

[JEE(Advanced)-2017, 4(–2)]

(A) $P(X'|Y)=\frac{1}{2}$ (B) $P(X\cap Y)=\frac{1}{5}$ (C) $P(X\cup Y)=\frac{2}{5}$ (D) $P(Y)=\frac{4}{15}$

PB0195

22. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation x + y + z = 10. Then the probability that z is even, is **[JEE(Advanced)-2017, 3(–1)]**

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$ (C) $\frac{5}{11}$ (D) $\frac{1}{2}$

PB0196

Paragraph For Questions 23 and 24

There are five students S_1, S_2, S_4 and S_5 in a music class and for them there are five sets R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to the student S_i, i = 1, 2, 3, 4, 5. But, on the examination day, the five students are randomly allotted the five seats.

(*There are two questions based on Paragraph "A". the question given below is one of them*)

23. The probability that, on the examination day, the student S_1 gets the previously allotted seat R_1 and **NONE** of the remaining students gets the seat previously allotted to him/her is -

[JEE(Advanced)-2018, 3(–1)]

(A) $\frac{3}{40}$ (B) $\frac{1}{8}$ (C) $\frac{7}{40}$ (D) $\frac{1}{5}$

PB0197

24. For i = 1, 2, 3, 4, let T_i denote the event that the students S_i and S_{i+1} do **NOT** sit adjacent to each other on the day of the examination. Then the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is -

[JEE(Advanced)-2018, 3(–1)]

(A) $\frac{1}{15}$ (B) $\frac{1}{10}$ (C) $\frac{7}{60}$ (D) $\frac{1}{5}$

PB0197

25. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls, Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}, \frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

[JEE(Advanced)-2019, 4(–1)]

(1) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(2) Probability that the chosen ball is green equals $\frac{39}{80}$

(3) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(4) Probability that the selected bag is B_3, given that the chosen balls is green, equals $\frac{5}{13}$

PB0198

26. Let S be the sample space of all 3 × 3 matrices with entries from the set {0, 1}. Let the events E_1 and E_2 be given by

$$E_1 = \{A \in S : \det A = 0\} \text{ and}$$
$$E_2 = \{A \in S : \text{sum of entries of A is } 7\}.$$

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ____

[JEE(Advanced)-2019, 3(0)]

PB0199

27. Let |X| denote the number of elements in set X. Let S = {1,2,3,4,5,6} be a sample space, where each element is equally likely to occur. If A and B are independent events associated with S, then the number of ordered pairs (A,B) such that $1 \le |B| < |A|$, equals **[JEE(Advanced)-2019, 3(0)]**

PB0200

ANSWER KEY

EXERCISE (O-1)

PART # 1

1. A **2.** A **3.** C **4.** 23/168

5. (i) 7/13, (ii) 1/2, (iii) 2/13, (iv) 2/13, (v) 1/2, (vi) 9/13 **6.** 1/56 **7.** 1/2 ; 1/2

8. 5 : 1 **9.** 952 to 715 **10.** A **11.** 4/21 **12.** **(a)** 2/3, **(b)** 1/2

13. B **14.** B **15.** C

PART # 2

1. C **2.** D **3.** A **4.** B **5.** C **6.** A **7.** A

8. 2/3 **9.** 3/4, 1/4; 15/16 **10.** (i) 0.18, (ii) 0.12, (iii) 0.42, (iv) 0.28, (v) 0.72

11. (i) 0.6, (ii) 0.5, (iii) 0.25 **12.** (i) 1/36, (ii) 5/108, (iii) 53/54 **13.** 11/20

14. 3/5 **15.** 2/7 **16.** A **17.** 12/25 **18.** (a) 1/18, (b) 43/90, (c) 5/18, (d) NO

PART # 3

1. C **2.** C **3.** D **4.** D **5.** B **6.** A **7.** A

8. C **9.** A **10.** D **11.** B **12.** A **13.** C **14.** B

15. D

PART # 4

1. D **2.** A **3.** B **4.** C **5.** C **6.** A **7.** B

8. C **9.** C **10.** A **11.** B **12.** D

PART # 5

1. D **2.** B **3.** B **4.** C **5.** C **6.** B **7.** B

8. A **9.** C **10.** C **11.** C **12.** C **13.** B **14.** D

15. B

PART # 6

1. A **2.** B **3.** C **4.** D **5.** C **6.** A **7.** B

8. B **9.** D **10.** A **11.** B **12.** D **13.** A

EXERCISE (O-2)

1. B **2.** B **3.** A **4.** C **5.** B **6.** A **7.** C

8. C **9.** **(a)** D, **(b)** A, **(c)** B **10.** B,C,D **11.** B,C,D **12.** C, D **13.** B,C,D

14. A,B,C,D **15.** A,B,D **16.** A,C,D **17.** B,C,D **18.** A,D **19.** B,C,D **20.** A,D

21. A,C,D **22.** A,D **23.** (A) Q; (B) P; (C) S

EXERCISE (S-1)

1. **(i)** 3/56; **(ii)** 9/28 **2.** 13/24 **3.** 5/9 **4.** 319/420 **5.** 120

6. $P(\bar{H}/S) = 1/2$ **7.** $\dfrac{\sqrt{5}-1}{2}$ **8.** $\dfrac{\sqrt{2}}{2}$ **9.** 2065 **10.** $\dfrac{29}{30}$

11. 17 **12.** 17/105 **13.** 2/5 **14.** 37

15. **(i)** A,B,C are pairwise independent **(ii)** A,B,C are not independent **16.** 3

17. $\frac{328}{625}$ **18.** $\frac{1}{9}, \frac{1}{3}$ **19.** 301 **20.** 0.6976 **21.** 19/42 **22.** 407/576

23. (i) $\frac{125}{16^3}$; (ii) $\frac{63}{64}$ **24.** 165/193 **25.** $(1-p)^{n-1}.\ [p_0\ (1-p) + np(1-p_0)]$

26. (a) 7/8, (b) 1/3 **27.** B = 2/5 ; C = 4/15 **28.** 13 to 5 **29.** 5/8

30. 13/14 **31.** 179

EXERCISE (S-2)

1. 5 **2.** Option B **3.** 1/2, 1/2, 1/12 **4.** 4/35 **5.** 262

6. 5 **7.** **(a)** $\frac{9}{50}$ **(b)** $\frac{(5n-3)}{(9n-3)}$ **8.** 79 **9.** 79 **10.** $\frac{98}{243}$ **11.** 20

12. Both are equally likely **13.** 4 **14.** 6

EXERCISE (JM)

1. 4 **2.** 1 **3.** 1 **4.** 3 **5.** 3 **6.** 3 **7.** 1 **8.** 2

9. 2 **10.** 3 **11.** 1 **12.** 2 **13.** 3 **14.** 2 **15.** 2 **16.** 2

17. 3 **18.** 1 **19.** 2 **20.** 3 **21.** 3 **22.** 2

23. NTA Ans. (3); Correct Ans. is $\frac{17 \times 945}{2^{15}}$ **24.** 1

EXERCISE (JA)

1. **(a)** C; **(b)** C **2.** B **3.** D **4.** A,D **5.** B,D **6.** A **7.** A,B

8. A **9.** 6 **10.** D **11.** A **12.** A **13.** B **14.** C **15.** 8

16. A,B **17.** C,D **18.** **C** **19.** **B** **20.** **C** **21.** A,D **22.** B **23.** A

24. C **25.** 2,3 **26.** 0.50 **27.** 422.00

ALLEN CAREER INSTITUTE KOTA (RAJASTHAN)

Chapter 03 Contents

JEE-MAIN
(STATISTICS & REASONING)

STATISTICS

MATHEMATICAL REASONING

JEE (Main/Advanced) Syllabus

JEE (Main) Syllabus :

STATISTICS : *Measures of Dispersion: Calculation of mean, median, mode of grouped and ungrouped data calculation of standard deviation, variance and mean deviation for grouped and ungrouped data.*

MATHEMATICAL REASONING : *Statements, logical operations and, or, implies, implied by, if and only if. Understanding of tautology, contradiction, converse and contrapositive*

Important Notes

STATISTICS

MEASURES OF CENTRAL TENDENCY :

An average value or a central value of a distribution is the value of variable which is representative of the entire distribution, this representative value are called the measures of central tendency.

Generally the following five measures of central tendency.

(a) Mathematical average

(i) Arithmetic mean (ii) Geometric mean (iii) Harmonic mean

(b) Positional average

(i) Median (ii) Mode

1. ARITHMETIC MEAN :

(i) For ungrouped dist. : If $x_1, x_2, ... x_n$ are n values of variate x_i then their A.M. $\bar{x}$ is defined as

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\Rightarrow$ $\Sigma x_i = n\bar{x}$

(ii) For ungrouped and grouped freq. dist. : If $x_1, x_2, ... x_n$ are values of variate with corresponding frequencies $f_1, f_2, ... f_n$ then their A.M. is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{N}, \quad \text{where } N = \sum_{i=1}^{n} f_i$$

Ex.1 Find the A.M. of the following freq. dist.

x_i	5	8	11	14	17
f_i	4	5	6	10	20

Sol. Here $N = \Sigma f_i = 4 + 5 + 6 + 10 + 20 = 45$

$\Sigma f_ix_i = (5 \times 4) + (8 \times 5) + (11 \times 6) + (14 \times 10) + (17 \times 20) = 606$

$$\therefore \quad \bar{x} = \frac{\Sigma f_ix_i}{N} = \frac{606}{45} = 13.47$$

(iii) By short method : If the value of x_i are large, then the calculation of A.M. by using previous formula is quite tedious and time consuming. In such case we take deviation of variate from an arbitrary point a.

Let $d_i = x_i - a$

$$\therefore \quad \bar{x} = a + \frac{\Sigma f_id_i}{N}, \quad \text{where a is assumed mean}$$

(iv) By step deviation method : Sometime during the application of short method of finding the A.M. If each deviation d_i are divisible by a common number h(let)

Let $u_i = \frac{d_i}{h} = \frac{x_i - a}{h}$

$$\therefore \quad \bar{x} = a + \left(\frac{\Sigma f_iu_i}{N}\right)h$$

Ex.2 Find the mean of the following freq. dist.

x_i	5	15	25	35	45	55
f_i	12	18	27	20	17	6

Sol. Let assumed mean a = 35, h = 10

here $N = \Sigma f_i = 100$, $u_i = \dfrac{(x_i - 35)}{10}$

$\therefore \ \Sigma f_i u_i = (12 \times -3) + (18 \times -2) + (27 \times -1) + (20 \times 0) + (17 \times 1) + (6 \times 2) = -70$

$\therefore \ \bar{x} = a + \left(\dfrac{\Sigma f_i u_i}{N}\right) h = 35 + \dfrac{(-70)}{100} \times 10 = 28$

(v) Weighted mean : If $w_1, w_2, \ldots\ldots w_n$ are the weights assigned to the values $x_1, x_2, \ldots\ldots x_n$ respectively then their weighted mean is defined as

$$\text{Weighted mean} = \frac{w_1x_1 + w_2x_2 + \ldots\ldots + w_nx_n}{w_1 + \ldots\ldots + w_n} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Ex.3 Find the weighted mean of first n natural numbers when their weights are equal to their squares respectively

Sol. $$\text{Weighted Mean} = \frac{1.1^2 + 2.2^2 + \ldots + n.n^2}{1^2 + 2^2 + \ldots + n^2} = \frac{1^3 + 2^3 + \ldots\ldots + n^3}{1^2 + 2^2 + \ldots + n^2} = \frac{[n(n+1)/2]^2}{[n(n+1)(2n+1)/6]} = \frac{3n(n+1)}{2(2n+1)}$$

(vi) Combined mean : If $\bar{x}_1$ and $\bar{x}_2$ be the means of two groups having n_1 and n_2 terms respectively then the mean (combined mean) of their composite group is given by

$$\text{combined mean} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

If there are more than two groups then, $\quad \text{combined mean} = \dfrac{n_1\bar{x}_1 + n_1\bar{x}_2 + n_3\bar{x}_3 + \ldots}{n_1 + n_2 + n_3 + \ldots}$

Ex.4 The mean income of a group of persons is Rs. 400 and another group of persons is Rs. 480. If the mean income of all the persons of these two groups is Rs. 430 then find the ratio of the number of persons in the groups.

Sol. Here $\bar{x}_1 = 400$, $\bar{x}_2 = 480$, $\bar{x} = 430$

$\because \ \bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} \Rightarrow 430 = \dfrac{400n_1 + 480n_2}{n_1 + n_2}$

$\Rightarrow \ \dfrac{n_1}{n_2} = \dfrac{5}{3}$

(vii) Properties of Arithmetic mean :

- Sum of deviations of variate from their A.M. is always zero i.e. $\Sigma(x_i - \bar{x}) = 0$, $\Sigma f_i(x_i - \bar{x}) = 0$
- Sum of square of deviations of variate from their A.M. is minimum i.e. $\Sigma(x_i - \bar{x})^2$ is minimum
- If $\bar{x}$ is the mean of variate x_i then A.M. of $(x_i + \lambda) = \bar{x} + \lambda$
  A.M. of $(\lambda x_i) = \lambda \bar{x}$
  A.M. of $(ax_i + b) = a\bar{x} + b$ (where λ, a, b are constant)
- A.M. is independent of change of assumed mean i.e. it is not effected by any change in assumed mean.

2. MEDIAN :

The median of a series is the value of middle term of the series when the values are written in ascending order. Therefore median, divided an arranged series into two equal parts.

Formulae of median :

(i) For ungrouped distribution : Let n be the number of variate in a series then

$$\text{Median} = \begin{cases} \left(\dfrac{n+1}{2}\right)^{th} \text{term}, \ (\text{when } n \text{ is odd}) \\ \text{Mean of} \left(\dfrac{n}{2}\right)^{th} \text{and} \left(\dfrac{n}{2}+1\right)^{th} \text{terms}, \ (\text{when } n \text{ is even}) \end{cases}$$

(ii) For ungrouped freq. dist. : First we prepare the cumulative frequency (c.f.) column and Find value of N then

$$\text{Median} = \begin{cases} \left(\dfrac{N+1}{2}\right)^{th} \text{term}, \ (\text{when } N \text{ is odd}) \\ \text{Mean of} \left(\dfrac{N}{2}\right)^{th} \text{and} \left(\dfrac{N}{2}+1\right)^{th} \text{terms}, \ (\text{when } N \text{ is even}) \end{cases}$$

(iii) For grouped freq. dist : Prepare c.f. column and find value of $\frac{N}{2}$ then find the class which contain value of c.f. is equal or just greater to N/2, this is median class

$$\therefore \ \text{Median} = \ell + \frac{\left(\frac{N}{2}-F\right)}{f} \times h$$

where ℓ — lower limit of median class

f — freq. of median class

F — c.f. of the class preceeding median class

h — Class interval of median class

Ex.5 Find the median of following freq. dist.

class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
f	8	30	40	12	10

Sol.

class	f_i	c.f.
0 – 10	8	8
10 – 20	30	38
20 – 30	40	78
30 – 40	12	90
40 – 50	10	100

Here $\frac{N}{2} = \frac{100}{2} = 50$ which lies in the value 78 of c.f. hence corresponding class of this c.f. is 20-30 is the median class, so $\ell = 20, \ f = 40, F = 38, h = 10$

$$\therefore \ \text{Median} = \ell + \frac{\left(\frac{N}{2}-F\right)}{f} \times h = 20 + \frac{(50-38)}{40} \times 10 = 23$$

3. MODE :

In a frequency distribution the mode is the value of that variate which have the maximum frequency

Method for determining mode :

(i) For ungrouped dist. : The value of that variate which is repeated maximum number of times

(ii) For ungrouped freq. dist. : The value of that variate which have maximum frequency.

(iii) For grouped freq. dist. : First we find the class which have maximum frequency, this is model calss

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where ℓ — lower limit of model class

f_0 — freq. of the model class

f_1 — freq. of the class preceeding model class

f_2 — freq. of the class succeeding model class

h — class interval of model class

Ex. 6 Find the mode of the following frequecy dist

class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
f_i	2	18	30	45	35	20	6	3

Sol. Here the class 30–40 has maximum freq. so this is the model class

$\ell = 30,\ f_0 = 45,\ f_1 = 30,\ f_2 = 35,\ h = 10$

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h = 30 + \frac{45 - 30}{2 \times 45 - 30 - 35} \times 10 = 36$$

4. RELATION BETWEEN MEAN, MEDIAN AND MODE :

In a moderately asymmetric distribution following relation between mean, median and mode of a distribution. It is known as imprical formula.

Mode = 3 Median – 2 Mean

Note (i) Median always lies between mean and mode

(ii) For a symmetric distribution the mean, median and mode are coincide.

5. MEASURES OF DISPERSION :

The dispersion of a statistical distribution is the measure of deviation of its values about the their average (central) value.

It gives an idea of scatteredness of different values from the average value.

Generally the following measures of dispersion are commonly used.

(i) Range (ii) Mean deviation (iii) Variance and standard deviation

(i) Range : The difference between the greatest and least values of variate of a distribution, are called the range of that distribution.

If the distribution is grouped distribution, then its range is the difference between upper limit of the maximum class and lower limit of the minimum class.

$$\text{Also, coefficient of range} = \frac{\text{difference of extreme values}}{\text{sum of extreme values}}$$

Ex.7 Find the range of following numbers 10, 8, 12, 11, 14, 9, 6

Sol. Here greatest value and least value of the distribution are 14 and 6 resp. therefore

Range = 14 – 6 = 8

(ii) Mean deviation (M.D.) : The mean deviation of a distribution is, the mean of absolute value of deviations of variate from their statistical average (Mean, Median, Mode).

If A is any statistical average of a distribution then mean deviation about A is defined as

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} |x_i - A|}{n} \qquad \text{(for ungrouped dist.)}$$

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} f_i |x_i - A|}{N} \qquad \text{(for freq. dist.)}$$

Note :- is minimum when it taken about the median

$$\text{Coefficient of Mean deviation} = \frac{\text{Mean deviation}}{A}$$

(where A is the central tendency about which Mean deviation is taken)

Ex.8 Find the mean deviation of number 3, 4, 5, 6, 7

Sol. Here n = 5, $\bar{x} = 5$

$$\therefore \quad \text{Mean deviation} = \frac{\Sigma |x_i - \bar{x}|}{n}$$

$$= \frac{1}{5}[|3-5| + |4-5| + |5-5| + |6-5| + |7-5|]$$

$$= \frac{1}{5}[2 + 1 + 0 + 1 + 2] = \frac{6}{5} = 1.2$$

Ex.9 Find the mean deviation about mean from the following data

x_i	3	9	17	23	27
f_i	8	10	12	9	5

Sol.

x_i	f_i	$f_i x_i$	$\|x_i - \bar{x}\|$	$f_i \|x_i - \bar{x}\|$
3	8	24	12	96
9	10	90	6	60
17	12	204	2	24
23	9	207	8	72
27	5	135	12	60
	N = 44	$\Sigma f_i x_i = 660$		$\Sigma f_i \|x_i - \bar{x}\| = 312$

$$\text{Mean } (\bar{x}) = \frac{\Sigma f_i x_i}{N} = \frac{660}{44} = 15$$

$$\text{Mean deviation} = \frac{\Sigma f_i |x_i - \bar{x}|}{N} = \frac{312}{44} = 7.09$$

(iii) Variance and standard deviation : The variance of a distribution is, the mean of squares of deviation of variate from their mean. It is denoted by σ^2 or var(x).

The positive square root of the variance are called the standard deviation. It is denoted by σ or S.D.

Hence standard deviation = $+\sqrt{\text{variance}}$

Formulae for variance :

(i) for ungrouped dist. :

$$\sigma_x^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$\sigma_x^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2, \quad \text{where } d_i = x_i - a$$

(ii) For freq. dist. :

$$\sigma_x^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{N}$$

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - (\bar{x})^2 = \frac{\Sigma f_i x_i^2}{N} - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\sigma_u^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] \quad \text{where } u_i = \frac{d_i}{h}$$

(iii) Coefficient of S.D. = $\dfrac{\sigma}{\bar{x}}$

Coefficient of variation = $\dfrac{\sigma}{\bar{x}} \times 100$ (in percentage)

Note :- $\sigma^2 = \sigma_x^2 = \sigma_d^2 = h^2\sigma_u^2$

Ex.10 Find the variance of first n natural numbers

Sol. $$\sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 = \frac{\Sigma n^2}{n} - \left(\frac{\Sigma n}{n}\right)^2 = \frac{n(n+1)(2n+1)}{6n} - \left\{\frac{n(n+1)}{2n}\right\}^2 = \frac{n^2-1}{12}$$

Ex.11 If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$, then find the standard deviation of $x_1, x_2, \ldots. x_{18}$

Sol. Let $(x_i - 8) = d_i$

$$\therefore \sigma_x = \sigma_d = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} = \sqrt{\frac{45}{18} - \left(\frac{9}{18}\right)^2} = \sqrt{\frac{5}{2} - \frac{1}{4}} = \frac{3}{2}$$

Ex.12 Find the coefficient of variation of first n natural numbers

Sol. For first n natural numbers.

$$\text{Mean } (\bar{x}) = \frac{n+1}{2}, \text{ S.D.}(\sigma) = \sqrt{\frac{n^2-1}{12}}$$

$$\therefore \text{ coefficient of variance} = \frac{\sigma}{\bar{x}} \times 100 = \sqrt{\frac{n^2-1}{12}} \times \frac{1}{\left(\frac{n+1}{2}\right)} \times 100 = \sqrt{\frac{(n-1)}{3(n+1)}} \times 100$$

6. MEAN SQUARE DEVIATION :

The mean square deviation of a distrubution is the mean of the square of deviations of variate from assumed mean. It is denoted by S^2

Hence $$S^2 = \frac{\Sigma(x_i - a)^2}{n} = \frac{\Sigma d_i^2}{n} \quad \text{(for ungrouped dist.)}$$

$$S^2 = \frac{\Sigma f_i (x_i - a)^2}{N} = \frac{\Sigma f_i d_i^2}{N} \quad \text{(for freq. dist.), where } d_i = (x_i - a)$$

7. RELATION BETWEEN VARIANCE AND MEAN SQUARE DEVIATION :

$$\because \ \sigma^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\Rightarrow \sigma^2 = s^2 - d^2, \quad \text{where } d = \bar{x} - a = \frac{\Sigma f_i d_i}{N}$$

$$\Rightarrow s^2 = \sigma^2 + d^2 \Rightarrow s^2 \geq \sigma^2$$

Hence the variance is the minimum value of mean square deviation of a distribution

Ex.13 Determine the variance of the following frequency dist.

class	0 – 2	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12
f_i	2	7	12	19	9	1

Sol. Let a = 7, h = 2

class	x_i	f_i	$u_i = \frac{x_i - a}{h}$	$f_i u_i$	$f_i u_i^2$
0 – 2	1	2	–3	–6	18
2 – 4	3	7	–2	–14	28
4 – 6	5	12	–1	–12	12
6 – 8	7	19	0	0	0
8 – 10	9	9	1	9	9
10 – 12	11	1	2	2	4
		N = 50		$\Sigma f_i u_i = -21$	$\Sigma f_i u_i^2 = 71$

$$\therefore \ \sigma^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] = 4\left[\frac{71}{50} - \left(\frac{-21}{50}\right)^2\right] = 4[1.42 - 0.1764] = 4.97$$

8. MATHEMATICAL PROPERTIES OF VARIANCE :

- $\text{Var.}(x_i + \lambda) = \text{Var.}(x_i)$

  $\text{Var.}(\lambda x_i) = \lambda^2.\text{Var}(x_i)$

  $\text{Var}(ax_i + b) = a^2.\text{Var}(x_i)$

  where λ, a, b, are constant

- If means of two series containing n_1, n_2 terms are $\bar{x}_1, \bar{x}_2$ and their variance's are σ_1^2, σ_2^2 respectively and their combined mean is $\bar{x}$ then the variance σ^2 of their combined series is given by following formula

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{(n_1 + n_2)} \qquad \text{where } d_i = \bar{x}_1 - \bar{x},\ d_2 = \bar{x}_2 - \bar{x}$$

i.e.

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2$$

SOLVED EXAMPLES

Ex.1 If in an examination different weights are assigned to different subjects Physics (2), Chemistry (1), English (1), Mathematics (2) A student scores 60 in Physics, 70 in Chemistry, 70 in English and 80 in Mathematics, then weighted mean is-

(1) 60 (2) 70 (3) 80 (4) 85

Sol.(2) Weighted mean = $\dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} = \dfrac{2\times 60+1\times 70+1\times 70+2\times 80}{6} = 70$

Ex.2 The mean of two groups of sizes 200 and 300 are 25 and 10 respectively.Their standard deviation are 3 and 4 respectively. The variance of combined sample of size 500 is-

(1) 64 (2) 65.2 (3) 67.2 (4) 64.2

Sol.(3) Combined mean $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1+n_2} = \dfrac{200\times 25+300\times 10}{500} = 16$

Here $d_1 = \bar{x}_1 - \bar{x} = 25-16 = 9$ and $d_2 = \bar{x}_2 - \bar{x} = 10-16 = -6$

We know that $\sigma^2 = \dfrac{n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)}{n_1+n_2} = \dfrac{200(9+81)+300(16+36)}{500} = \dfrac{33600}{500} = 67.2$

Ex.3 If the mean of the series $x_1, x_2, ..., x_n$ is $\bar{x}$, then the mean of the series $x_i + 2i$, i = 1, 2, ..., n will be-

(1) $\bar{x}+n$ (2) $\bar{x}+n+1$ (3) $\bar{x}+2$ (4) $\bar{x}+2n$

Sol.(2) As given $\bar{x} = \dfrac{x_1+x_2+.....+x_n}{n}$ ....(1)

If the mean of the series $x_i + 2i$, i = 1, 2, ..., n be $\bar{X}$, then

$$\bar{X} = \frac{(x_1+2)+(x_2+2.2)+(x_3+2.3)+.....+(x_n+2.n)}{n}$$

$$= \frac{x_1+x_2+.....+x_n}{n} + \frac{2(1+2+3+.....+n)}{n}$$

$$= \bar{x} + \frac{2n(n+1)}{2n} \quad \text{from (1)}$$

$$= \bar{x} + n + 1$$

Ex.4 The variance of first 20-natural numbers is-

(1) $\dfrac{133}{4}$ (2) $\dfrac{379}{12}$ (3) $\dfrac{133}{2}$ (4) $\dfrac{399}{4}$

Sol.(1) $\because\ \sigma^2 = \dfrac{\Sigma x_i^2}{n} - \left(\dfrac{\Sigma x_i}{n}\right)^2$

$$= \frac{1}{20}[1^2 + 2^2 + ... + 20^2] - \left[\frac{1}{20}(1 + 2 + ... + 20)\right]^2$$

$$= \frac{1}{20}\frac{20 \times 21(2 \times 20 + 1)}{6} - \left[\frac{1}{20}\frac{20 \times 21}{2}\right]^2 = \frac{7 \times 41}{2} - \frac{441}{4} = \frac{133}{4}.$$

In fact, the variance of first n-natural numbers is $\dfrac{n^2 - 1}{12}$

Ex.5 The mean of the following freq. table is 50 and $\Sigma f = 120$

class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
f	17	f_1	32	f_2	19

the missing frequencies are-

(1) 28, 24 (2) 24, 36 (3) 36, 28 (4) None of these

Sol.(1) $\Sigma f = 120 = 17 + f_1 + 32 + f_2 + 19$

$\Rightarrow\ f_1 + f_2 = 52$ ...... (1)

and $\Sigma fx = (10 \times 17) + (30 \times f_1) + (50 \times 32) + (70 \times f_2) + (90 \times 19) = 30f_1 + 70f_2 + 3480$

$$\therefore\ \bar{x} = \frac{\Sigma fx}{\Sigma f} \Rightarrow 50 = \frac{30f_1 + 70f_2 + 3480}{120}$$

$\Rightarrow\ 30f_1 + 70f_2 = 2520 \Rightarrow 3f_1 + 7f_2 = 252$ ..... (2)

by (1) and (2) we get $f_1 = 28$, $f_2 = 24$

Ex.6 A student obtained 75%, 80%, 85% marks in three subjects. If the marks of another subject are added then his average marks can not be less than-

(1) 60% (2) 65% (3) 80% (4) 90%

Sol.(1) Total marks obtained from three subjects out of 300 = 75 + 80 + 85 = 240

if the marks of another subject is added then total marks obtained out of 400 is greater than 240

if marks obtained in fourth subject is 0 then

$$\text{minimum average marks} = \frac{240}{400} \times 100 = 60\%$$

Ex.7 The mean and variance of a series containing 5 terms are 8 and 24 respectively. The mean and variance of another series containing 3 terms are also 8 and 24 respectively. The variance of their combined series will be-

(1) 20 (2) 24 (3) 25 (4) 42

Sol.(2) Using $\sigma^2 = \dfrac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \dfrac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2 \Rightarrow \sigma^2 = \dfrac{5(24) + 3(24)}{5 + 3} + \dfrac{5(3)}{(5 + 3)^2}(8 - 8)^2 = 24$

Ex.8 The mean deviation about median from the following data 340, 150, 210, 240, 300, 310, 320, is-

(1) 52.4 (2) 52.5 (3) 52.8 (4) none of these

Sol.(3) Arranging the observations in ascending order of magnitude, we have 150, 210, 240, 300, 310, 320, 340. Clearly, the middle observation is 300. So, median = 300

Calculation of Mean deviation

x_i	$\|x_i - 300\|$
340	40
150	150
210	90
240	60
300	0
310	10
320	20
Total	$\sum \|x_i - 300\| = 370$

Mean deviation from median = $\frac{1}{7}\sum |x_i - 300| = \frac{370}{7} = 52.8$

Ex.9 Variance of the data given below is

Size of item	3.5	4.5	5.5	6.5	7.5	8.5	9.5
Frequency	3	7	22	60	85	32	8

(1) 1.29 (2) 2.19 (3) 1.32 (4) none of these

Sol.(3) Let the assumed mean be a = 6.5

Calculation of variance

x_i	f_i	$d_i = x_i - 6.5$	$f_i d_i$	$f_i d_i^2$
3.5	3	–3	–9	27
4.5	7	–2	–14	28
5.5	22	–1	–22	22
6.5	60	0	0	0
7.5	85	1	85	85
8.5	32	2	64	128
9.5	8	3	24	72
	$N = \sum f_i = 217$		$\sum f_i d_i = 128$	$\sum f_i d_i^2 = 362$

Here N = 217, $\sum f_i d_i = 128$ and $\sum f_i d_i^2 = 362$

$$\therefore \quad \text{Var (X)} = \left(\frac{1}{N}\sum f_i d_i^2\right) - \left(\frac{1}{N}\sum f_i d_i\right)^2 = \frac{362}{217} - \left(\frac{128}{217}\right)^2 = 1.668 - 0.347 = 1.321$$

Ex.10 If a variable takes the value 0, 1, 2......n with frequencies proportional to the bionomial coefficients $^nC_0$, $^nC_1$,.....,^nC_n then the mean of the distribution is-

(1) $\frac{n(n+1)}{4}$ (2) $\frac{n}{2}$ (3) $\frac{n(n-1)}{2}$ (4) $\frac{n(n+1)}{2}$

Sol.(2) $N = \sum f_i = k\,[^nC_0 + {}^nC_1 + ... + {}^nC_n] = k2^n$

$$\sum f_i x_i = k\,[1.^nC_1 + 2.\,^nC_2 + ... + n\,^nC_n] = k\sum_{r=1}^{n} r.^nC_r = kn\sum_{r=1}^{n} {}^{n-1}C_{r-1} = kn2^{n-1}$$

Thus $\bar{x} = \frac{1}{2^n}(n\ \ 2^{n-1}) = \frac{n}{2}$.

Ex.11 The mean and variance of 5 observations of an experiment are 4 and 5.2 respectively. If from these observations three are 1, 2 and 6, then the remaining will be-

(1) 2, 9 (2) 5, 6 (3) 4, 7 (4) 3, 8

Sol.(3) As given $\bar{x} = 4$, n = 5 and $\sigma^2 = 5.2$. If the remaining observations are x_1, x_2 then

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = 5.2$$

$$\Rightarrow \frac{(x_1-4)^2 + (x_2-4)^2 + (1-4)^2 + (2-4)^2 + (6-4)^2}{5} = 5.2$$

$\Rightarrow (x_1 - 4)^2 + (x_2 - 4)^2 = 9$ ....(1)

Also $\bar{x} = 4 \Rightarrow \frac{x_1 + x_2 + 1 + 2 + 6}{5} = 4 \Rightarrow x_1 + x_2 = 11$ ....(2)

from eq.(1), (2) $x_1, x_2 = 4, 7$

Ex.12 The mean deviation of the series a, a + d, a + 2d, ..., a + 2nd from its mean is-

(1) $\frac{n+1}{2n+1}|d|$ (2) $\frac{n(n+1)}{2n+1}|d|$ (3) $\frac{n(n-1)}{2n+1}|d|$ (4) none of these

Sol.(2) Number of terms in the series = 2n + 1

$$\therefore \text{ mean } \bar{x} = \frac{a+(a+d)+(a+2d)+.....+(a+2nd)}{2n+1} = \frac{1}{2n+1}\left[\frac{2n+1}{2}(a+a+2nd)\right] = a+nd$$

Also $\sum|x_i - \bar{x}| = |-nd| + |(1-n)d| + ... + |-d| + 0 + |d| + ... + |nd|$

$$= 2|d|[n+(n-1) + ... + 1] = 2|d|\frac{n(n+1)}{2} = n(n+1)|d|$$

$$\therefore \text{ mean deviation from mean} = \frac{\sum|x_i - \bar{x}|}{N} = \frac{n(n+1)}{2n+1}|d|$$

Ex.13 Let $x_1, x_2 ....., x_n$ be values taken by a variable X and $y_1, y_2, ..., y_n$ be the values taken by a variable Y such that $y_i = ax_i + b$, $i = 1, 2, ..., n$. Then-

(1) Var (Y) = a^2Var(X) (2) Var(Y) = a^2Var(X) + b

(3) Var(Y) = Var(X) + b (4) None of these

Sol.(1) We have,

$$\text{Var}(Y) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \qquad [\because y_i = ax_i + b; i = 1, 2, ..., n \Rightarrow \bar{Y} = a\bar{X} + b]$$

$$\Rightarrow \text{Var}(Y) = \frac{1}{n}\sum_{i=1}^{n}a^2(x_i - \bar{X})^2$$

$$\Rightarrow \text{Var}(Y) = a^2\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2\right\} = a^2\,\text{Var}(X)$$

Ex.14 The mean square deviation of a set of n observations $x_1, x_2, ..., x_n$ about a point c is defined as $\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$ The mean square deviation about –2 and 2 are 18 and 10 respectively, then standard deviation of this set of observations is-

(1) 3 (2) 2 (3) 1 (4) None of these

Sol.(1) $\because \frac{1}{n}\Sigma(x_i + 2)^2 = 18$ and $\frac{1}{n}\Sigma(x_i - 2)^2 = 10$

$\Rightarrow \Sigma(x_i + 2)^2 = 18n$ and $\Sigma(x_i - 2)^2 = 10n$

$\Rightarrow \Sigma(x_i + 2)^2 + \Sigma(x_i - 2)^2 = 28\,n$ and $\Sigma(x_i + 2)^2 - \Sigma(x_i - 2)^2 = 8\,n$

$\Rightarrow 2\Sigma x_i^2 + 8n = 28\,n$ and $8\Sigma x_i = 8n$

$\Rightarrow \Sigma x_i^2 = 10\,n$ and $\Sigma x_i = n$

$$\Rightarrow \frac{\Sigma x_i^2}{n} = 10 \quad \text{and} \quad \frac{\Sigma x_i}{n} = 1$$

$$\therefore \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2} = \sqrt{10 - (1)^2} = 3$$

CHECK YOUR GRASP STATISTICS EXERCISE-I

ARITHMETIC MEAN, WEIGHTED MEAN, COMBINED MEAN

1. Mean of the first n terms of the A.P. a, (a + d), (a + 2d), ... is-

(1) $a+\frac{nd}{2}$ (2) $a+\frac{(n-1)d}{2}$

(3) a + (n – 1) d (4) a + nd

SI0001

2. The A.M. of first n even natural number is -

(1) n(n + 1) (2) $\frac{n+1}{2}$ (3) $\frac{n}{2}$ (4) n + 1

SI0002

3. The A.M. of ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ...${}^nC_n$ is -

(1) $\frac{2^n}{n}$ (2) $\frac{2^{n+1}}{n}$ (3) $\frac{2^n}{n+1}$ (4) $\frac{2^{n+1}}{n+1}$

SI0003

4. If the mean of numbers 27, 31, 89, 107, 156 is 82, then the mean of numbers 130, 126, 68, 50, 1 will be-

(1) 80 (2) 82 (3) 75 (4) 157

SI0004

5. If the mean of n observations x_1, x_2, ...x_n is $\bar{x}$, then the sum of deviations of observations from mean is :-

(1) 0 (2) $n\bar{x}$

(3) $\frac{\bar{x}}{n}$ (4) None of these

SI0005

6. The mean of 9 terms is 15. if one new term is added and mean become 16, then the value of new term is :-

(1) 23 (2) 25 (3) 27 (4) 30

SI0006

7. If the mean of first n natural numbers is equal to $\frac{n+7}{3}$, then n is equal to-

(1) 10 (2) 11

(3) 12 (4) none of these

SI0007

8. The mean of first three terms is 14 and mean of next two terms is 18. The mean of all the five terms is-

(1) 15.5 (2) 15.0

(3) 15.2 (4) 15.6

SI0008

9. If the mean of five observations x, x + 2, x + 4, x+ 6 and x + 8 is 11, then the mean of last three obsevations is-

(1) 11 (2) 13 (3) 15 (4) 17

SI0009

10. The mean of a set of numbers is $\bar{x}$. If each number is decreased by λ, the mean of the new set is-

(1) $\bar{x}$ (2) $\bar{x}+\lambda$ (3) $\lambda-\bar{x}$ (4) $\bar{x}-\lambda$

SI0010

11. The mean of 50 observations is 36. If its two observations 30 and 42 are deleted, then the mean of the remaining observations is-

(1) 48 (2) 36

(3) 38 (4) none of these

SI0011

12. In a frequency dist. , if d_i is deviation of variates from a number ℓ and mean = $\ell + \frac{\Sigma f_i d_i}{\Sigma f_i}$, then ℓ is :-

(1) Lower limit

(2) Assumed mean

(3) Number of observation

(4) Class interval

SI0012

13. The A.M. of n observation is $\bar{x}$. If the sum of n – 4 observations is K, then the mean of remaining observations is-

(1) $\frac{\bar{x}-K}{4}$ (2) $\frac{n\bar{x}-K}{n-4}$

(3) $\frac{n\bar{x}-K}{4}$ (4) $\frac{n\bar{x}-(n-4)K}{4}$

SI0013

14. The mean of values $1, \frac{1}{2}, \frac{1}{3}, ... \frac{1}{n}$ which have frequencies 1, 2, 3, ... n resp., is :-

(1) $\frac{2n+1}{3}$ (2) $\frac{2}{n}$ (3) $\frac{n+1}{2}$ (4) $\frac{2}{n+1}$

SI0014

15. The sum of squares of deviation of variates from their A.M. is always :-

(1) Zero
(2) Minimum
(3) Maximum
(4) Nothing can be said

SI0015

16. If the mean of following feq. dist. is 2.6, then the value of f is :-

x_i	1	2	3	4	5
f_i	5	4	f	2	3

(1) 1 (2) 3
(3) 8 (4) None of these

SI0016

17. The weighted mean (W.M.) is computed by the formula ?

(1) W.M. = $\frac{\Sigma x_i}{\Sigma w_i}$ (2) W.M. = $\frac{\Sigma w_i}{\Sigma x_i}$

(3) W.M. = $\frac{\Sigma w_i x_i}{\Sigma x_i}$ (4) W.M. = $\frac{\Sigma w_i x_i}{\Sigma w_i}$

SI0017

18. The weighted mean of first n natural numbers when their weights are equal to corresponding natural number, is :-

(1) $\frac{n+1}{2}$ (2) $\frac{2n+1}{3}$

(3) $\frac{(n+1)(2n+1)}{6}$ (4) None of these

SI0018

19. The average income of a group of persons is $\overline{x}$ and that of another group is $\overline{y}$. If the number of persons of both group are in the ratio 4 : 3, then average income of combined group is :-

(1) $\frac{\overline{x}+\overline{y}}{7}$ (2) $\frac{3\overline{x}+4\overline{y}}{7}$

(3) $\frac{4\overline{x}+3\overline{y}}{7}$ (4) None of these

SI0019

20. In a group of students, the mean weight of boys is 65 kg. and mean weight of girls is 55 kg. If the mean weight of all students of group is 61 kg, then the ratio of the number of boys and girls in the group is :-

(1) 2 : 3 (2) 3 : 1 (3) 3 : 2 (4) 4 : 3

SI0020

MEDIAN, MODE

21. The median of an arranged series of n even observations, will be :-

(1) $\left(\frac{n+1}{2}\right)$ th term

(2) $\left(\frac{n}{2}\right)$ th term

(3) $\left(\frac{n}{2}+1\right)$ th term

(4) Mean of $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th terms

SI0021

22. The median of the numbers 6, 14, 12, 8, 10, 9, 11, is :-

(1) 8 (2) 10 (3) 10.5 (4) 11

SI0022

23. Median of the following freq. dist.

x_i	3	6	10	12	7	15
f_i	3	4	2	8	13	10

(1) 7 (2) 10
(3) 8.5 (4) None of these

SI0023

24. Median is independent of change of :-

(1) only Origin
(2) only Scale
(3) Origin and scale both
(4) Neither origin nor scale

SI0024

25. A series which have numbers three 4's, four 5's, five 6's, eight 7's, seven 8's and six 9's then the mode of numbers is :-

(1) 9 (2) 8 (3) 7 (4) 6

SI0025

26. Mode of the following freqency distribution

x :	4	5	6	7	8
f :	6	7	10	8	3

(1) 5 (2) 6 (3) 8 (4) 10

SI0026

27. The mode of the following freq. dist is :-

Class	1–10	11–20	21–30	31–40	41–50
f_i	5	7	8	6	4

(1) 24 (2) 23.83
(3) 27.16 (4) None of these

SI0027

SYMMETRIC AND ASYMMETRIC DISTRIBUTION, RANGE

28. For a normal dist :-

(1) mean = median
(2) median = mode
(3) mean = mode
(4) mean = median = mode

SI0028

29. The relationship between mean, median and mode for a moderately skewed distribution is-

(1) mode = median – 2 mean
(2) mode = 2 median – mean
(3) mode = 2 median – 3 mean
(4) mode = 3 median – 2 mean

SI0029

30. The range of observations 2, 3, 5, 9, 8, 7, 6, 5, 7, 4, 3 is :-

(1) 6 (2) 7 (3) 5.5 (4) 11

SI0030

MEAN DEVIATION

31. The mean deviation of a frequency dist. is equal to :-

(1) $\frac{\Sigma d_i}{\Sigma f_i}$ (2) $\frac{\Sigma|d_i|}{\Sigma f_i}$ (3) $\frac{\Sigma f_i d_i}{\Sigma f_i}$ (4) $\frac{\Sigma f_i|d_i|}{\Sigma f_i}$

SI0031

32. Mean deviation from the mean for the observation –1, 0, 4 is-

(1) $\sqrt{\frac{14}{3}}$ (2) $\frac{2}{3}$
(3) 2 (4) none of these

SI0032

33. Mean deviation of the observations 70, 42, 63, 34, 44, 54, 55, 46, 38, 48 from median is :-

(1) 7.8 (2) 8.6
(3) 7.6 (4) 8.8

SI0033

34. Mean deviation of 5 observations from their mean 3 is 1.2, then coefficient of mean deviation is :-

(1) 0.24 (2) 0.4
(3) 2.5 (4) None of these

SI0034

35. The mean deviation from median is

(1) greater than the mean deviation from any other central value
(2) less than the mean deviation from any other central value
(3) equal to the mean deviation from any other central value
(4) maximum if all values are positive

SI0035

VARIANCE AND STANDARD DEVIATION

36. The variate x and u are related by $u = \frac{x-a}{h}$ then correct relation between σ_x and σ_u is :-

(1) $\sigma_x = h\sigma_u$ (2) $\sigma_x = h + \sigma_u$
(3) $\sigma_u = h\sigma_x$ (4) $\sigma_u = h + \sigma_x$

SI0036

37. The S.D. of the first n natural numbers is-

(1) $\sqrt{\frac{n^2-1}{2}}$ (2) $\sqrt{\frac{n^2-1}{3}}$
(3) $\sqrt{\frac{n^2-1}{4}}$ (4) $\sqrt{\frac{n^2-1}{12}}$

SI0037

38. The variance of observations 112, 116, 120, 125, 132 is :-
(1) 58.8 (2) 48.8
(3) 61.8 (4) None of these
SI0038

39. If $\sum_{i=1}^{10}(x_i - 15) = 12$ and $\sum_{i=1}^{10}(x_i - 15)^2 = 18$ then the S.D. of observations x_1, x_2 .... x_{10} is :-
(1) $\frac{2}{5}$ (2) $\frac{3}{5}$
(3) $\frac{4}{5}$ (4) None of these
SI0039

40. The S.D. of 7 scored 1, 2, 3, 4, 5, 6, 7 is-
(1) 4 (2) 2
(3) $\sqrt{7}$ (4) none of these
SI0040

41. The variance of series a, a + d, a + 2d, ..., a + 2nd is :-
(1) $\frac{n(n+1)}{2}d^2$ (2) $\frac{n(n+1)}{3}d^2$
(3) $\frac{n(n+1)}{6}d^2$ (4) $\frac{n(n+1)}{12}d^2$
SI0041

42. Variance is independent of change of-
(1) only origin
(2) only scale
(3) origin and scale both
(4) none of these
SI0042

43. If the coefficient of variation and standard deviation of a distribution are 50% and 20 respectively, then its mean is-
(1) 40 (2) 30
(3) 20 (4) None of these
SI0043

44. If each observation of a dist. whose S.D. is σ, is increased by λ, then the variance of the new observations is -
(1) σ (2) $\sigma + \lambda$ (3) σ^2 (4) $\sigma^2 + \lambda$
SI0044

45. The variance of 2, 4, 6, 8, 10 is-
(1) 8 (2) $\sqrt{8}$
(3) 6 (4) none of these
SI0045

46. If each observation of a dist., whose variance is σ^2, is multiplied by λ, then the S.D. of the new new observations is-
(1) σ (2) $\lambda\sigma$
(3) $|\lambda|\sigma$ (4) $\lambda^2 \sigma$
SI0046

47. The standard deviation of variate x_i is σ. Then standard deviation of the variate $\frac{ax_i + b}{c}$, where a , b, c are constants is-
(1) $\left(\frac{a}{c}\right)\sigma$ (2) $\left|\frac{a}{c}\right|\sigma$
(3) $\left(\frac{a^2}{c^2}\right)\sigma$ (4) None of these
SI0047

CHECK YOUR GRASP **ANSWER-KEY** **EXERCISE-I**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	2	4	3	3	1	2	2	4	2	4	2	2	3	4	2	1	4	2	3	3
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Ans.	4	2	3	4	3	2	2	4	4	2	4	3	2	2	2	1	4	2	2	2
Que.	41	42	43	44	45	46	47													
Ans.	2	1	1	3	1	3	2													

BRAIN TEASERS STATISTICS EXERCISE-II

1. The A.M. of the series 1, 2, 4, 8, 16, ..., 2^n is-

(1) $\frac{2^n-1}{n}$ (2) $\frac{2^{n+1}-1}{n+1}$

(3) $\frac{2^n-1}{n+1}$ (4) $\frac{2^{n+1}-1}{n}$

SI0048

2. If the mean of n observations $1^2, 2^2, 3^2$, ... n^2 is $\frac{46n}{11}$, then n is equal to-

(1) 11 (2) 12

(3) 23 (4) 22

SI0049

3. The weighted mean of first n natural numbers whose weights are equal, is :-

(1) $\frac{n+1}{2}$ (2) $\frac{2n+1}{2}$

(3) $\frac{2n+1}{3}$ (4) $\frac{(2n+1)(n+1)}{6}$

SI0050

4. The average age of a group of men and women is 30years. If average age of men is 32 and that of women is 27, then the percentage of women in the group is-

(1) 60 (2) 50

(3) 40 (4) 30

SI0051

5. Mean and median of four numbers a, b, c and d ($b < a < d < c$) is 35 and 25 respectively then the value of $b + c - a - d$ will be :-

(1) 90 (2) 115 (3) 40 (4) 10

SI0052

6. Variance of the group α, $\alpha + 2$, $\alpha + 4$, $\alpha + 6$, ... upto n terms ($\alpha \neq 0$) is :-

(1) $\frac{n^2-1}{12}+2n+\alpha$ (2) $\frac{n^2-1}{3}+\alpha$

(3) $\frac{n^2-1}{3}$ (4) None

SI0053

7. Product of n positive numbers is unit. The sum of these numbers can not be less than-

(1) 1 (2) n

(3) n^2 (4) none of these

SI0054

8. The A.M. of first n terms of the series 1.3.5, 3.5.7, 5.7.9,....., is-

(1) $3n^3 + 6n^2 + 7n - 1$ (2) $n^3 + 8n^2 + 7n - 1$

(3) $2n^3 + 8n^2 - 7n - 2$ (4) $2n^3 + 8n^2 + 7n - 2$

SI0055

9. The observations 29, 32, 48, 50, x, x + 2, 72, 78, 84, 95 are arranged in ascending order and their median is 63 then the value of x is :-

(1) 61 (2) 62 (3) 62.5 (4) 63

SI0056

10. If the mode of a distribution is 18 and the mean is 24, then median is-

(1) 18 (2) 24 (3) 22 (4) 21

SI0057

11. If the mean and S.D. of n observations $x_1, x_2, ...x_n$ are $\bar{x}$ and σ resp, then the sum of squares of observations is :-

(1) $n(\sigma^2 + \bar{x}^2)$ (2) $n(\sigma^2 - \bar{x}^2)$

(3) $n(\bar{x}^2 - \sigma^2)$ (4) None of these

SI0058

12. The variance of observations 8, 12, 13, 15, 22, is :-

(1) 21 (2) 21.2

(3) 21.4 (4) None of these

SI0059

13. If the mean of a set of observations $x_1, x_2,......., x_{10}$ is 20, then the mean of x_1+4, x_2+8, $x_3+12,......,x_{10}+40$ is-
(1) 34 (2) 42 (3) 38 (4) 40

SI0060

14. The mean of values 0, 1, 2, ..., n when their weights are $1, {}^nC_1, {}^nC_2, ..., {}^nC_n$, resp., is

(1) $\frac{2^n}{n+1}$ (2) $\frac{n+1}{2}$

(3) $\frac{2^{n+1}}{n(n+1)}$ (4) $\frac{n}{2}$

SI0061

15. For 15 observations of x, mean and median were found to be 12 and 20 respectively. Later an observation which was 25 found to be wrong then replaced by its correct value 55, then new mean and median will be :-
(1) 14 and 50 respectively
(2) 12 and 20 respectively
(3) 14 and 20 respectively
(4) Mean is 14 but median can't be determined.

SI0062

16. If a variable takes the discrete values $\alpha + 4$, $\alpha - \frac{7}{2}$, $\alpha - \frac{5}{2}$, $\alpha - 3$, $\alpha - 2$, $\alpha + \frac{1}{2}$, $\alpha - \frac{1}{2}$, $\alpha + 5 (\alpha > 0)$, then the median of these values-

(1) $\alpha - \frac{5}{4}$ (2) $\alpha - \frac{1}{2}$

(3) $\alpha - 2$ (4) $\alpha + \frac{5}{4}$

SI0063

17. The S.D. of first n odd natural numbers is :-

(1) $\sqrt{\frac{n^2-1}{2}}$ (2) $\sqrt{\frac{n^2-1}{3}}$

(3) $\sqrt{\frac{n^2-1}{6}}$ (4) $\sqrt{\frac{n^2-1}{12}}$

SI0064

18. If the sum and sum of squares of 10 observations are 12 and 18 resp., then, The S.D. of observations is :-

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

SI0065

19. The mean of n values of a distribution is $\bar{x}$. If its first value is increased by 1 , second by 2, ... then the mean of new values will be-

(1) $\bar{x}+n$ (2) $\bar{x}+n/2$

(3) $\bar{x}+\left(\frac{n+1}{2}\right)$ (4) None of these

SI0066

20. The mean of the series $x_1, x_2, ..., x_n$ is $\bar{X}$. If x_2 is replaced by λ, then the new mean is-

(1) $\frac{\bar{X}-x_2+\lambda}{n}$ (2) $\frac{n\bar{X}+x_2-\lambda}{n}$

(3) $\frac{(n-1)\bar{X}+\lambda}{n}$ (4) $\frac{n\bar{X}-x_2+\lambda}{n}$

SI0067

21. The mean square deviation about –1 and +1 of a set of observations are 7 and 3 respectively then standard deviation of the set is :-

(1) $\sqrt{2}$ (2) $\sqrt{3}$ (3) 2 (4) None

SI0068

22. The mean deviation of the numbers 1, 2, 3, 4, 5 is-
(1) 0 (2) 1.2
(3) 2 (4) 1.4

SI0069

23. If mean = (3 median – mode) x, then the value of x is-
(1) 1 (2) 2 (3) 1/2 (4) 3/2

SI0070

24. A man spends equal ammount on purchasing three kinds of pens at the rate 5 Rs/pen, 10 Rs/pen, 20 Rs/pen, then average cost of one pen is :-

(1) 10 Rs (2) $\frac{35}{3}$ Rs

(3) $\frac{60}{7}$ Rs (4) None of these

SI0071

25. The median of 21 observation is 40. if each observations greater than the median are increased by 6, then the median of the observations will be-

(1) 40 (2) 46

(3) 46 + 40/21 (4) 46 – 40/21

SI0072

26. The coefficient of range of the following distribution 10, 14, 11, 9, 8, 12, 6

(1) 0.4 (2) 2.5

(3) 8 (4) 0.9

SI0073

27. The S.D. of the following freq. dist. :-

Class	0 – 10	10 – 20	20 – 30	30 – 40
f_i	1	3	4	2

(1) 7.8 (2) 9

(3) 8.1 (4) 0.9

SI0074

28. The mean of a dist. is 4. if its coefficient of variation is 58%. Then the S.D. of the dist. is:-

(1) 2.23 (2) 3.23

(3) 2.32 (4) None of these

SI0075

29. The mean of a set of observations is $\overline{x}$. If each observation is divided by α, (α ≠ 0) and then is increased by 10, then the mean of the new set is

(1) $\frac{\overline{x}}{\alpha}$ (2) $\frac{\overline{x}+10}{\alpha}$

(3) $\frac{\overline{x}+10\alpha}{\alpha}$ (4) $\frac{\alpha\overline{x}+10}{\alpha}$

SI0076

30. The average age of a teacher and three students is 20 years. If all students are of equal age and the difference between the age of the teacher and that of a student is 20 years, then the age of the teacher is-

(1) 25 years (2) 30 years

(3) 35 years (4) 45 years

SI0077

31. Median of 5 observations i.e.

$3^{\log_9 4}, 5^{\log_{1/2} 8}, e^{2\ell n 3}, \ell n\left(\frac{1}{e}2\right)+3, e^{2\ell n 3+\frac{1}{\log_4 e}}$:-

(1) 1 (2) 2 (3) 9 (4) 36

SI0078

32. Median of ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, \ldots, {}^{2n}C_n$ (when n is even) is-

(1) ${}^{2n}C_{\frac{n-1}{2}}$ (2) ${}^{2n}C_{\frac{n}{2}}$

(3) ${}^{2n}C_{\frac{n+1}{2}}$ (4) None of these

SI0079

33. The mean deviation from mean of observations 5, 10, 15, 20, ...85 is :-

(1) 43.71 (2) 21.17

(3) 38.7 (4) None of these

SI0080

34. If standard deviation of variate x_i is 10, then variance of the variate $(50 + 5x_i)$ will be-

(1) 50 (2) 250

(3) 500 (4) 2500

SI0081

35. The S.D. of the numbers 31, 32, 33, ... 47 is-

(1) $2\sqrt{6}$ (2) $4\sqrt{3}$

(3) $\sqrt{\frac{47^2-1}{12}}$ (4) None of these

SI0082

36. The sum of the squares of deviation of 10 observations from their mean 50 is 250, then coefficient of variation is-

(1) 10% (2) 40%

(3) 50% (4) None of these

SI0083

37. The median and standard deviation (S.D.) of a distribution will be, If each term is increased by 2 -

(1) median and S.D. will increased by 2

(2) median will increased by 2 but S.D. will remain same

(3) median will remain same but S.D. will increased by 2

(4) median and S.D. will remain same

SI0084

38. If $\bar{X}_1$ and $\bar{X}_2$ are the means of two series such that $\bar{X}_1 < \bar{X}_2$ and $\bar{X}$ is the mean of the combined series, then-

(1) $\bar{X} < \bar{X}_1$ (2) $\bar{X} > \bar{X}_2$

(3) $\bar{X}_1 < \bar{X} < \bar{X}_2$ (4) $\bar{X} = \dfrac{\bar{X}_1 + \bar{X}_2}{2}$

SI0085

39. The median of 19 observations of a group is 30. If two observations with values 8 and 32 are further included, then the median of the new group of 21 observation will be

(1) 28 (2) 30 (3) 32 (4) 34

SI0086

40. The coefficient of mean deviation from median of observations 40, 62, 54, 90, 68, 76 is :-

(1) 2.16 (2) 0.2

(3) 5 (4) None of these

SI0087

41. A group of 10 observations has mean 5 and S.D. $2\sqrt{6}$. another group of 20 observations has mean 5 and S.D. $3\sqrt{2}$, then the S.D. of combined group of 30 observations is :-

(1) $\sqrt{5}$ (2) $2\sqrt{5}$

(3) $3\sqrt{5}$ (4) None of these

SI0088

42. For the values $x_1, x_2 \ldots\ldots x_{101}$ of a distribution $x_1 < x_2 < x_3 < \ldots\ldots < x_{100} < x_{101}$. The mean deviation of this distribution with respect to a number k will be minimum when k is equal to-

(1) x_1 (2) x_{51}

(3) x_{50} (4) $\dfrac{x_1 + x_2 + \ldots + x_{101}}{101}$

SI0089

43. In any discrete series (when all the value are not same) the relationship between M.D. about mean and S.D. is-

(1) M.D. = S.D. (2) M.D. > S.D.

(3) M.D. < S.D. (4) M.D. ≤ S.D.

SI0090

44. Median of observations x_i such that $(x_i^2 - 7x_i + 12)(x_i^3 - x_i^2 - 4x_i + 4) = 0$ will be :-

(1) 1 (2) 2 (3) 3 (4) None

SI0091

BRAIN TEASERS **ANSWER-KEY** **EXERCISE-II**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	2	1	1	3	3	3	2	4	2	3	1	2	2	4	3	1	2	3	3	4
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Ans.	2	2	3	3	1	1	2	3	3	3	3	2	2	4	1	1	2	3	2	2
Que.	41	42	43	44																
Ans.	2	2	3	2																

PREVIOUS YEAR QUESTIONS STATISTICS EXERCISE-III

1. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given? **[JEE(Main)-2013]**

(1) mean (2) median
(3) mode (4) variance

SI0092

2. The variance of first 50 even natural numbers is :- **[JEE(Main)-2014]**

(1) $\frac{833}{4}$ (2) 833
(3) 437 (4) $\frac{437}{4}$

SI0093

3. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is : **[JEE(Main)-2015]**

(1) 15.8 (2) 14.0
(3) 16.8 (4) 16.0

SI0094

4. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true ? **[JEE(Main)-2016]**

(1) $3a^2 - 23a + 44 = 0$
(2) $3a^2 - 26a + 55 = 0$
(3) $3a^2 - 32a + 84 = 0$
(4) $3a^2 - 34a + 91 = 0$

SI0095

5. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2,..., x_9$ is - **[JEE(Main)-2018]**

(1) 4 (2) 2 (3) 3 (4) 9

SI0096

6. 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is: **[JEE(Main)-19]**

(1) 22 (2) 20 (3) 16 (4) 18

SI0097

7. A data consists of n observations : $x_1, x_2, ..., x_n$. If $\sum_{i=1}^{n}(x_i+1)^2 = 9n$ and $\sum_{i=1}^{n}(x_i-1)^2 = 5n$, then the standard deviation of this data is : **[JEE(Main)-19]**

(1) 5 (2) $\sqrt{5}$
(3) $\sqrt{7}$ (4) 2

SI0098

8. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is : **[JEE(Main)-19]**

(1) 4 : 9 (2) 6 : 7
(3) 5 : 8 (4) 10 : 3

SI0099

9. The outcome of each of 30 items was observed; 10 items gave an outcome $\frac{1}{2} - d$ each, 10 items gave outcome $\frac{1}{2}$ each and the remaining 10 items gave outcome $\frac{1}{2} + d$ each. If the variance of this outcome data is $\frac{4}{3}$ then |d| equals :- **[JEE(Main)-19]**

(1) 2 (2) $\frac{\sqrt{5}}{2}$
(3) $\frac{2}{3}$ (4) $\sqrt{2}$

SI0100

10. The mean and the variance of five observation are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4; then then absolute value of the difference of the other two observations, is : **[JEE(Main)-19]**

(1) 1 (2) 3

(3) 7 (4) 5

SI0101

11. The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is : **[JEE(Main)-19]**

(1) 40 (2) 49

(3) 48 (4) 45

SI0102

12. If the standard deviation of the numbers –1, 0, 1, k is $\sqrt{5}$ where k > 0, then k is equal to

(1) $2\sqrt{\frac{10}{3}}$ (2) $2\sqrt{6}$

(3) $4\sqrt{\frac{5}{3}}$ (4) $\sqrt{6}$

SI0103

13. If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is : **[JEE(Main)-19]**

Marks	2	3	5	7
Frequencey	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is :

(1) 2.8 (2) 3.2

(3) 3.0 (4) 2.5

SI0104

14. If both the mean and the standard deviation of 50 observations $x_1, x_2 ..., x_{50}$ are equal to 16, then the mean of $(x_1 - 4)^2, (x_2 - 4)^2, ...(x_{50} - 4)^2$ is : **[JEE(Main)-19]**

(1) 525 (2) 380

(3) 480 (4) 400

SI0105

15. If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then x·y is equal to_____ **[JEE(Main)-20]**

SI0106

16. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then m + n is equal to ________. **[JEE(Main)-20]**

SI0107

17. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is **[JEE(Main)-20]**

(1) 3.99 (2) 3.98

(3) 4.02 (4) 4.01

SI0108

18. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 resepectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to **[JEE(Main)-20]**

(1) –20 (2) 10

(3) –10 (4) –5

SI0109

19. Let the observations $x_i (1 \le i \le 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i - 5) = 10$ and $\sum_{i=1}^{10}(x_i - 5)^2 = 40$.

If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, ..., x_{10} - 3$, then the ordered pair (μ, λ) is equal to : **[JEE(Main)-20]**

(1) (6, 6) (2) (3, 6)

(3) (6, 3) (4) (3, 3)

SI0110

PREVIOUS YEARS QUESTIONS									ANSWER-KEY								EXERCISE-III		
Que.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
Ans.	4	2	2	3	2	2	2	1	4	3	3	2	1	4	54.00	18	1	1	4

MATHEMATICAL REASONING

1. STATEMENT :

A sentence which is either true or false but cannot be both are called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F

For ex.

(i) "New Delhi is the capital of India", a true statement

(ii) "3 + 2 = 6", a false statement

(iii) "Where are you going ?" not a statement beasuse it connot be defined as true or false

Note : A statement cannot be both true and false at a time

2. SIMPLE STATEMENT :

Any statement whose truth value does not depend on other statement are called simple statement

For ex. (i) " $\sqrt{2}$ is an irrational number" (ii) "The set of real number is an infinite set"

3. COMPOUND STATEMENT :

A statement which is a combination of two or more simple statements are called compound statement

Here the simple statements which form a compound statement are known as its sub statements

For ex.

(i) "If x is divisible by 2 then x is even number"

(ii) "ΔABC is equilatral if and only if its three sides are equal"

4. LOGICAL CONNECTIVES :

The words or phrases which combined simple statements to form a compound statement are called logical connectives.

In the following table some possible connectives, their symbols and the nature of the compound statement formed by them

S.N.	Connectives	symbol	use	operation
1.	and	$\wedge$	$p \wedge q$	conjunction
2.	or	$\vee$	$p \vee q$	disjunction
3.	not	$\sim$ or '	$\sim p$ or p'	negation
4.	If .... then	$\Rightarrow$ or $\rightarrow$	$p \Rightarrow q$ or $p \rightarrow q$	Implication or conditional
5.	If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	$p \Leftrightarrow q$ or $p \leftrightarrow q$	Equivalence or Bi-conditional

Explanation :

(i) $p \wedge q \equiv$ statement p and q

($p \wedge q$ is true only when p and q both are true otherwise it is false)

(ii) $p \vee q \equiv$ statement p or q

($p \vee q$ is true if at least one from p and q is true i.e. $p \vee q$ is false only when p and q both are false)

(iii) $\sim p \equiv$ not statement p

(~ p is true when p is false and ~ p is false when p is true)

(iv) $p \Rightarrow q \equiv$ statement p then statement q

($p \Rightarrow q$ is false only when p is true and q is false otherwise it is true for all other cases)

(v) $p \Leftrightarrow q \equiv$ statement p if and only if statement q

($p \Leftrightarrow q$ is true only when p and q both are true or false otherwise it is false)

5. TRUTH TABLE :

A table which shows the relationship between the truth value of compound statement S(p, q, r ....) and the truth values of its sub statements p, q, r, ... is said to be truth table of compound statement S

If p and q are two simple statements then truth table for basic logical connectives are given below

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	(~ p)
T	F
F	T

Conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note : If the compound statement contain n sub statements then its truth table will contain 2^n rows.

6. LOGICAL EQUIVALENCE :

Two compound statements S_1(p, q, r...) and S_2(p, q, r ....) are said to be logically equivalent or simply equivalent if they have same truth values for all logically possibilities

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

For ex. The truth table for $(p \to q)$ and $(\sim p \vee q)$ given as below

p	q	(~ p)	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that last two columns of the above truth table are identical hence compound statements

$(p \to q)$ and $(\sim p \vee q)$ are equivalent

i.e. $\boxed{p \to q \equiv \sim p \vee q}$

7. TAUTOLOGY AND CONTRADICTION :

(i) Tautology : A statement is said to be a tautology if it is true for all logical possibilities i.e. its truth value always T. it is denoted by t.

For ex. the statement $p \vee \sim (p \wedge q)$ is a tautology

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Clearly, The truth value of $p \vee \sim (p \wedge q)$ is T for all values of p and q. so $p \wedge \sim (p \wedge q)$ is a tautology

(ii) Contradiction : A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. It is denoted by c.

For ex. The statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Clearly, then truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$ is F for all value of p and q. So $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Note : The negation of a tautology is a contradiction and negation of a contradiction is a tautology

8. DUALITY :

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

If a compound statement contains the special variable t (tautology) and c (contradiction) then obtain its dual we replaced t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Note :

(i) the connectives $\wedge$ and $\vee$ are also called dual of each other.

(ii) If $S^*(p, q)$ is the dual of the compound statement $S(p, q)$ then

(a) $S^*(\sim p, \sim q) \equiv \sim S(p, q)$ (ii) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

For ex. The duals of the following statements

(i) $(p \wedge q) \vee (r \vee s)$ (ii) $(p \vee t) \wedge (p \vee c)$

(iii) $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim s)]$

are as given below

(i) $(p \vee q) \wedge (r \wedge s)$

(ii) $(p \wedge c) \vee (p \wedge t)$

(iii) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim s)]$

9. CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL STATEMENT ($p \rightarrow q$):

(i) Converse : The converse of the conditional statement $p \rightarrow q$ is defined as $q \rightarrow p$

(ii) Inverse : The inverse of the conditional statement $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$

(iii) Contrapositive : The contrapositive of conditional statement $p \rightarrow q$ is defined as $\sim q \rightarrow \sim p$

10. NEGATION OF COMPOUND STATEMENTS :

If p and q are two statements then

(i) **Negation of conjunction :** $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(ii) Negation of disjunction : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \vee q)$	$(\sim p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(iii) Negation of conditional : $\sim(p \rightarrow q) \equiv p \wedge \sim q$

p	q	$\sim q$	$(p \rightarrow q)$	$\sim(p \rightarrow q)$	$(p \wedge \sim q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(iv) Negation of biconditional : $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

we know that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\therefore \sim(p \leftrightarrow q) \equiv \sim[(p \rightarrow q) \wedge (q \rightarrow p)]$

$\equiv \sim(p \rightarrow q) \vee \sim(q \rightarrow p)$

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Note : The above result also can be proved by preparing truth table for $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

11. ALGEBRA OF STATEMENTS :

If p, q, r are any three statements then the some low of algebra of statements are as follow

(i) Idempotent Laws :

(a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$

i.e. $p \wedge p \equiv p \equiv p \vee p$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

(ii) Comutative laws :

(a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

(iii) Associative laws :

(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can proved result (b)

(iv) Distributive laws : (a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (c) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (d) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

p	q	r	$(q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Similarly we can prove result (b), (c), (d)

(v) De Morgan Laws : (a) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly we can proved resulty (b)

(vi) Involution laws (or Double negation laws) : $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

(vii) Identity Laws : If p is a statement and t and c are tautology and contradiction respectively then

(a) $p \wedge t \equiv p$ (b) $p \vee t \equiv t$ (c) $p \wedge c \equiv c$ (d) $p \vee c \equiv p$

p	t	c	$(p \wedge t)$	$(p \vee t)$	$(p \wedge c)$	$(p \vee c)$
T	T	F	T	T	F	T
F	T	F	F	T	F	F

(viii) Complement Laws :

(a) $p \wedge (\sim p) \equiv c$ (b) $p \vee (\sim p) \equiv t$ (c) $(\sim t) \equiv c$ (d) $(\sim c) \equiv t$

p	$\sim p$	$(p \wedge \sim p)$	$(p \vee \sim p)$
T	F	F	T
F	T	F	T

(ix) Contrapositive laws : $p \to q \equiv \sim q \to \sim p$

p	q	~ p	~ q	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

12. QUANTIFIED STATEMENTS AND QUANTIFIERS :

The words or phrases "All", "Some", "None", "There exists a" are examples of quantifiers.

A statement containing one or more of these words (or phrases) is a quantified statement.

E.g. (1) All dogs are poodles

(2) Some books have hard covers

(3) There exists an odd number which is prime.

Note : Phrases "There exists a" and "Atleast one" and the word "some" have the same meaning.

NEGATION OF QUANTIFIED STATEMENTS :

(1) '**None**' is the negation of '**at least one**' or '**some**' or '**few**'

Statement : Some dogs are poodles.

Negation : No dogs are poodles.

Similarly negation of '**some**' is '**none**'

(2) The negation of "**some A are B**" or "**There exist A which is B**" is "**No A are (is) B**" or "**There does not exist any A which is B".**

Statement-1 : Some boys in the class are smart

Statement-2 : There exists a boy in the class who is smart

Statement-3 :Alteast one boy in the class is smart

All the three above statements have same meaning as they all indicate "**existence**" of smart boy in the class.

Negation of these statements is

No boy in the class is smart.

or

There does not exist any boy in the class who is smart.

(3) Negation of "**All A are B**" is "**Some A are not B**".

Statement : All boys in the class are smart.

Negation : Some boys in the class are not smart.

or

There exists a boy in the class who is not smart.

SOLVED EXAMPLES

Ex.1 Which of the following is correct for the statements p and q ?

(1) $p \wedge q$ is true when at least one from p and q is true

(2) $p \rightarrow q$ is true when p is true and q is false

(3) $p \leftrightarrow q$ is true only when both p and q are true

(4) $\sim (p \vee q)$ is true only when both p and q are false

Sol.(4) We know that $p \wedge q$ is true only when both p and q are true so option (1) is not correct

we know that $p \rightarrow q$ is false only when p is true and q is false so option (2) is not correct

we know that $p \leftrightarrow q$ is true only when either p and q both are true or both are flase

so option (3) is not correct

we know that $\sim(p \vee q)$ is true only when $(p \vee q)$ is false

i.e. p and q both are false

So option (4) is correct

Ex.2 $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to-

(1) p (2) ~p (3) q (4) ~q

Sol.(2) $\because\ \sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ (By Demorgan Law)

$\equiv \sim p \wedge (\sim q \vee q)$ (By distributive laws)

$\equiv \sim p \wedge t$ (By complement laws)

$\equiv \sim p$ (By Identity Laws)

Ex.3 Which of the following is logically equivalent to $(p \wedge q)$?

(1) $p \rightarrow \sim q$ (2) $\sim p \vee \sim q$ (3) $\sim(p \rightarrow \sim q)$ (4) $\sim(\sim p \wedge \sim q)$

Sol.(3) $\because\ p \rightarrow \sim q \equiv \sim p \vee \sim q \equiv \sim(p \wedge q)$ $(\because\ p \rightarrow q \equiv \sim p \vee q)$

i.e. $\sim(p \rightarrow \sim q) \equiv p \wedge q$

$\because\ \sim p \vee \sim q \equiv \sim(p \wedge q)$

and $\sim(\sim p \wedge \sim q) \equiv p \vee q$

Ex.4 If $p \rightarrow (q \vee r)$ is false, then the truth values of p, q, r respectively are-

(1) T, F, F (2) F, F, F (3) F, T, T (4) T, T, F

Sol.(1) We know $p \rightarrow (q \vee r)$ is false only when p is true and $(q \vee r)$ is false. but $(q \vee r)$ is false only when q and r both are false

Hence truth values of p, q, r are respectively T, F, F

Ex.5 Statement $(p \wedge \sim q) \wedge (\sim p \vee q)$ is

(1) a tautology (2) a contradiction

(3) neither a tautology not a contradiction (4) None of these

Sol.(2) $\because\ (p \wedge \sim q) \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim q) \wedge \sim(p \wedge \sim q)$ (By Demargon Laws)

$\equiv c$, where c is contradiction (By complement laws)

Ex.6 Negation of the statement $p \rightarrow (q \wedge r)$ is-

(1) $\sim p \rightarrow \sim(q \wedge r)$ (2) $\sim p \vee (q \wedge r)$ (3) $(q \wedge r) \rightarrow p$ (4) $p \wedge (\sim q \vee \sim r)$

Sol.(4) $\sim(p \rightarrow (q \wedge r)) \equiv p \wedge \sim(q \wedge r)$ $(\because\ \sim(p \rightarrow q) \equiv p \wedge \sim q)$

$\equiv p \wedge (\sim q \vee \sim r)$

Ex.7 If $x = 5$ and $y = -2$ then $x - 2y = 9$. The contrapositive of this statement is-

(1) If $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$ (2) If $x - 2y \neq 9$ then $x \neq 5$ and $y \neq -2$

(3) If $x - 2y = 9$ then $x = 5$ and $y = -2$ (4) None of these

Sol.(1) Let p, q, r be the three statements such that

$p : x = 5,\quad q : y = -2$ and $r : x - 2y = 9$

Here given statement is $(p \wedge q) \rightarrow r$ and its contrapositive is $\sim r \rightarrow \sim(p \wedge q)$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$

Ex.8 Which of the following is wrong ?

(1) $p \rightarrow q$ is logically equivalent to $\sim p \vee q$

(2) If the $(p \vee q) \wedge (q \vee r)$ is true then truth values of p, q, r are T, F, T respectively

(3) $\sim(p \wedge (q \vee r)) \equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$

(4) The truth value of $p \wedge \sim(p \vee q)$ is always T

Sol.(4) We know that $p \rightarrow q \equiv \sim p \vee q$

If $(p \vee q) \wedge (q \vee r)$ is true then

$(p \vee q)$ and $(q \vee r)$ both are true.

i.e. truth values of p, q, r may be T, F, T respectively

$\because\ \sim(p \wedge (q \vee r)) \equiv \sim((p \wedge q) \vee (p \wedge r) \equiv \sim(p \wedge q) \wedge \sim(p \wedge r) \equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$

If p is true and q is false then $\sim(p \vee q)$ is false i.e. $p \wedge \sim(p \vee q)$ is false

Ex.9 If $S^*(p, q, r)$ is the dual of the compound statement $S(p, q, r)$ and $S(p, q, r) = \sim p \wedge [\sim(q \vee r)]$ then $S^*(\sim p, \sim q, \sim r)$ is equivalent to-

(1) $S(p, q, r)$ (2) $\sim S(\sim p, \sim q, \sim r)$ (3) $\sim S(p, q, r)$ (4) $S^*(p, q, r)$

Sol.(3) $\because\ S(p, q, r) = \sim p \wedge [\sim(q \vee r)]$

So $S(\sim p, \sim q, \sim r) \equiv \sim(\sim p) \wedge [\sim(\sim q \vee \sim r)] \equiv p \wedge (q \wedge r)$

$S^*(p, q, r) \equiv \sim p \vee [\sim(q \wedge r)]$

$S^*(\sim p, \sim q, \sim r) \equiv p \vee (q \vee r)$

Clearly $S^*(\sim p, \sim q, \sim r) \equiv \sim S(p, q, r)$

Ex.10 The negation of the statement "If a quadrilateral is a square then it is a rhombus"

(1) If a quadrilateral is not a square then is a rhombus it

(2) If a quadrilateral is a square then it is not a rhombus

(3) a quadrilateral is a square and it is not a rhombus

(4) a quadritateral is not a square and it is a rhombus

Sol.(3) Let p and q be the statements as given below

p : a quadrilateral is a square

q : a quadritateral is a rhombus

the given statement is $p \rightarrow q$

$\because\ \sim(p \rightarrow q) \equiv p \wedge \sim q$

Therefore the negation of the given statement is a quadrilateral is a square and it is not a rhombus

CHECK YOUR GRASP MATHEMATICAL REASONING EXERCISE-I

1. The inverse of the statement $(p \wedge \sim q) \rightarrow r$ is-

(1) $\sim(p \vee \sim q) \rightarrow \sim r$ (2) $(\sim p \wedge q) \rightarrow \sim r$
(3) $(\sim p \vee q) \rightarrow \sim r$ (4) None of these

MR0001

2. $(\sim p \vee \sim q)$ is logically equivalent to-

(1) $p \wedge q$ (2) $\sim p \rightarrow q$
(3) $p \rightarrow \sim q$ (4) $\sim p \rightarrow \sim q$

MR0002

3. The equivalent statement of $(p \leftrightarrow q)$ is-

(1) $(p \wedge q) \vee (p \vee q)$
(2) $(p \rightarrow q) \vee (q \rightarrow p)$
(3) $(\sim p \vee q) \vee (p \vee \sim q)$
(4) $(\sim p \vee q) \wedge (p \vee \sim q)$

MR0003

4. If the compound statement $p \rightarrow (\sim p \vee q)$ is false then the truth value of p and q are respectively-

(1) T, T (2) T, F (3) F, T (4) F, F

MR0004

5. The statement $(p \rightarrow \sim p) \wedge (\sim p \rightarrow p)$ is-

(1) a tautology
(2) a contradiction
(3) neither a tautology nor a contradiction
(4) None of these

MR0005

6. Negation of the statement $(p \wedge r) \rightarrow (r \vee q)$ is-

(1) $\sim(p \wedge r) \rightarrow \sim(r \vee q)$
(2) $(\sim p \vee \sim r) \vee (r \vee q)$
(3) $(p \wedge r) \wedge (r \wedge q)$
(4) $(p \wedge r) \wedge (\sim r \wedge \sim q)$

MR0006

7. The dual of the statement $\sim p \wedge [\sim q \wedge (p \vee q) \wedge \sim r]$ is-

(1) $\sim p \vee [\sim q \vee (p \vee q) \vee \sim r]$
(2) $p \vee [q \vee (\sim p \wedge \sim q) \vee r]$
(3) $\sim p \vee [\sim q \vee (p \wedge q) \vee \sim r]$
(4) $\sim p \vee [\sim q \wedge (p \wedge q) \wedge \sim r]$

MR0007

8. Which of the following is correct-

(1) $(\sim p \vee \sim q) \equiv (p \wedge q)$
(2) $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$
(3) $\sim(p \rightarrow \sim q) \equiv (p \wedge \sim q)$
(4) $\sim(p \leftrightarrow q) \equiv (p \rightarrow q) \vee (q \rightarrow p)$

MR0008

9. The contrapositive of $p \rightarrow (\sim q \rightarrow \sim r)$ is-

(1) $(\sim q \wedge r) \rightarrow \sim p$ (2) $(q \rightarrow r) \rightarrow \sim p$
(3) $(q \vee \sim r) \rightarrow \sim p$ (4) None of these

MR0009

10. The converse of $p \rightarrow (q \rightarrow r)$ is-

(1) $(q \wedge \sim r) \vee p$ (2) $(\sim q \vee r) \vee p$
(3) $(q \wedge \sim r) \wedge \sim p$ (4) $(q \wedge \sim r) \wedge p$

MR0010

11. If p and q are two statement then $(p \leftrightarrow \sim q)$ is true when-

(1) p and q both are true
(2) p and q both are false
(3) p is false and q is true
(4) None of these

MR0011

12. Statement $(p \wedge q) \rightarrow p$ is-

(1) a tautology (2) a contradiction
(3) neither (1) nor (2) (4) None of these

MR0012

13 If statements p, q, r have truth values T, F, T respectively then which of the following statement is true-

(1) $(p \rightarrow q) \wedge r$ (2) $(p \rightarrow q) \vee \sim r$
(3) $(p \wedge q) \vee (q \wedge r)$ (4) $(p \rightarrow q) \rightarrow r$

MR0013

14. If statement $p \rightarrow (q \vee r)$ is true then the truth values of statements p, q, r respectively-

(1) T, F, T (2) F, T, F
(3) F, F, F (4) All of these

MR0014

15. Which of the following statement is a contradiction-

(1) $(p \wedge q) \wedge (\sim(p \vee q))$ (2) $p \vee (\sim p \wedge q)$

(3) $(p \rightarrow q) \rightarrow p$ (4) $\sim p \vee \sim q$

MR0015

16. The negative of the statement "If a number is divisible by 15 then it is divisible by 5 or 3"

(1) If a number is divisible by 15 then it is not divisible by 5 and 3

(2) A number is divisible by 15 and it is not divisible by 5 or 3

(3) A number is divisible by 15 or it is not divisible by 5 and 3

(4) A number is divisible by 15 and it is not divisible by 5 and 3

MR0016

17. If x = 5 and y = –2 then x – 2y = 9. The contrapositive of this statement is-

(1) If $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$

(2) If $x - 2y \neq 9$ then $x \neq 5$ and $y \neq -2$

(3) If $x - 2y = 9$ then $x = 5$ and $y = -2$

(4) None of these

MR0017

18. The negation of the statement "2 + 3 = 5 and 8 < 10" is-

(1) $2 + 3 \neq 5$ and $8 \nless 10$

(2) $2 + 3 \neq 5$ or $8 > 10$

(3) $2 + 3 \neq 5$ or $8 \geq 10$

(4) None of these

MR0018

19. For any three simple statement p, q, r the statement $(p \wedge q) \vee (q \wedge r)$ is true when-

(1) p and r true and q is false

(2) p and r false and q is true

(3) p, q, r all are false

(4) q and r true and p is false

MR0019

20. Which of the following statement is a tautology-

(1) $(\sim p \vee \sim q) \vee (p \vee \sim q)$

(2) $(\sim p \vee \sim q) \wedge (p \vee \sim q)$

(3) $\sim p \wedge (\sim p \vee \sim q)$

(4) $\sim q \wedge (\sim p \vee \sim q)$

MR0020

21. Which of the following statement is a contradiction-

(1) $(\sim p \vee \sim q) \vee (p \vee \sim q)$

(2) $(p \rightarrow q) \vee (p \wedge \sim q)$

(3) $(\sim p \wedge q) \wedge (\sim q)$

(4) $(\sim p \wedge q) \vee (\sim q)$

MR0021

22. The negation of the statement $q \vee (p \wedge \sim r)$ is equivalent to-

(1) $\sim q \wedge (p \rightarrow r)$ (2) $\sim q \wedge \sim(p \rightarrow r)$

(3) $\sim q \wedge (\sim p \wedge r)$ (4) None of these

MR0022

23. Let Q be a non empty subset of N. and q is a statement as given below :-

q : There exists an even number $a \in Q$.

Negation of the statement q will be :-

(1) There is no even number in the set Q.

(2) Every $a \in Q$ is an odd number.

(3) (1) and (2) both

(4) None of these

MR0023

24. The statement $\sim(p \rightarrow q) \leftrightarrow (\sim p \vee \sim q)$ is-

(1) a tautology

(2) a contradiction

(3) neither a tautology nor a contradiction

(4) None of these

MR0024

25. Which of the following is equivalent to $(p \wedge q)$

(1) $p \rightarrow \sim q$ (2) $\sim(\sim p \wedge \sim q)$

(3) $\sim(p \rightarrow \sim q)$ (4) None of these

MR0025

26. The dual of the following statement "Reena is healthy and Meena is beautiful" is-

(1) Reena is beaufiful and Meena is healthy

(2) Reena is beautiful or Meena is healthy

(3) Reena is healthy or Meena is beutiful

(4) None of these

MR0026

27. If p is any statement, t and c are a tautology and a contradiction respectively then which of the following is not correct-

(1) $p \wedge t \equiv p$ (2) $p \wedge c \equiv c$

(3) $p \vee t \equiv c$ (4) $p \vee c \equiv p$

MR0027

28. If S*(p, q) is the dual of the compound statement S(p, q) then S*(~p, ~q) is equivalent to-

(1) S(~p, ~q) (2) ~S(p, q)

(3) ~S*(p, q) (4) None of these

MR0028

29. If p is any statement, t is a tautology and c is a contradiction then which fo the following is not correct-

(1) $p \wedge (\sim c) \equiv p$

(2) $p \vee (\sim t) \equiv p$

(3) $t \vee c \equiv p \vee t$

(4) $(p \wedge t) \vee (p \vee c) \equiv (t \wedge c)$

MR0029

30. If p, q, r are simple statement with truth values T, F, T respectively then the truth value of $((\sim p \vee q) \wedge \sim r) \rightarrow p$ is-

(1) True

(2) False

(3) True if r is false

(4) True if q is true

MR0030

31. Which of the following is wrong-

(1) $p \vee \sim p$ is a tautology

(2) $\sim(\sim p) \leftrightarrow p$ is a tautology

(3) $p \wedge \sim p$ is a contradiction

(4) $((p \wedge p) \rightarrow q) \rightarrow p$ is a tautology

MR0031

32. The statement "If $2^2 = 5$ then I get first class" is logically equivalent to-

(1) $2^2 = 5$ and I do not get first class

(2) $2^2 = 5$ or I do not get first class

(3) $2^2 \neq 5$ or I get first class

(4) None of these

MR0032

33. If statement $(p \vee \sim r) \rightarrow (q \wedge r)$ is false and statement q is true then statement p is-

(1) true (2) false

(3) may be true or false (4) None of these

MR0033

34. Which of the following statement are not logically equivalent-

(1) $\sim(p \vee \sim q)$ and $(\sim p \wedge q)$

(2) $\sim(p \rightarrow q)$ and $(p \wedge \sim q)$

(3) $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$

(4) $(p \rightarrow q)$ and $(\sim p \wedge q)$

MR0034

35. Consider the following statements

p : Virat kohli plays cricket.

q : Virat kohli is good at maths

r : Virat kohli is successful.

then negation of the statement "If virat kohli plays cricket and is not good at maths then he is successful" will be :-

(1) $\sim p \wedge (q \wedge r)$ (2) $(\sim p \vee q) \wedge r$

(3) $p \wedge (\sim q \wedge \sim r)$ (4) None of these

MR0035

36. Let p statement "If 2 > 5 then earth will not rotate" and q be the statement "$2 \ngtr 5$ or earth will not rotate".

Statement–1 : p and q are equivalent.

Statement–2 : $m \rightarrow n$ and $\sim m \vee n$ are equivalent.

(1) Statement–1 is true, Statement–2 is true; Statement–2 is not the correct explanation of Statement–1.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is true, Statement–2 is true; Statement–2 is the correct explanation of Statement–1.

MR0036

37. Which of the following is a tautology :-

(1) $[(\sim p \wedge p) \rightarrow q] \longrightarrow (p \wedge p)$

(2) $[(\sim p \wedge p) \rightarrow q] \longrightarrow (\sim p \rightarrow p)$

(3) $[(\sim p \wedge p) \rightarrow q] \longrightarrow (p \rightarrow p)$

(4) None of these

MR0037

38. Negation of the statement "No one in the class is fond of music" is :-

(1) everyone in the class is fond of music.

(2) Some of the students in the class are fond of music.

(3) There exists a student in the class who is fond of music.

(4) (2) and (3) both

MR0038

CHECK YOUR GRASP **ANSWER-KEY** **EXERCISE-I**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	3	3	4	2	2	4	3	2	1	1	3	1	4	4	1	4	1	3	4	1
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		
Ans.	3	1	3	3	3	3	3	2	4	1	4	3	3	4	3	4	3	4		

PREVIOUS YEAR QUESTIONS STATISTICS EXERCISE-II

1. The negation of the statement **[JEE(Main)-2012]**

"If I become a teacher, then I will open a school", is :

(1) I will not become a teacher or I will open a school.

(2) I will become a teacher and I will not open a school.

(3) Either I will not become a teacher or I will not open a school.

(4) Neither I will become a teacher nor I will open a school.

MR0039

2. Consider :

Statement-I : $(p\wedge \sim q)\wedge(\sim p\wedge q)$ is a fallacy.

Statement-II : $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tuatology. **[JEE(Main)-2013]**

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true.

MR0040

3. The statement $\sim(p \leftrightarrow \sim q)$ is : **[JEE(Main)-2014]**

(1) equivalent to $p \leftrightarrow q$

(2) equivalent to $\sim p \leftrightarrow q$

(3) a tautology

(4) a fallacy

MR0041

4. The negation of $\sim s\vee(\sim r\wedge s)$ is equivalent to : **[JEE(Main)-2015]**

(1) $s\vee(r\vee \sim s)$ (2) $s\wedge r$

(3) $s\wedge \sim r$ (4) $s\wedge(r\wedge \sim s)$

MR0042

5. The Boolean Expression $(p\wedge\sim q)\vee q\vee(\sim p\wedge q)$ is equivalent to :- **[JEE(Main)-2016]**

(1) $p\vee\sim q$ (2) $\sim p\wedge q$ (3) $p\wedge q$ (4) $p\vee q$

MR0043

6. The following statement

$(p \rightarrow q) \rightarrow [(\sim p \rightarrow q) \rightarrow q]$ is :

[JEE(Main)-2017]

(1) a fallacy

(2) a tautology

(3) equivalent to $\sim p \rightarrow q$

(4) equivalent to $p \rightarrow \sim q$

MR0044

7. The Boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to : **[JEE(Main)-2018]**

(1) p (2) q (3) ~q (4) ~p

MR0045

8. If the Boolean expression $(p \oplus q) \wedge (\sim p \odot q)$ is equivalent to $p \wedge q$, where $\oplus, \odot \in \{\wedge,\vee\}$, then the ordered pair $(\oplus, \odot)$ is:

[JEE(Main)-19]

(1) $(\wedge,\vee)$ (2) $(\vee,\vee)$ (3) $(\wedge,\wedge)$ (4) $(\vee,\wedge)$

MR0046

9. The logical statement

$$[\sim(\sim p\vee q)\vee(p\wedge r)]\wedge(\sim q\wedge r)$$

is equivalent to : **[JEE(Main)-19]**

(1) $(p \wedge r) \wedge \sim q$ (2) $(\sim p \wedge \sim q) \wedge r$

(3) $\sim p \vee r$ (4) $(p \wedge \sim q) \vee r$

MR0047

10. Consider the following three statements :

P : 5 is a prime number.

Q : 7 is a factor of 192.

R : L.C.M. of 5 and 7 is 35.

Then the truth value of which one of the following statements is true ?

[JEE(Main)-19]

(1) $(P \wedge Q) \vee (\sim R)$ (2) $(\sim P) \wedge (\sim Q \wedge R)$

(3) $(\sim P) \vee (Q \wedge R)$ (4) $P \vee (\sim Q \wedge R)$

MR0048

11. If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology?

[JEE(Main)-19]

(1) $(p \vee r) \rightarrow (p \wedge r)$ (2) $p \vee r$

(3) $p \wedge r$ (4) $(p \wedge r) \rightarrow (p \vee r)$

MR0049

12. Contrapositive of the statement "If two numbers are not equal, then their squares are not equal." is :-

[JEE(Main)-19]

(1) If the squares of two numbers are equal, then the numbers are equal.

(2) If the squares of two numbers are equal, then the numbers are not equal.

(3) If the squares of two numbers are not equal, then the numbers are equal.

(4) If the squares of two numbers are not equal, then the numbers are not equal.

MR0050

13. The contrapositive of the statement "If you are born in India, then you are a citizen of India", is : **[JEE(Main)-19]**

(1) If you are born in India, then you are not a citizen of India.

(2) If you are not a citizen of India, then you are not born in India.

(3) If you are a citizen of India, then you are born in India.

(4) If you are not born in India, then you are not a citizen of India.

MR0051

14. For any two statements p and q, the negation of the expression $p \vee (\sim p \wedge q)$ is

[JEE(Main)-19]

(1) $p \wedge q$ (2) $p \leftrightarrow q$

(3) $\sim p \vee \sim q$ (4) $\sim p \wedge \sim q$

MR0052

15. If the truth value of the statement $p \rightarrow (\sim q \vee r)$ is false(F), then the truth values of the statements p, q, r are respectively : **[JEE(Main)-19]**

(1) F, T, T (2) T, F, F

(3) T, T, F (4) T, F, T

MR0053

16. The Boolean expression $\sim(p \Rightarrow (\sim q))$ is equivalent to : **[JEE(Main)-19]**

(1) $(\sim p) \Rightarrow q$ (2) $p \vee q$

(3) $q \Rightarrow \sim p$ (4) $p \wedge q$

MR0054

17. Let A, B, C and D be four non-empty sets. The contrapositive statement of "If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$" is : **[JEE(Main)-20]**

(1) If $A \subseteq C$, then $B \subset A$ or $D \subset B$

(2) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$

(3) If $A \not\subseteq C$, then $A \subseteq B$ and $B \subseteq D$

(4) If $A \not\subseteq C$, then $A \not\subseteq B$ and $B \subseteq D$

MR0055

18. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to : **[JEE(Main)-20]**

(1) p (2) q (3) ~p (4) ~q

MR0056

19. Which of the following statements is a tautology? **[JEE(Main)-20]**

(1) $\sim(p \vee \sim q) \rightarrow p \vee q$

(2) $\sim(p \wedge \sim q) \rightarrow p \vee q$

(3) $\sim(p \vee \sim q) \rightarrow p \wedge q$

(4) $p \vee (\sim q) \rightarrow p \wedge q$

MR0057

20. Which one of the following is a tautology ? **[JEE(Main)-20]**

(1) $P \wedge (P \vee Q)$

(2) $P \vee (P \wedge Q)$

(3) $Q \rightarrow (P \wedge (P \rightarrow Q))$

(4) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

MR0058

21. If $p \rightarrow (p \wedge \sim q)$ is false, then the truth values of p and q are respectively : **[JEE(Main)-20]**

(1) F, T (2) T, T (3) F, F (4) T, F

MR0059

22. Negation of the statement : **[JEE(Main)-20]**

$\sqrt{5}$ is an integer or 5 is irrational is :

(1) $\sqrt{5}$ is irrational or 5 is an integer.

(2) $\sqrt{5}$ is not an integer and 5 is not irrational.

(3) $\sqrt{5}$ is an integer and 5 is irrational.

(4) $\sqrt{5}$ is not an integer or 5 is not irrational.

MR0060

PREVIOUS YEARS QUESTIONS ANSWER-KEY EXERCISE-II

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	2	2	1	2	4	2	4	1	1	4	4	1	2	4	3
Que.	16	17	18	19	20	21	22								
Ans.	4	2	3	1	4	2	2								