

## FUNDAMENTALS OF MATHEMATICS # 02

### SIMILAR TRIANGLES

We call two figures similar if one is simply a blown-up or scaled-up, and possibly rotated and/or flipped, version of the other.

#### Sufficient Conditions For Similarity of Triangles.

Two triangles are said to be similar if any one of the following conditions is satisfied.

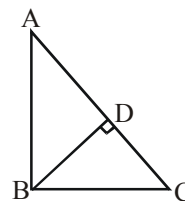
- (i) their corresponding angles are equal.
- (ii) their corresponding sides are proportional.

Based on the above, there are three axioms for similarity of two triangles.

1. A.A. (Angle-Angle) Axiom of Similarity.
2. S.A.S (Side-Angle-Side) Axiom of Similarity.
3. S.S.S. (Side-Side-Side) Axiom of Similarity.

Note :

- (i) Congruent triangles are necessarily similar but the similar triangles may not be congruent.
- (ii) If  $ABC$  is a triangle, right-angled at  $B$  and  $BD \perp AC$ , then  
 $\triangle ABC \sim \triangle ADB \sim \triangle BDC$
- (iii) If two triangles are similar to a third triangle, then they are similar to each other.



#### Similar Polygons

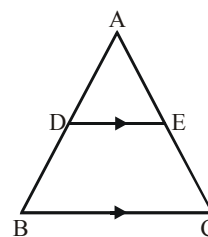
Two polygons with the same number of sides are said to be similar if their corresponding angles are equal as well as their corresponding sides are proportional.

#### Theorem 1 :

##### Basic Proportionality Theorem (B.P.T.)

If a line is drawn parallel to a side of a triangle intersecting the other two sides, then it divides the two sides in the same

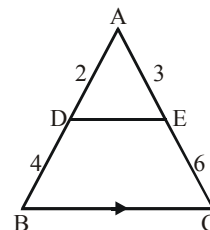
ratio. i.e., if in a  $\triangle ABC$ ,  $DE \parallel BC$  then,  $\frac{AD}{DB} = \frac{AE}{EC}$ .



**Theorem 2 :****Converse of B.P.T.**

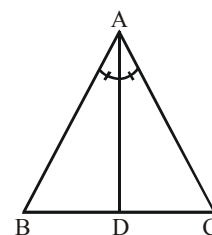
If a line divides any two sides of a triangle proportionally, then it must be parallel to the third side.

i.e., If in a  $\triangle ABC$ ,  $\frac{AD}{DB} = \frac{AE}{EC}$  then  $DE \parallel BC$ .

**Theorem 3 :****Angle Bisector Theorems****(i) Angle Bisector Theorem (Internal)**

The internal bisector of an angle of a triangle divides the opposite side, internally, in the ratio of the sides containing the angle. i.e., If in a  $\triangle ABC$ , AD bisects  $\angle A$ ,

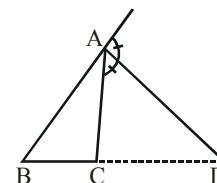
$$\text{then } \frac{AB}{AC} = \frac{BD}{DC}.$$

**(ii) Angle Bisector Theorem (External)**

The bisector of an exterior angle of a triangle divides the opposite side (provided bisector and opposite side are not parallel) externally, in the ratio of the sides containing the angle.

i.e., If in a  $\triangle ABC$ , AD bisects the exterior angle A and intersects side BC produced in D, then

$$\frac{BD}{CD} = \frac{AB}{AC}$$

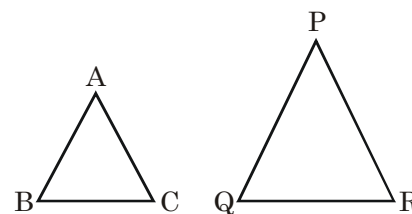


Converse of both the angle bisector theorems are also true.

**Theorem 4 :****Proportion Applied to Area**

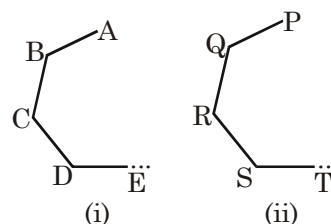
The areas of similar triangles are proportional to the squares of corresponding sides.

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CA^2}{RP^2} \quad (\triangle ABC \sim \triangle PQR)$$

**Theorem 5 :**

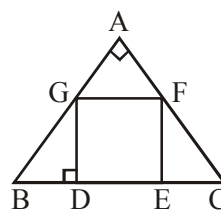
The areas of similar polygons are proportional to the squares of corresponding sides.

$$\frac{\text{Area 1}^{\text{st}} \text{ poly}}{\text{Area 2}^{\text{nd}} \text{ poly}} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{CD^2}{RS^2} = \dots$$



**Illustration-1 :** In the given figure, DEFG is a square and  $\angle BAC = 90^\circ$ . Prove that :

- (i)  $\triangle AGF \sim \triangle DBG$
- (ii)  $\triangle AGF \sim \triangle EFC$
- (iii)  $\triangle DBG \sim \triangle EFC$
- (iv)  $DE^2 = BD \times EC$



**Solution :**

- (i) In  $\triangle AGF$  and  $\triangle DBG$ , we have :  
 $\angle GAF = \angle BDG = 90^\circ$  and  $\angle AGF = \angle DBG$  [corresponding  $\angle$ s]  
 $\therefore \triangle AGF \sim \triangle DBG$  [by A.A. similarity]
- (ii) In  $\triangle AGF$  and  $\triangle EFC$ , we have :  
 $\angle FAG = \angle CEF = 90^\circ$  and  $\angle AFG = \angle ECF$  [corresponding  $\angle$ s]  
 $\therefore \triangle AGF \sim \triangle EFC$  [by A.A. similarity]
- (iii) Since,  $\triangle AGF \sim \triangle DBG$  and  $\triangle AGF \sim \triangle EFC$ , it follows that  
 $\triangle DBG \sim \triangle EFC$ .

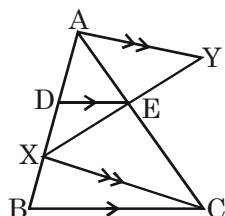
**Note :** If two triangles are similar to a third triangle, then they are similar to each other.

- (iv) Since,  $\triangle DBG \sim \triangle EFC$ , we have :

$$\frac{BD}{EF} = \frac{DG}{EC} \text{ or } \frac{BD}{DE} = \frac{DE}{EC} \quad [\because EF = DE, DG = DE \text{ (Sides of a sq.)}]$$

Hence,  $DE^2 = BD \times EC$ .

**Illustration-2 :** Given the  $\overline{DE} \parallel \overline{BC}$  and  $\overline{AY} \parallel \overline{XC}$ , prove that  $\frac{EY}{EX} = \frac{AD}{DB}$ .



**Solution :**

The ratios of side lengths in the problem suggest we look for similar triangles.

Since  $\overline{AY} \parallel \overline{XC}$ , we have  $\triangle AYE \sim \triangle CXE$ .

$\Rightarrow EY/EX = AE/EC$ . All we have left is to show that  $AE/EC = AD/DB$ .

Since  $\overline{DE} \parallel \overline{BC}$ , we have  $\triangle ADE \sim \triangle ABC$ . Therefore,  $AD/AB = AE/AC$ , which is almost what we want! We break AB and AC into  $AD + DB$  and  $AE + EC$ , hoping we can do a little algebra to finish :

$$\frac{AD}{AD + DB} = \frac{AE}{AE + EC}$$

If only we could get rid of the AD and AE in the denominators-then we would have  $AD/DB = AE/EC$ .

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

Therefore,  $\frac{AD}{AD} + \frac{DB}{AD} = \frac{AE}{AE} + \frac{EC}{AE}$ , so  $1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$ , which gives us  $\frac{DB}{AD} = \frac{EC}{AE}$

Flipping these fractions back over gives us  $AD/DB = AE/EC$ . Therefore, we have  $EY/EX = AE/EC = AD/DB$ , as desired.

**Illustration-3 :** In  $\triangle ABC$ ,  $\angle B = 2 \angle C$  and the bisector of  $\angle B$  intersects  $AC$  at  $D$ . Prove that  $\frac{BD}{DA} = \frac{BC}{BA}$ .

**Solution :** Given : In  $\triangle ABC$ ,  $\angle B = 2 \angle C$  and  $BD$  is bisector of  $\angle B$ .

To Prove :  $\frac{BD}{DA} = \frac{BC}{BA}$

$BD$  is bisector of  $\angle CBA$ .

$$\therefore \frac{BC}{BA} = \frac{CD}{AD} \quad \dots(1)$$

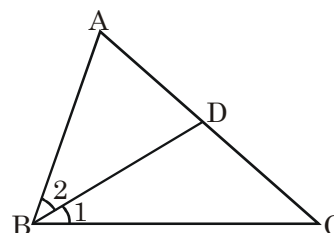
$$2 \angle C = \angle 1 + \angle 2 \quad [\text{given}]$$

$$\text{But } \angle 1 = \angle 2$$

$$\therefore 2 \angle C = 2 \angle 1$$

$$\Rightarrow \angle C = \angle 1 \Rightarrow BD = CD \quad \dots(2)$$

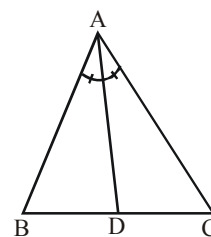
$$\text{from (1) \& (2) } \frac{BC}{BA} = \frac{BD}{DA}$$



**Illustration-4 :** In a  $\triangle ABC$ ,  $AD$  is the bisector of  $\angle BAC$ .

If  $AB = 3.5$  cm,  $AC = 4.2$  cm and  $DC = 2.4$  cm

Find  $BD$ .



**Solution :** Since,  $AD$  is the bisector  $\angle A$ , we have  $\frac{BD}{DC} = \frac{AB}{AC}$

$$\therefore \frac{BD}{2.4} = \frac{3.5}{4.2} \Rightarrow BD = \frac{2.4 \times 3.5}{4.2} = 2 \text{ cm.}$$

**Illustration-5 :**  $O$  is any point inside a triangle  $ABC$ . The bisector of  $\angle AOB$ ,  $\angle BOC$  and  $\angle COA$  meet the sides  $AB$ ,  $BC$  and  $CA$  in points  $D, E, F$  respectively. Prove that

$AD \times BE \times CF = DB \times EC \times FA$ .

**Solution :** Given :  $O$  is any point inside a  $\triangle ABC$ . The bisectors of  $\angle AOB$ ,  $\angle BOC$  and  $\angle COA$  meet the sides  $AB$ ,  $BC$  and  $CA$  in points  $D, E, F$  respectively.

To prove :  $AD \cdot BE \cdot CF = DB \cdot EC \cdot FA$

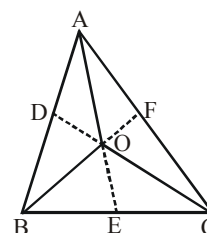
Proof : In  $\triangle AOB$ ,  $OD$  is the bisector of  $\angle AOB$ .

$$\therefore \frac{OA}{OB} = \frac{AD}{DB} \quad \dots(i)$$

In  $\triangle BOC$ ,  $OE$  is the bisector of  $\angle BOC$

$$\therefore \frac{OB}{OC} = \frac{BE}{EC} \quad \dots(ii)$$

In  $\triangle COA$ ,  $OF$  is the bisector of  $\angle COA$ .



$$\therefore \frac{OC}{OA} = \frac{CF}{FA} \quad \dots(iii)$$

Multiplying the corresponding sides of (i), (ii) and (iii), we get :

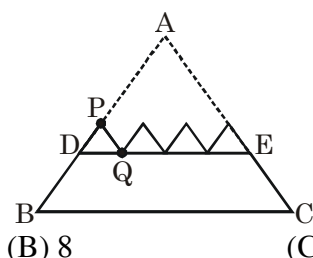
$$\frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA} = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow 1 = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$\Rightarrow DB \times EC \times FA = AD \times BE \times CF$$

$$\text{or } AD \times BE \times CF = DB \times EC \times FA.$$

**Illustration-6 :** All the triangles in the diagram below are similar to isosceles triangle ABC in which  $AB = AC$ . Each of the smaller triangle has area  $1 \text{ unit}^2$  and  $\triangle ABC$  has area  $40 \text{ unit}^2$ , then area of trapezium DBCE is



(A) 16

(B) 8

(C) 24

(D) 32

**Solution :**

$$\triangle PDQ \sim \triangle ABC$$

$$\frac{\text{ar.} \triangle PDQ}{\text{ar.} \triangle ABC} = \frac{DQ^2}{BC^2} = \frac{1}{40}$$

$$\therefore DQ = \frac{BC}{2\sqrt{10}}$$

$$\text{Also } DE = 4.DQ$$

$$DE = \frac{4.BC}{2\sqrt{10}} \text{ Hence } \frac{DE}{BC} = \frac{2}{\sqrt{10}}$$

$$\frac{\text{ar} \triangle ADE}{\text{ar} \triangle ABC} = \frac{DE^2}{BC^2} = \frac{4}{10} = \frac{2}{5}$$

$$\text{area } \triangle ADE = 16$$

$$\text{area trap DBCE} = 40 - 16 = 24$$

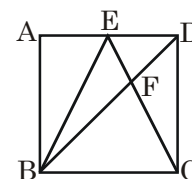
**Illustration-7 :** ABCD is a square and E is mid point of AD. If area of square is given by  $A_1$ . Prove following :

$$(A) \frac{\text{area of } \triangle EFD}{\text{area of } \triangle BFC} = \frac{1}{4}$$

$$(B) \frac{\text{area of } \triangle BEF}{\text{area of } \triangle DFC} = 1$$

$$(C) \text{ area of } \triangle EFD = \frac{A_1}{12}$$

$$(D) \text{ area of } \triangle BEF = \frac{A_1}{6}$$



**Solution :**

Let side of square = a

$$A_1 = a^2$$

$$\text{area of } \triangle BED = \frac{1}{2} \times a \times \frac{a}{2} = \frac{1}{4} a^2 = \frac{A_1}{4}$$

Since  $\triangle EFD$  is similar to  $\triangle CFB$ 

$$\Rightarrow \frac{ED}{BC} = \frac{1}{2} = \frac{EF}{FC} = \frac{BF}{FD}$$

$$\Rightarrow \frac{\text{area of } \triangle EFD}{\text{area of } \triangle BFC} = \frac{1}{4}$$

Let area of  $\triangle EFD = \Delta$ 

$$\Rightarrow \text{area of } \triangle BFC = 4\Delta$$

Also let area of  $\triangle BEF = x$  and let area of  $\triangle DFC = y$ 

$$\Rightarrow x + \Delta = \frac{A_1}{4} \text{ and } y + 4\Delta = \frac{A_1}{2}$$

Area of  $\triangle BED = \text{Area of } \triangle CED$ ( $\because$  triangles on the same base ED and between the same parallel lines ED and BC)

$$\Rightarrow \text{Area of } \triangle BEF + \text{Area of } \triangle EFD = \text{Area of } \triangle CDF + \text{Area of } \triangle EFD$$

$$\Rightarrow x + \Delta = y + \Delta \Rightarrow x = y$$

$$\text{Now } x = y \text{ therefore } x + \Delta = \frac{A_1}{4} \text{ and } x + 4\Delta = \frac{A_1}{2} \text{ on solving these equations}$$

$$\Rightarrow \Delta = \frac{A_1}{12} \Rightarrow x = \frac{A_1}{6}$$

$$\text{Also } y + \Delta = \frac{A_1}{4} \Rightarrow y = \frac{A_1}{6}$$

**Illustration-8 :** In a right angle triangle ABC. Let k is maximum possible area of a square that can be inscribed when one of its vertices coincide with the vertex of right angle of triangle then find the value of k

**Solution :** $\triangle AMN$  &  $\triangle NLB$ 

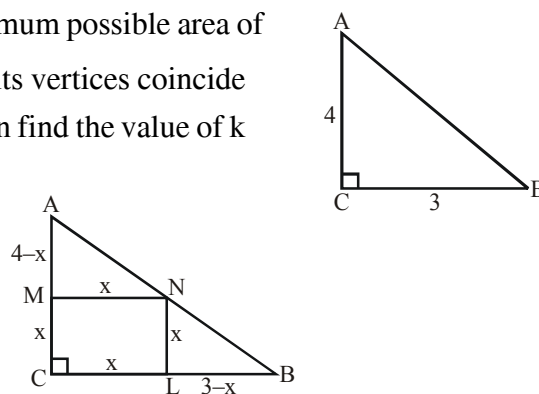
are similar.

$$\frac{4-x}{x} = \frac{x}{3-x}$$

$$\Rightarrow 12 - 7x + x^2 = x^2$$

$$x = \frac{12}{7}$$

$$k = \frac{144}{49}$$



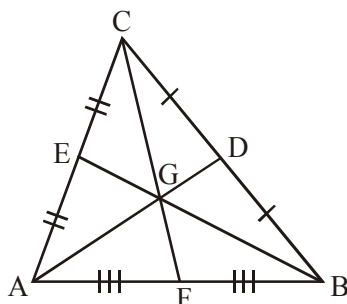
## SOME IMPORTANT POINTS AND THEIR MEANING :

### MEDIANS AND CENTROID :

A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

In the figure below,  $\overline{AD}$ ,  $\overline{BE}$  and  $\overline{CF}$  are all medians.

- The medians of a triangle are concurrent at a point called the **centroid** of the triangle. The centroid of the triangle is usually labelled by G.



- The medians of a triangle divide the triangle into six little triangles of equal area.
- The centroid of a triangle cuts its medians into 2 : 1 ratio. For example, for the triangle shown,

$$\text{we have } \frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}.$$

- The median to the hypotenuse of a right triangle is equal in length to half the hypotenuse.

### PERPENDICULAR BISECTORS AND CIRCUMCENTRE :

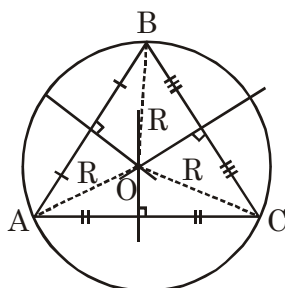
The perpendicular bisector of a segment is the straight line consisting of all the points that are equidistant from the end points of the segment.

#### Circumcentre :

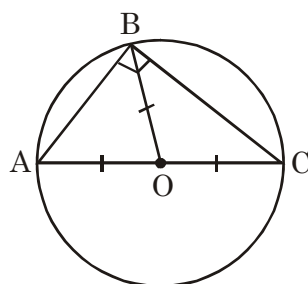
The perpendicular bisectors of the sides of a triangle are concurrent at a point called the **circumcenter**. The circle centered at the circumcenter that passes through the vertices of the triangle is called the **circumcircle** of the triangle, because it is **circumscribed** about the triangle (meaning it passes through all the vertices of the triangle).

Finally, the radius of this circle is called the **circumradius**, the circumcenter is usually labelled with the letter O, and the circumradius is usually denoted by R.

$$OA = OB = OC = R$$



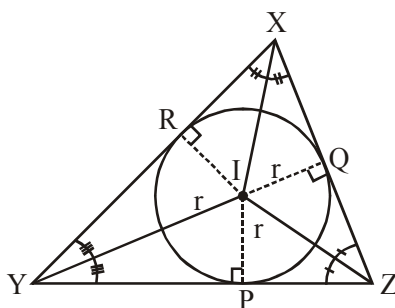
- The circumcenter of a right triangle is the midpoint of the hypotenuse and the circumradius equals half the length of the hypotenuse.



- Just as two points determine a line, we have now shown that three non-collinear points determine a circle. This means that given any three non-collinear points, there is exactly one circle that passes through all three.

### ANGLE BISECTORS, INCENTRE & EXCENTRE :

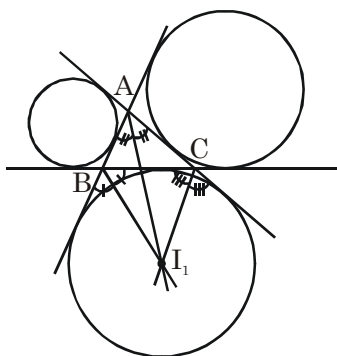
- The angle bisector of an angle consists of all points that are equidistant from the lines forming the angle.
- The angle bisectors of interior angles of a triangle are concurrent at a point called the **incentre**. This point is equidistant from the sides of the triangle. This common distance from the incentre to the sides of a triangle is called the **inradius**. Thus the circle with centre I and this radius is tangent to all three sides of the triangle. This circle is called the **incircle** because it is inscribed in the triangle (meaning it is tangent to all the sides of the triangle). The incentre is usually denoted I, and the inradius is usually written as  $r$ .



**Escribed Circle :** It is a circle touching one side of a triangle internally and the other two externally. A triangle has three escribed circles. The centre of an escribed circle is the point of concurrence of the bisectors of the two exterior angles and the bisector of the third interior angle.

The centres of these escribed circles are called excentres and are labelled with  $I_1$ ,  $I_2$  and  $I_3$ . Also, their corresponding radii called exradii are denoted by  $r_1$ ,  $r_2$  and  $r_3$ .

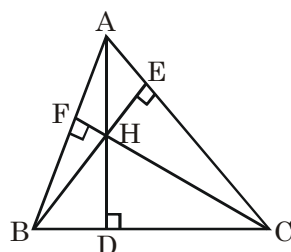




## ALTITUDES & ORTHOCENTRE :

**Altitude :** It is the perpendicular dropped on any side of a triangle from the opposite vertex.

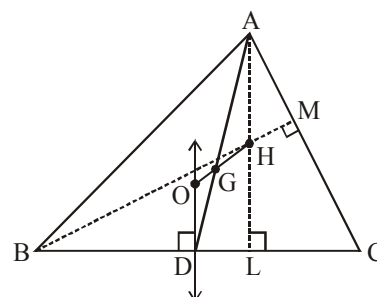
**Orthocentre :** The three altitudes of a triangle are concurrent and the point of concurrence is the orthocentre, which is denoted by H.



## EULER'S LINE :

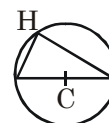
**In the figure**

- AD is a median and G the centroid.  
 $AG : GD = 2 : 1$ .
- BM is the altitude to AC from B.  
Here H is the orthocentre.
- OD is the perpendicular bisector of side BC.  
O is the circumcentre.



**Note :** O, G and H are collinear. This line is known as Euler's line and  $HG : GO = 2 : 1$ .

- In an equilateral triangle these four centres (centroid, orthocentre, incentre, circumcentre) are coincident.
- In an isosceles triangle they are collinear.
- The incentre and centroid are always within the triangle.
- In an acute angled triangle circumcentre and orthocentre also lie inside the triangle
- In a right triangle circumcentre will be the midpoint of the hypotenuse and the orthocentre is the vertex containing the right angle.
- In obtuse angled triangle orthocentre and circumcentre lie outside the triangle.



**Illustration-9 :** For any triangle, the area of triangle formed by medians of the triangle is  $\left(\frac{3}{4}\right)^{\text{th}}$  the area of given triangle.

**Solution :** Consider  $\triangle ABC$ , AD, BE and CF are length of median say  $\ell_1$ ,  $\ell_2$  and  $\ell_3$  respectively. We have to find area of triangle, the length of whose sides are  $\ell_1$ ,  $\ell_2$  and  $\ell_3$ .

Let G be the centroid of  $\triangle ABC$

Let us produce AG to K

Such that  $GD = DK$

and join CK and BK

We know that

$BD = DC$  and  $GD = DK$

$\therefore$  BKCG is a parallelogram and hence

$BK = GC$  and  $CK = BG$

$BK = \frac{2}{3} CF$  and  $CK = \frac{2}{3} BE$

$BK = \frac{2}{3} \ell_3$  and  $CK = \frac{2}{3} \ell_2$

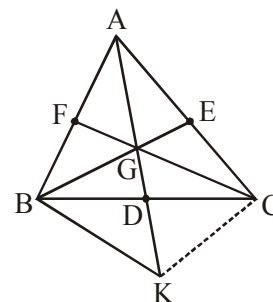
and  $GK = 2GD = 2 \cdot \left(\frac{1}{3} AD\right) = \frac{2}{3} \ell_1$

Now say PQR is triangle formed by length  $\ell_1$ ,  $\ell_2$  and  $\ell_3$  as sides.  $\triangle PQR \sim \triangle GKC$

$$\frac{\text{area } \triangle PQR}{\text{area } \triangle GKC} = \frac{(\ell_1)^2}{\left(\frac{2\ell_1}{3}\right)^2} = \frac{9}{4}$$

$$\text{area } \triangle PQR = \frac{9}{4} \text{ area } \triangle GKC = \frac{9}{4} \times \frac{1}{3} \text{ area } \triangle ABC$$

$$\text{area } \triangle PQR = \frac{3}{4} \text{ area } \triangle ABC$$



**Illustration-10 :** If I is the incenter of  $\triangle ABC$  then  $\angle BIC = 90^\circ + \frac{1}{2} A$ .

**Solution :**

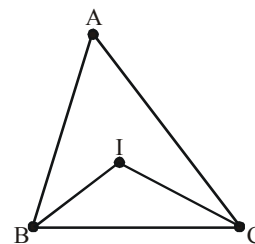
We have

$$\angle BIC = 180^\circ - (\angle IBC + \angle ICB)$$

$$= 180^\circ - \frac{1}{2}(B + C)$$

$$= 180^\circ - \frac{1}{2}(180^\circ - A)$$

$$= 90^\circ + \frac{1}{2} A.$$



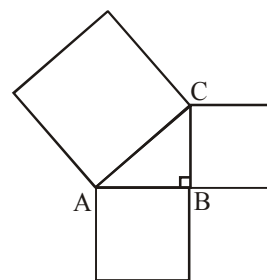
# SOME THEOREMS ON SIDE LENGTH OF A TRIANGLE

## Baudhayana Theorem (Pythagorean/Pythagoras' Theorem) :

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.

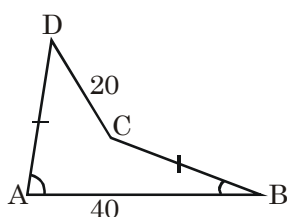
In Right Angle Triangle ABC, right angled at B,

$$AC^2 = AB^2 + BC^2$$



**Illustration-11 :** ABCD is a quadrilateral (shown in figure)  $AD = BC$ ,  $AB = 40$ ,  $CD = 20$ .

$\angle DAB + \angle ABC = 90^\circ$ . If area of quadrilateral is N then  $\frac{N}{100}$  is



**Solution :** Given :  $AD = BC$ ,  $AB = 40$

$CD = 20$ ,  $\angle DAB + \angle ABC = 90^\circ$

Construction :

Extend BC to intersect AD in X.

$$\Rightarrow \angle AXB = 180^\circ - (\angle DAB + \angle ABC)$$

$$\Rightarrow \angle AXB = 180^\circ - 90^\circ = 90^\circ$$

Let  $CX = c$ ,  $AX = a$ ,  $XD = b$

$CB = d$

$BC = AD$

$BC = AX + XD$

$$\boxed{BC = a + b}$$

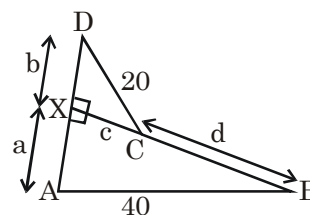
$$\therefore \boxed{d = a + b}$$

$$\text{area (quad)} = \frac{1}{2} a(c + d) + \frac{1}{2} bc$$

$$= \frac{1}{2} [ac + ad + bc]$$

$$= \frac{1}{2} [ad + c(a + b)]$$

$$= \frac{1}{2} [ad + cd]$$



In  $\triangle BXA$

$$a^2 + (c + d)^2 = 40^2 \quad \dots(1)$$

In  $\triangle CXD$

$$c^2 + b^2 = 20^2 \quad \dots(2)$$

On subtracting (2) from (1)

$$a^2 + 2cd + d^2 - b^2 = 1200$$

$$a^2 + 2cd + (a + b)^2 - b^2 = 1200 \quad (d = a + b)$$

$$2a^2 + 2cd + 2ab = 1200$$

$$2a(a + b) + 2cd = 1200$$

$$2ad + 2cd = 1200$$

$$\frac{1}{2}(ad + cd) = 300 = \text{Area of quad. ABCD}$$

**Illustration-12:** In an equilateral triangle ABC, the side BC is trisected at D. Prove that :

$$9 AD^2 = 7 AB^2$$

**Solution :** Given : An equilateral  $\triangle ABC$ , in which the side BC is trisected at D.

To Prove :  $9 AD^2 = 7 AB^2$

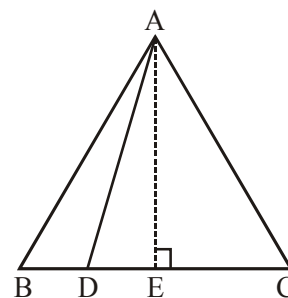
Const. : Draw  $AE \perp BC$

Proof : We know that in an equilateral triangle perpendicular from a vertex bisects the base.

$$\therefore BE = EC = \frac{1}{2} BC \quad \dots (i)$$

Also, the side BC is trisected at D. [given]

$$\Rightarrow BD = \frac{1}{3} BC \quad [\text{from figure}] \quad \dots (ii)$$



Now, in a right-triangle AED, We have :

$$AD^2 = AE^2 + DE^2$$

$$= AE^2 + (BE - BD)^2 \quad [\because DE = BE - BD]$$

$$= AE^2 + BE^2 + BD^2 - 2 BE \cdot BD$$

$$= AB^2 + BD^2 - 2 BE \cdot BD \quad [\because AB^2 = AE^2 + BE^2]$$

$$= BC^2 + \left(\frac{1}{3} BC\right)^2 - 2\left(\frac{1}{2} BC\right) \cdot \left(\frac{1}{3} BC\right)$$

$$[\because AB = BC, BD = \frac{1}{3} BC \text{ and } BE = \frac{1}{2} BC \text{ see (i) and (ii)}]$$

$$= BC^2 + \frac{BC^2}{9} - \frac{BC^2}{3} = \frac{9BC^2 + BC^2 - 3BC^2}{9}$$

$$\Rightarrow AD^2 = \frac{7BC^2}{9} \text{ or } 9AD^2 = 7BC^2$$

$$\text{or } 9AD^2 = 7AB^2 \quad (AB = BC = CA)$$

**Illustration-13 :** The perpendicular AD on the base BC of a  $\Delta ABC$  intersects BC at D so that  $DB = 3 CD$ .

Prove that :  $2AB^2 = 2AC^2 + BC^2$ .

**Solution :** Given : In figure,  $AD \perp BC$  and  $DB = 3CD$ .

To Prove :  $2AB^2 = 2AC^2 + BC^2$ .

Proof : From right triangles ADB and ADC, we have :

$$AB^2 = AD^2 + BD^2$$

$$\text{and } AC^2 = AD^2 + CD^2$$

$$\therefore AB^2 - AC^2 = BD^2 - CD^2$$

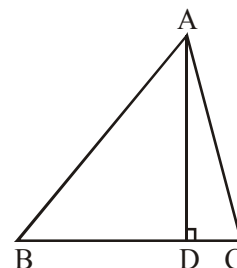
$$= (3CD)^2 - CD^2 \quad [\because BD = 3CD]$$

$$= 9CD^2 - CD^2 = 8CD^2 = 8 \times \left(\frac{1}{4}BC\right)^2 = 8 \times \frac{BC^2}{16} = \frac{BC^2}{2}$$

$$\left(\because \frac{BD}{CD} = \frac{3}{1} \Rightarrow \frac{BD+CD}{CD} = \frac{3+1}{1} \Rightarrow \frac{BC}{CD} = \frac{4}{1} \Rightarrow CD = \frac{1}{4}BC\right)$$

$$\therefore 2AB^2 - 2AC^2 = BC^2$$

$$\text{Hence, } 2AB^2 = 2AC^2 + BC^2.$$

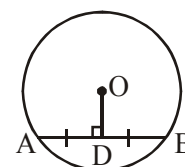


## CIRCLE

### Theorem 1 :

If a straight line drawn from the center of a circle bisects a chord, not passing through the centre, then it cuts the chord at right angles.

Conversely, if it cuts the chord at right angles, then it bisects the chord.



### Theorem 2 :

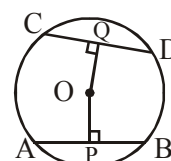
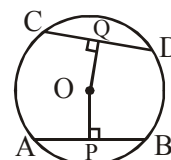
Equal chords of a circle are equidistant from the center.

$$AB = CD, OP \perp AB \text{ and } OQ \perp CD \Rightarrow OP = OQ$$

**Conversely,**

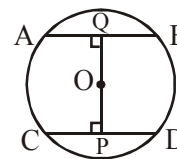
Chords which are equidistant from the centre are equal.

$$OP = OQ, OP \perp AB \text{ and } OQ \perp CD \Rightarrow AB = CD$$



**Theorem 3 :**

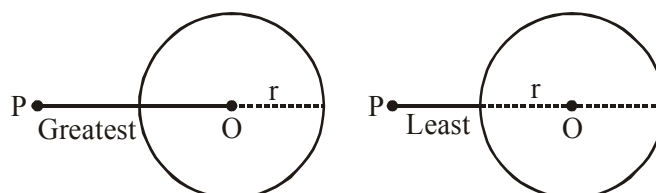
Of any two chords of a circle, that which is nearer to the centre is greater than one lying remote. Conversely, the greater of two chords is nearer to the centre.



$$OP > OQ \Rightarrow CD < AB$$

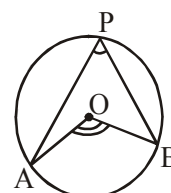
**Theorem 4 :**

If P be any point (interior, exterior or on the circle) then greatest distance between point P and circumference of the circle is  $PO + r$  and the smallest distance between point P and circumference of the circle is  $|PO - r|$  (where r is radius and O is centre)

**ANGLES IN A CIRCLE****Theorem 1 :**

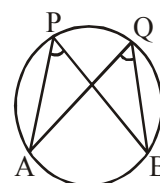
The angle at the centre of a circle is double of an angle at the circumference subtended by the same arc.

$$\angle AOB = 2\angle APB$$

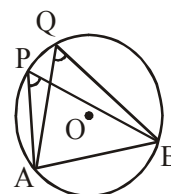
**Theorem 2 :**

Angles in the same segment of a circle are equal.

$$\angle APB = \angle AQB$$

**Converse of Theorem 2 :**

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, then four points lie on a circle (i.e. they are concyclic). If  $\angle APB = \angle AQB \Rightarrow AB$  is a chord of circle and A, P, Q, B are concyclic points



**Theorem 3 :**

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.

$$\angle P + \angle R = 180^\circ \text{ and } \angle Q + \angle S = 180^\circ$$

**Converse of Theorem 3 :**

If a pair of opposite angles of a quadrilateral are supplementary, then its vertices are concyclic.

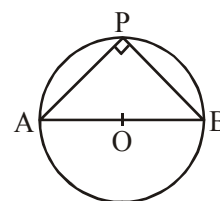
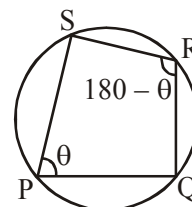
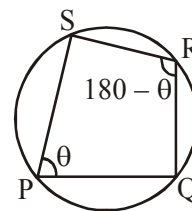
$$\text{If } \angle P + \angle R = 180^\circ \text{ or } \angle Q + \angle S = 180^\circ$$

$\Rightarrow$  Then points P, Q, R and S are concyclic.

**Theorem 4 :**

The angle in a semi-circle is a right angle.

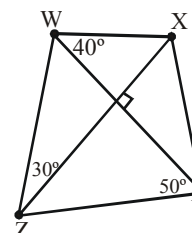
$$\angle APB = 90^\circ$$



**Illustration-14 :** In quadrilateral WXYZ with perpendicular diagonals, we are given  $\angle WZX = 30^\circ$ ,  $\angle XWY = 40^\circ$ , and  $\angle WYZ = 50^\circ$ .

(a) Compute  $\angle Z$ .

(b) Compute  $\angle X$ .



**Solution:**

Let P be the intersection of the diagonals.

In  $\triangle PZY$

$$\angle PZY = 90^\circ - 50^\circ = 40^\circ.$$

Now consider angles.

$$\angle XWY \text{ and } \angle XZY$$

$$\text{Since } \angle XWY = \angle XZY = 40^\circ$$

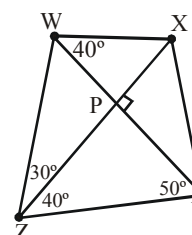
Using theorem : WXYZ is cyclic quadrilateral.

$$\angle Z = \angle WZX + \angle PZY$$

$$= 30^\circ + 40^\circ = 70^\circ$$

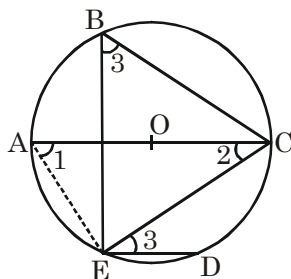
$$\text{Now } \angle Z + \angle X = 180^\circ \quad \{\text{opposite angle of cyclic quadrilateral}\}$$

$$\angle X = 180^\circ - 70^\circ = 110^\circ$$



**Illustration-15 :** In the given figure, the chord ED is parallel to the diameter AC and  $\angle CBE = 50^\circ$ . Find  $\angle CED$ .

**Solution:**



$$\angle CBE = \angle CAE \quad (\because \text{angles in the same segment})$$

$$\angle CAE = \angle 1 = 50^\circ \quad \dots(1) \quad (\because \angle CBE = 50^\circ)$$

$$\angle AEC = 90^\circ \quad \dots(2) \quad (\because \text{Angle in a semi circle is a right angle})$$

Now in  $\triangle AEC$

$$\angle 1 + \angle AEC + \angle 2 = 180^\circ \quad [\because \text{sum of angles of a } \Delta = 180^\circ]$$

$$\therefore 50^\circ + 90^\circ + \angle 2 = 180^\circ$$

$$\Rightarrow \angle 2 = 40^\circ \quad \dots(3)$$

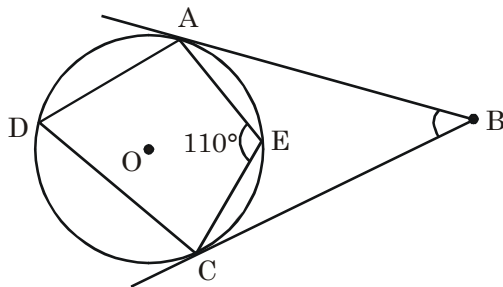
Also,  $ED \parallel AC$  (Given)

$$\therefore \angle 2 = \angle 3 \quad (\text{Alternate angles})$$

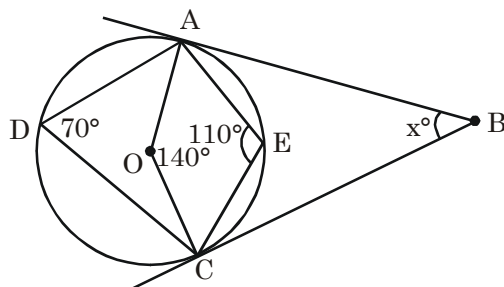
$$\therefore 40^\circ = \angle 3 \text{ i.e., } \angle 3 = 40^\circ$$

Hence  $\angle CED = 40^\circ$

**Illustration-16 :** In the figure below, BA and BC are tangents to the circle at points A and C, respectively. If  $\angle AEC = 110^\circ$ . Find  $\angle ABC$ .



**Solution :**



Consider the diagram to the right.

- Opposite angles of a cyclic quadrilateral sum up to  $180^\circ$ .

$$\text{So } \angle ADC = 180^\circ - 110^\circ = 70^\circ.$$



- From the inscribed angle theorem, we can see that

$$\angle AOC = 2\angle ADC = 2(70^\circ) = 140^\circ$$

Since the sum of interior angle of a quadrilateral is  $360^\circ$ , we have

$$\begin{aligned}\angle ABC &= 360^\circ - \angle BCO - \angle BAO - \angle AOC \\ &= 360^\circ - 90^\circ - 90^\circ - 140^\circ \\ &= 40^\circ\end{aligned}$$

**Illustration-17 :** An acute isosceles triangle, ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at point D. If  $\angle ABC = \angle ACB = 2\angle D$ , find the measure of  $\angle A$ .

**Solution :**

$$\text{Let } \angle D = \theta$$

$$\Rightarrow \angle ABC = \angle ACB = 2\theta$$

In  $\triangle BCD$ .

$$180^\circ - 4\theta + 180^\circ - 4\theta + \theta = 180^\circ$$

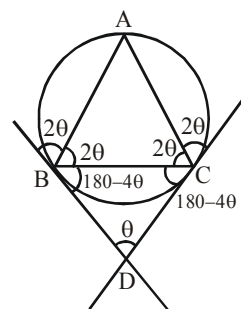
$$\Rightarrow 7\theta = 180^\circ$$

$$\Rightarrow \theta = \frac{180^\circ}{7}$$

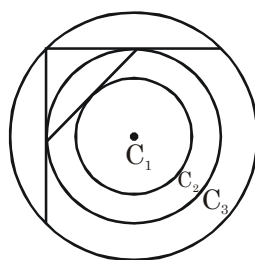
$$\therefore \angle A = 180^\circ - 4\theta$$

$$= 180^\circ - \frac{4 \times 180^\circ}{7}$$

$$\angle A = \frac{3 \times 180^\circ}{7}$$



**Illustration-18 :** In the given figure if  $C_1, C_2, C_3$  are three concentric circles such that radius of  $C_1$  and  $C_2$  is 1 and 3 unit respectively, then radius of  $C_3$  is



**Solution :**

$$\text{Let } \angle BDC = \theta, \text{ then}$$

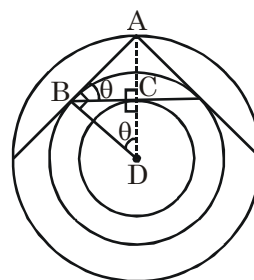
$$\angle DBC = \frac{\pi}{2} - \theta$$

$$\therefore \angle CBA = \theta, \text{ So}$$

$$\triangle ABD \sim \triangle BCD, \text{ so}$$

$$\frac{AD}{BD} = \frac{BD}{CD}$$

$$\Rightarrow R_3 = \frac{(3)^2}{1} = 9$$



**Illustration-19 :** In the given fig., O is the centre of the circle. Prove that  $\angle x + \angle y = \angle z$ .

**Solution :** Given : In fig., O is the centre of the circle.

To Prove :  $\angle x + \angle y = \angle z$

Proof :  $\angle z = 2 \angle 2$  [angle at the centre is double to the angle in the remaining part.]

or  $\angle z = \angle 2 + \angle 2$

But,  $\angle 2 = \angle 3$  [angles in the same segment]

$\Rightarrow \angle z = \angle 2 + \angle 3$  ... (i)

Now, we will determine the values of  $\angle 2$  and  $\angle 3$  in terms of  $\angle x$  and  $\angle y$ .

$\therefore \angle 3$  is an exterior angle of  $\triangle AEB$ ,

$\therefore \angle 3 = \angle 1 + \angle x$  ... (ii) [ext.  $\angle$  = sum of two int. opp.  $\angle$ s]

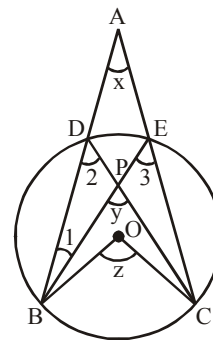
Again,  $\angle y$  is an exterior angle of  $\triangle DEP$

$\therefore \angle y = \angle 1 + \angle 2 \Rightarrow \angle 2 = \angle y - \angle 1$  ... (iii)

(i), (ii) and (iii)

$\Rightarrow \angle z = (\angle y - \angle 1) + (\angle 1 + \angle x) = \angle x + \angle y$

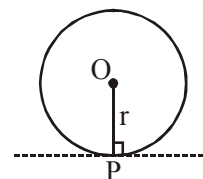
Hence,  $\angle x + \angle y = \angle z$ .



## TANGENT

### Theorem 1 :

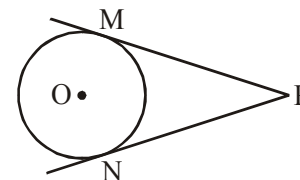
The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



### Theorem 2 :

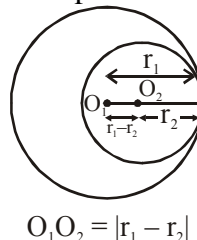
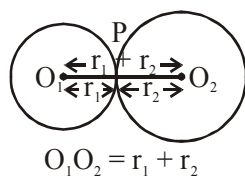
Two tangents can be drawn to a circle from an external point and length of these tangents are equal.

So,  $PM = PN$



### Theorem 3 :

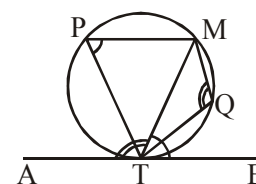
If two circles touch one another, the centres and the point of contact are collinear.



### Theorem 4 (Alternate Segments Theorem) :

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

$\angle MTB = \angle TPM$ ,  $\angle MTA = \angle MQT$



## LENGTH OF COMMON TANGENTS

### Direct Common Tangent (External Common Tangent) :

Consider two circle with centre A and B and line ' $\ell$ ' touching the two circles at P and Q. Let radius of the circles be  $r_1$  and  $r_2$ .

Now draw perpendicular from 'B' to AP meeting at 'C'.

Thus, CPQB is a rectangle and hence  $PQ = CB$ ;  $PC = BQ$

Now, in  $\triangle ACB$

$$AB^2 = AC^2 + BC^2$$

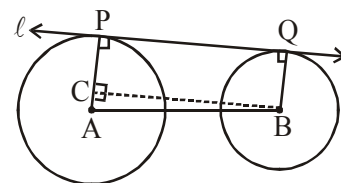
$$\Rightarrow BC^2 = AB^2 - AC^2$$

$$\Rightarrow PQ^2 = AB^2 - AC^2 \quad [BC = PQ]$$

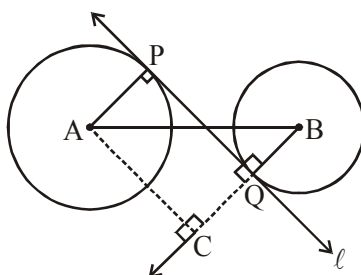
$$\Rightarrow PQ^2 = AB^2 - (r_1 - r_2)^2 \quad [AC = AP - PC = r_1 - r_2]$$

Hence, length of direct common tangent

$$PQ = \sqrt{d^2 - (r_1 - r_2)^2} \quad [\text{where } AB = d = \text{distance between the centres}]$$



### Transverse Common Tangent (Internal Common Tangent) :



Drop perpendicular from A to meet BQ produced in 'C'

Now  $AP = r_1$  ;  $BQ = r_2$  ;  $AB = d$

$AP = QC$  and  $PQ = AC$

Also  $BC = BQ + QC$

So  $BC = r_2 + r_1$

Now in  $\triangle ACB$

$$AB^2 = AC^2 + BC^2$$

$$\Rightarrow d^2 = AC^2 + (r_1 + r_2)^2$$

$$\Rightarrow AC^2 = d^2 - (r_1 + r_2)^2$$

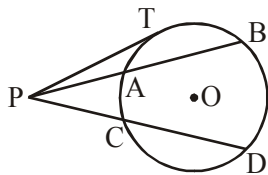
$$\Rightarrow PQ^2 = d^2 - (r_1 + r_2)^2$$

$$\Rightarrow PQ = \sqrt{d^2 - (r_1 + r_2)^2}$$

Length of transverse common tangent is  $\sqrt{d^2 - (r_1 + r_2)^2}$

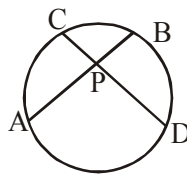
## POWER OF A POINT

Suppose a line through a point P intersects a circle in two points, A and B. The **power of a point theorem** states that for all such lines, the product (PA)(PB) is constant. We call this product the power of point P.



**P lies outside the circle**

$$PT^2 = (PA)(PB) = (PC)(PD) \\ = \text{The power of point P}$$



**P lies inside the circle**

$$PA \cdot PB = PC \cdot PD$$

**Proof :** Given : A secant PAB to a circle C(O, r) intersecting it in points A and B and PT is a tangent segment of the circle.

To Prove :  $PA \times PB = PT^2$ .

Construction: Draw  $OD \perp AB$ . Join OP, OT and OA.

Proof :  $OD \perp AB \Rightarrow AD = DB$

$$\text{Now, } PA \times PB = (PD - AD)(PD + DB)$$

$$= (PD - AD)(PD + AD) \quad [\because DB = AD]$$

$$= PD^2 - AD^2$$

$$= (OP^2 - OD^2) - AD^2 \quad [OP^2 = OD^2 + PD^2]$$

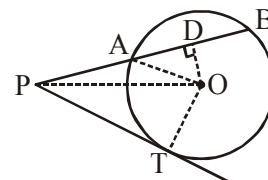
[Pythagoras Theorem]

$$= OP^2 - (OD^2 + AD^2)$$

$$= OP^2 - OA^2 \quad [\because OA^2 = OD^2 + AD^2]$$

$$= OP^2 - OT^2 \quad [\because OA = OT] \text{ [radii of same circle]}$$

$$= PT^2 \quad [\because \angle OTP = 90^\circ, OP^2 = OT^2 + PT^2]$$



**Illustration-20 :** Chords  $\overline{TY}$  and  $\overline{OP}$  meet at point K such that  $TK = 2$ ,  $KY = 16$  and  $KP = 2(KO)$ .

Find OP.

**Solution :** A quick sketch suggests how to apply power of a point.

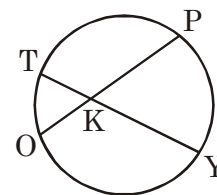
From the power of point K, we have

$$(KP)(KO) = (KT)(KY).$$

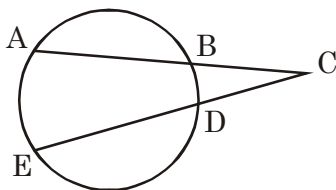
Substituting the given information in this equation yields

$$2(KO)(KO) = 2(16),$$

from which we find  $KO = 4$ . Therefore,  $KP = 2KO = 8$ , so  $OP = KO + KP = 12$ .



**Illustration-21 :** In the diagram,  $CB = 9$ ,  $BA = 11$  and  $CE = 18$ . Find  $DE$ .

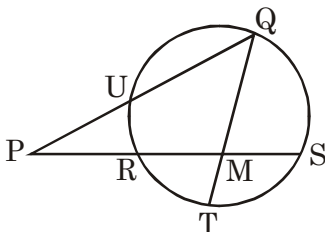


**Solution :** We have two intersecting secants, so we apply power of a point, which gives

$$(CD)(CE) = (CB)(CA).$$

Therefore,  $(CD)(18) = 9(20)$ , so  $CD = 10$  and  $DE = CE - CD = 8$ .

**Illustration-22 :** Points  $R$  and  $M$  trisect  $\overline{PS}$ , so  $PR = RM = MS$ . Point  $U$  is the midpoint of  $\overline{PQ}$ ,  $TM = 2$  and  $MQ = 8$ . Find  $PU$ .



**Solution :** The power of point  $M$  gives us

$$(MR)(MS) = (MT)(MQ).$$

We know that  $RM = MS$ , so substitution gives  $MR^2 = (2)(8)$ , i.e.,  $MR = 4$ . Therefore,

$PR = MR = 4$  and  $PS = 3(MR) = 12$ . Since  $U$  is the midpoint of  $\overline{PQ}$ , we have  $PQ = 2PU$ .

Now we can apply the power of point  $P$  to find :

$$(PU)(PQ) = (PR)(PS).$$

Substitution gives  $(PU)(2PU) = 4(12)$ , so  $PU = 2\sqrt{6}$ .

**Illustration-23 :** In the diagram,

we have  $BP = 8$ ,  $AB = 10$ ,

$CD = 7$  and  $\angle APC = 60^\circ$ .

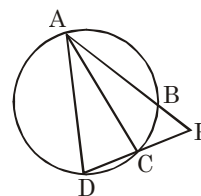
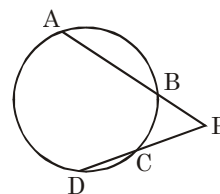
Find the area of the circle.

**Solution :** It's not immediately obvious how

we will find the radius,

so we start by finding what we can. The power of a point  $P$  gives us

$$(PC)(PD) = (PB)(PA),$$

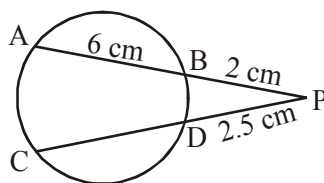


so  $(PC)(PC + 7) = 144$ . Therefore,  $PC^2 + 7PC - 144 = 0$ , so  $(PC + 16)(PC - 9) = 0$ . PC must be positive, so  $PC = 9$ .

Seeing that  $\angle APC = 60^\circ$  makes us wonder if there are any equilateral or 30-60-90 triangle lurking about. Since  $CP = AP/2$  and the angle between these sides is  $60^\circ$ , the sides adjacent to the  $60^\circ$  angle in  $\triangle ACP$  are in the same ratio as the sides adjacent to the  $60^\circ$  angle in a 30-60-90 triangle with right angle at  $\angle ACP$ !  $\Rightarrow \angle ACP = 90^\circ \Rightarrow \angle ACD = 90^\circ$

Since  $\angle ACD$  is right and inscribed in  $\widehat{AD}$ , we know  $\widehat{AD}$  is a semicircle. Therefore,  $\overline{AD}$  is a diameter of the circle. Since  $AC = CP\sqrt{3} = 9\sqrt{3}$  from our 30-60-90 triangle, we have  $AD = \sqrt{AC^2 + CD^2} = \sqrt{243 + 49} = 2\sqrt{73}$ . Finally, the radius of the circle is  $AD/2 = \sqrt{73}$ , so the area is  $(\sqrt{73})^2 \pi = 73\pi$ .

**Illustration-24:** Chords AB and CD of a circle intersect each other at P as shown in fig., if AB = 6 cm, PB = 2 cm, PD = 2.5 cm, find CD.



**Solution :** Since chords AB and CD of a circle meet in P (when produced),

$$\therefore PA \cdot PB = PC \cdot PD$$

$$\Rightarrow (6 + 2) \cdot 2 = PC (2.5)$$

$$\Rightarrow PC = \frac{16}{2.5} = \frac{16 \times 2}{5} = \frac{32}{5} = 6.4 \text{ cm}$$

$$\therefore CD = PC - PD = 6.4 - 2.5 = 3.9 \text{ cm.}$$

**Illustration-25:** If two circles of radius 1 cm and 4 cm touch each other, then find the length of common tangent.

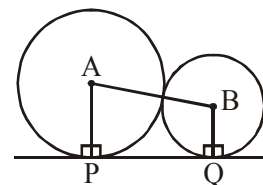
**Solution :** Distance between centres  $AB = r_1 + r_2 = 5$

Length of tangent

$$PQ = \sqrt{d^2 - (r_1 - r_2)^2}$$

$$= \sqrt{5^2 - (4 - 1)^2}$$

$$PQ = 4 \text{ cm}$$



## INEQUALITIES IN TRIANGLE

In any triangle, the longest side is opposite the largest angle and the shortest side is opposite the smallest angle. The middle side, of course, is therefore opposite the middle angle.

In other words, in  $\triangle ABC$ ,  $AB \geq AC \geq BC$  if and only if  $\angle C \geq \angle B \geq \angle A$ .

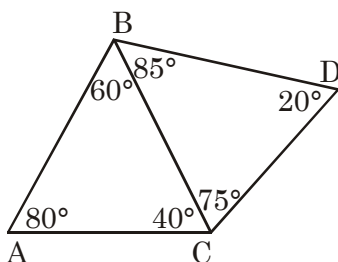
### Important :

- $\angle C$  of  $\triangle ABC$  is acute if and only if  $AB^2 < AC^2 + BC^2$ .  
 $\angle C$  of  $\triangle ABC$  is right if and only if  $AB^2 = AC^2 + BC^2$ .  
 $\angle C$  of  $\triangle ABC$  is obtuse if and only if  $AB^2 > AC^2 + BC^2$ .
- The Triangular Inequality states that for any three points, A, B and C, we have

$$AB + BC \geq AC,$$

where equality holds if and only if B lies on  $\overline{AC}$ . Therefore, for nondegenerate triangles (i.e., those in which the vertices are not collinear),  $AB + BC > AC$ .

**Illustration-26 :** Given the angles as shown, order the lengths AB, BD, CD, BC and AC from greatest to least. (Note : the diagram is not drawn to scale!)



### Solution :

First we focus on  $\triangle ABC$ . Since  $\angle A > \angle B > \angle C$ , we have  $BC > AC > AB$ . Then, we turn to  $\triangle BDC$ . Since  $\angle B > \angle C > \angle D$ , we have  $CD > BD > BC$ . BC is the smallest in one inequality string and the largest in the other, so we can put the inequalities together :

$$CD > BD > BC > AC > AB.$$

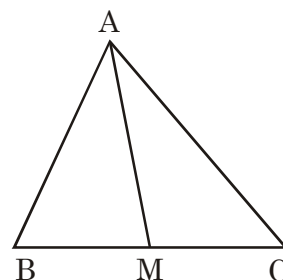
**Illustration-27 :** In  $\triangle ABC$  the median  $\overline{AM}$  is longer than  $BC/2$ . Prove that  $\angle BAC$  is acute.

### Solution :

We start with a diagram. We'd like to prove something about an angle, but all we are given is an inequality regarding lengths. So, we use the length inequality to get some angle inequalities to work with. Specifically, since  $AM > BM$  in  $\triangle ABM$  and  $AM > MC$  in  $\triangle ACM$  (because  $AM > BC/2$  and  $BM = MC = BC/2$ ), we have

$$\angle B > \angle BAM$$

$$\angle C > \angle CAM.$$



We want to prove something about  $\angle BAC$ , which equals  $\angle BAM + \angle CAM$ , so we add these two inequalities to give  $\angle B + \angle C > \angle BAC$ . Since  $\angle B + \angle C + \angle BAC = 180^\circ$ , we can write  $\angle B + \angle C > \angle BAC$  as

$$180^\circ - \angle BAC > \angle BAC.$$

This gives us  $180^\circ > 2\angle BAC$ , so  $90^\circ > \angle BAC$ . Therefore,  $\angle BAC$  is acute.

**Illustration-28 :** In  $\triangle XYZ$ , we have  $XY = 11$  and  $YZ = 14$ . For how many integer values of  $XZ$  is  $\triangle XYZ$  acute ?

**Solution :** In order for  $\triangle XYZ$  to be acute, all three of its angles must be acute. Using the given side lengths and our inequalities above, we see that we must have

$$121 + 196 > XZ^2$$

$$196 + XZ^2 > 121$$

$$121 + XZ^2 > 196$$

Simplifying these three yields :

$$317 > XZ^2$$

$$XZ^2 > -75$$

$$XZ^2 > 75$$

The middle inequality is clearly always true. (We really didn't even have to include it – clearly 11 couldn't ever be the largest side!) Combining the other two and noting that we seek integer values of  $XZ$ , we have  $9 \leq XZ \leq 17$  (since  $8^2 < 75 < 9^2$  and  $18^2 > 317 > 17^2$ ). So, there are 9 integer values of  $XZ$  such that  $\triangle XYZ$  is acute.

**Illustration-29 :** In how many ways can we choose three different numbers from the set  $\{1, 2, 3, 4, 5, 6\}$  such that the three could be the sides of a nondegenerate triangle ?

(Note : The order of the chosen numbers doesn't matter; we consider  $\{3, 4, 5\}$  to be the same as  $\{4, 3, 5\}$ .)

**Solution :** We first notice that if we have three numbers to consider as possible side lengths of a triangle, we only need to make sure that the sum of the smallest two is greater than the third. (Make sure you see why!) We could just start listing all the ones we see that work, but we should take an organized approach to make sure we don't miss any. We can do so by classifying sets of three numbers by the smallest number.

**Case 1 :** Smallest side has length 1. No triangles can be made with three different lengths from our set if we include one of length 1.

**Case 2 :** Smallest side has length 2. The other two sides must be 1 apart, giving the sets  $\{2, 3, 4\}$ ,  $\{2, 4, 5\}$ , and  $\{2, 5, 6\}$ .



**Case 3 :** Smallest side has length 3. There are only three possibilities and they all work :  
 $\{3, 4, 5\}$ ,  $\{3, 4, 6\}$ ,  $\{3, 5, 6\}$ .

**Case 4 :** Smallest side has length 4. The only possibility is  $\{4, 5, 6\}$ , which works.

Adding them all up, we have 7 possibilities.

We sometimes have to use some other tools in addition to triangle inequality.

**Illustration-30 :** Can the lengths of the altitudes of a triangle be in the ratio 2 : 5 : 6 ? Why or why not ?

**Solution :** We don't know anything about how the lengths of the altitudes of a triangle are related to each other. We do, however, know a whole lot about how the lengths of the sides of a triangle are related to each other. Therefore, we turn the problem from one involving altitudes into one involving side lengths. We let the area be  $K$ , let the side lengths be  $a$ ,  $b$ ,  $c$  and the lengths of the altitudes to these sides be  $h_a$ ,  $h_b$ ,  $h_c$ , respectively. Therefore, we have  $K = ah_a/2 = bh_b/2 = ch_c/2$ , so the sides of the triangle have lengths

$$\frac{2K}{h_a}, \frac{2K}{h_b}, \frac{2K}{h_c}.$$

If our heights are in the ratio 2 : 5 : 6, then for some  $x$ , our heights are  $2x$ ,  $5x$  and  $6x$ . Then, our sides are

$$\frac{2K}{2x}, \frac{2K}{5x}, \frac{2K}{6x}.$$

However, the sum of the smallest two sides is then

$$\frac{2K}{5x} + \frac{2K}{6x} = \frac{12K}{30x} + \frac{10K}{30x} = \frac{22K}{30x},$$

which is definitely less than the largest side, which is  $2K/2x = K/x$ . Therefore, the sides don't satisfy the triangle inequality, which means it is impossible to have a triangle with heights in the ratio 2 : 5 : 6.

**Concept :** When facing problems involving lengths of altitudes of a triangle, consider using area as a tool.

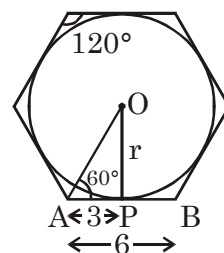
**Illustration-31 :** If a circle is inscribed in regular hexagon of side 6 cm, then area of inscribed circle.

**Solution :** Since we know that sum of angles of polygon =  $(n - 2)180^\circ$

$$\text{Each interior angle of polygon} = \frac{(n - 2)180^\circ}{n}$$

For hexagon  $n = 6$

$$\therefore \text{Each angle} = \frac{(6 - 2)180^\circ}{6} = 120^\circ$$

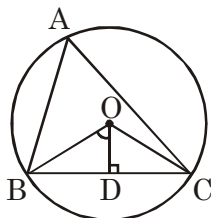


In  $\triangle OPA$

$$\frac{r}{3} = \tan 60^\circ \therefore r = 3\sqrt{3}$$

$$\text{Area of circle} = \pi r^2 = \pi (3\sqrt{3})^2 = 27\pi$$

**Illustration-32 :** In given circle with centre O a triangle ABC is inscribed,  $\angle A = 60^\circ$ . OD is perpendicular to BC, then what is  $\angle BOD$  ?



**Solution :** In  $\triangle OBD$  and  $\triangle ODC$

$OD \perp BC$

$$\therefore \angle ODB = \angle ODC = 90^\circ$$

$OB = OC$  (radius of circle)

$BD = DC$  [ $\perp$  from centre bisects chord]

$$\therefore \triangle OBD \cong \triangle OCD$$

$$\text{Hence } \angle BOD = \angle COD = \frac{1}{2} \angle BOC \quad \dots (1)$$

$$\text{Again } \angle BOC = 2\angle BAC$$

$$\angle BOC = 2[\angle A] \quad \dots (2)$$

From (1) and (2)

$$\angle BOD = \frac{1}{2} (2\angle A) = 60^\circ$$

**Illustration-33 :** In right  $\triangle ABC$  right angled at 'B' (As in figure).  
Prove that distance between circumcentre and centroid

$$GD = \frac{1}{3}(BD) = \frac{1}{3}\left(\frac{1}{2}AC\right). \text{ If } AC = 10, \text{ then find } GD$$

**Solution. :** **Property :** We know that mid-point of hypotenuse is equidistance from all the three vertices of triangle

$$\therefore AD = CD = BD$$

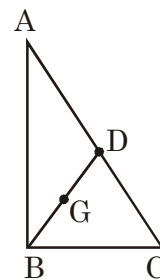
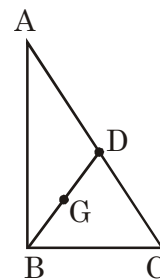
$$\therefore BD = \frac{1}{2} AC$$

$$BD = \frac{1}{2} (10) = 5$$

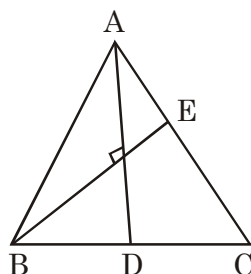
Again centroid divides median in ratio 2 : 1

$$\therefore GD = \frac{1}{3} BD = \frac{5}{3}$$

[D is circumcentre of  $\triangle ABC$  since mid-point of hypotenuse is circumcentre of triangle]



**Illustration-34 :** In  $\triangle ABC$ , AD and BE are medians such that they intersect at right angles. If  $AD = 6$  and  $BE = \frac{9}{2}$  then area of  $\triangle ABC$  is



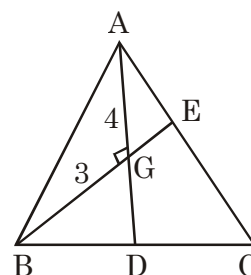
**Solution. :** Let G be the centroid of  $\triangle ABC$  i.e. point of intersection of medians AD and BE

$$\text{Now } AG = \frac{2}{3}AD = \frac{2}{3}[6] = 4$$

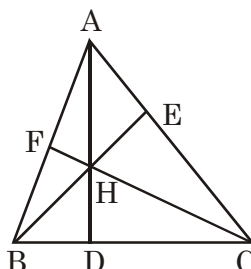
$$BG = \frac{2}{3}BE = \frac{2}{3} \times \frac{9}{2} = 3$$

$$\text{Thus Area of } \triangle AGB \text{ is } \frac{1}{2} \times 3 \times 4 = 6 \text{ unit}^2$$

$$\begin{aligned} \text{Now we know that area of } \triangle ABC \\ = 3 \times \text{Area}(\triangle AGB) = 3(6) = 18 \text{ unit}^2 \end{aligned}$$



**Illustration-35 :** In  $\triangle ABC$ , AD, BE and CF are altitudes and H is orthocentre. If  $\angle A = 40^\circ$  then find  $\angle FDE$ .



**Solution. :** It is very clear that  $\angle HFB = 90^\circ$  and  $\angle HDB = 90^\circ$

(Since CF and AD are altitudes)

Thus for quadrilateral HFBD

sum of opposite angle is  $180^\circ$

Hence it is cyclic

In  $\triangle AEB$

$$\angle ABE = 90^\circ - A = \angle FBH$$

and using the property of circle

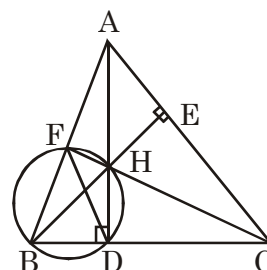
$$\angle FBH = \angle FDH$$

$$\therefore \angle FDH = 90^\circ - A$$

$$\text{Similarly } \angle HDE = 90^\circ - A$$

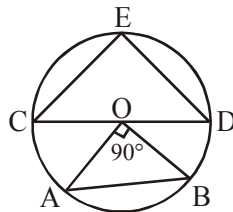
$$\therefore \angle FDE = 180^\circ - 2A$$

$$\angle FDE = 180^\circ - 2(40^\circ) = 100^\circ$$

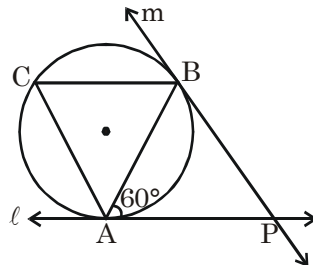


## EXERCISE

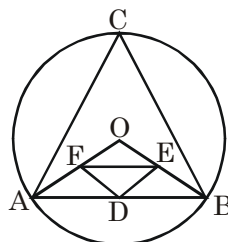
1. In the diagram 'O' is the centre of the circle. The point E lies on the circumference of the circle such that the area of the  $\triangle ECD$  is maximum. If  $\angle AOB$  is a right angle, then the ratio of the area of  $\triangle ECD$  to the area of the  $\triangle AOB$  is :



- (A) 2 : 1                      (B) 3 : 2                      (C) 4 : 3                      (D) 4 : 1
2. Let  $\triangle XOY$  be a right angled triangle with  $\angle XOY = 90^\circ$ . Let M and N be the midpoints of legs OX and OY, respectively. Given that  $XN = 19$  and  $YM = 22$ , the length XY is equal to
- (A) 24                      (B) 26                      (C) 28                      (D) 34
3. In the diagram below, if  $\ell$  and m are two tangents and AB is a chord making an angle of  $60^\circ$  with the tangent  $\ell$ , then the angle between  $\ell$  and m is

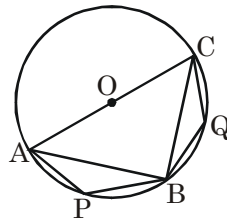


- (A)  $45^\circ$                       (B)  $30^\circ$                       (C)  $60^\circ$                       (D)  $90^\circ$
4. In the diagram, O is the centre of the circle and D, E and F are mid points of AB, BO and OA respectively. If  $\angle DEF = 30^\circ$ , then  $\angle ACB$  is

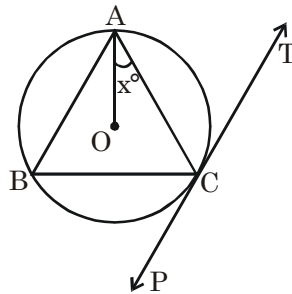


- (A)  $30^\circ$                       (B)  $60^\circ$                       (C)  $90^\circ$                       (D)  $120^\circ$

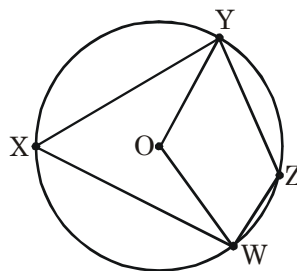
5. In the below diagram, O is the centre of the circle, AC is the diameter and if  $\angle APB = 120^\circ$ , then  $\angle BQC$  is



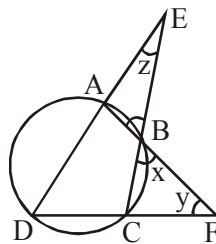
- (A)  $30^\circ$  (B)  $150^\circ$  (C)  $90^\circ$  (D)  $120^\circ$
6. In the adjoining figure, PT is a tangent at point C of the circle. O is the circumcentre of  $\triangle ABC$ . If  $\angle ACP = 118^\circ$ , then the measure of  $\angle x$  is



- (A)  $28^\circ$  (B)  $32^\circ$  (C)  $42^\circ$  (D)  $38^\circ$
7. In the cyclic quadrilateral WXYZ on the circle centered at O,  $\angle ZYW = 10^\circ$  and  $\angle YOW = 100^\circ$ . What is the measure of  $\angle YWZ$  ?

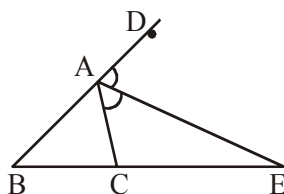


- (A)  $30^\circ$  (B)  $40^\circ$  (C)  $50^\circ$  (D)  $60^\circ$
8. In the adjacent figure, if  $\angle y + \angle z = 100^\circ$  then the measure of  $\angle x$  is :

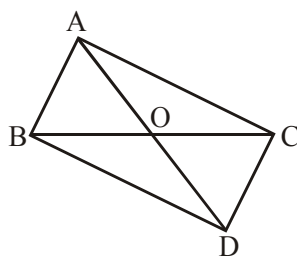


- (A)  $50^\circ$  (B)  $40^\circ$  (C)  $45^\circ$  (D) Cannot be determined

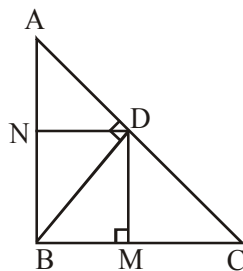
9. In a  $\triangle ABC$  if  $P$  be the point of intersection of the interior angle bisectors and  $Q$  be the point of intersection of the exterior angle bisectors of angles  $B$  and  $C$  respectively, then the figure  $BPCQ$  is a :  
 (A) Parallelogram (B) Rhombus (C) Trapezium (D) Cyclic quadrilateral
10.  $D$ ,  $E$  and  $F$  are points on sides  $BC$ ,  $CA$  and  $AB$  respectively of  $\triangle ABC$  such that  $AD$  bisects  $\angle A$ ,  $BE$  bisects  $\angle B$  and  $CF$  bisects  $\angle C$ . If  $AB = 5$  cm,  $BC = 8$  cm and  $CA = 4$  cm, determine  $AF$ ,  $CE$  and  $BD$ .
11. In the adjoining figure,  $AE$  is the bisector of exterior  $\angle CAD$  meeting  $BC$  produced in  $E$ . If  $AB = 10$  cm,  $AC = 6$  cm and  $BC = 12$  cm, find  $CE$ .



12. In fig.  $\triangle ABC$  and  $\triangle DBC$  are two triangles on the same base  $BC$ . Prove that  $\frac{\text{ar}(\triangle ABC)}{\text{ar}(\triangle DBC)} = \frac{AO}{DO}$ .

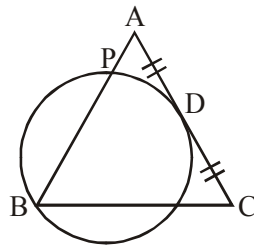


13.  $ABC$  is a right-triangle with  $\angle ABC = 90^\circ$ ,  $BD \perp AC$ ,  $DM \perp BC$  and  $DN \perp AB$ . Prove that  
 (i)  $DM^2 = DN \times MC$   
 (ii)  $DN^2 = DM \times AN$

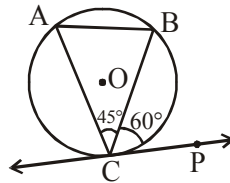


14. Prove that sum of squares of diagonals of a parallelogram is equal to sum of squares of its sides.

15. In fig., ABC is a triangle in which  $AB = AC$ . A circle through B touches AC at D and intersects AB at P. If D is the mid-point of AC, show that  $4 AP = AB$ .



16. In fig., CP is a tangent to a circle. If  $\angle PCB = 60^\circ$  and  $\angle BCA = 45^\circ$ , find  $\angle ABC$ .



17. The inscribed circle of  $\triangle ABC$  touches BC, CA and AB at X, Y and Z respectively. If  $\angle A = 64^\circ$ ,  $\angle C = 52^\circ$ , find  $\angle XYZ$  and  $\angle XZY$ .
18. Let  $\overline{AM}$  be a median of  $\triangle ABC$ . Prove that  $AM > (AB + AC - BC)/2$ .

# ANSWER KEY

- |                  |                             |   |   |      |      |
|------------------|-----------------------------|---|---|------|------|
| 1. A             | 2. B                        | 3. C  | 4. B  | 5. B | 6. A |
| 7. B             | 8. B                        | 9. D  | 10. $AF = \frac{5}{3}$ , $CE = \frac{32}{13}$ , $BD = \frac{40}{9}$ |      |      |
| 11. $CE = 18$ cm | 16. $\angle ABC = 75^\circ$ | 17. $\angle XYZ = 58^\circ$ , $\angle XZY = 64^\circ$ |   |      |      |