The Value Of Π

Mathematics Extended Essay

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International Baccalaureate Diploma Program

Research question - Which infinite series is the most efficient in calculating the value of π? (Efficiency will be compared with the aid of number of accurate decimal places, the run-time calculation and the graphical convergence)

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1 Introduction

1.1 Background and personal connection

The value of π has been a topic of debate for centuries. *pi* is an irrational constant, meaning it cannot be expressed as a ratio between any two real numbers a and b. Tere have been many world records related to it, like the most decimal places memorized, or the most decimal places ever calculated. This calculation of π is done on computers by using formulas, some of which I will also use in this essay.

π is explained as the ratio between the Circumference and Diameter of any circle. Circumference is the perimeter or the length of the outline of the circle, and the Diameter is a straight line inside the circle, which passes through the center of the circle.

The first known calculation of π was done way back in the 200 BC, by Archimedes. Moreover, the Greek letter and the most common symbol for Pi (π) was first used in the 1700's. This Greek letter was first used by William Jones in 1706 and brought into fame by Leonard Euler in 1737.

In this essay the value of pie will be found by different formulas and at different rates. Hence, the research will be based on which formula gives the value of π, to the most accurate decimal places, and in the least number of terms used. This makes the test fair and using the code, the results can be found efficiently.

I got introduced to π back in 8th grade, and it was explained as an irrational constant which mathematicians consider the biggest superstar of modern day mathematics. We started learning some basic trigonometry and the common functions like sine, cosine and tangent came up. My teacher used the Unit circle as well to really show what these values mean. The application of having accurate value of π are crucial and will be discussed later in this essay.

I was keen to this essay even being an SL math student, because it has always interested since a very young age. I am deliberately not using the Ramanujan formula and Chudnovsky's formula, because they're way too fast and will be incomparable with the series that I have.

1.2 Attempts of calculating π in the past.

The oldest known calculation of π was done between 1900 and 1680 BC, by Babylonians and the approximate value found was 3.215. Then came the Egyptians who calculated the value of π using a formula that gave value 3.1605. After that came Archimedes and made the first most accurate at the time approximation of π. He was using 1 circle which is circumscribed and inscribed inside other Polygons, where the more the number of sides of the polygon, the closer the estimated value to the real value. He essentially found an upper and lower bound to the value of π, not the exact value. Upper bound was 3 10/71 and the lower bound was 3 1/7. These values are extremely close to each other and only goes to show that the polygons Archimedes was using had a big number of sides.

After that during the 16th-19th centuries, many formulas to calculate π were known. A few examples are also the series made by Leibniz and Viete. Many other mathematicians calculated by hand the decimal places of π and were breaking records for accuracy. For example - Ludolph Van Ceulen who calculated the first 35 decimal places by using the traditional Archimedes method and used a 2^{62}-gon polygons to calculate the area.

In the modern day we have realized that the decimal places of π can be calculated a lot faster using electronic signals and hence the value of π is being calculated by computers which use the formulas mentioned later in the essay to calculate the value to trillions of digits and 100 percent accuracy. These values are calculated in milliseconds. Since it is irrational, the decimal places are purely random and do repeat occasionally, although out of randomness and not because they are meant to.

The current world record of calculating the most number of decimal places is held by Emma Haruko Iwao, who managed to compute it to 31 trillion digits. it's said that she used 25 virtual machines and 121 days of work to calculate the decimal places to that extent. Another interesting fact about the number is that assuming a person could say 1 decimal place in one second it would take them more than 330,000 years to get to the 31st trillionth digit.

1.3 The transcendence of π

Transcendental essentially refers to how π isn't algebraic, which means that it can't be reduced to zero using regular algebraic operations. Even all the complex or imaginary numbers can be reduced to zero using just the regular algebraic operations, however, transcendental numbers is a different category all together. Other transcendental numbers include e, $\sqrt{2}$, $\sqrt{3}$ etc.

Transcendental numbers can still be computed and that's what this research is based on. Π is one of the most famous transcendental numbers that we know of and it's value only gets better and better with more the number of terms that we use, so it is computable. We can use technology to do the coding and get the value of π to any desired accuracy. This also means that as technology gets better and better, and soon with the invention of supercomputers we will be able to calculate the actual decimal places of π to an even higher accuracy, possibly something we've never reached before.

As of now we only know about a finite number of transcendental numbers, the list does have an end. For comparison something like the number of even numbers there are, doesn't have an end. That list is endless, because there are infinitely many even numbers,

1.4 Important note about the graphs

The serial number mentioned in the graphs in this essay is the n value in the infinite series. The serial number is essentially just the series of natural numbers and is coded just by adding 1 to the previous number in the series.

2 Leibniz series

2.1 The derivation of Leibniz series

A set of numbers is a sequence of numbers, however if they are paired with + signs between them, it becomes a series. The proof starts from a convergent series. convergent series is a kind of series whose infinite sum approaches a constant or a particular value.

The equation for this is :

$$S = 1 + a + a^2 + a^3 + a^4 \dots$$

Let's multiply both sides by a :

$$aS = a + a^2 + a^3 + a^4 \;.....$$

If we now calculate the difference between them we get :

$$S - aS = 1$$

Taking S common out of the bracket we get :

$$S\ (\ 1 - a\ ) = 1$$

Now to make S the subject of the formula we get :

$$S = \frac{1}{(1-a)}$$

To compare this with the first equation, from the infinite series we get :

$$\frac{1}{(1-a)} = 1 + a + a^2 + a^3 \;.....$$

Using fundamental theorem of calculus and a lot of back calculation we come to a situation where we can replace the a value with -t^2. We get :

$$\frac{1}{(1+t)^2} = 1 - t^2 + t^4 - t^6 \;.....$$

Now we integrate both sides with respect to t :

$$\int_0^x \frac{1}{(1+t)^2}\, dt = \int_0^x (1 - t^2 + t^4 - t^6 \;...)dt$$

Let t be equal to $\arctan(\theta)$. We integrate the $\arctan(\theta)$ term now. We would need to prove that the derivative function for the arctan is actually equal to:

$$\frac{1}{1+t^2} \tag{1}$$

Let's assume that arctan of some x is equal to y. This gives us:

$$arctan(x) = y \tag{2}$$

Thie means that x = tan(y)
From this equation we get:

$$\frac{dx}{dy} = sec^2(y) \tag{3}$$

Using the pythagoream identity we know that $sec^2(y) = 1 + tan^2(y)$

Earlier we said that x = tan(y), this means that $tan^2(y)$ is just x^2

From here we put $\frac{dy}{dx} = \frac{1}{1+x^2}$ This is just the reciprocal of the equation above.

Now we substitute x for in the equation $-$t^2

$$arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Let's substitute 1 as x

$$arctan(1) = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + ...$$

We know from the unit circle that the value of $Tan^{-1} = \frac{\pi}{4}$. So we can say that :

$$\frac{\pi}{4} = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + ...$$

Taking 4 on the other side of the equation leaving only π as the subject of the formula gives us:

$$\pi = 4(1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + ...)$$

We know that 1 raised to any positive whole number power always remains 1. Hence we can write it without the power on the numerator as well. This gives us :

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...)$$

The code that was used to perform this function in Excel can be seen below. Programming the Leibniz series on google sheets, which works exactly like Microsoft Excel, was straight forward, but it took me some time to come up with the code.

For this I used 4 columns, and they were all separate in their functions. Hence, four pictures of each of the columns are shown below.

2.2 The excel sheets for the Leibniz series code

`=A2+1`

A
S No
0
1
2
3
4
5
6
7
8

`=(((-1)^A3)/(2*A3+1))`

A	B
S No	Term
0	1
1	-0.3333333333
2	0.2
3	-0.1428571429
4	0.1111111111
5	-0.09090909091
6	0.07692307692
7	-0.06666666667
8	0.05882352941

`=B3+C2`

A	B	C
S No	Term	Pi/4
0	1	1
1	-0.3333333333	0.6666666667
2	0.2	0.8666666667
3	-0.1428571429	0.7238095238
4	0.1111111111	0.8349206349
5	-0.09090909091	0.744011544
6	0.07692307692	0.8209346209
7	-0.06666666667	0.7542679543
8	0.05882352941	0.8130914837

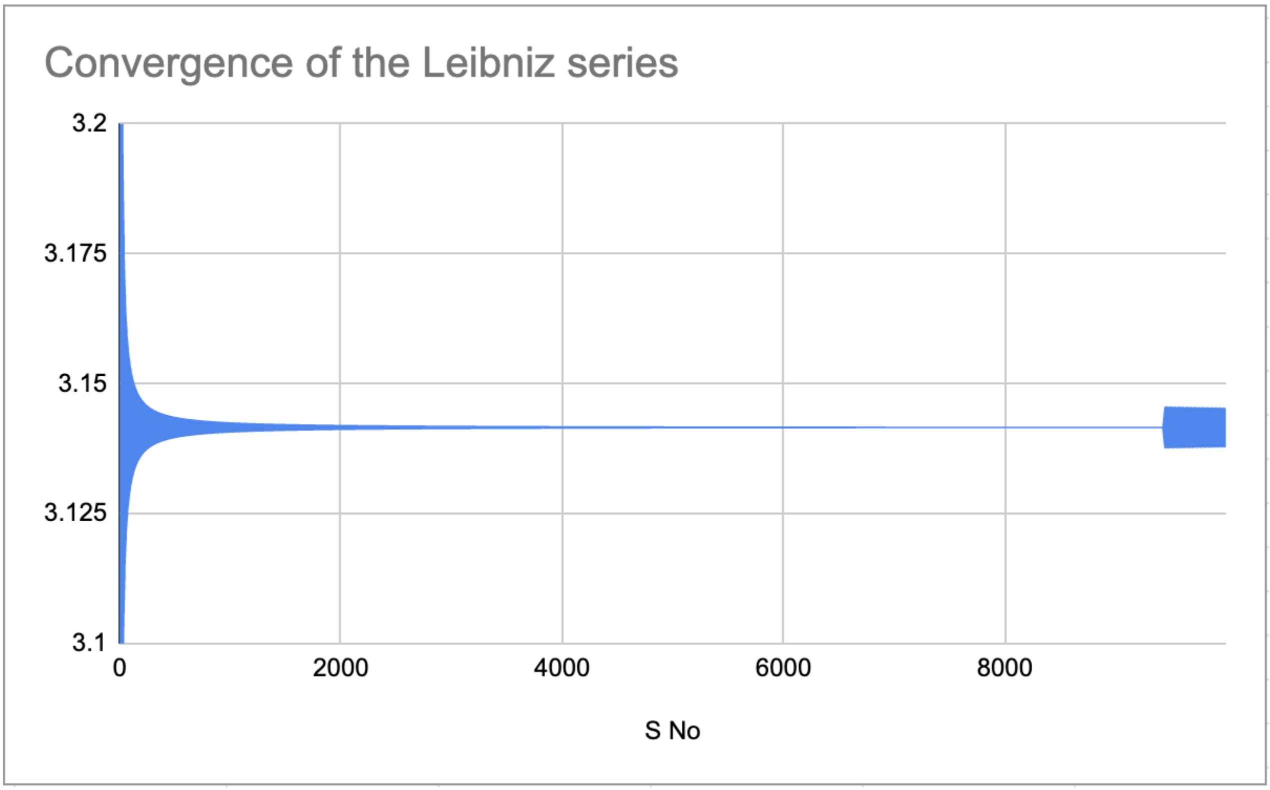
`=C3*4`

A	B	C	D
S No	Term	Pi/4	Pi
0	1	1	4
1	-0.3333333333	0.6666666667	2.666666667
2	0.2	0.8666666667	3.466666667
3	-0.1428571429	0.7238095238	2.895238095
4	0.1111111111	0.8349206349	3.33968254
5	-0.09090909091	0.744011544	2.976046176
6	0.07692307692	0.8209346209	3.283738484
7	-0.06666666667	0.7542679543	3.017071817
8	0.05882352941	0.8130914837	3.252365935

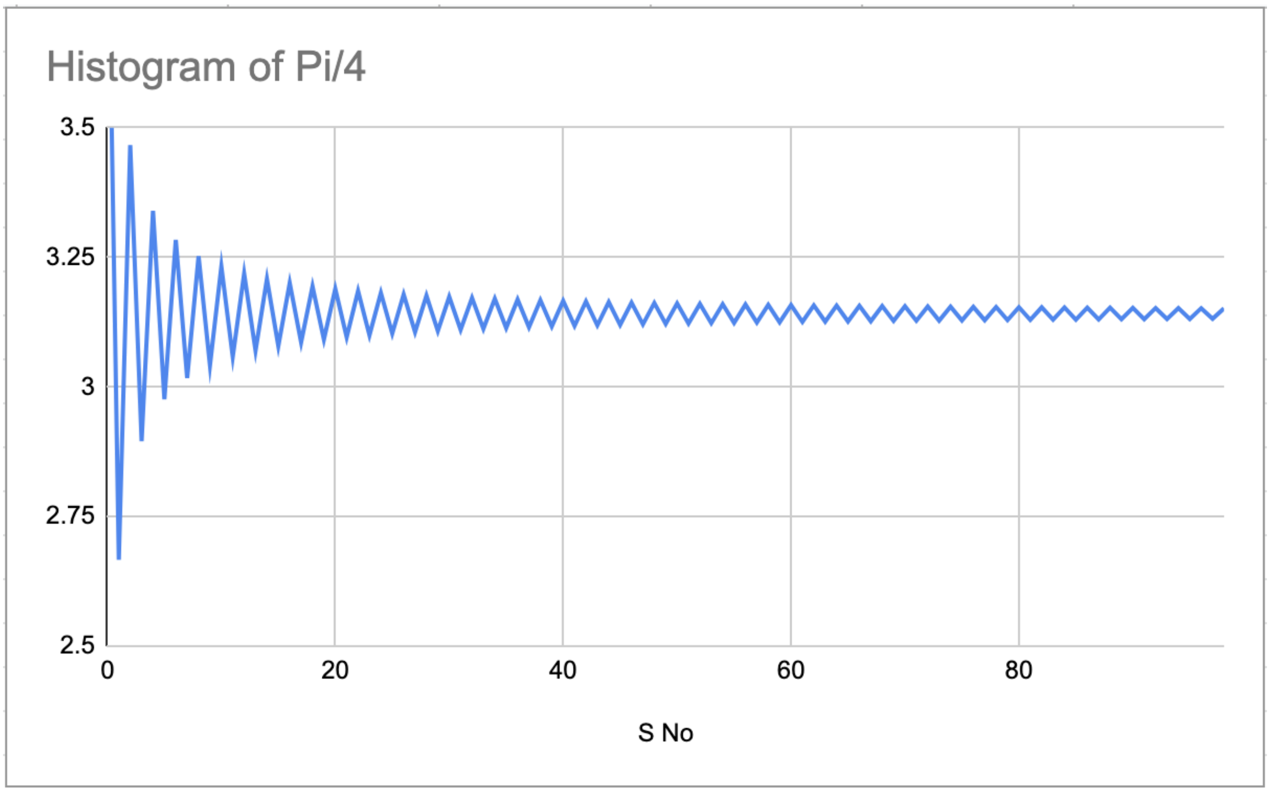
To describe the code we can just say that each of the column performs a distinct function of the Leibniz formula for π. A column increases the n value, the serial number by 1 each cell. B column is the actual function, the alternating (-) and (+) signs are due to the powers of (-1). This alternating sign performs the function of adding and subtracting the values of the odd fractions each time. C column takes the previous C and the corresponding B cells and adds them. this ensures that the terms are added, even with the (-) sign in which case it'll just mean that the term is subtracted. The last column, takes the corresponding C value and multiplies it by 4. This is the last step in the formula (subsection Leibniz series).

There are two graphs, 10,000 terms and 100 terms. The zoomed in graph allows us to see the convergence in more detail and hence we can easily compare the growth and the convergence.

2.3 The graph to show the convergence through the Leibniz series all 10,000 terms



2.4 The graph showing convergence through the Leibniz series, first 100 terms



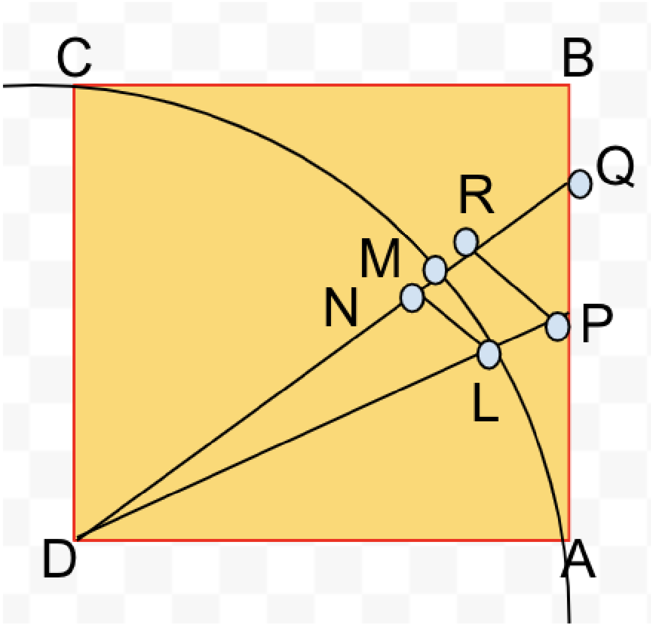
3 Nilakantha series.

3.1 The derivation for Nilkantha series

This is a diagrammatic proof for the Nilkantha series and how he showed the value of π, using this diagram. As we can see in the diagram below, we have a quarter of a circle inside a square. We can call the quarter of the circle inscribed inside the square.

It is important to understand the Leibniz proof for the value of π before this because this eventually gets us to that equation. So, once the proof starts we can see that the equations, found by Nilkantha, lead us to the Leibniz series. For this reason it is also sometimes called the Nilkantha-Leibniz series for π.

We start the proof with



In this image we have a square ABCD, with side length 1. Additionally, the quarter circle we can see has a unit radius. This would straight away imply that the length of the arc is $\pi/2$. We can derive the length of the arc in many different ways, including using the formula for the length of the arc = 2πr $\theta/$ 360.

Now we draw two points on the side AB, let's call them P and Q, such that they appear from A to B as seen in the diagram. It is of paramount importance to note at this stage that the distance between P and Q is negligible, i.e. $PQ << 1$. Let's draw the lines connecting them to the origin now. We can call point D the origin of the circle, and as the diagram shows the lines DP and DQ meet at L and M respectively. This now means that triangles DLN and DPR are similar, and triangles QPR and QAD are also similar.

We can express this mathematically also -

$$\Delta DLN \sim DPR \text{ and } \Delta QPR \sim QAD$$

We can now write the equations of the ratios between the sides of these similar triangles.

$$LN/DL = PR/DP \text{ and } PR/PQ = DA/DQ$$

We know that the radius od the circle is 1. Therefore it can be said that -

$$DL = DA = 1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8...$$

Substituting the value of 1 in the ratios above gives us -

$$\frac{LN}{1} = \frac{PR}{DP} \text{ and } \frac{PR}{PQ} = \frac{1}{DQ}$$

Rearranging for the line PR we get

$$PR = PQ/DQ$$

Which then follows

$$\frac{LN}{1} = \frac{PQ}{DP.DQ}$$

Using Pythagoras theorem we can say that

$$DP.DQ \approx DQ^2 = 1 + AQ^2$$

Since we know that the distance between P and Q is negligible we can say that the arc length LM is approximately equal to the line LN.

This gives us

$$LM \approx \frac{PQ}{1/AQ^2}$$

Since we know that the angle made by the line DB to the line DA is 45 degrees, we know that it is equal to $\frac{\pi}{4}$ radians. Before we move on let's first divide the line AB into n parts each of length $\frac{1}{n}$. In this scenario the value of n Is large, and that will be helpful soon. Let the points on the side AB be $P_1, P_2, P_3...P_{n-1}, P_n$. This means that for all m such that m = 1, 2, 3 ... n-1, we have $P_m P_{m-1} = \frac{1}{n}$. If we now sum the same relation n times, we get

$$\pi/4 \approx \sum_{m=1}^{n} \frac{1/n}{1+m^2/n^2}$$

We can manipulate the equation slightly to get

$$\pi/4 = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{1/n}{1+m^2/n^2}$$

Since the integral has been discretized now, which refers to non-continuous, we can link it back to the integral in the Leibniz series. The link to the Leibniz series is just to prove that this series is in fact proven to be equal to the value of π. We write the fraction $\frac{1/n}{1+m^2/n^2}$ as an infinite series using the fact that $\frac{1}{(1+x)^2} = 1 - x^2 + x^4 - x^6 + x^8 - ....$ It is also important to note at this stage that this works only for the absolute value of x less than 1, and we do this so as to evaluate the limit.

This gives us the equation

$$\frac{1/n}{1+m^2/n^2} = \frac{1}{n}(1 - \frac{m^2}{n^2} + \frac{m^4}{n^4} - \frac{m^6}{n^6} + ...)$$

Therefore

$$\frac{\pi}{4} = \lim_{n\to\infty} \frac{1}{n}(\sum_{m=1}^{n} 1 - \sum_{m=1}^{n} \frac{m^2}{n^2} + \sum_{m=1}^{n} \frac{m^4}{n^4} - \sum_{m=1}^{n} \frac{m^6}{n^6} + ...)$$

$$= \lim_{n\to\infty} (1 - \frac{\sum m^2}{n^3} + \frac{\sum m^4}{n^5} - \frac{\sum m^6}{n^7} + ...)$$

Every summation in this equation is from m=1 to m=n. Since we know that $\sum_{m=1}^{n} m = \frac{n(n+1)}{2}$ and this leads to the $\sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}$ which logically follows $\sum_{m=1}^{n} m^3 = (\frac{n(n+1)}{2})^2$. This can keep going on and on however, we are only interested in the second term. It is known that as $n \to \infty$ the value of $\frac{1}{n} \to 0$. This removes the summation sign and we get the following

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{(6n^3)}$$

We can take a constant of $\frac{1}{3}$ out of the limits and modify this

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{(6n^3)} = \frac{1}{3} + \lim_{n\to\infty} (\frac{1}{2n} \frac{1}{6n^2})$$

Since $n \to \infty$ the limit goes to 0, which leaves us with the $\frac{1}{3}$. This means that the second term as evaluated using the limits gives us $\frac{1}{3}$. This also happens to be the second term in the Leibniz series and as we go on, evaluating all the terms in this sequence we keep getting terms for the Leibniz series. This ends up with

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} ...$$

This is the Leibniz series and since this proof had been done by Nilkantha it is also sometimes called the Gregory-Nilkantha series. It is important to notice that Nilkantha's series is still

$$\pi/4 = \lim_{n\to\infty} \sum_{m=1}^{n} \frac{1/n}{1+m^2/n^2}$$

However, to prove that this is true he connected it with the Leibniz series as it already had the term $\frac{\pi}{4}$ proven to be equal to the alternating sum of all the odd fractions.

The code in Excel for this can be seen below. Nilkantha's series was by far the hardest to code and it took me a long time to be able to do it properly. I have used 8 columns for this and divided the functions equally. It is however, possible to merge some of the functions together and obtain the result more efficiently.

In the proof we got to know that this series ends up being equal to Leibniz series, which isn't true. That was just a method to prove that series, and how after we get to the Leibniz series that the series has been proved. The series we need to code is :

$$\pi = 3 + (4/(2.3.4) - (4/(4.5.6) + (4/(6.7.8)...)))$$

In this we can observe that 4 is a common denominator and we can take common factor of 4 outside the brackets. Which will give us -

$$\pi = 3 + 4(1/(2.3.4) - (1/(4.5.6) + (1/(6.7.8)...)))$$

The columns and their respective codes can be seen below.

3.2 The excel sheets to show the Nilkantha series code

=A2+1

A
1
2
3
4
5
6
7
8
9
10

=B2*-1

A	B
1	1
2	-1
3	1
4	-1
5	1
6	-1
7	1
8	-1
9	1
10	-1

=2*A2

Search documents and filenames fo

A		
1	1	2
2	-1	4
3	1	6
4	-1	8
5	1	10
6	-1	12
7	1	14
8	-1	16
9	1	18
10	-1	20

=C2+1

A	B	C	D
1	1	2	3
2	-1	4	5
3	1	6	7
4	-1	8	9
5	1	10	11
6	-1	12	13
7	1	14	15
8	-1	16	17
9	1	18	19
10	-1	20	21

fx =D3+1

	A	B	C	D	E
1					
2	1	1	2	3	4
3	2	-1	4	5	6
4	3	1	6	7	8
5	4	-1	8	9	10
6	5	1	10	11	12
7	6	-1	12	13	14
8	7	1	14	15	16
9	8	-1	16	17	18
10	9	1	18	19	20

fx =C2*D2*E2

	A	B	C	D	E	F
1						
2	1	1	2	3	4	24
3	2	-1	4	5	6	120
4	3	1	6	7	8	336
5	4	-1	8	9	10	720
6	5	1	10	11	12	1320
7	6	-1	12	13	14	2184
8	7	1	14	15	16	3360
9	8	-1	16	17	18	4896
10	9	1	18	19	20	6840

= 4*(B2/F2)

A	B	C	D	E	F	G
1	1	2	3	4	24	0.1666666667
2	-1	4	5	6	120	-0.03333333333
3	1	6	7	8	336	0.0119047619
4	-1	8	9	10	720	-0.00555555555
5	1	10	11	12	1320	0.00303030303
6	-1	12	13	14	2184	-0.00183150183
7	1	14	15	16	3360	0.00119047619
8	-1	16	17	18	4896	-0.00081699346
9	1	18	19	20	6840	0.000584795321
10	-1	20	21	22	9240	-0.00043290043

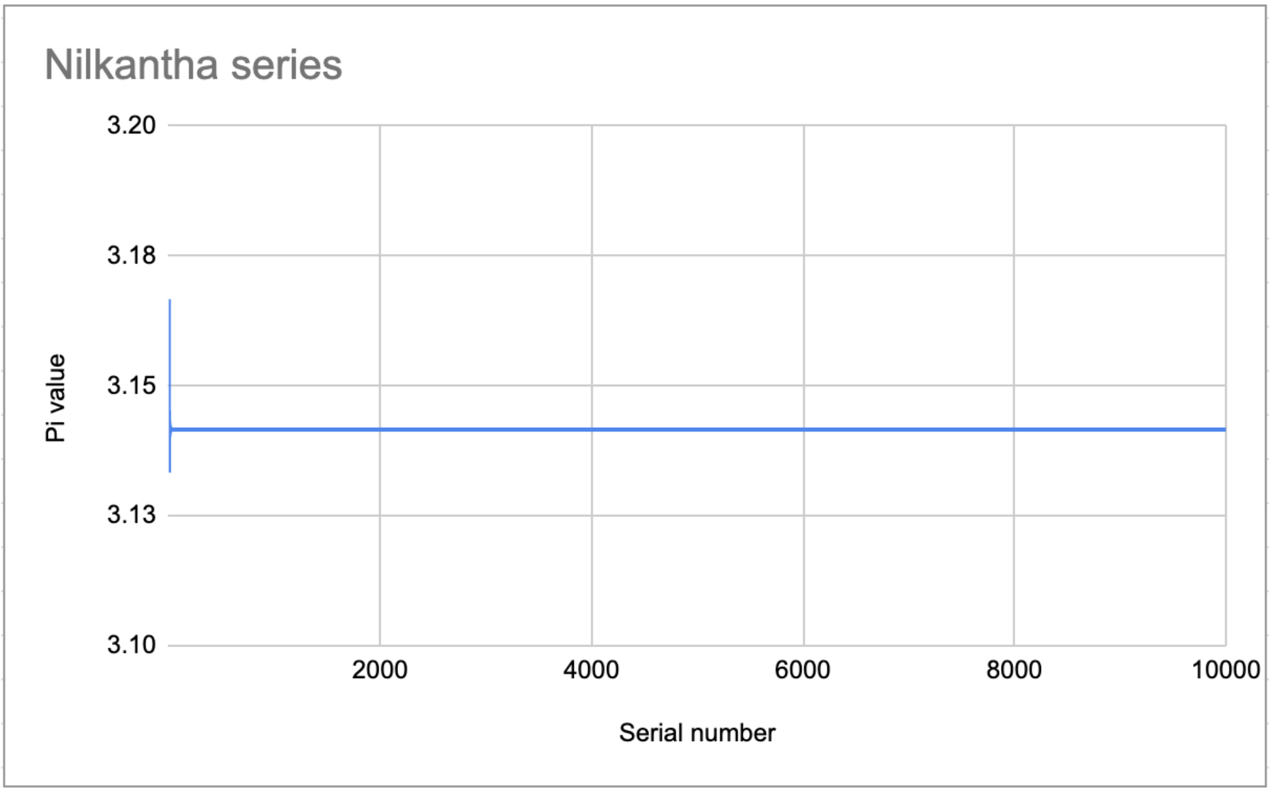
=H1+G2

A	B	C	D	E	F	G	H
							3.00000000000000000000
1	1	2	3	4	24	0.1666666667	3.16666666666667000000
2	-1	4	5	6	120	-0.03333333333	3.13333333333333000000
3	1	6	7	8	336	0.0119047619	3.14523809523810000000
4	-1	8	9	10	720	-0.00555555555	3.13968253968254000000
5	1	10	11	12	1320	0.00303030303	3.14271284271284000000
6	-1	12	13	14	2184	-0.00183150183	3.14088134088134000000
7	1	14	15	16	3360	0.00119047619	3.14207181707182000000
8	-1	16	17	18	4896	-0.00081699346	3.14125482360776000000
9	1	18	19	20	6840	0.000584795321	3.14183961892940000000
10	-1	20	21	22	9240	-0.00043290043	3.14140671849650000000

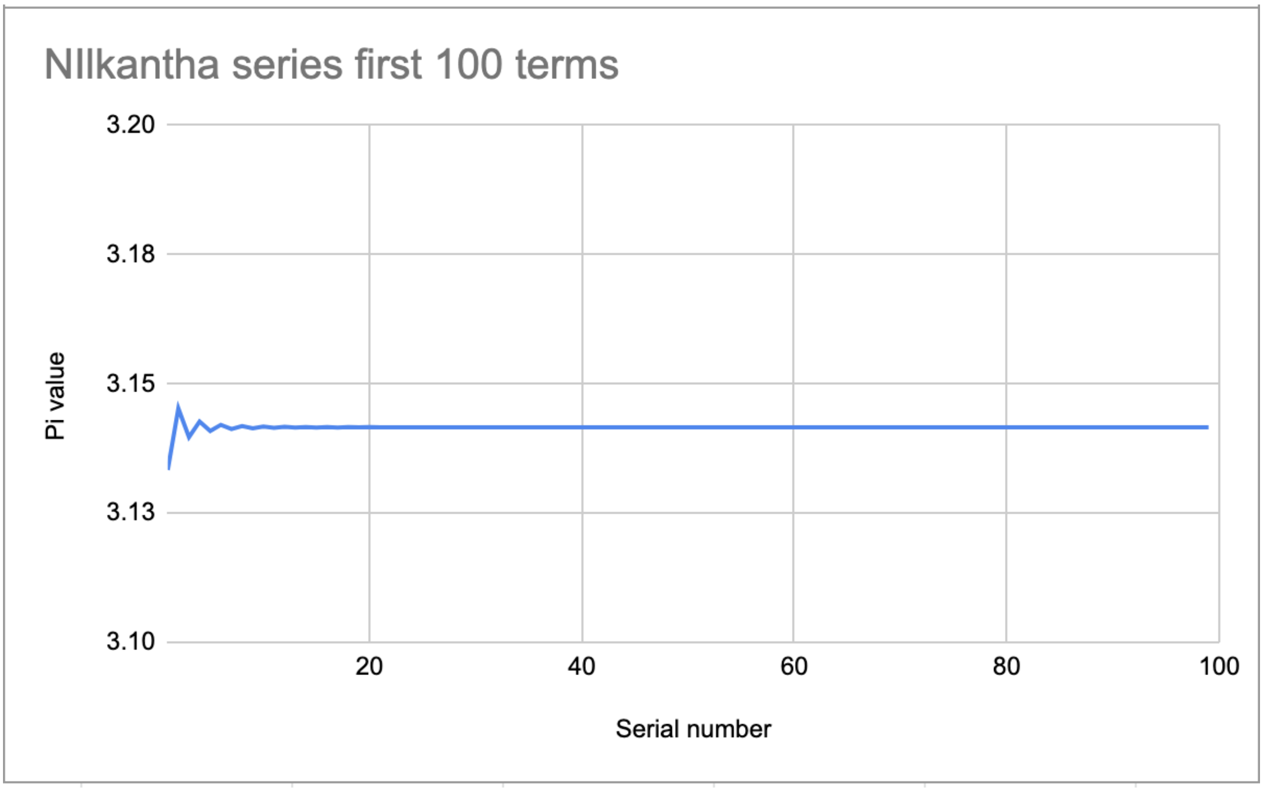
About the description of the code Let's start with A column whose function is the same as the A column in the Leibniz series and that is to increase the serial number by 1. Th B column has the job of creating an alternative series of (+) and (-) 1's so that it'll be used in the formula later on. We can see the effect of that as the terms are alternatively added. The C column doubles the serial number which is necessary for starting the denominator part of the term. We know that the denominator is the product between the 3 consecutive numbers which start from double the serial number value. This is helpful in defining the code for the next line, which we will see later. Column D and E define the consecutive numbers that come after the first number in the denominator part of the term. F column performs the multiplication between the 3 consecutive numbers. G column has the function of putting 4 as the numerator value so that the term becomes 4/(2.3.4) for example. The H cell does the final addition of 3 in each term, giving the π value on the other side.

The graphs to see the convergence of the series can be seen below.

3.3 The graph showing the convergence through Nilkantha series, 10,000 terms



3.4 The graph showing the convergence through Nilkantha series, first 100 terms



4 Euler's series

4.1 The derivation of Euler's series

Euler's formula for Pi was at the time the most advanced and controversial formula. This was mainly due to the fact that there was no involvement of any sort of circles in the proof, which meant that nobody could see the proof themselves. He starts off by proving that the value of the series he claims to be equal to $\frac{\pi^2}{6}$ converges to a finite value.

The series that Euler came up with is the Zeta function for 2. Since we know that the Zeta function for any x is -

$$\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x} + \frac{1}{6^x} + ...$$

This gives us the Zeta function for 2 which is directly -

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ...$$

Euler managed to prove that this series is equal to $\frac{\pi^2}{6}$

We can write this as

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{6}$$

We can rewrite this as in a slightly expanded form like this.

$$\zeta(2) = \frac{1}{1}\frac{1}{1} + \frac{1}{2}\frac{1}{2} + \frac{1}{3}\frac{1}{3} + \frac{1}{4}\frac{1}{4} + \frac{1}{5}\frac{1}{5} + ...$$

Since Euler was trying to prove that this series converge to a finite value he made a new series from this one by shifting one part of the squared terms to the next term, infinitely many times. This gives a new equation which is

$$\frac{1}{1} + \frac{1}{1}\frac{1}{2} + \frac{1}{2}\frac{1}{3} + \frac{1}{3}\frac{1}{4} + \frac{1}{4}\frac{1}{5} + ...$$

Since the sum of the partial sums in this series are bigger than the partial sums in the previous series we now know for sure that this series has a bigger value than the previous one. Now to prove the convergence Euler had an ingenious idea and showed that a fraction multiplied by another equals the absolute value of the difference between the two. Using this logic we can rewrite the equation above as

$$\frac{1}{1} + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + ...$$

As we can see that if the same term is added and subtracted it gets canceled or equals to an addition of 0.

The terms that cancel out are shown in brackets in the step below

$$\frac{1}{1} + \frac{1}{1}(-\frac{1}{2} + \frac{1}{2}) + (-\frac{1}{3} + \frac{1}{3}) + (-\frac{1}{4} + \frac{1}{4}) + ...$$

This gives us a total of 2 and a fractional value that keeps decreasing as the terms increase. Therefore, at infinity the fractional value will be 0 and the sum will be equal to 2. From this step Euler deduced that the value for the convergence of his series equals a finite number smaller than 2.

Another way of looking at this is simply calculating the value of $\frac{\pi^2}{6}$. Using a calculator this can be easily done and we observe that it equals to 1.645. This further hints that Euler's claim about the zeta function of 2 was in fact correct.

Before we actually start the proof it is important to know that the equation Euler obtained for the zeta function of 2 is just a part of a big family of such series. In fact there are infinitely many members in this family. Some of them are listed below.

$$\frac{\pi^4}{90} = \zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + ...$$
$$\frac{\pi^6}{945} = \zeta(6) = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + ...$$

In this process Euler is essentially just trying to map a polynomial on to the sin(x) graph. For this, his process was essentially just trying come up with a polynomial equation which maps the closest to the graph of sin(x). It is also important to mention that he did it section by section for the sin(x) graph which means that the equation he obtained would be mapped onto the sin(x) graph at infinity. Euler knew the Mclauren series for sin(x) beforehand and that meant that he had to come up with some sort of an equation which would look similar to the Mclauren series for sin(x). The Mclauren series for the sin(x) looks something like this

$$sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}...$$

The first section he took was the part of the sin curve that looks like a cubic curve, meaning that the zeros of this section or the x - coordinates for this section are (π, 0, -π). Just by looking at the graph it is easy to manipulate the cubic curve in such a way that it comes closest to the sin curve. The equation $(\pi - x)x(\pi + x)$ already has the same three zeros. From here he needed to find the constant outside this function that would come very close to the sin curve but not exactly. He used graphing methods and I used the website desmos to find out that this constant is exactly $\dfrac{1}{\pi^2}$.

There was no sign of any sort of a circle or anything trigonometric until now. Since there is a sin function involved we can see whether or not π is in the equation. Clearly the sin function is derived from the unit circle as we all know. At this stage we can use π in the equation as well. Let x = π.

From this we know that replacing with π in the Mclauren series would give us -

$$sin(\pi) = \frac{\pi^1}{1!} - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!}...$$

The sin function at π is equal to 0, so on the other side, we'd get an entire infinite sum equal to 0.

Let's now assume that the sin curve can be written in the form shown below.

$$sin(x) = a + bx + cx^2 + dx^3 + ex^4...$$

We would need to find the coefficients of the sum which are (a, b, c, d, e, ...). To do this we can set the value for x = 0. This would give us the (a) value directly and for the values of (b, c, d, e ...) we can simply keep differentiating the infinite sum, set the value of x = 0, and keep getting our answers.

It isn't hard to see after doing the calculations for the values of (a, b, c and d) that they are equal to the values shown below.

$$a=0,\ b=1,\ c=0,\ d = \tfrac{-1}{2.3}\ ...$$

From this we can simplify to

$$sin(x) = x - \frac{1}{2.3}x^3 + ...$$

We know that Euler is only forming polynomials that match up with the sin curve at any particular given number of zeros. For the first three zeros of the sin curve we already know that the polynomial will have the equation

$$x(\pi - x)(\pi + x)$$

We can multiply the terms out to give us

$$x(1 - \frac{x^2}{\pi^2})$$

This forms its own infinite sum as we know that the next section of the sin curve will have 5 zeros and then 7 zeros. The equations for both of those polynomials are given below.

$$sin(x) = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2\pi^2})...$$
$$sin(x) = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2\pi^2})(1 - \frac{x^2}{3\pi^2})...$$

As we can see there is a clear pattern here and so we now have two infinite equations that we can work with.

$$sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}...$$
$$sin(x) = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2\pi^2})(1 - \frac{x^2}{3\pi^2})...$$

Equating these two at this stage isn't the right thing to do, as the partial sums of an infinite product and an infinite sum can't compared. Here we case simplify the terms of the infinite product by multiplying them out and turn it into an infinite sum. This way it'll be easier to compare the values.

We can start multiplying the terms in the infinite product term by term. For the first two terms we'd get.

$$x(1 - \frac{x^2}{\pi^2}) = (x - \frac{1}{\pi^2}x^3)$$

We can rewrite this term in the equation

$$(x - \frac{1}{\pi^2}x^3)(1 - \frac{x^2}{2\pi^2})(1 - \frac{x^2}{3\pi^2})...$$

We can multiply the next two terms

$$(x - \frac{1}{\pi^2}x^3)(1 - \frac{x^2}{2\pi^2})$$

This can again be written in the equation like this

$$(x + (-\frac{1}{\pi^2}(-\frac{1}{(2\pi)^2}))x^3 + \frac{1}{\pi^2}\frac{1}{(2\pi)^2}x^5)...$$

The coefficients of the x^3 term form a pattern, which we will be able to recognize as we go through this. Now we can multiply the next two terms.

$$(x + (-\frac{1}{\pi^2}(-\frac{1}{(2\pi)^2}(-\frac{1}{(3\pi)^2})x^3 + (...)x^5 + (...)x^7...$$

From this we can see that at infinity the coefficient for the x^3 term will be

$$(-\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \frac{1}{(4\pi)^2}...)$$

From comparing the two infinite sums we can know that the coefficients should be equal to each other and the coefficient for the x^3 term can be directly compared which gives us -

$$\frac{-1}{3!} = -\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \frac{1}{(4\pi)^2}...$$

Multiplying both sides by -1 gives

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \frac{1}{(4\pi)^2}...$$

Multiplying both sides by π^2 gives us

$$\frac{\pi^2}{3!} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ...$$

There we go, since we know that 3! is just 6, we get

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + ...$$

The code that performs this in excel is given below. This was relatively easier to code, due to its simplicity in the functions. 5 columns were required to code this.

The columns and their respective codes can be seen below.

4.2 The excel sheets showing the Euler series code

=A2+1

A
1
2
3
4
5
6
7
8
9
10

=1/(A3^2)

A	B
	0
1	1
2	0.25
3	0.1111111111
4	0.0625
5	0.04
6	0.02777777778
7	0.02040816327
8	0.015625
9	0.01234567901
10	0.01

=B3+C2

A	B	C
	0	0
1	1	1
2	0.25	1.25
3	0.1111111111	1.361111111
4	0.0625	1.423611111
5	0.04	1.463611111
6	0.02777777778	1.491388889
7	0.02040816327	1.511797052
8	0.015625	1.527422052
9	0.01234567901	1.539767731
10	0.01	1.549767731

=C3*6

A	B	C	D
	0	0	
1	1	1	6
2	0.25	1.25	7.5
3	0.1111111111	1.361111111	8.166666667
4	0.0625	1.423611111	8.541666667
5	0.04	1.463611111	8.781666667
6	0.02777777778	1.491388889	8.948333333
7	0.02040816327	1.511797052	9.070782313
8	0.015625	1.527422052	9.164532313
9	0.01234567901	1.539767731	9.238606387
10	0.01	1.549767731	9.298606387
11	0.00826446281	1.558032194	9.348193164

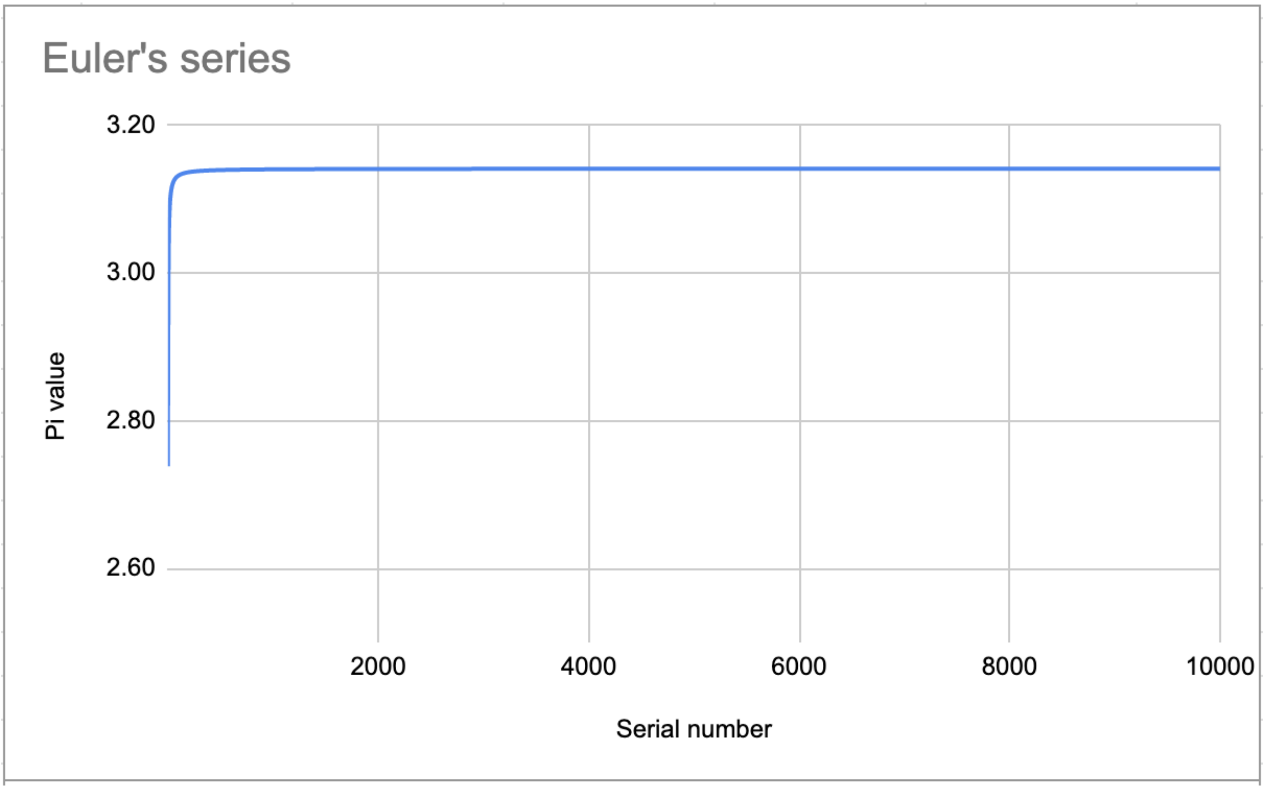
=SQRT(D3)

A	B	C	D	E
	0	0		
1	1	1	6	2.74
2	0.25	1.25	7.5	2.86
3	0.1111111111	1.361111111	8.166666667	2.86
4	0.0625	1.423611111	8.541666667	2.92
5	0.04	1.463611111	8.781666667	2.96
6	0.02777777778	1.491388889	8.948333333	2.99
7	0.02040816327	1.511797052	9.070782313	3.01
8	0.015625	1.527422052	9.164532313	3.03
9	0.01234567901	1.539767731	9.238606387	3.04

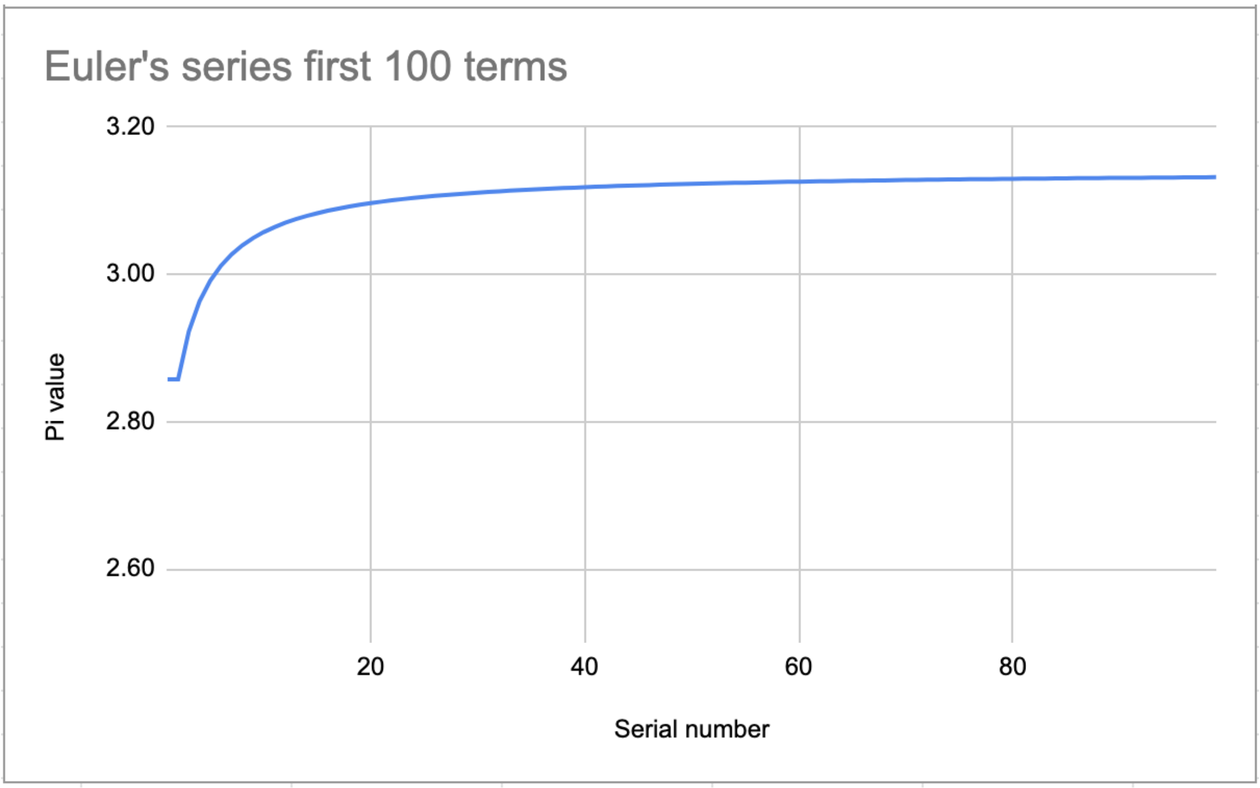
As always we have the serial number column A. Column B has the function of executing the $1/(S.no^2)$ function. The C column then adds the corresponding B and the previous C cell, to perform the repeated addition of the reciprocal of the square numbers. The D column multiplies the result in the previous step by 6 to give us π^2 value. Then finally the E cell finds the square root of it and gives us the π value.

The graph to show the convergence can be seen below

4.3 The graph showing the convergence through the Euler series, all 10,000 terms



4.4 The graph showing the convergence through the Euler series, first 100 terms



5 Wallis series

5.1 The derivation of Wallis' series

Let's first see what the Wallis product is before proving it. Wallis was trying back in 1655 to finish his research work when he came across this and a few other interesting results, that will be shown later. The infinite product is

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}...$$

This can be written in the product notation like this

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}(\frac{2k}{2k-1}\frac{2k}{2k+1})$$

Wallis really was essentially trying to study the integrals in this form shown below.

$$\int_0^1 (1-x^{\frac{1}{p}})^q dx$$

He used trial and error method to find that this equation is equal to

$$\frac{p!q!}{(p+q)!}$$

Which is only true for positive integers.

At this stage if we equate p to q to 1/2, we can get something rather familiar.

$$\int_0^1 \sqrt{1-x^2}dx$$

This equation above is also the formula for the area of the quarter unit circle, which means we can equate this to $\frac{\pi}{4}$. One remarkable result that he found can be obtained by equating p to q to 1/2 in the formula $\frac{p!q!}{(p+q)!}$

We can see that this is equal to

$$(\frac{1}{2}!)^2 = \frac{\pi}{4} \text{ Therefore } (\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$$

The modern proof as shown by mathematicians today starts with the integral of the form below

$$\int_0^1 (1-x^2)^{\frac{n}{2}}dx$$

Let x = cosθ, we will need to change the limits of integration so that it now is from $\frac{\pi}{2}$ to 0. This is equal to

$$\int_{\frac{\pi}{2}}^0 (1-cos^2\theta)^{\frac{n}{2}}(-sin\theta)d\theta$$

Here it is obvious that we can use the identity about the square sin and cos function to equate $1-cos^2\theta$ to $sin^2\theta$. The negatives signs are removed, and the limits are flipped. The new equation we get from this is

$$\int_0^{\frac{\pi}{2}} sin^{n+1}\theta d\theta$$

As we know the sin graph has zeros at the integral multiples of π and 0, and to make our integration easier we change the limits to 0 and π. For some Δ_n we have

$$\Delta_n = \int_0^{\pi} sin^n x dx$$

We have to solve this integration by parts. Consider the following values of u and v and their derivative functions below.

$u = sin^{n-1}x$
$du = (n-1)sin^{n-2}xcosxdx$
$v = -cosx$
$dv = sinxdx$

This means that the integral becomes $uv - \int_0^{\pi} vdu$ From 0 to π. We substitute the values in this to obtain the following

$$[sin^{n-1}x - cosx]_0^{\pi} - \int_0^{\pi}(-cosx(n-1)sin^{n-2}xcosxdx)$$

We can see very easily that the first term equals zero, because the sin function is zero for x = 0 and x = π, and hence sin^{n-1} is also zero for both. The new equation is now

$$= \ 0 + (n-1)\int_0^{\pi} cos^2 x sin^{n-2} x dx$$

Using the identity $cos^2x = 1 - sin^2x$.

$$= \ (n-1)\int_0^{\pi} (1 - sin^2x)sin^{n-2}xdx$$

We can now split the integration sign and since $(n-1)$ is constant it stays outside the integration. Multiply both terms inside the brackets with $(n-1)$.

$$= \ (n-1)\int_0^{\pi} (sin^{n-2})xdx - (n-1)\int_0^{\pi} (sin^n x)dx$$

This in terms of the Δ_n is equal to the following

$$\Delta_n = (n-1)\Delta_{n-2} - (n-1)\Delta_n$$

We take the $(n-1)$ common term outside the brackets which gives us

$$\Delta_n = \frac{n-1}{n}\Delta_{n-2}$$

From this equation we can divide both sides by the term Δ_{n-2} to get an important ratio, which will be useful later in the proof. The ratio we get is

$$\frac{\Delta_n}{\Delta_{n-2}} = \frac{n-1}{n}$$

Considering the following equation we can start by spiting the process for even and odd numbers

$$\Delta_n = \int_0^\pi sin^n x dx$$

We know that $\frac{\Delta_n}{\Delta_{n-2}} = \frac{n-1}{n}$ and that the formula for even numbers in just 2n and for odd numbers its 2n+1. We can use this to substitute the formulae for even and odd numbers in the ratio.

For even numbers we get

$$\frac{\Delta_{2n}}{\Delta_{2n-2}} = \frac{2n-1}{2n}$$

For odd numbers it is

$$\frac{\Delta_{2n+1}}{\Delta_{2n-1}} = \frac{2n}{2n+1}$$

Let's solve the even number formula first.

For $x = 0$ we know $sin^0 = 1$, we can write this as -

$$\Delta_0 = \int_0^\pi 1dx = \pi$$

We can now make Δ_{2n} the subject of the formula. This gives us

$$\Delta_{2n} = \frac{2n-1}{2n}\Delta_{2n-2}$$

From here we can solve for the Δ_{2n-2} term by subtracting 2 from the denominator and the numerator, and that gives us $\dfrac{2n-3}{2n-2}$ Since this is a part of the entire series of values of n, by doing this we get the new equation below

$$\Delta_{2n} = \frac{2n-1}{2n}\frac{2n-3}{2n-2}\Delta_{2n-4}$$

We can keep reducing the term and keep evaluating more and more terms and we'll eventually get a series that can be written as the following after the integration. When $n = 0$ we get $\Delta_0 = \pi$ we can equate this with the series above.

$$\Delta_{2n} = \pi \prod_{k=1}^{n} \frac{2k-1}{2k}$$

It is important top note at this point that $\Delta_1 = 2$ since the integrated value of $sinx$ from 0 tp π, is 2. When $n = 1$ $\Delta_1 = sinxdx$

For odd numbers we can start by making the Δ_{2n+1} term the subject of the formula. This will give us the following

$$\Delta_{2n+1} = \frac{2n}{2n+1}\Delta_{2n-1}$$

Same procedure, of repeated reduction until Δ_1 as the even numbers, brings us to the intermediate equation

$$\Delta_{2n+1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\Delta_{2n-3}$$

Hence the general formula for this series would be

$$\Delta_{2n+1} = 2\prod_{k=1}^{n}\frac{2k}{2k+1}$$

Now we can compare the ratios of the two formulae achieved

$$\Delta_{2n} = \prod_{k=1}^{n}\frac{2k-1}{2k} \text{ and } \Delta_{2n+1} = 2\prod_{k=1}^{n}\frac{2k}{2k+1}$$

As the inequality $0 \leq sinx \leq 1$ for the values of x in the inequality $0 \leq x \leq \pi$ we have the following equation below. We know that the $sinx$ value between 0 and π is always between 0 and 1, and when any number between 0 and 1 is squared, the number gets smaller. This means -

$$sin^{2n+1}x \leq sin^{2n}x \leq sin^{2n-1}x$$

In the terms of the Δ we get

$$\Delta_{2n+1} \leq \Delta_{2n} \leq \Delta_{2n-1}$$

From here we can see that this equation heads toward the following when we divide all the terms by Δ_{2n+1}

$$1 \leq \frac{\Delta_{2n}}{\Delta_{2n+1}} \leq \frac{\Delta_{2n-1}}{\Delta_{2n+1}} = \frac{2n+1}{2n}$$

From these inequalities we can say that as $n \to infinity$ the value of $\frac{2n+1}{2n} \to 1$. This gives us

$$\lim_{n\to\infty}\frac{\Delta_{2n}}{\Delta_{2n+1}} = 1$$

And from there we can finish by saying that this equal to

$$\frac{\pi}{2}\prod_{k=1}^{\infty}(\frac{2k-1}{2k})(\frac{2k+1}{2k}) = 1$$

When we find the reciprocal of the terms we get -

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}(\frac{2k}{2k-1})(\frac{2k}{2k+1})$$

The code required to perform this infinite series can be seen below

5.2 The excel sheets showing the Wallis series code

=A1+1

A
1
2
3
4
5
6
7
8
9
10

=2*A2

A	B
1	2
2	4
3	6
4	8
5	10
6	12
7	14
8	16
9	18
10	20

=(2*A1)+1

A	B	C
1	2	1
2	4	3
3	6	5
4	8	7
5	10	9
6	12	11
7	14	13
8	16	15
9	18	17
10	20	19

=((B2/C2)*(B2/C3))

A	B	C	D
1	2	1	1.333333333
2	4	3	1.066666667
3	6	5	1.028571429
4	8	7	1.015873016
5	10	9	1.01010101
6	12	11	1.006993007
7	14	13	1.005128205
8	16	15	1.003921569
9	18	17	1.003095975
10	20	19	1.002506266
11	22	21	1.002070393

=E1*D2

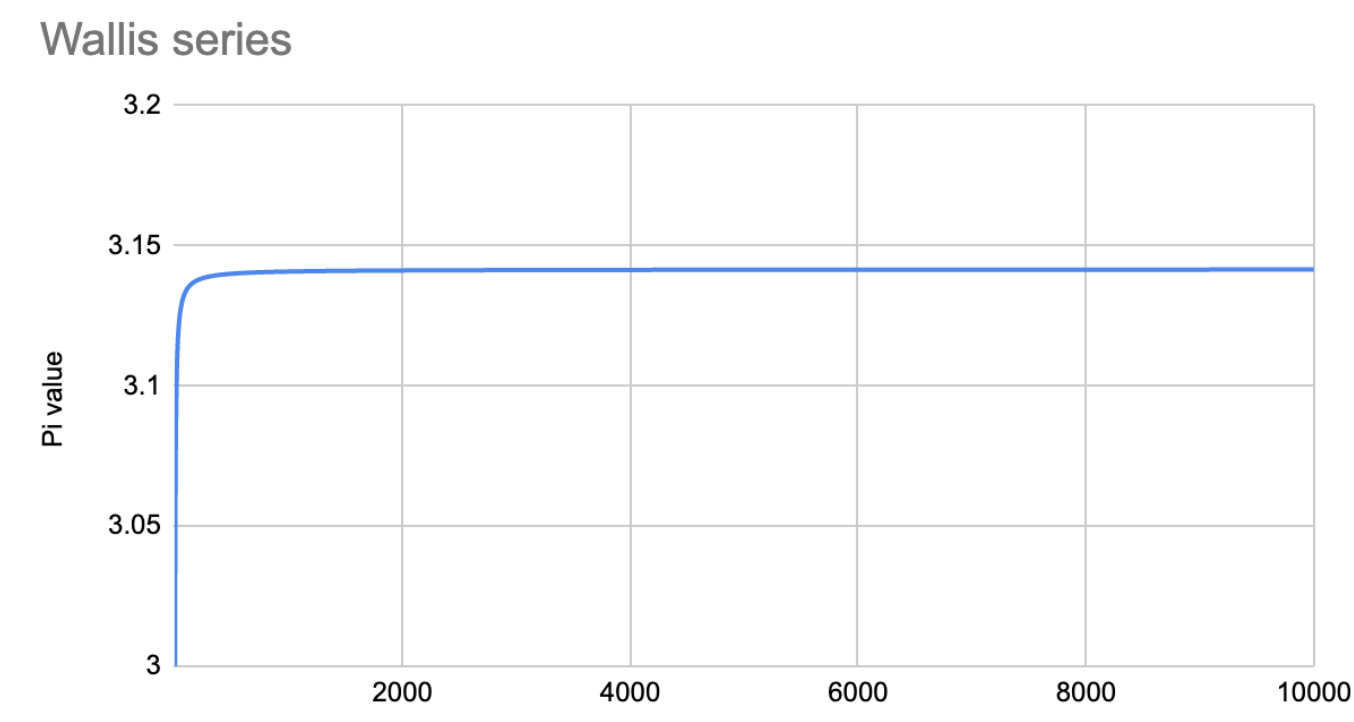
A	B	C	D	E
1	2	1	1.333333333	1.333333333
2	4	3	1.066666667	1.422222222
3	6	5	1.028571429	1.462857143
4	8	7	1.015873016	1.486077098
5	10	9	1.01010101	1.501087977
6	12	11	1.006993007	1.511585096
7	14	13	1.005128205	1.519336814
8	16	15	1.003921569	1.525294998
9	18	17	1.003095975	1.530017274
10	20	19	1.002506266	1.533851903

=2*E2

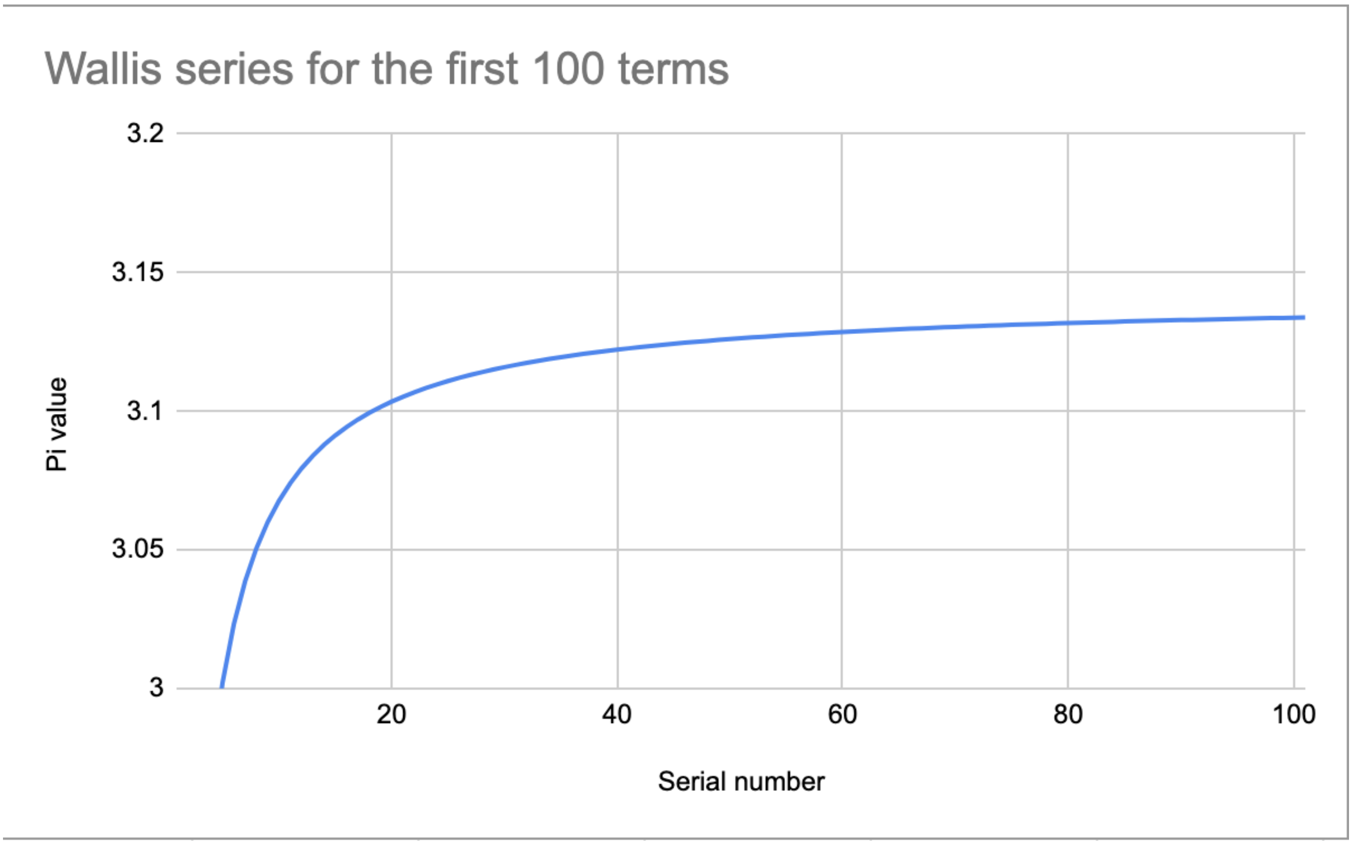
A	B	C	D	E	F
1	2	1	1.333333333	1.333333333	2.666666667
2	4	3	1.066666667	1.422222222	2.844444444
3	6	5	1.028571429	1.462857143	2.925714286
4	8	7	1.015873016	1.486077098	2.972154195
5	10	9	1.01010101	1.501087977	3.002175955
6	12	11	1.006993007	1.511585096	3.023170192
7	14	13	1.005128205	1.519336814	3.038673629
8	16	15	1.003921569	1.525294998	3.050589996
9	18	17	1.003095975	1.530017274	3.060034547
10	20	19	1.002506266	1.533851903	3.067703807

The following graph can be used to compare the convergence

5.3 The graph showing the convergence through the Wallis' series, all 10,000 terms



5.4 The graph showing the convergence through the Wallis' series, first 100 terms



6 Run time codes

The code is shown only for the Euler's series however, it has been applied to all and hence we can see the different runtimes for all the series and their obtained value of π below.

6.1 Euler's series code (calculation of runtime)

```
#include <stdio.h>
#include <time.h>       // for clock_t, clock(), CLOCKS_PER_SEC
#include <unistd.h>     // for sleep()
#include <stdlib.h>
#include <math.h>

int main()
{
    // to store execution time of code
    double time_spent = 0.0;

   clock_t begin = clock();
   double n, i;     // Number of iterations and control variables
   double f = 0;            // factor that repeats
   double pi = 0;

   printf("Approximation of the number PI through the Euler's series\n");
   printf("\nEnter the number of iterations: ");
   scanf("%lf",&n);
   printf("\nPlease wait. Running...\n");
   for(i = 1; i <= (n); i+=1)
      f = f + 1/(i*i);
      pi = sqrt (6*f);
      printf("\nAproximated value of PI = %1.16lf\n", pi);

```

6.2 Euler's series code (calculation for the value of π)

```

   printf("Approximation of the number PI through the Euler's series\n");
   printf("\nEnter the number of iterations: ");
   scanf("%lf",&n);
   printf("\nPlease wait. Running...\n");
   for(i = 1; i <= (n); i+=1)
      f = f + 1/(i*i);
      pi = sqrt (6*f);
      printf("\nAproximated value of PI = %1.16lf\n", pi);

    // do some stuff here
    sleep(3);

    clock_t end = clock();

    // calculate elapsed time by finding difference (end - begin) and
    // dividing the difference by CLOCKS_PER_SEC to convert to seconds
    time_spent += (double)(end - begin) / CLOCKS_PER_SEC;

    printf("Time elpased is %f seconds", time_spent);

    return 0;
}

```

6.3 Time calculated for the Euler's series

```
Please wait. Running...

Aproximated value of PI = 3.1414971639472147
Time elpased is 0.000243 seconds

...Program finished with exit code 0
Press ENTER to exit console.
```

6.4 Time calculated for the Leibniz series

```
Please wait. Running...

Aproximated value of PI = 3.1414926535900345
Time elpased is 0.000288 seconds

...Program finished with exit code 0
Press ENTER to exit console.
```

6.5 Time calculated for the Wallis' series

```
Please wait. Running...

Aproximated value of PI = 3.1415926535895382
Time elpased is 0.000204 seconds

...Program finished with exit code 0
Press ENTER to exit console.
```

6.6 Time calculated for the Nilkantha's series

```
Please wait. Running...

Aproximated value of PI = 3.1417497057380084
Time elpased is 0.000238 seconds

...Program finished with exit code 0
Press ENTER to exit console.
```

7 Comparison and reflection

For the comparison we need to look at the 10,000th term in each series and that will tell us how accurate the decimal places are. Since the number of terms are all the same for each one we can say that it is a 'fair' test and that means that the convergence is more comparable like that.

7.1 Comparing the accuracy of the series

Leibniz series at the end of the 10,000 term gives us only 3 accurate decimal places, which means that the convergence is extremely slow and not reliable at all. The Nilkantha series on the other hand gives us 7 accurate decimal places, compared to Leibniz series this is more reliable and has a faster convergence. Euler series again only gives us 3 decimal places at the 10,000th term, this just like the Leibniz series is much slower than the Nilkantha series, very unreliable and overall not a good series to calculate π in general. Wallis series gives 4 accurate decimal places at the 10,000th term. This means that it is better than the Euler series and Leibniz series, although it's not as good as the Nilkantha series, meaning that the convergence is faster than Euler and Leibniz but not as fast as Nilkantha.

7.2 Graphical comparison

With the aid of the graphs also we can say that the graph for the Nilkantha series flattens out much faster and in lesser number of terms when compared with the others. Since the Wallis series is converging faster than Euler series and Leibniz series, we can notice how the values get closer and closer to a straight line i.e. π value. Euler and Leibniz series are much slower than both the other series and we can see from the graph also that the Leibniz series keeps bouncing above and below the π value for a long time, and Euler series reaches the straight line, which is the π value much later on.

7.3 Run-time comparison

Lastly the comparison for the run-times of the series. As shown below the runtimes of the codes were varying by a fraction of a second. This is due to the fact that the code isn't hard to run, and can be run to way more than 10,000 terms in some seconds. The longest time I have seen the code run was 2 minutes, rounded down and that was when I asked it to run 10,000,000,000,000 iterations. Anyway the final ranking observed is something like this. Walllis' series is the fastest and most efficient series as it took the online compiler the least amount of time to run it, in just

2.0 milli-seconds. Second best was Nilkantha's series, this took 2.3 milli-seconds. As mentioned earlier the series only differ in their run-times by a fraction of second. Third was Leibniz' series, as it took the compiler 2.4 seconds. The competition was good between the Leibniz and Nilkantha's series. This is shown as their run-times differ only by 1 milli-second. Last in this comparison is the Euler's series which took 2.8 milli-seconds to run.

8 Limitations

8.1 Run-time code limitations

In the code run-time calculations, I used an online compiler and an online C++ code. This means that instead of the run-time depending on the series or the complexity of the code, it was now depending on the wi-fi speed. To minimize this effect I tried using a LAN wire, but even then the internet didn't have a constant value. This means that everytime I run the code I was getting a different run-time value in milli-seconds for all the codes.

8.2 Google sheets limitations

When I was working on the google sheets, I had to hold an drag the cell to all the way down to 10,000th cell, this means that it took me approximately a minute just to scroll down to the 10,000th cell. Eventhough it is accurate, and relatively efficient it still took me sometime.

8.3 Graphical limitations

Furthermore, the actual graphs that were processed made it harder to see which graph converges faster. That's why I had to make a graph for the first 100 terms as well. Using this we can compare the convergence in the first 200 terms. This is because the convergence is more easily visible in the first 100 terms and less so in the 10,000 term graphs. I tried plotting a constant π graph as well, however, in the 10,000 terms graph, the two lines became one very quickly and I couldn't compare them.

9 Conclusion

From the above comparisons we can find that Wallis' series is the most efficient and the fastest series when compared with the other series, in terms of the run-time calculations. This is because it took the compiler the least amount of time to calculate it's value, as we can see in the comparison section. However, in order of graphical comparison Nilkantha's series is the best as it converges closer and closer to the value of π. Furthermore, it was also found while comparing the accuracy of the π value that Nilkantha's series was the best at that too, as it was able to give 7 accurate decimal places of π, which no other series could do.

Through this we can conclude with Nikantha's series being the best as it proved to be the best series in two ways of comparison, and as mentioned earlier, the run-time calculation of the code was anyway not reliable, due to fluctuating internet speeds.

It is important to note that none of these series are being used to calculate the actual value or π by modern day super-computers. This is because they are all extremely slow. Even 7 decimal places at 10,000th n value is extremely slow, and the series mentioned below are much better than the ones used in the research. These series are used to calculate the value of π by modern day super-computers.

Ramanujan's series - $\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}.\frac{26290n+1103}{396^n}$

Chudnovsky's series - $\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(545140134k+5909)}{(3k)!(k)!^3(640320)^{\frac{3k+3}{2}}}$

10 Applications

One of the most well-known uses of the value of pi is by NASA, when they have to measure the size of crater. By measuring the size of any crater, they can use it to figure out, what size of an asteroid made it, or how fast it must have been travelling, and other important features related to its trajectory in space and its mass and density and volume and momentum. They can essentially circle the crate, finding its circumference in the process, and from there use the value of π in the 2πr formula to figure out the radius.

Another use of π is while changing the CASSINI's direction of orbit. CASSINI is a satellite orbiting, Titan, which is Saturn's largest moon. There is a process called π transfer, which essentially alters the orbit of the CASSINI π radians, using Titan's own gravity. This changes its 'persepective' of Saturn and allows the Saturn facing side to know more about the features of Saturn, for instance, what are the rings made out of, etc.

While in search of exoplanets, π comes up regularly. This is because whenever we want to calculate the size of the exoplanet, we can do this using the comparison of the ratios of the light given off by its sun with no interference, and the deviation that occurs when the light given off is curved by the planet's gravity. In these calculations π features heavily.

11 Implications

I'd like to carry this research on in the future as it has so many real life applications. Especially things related to the super computing of larger and larger amounts of data. Super computing has always been an area of my keen interest and people many times forget how important it is to be able to calculate the pi value accurately. Not only does this bring fame, as mentioned above, to the people who find it, but the calculations are much more accurate and that means that larger and larger operations and number of calculations can be performed per unit time.

Astrophysics has also been an area of my deepest interest and considering the number of applications this has in that field, as mentioned above, it'll be my pleasure to work on this later on. In the future if I get the chance of working in NASA and possibly on the mentioned areas of work I'll be more than content to carry on this research and find bigger and more accurate decimal places of π.

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