Introduction

Mersenne prime numbers are a different set of prime numbers that can be written in the form 2^n-1[1]. Their effectiveness comes from the ease with which they can be computed, compared to other sorts of prime numbers. Mersenne primes guarantee that a number found this way will always be a prime number and further checking isn't required. The current world record for the largest prime number is $2^{82589933}$-1.[2] This is a kind of a Mersenne prime and it was a result of the GIMPS (great internet Mersenne Prime search) race.[3]

This essentially just means that we can use the standard notation 2^n-1 to express any kind of Mersenne prime number. This only works if n in the equation meets some requirements. One of which is that, it has to be a prime number itself. Another is that it should be a part of the prime number factorization of a perfect number.[4]

Perfect number by definition is a number that can be written as a product and sum of the prime numbers that make it up, as seen from its prime number factorization[5]. For instance, 6 can be written as 3+2+1, which are also the prime numbers that multiply to give 6. The debate about including 1 as a prime number still exists today, but we are considering it as a prime number in this research paper.

There have only been even perfect numbers found so far, although the existence of an odd perfect number is still unknown. Perfect numbers are rare and also have numerous real life applications.

Arguably, 6 is also a factor of 6, however, this increases the value found in the addition process, and essentially increases the perfect number 2 times. We can also write that as an equation: $(2^n-1) \times (2^n) = 2P.N$. Where P.N is the perfect number being assessed. In this research paper, I will compare the frequencies and the growth rates of the two series: prime numbers and Mersenne prime numbers.

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[1] "Mersenne Prime." *Wikipedia, the Free Encyclopedia*, Wikimedia Foundation, Inc, 7 Oct. 2001, en.wikipedia.org/wiki/Mersenne_prime. Accessed 7 Apr. 2020.

[2] "Mersenne Primes: History, Theorems and Lists." *The Prime Pages (prime Number Research, Records and Resources)*, primes.utm.edu/mersenne/.

[3] *Great Internet Mersenne Prime Search - PrimeNet*, www.mersenne.org/.

[4] "Mersenne Prime." *Wolfram MathWorld: The Web's Most Extensive Mathematics Resource*, 19 Mar. 2020, mathworld.wolfram.com/MersennePrime.html.

[5] "Perfect Number." *Wikipedia, the Free Encyclopedia*, Wikimedia Foundation, Inc, 7 Aug. 2001, en.wikipedia.org/wiki/Perfect_number. Accessed 7 Apr. 2020.

Thesis statement

This study focuses on a comparison between the growth rates of the primes numbers and Mersenne prime numbers.

Personal connection

I chose this topic because it was fascinating to me how the Mersenne primes work, and were earlier used, just to get to the perfect numbers. Since Mersenne primes are so connected to perfect numbers, deriving the formula was also interesting to me. I have known about this topic for 3 years now, and I never stop getting intrigued by Mersenne prime numbers and the GIMPS race. Moreover, when I first got to know about the existence of a whole new type of prime numbers that don't need to be checked, I was extremely curious to know how that happens, I started learning more about it.

Methodology

There are two series, one is the regular prime numbers and the other is the Mersenne prime numbers. So the comparison will be between the variances of the two to know what kind of growth they are going under and that will be done using some basic linear modelling methods.

Once we have the variance values of the two, this will tell us how fast the series is growing. The applications of this will be discussed later on.

The first 10 numbers of the two series will be used, as it will be a good enough comparison point for the growth rates of the two series, and also, using a graphical calculator, we can calculate the variance. The variance values provide further comparison between the two series. Graphing the two series against the natural numbers, will also provide more comparison of the data, as the two graphs won't be similar, in fact the gradient and the growth of the graph can also be used to compare the growth and frequency.

Derivation[6]

A clear relation between Mersenne primes and perfect numbers is seen. It is known that only one Mersenne prime number can be used for the prime number factorization of a perfect number. From the table below we can obtain an equation, since it is known that the product of a Mersenne prime number with its corresponding 2^n-1 value, will always give a perfect number.

[6] "Mersenne Prime." *Wikipedia, the Free Encyclopedia*, Wikimedia Foundation, Inc, 7 Oct. 2001, en.wikipedia.org/wiki/Mersenne_prime. Accessed 7 Apr. 2020.

n	2^n-1	2^{n-1}	series
2	3	2	2^1
3	7	4	2^2
5	31	16	2^4
7	127	64	2^6

We can start the derivation of this formula now: $(2^n-1) \times (2^{n-1}) = P.N$

The sum of all the prime numbers of a perfect number add up to give that same perfect number, or if the number itself is added, then twice the perfect number is obtained, as a result of the summation.

So, we just need to prove algebraically that the factors of $(2^n-1) \times (2^{n-1})$ add up to it, or twice it, if the formula is added back in the sum.

We start with the fact that when writing the sum of all the prime numbers to give a perfect number, we can notice something interesting. Let's discuss this as an example as it'll be useful for the proof later on. Let's take a perfect number like 28, this can be written as - 1+2+4+7+14. Here we see that we start with the addition of the powers of two, which is the 1+2+4 part. When we reach the Mersenne prime we add it and then start doubling it to get to the perfect number. That is the 7+14 part. However, if we now add 28 in the list, it doubles the number 28. So we can write this as 2(28) = 1+2+4+7+14+28. We can get back to the perfect number by removing the number in its factors, or simply dividing by 2.

Rewriting this expression algebraically, gives us -

$$2P.N = 2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1} \; + \; 2^0(2^n-1)+2^1(2^n-1)+2^2(2^n-1)+...+2^{n-3}(2^n-1)+2^{n-2}(2^n-1)+2^{n-1}(2^n-1)$$

We can simplify this by taking the common term (2^n-1) out of the second part, and taking a common 1 out of the first part, which doesn't change anything inside the bracket, but is useful in factoring the expression and hopefully see the correlation.

$$2P.N = 1(2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1}) + (2^n-1)(2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1})$$

We can now group the common terms together, this gives us -

$$2P.N = (1+2^n-1)(2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1})$$

We can cancel the 1 and -1 inside the first brackets -

$2P.N = (2^n)(2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1})$[7]

The number of terms inside the second bracket is unknown, hence we will need to prove its total and carry on. Let's assume the grand total of the second bracket to be some G for grand total. This can be rewritten as -

$G = (2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1})$

This also means that 2G.T would double the value inside the bracket, so-

$2G = (2^1+2^2+2^3+...+2^{n-3}+2^{n-2}+2^{n-1}+2^n)$

Here we can notice that the terms inside the brackets are similar, apart from the 2^0 term in the G equation and 2n in the 2G equation. To isolate these two terms we must subtract the original G equation from the 2G equation.
This would give -

$2G - G =(2^1+2^2+2^3+...+2^{n-3}+2^{n-2}+2^{n-1}+2^n) - (2^0+2^1+2^2+...+2^{n-3}+2^{n-2}+2^{n-1})$

The common terms cancel out and the result would be just the G term as we know 2G - G can only equal G.

$G = 2^n-1$

Now that we know that the sum of all the terms inside the bracket is actually equal to 2^n-1. We can finish our proof by equation back the P.N equation here, which gives us-

$2P.N = (2^n-1)2^n$

When we divide both sides by two to isolate P.N. we get-

$P.N = (2^n-1)2^{n-1}$

Data table

The first 10 numbers in both the series provide a good enough comparison, between the two prime numbers and the growth of the two series. This can be shown in the table below.

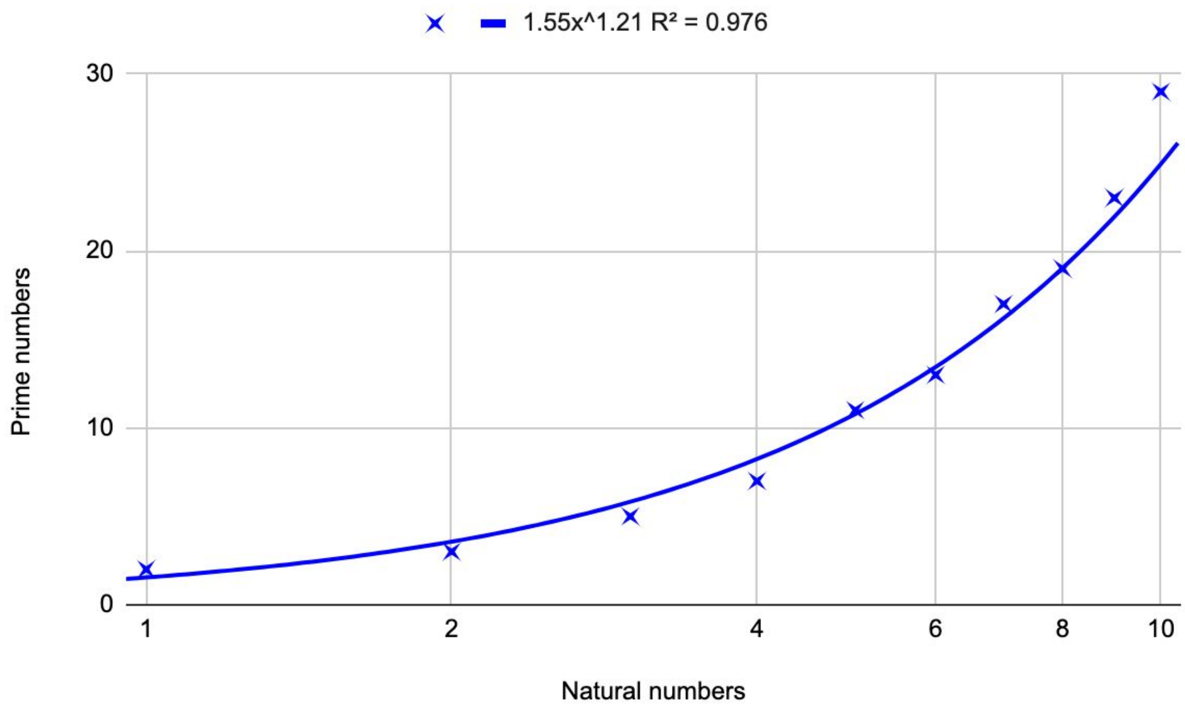
[7] "Mersenne Prime." *Wikipedia, the Free Encyclopedia*, Wikimedia Foundation, Inc, 7 Oct. 2001, en.wikipedia.org/wiki/Mersenne_prime. Accessed 7 Apr. 2020.

S. No.	Prime numbers	Mersenne prime numbers	Perfect numbers[8]
1	2	3	6
2	3	7	28
3	5	31	496
4	7	127	8128
5	11	8191	33550336
6	13	131071	8589869056
7	17	524827	137438691328
8	19	2147483647	2305843008139952128
9	23	2305843009213693951	2658455991569831744654 692615953842176
10	29	618970019642690137449 562111	1915619426082361072947 9337808430363813099732 1 548169216

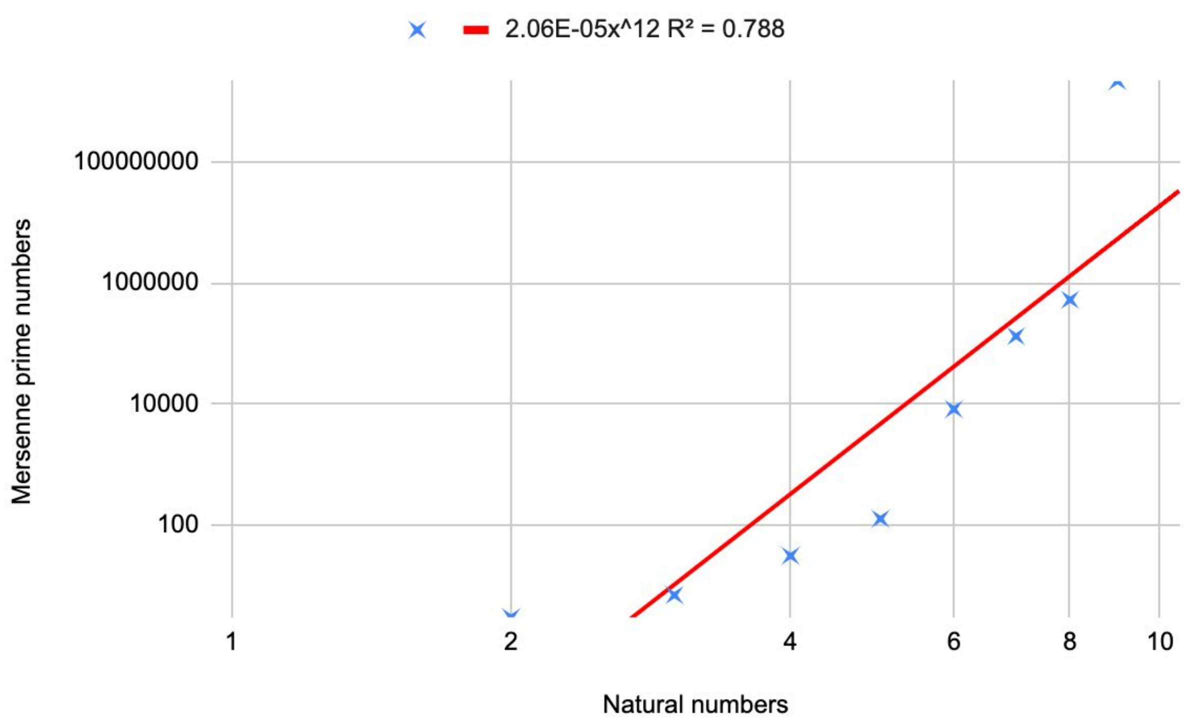
Graphing the results

[8] "1 Computability CS 101E Spring 2007 Michele Co. 2 So Far In Class We've Seen the Following Programming Constructs –if, If-else, If-else-if –switch –for." *SlidePlayer - Upload and Share Your PowerPoint Presentations*, slideplayer.com/slide/8729218/.

Graph - 1



Graph - 2



Analysis

As we can see from the graphs above. The Graph that shows the Natural numbers versus Prime numbers has a much higher variance value, when compared with the graph that shows the Natural numbers versus Mersenne Prime numbers. This is because of the fact that there is barely any difference within the first 10 prime numbers, however, there is a huge difference between the first 10 Mersenne prime numbers. In fact, the 10th Mersenne prime number is 10^{26} times larger than the first.

We can also see that the line of best fit on Graph - 2 appears to be more or less a straight line, even though it isn't. It is an exponential curve, but the Mersenne prime values are so big that it becomes an exaggerated view of the graph at that point. That appears to be a straight line, but if we zoom out we can see that it's an exponential curve. The line of best fit for graph - 1 on the other hand, is a smooth curve. This is because of the negligible difference between the data points on that graph when compared to graph - 2.

We can also compare the equations that are found. Graph - 1 has equation - $1.55x^{1.21}$. However graph - 2 has equation $2.06x10^{-5}$ x ^{12}. Through this we can see the overall difference in the growth rates, as the base value on the equation for graph - 2 is much smaller, than the base value for graph - 1. However, if we take into account the value of the power, which is much higher in graph - 2, when compared with graph - 1, we can easily see why graph - 2 grows so much faster than graph - 1.

Overall, we can conclude by saying that Mersenne prime numbers grow much faster than normal prime numbers, and this can be seen through the equation, the line of best fit and the data points. This can however, lead to a significant realisation, which is mentioned in the (limitation in using Mersenne prime numbers to find bigger prime numbers) section.

Limitations

The two sets could be compared, however, a lot of the times, it might not be a fair comparison. For example, both graphs had to be plotted on a logarithmic scale because there couldn't be a fair comparison between the two, on a regular scale. This would make it unfair, because the mersenne prime graph now looks like a straight line on the exponential X-axis, whereas, the regular prime graph looks like an exponential graph, on an exponential X-axis. Another point is that the two graphs don't have the same X-axis, even though they are both plotted on a logarithmic scale. This would again, make it unfair because the two graphs might look similar in terms of them growing on a graph, but the Mersenne prime graph goes about a thousand times faster.

Mathematically the comparisons can also be unfair sometimes. For example when comparing the S.D value of the two, we notice that the largest value in the Mersenne prime graph, pulls the value of the S.D too much. This means that the S.D value is more or less pointless in serving the purpose of finding the center, even though it provided the comparison between the growth rates of the two graphs.

The point above is actually linked with mean value. Since the S.D value is dependent on the mean value, and the mean value for the second set, is a number to the 25th power, because of the 10th data point in the Mersenne value set, we can observe that that's what deviates the average and the S.D value from the center.

Limitations of using Mersenne Prime numbers to find bigger prime numbers

The one major drawback of using Mersenne prime numbers, is that this process skips over a huge amount of prime numbers, and at the biggest prime number scale, there may even be thousands of these, which we don't know about.[9] Mersenne primes are a very helpful tool, in the GIMPS competition however, this limitation has always held them back. To resolve this problem, the coders who are currently working on the Mersenne primes are trying to find a way through which they have multiple prime number checks, so that even if the Mersenne primes method says that a number isn't a prime, other methods might say it is. And in that case, Mersenne prime's decision will have to be overruled.[10] This happens because Mersenne prime numbers

[9] "Mersenne Prime." *Wikipedia, the Free Encyclopedia*, Wikimedia Foundation, Inc, 7 Oct. 2001, en.wikipedia.org/wiki/Mersenne_prime. Accessed 7 Apr. 2020.

[10] "1 Computability CS 101E Spring 2007 Michele Co. 2 So Far In Class We've Seen the Following Programming Constructs –if, If-else, If-else-if –switch –for." *SlidePlayer - Upload and Share Your PowerPoint Presentations*, slideplayer.com/slide/8729218/.

check for a very specific format of prime numbers, and thousands of prime numbers don't fall in that format.

Application

One of the biggest applications of Mersenne prime numbers is finding the biggest prime number known today.[11] This is ensured through a very distinct property of Mersenne prime numbers, which mentions that, once a prime number is found in the form 2^n-1, it is guaranteed to be a Mersenne prime, and therefore a prime number. Consequently, this follows that Mersenne prime numbers skip through a whole bunch of prime numbers which aren't in the form 2^n-1. As we have seen this to be true in the table itself.[12]

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[11] "1 Computability CS 101E Spring 2007 Michele Co. 2 So Far In Class We've Seen the Following Programming Constructs –if, If-else, If-else-if –switch –for." *SlidePlayer - Upload and Share Your PowerPoint Presentations*, slideplayer.com/slide/8729218/.

[12] "Mersenne Prime." *Wikipedia, the Free Encyclopedia*, Wikimedia Foundation, Inc, 7 Oct. 2001, en.wikipedia.org/wiki/Mersenne_prime. Accessed 7 Apr. 2020.