Part 1: Analytical Option Formulae

Aim of this part is to derive the option pricing formulae using **Black-Scholes, Bachelier, Black76 and Displaced-diffusion model.**

We have following European Options:

- Vanilla Call/Put
- Digital cash-or-nothing call/put
- Digital asset-or-nothing call/put

Black-Scoles Model

Black-Scholes model, uses the stock price as follows-

$$S_T = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T\right], \quad W_T \sim N(0, T).$$

Option pricing formulae

Option	***Call***	***Put***
Vanilla	$S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$Ke^{-rT}\Phi\left(\frac{\log\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S_0\Phi\left(\frac{\log\left(\frac{K}{S_0}\right)-\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$
Cash-or-Nothing	$e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$e^{-rT}\Phi\left(\frac{\log\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$
Asset-or-Nothing	$S_0\Phi\left(\frac{\log\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$S_0\Phi\left(\frac{\log\left(\frac{K}{S_0}\right)-\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

Bachelier Model

Bachelier model, uses the stock price as follows-

$$S_T = S_0 + \sigma W_T, \quad W_T \sim N(0, T).$$

Option pricing formulae

Option	***Call***	***Put***
Vanilla	$(S_0 - K)\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) + S_0\sigma\sqrt{T}\phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right)$	$(K - S_0)\Phi\left(\frac{K - S_0}{S_0\sigma\sqrt{T}}\right) + S_0\sigma\sqrt{T}\phi\left(\frac{K - S_0}{S_0\sigma\sqrt{T}}\right)$
Cash-or-Nothing	$\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right)$	$\Phi\left(\frac{K - S_0}{S_0\sigma\sqrt{T}}\right)$
Asset-or-Nothing	$S_0\left[\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right)\right]$	$S_0\left[\Phi\left(\frac{K - S_0}{S_0\sigma\sqrt{T}}\right) - \sigma\sqrt{T}\phi\left(\frac{K - S_0}{S_0\sigma\sqrt{T}}\right)\right]$

Black76 Model

Black model uses modelling the forward price instead of the underlying price. We have the definition of the forward price

$$F_t = e^{r(T-t)} S_t$$

and the underlying price process of

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Option pricing formulae

Option	***Call***	***Put***
Vanilla	$e^{-rT}\left[F_0\Phi\left(\frac{\log\left(\frac{F_0}{K}\right)+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)-K\Phi\left(\frac{\log\left(\frac{F_0}{K}\right)-\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)\right]$	$e^{-rT}\left[K\Phi\left(\frac{\log\left(\frac{K}{F_0}\right)+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)-F_0\Phi\left(\frac{\log\left(\frac{K}{F_0}\right)-\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\right)\right]$
Cash-or-Nothing	$e^{-rT}\Phi\left(\frac{\log\left(\frac{F_0}{K}\right)-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$	$e^{-rT}\Phi\left(\frac{\log\left(\frac{K}{F_0}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$
Asset-or-Nothing	$e^{-rT}F_0\Phi\left(\frac{\log\left(\frac{F_0}{K}\right)-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$	$e^{-rT}F_0\Phi\left(\frac{\log\left(\frac{K}{F_0}\right)-\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$

Displaced-Diffusion Model

We say that F_T follows a lognormal distribution. Based on this definition, we call the following a shifted lognormal (or displaced-diffusion) process:

$$d(F_t + \alpha) = dF_t = \sigma(F_t + \alpha)dW_t$$

After solving the SDE using Ito's formula we obtain the following formula which is somewhat similar to black model and hence we can directly price the option using black model with some changes.

$$F_T = \frac{F_0}{\beta}\exp\left[-\frac{\beta^2\sigma^2 T}{2} + \beta\sigma W_T\right] - \frac{1-\beta}{\beta}F_0.$$

Hence,

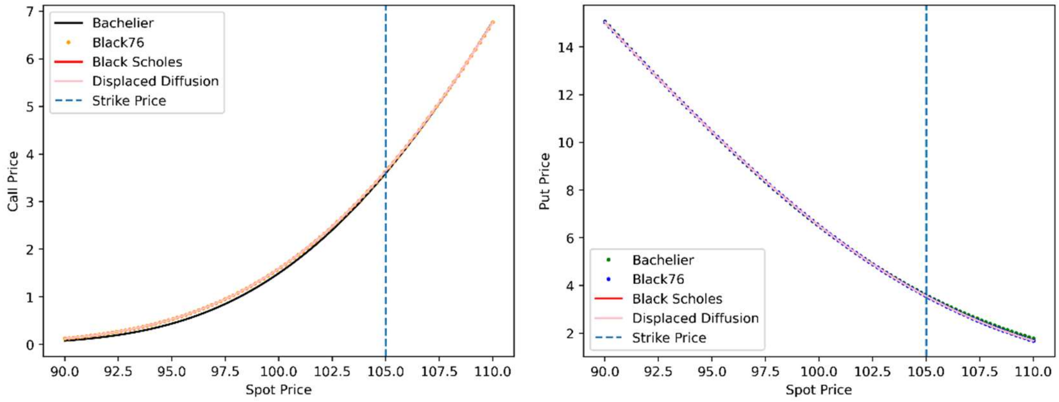
The option price under the displaced-diffusion model is

$$\text{Displaced-Diffusion} = \text{Black}\left(\frac{F_0}{\beta},\ K + \frac{1-\beta}{\beta}F_0,\ \sigma\beta,\ T\right)$$

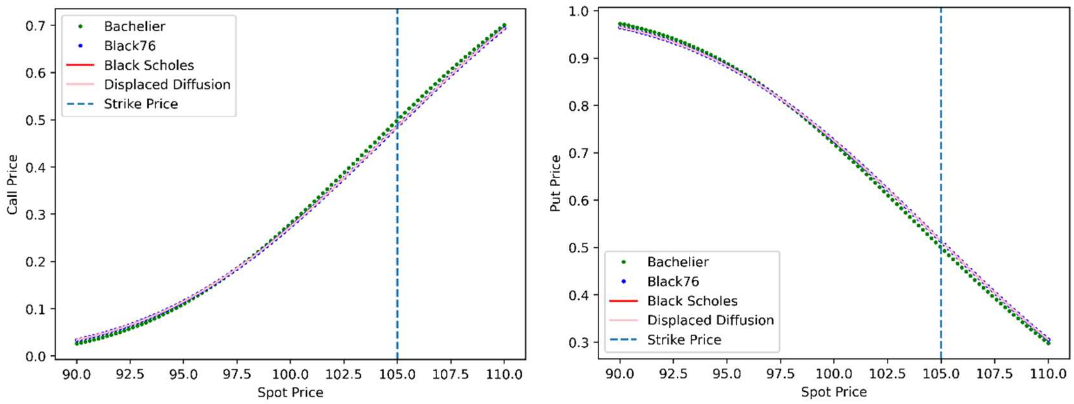
Consider the following option parameters for pricing,

T = 30/365, sigma = 0.3, K = 105, r = 0.01, S = 90 to 110, beta = 0.9999

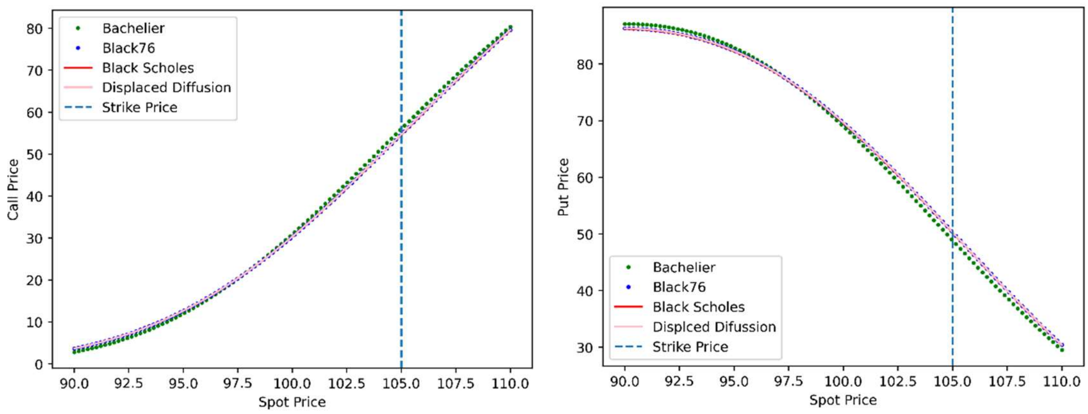
Vanilla Option



Cash-or-Nothing



Asset-or-Nothing



Part 2: Model Calibration

So far, we derived option pricing formulae for different models but it is necessary to calibrate them using market data. In this section we will look at model calibration methods for Displaced Diffusion model and SABR model.

We will use S&P 500 index (SPX) and SPDR ETF (SPY) valued at 3662.45 and 366.02 respectively.

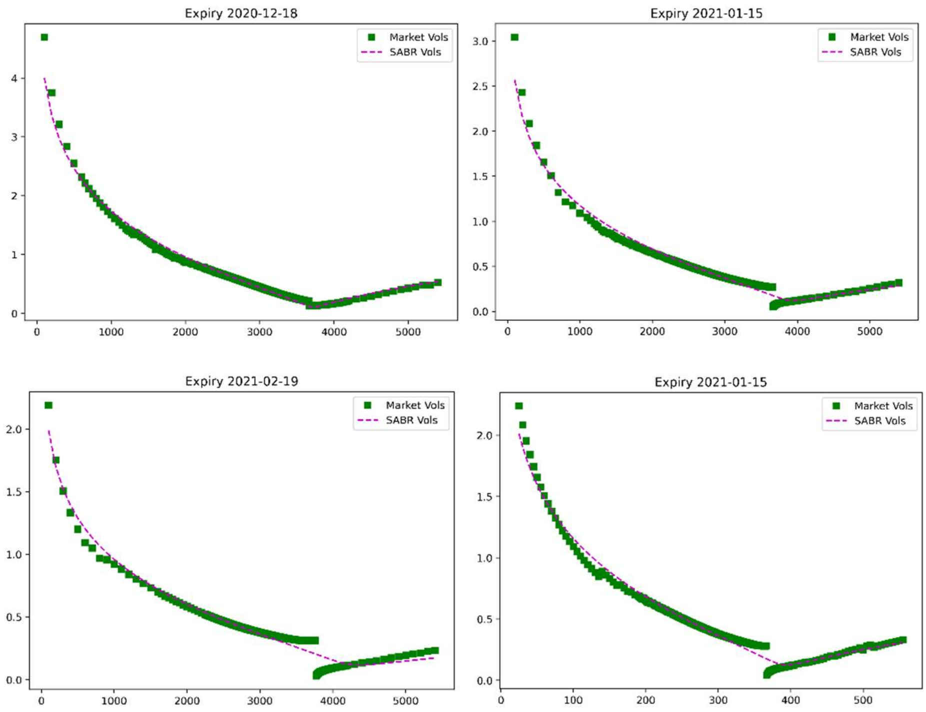
SABR Model

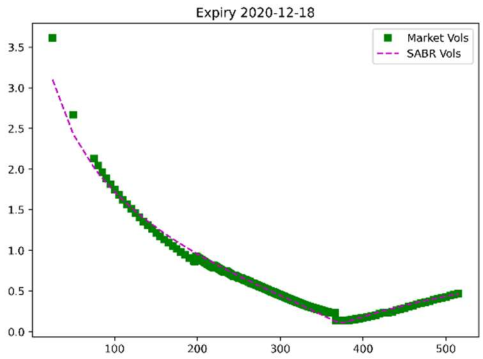
Methodology for SABR:

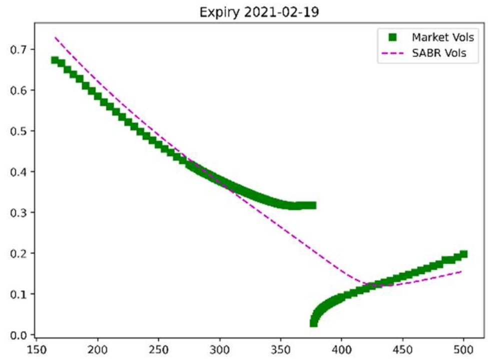
- We take the market price data for different maturities and calculate the respective Implied Volatility using root search method in python and Black Scholes as a reporting model.
- Using that market implied volatility, we use the least squares method to minimise the error between SABR and market IV.
- We calibrated α, ρ and ν by keeping β = 0.7

Calibrated Data and Volatility smiles as follows:

	Maturity	**α (Alpha)**	**ρ (rho)**	**ν (nu)**
SPX	2020-12-18	1.211	-0.364	5.452
	2021-01-15	1.725	-0.667	2.862
	2021-02-19	2.272	-0.851	1.844
SPY	2020-12-18	0.670	-0.452	5.245
	2021-01-15	0.877	-0.620	2.795
	2021-02-19	1.154	-0.846	1.940







Displaced-Diffusion Model

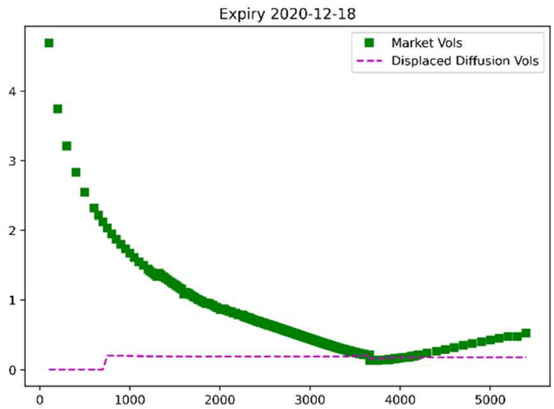
Methodology for SABR:

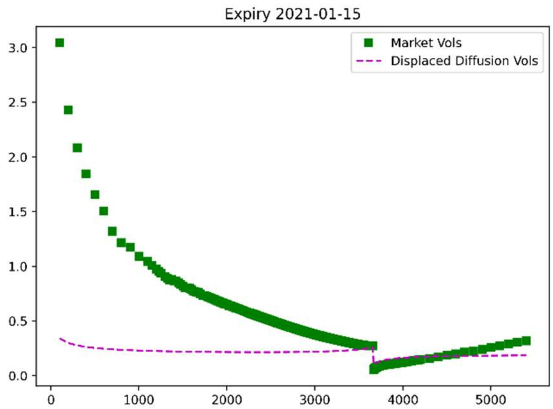
- Since we don't have formula to calculate the volatility in Displaced Diffusion model similar to SABR model, we try to calibrate the parameters by minimising the error between market prices and the prices from Displaced diffusion model using least squares method.
- We get the calibrated σ and β parameters which we use to calculate the implied volatility. Displaced diffusion has less degree of freedom as against SABR model hence it does not fit the market data perfectly.

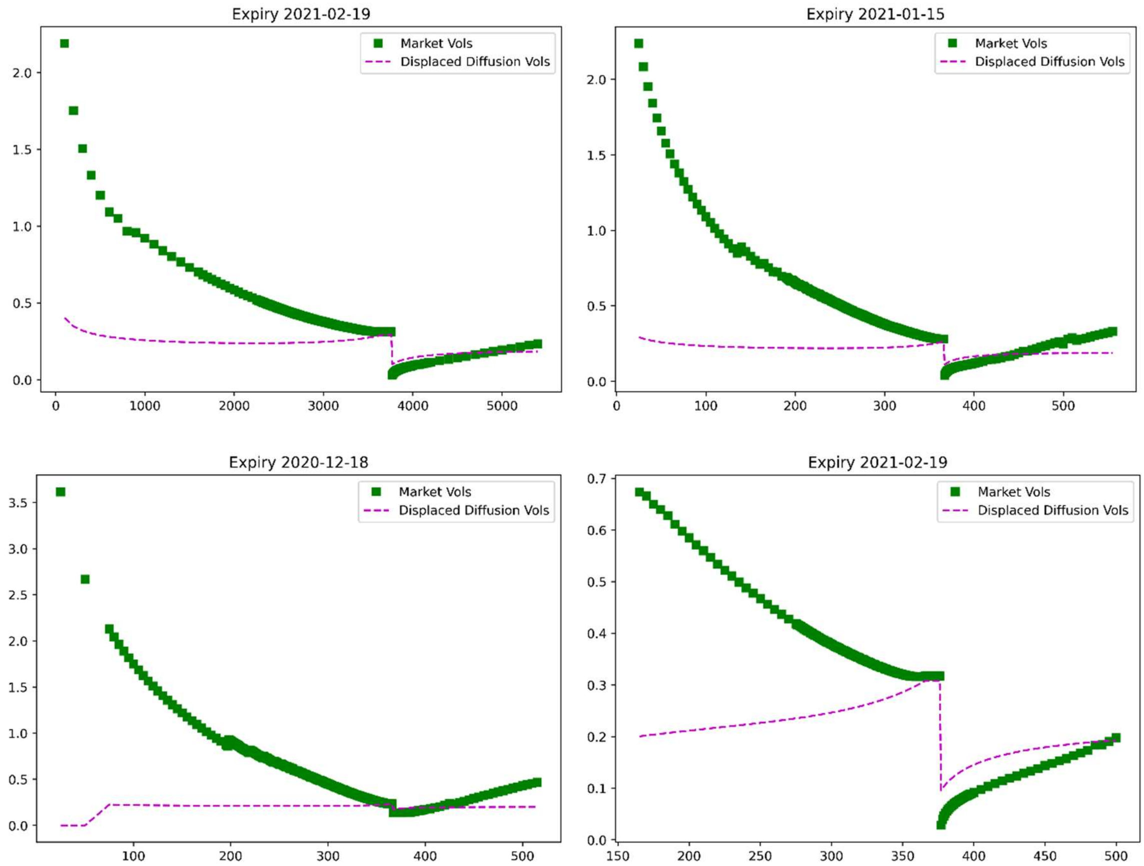
Calibrated Data and Volatility smiles as follows:

	Maturity	**σ (sigma)**	**β (beta)**
SPX	2020-12-18	0.183337	0.9367989
	2021-01-15	0.194327	0.878215
	2021-02-19	0.202368	0.838221
SPY	2020-12-18	0.198546	0.876404
	2021-01-15	0.203280	0.932851
	2021-02-19	0.206402	1.256487

β (beta) ranges from 0 to 1 where 0 being the complete normal Bachelier model and 1 being the lognormal Black Scholes Model. In our calibration we are getting **β (beta) value higher than 1** which is because we haven't set the parameter range while calibrating so model is trying to fit the data with possible beta value.



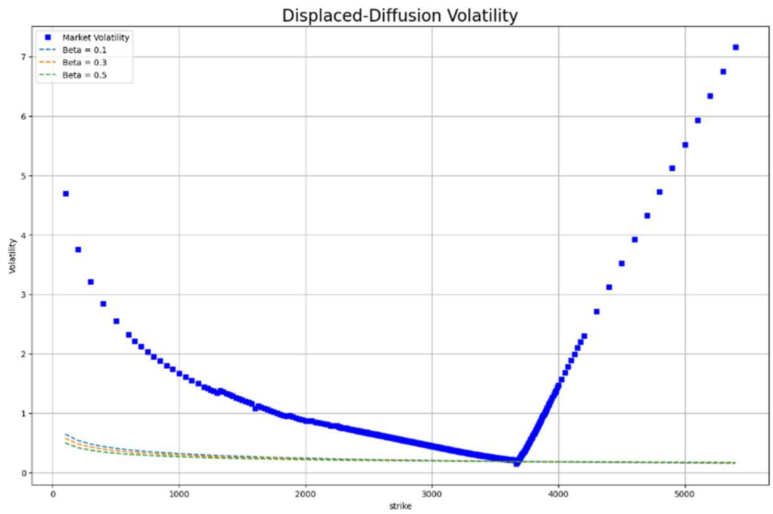




Model Parameter Sensitivity

Displaced Diffusion -

β (beta) parameter in Displaced Diffusion model decides the shift between normal and lognormal model. β (beta) = 1 is equal to lognormal model while β (beta) = 0 is normal model.

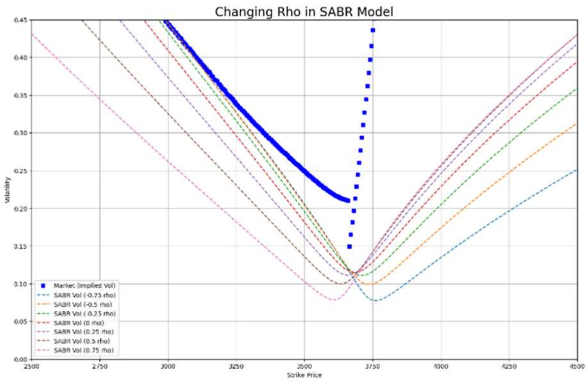


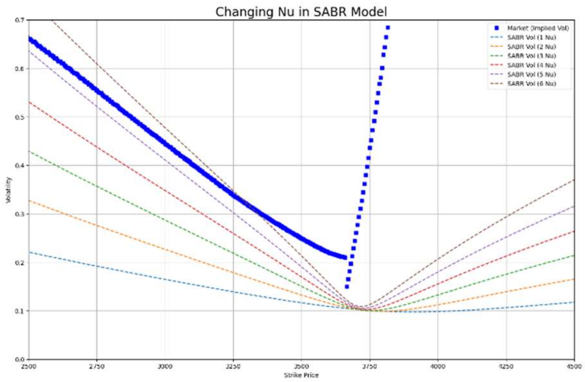
From the graph we can see that as we increase the beta value the curve becomes flatter suggesting the lognormal model.

SABR Model –

ρ (rho) is proportional to the **skewness of stock returns**. Positive correlation between stock and volatility is associated with positive skew in return distribution. Negative correlation between stock and volatility is associated with negative skew in return distribution.

v (nu) increases the **kurtosis of stock returns**, creating two fat tails in both ends of the distribution. This has the effect of raising out-of-the-money puts and out-of-the-money call prices.





Part 3: Static Replication

In this section we will derive the pricing formula for an exotic European option having following payoff

$$S_T^{1/3} + 1.5 \times \log(S_T) + 10.0$$

We will use Black Scholes, Bachelier and Carr-Madan Static Replication formula and compare the results.

For pricing the option, we will use the ATM implied volatility from the market since it's the most liquid option

Option Pricing Formulae

	Black Scholes	**Bachelier**
Formula	$V = e^{-rT}[S_0^{1/3} e^{rT/3 - \sigma^2 T/9} + 3/2 log(S_0) + 3/2(r - \sigma^2/2)T + 10]$	$V = e^{-rT}[S_0^{1/3} + 1.5 log S_0 - \frac{3\sigma^2 T}{4 S_0^2} + 10]$
Option Price	36.94057587	36.78118095

For any twice-differentiable payoff $h(St)$, Breeden-Litzenberger states that,

$$V_0 = e^{-rT} h(F) + \underbrace{\int_0^F h''(K) P(K)\, dK}_{\text{put integral}} + \underbrace{\int_F^\infty h''(K) C(K)\, dK}_{\text{call integral}}$$

$$h''(S_T) = -\frac{2}{9 S_T^{5/3}} - \frac{3}{2 S_T^2}$$

For static replication formula we will use the SABR volatility using the calibrated parameters. We get the option price 36.9319490 which is close to the Black and Bachelier model.

Part 4: Dynamic Hedging

The insight behind the Black-Scholes formula for options valuation is the recognition that, if you know the future volatility of a stock, you can replicate an option payoff exactly by a continuous rebalancing of a portfolio consisting of the underlying stock and a risk-free bond. If no arbitrage is possible, then the value of the option should be the cost of the replication strategy.

Black-Scholes provided with a statement about dynamic hedging/trading strategies as below:

$$C(S,K,\sigma,r,T) = \underbrace{S\Phi\left(\frac{\log\frac{S}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)}_{\text{stock holding}} - \underbrace{Ke^{-rT}\Phi\left(\frac{\log\frac{S}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)}_{\text{bond holding}}$$

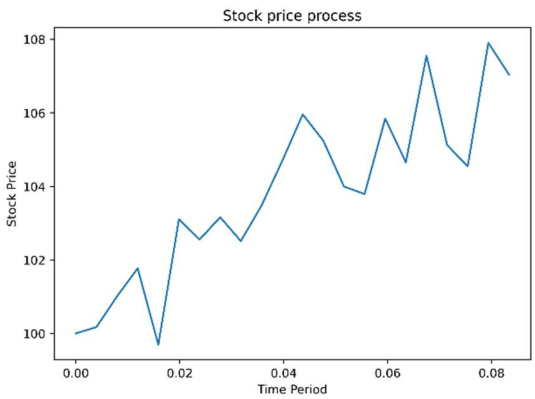
In this section, our goal is to calculate the hedging error of a dynamic delta hedging strategy. We will use Monte-Carlo simulation to simulate 50000 stock price paths and hedge the option at 21 and 84 discrete times.

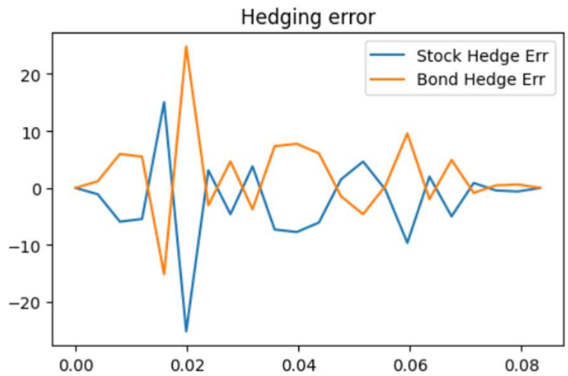
We will simulate the following call option

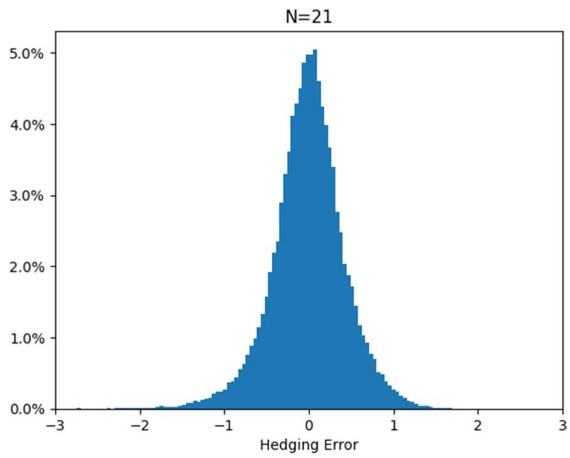
$S_0 = 100$, $K = 100$, $\sigma = 0.2$, $r = 5\%$, $T = 1/12$ i.e., 1 Month

Call option Price using Black Scholes = 2.512067

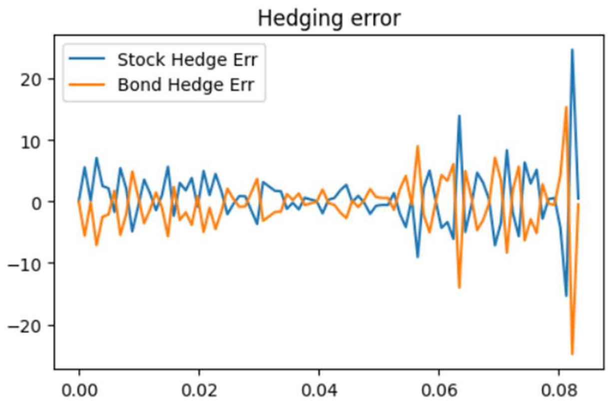
N = 21

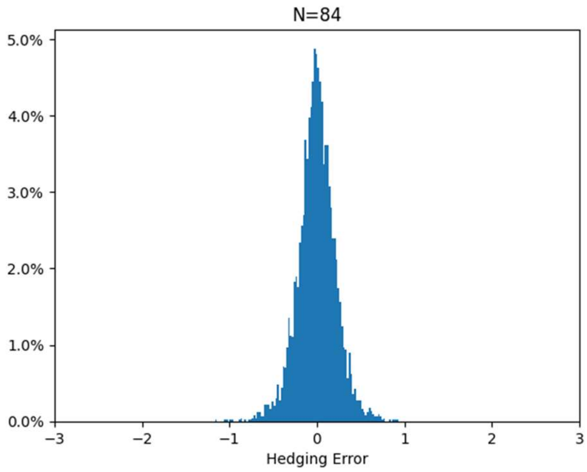






N = 84





Results

	Number of Trades	Mean Error	Std of Error	Std of Error as % of Premium
1	21	-0.002	0.427	17.003134
2	84	-0.001	0.215	8.567699

From results we can conclude that hedging more frequently reduces the std of error. The std as a fraction of the option premium is around 17% for N = 21 and 8.5% for N = 84; hedging four times as frequently roughly halves the std of the replication error.

Had we continuously hedged our portfolio, the error would have been perfect zero.

Reference: `http://pricing.free.fr/docs/when you cannot hedge.pdf`