EE386 Digital Signal Processing Lab	**Jul-Dec 2021**

Experiment 7

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1 Introduction

The exercise is based on FIR filter design. We begin by investigating window functions and the variation of their spectrum with window length after which we design a low pass FIR using Rectangular and Hanning windows and compare their impulse responses and bode plots. Next we extract the fundamental frequency from an instrument track and conclude by differentiating time domain windowing and window-based FIR filter design.
In our example we have $\alpha = 1 + mod(260, 3) = 3$.
All the code for this exercise and the relevant files are included in the `.zip` file submitted along with this report.

2 Problems

1. Window Functions
(Part 1: Comparing Window Functions)
(Solution)
The finite impulse response (FIR) filter designed using window method is a very popular, simpler and well working for various applications. Here we investigate the types of window functions for designing FIR filters.
In signal processing, the window function is a mathematical function that is zero-valued outside of certain interval. In engineering terms, a window is a finite array, consists of coefficients to satisfy the desirable requirements. Windowing method of FIR filter design involves the desired frequency response $h_d(\omega)$ and can be defined as:

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n} \tag{2.1}$$

where $h_d(n) = \frac{1}{2}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}$. In an ideal filter, length of $h_d(n)$ is infinite but in practical filter, shown in Figure 1 it must be truncated to some points say $n = M - 1$, where M is an integer. This truncation causes ripples in pass-band and stop-band section.

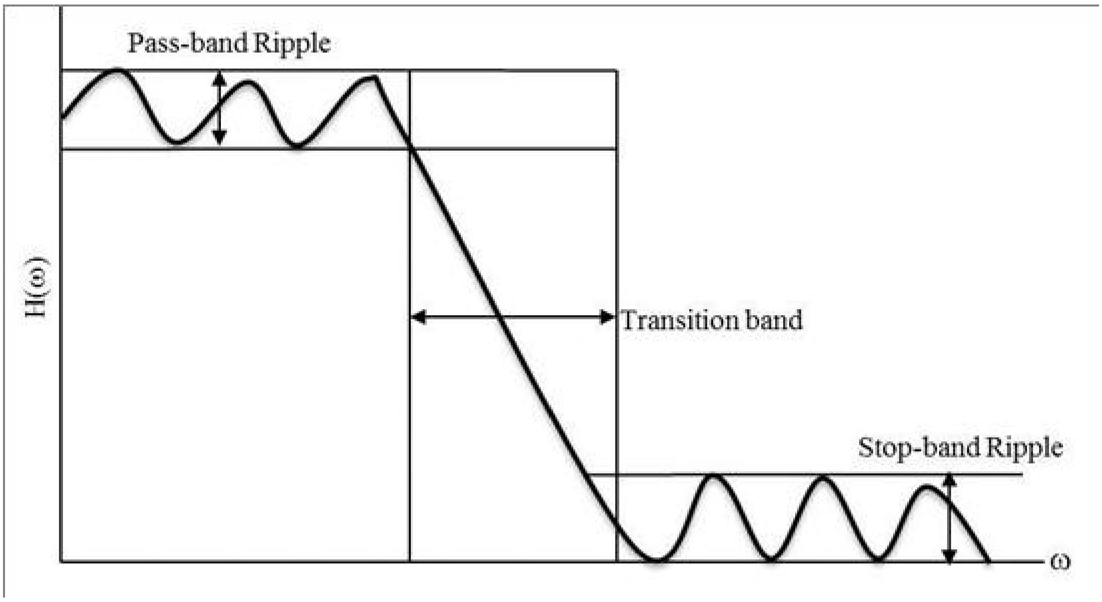

Figure 1: Magnitude characteristics of physically realizable filters

The band edge frequency ω_p defines the edge of the pass- band while the frequency ω_s denotes the beginning of the stop-band. Thus the width of transition band is $(\omega_p - \omega_s)$. The width of the pass-band is usually called the bandwidth of the filter. If a window has narrower main lobe width, its stop band attenuation is worse and vice-versa. Now we compare the performance of different types of window methods used for designing FIR low-pass filter.

- **Rectangular Window**

  Rectangular window, also called the boxcar window, is the simplest window function for FIR filter design. The function can be stated as follows

  $$w(n) = \begin{cases} 1, & n = 0, 1, 2, ..., M-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

  The unit sample response of the FIR filter becomes

  $$h(n) = h_d(n)w(n) \tag{2.3}$$

  where $h_d(n)$ is desired impulse response of the filter.

  $$h(n) = \begin{cases} h_d(n), & n = 0, 1, 2, ..., M-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

  The Fourier Transform of $w(n)$ gives the frequency domain representation of the window which is given by as follows

$$w(\omega) = \sum_{n=0}^{M-1} w(n)e^{-jwn} \tag{2.5}$$

$$w(\omega) = \sum_{n=0}^{M-1} e^{-jwn} \tag{2.6}$$

Figure 2 shows the rectangular window of Length 61 and its frequency response.

- **Hamming Window** Hamming window is the most popular and most commonly used window function. This window function belongs to cosine family. Hamming window function can be expressed as follows

$$w(n) = 0.54 - 0.42\cos(\frac{2n\pi}{M-1}) \tag{2.7}$$

  where M is the window length. The starting and ending points of this window do not touch the X-axis. This means that the coefficients of the hamming window are always greater than zero. Figure 2 shows the Hamming window of Length 61 and its frequency response.

- **Hanning Window** Hanning window belongs to a class of general cosine windows. The Hanning window co-efficient can be expressed as

$$w(n) = 0.5[1 - \cos(\frac{2n\pi}{M-1})] \tag{2.8}$$

  where M is the window length. Figure 2 shows the Hanning window of Length 61 and its frequency response. From the figure it is seen that the Hanning window has slightly less width with respect to the Hamming window.

From Figure 2, it is seen that the stop-band attenuation for Rectangular window is about -12 dB. On the other hand, Rectangular achieves minimum main-lobe width. It is observed that, Hamming window performs better in respect to normalized side-lobe peak. Hanning and the Hamming windows have the same main lobe width. The first side lobe attenuation of the Hanning window is about -34 dB and it decreases sharply as the frequency increases. Table 1 summarises our observations. Hence if frequency selectivity is a main priority the Rectangular window is preferable for FIR filter design and if stop-band attenuation is priority then Hamming or Hanning window can be used.

Window	Length	Mainlobe Width	Stopband Attenuation
Rectangular	61	0.062015	-12
Hamming	61	0.12818	-58
Hanning	61	0.12818	-34

Table 1: Characteristics of different window functions

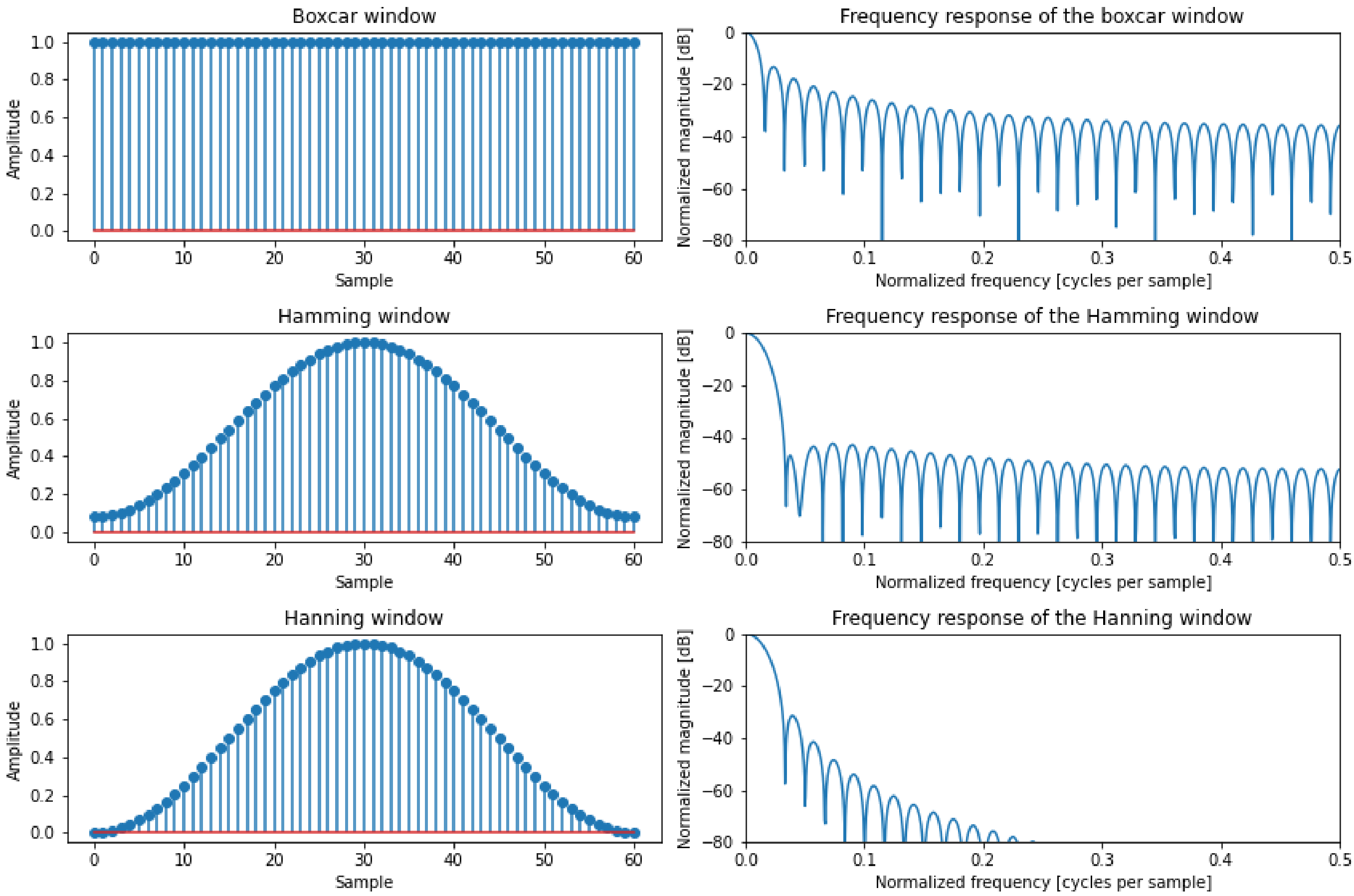

Figure 2: Magnitude characteristics of physically realizable filters

(Part 2: Spectrum of Hanning Window)

(Solution)

As seen in Part 1 of the problem, the frequency response of a filter designed with the window design method approximates a desired response $h_d(\omega)$, is determined by the two factors

- The width of the main lobe of $w(\omega)$
- The peak side-lobe amplitude of $w(\omega)$

Ideally the main-lobe width should be narrow for good frequency selectivity, and the side-lobe amplitude should be small for better stop-band attenuation. However for a fixed-length window, these cannot be minimized independently. Here we investigate the effect of variation of N on the frequency response. We have chosen the Hanning window and taken a 1024 point DFT for the different values of N. Figure 3 shows the magnitude response of the Hanning window for different values of N.

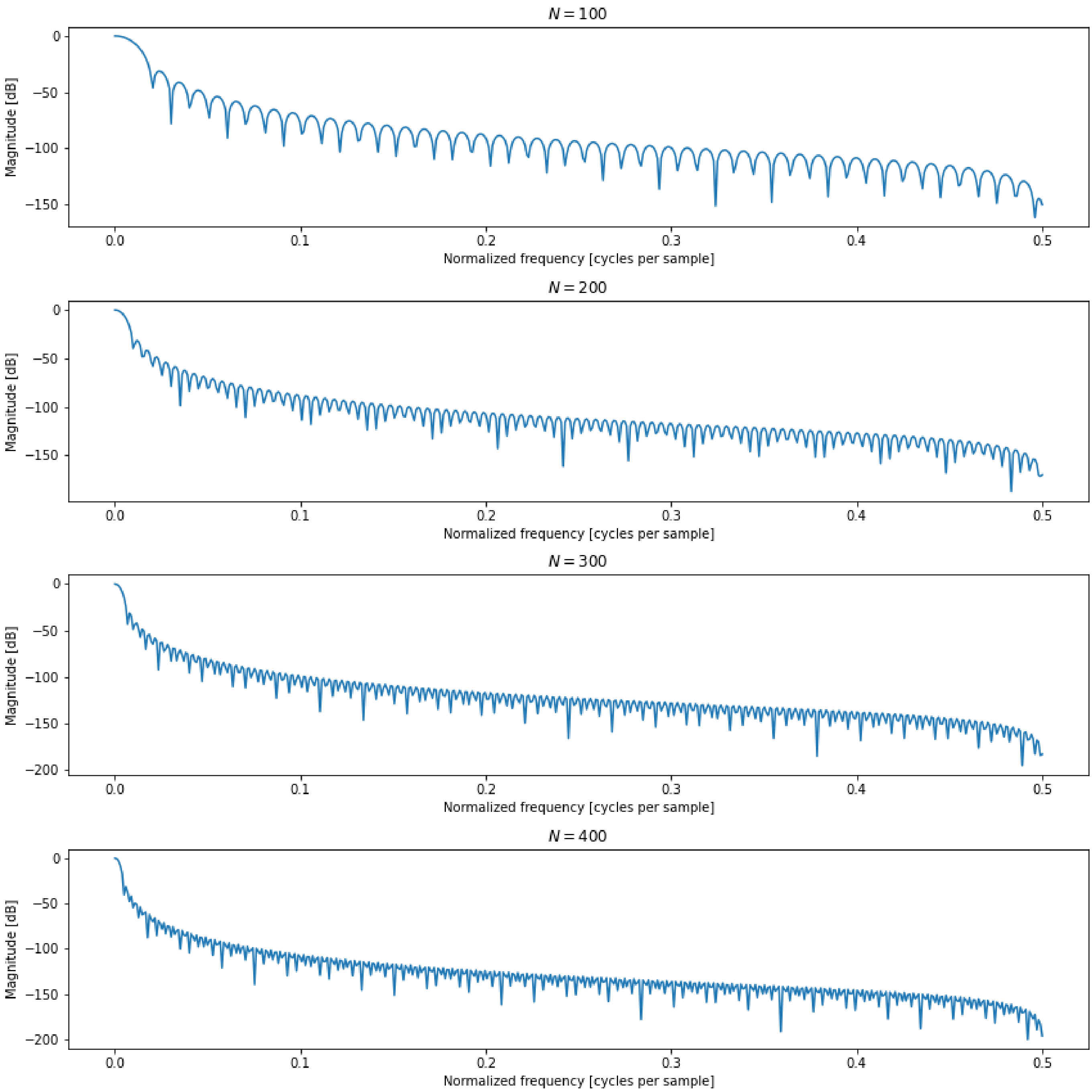

Figure 3: Magnitude Spectrum of Hanning window for different lengths

We observe that as the length N of the window increases, the width of the main lobe decreases. This results in decrease in the transition width between the pass-bands and stop-bands. This relationship is given approximately by

$$N\Delta f = c \tag{2.9}$$

where Δf is the transition width, and c is a parameter that depends on the window, $c = 3.1$ for Hanning window. As smaller transition width leads to better frequency selectivity, it can be said that for a given window as N increases, the frequency selectivity of the filter increases. The peak side-lobe amplitude of the window is determined by the shape of the window, and it is essentially independent of the window length. Hence we can observe in Figure 3 that the amplitude of the peak side-lobe has remained constant for different values of N.

2. FIR Filter Design

(Part 1: Impulse response of FIR filters)

(Solution)

We use the built in SciPy function `sp.firwin` to obtain the coefficients of the finite impulse response filter using the Rectangular (Boxcar) and Hanning window functions. The window length is 21 and cutoff frequency $\omega_c = \frac{\pi}{4}$ rad/sample, $f_c = 0.125$ cycles/sample. Figure 5 and Figure 6 show the Bode plots of the two filters. Figure 4 illustrates the impulse response of the two filters. The impulse response is just the set of FIR coefficients, in other words the output of the filter will be simply the set of coefficients as the 1 valued sample moves past each coefficient in turn to form the output. We observe that for the rectangular window, the impulse response is essentially $h_d(n) = \mathcal{F}^{-1}\{e^{-j\alpha\omega}H_d(e^{j\omega})\}$ where $\alpha = 10$. For the Hanning window, $h(n) = h_d(n)w(n)$ and so we observe the tapering of the response towards the ends following the hanning window shape.

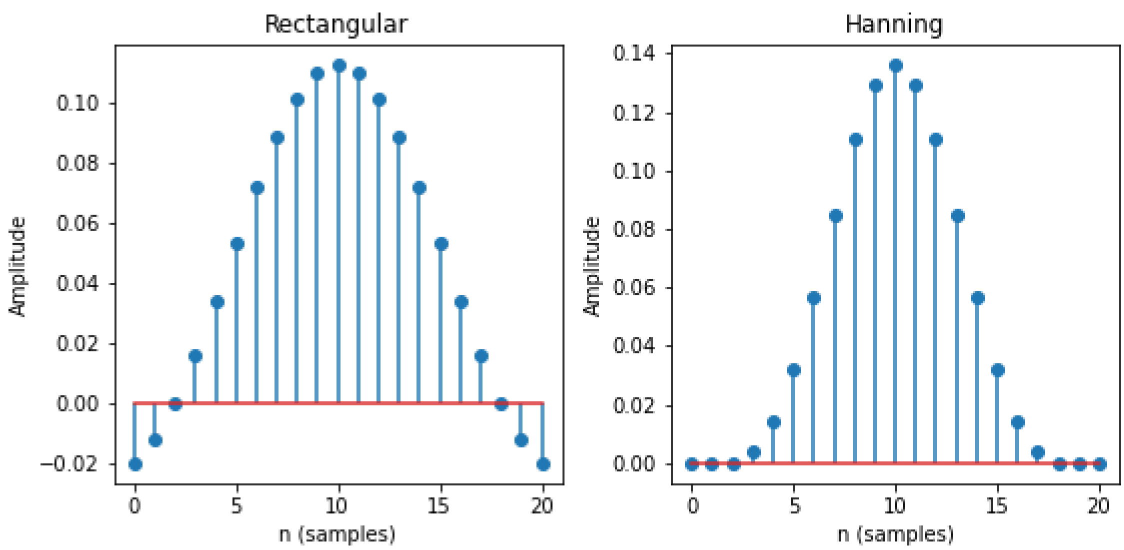

Figure 4: Impulse response of the two FIR filters

(Part 2: Bode plot of FIR filters)

(Solution)

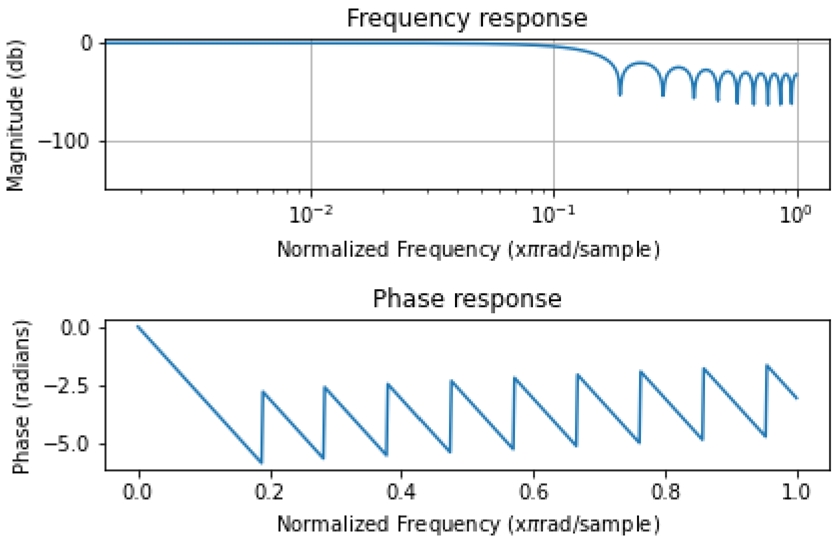

Figure 5: Bode plot of FIR filter using Rectangular Window

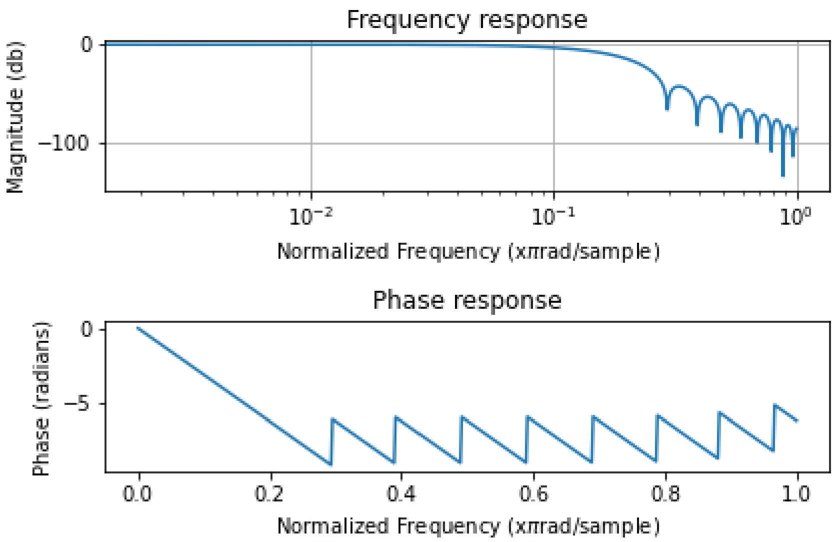

Figure 6: Bode plot of FIR filter using Hanning Window

Figure 5 and Figure 6 show the Bode plots of the FIR filters using Rectangular and Hanning window respectively. As investigated in Problem 1: Part 1, we observe that the width of the main lobe of the Hanning window is wider than that of Rectangular window. Thus the transition width is narrow and frequency selectivity is higher in the FIR filter using the Rectangular window. Conversely the amplitude of the side lobes is smaller in Hanning window and rolls off as frequency is increased as opposed to rectangular window. Hence the stop-band attenuation is better in the FIR filter using Hanning window. From Figure 4 we can see that the impulse response of the FIR filter is symmetric. That is

$$h(n) = h(N - 1 - n), \ n = 0, 1, 2, ..., 20 \tag{2.10}$$

Every real symmetric impulse response corresponds to a real frequency response times a linear phase term $e^{-j\alpha\omega T}$, where $\alpha = \frac{N-1}{2}$ is the slope of the linear phase. A linear phase of this form corresponds to a phase delay. That is, both the phase and group delay of the linear-phase filter are equal to $\frac{N-1}{2}$ and can be observed from the phase spectrum in Figure 5 and 6. Hence FIR filters make an excellent choice when a causal filter is needed to modify a signal's magnitude-spectrum while preserving the signal's time domain waveform as much as possible.

3. Filtering using FIR Filters

(Extracting the fundamental frequency)

(Solution)

Figure 7 shows the spectrogram of the instrument signal using Hanning window of length 512 samples and 25% overlap.

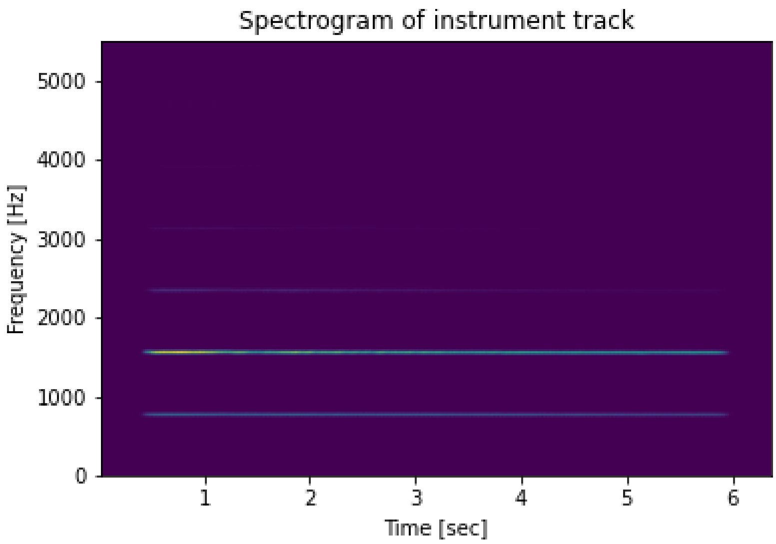

Figure 7: Spectrogram of instrument track

In order to extract the fundamental frequency, we design a Band-pass digital FIR filter with $F_l = 500$ Hz and $F_u = 900$ Hz using the Blackman-Harris window. Figure 8 shows the frequency response of the filter.

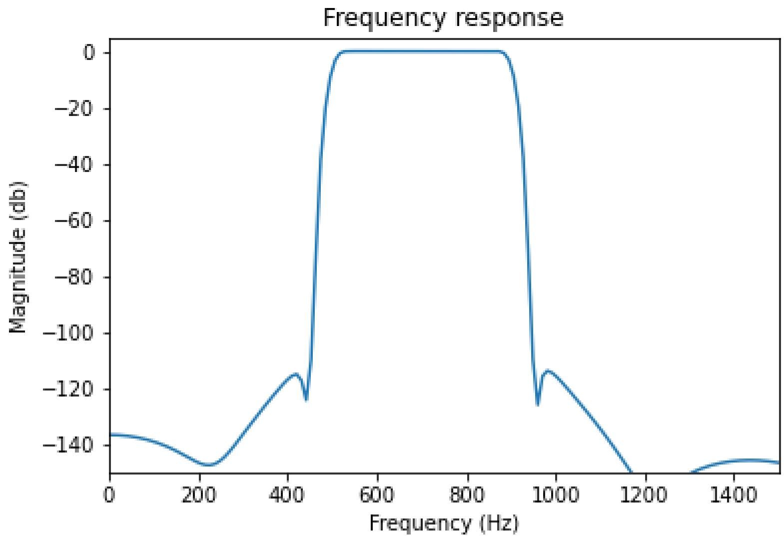

Figure 8: Frequency response of Band-pass FIR filter

In music theory, pitch is the perceived fundamental frequency of a sound, however the actual fundamental frequency may differ from the perceived because of the overtones. It is these overtones together that give an instrument its timbre. When we listen to the audio after filtering, it sounds almost like computer generated audio. The timbre of the instrument is lost indicating the attenuation of overtones. Figure 9 shows the spectrogram of the filtered signal for the same specifications as the original and it confirms that only the fundamental is present in the signal.

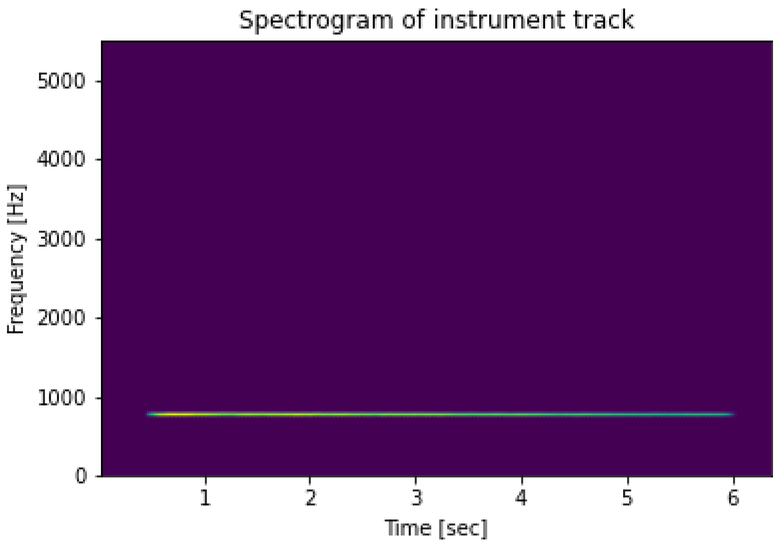

Figure 9: Spectrogram of the filtered instrument track

4. Time-domain windowing and the window-based FIR filter design

(Difference between the two)

(Solution)

Both Time-domain windowing and window-based FIR filter design are essentially two sides of the same coin where the properties of the frequency response of various window functions are exploited to achieve completely different purposes. Time-domain windowing is mainly used in spectral analysis. The problem there is that DFT implicitly assumes that the signal essentially repeats itself after the measured interval. However, many times, the measured signal isn't an integer number of periods. Therefore, the finiteness of the measured signal may result in a truncated waveform with different characteristics from the original continuous-time signal introducing sharp transition changes into the measured signal. The sharp transitions are discontinuities. These artificial discontinuities show up in the DFT as frequency components not present in the original signal. The spectrum we get by using a the DFT, therefore, is not the actual spectrum of the original signal, but a smeared version. This phenomenon is known as spectral leakage. Windowing is used to minimize the effects of performing a DFT over a noninteger number of cycles as it reduces the amplitude of the discontinuities at the boundaries of each finite sequence by multiplying the time record by a finite-length window with an amplitude that varies smoothly and gradually toward zero at the edges. This makes the endpoints of the waveform meet and, therefore, results in a continuous waveform without sharp transitions. Hence we choose a window that has the side lobe attenuation that meets our need to reduce spectral leakage. Spectral resolution or frequency resolution is an important property of the DFT and is its ability to resolve two signals with similar spectral content. The narrower the width of the main-lobe, the better the spectral resolution and it can be improved by increasing the window length. Thus frequency resolution and spectral leakage related to main lobe wide and amplitude of side lobes respectively become our main concern in time domain windowing.

In FIR filter design, the ideal impulse response $h_d(n)$ will generally be infinite in length, necessitating an FIR approximation. With the window design method, the filter is designed by windowing the unit sample response $h(n) = h_d(n)w(n)$ where $w(n)$ is a finite length window that is equal to zero outside the interval $0 \leq n \leq N$ and is symmetric about its midpoint. The effect of the window on the frequency response may be seen from the complex convolution theorem,

$$H(e^{j\omega}) = \frac{1}{2\pi} H_d(e^{j\omega}) * W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta \tag{2.11}$$

Thus, the ideal frequency response is smoothed by the discrete-time Fourier transform of the window, $W(e^{j\omega})$. How well the frequency response of a filter designed with the window design method approximates a desired response is determined by frequency selectivity which depends on the width of the main lobe and stop-band attenuation which is determined by the peak side-lobe amplitude of the frequency response of the window, becoming design considerations.