This week is about modelling some real-world data — the growth, stabilisation and saturation of epidemics and studying effects such as virulence and social distancing.

Problem 1. (First-Order Model)
The propagation mechanism of an epidemic, such as the one caused by the SARS-CoV-2 virus, can be modelled, at least in its initial phase, as a process in which each infected individual will eventually transmit the disease to an average of R_0 healthy people; these newly infected patients will, in turn, infect R_0 healthy individuals each, and so on, creating a pernicious positive feedback in the system. The constant R_0 is called the *basic reproduction number* for a virus.

In signal processing terms, the infection mechanism is equivalent to a first-order recursive filter. Assume that each infected person spreads the virus over a single day and then recovers and assume that an initial patient zero appears at day $n = 0$. The number of newly infected people per day is described by the difference equation

$$y[n] = \delta[n] + R_0 y[n-1]. \tag{1}$$

1. What is the transfer function $H_1(z)$ of the above system? Plot the pole-zero plot of the system.

2. Solve the difference equation and give the time-domain equation for the number of newly infected people. Note that this depends on the parameter R_0. Comment on the effect of the parameter R_0. Can this be inferred from the pole-zero plot?

3. Suppose $R_0 = 2.5$, how many days will it take to reach 1 million new daily infections?

4. Using a similar one-point trick, and the data available from Covid-19 in India, estimate the value of R_0 in the initial phase of the first wave of infections in India. Note that this is not a reliable method to estimate R_0 (why?). *Bonus:* Comment on a reliable technique to estimate R_0.

5. With $R_0 = 2.5$, plot the new daily infections for the first $n = 20$ days. Design an integrator filter. Use this filter to obtain the total number of infections for the first $n = 20$ days.

Problem 2. (Increasing the Complexity)
The actual infectiousness of SARS-CoV-2, however, is not concentrated in a single day but follows a curve that rises and falls with the duration of the sickness (cf. Figure 1).
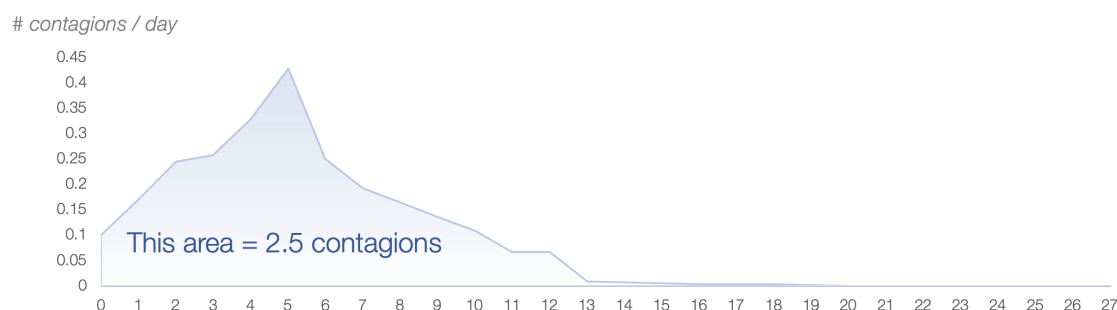

Figure 1: Virulence of SARS-CoV-2 [source].

Figure 1 scientifically substantiates the 14-day quarantine rules. Also note, the virulence decreases from day-7 on. The system, now, is a multi-pole IIR filter with the

transfer function

$$H_M(z) = \frac{1}{1 - \sum_{k=1}^{M} a_k z^{-k}}. \tag{2}$$

The coefficients $(a_k)_{k=1}^{M}$ are the daily values that can be obtained from Figure 1 and for $M = 12$ are given by $\{.1, .15, .25, .26, .34, .42, .25, .2, .15, .1, .1, .1\}$.

1. Plot the new daily infections for the first $n = 100$ days by implementing the filter given in Eq. (2) with the Kronecker delta as the input. Use an integrator filter to obtain the total number of infections for $n = 100$ days.

2. Comment on the differences between the trends that are obtained with the first-order model. How many days will it take to reach 1 million new daily infections?

3. *Bonus:* Comment on a reliable technique to estimate the coefficients $(a_k)_{k=1}^{M}$.

Problem 3. (Effects of Social Distancing)
The idea of social distancing is predicated on the fact that the coefficients $(a_k)_{k=1}^{M}$ are dependent on the number of encounters between a sick individual and healthy people. By reducing the number of encounters, we proportionally reduce the coefficients. Let's assume that we can reduce the social interaction of every individual by a factor $0 < \rho \le 1$. Then the coefficients of the filter will be scaled and we get an IIR filter with transfer function

$$H_M(z; \rho) = \frac{1}{1 - \sum_{k=1}^{M}(1 - \rho)a_k z^{-k}}. \tag{3}$$

1. Comment on the role of ρ. What does $\rho = 1$ indicate?

2. Plot the new daily infections for the first $n = 100$ days by implementing the filter given in Eq. (3) with the Kronecker delta as the input for $\rho = 0.25, 0.50, 0.75$. Use an integrator filter to obtain the total number of infections for $n = 100$ days.

3. Comment on the effects of social distancing by inferring the results. Does social distancing reduce the number of total infections? How does social distancing help?

Problem 4. (Saturation and Towards Normality)
No natural system can support a purely exponential growth. In the case of a viral epidemic, as more and more people contract the disease and achieve immunity, the rate of transmission for the infection progressively decreases. If the rate of diffusion is assumed to be inversely proportional to the fraction of healthy people in a population, the evolution of the cumulative number of infections $x[n] = \sum_{k=0}^{n} y[k]$ since the beginning of the disease can be modelled by a logistic function:

$$x[n] = \frac{K}{1 + [K(R_0 - 1) - R_0]R_0^{-(n+1)}} - \frac{1}{R_0 - 1}, \tag{4}$$

where K is the population size. The notable thing about the logistic function is that it has a clear inflection point, after which the epidemic starts to level out; this corresponds to the moment in which the implicit reproduction number becomes less than one. It would be useful to detect the inflexion point because in that case some of the more restrictive measures could start to be relaxed gradually.

The inflection point corresponds to the global maximum of the first derivative of the logistic function. We can approximate the derivative with a simple two-tap FIR filter of the form $D_1(z) = 1 - z^{-1}$. We can also look directly at the zero-crossing of the second derivative, approximated with the FIR filter $D_2(z) = 1 - 2z^{-1} + z^{-2}$.

1. With $R_0 = 1.15$ and $K = 10^6$, plot the total number of infections in the first-order model versus the logistic evolution for a population of one million for $n = 100$ days. Does the first order model follow the logistic hypothesis?

2. Find the point of inflection using the global maximum of the first derivative and the zero-crossing of the second derivative.

Congratulations! This is the last experiment in the course. You did it!