EE386 Digital Signal Processing Lab **Jul-Dec 2021**

Experiment 4

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1 Introduction

The exercise is based on spectral analysis using DFT. We begin by computing DFT for various samples sizes and witness the phenomenon of spectral leakage, then the variation of frequency resolution with sample size is explored, windowing using Blackman and Hamming windows and estimation of frequency components in a given signal is done.
In our example we have $\alpha = 1 + mod(260, 3) = 3$.
All the code for this exercise and the relevant files are included in the `.zip` file submitted along with this report.

2 Problems

1. Computing DFT

(Part 1: Plotting magnitude spectrum)

(Solution)

Discrete Fourier Transform

The Discrete-Time Fourier Transform (DTFT) is the primary theoretical tool for understanding the frequency content of a discrete-time (sampled) signal. The DTFT is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{2.1}$$

The inverse DTFT (IDTFT) is defined by an integral formula, because it operates on a continuous-frequency DTFT spectrum:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(k)e^{j\omega n}d\omega \tag{2.2}$$

The DTFT is very useful for theory and analysis, but is not practical for numerically computing a spectrum digitally, because infinite time samples means infinite computation and delay. For practical computation of the frequency content of real-world signals, the Discrete Fourier Transform (DFT) is used.

The DFT transforms N samples of a discrete-time signal to the same number of discrete frequency samples, and is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} \tag{2.3}$$

The DFT is invertible by the inverse discrete Fourier transform (IDFT):

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi nk}{N}} \tag{2.4}$$

The DFT and IDFT are a self-contained, one-to-one transform pair for a length-N discrete-time signal. However, the DFT is very often used as a practical approximation to the DTFT.
When a discrete-time sequence happens to equal zero for all samples except for those between 0 and $N-1$, the infinite sum in the DTFT equation becomes the same as the finite sum from 0 to $N-1$ in the DFT equation. That is, the DFT computes exact samples of the DTFT at N equally spaced frequencies $\omega_k = \frac{2\pi k}{N}$.
In most cases, the signal is neither exactly periodic nor truly of finite length; in such cases, the DFT gives frequency samples of a windowed (truncated) DTFT.
The goal of our spectrum analysis is to determine the frequency content of an analog (continuous-time) signal. We accomplish this by sampling the analog signal, windowing (truncating) the data, and computing and plotting the magnitude of its DFT. It is thus essential to relate the DFT frequency samples back to the original analog frequency. Assuming that the analog signal is bandlimited and the sampling frequency F_s is such that no frequency aliasing occurs, the relationship between the continuous-time Fourier frequency Ω (in radians) and the DTFT frequency ω imposed by sampling $\omega = \Omega T$ where T is the sampling period. Through the relationship $\omega_k = \frac{2\pi k}{N}$ between the DTFT frequency ω and the DFT frequency index k, the correspondence between the DFT frequency index and the original analog frequency can be found:

$$\Omega = \frac{2\pi k}{NT} \tag{2.5}$$

or in terms of analog frequency F in Hertz (cycles per second rather than radians)

$$F = \frac{k}{NT} \tag{2.6}$$

When using DFT to study the frequency domain characteristics of a signal, there are two limits

- The detect-ability of a small signal in the presence of a larger one
- Frequency resolution – which distinguishes two different frequencies

In reality, signals are of time-limited nature and nothing can be known about the signal beyond the measured interval. DFT implicitly assumes that the signal essentially repeats itself after

the measured interval. When we choose number of samples equal to integral multiple of signal period, this periodicity is maintained during the repeated juxtaposition.
In our example, a unit amplitude sinusoidal signal of frequency $F = 45$ Hz is sampled at $F_s = 120$ samples/second for a duration of 2 seconds giving $x(n) = \sin(0.75\pi n)$, normalized frequency $f = \frac{3}{8}$ cycles/sample and period $N = 8$. The blue plot in Figure 1 is the DFT for the first 120 samples and is zero for all values other than 45 Hz.
When the number of samples is purposefully taken to be a non-integral multiple of the actual signal period, it leads to glitches in the signal assumed by DFT during the repeated juxtaposition. These sharp discontinuities will spread out in the frequency domain leading to the phenomenon of spectral leakage. The orange plot in Figure 1 illustrates spectral leakage in the DFT for the first 130 samples of the signal. We observe that the DFT is non zero for all frequencies and spreads about the 45 Hz point. In addition we also note the decrease in the magnitude compared to the blue plot.

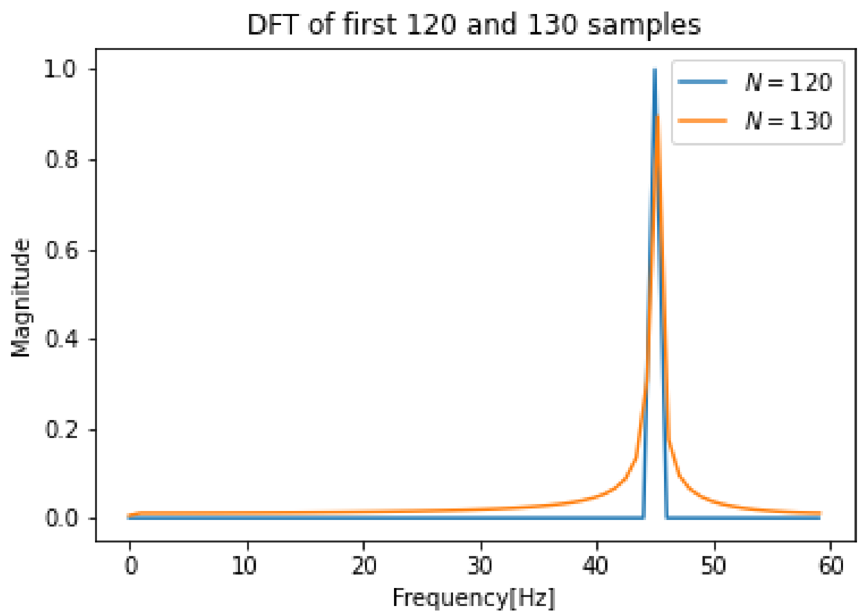

Figure 1: DFT of the first 120 and 130 samples of the signal

(Part 2: Determining the value of N to match DFT of first 120 samples)
(Solution)
As mentioned in Part 1 of this problem, when we choose number of samples equal to integral multiple of signal period, periodicity is maintained during the repeated juxtaposition. Hence for $N = 240$ we can observe in Figure 2 that the indeed the DFTs of the two sequences are the same.

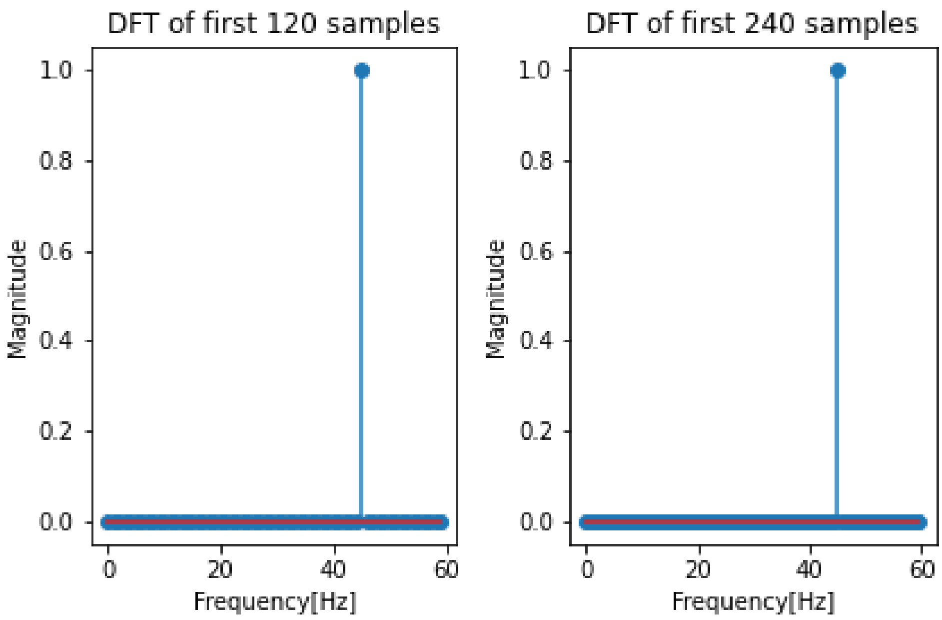

Figure 2: Identical DFT for 120 and 240 samples

2. Resolution of DFT)

(Plotting DFT for different sample sizes)
(Solution)

Spectral resolution or frequency resolution is an important property of the DFT and is its ability to resolve two signals with similar spectral content. The frequency resolution is the sample frequency divided by the number of samples

$$\Delta F = \frac{F_s}{N} \tag{2.7}$$

When taking the DFT of a signal, the size of the transform is equivalent to the number of frequency bins that will be created. Each bin represents the amount of energy that the signal has at that particular frequency. The frequency resolution is the difference in frequency between each bin, and thus sets a limit on how precise the results can be. The most intuitive way to increase the frequency resolution of the DFT is to increase the size while keeping the sampling frequency constant. Doing this will increase the number of frequency bins that are created, decreasing the frequency difference between each.

In our example we sample the analog signal having frequencies $F = 80$ Hz for the sine and $F = 83$ Hz for the cosine signal $x_a(t) = 0.1\sin(160\pi t) + \cos(166\pi t)$ at a rate of $F_s = 200$ samples/second for a duration of 10 seconds giving $x(n) = 0.1\sin(0.8\pi n) + \cos(0.83\pi n)$, $f = \frac{2}{5}$ cycles/sample and $N = 5$ for the sine signal and $f = \frac{83}{200}$ cycles/sample and $N = 200$ for the cosine signal and overall $N = 200$ for the signal.

Figure 3 shows the DFT of the signal for 215, 415, 1115, 1515, 1915 samples. We immediately notice spectral leakage in the 215 sample DFT plot and the presence of only one peak at 83 Hz. As we increase the number of the samples, we are able to distinguish the other smaller peak at 80 Hz. We also notice that the peaks get thinner and sharper thus confirming that the frequency resolution improves as we increase the number of samples.

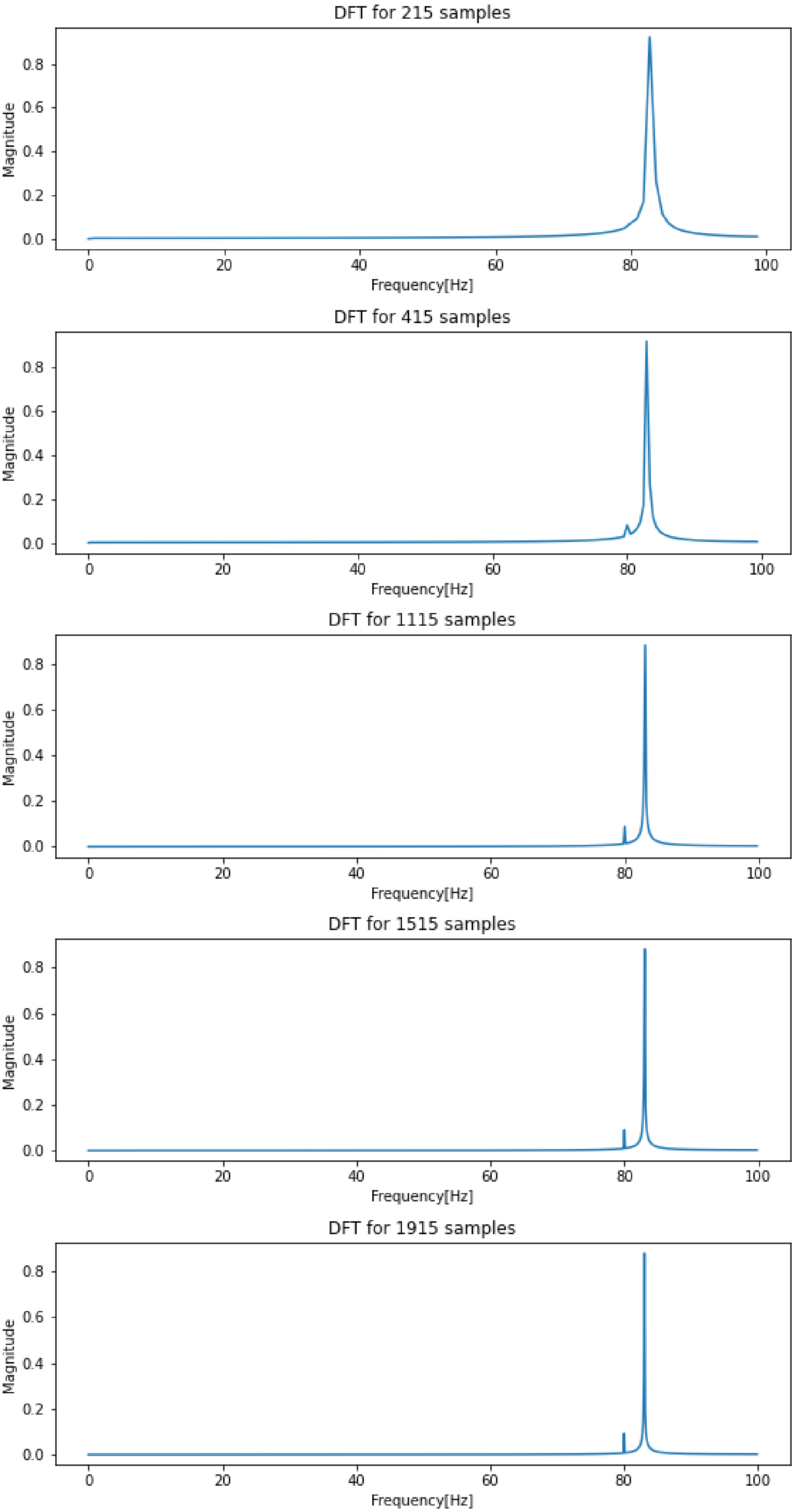

Figure 3: DFT of the signal for various sample sizes

3. Resolution of DFT with windowing

(Using Blackman window)

(Solution)

As mentioned in part 1 of Problem 1, DFT implicitly assumes that the signal essentially repeats itself after the measured interval. However, many times, the measured signal isn't an integer number of periods. Therefore, the finiteness of the measured signal may result in a truncated waveform with different characteristics from the original continuous-time signal, and the finiteness can introduce sharp transition changes into the measured signal. The sharp transitions are discontinuities. When the number of periods in the acquisition is not an integer, the endpoints are discontinuous. These artificial discontinuities show up in the DFT as frequency components not present in the original signal. The spectrum we get by using a the DFT, therefore, is not the actual spectrum of the original signal, but a smeared version. It appears as if energy at one frequency leaks into other frequencies. This phenomenon is known as spectral leakage, which causes the fine spectral lines to spread into wider signals. We can observe spectral leakage in the plots of Figure 3 as the frequency peaks spread. We can minimize the effects of performing a DFT over a noninteger number of cycles by using a technique called windowing. Windowing reduces the amplitude of the discontinuities at the boundaries of each finite sequence by multiplying the time record by a finite-length window with an amplitude that varies smoothly and gradually toward zero at the edges. This makes the endpoints of the waveform meet and, therefore, results in a continuous waveform without sharp transitions.

There are several different types of window functions that can be applied depending on the signal, however for our example we choose the Blackman window which is a taper formed by using the first three terms of a summation of cosines. It was designed to have close to the minimal leakage possible.

The Blackman window is a taper formed by using the first three terms of a summation of cosines. It was designed to have close to the minimal leakage possible. The Blackman Window is defined as

$$w(n) = 0.42 - 0.5\cos(\frac{2\pi n}{M}) + 0.08\cos(\frac{4\pi n}{M}) \tag{2.8}$$

Figure 4 shows the DFT of the signal in Problem 2 multiplied by Blackman Window of corresponding size for 215, 415, 1115, 1515, 1915 samples. At once we can notice the reduction in spectral leakage compared to Figure 3. For the 215 samples we can even notice the presence of another peak as opposed to Figure 3. With the increase in the samples the frequency resolution improves giving us the two peaks with almost zero spectral leakage for 1915 samples. However we notice the reduction in magnitude of the peaks and the peaks being wider in the 215 plot indicating that the for a given sample size the frequency resolution has reduced in comparison to Figure 3. Hence windowing helps with spectral leakage at the cost of frequency resolution.

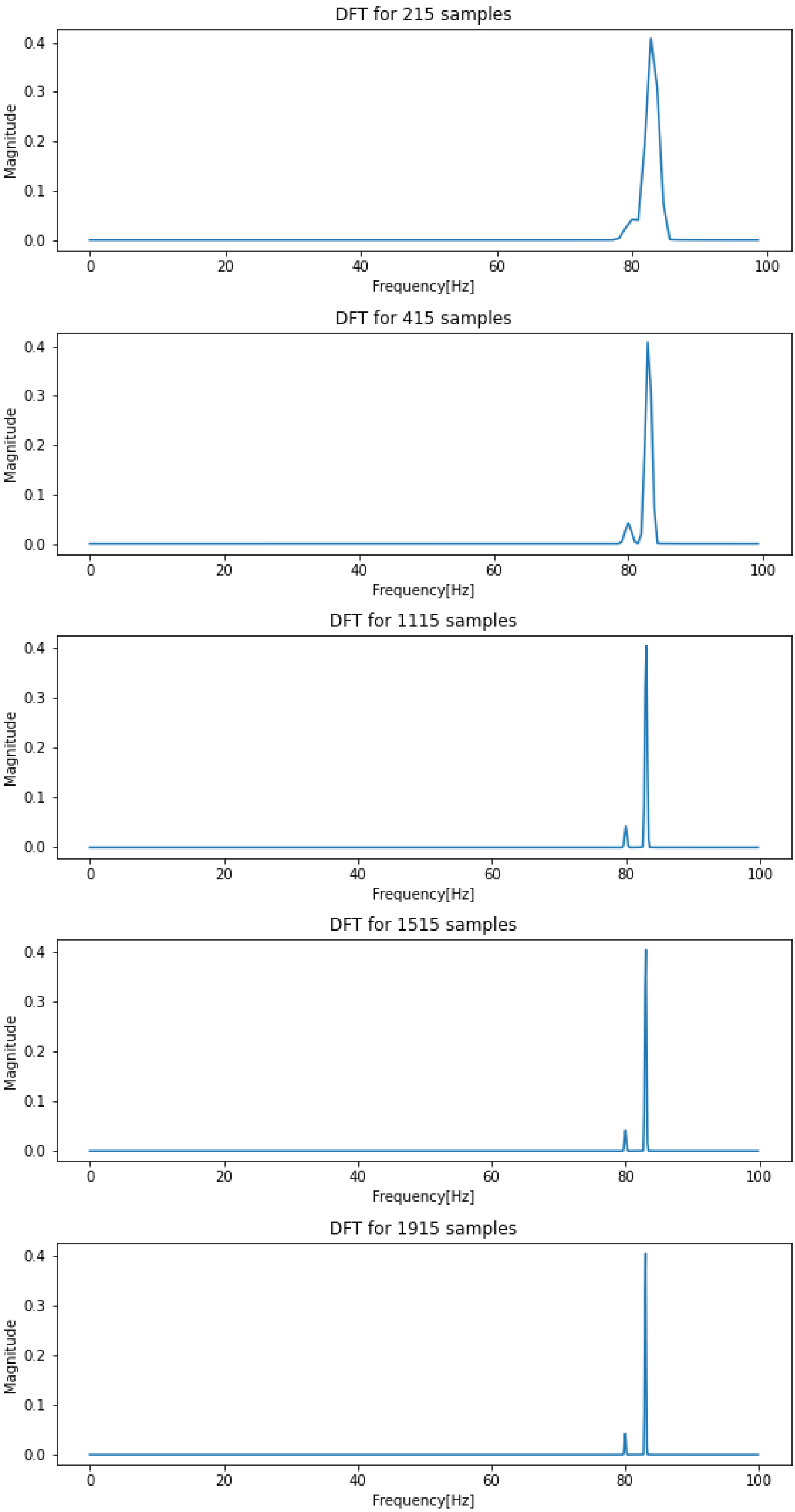

Figure 4: DFT of the signal for various sample sizes using Blackman window

4. Frequency estimation using windowing

(Part 1: Using Hamming Window)

(Solution)

Zero-Padding

In order to get a clear picture of the spectrum we can pad the discrete-time signal with zeros and compute the DFT of the longer length. Zero-padding interpolates the spectrum and hence no new data is added in the process. `np.fft.fft` does zero-padding whenever the the input for DFT size exceeds the signal length. Our aim is to estimate the frequency components of a dual tone signal using the Hamming Window which is defined as

$$w(n) = 0.54 - 0.46\cos(\frac{2\pi n}{M-1}) \text{ for } 0 \leq n \leq M-1 \tag{2.9}$$

We begin by multiplying the signal with Hamming window of corresponding length and taking the 10000 point DFT at a sampling rate of $F_s = 1$ samples/second by zero padding. Figure 5 shows the obtained DFT and a zoomed in view near the two peaks.

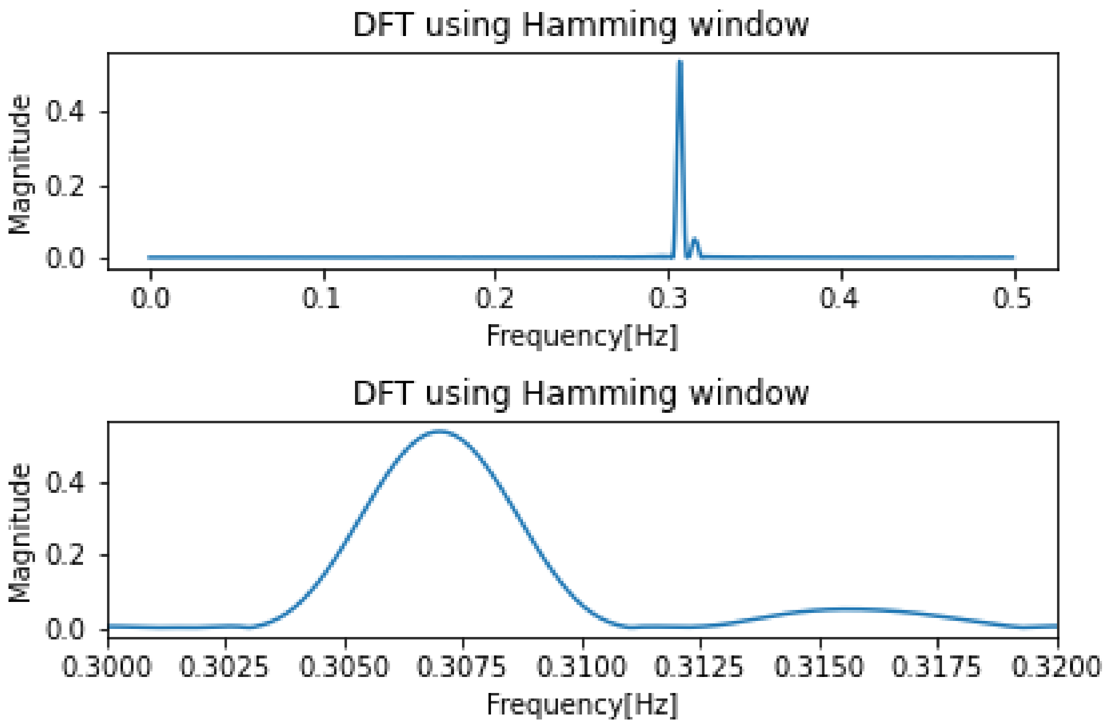

Figure 5: DFT of the signal using Hamming Window

We use the `find_peaks` function in order to obtain the frequency in terms of F_s and obtain $F_1 = 0.307F_s$ and

$F_2 = 0.3156F_s$

(Part 2: Using Rectangular Window)

(Solution)

The boxcar window or rectangular window is equivalent to having no window at all. Hence we can directly work with the given signal. We follow the exact same procedure as we did in Part 1 of this problem by obtaining the 10000 DFT of the signal at sampling rate $F_s = 1$ samples/second. Figure 6 shows the obtained DFT along with a zoomed in view near the peak.

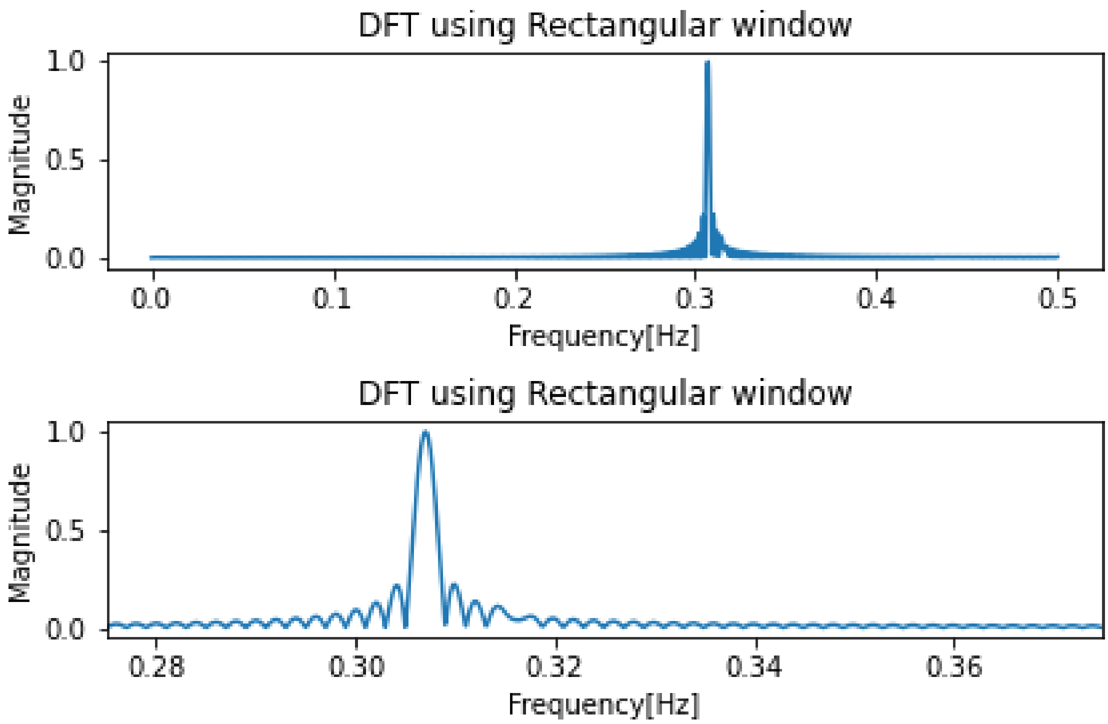

Figure 6: DFT of the signal using Rectangular Window

We apply the same procedure of using `find_peaks` to obtain the peaks of the signal. However here we obtain $F_1 = 0.307F_s$ and $F_2 = 0.3099F_s$

While the first answer maybe the same, we get different answers for F_2 because of spectral leakage in the DFT with rectangular window. Windowing improves spectral leakage as seen in Figure 5, the peaks are distinct and do not spread as much. Hence the answer obtained with the rectangular window is not reliable due to possible overshadowing of the correct frequency due to the spectral leakage. Hence we can agree with the answer obtained in Part 1.