FUNDAMENTALS OF CONVECTIVE HEAT TRANSFER

Prof. Amaresh Dalal

Mechanical Engineering

IIT Guwahati

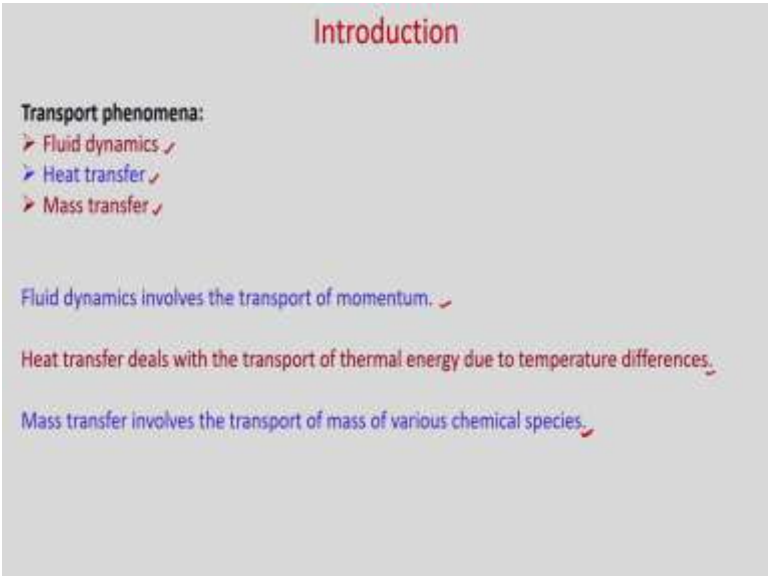
INDEX

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 01
Introduction
Lecture - 01
Application of convective heat transfer

Hello everyone. Welcome to this course on Fundamentals of Convective Heat Transfer. I am Professor Amaresh Dalal from the Department of Mechanical Engineering of IIT, Guwahati. In today's lecture, we will introduce convective heat transfer, then we will see few Applications of convective heat transfer, then finally, we will discuss about the course contents.

(Refer Slide Time: 00:58)



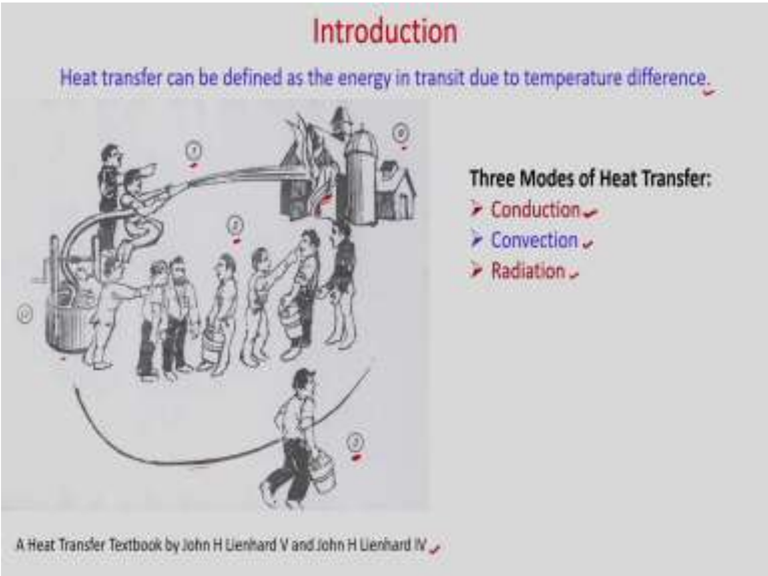
Transport phenomena: what is transport phenomena? Transport phenomena is the subject which deals with the movement of physical quantities in mechanical and chemical processes and describes the principles and laws of transport.

Transport phenomena includes three closely related topics; these are fluid dynamics, heat transfer and mass transfer. You can see fluid dynamics involves the transport of momentum. If you consider flow inside a pipe, then you know that fluid flow occurs from high pressure region to low pressure region.

So, pressure difference is the driving force for this fluid flow. Next, heat transfer deals with the transport of thermal energy due to temperature difference. So, heat transfer takes place from higher temperature region to lower temperature region. So, obviously you can see that temperature difference is the driving force for heat transfer. Mass transfer involves the transport of mass of various chemical species.

So, if there is a concentration difference, then mass transfer will take place from higher concentration region to lower concentration region. So, here concentration difference is the driving force for mass transfer.

(Refer Slide Time: 02:36)



So, as we discussed, heat transfer can be defined as the energy in transit due to temperature difference. We all know that there are basically three fundamental modes of heat transfer namely; conduction, convection and radiation. You can see these illustrations. These we have taken from this book. From these illustrations, it will be easy to describe this heat transfer mechanism. This is the building which is on fire and this is the closely available well where water is available.

Let us assume that there is no quick availability of the fire engine, so this group of people is trying to extinguish this fire. So, here you can see that this group of people in three different ways, they are trying to extinguish this fire, taking water from this well. You can see in the first one, these people are taking the water from the well directly and with using some pump through this hosepipe, they are directly putting this water on the fire.

So, you can see there is no movement of the people. They are directly just using the water and putting this water on this fire. In the second mode, you can see that there are group of people and the person who is near to the well; he is taking the water in the bucket and passing to the next and next person passing this bucket to the next. And that way they are actually passing this water bucket to the person who is near to the fire and he is putting this water in this fire.

So, you can see that there is no movement of these people, but they are actually passing the water bucket from one to the next and they were trying to extinguish this fire using this bucket of water. In third way, you can see this man is very energetic and he alone is taking this water from the well in the bucket and running to this building, putting the water on the fire to extinguish the fire.

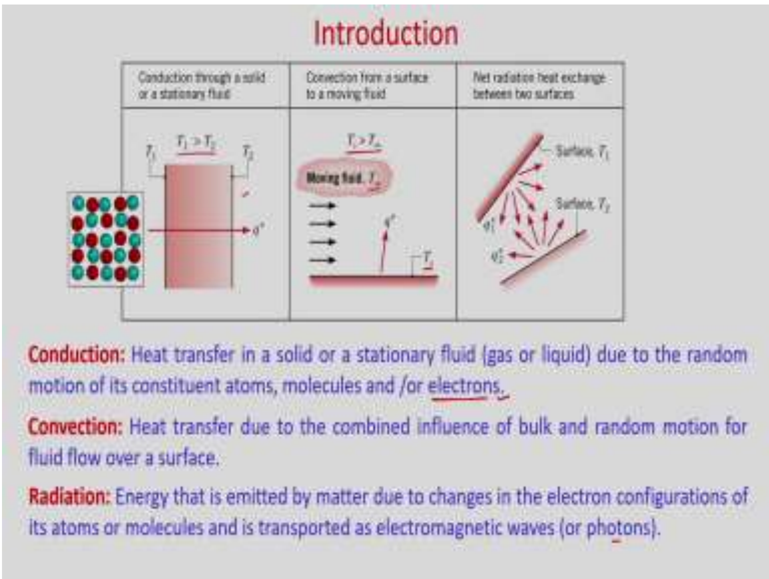
So, you can see there are three different ways they are trying to extinguish this fire. In the first mode, they are not moving, just directly putting the water. In the second mode, these group of people they are passing the bucket to the next and trying to extinguish this fire. And in the third way, the person himself is taking the water and running to this building and putting the water on the fire.

So, now you can compare these three ways with the three modes of heat transfer. First, let us discuss about the second mode. You can see they are passing the bucket from one to another and these are comparable with the conduction mode of heat transfer, because in conduction mode of heat transfer lattice vibrates. And due to the movement of these atoms heat transfer actually takes place from one atom to another and it passes the heat from one end to the other end.

So, you can compare this second way which way they are actually trying to extinguish the fire with the conduction. Now, you consider this third mode here. So, this person is actually moving and trying to extinguish this fire. So, this is kind of convection. So, convection, takes place due to the presence of fluid flow and you can compare this mode with the convection.

In the first mode, you can see here. Actually, they are directly putting the water on the fire and there is no movement. So, this mode you can compare with the radiation.

(Refer Slide Time: 07:46)



So, now let us define these three modes of heat transfer. Conduction: so, heat transfer in a solid or a stationary fluid gas or liquid takes place due to the random motion of its constituent atoms, molecules and or electrons.

You can see the conduction through a solid or a stationary fluid. This wall temperature is T_1 and the this wall temperature is T_2 and, let us assume $T_1 > T_2$. So, obviously, heat transfer takes place from a higher temperature region to the lower temperature region, so in this direction heat transfer will takes place. And as it is a conduction mode of heat transfer, you can see there will be vibration of lattices. And these atoms when they vibrate, they actually carry the heat from one to the other.

So, generally in lattice vibration, due to this lattice vibration this conduction takes place, but if it is a conductor, then there will be a movement or translation of pre electrons and due to that there will be heat transfer. Next, let us discuss about the convection. Heat transfer due to the combined influence of bulk and random motion for fluid flow over a surface. So, you can see this convection heat transfer takes place due to the combined influence of bulk and random motion of fluid flow over a surface.

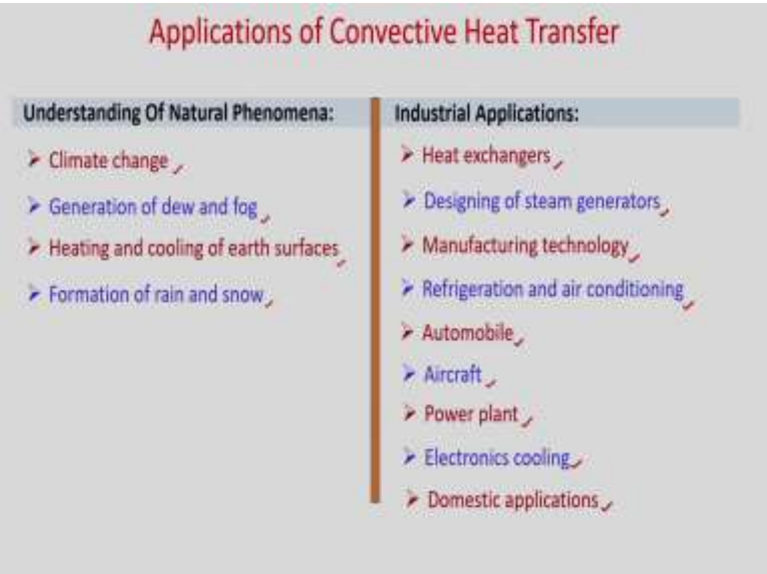
So, this is the surface that maintains the temperature T_s and there is a moving fluid over the surface and this moving fluid temperature is infinity. Let us assume that surface temperature is higher than the fluid temperature, then obviously, heat transfer will take place from surface to the fluid. In this mode, you see that the fluid which is just residing on the top of the surface, it is in direct contact with the surface. And here, conduction mode of heat transfer will take place; from the surface to the immediate fluid molecule.

From those fluid molecules, due to the advection there will be heat transfer. So, you can see in convection heat transfer, there are two different ways heat transfer is taking place. First from the surface to the immediate fluid molecules heat transfer is taking place due to the conduction, and then due to the fluid flow means advection. So, you can say that it is advection influenced conduction. Now, you see the radiation.

So, energy that is emitted by matter due to the changes in the electron configurations of its atoms or molecules and is transported as electromagnetic waves or photons. So, here you can see that net radiation heat exchange between two surfaces. So, this surface is maintained at temperature T_1 and surface temperature T_2. So, there will be just heat transfer due to radiation. Any matter at any temperature can emit radiation and generally radiation does not need any medium to transfer the heat.

It actually effectively transfers the heat in vacuum; however, in conduction and convection you need medium for heat transfer.

(Refer Slide Time: 11:52)



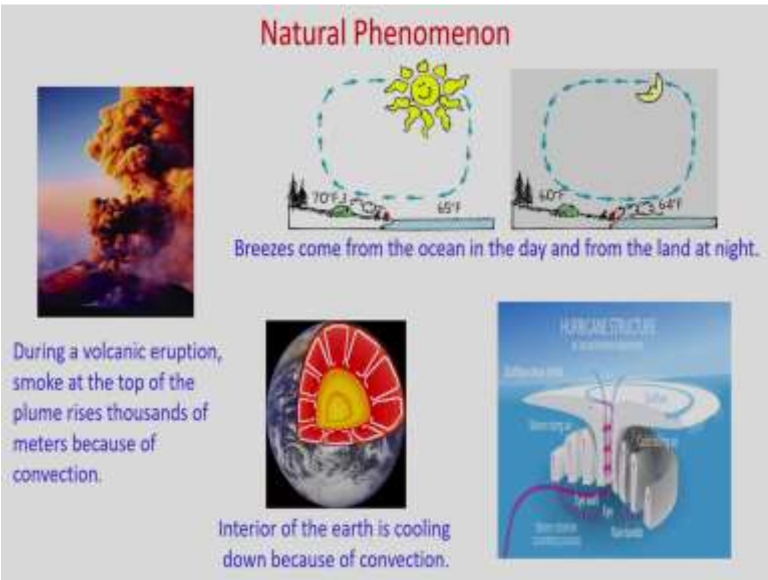
Now, let us see few applications of convective heat transfer. You can see there are many applications in natural phenomena as well as in industrial applications. And in daily life also you can see the application of convective heat transfer. Let us take a simple example. When you take a tea in a cup and you are in hurry so you need to drink the tea quickly and tea is very hot. So, what do you do?

So, generally we put this hot tea on the plate and we blow the air through the mouth, so that it cools down. So, you can see it is a simple example of convective heat transfer; one way you are increasing the surface area putting the tea on the plate. So, on the plate you can see that you have more heat transfer area and you are blowing air through your mouth.

So, that means, forced convection is taking place. So, there is heat transfer from the hot tea to the air. So, it is a simple example of convective heat transfer. And in daily life you can see there are many applications of convective heat transfer. Now, you can see in understanding of natural phenomena like; climate change, generation of dew and fog, heating and cooling of earth surfaces, formation of rain and snow, you can see that these are examples of convective heat transfer.

In industrial applications, you can see this in heat exchangers, designing of steam generators, manufacturing technology, refrigeration and air conditioning, automobile, aircraft, power plant, electronics cooling, in domestic applications and many more. Now, let us see some pictorial examples of these applications of heat transfer.

(Refer Slide Time: 14:10)



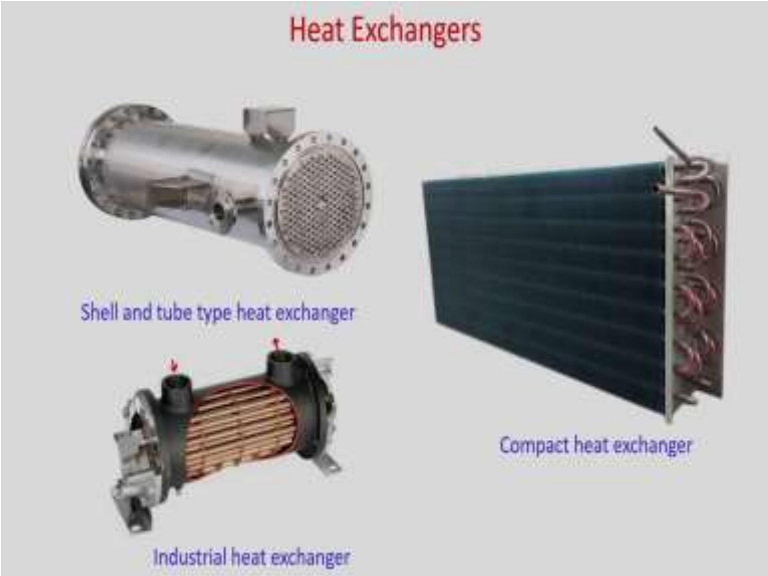
So, you see this is the natural phenomena. Breezes come from the ocean in the day and from the land at night. You consider this figure.

So, in the daytime you can see the land is at higher temperature than the ocean. So, obviously, the air which is in contact with the land will have the lower density due to the higher temperature and it will go up. And to fill up this place, the air will come from the lower temperature region which is having high density or higher density and it will move from here to the land and that way there will be formation of breeze, but at night this direction of breeze reverses.

You can see at night the ocean temperature will be higher than the land temperature. So, obviously, you can see the density will become low and it will go up and higher density fluid will try to fill up the space and there will be formation of breeze. This is one example of natural phenomena. Here you can see, during a volcanic eruption smoke at the top of the plume rises thousands of meters, because of convection.

Here you can see, interior of the earth is cooling down, because of convections. There will be natural convection here. And in different types of cyclones, it moves from one place to other and it rotates. So, this is also one example of convective heat transfer, because the cyclone is actually formed due to the change of temperature.

(Refer Slide Time: 16:08)

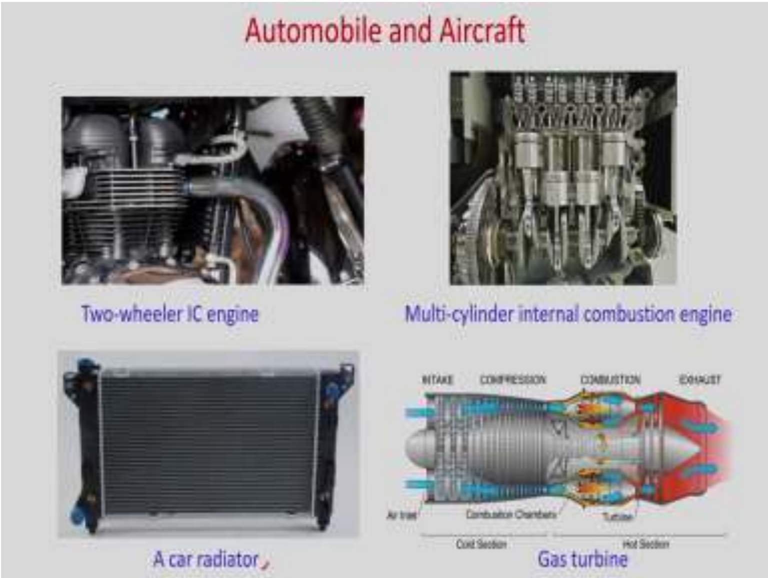


Now, let us see few examples in heat exchangers. This is cell and tube type heat exchanger. You can see there are small tubes. So, hot fluid goes inside and it is to be cooled. So, there will be some openings like here and here. So, the cold fluid will go inside from here and it will exit from other side, you can see in this picture. This is one example of industrial heat exchanger.

So, you can see inside these tubes are seen and hot fluid flows inside these tubes and cold fluids come in here and goes out from here, so there will be heat exchange between the cold fluid and hot fluid. So, this is one example of convective heat transfer.

This is one example of compact heat exchanger. So, in compact heat exchanger you can see that heat transfer area to the volume ratio is very high. It is more than 700 m^2/m^3. Here, you can see there are fins and heat transfer area is increased and through these tube hot fluid flows and it is to be cooled. And from outside maybe there will be an external fan or forced convection which actually cools these fins effectively cooling down this hot fluid inside the tubes.--

(Refer Slide Time: 17:48)



Now, let see some examples in automobiles and aircraft. You can see in your bike or two-wheeler. So, there will be IC engine, it has single cylinder and over it there will be some extended metal strip. So, those are known as fins. So, it increases the surface area and due to conduction from the cylinder surface actually, heat transfer takes place due to conduction. Inside the IC engine you know due to the combustion there will be high temperature and that is to be cooled.

If you notice here in a two-wheeler IC engine, so you can see these are the fins attached with the cylinders of IC engine and when it moves, these bike moves obviously, this air will pass through these fins and this is one example of convective heat transfer.

So, these fins will be cooled down due to the forced convection. There will be multi-cylinder internal combustion engine. Here, sometime these cylinders are cooled using cooling fluid and this is also example of convective heat transfer.

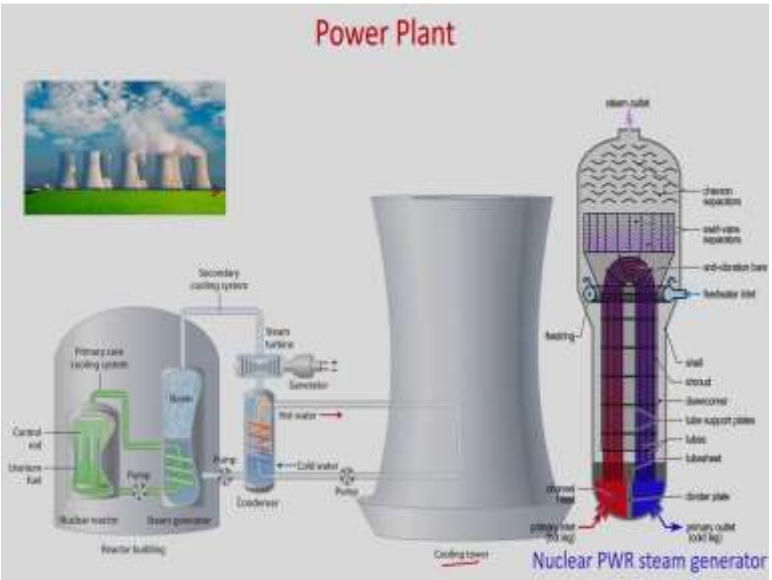
You can see this is the car radiator. Generally, it is placed in front of the four-wheeler. So, engine in a car is cooled by a cooling fluid and this cooling fluid is passed through this radiator to cool down. So, once the hot cooling fluid is coming out from the engine, it is passed through this radiator.

And in this radiator you can see, there are multiple fins and it is placed in front of the car and when the car moves, air will pass through these fins and it will be cooled and this hot cooling fluid when it comes out, it will become cold.

So, this is one example of convective heat transfer. In gas turbine of aircraft engine you can see that there will be cold air inlet and it will pass through this compression and it will go to the combustion chamber and in the combustion there will be high heat generation. And there will be a high temperature and the fluid which is hot; it will go out through this exhaust. And this combustion chamber is cooled using some cooling fluid.

So, you can see in this hot section, this is cooled using some cooling fluid and this is one example of this convective heat transfer. In addition, you can see in automobile and aircraft, for the passengers comfort, inside you have this flow of cold air which is coming from the air conditioner. So, obviously, you need to have the knowledge of convective heat transfer, so that you can design the seats and the interior, so that this cooling fluid passes through all the passengers. So, that is also one example of convective heat transfer.

(Refer Slide Time: 21:20)



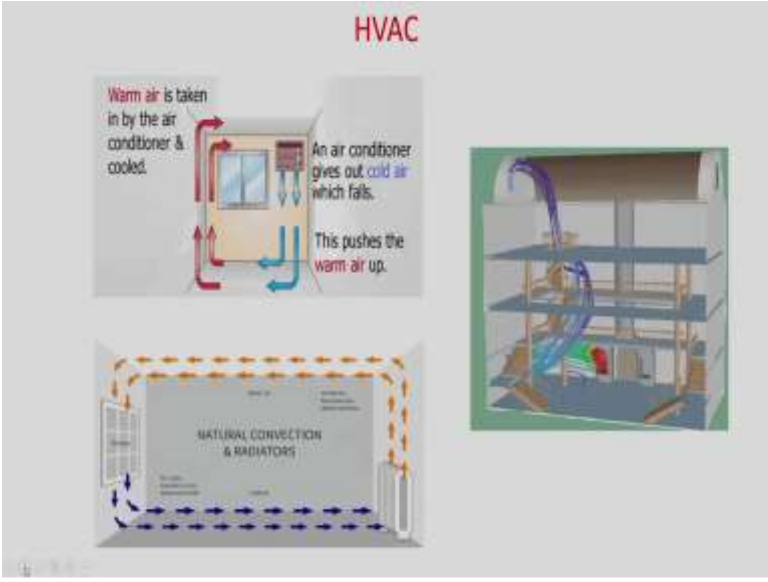
So, now, let us see their application of convective heat transfer in power plant. This is one typical nuclear power plant. Here in the nuclear reactor this cooling system, hot fluid comes through here and this water actually becomes steam. So, this steam passes through this steam turbine and in the steam turbine; obviously, it generates power and this steam now, is to be cooled. So, now, this steam is passed through this cooling tower.

So, you can see the cooling tower in any power plant. And when it is passed through the cooling tower, this hot water becomes cold and again it is pumped to the steam generator. So, you can see in many processes in this power plant, you have the examples of convective heat transfer.

(Refer Slide Time: 22:17)

In household applications, we have air conditioner, air cooler and refrigerator. And in these processes you know that there are many applications of convective heat transfer. So, we are not going to discuss in detail, but you know that all these cycles if you study; obviously, you will find the application of convective heat transfer.

(Refer Slide Time: 22:44)



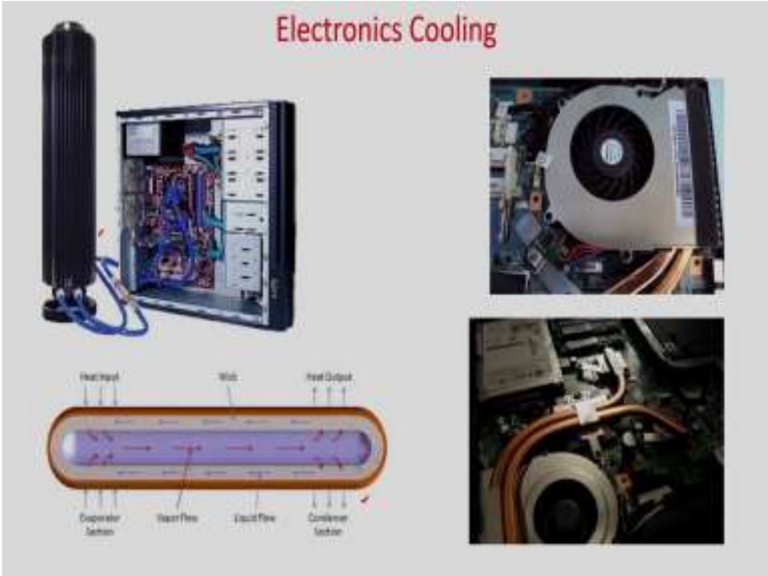
Now, we will discuss about the HVAC, Heating Ventilation and Air Conditioning. So, you can see the room where AC is there, so obviously this air condition gives out cold air and as it is cold air it is having high density. So, it will move inside the house and these pushes the warm air up.

So, warm air will go up due to the low density and this warm air is taken in by the air conditioner and cooled. So, this way there will be a natural circulation and this is the natural convection taking place inside the room. Similarly, if you use heater, room heaters or radiator.

So, here also the fluid which is coming into contact with the radiators, the density will become low and it will go up and this warm air will flow. And when it will come to some cold place like window, its density will become higher and it will go down and this way cold air will come. So, there will be a natural convection.

In any building, how effectively you can cool you need to have this knowledge of convective heat transfer, so that effectively you can cool the building.

(Refer Slide Time: 24:09)



There are many applications of convective heat transfer in electronics cooling. You can see that day by day the size of the chip is decreasing and heat generation inside the chip is increasing.

Hence, you need to remove the heat from this smaller chip. If you notice that in a desktop the board is placed vertically and different electronics components are cooled in natural way by natural convection. But the chip which is your Intel chip, this generates very high temperature. It generates heat and the temperature will be very high on the surface of the chip.

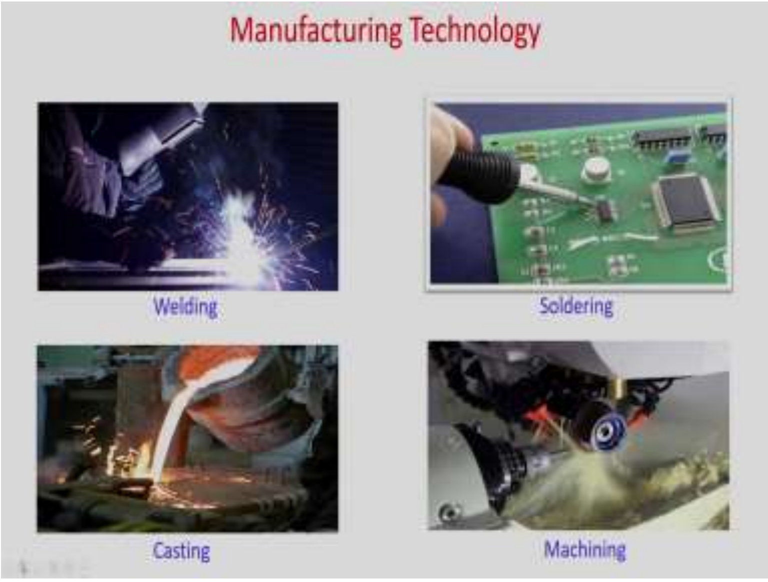
So, it has to be cooled. If you notice that in the PC generally, there will be fins which are cooled by a fan. So, you can see that there will be forced convection and due to the forced convection, there will be heat transfer and heat will be removed from the pin and that way the chip is cooled. If you see the laptop or any mobile you can see that it is very thin and it is very difficult to put the fan over the chip. So, for that generally heat pipes are used.

So, you can see this is the fan over this chip and in forced convection these are cooled. And in laptop you can see these are the heat pipes. How the heat pipes works? In heat pipes this evaporator section is put on the chip. So, due to the high temperature the fluid will become vapor and this vapor will pass inside this hollow space and it will come to the condenser section which is put outside the ambient.

So, due to this condensation this vapor will become liquid and this liquid will pass through this wick. And it will pass through this wick and it will go to the evaporator section. So, you can see it is natural way this fluid flow is taking place, so there is no external force and the advantage is that you can bend it. So, here you can see you can bend it and put it towards the outside and the evaporator section you can put on the chip. And that way naturally this heat is removed from the chip to the ambient.

So, this is one example of convective heat transfer. Nowadays, these chips are also cooled using some fluid and this fluid is passed through these tube. And for more heat transfer or effective heat transfer these fluids are used to remove the heat from the chip.

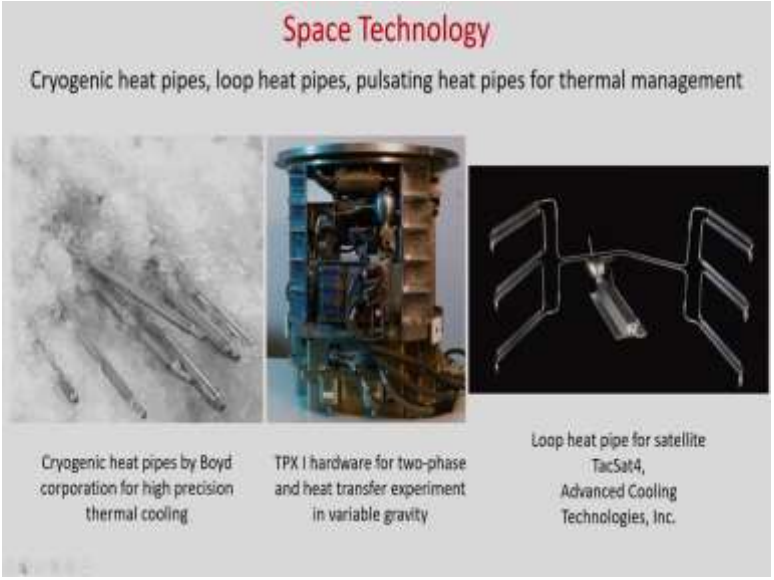
(Refer Slide Time: 27:24)



In addition, you will find there are many applications in manufacturing technology. So, in machining process when the tool cuts the metal you can see there will be more heat generation and these are cooled using some cooling fluid.

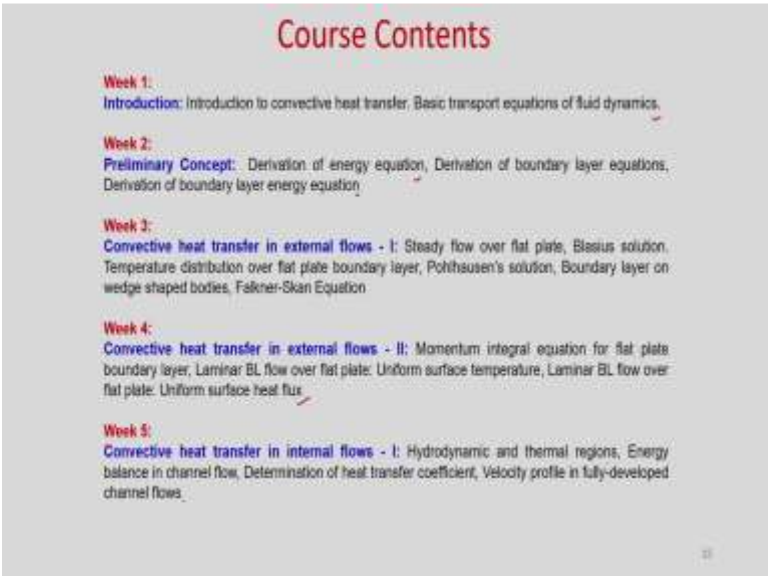
So, these cooling fluids are put near to the tool and this heat is removed. In addition, you can see in a welding, casting and soldering processes there will be phase changes and also there will be heat transfer due to convection.

(Refer Slide Time: 28:05)



In space technology, you can see cryogenic heat pipes, loop heat pipes, pulsating heat pipes for thermal management. These are examples of convective heat transfer in space technology.

(Refer Slide Time: 28:19)



Now, we will discuss about the course contents. In module 1, we will we are introducing the convective heat transfer, we discuss about the application of convective heat transfer and we will discuss basics transport equations of fluid dynamics.

In week 2, we will discuss about the preliminary concepts. We will derive the energy equation in general and then we will derive the hydrodynamic and thermal boundary layer equation.

In module 3, we will start the external flows. We will start with steady flow over a flat plate and we will solve the Blasius equation. Then, we will discuss about the temperature distribution over flat plate boundary layer and we will have the Pohlhausen's solutions and we will touch upon this boundary layer on wedge shaped bodies and we will discuss about this Falkner-Skan Equation.

Next, we will move to module 4. We will continue with the external flows, we will use the approximate method and we will use momentum integral equation for flat plate boundary layer. And we will also solve laminar boundary layer flow over a flat plate with two different thermal conditions: One is Uniform surface temperature and another is Uniform surface heat flux.

In module 5, we will start internal flows; first, we will discuss about hydrodynamic and thermal regions, then we will do the energy balance in channel flow, we will use scale analysis to find the heat transfer coefficients and finally, we will find the velocity profile in a fully-developed channel flows.

(Refer Slide Time: 30:12)

In module 6, we will continue with the internal flows; first, we will discuss about thermally fully developed laminar slug flow with uniform wall heat flux condition.

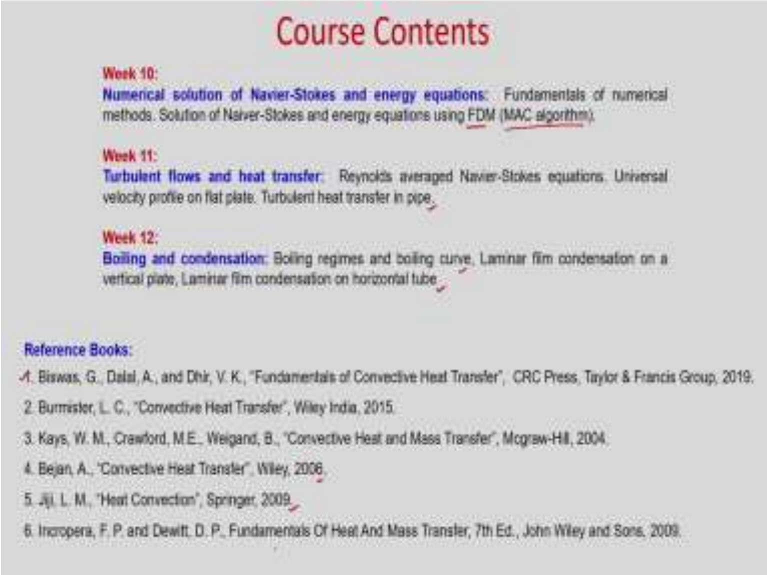
In slug flow, there will be uniform velocity. Considering that situation, we will find these temperature distribution. Then, we will consider a hydrodynamically and thermally fully developed flow with uniform wall heat flux condition in two different geometries: One is flow between parallel plates and circular pipe. Then, we will discuss about hydrodynamically and thermally fully developed flow through parallel plates channel flow with uniform wall temperature condition. And then, we will discuss hydrodynamically and thermally fully developed flow through circular pipe with uniform wall temperature.

In week 7, also we will continue with internal flows hydrodynamically developed and thermally developing flow through circular pipe with uniform wall heat flux. So, in this case you can see where you will consider hydrodynamically developed, but thermally developing flow. Next, we will consider hydrodynamically developed and thermally developing flow through circular pipe with uniform wall temperature. So, two different thermal conditions we will consider, and at last we will discuss about the heat transfer in plane Couette flow.

In week 8, we will start the natural convection, we will solve the free convection over a vertical plate using integral solution and similarity solution.

In week 9, we will continue with natural convection, we will discuss about internal natural convection and mixed or combined convection.

(Refer Slide Time: 32:08)



In module 10, we will solve this Navier-Stokes and energy equations, using numerical technique, we will use finite difference method and we will solve using marker and cell algorithm.

In module 11, we will introduce with turbulent flows and heat transfer. First, we will derive the Reynolds averaged Navier-Stokes equations, then we will discuss about universal velocity profile on flat plate and we will touch upon turbulent heat transfer in pipe.

In last module, module 12, we will start with boiling and we will also discuss about the condensation. First, we will discuss about different boiling regimes and we will discuss boiling curve which is your nukiyama curve. And next, we will discuss about Laminar film condensation on a vertical plate and Laminar film condensation on horizontal tube.

So, you can see the course content, it is a advanced level heat transfer, because we are discussing about one mode of heat transfer that is convective heat transfer. And it is more analytical in nature and rigorous derivation will be there.

So, you can refer these following books. Mostly I will follow the first book which is Fundamentals of Convective Heat Transfer and we will also refer these books. So, a Convective Heat Transfer by Burmister, Convective Heat and Mass Transfer by Kays, Crawford and Weigand. This is a good book on Convective Heat Transfer by Bejan and you can refer is for scale analysis.

This is also a good book Heat Convection by Jiji. And these book I think you have already referred in your undergraduate level; Fundamentals of Heat and Mass Transfer by Incropera and Dewitt. So, this book I think you will have, so you can refer in your convective books which is not mentioned here you might have other books also.

So, today we started with discussing about the transport phenomena and we introduced the convective heat transfer. Then we discussed about different modes of heat transfer: conduction, convection and radiation. And then, we show different applications of convective heat transfer in natural phenomena as well as in industrial applications.

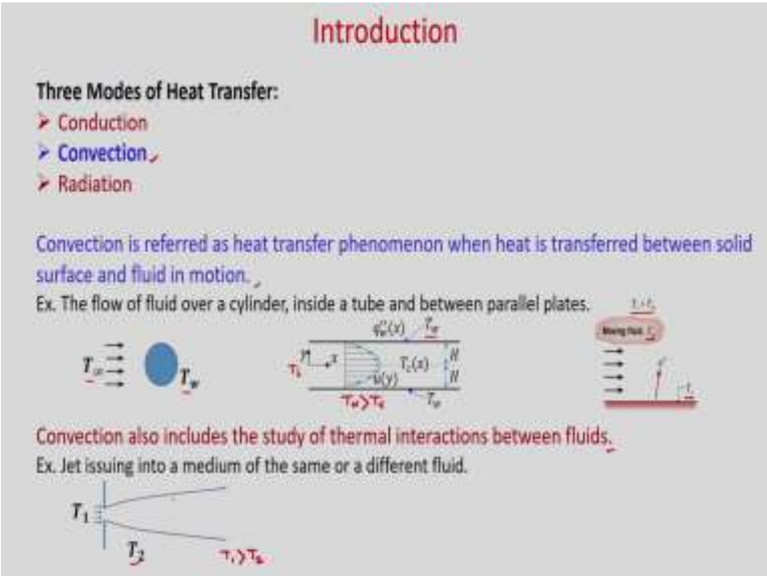
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 01
Introduction
Lecture - 02
Foundations of Heat Transfer

Hello everyone. So, in today's class we will first discuss about basic laws of 3 Modes of Heat Transfer. Then we will discuss about heat transfer coefficient which plays an important role in convective heat transfer, then we will discuss about some fluid dynamics equations which will be relevant in our course.

(Refer Slide Time: 00:57)



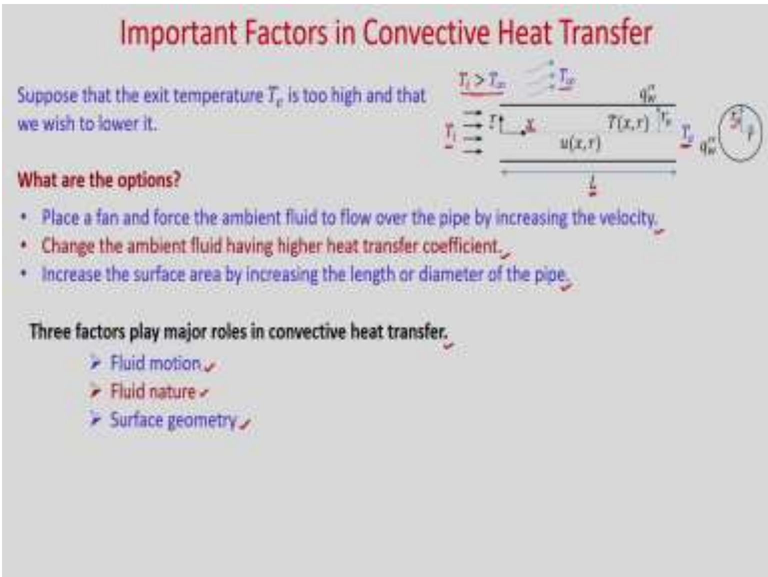
So, already we discussed that convection is one of the 3 modes of heat transfer. Convection is referred as heat transfer phenomena where heat is transferred between solid surface and fluid in motion. So, you can see there are some examples say the flow of fluid over a cylinder. Let us say cylinder is maintained at a higher temperature than the ambient temperature. So, $T_w > T_\infty$ and fluid flow is taking place. Obviously there will be heat transfer from the solid surface of the cylinder to the ambient fluid.

Similarly, if you see the flow inside a tube or parallel plates. So, in this case you can see fluid flow is happening inside 2 parallel plates. The wall is maintained at temperature T_w

and let us say at inlet you have temperature T_i and if $T_w > T_i$, then obviously when fluid flows inside these parallel plates there will be heat transfer from the solid surface to the fluid. Similarly, if you consider flow over a flat plate where your surface temperature is T_s and ambient temperature is T_∞ and $T_s > T_\infty$; obviously, there will be heat transfer from the solid surface to the fluid.

So, you can see in this phenomenon the heat is transferred between solid surface and moving fluid. Convection also includes the study of thermal interaction between fluids. So, you can see in this case jet issuing into a medium of the same or a different fluid. Let us say one fluid is entering through this jet whose temperature is T_1 and here another fluid is there which is having the temperature T_2 and let us say $T_1 > T_2$. Then when this fluid comes here, so in the jet you can see there will be a mixing and there will be heat transfer. So, you can see in this case the convection is taking place between the fluids.

(Refer Slide Time: 03:24)



Now, let us discuss some important factors in convective heat transfer. First let us consider flow inside a circular pipe. So, you can see this is a circular pipe of radius r_o. So, you can see that axial direction is x and radial direction is r and radial direction r is measured from the center of the cylinder. Here, let us say that steam is entering at high temperature T_i. The length of the pipe is L and the exit temperature is T_e that is to be determined. In the ambient temperature is T_∞ and obviously, as steam is entering here

$T_i > T_\infty$. Considering this situation let us say that when you measure the exit temperature T_e it is too high.

Now, you have to lower the temperature T_e. So, how can you do it? What are the possible ways you can decrease the exit temperature T_e? So, you can see here what the options are. Here, you have ambient temperature T_∞ and if you put a fan, then obviously its velocity will increase and more heat transfer is likely to take place. So, that is one possible way.

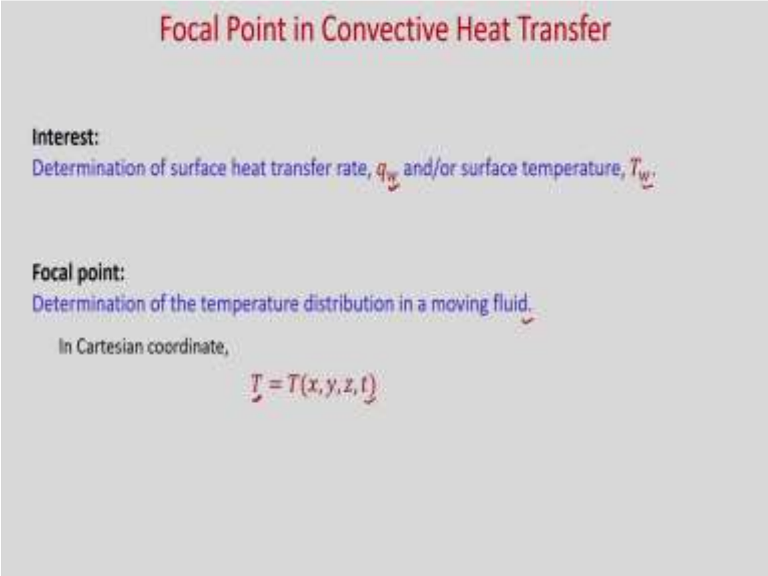
So, place a fan and force the ambient fluid to flow over the pipe by increasing the velocity. The other way is that whatever ambient fluid is there you just change that ambient fluid so that it can take away more heat from the pipe surface. So, obviously if you change the fluid you can get such a fluid whose heat transfer coefficient is high.

So, secondly you can change the ambient fluid having higher heat transfer coefficient. The other way to increase the heat transfer from the pipe surface, so that you can lower the temperature T_e is to increase the surface area.

So, which way you can increase the surface area? You can increase the diameter of the pipe or you can increase the length of the pipe. So, increase the surface area by increasing the length or diameter of the pipe. So, L you can increase or r_o you can increase so that your surface area increases and more heat transfer will take place, so that your exit temperature at T_e will decrease.

So, in this example you can see that 3 factors play major roles in convective heat transfer. One is fluid motion, then fluid nature or fluid properties and surface geometry. So, these are the 3 factors which play major roles in convective heat transfer.

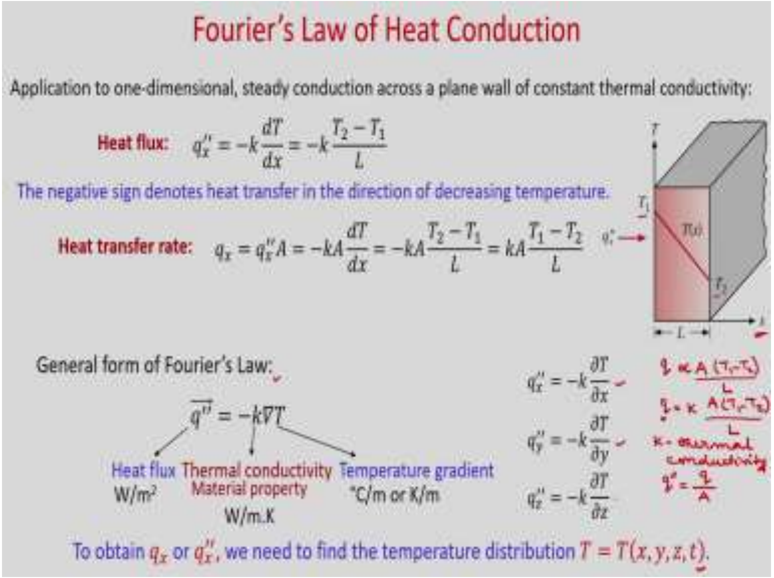
(Refer Slide Time: 06:36)



So, now next question is that why do we want to study this convective heat transfer? Why we are interested in studying this convective heat transfer and in which quantity we are interested? So, obviously you can see that we are more interested in finding the heat transfer rate while designing some industrial equipment or to know the surface temperature. And to determine the heat transfer rate or the surface temperature you need to know what the temperature distribution inside the domain is.

So, our interest is to determine the surface heat transfer rate q_w and or surface temperature T_w and focal point is the determination of the temperature distribution in a moving fluid. So, in Cartesian coordinate temperature will be function on the space x, y, z and may be with time. So, once you find the temperature distribution you will be able to calculate the heat transfer rate. Now, let us discuss the basic law in conduction. Already you have studied in the basic heat transfer course the Fourier's law of heat conduction.

(Refer Slide Time: 07:59)



So, if you consider the solid of thickness L, the temperature in the left surface is maintained at temperature T_1 and right surface at T_2 and other walls are insulated so that there will be no heat transfer and heat transfer will take place only in one direction in x direction in this case. So, in experiment it is shown that the heat transfer rate is proportional to the temperature difference and directly proportional to the heat transfer area and inversely proportional to the thickness of the solid.

So, in this case you can see q if it is a heat transfer rate, then it is directly proportional to the area, it is directly proportional to the temperature difference and it is inversely proportional to the thickness. And you can see that from here if you equate this heat transfer rate, then a proportionality constant will come and that proportionality constant is known as thermal conductivity k.

So, $q = kA\frac{(T_1 - T_2)}{L}$

So, this k is your thermal conductivity of the material. So, you can see that if it is heat transfer rate, then heat flux you can write as q'' is the heat transfer rate per unit area.

So, $\frac{q}{A}$ So, this is your heat flux.

So, you can see that heat flux in one dimensional steady conduction across a plane wall of constant thermal conductivity we can write $q''_z = -k\frac{dT}{dz}$

So, $\frac{dT}{dx}$ is the temperature gradient and you can see temperature gradient you can write $\frac{T_2 - T_1}{L}$

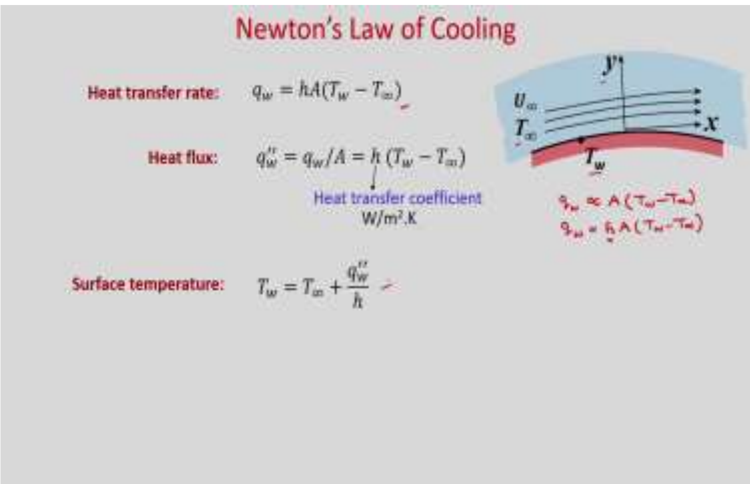
So, here you can see the negative sign these negative sign denotes heat transfer in the direction of decreasing temperature. So, heat transfer rate you can write as heat flux into the area.

So, it will be $-k\frac{dT}{dx}$ and hence you can write $kA\frac{T_1 - T_2}{L}$. In general form of Fourier's law you can write as a heat flux vector quantity.

So, ∇ $\vec{q''} = -k\nabla T$ where q'' is the heat flux in Watt per meter square, k is the thermal conductivity it is a material property. Its unit is W/mK and ∇T is the temperature gradient it is K/m or $^oC/m$.

So, we can see to obtain q_x or q''_x; that means, heat transfer rate and the heat flux we need to find the temperature distribution T, then only you can calculate the temperature gradient. So, in this case you can see in x direction you can write $q''_x = -k\frac{dT}{dx}$, in y direction $q''_y = -k\frac{dT}{dy}$ and in z direction $q''_z = -k\frac{dT}{dz}$

(Refer Slide Time: 11:41)

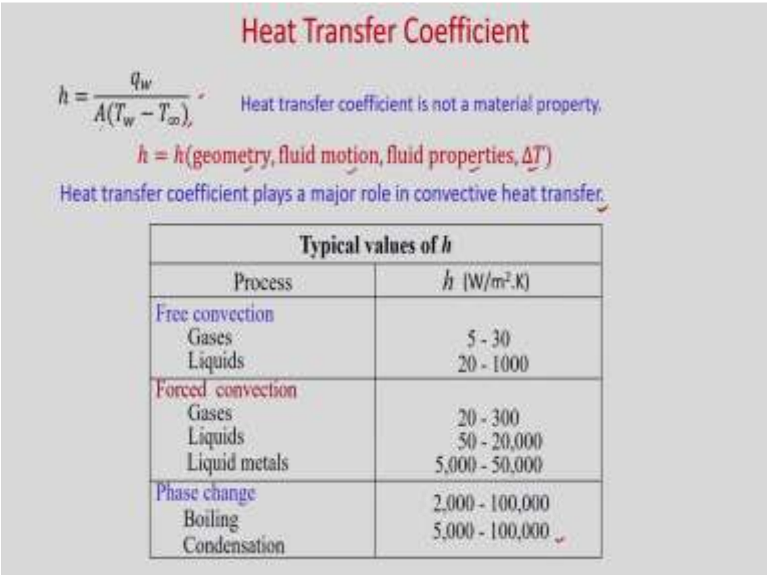


Now, in convection we have a basic law which is known as Newton's law of cooling. Consider this is a surface maintained at temperature T_w, x is along the surface and y is normal to the surface and you have fluid flow where free stream velocity is u_∞ and you have temperature T_∞. So, in experiments it is shown that your heat transfer rate at the wall is directly proportional to the area and the temperature difference and once you write equal to then you will get one proportionality constant into area into the temperature difference.

So, this proportionality constant is known as heat transfer coefficient. So, we can see heat transfer rate we can write $q_w = hA(T_w - T_\infty)$ and heat flux is the heat transfer rate per unit area. So, that you can write h (T_w -T_∞) where h is the proportionality constant and it is known as heat transfer coefficient its unit is W/m^{2}K. So, from here you can see that you can write the surface temperature $T_w = T_\infty + \frac{q''_w}{h}$.

So, from here you can define the heat transfer coefficient.

(Refer Slide Time: 13:17)



So, you can see heat transfer coefficient is the heat transfer rate per unit area per unit temperature difference. So, here you can see that heat transfer coefficient is not a material property. So, it is a transport property and it depends on many things. So, it depends on geometry, fluid motion, fluid properties and sometime on temperature difference. So, heat transfer coefficient plays a major role in convective heat transfer.

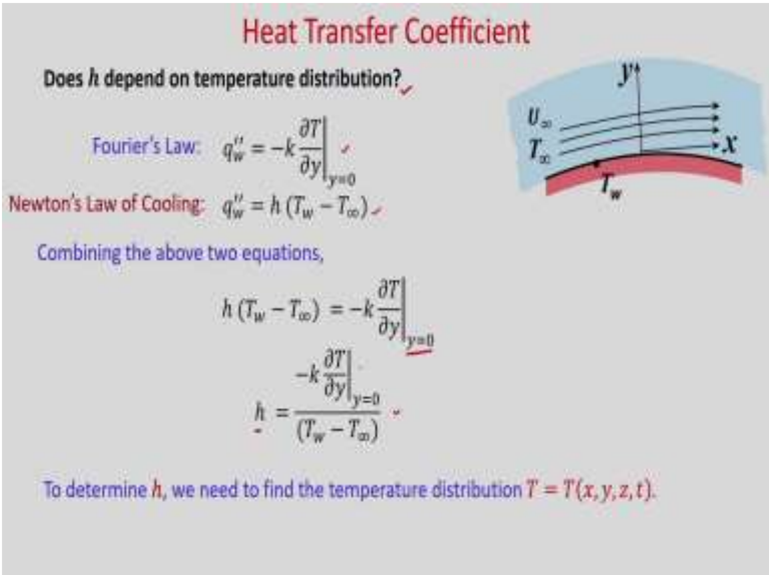
So, you can see that h is function of geometry, fluid motion, fluid properties and temperature difference. So, you will not get any particular value for any situation. So, some rough idea about the value of this heat transfer coefficient in different situation is tabulated here.

You can see if you have a free convection; that means, natural convections then for gases h is varies 5 to 30 W/m^{2}K. For liquids, obviously it is more it varies 20 to 1,000 W/m^{2}K. If it is a forced convection, then for gases it varies between 20 and 300 W/m^{2}K.

For liquids, in the range of 50 to 20,000 and in liquid metals 5,000 to 50,000 and if phase change takes place like boiling and condensation then you will get a very high heat transfer coefficient. You can see for boiling you can achieve the heat transfer coefficient in the range of 2,000 to 1 lakh and in condensation also 5,000 to 1 lakh you can achieve the heat transfer coefficient.

So, from here you can see that if you need to remove very high heat flux then you need to use phase change boiling or condensation.

(Refer Slide Time: 15:22)

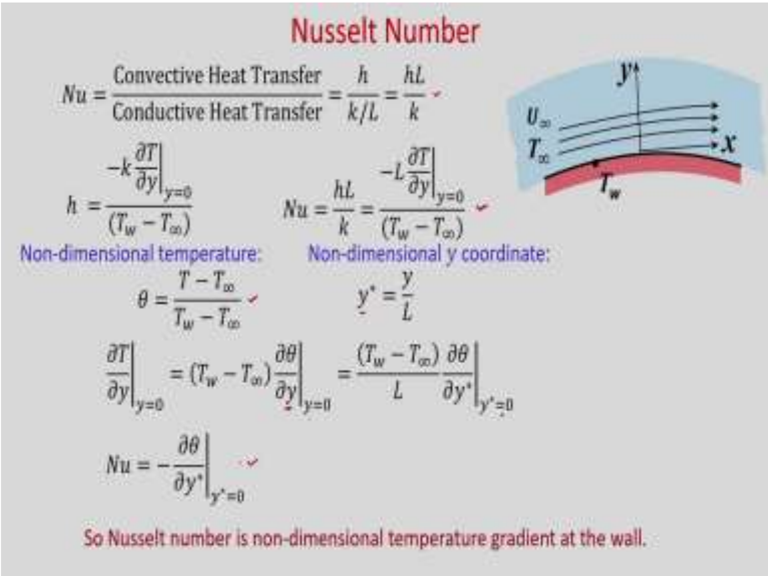


Now, the question is does h depend on temperature distribution. So, let us see in Fourier's law the heat flux at the wall we can write $q''_w = -k \frac{\partial T}{\partial y}|_{y=0}$. So, at the wall. From Newton's law of cooling also you can write the heat flux at the wall $q''_w = h(T_w -$

T_∞). So, if you equate these 2 you can write $h\left(T_w - T_\infty\right) = -k\frac{\partial T}{\partial y}\Big|_{y=0}$; that means, at the wall.

From here you can see h you can determine from this relation $h = \dfrac{-k\frac{\partial T}{\partial y}\Big|_{y=0}}{\left(T_w - T_\infty\right)}$. So, here you can see that you need to calculate the temperature gradient at the wall to find the heat transfer coefficient. So, obviously to determine the heat transfer coefficient we need to know the temperature distribution inside the fluid domain.

(Refer Slide Time: 16:24)



We have discussed about the heat transfer coefficient and very often we write this heat transfer in non-dimensional form and this non-dimensional number is known as Nusselt number. Nusselt number is the ratio of conductive to convective heat transfer in a fluid. So, you can see

Nusselt number, $N_u = \dfrac{ConvectiveHT}{ConductiveHT} = \dfrac{h}{k/L} = \dfrac{hL}{k}$

h is the heat transfer coefficient, k is the thermal conductivity and L; L is the characteristic length. So, this characteristic length varies depending on the different geometry.

So, if you consider flow over a flat plate, then your characteristic length will be the length of the plate, but if you consider flow inside a circular pipe then your characteristic length may be the diameter of the pipe. So, we have already shown that $h = \frac{-k \frac{\partial T}{\partial y}|_{y=0}}{(T_w - T_\infty)}$.

So, if you put this h in this expression, then you can write $N_u = \frac{hL}{k}$

So, $N_u = \frac{hL}{k} = \frac{-L \frac{\partial T}{\partial y}|_{y=0}}{(T_w - T_\infty)}$

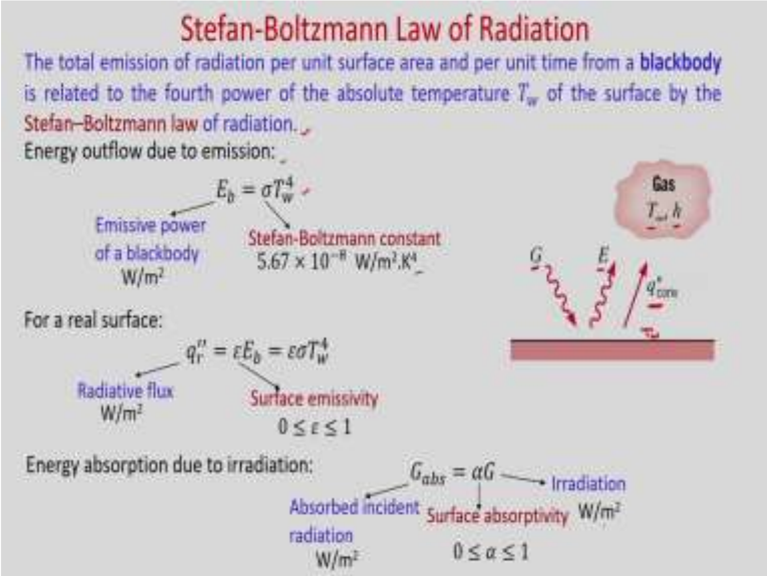
Now, let us write Nusselt number in terms of some non-dimensional quantities. So, now, let us define the non-dimensional temperature $\theta = \frac{(T - T_\infty)}{(T_w - T_\infty)}$ and non-dimensional y coordinate is $y^* = \frac{y}{L}$ where L is the characteristic length.

So, from this expression you can calculate the temperature gradient $\frac{\partial T}{\partial y}|_{y=0} = (T_w - T_\infty)\frac{\partial \theta}{\partial y}|_{y=0} = \frac{(T_w - T_\infty)}{L}\frac{\partial \theta}{\partial y^*}|_{y=0}$

So, this $\frac{\partial T}{\partial y}$ now if you put in this expression then you will get $N_u = -\frac{\partial \theta}{\partial y^*}|_{y^*=0}$

So, you can see that Nusselt number is non-dimensional temperature gradient at the wall. So, if you are solving non-dimensional equations and non-dimensional energy equations then you will get the Nusselt number directly as the temperature gradient because non-dimensional temperature we have considered. So, non-dimensional temperature gradient that you will give you the value of Nusselt number.

(Refer Slide Time: 19:31)



Now, let us discuss about the Stefan Boltzmann law of radiation. Here you can see this is the surface let us say the surface temperature is T_w. So, obviously at any temperature this plate will emit radiation and that is your E and some radiation will come to this surface and whatever is coming that is your irradiation G and ambient temperature is T_∞ and obviously your convective heat transfer coefficient is h.

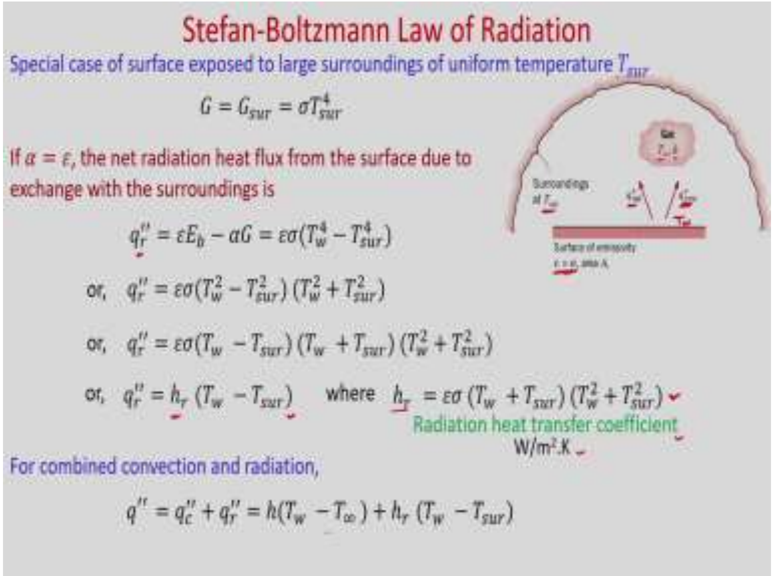
So, you can see there will be heat flux due to the radiation as well as you have heat flux due to convection. So, what is Stefan Boltzmann law? The total emission of radiation per unit surface area and per unit time from a black body is related to the fourth power of the absolute temperature T_w of the surface by the Stefan Boltzmann law of radiation.

So, we can see energy outflow due to emission due to Stefan Boltzmann law you can write $E_b = \sigma T^4_w$ where E_b is the emissive power of a black body and its unit is W/m^2 and T_w is the surface temperature and σ is the Stefan Boltzmann constant and its value is $5.67 \times 10^{-8}\ W/m^2.K^4$.

Now, for a real surface the radiative flux $q''_r = \varepsilon E_b$ where E_b is the emissive power of a black body and ε is the surface emissivity and it varies between 0 to 1 and for a black surface obviously, your emissivity is 1 and E_b if you put this then you will get $q''_r = \varepsilon\sigma T^4_w$

And if you see that whatever irradiation is coming G, so it will be absorbed some part of it by this surface and if you consider the surface absorptivity as α, then you can write energy absorption due to irradiation $G_{abs}= \alpha G$ where G_{abs} is the absorbed incident radiation W/m^2 and α is the surface absorptivity. Obviously, it varies between 0 and 1 and G is the irradiation in W/m^2.

(Refer Slide Time: 22:13)



Now, let us consider a special case. Surface exposed to large surrounding to uniform temperature T_{sur}. So, you have the surface whose temperature is T_w. So, due to convection there will be heat flux q''_{conv}. Due to radiation there will be heat flux q''_{rad}. Your ambient temperature is T_∞ heat transfer coefficient is h; surrounding temperature is maintained at T_{sur}.

Now, surface emissivity is ε and from Kirchhoff's law you can see that ε will be equal to the α and the area is A. So, obviously, you can see that whatever from irradiation is coming from the surrounding so that you can write $G= \sigma T^4_{sur}$ because it is coming from the surrounding and its fraction αG is absorbed by this surface.

So, now if you assume that $\alpha=\varepsilon$, then the net radiation heat flux from the surface due to exchange with the surrounding is, so this is your radiative heat flux. So, whatever emission is happening that is εE_b minus whatever radiation is absorbed that is αG. So, $q''_r= \varepsilon E_b - \alpha G$.

So, $\alpha = \varepsilon$ and G = sigma T^4_{sur} and $E_b = \sigma T^4_w$.

So, you can write $q''_r = \varepsilon\sigma(T^4_w - T^4_{sur})$

So, this you can write in this way $q''_r = \varepsilon\sigma(T^2_w - T^2_{sur})\,(T^2_w + T^2_{sur})$

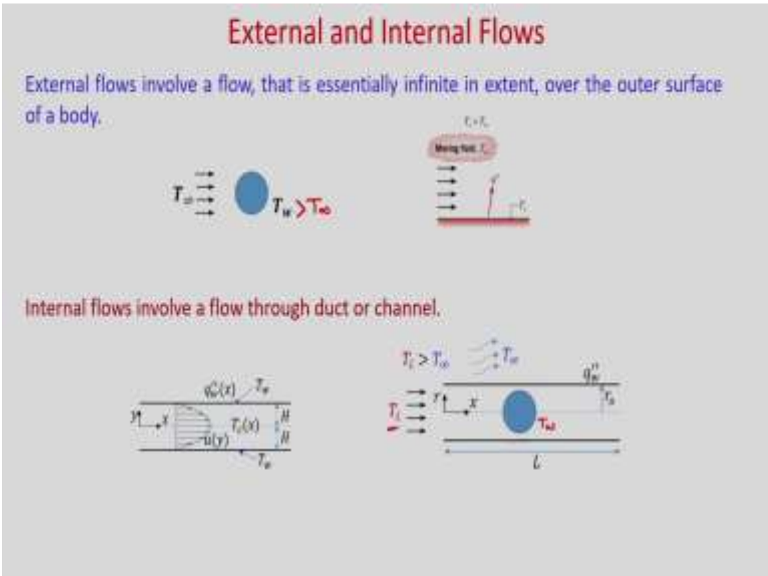
So, these quantity again you can write $(T_w - T_{sur})\,(T_w + T_{sur})$. So, now we can write equivalent to Newton's law of cooling you can write the radiation heat flux as,

$q''_r = h_r\,(T_w - T_{sur})$

So, this $h_r = \varepsilon\sigma(T_w + T_{sur})\,(T^2_w + T^2_{sur})$. So, where h_r is known as radiative heat transfer coefficient and its unit is also W/m^2K. So, you can see from the radiative heat flux we have written this expression similar to Newton's law of cooling, so that we can define radiative heat transfer coefficient and this is expression is this one.

So, now, for combined convection and radiation where it is taking place heat flux due to convection, heat flux due to radiation. So, $q'' = q''_c + q''_r = h(T_w - T_\infty) + (T_w - T_{sur})$

(Refer Slide Time: 25:22)

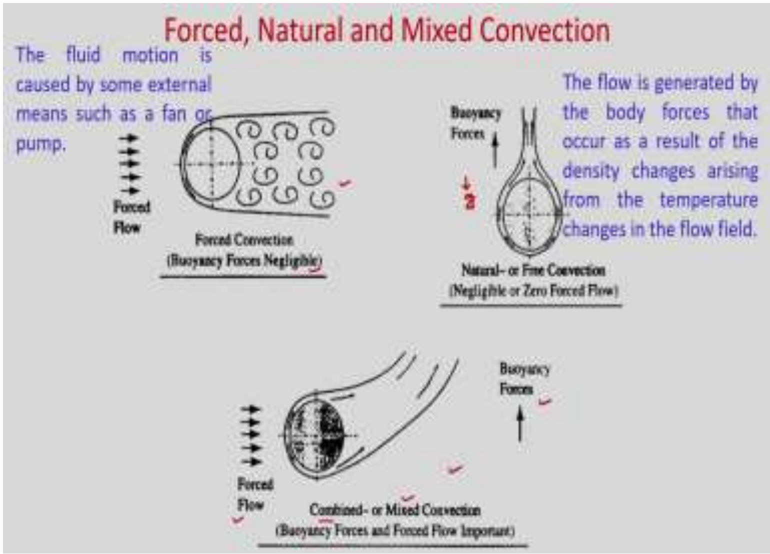


So, there are 2 types of flows external and internal. External flows involve a flow that is essentially infinite in extent over the outer surface of a body. So, you can see this example flow over a circular cylinder. Circular cylinder temperature is T_w which is greater than the ambient temperature T_∞. So, heat transfer will take place from the cylinder surface to the ambient and flow is taking place over this body. So, this is one example of external flows. You can see here that flow over a flat plate and heat transfer is taking place from the hot plate to the ambient fluid and in other direction it is infinite. So, this is external flow. Internal flows involve a flow through duct or channel.

So, if it is confined by wall, then it is internal flows you can see flow inside 2 parallel plates ok. So, here this is confined by 2 parallel plates and flow is taking place inside this domain. So, this is one example of internal flows. Here, this example you can see one cylinder is placed inside this pipe. This is a circular pipe r is the radius radial direction and r_o is the radius of the circular pipe. So, this is one example of sphere kept inside the circular pipe and this sphere you can see that it is maintained at some temperature T_w. Obviously, heat transfer will takes place from sphere to this fluid which is entering at temperature T_i.

Now, you can see this is also example of internal flows for the circular pipe, but at the same time the flow is taking place over this sphere. So, it is kind of external flows. So, in this case you cannot separately tell whether it is internal or external flows, but; obviously, it is confined flow. So, it is kind of internal flows.

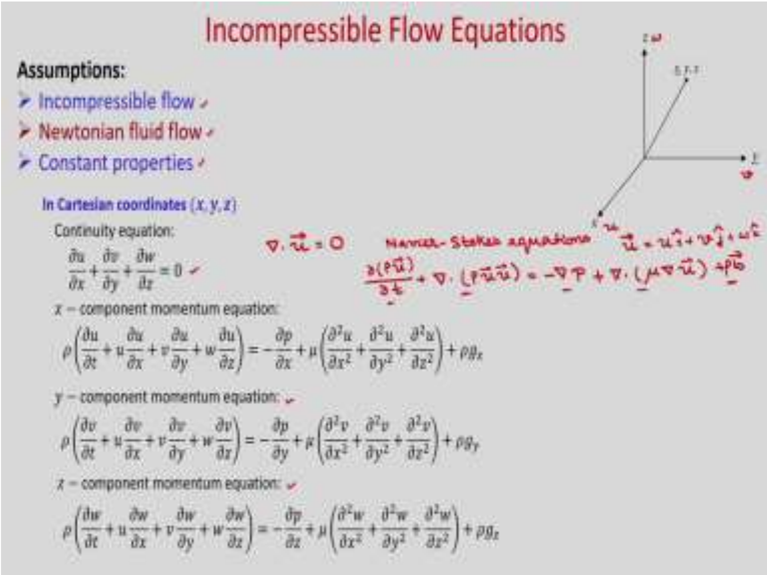
(Refer Slide Time: 27:31)



Convective heat transfer rate depends on the type of flow. So, you can see depending on the type of flow you can get forced, natural and mixed convection. In forced convection, externally there will be pump or fan from where this fluid flow will take place and buoyancy force is negligible. So, the fluid motion is caused by some external means such as a fan or pump. So, this is purely forced convection is taking place here, but if buoyancy force is present let us say you have a gravity is acting in negative y direction.

So, buoyancy force will act and this is known as natural or free convection, so and there is absence of externally induced flow. The flow is generated by the body forces that occur as a result of the density changes arising from the temperature changes in the flow field. Now, if you have both forced and natural; that means, buoyancy force is present as well as forced flow is present, then that is known as mixed convection or combined convection. So, buoyancy force and forced flow both are important in this case.

(Refer Slide Time: 28:51)



So, now, let us discuss some important governing equations in fluid mechanics. So, the main assumptions we are taking incompressible flow, Newtonian fluid flow and constant properties. So, in this case we are considering Cartesian coordinate. So, in x direction you have velocity u in y direction you have velocity v and in z direction you have velocity w.

So, if you define a vector velocity $\vec{u}=u\hat{i}+v\hat{j}+w\hat{k}$. So, in general you can write the continuity equation for incompressible flow as $\nabla\vec{u}=0$. So, u is a vector quantity and divergence of u you can be written in differential form as $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. So, this is the continuity equation.

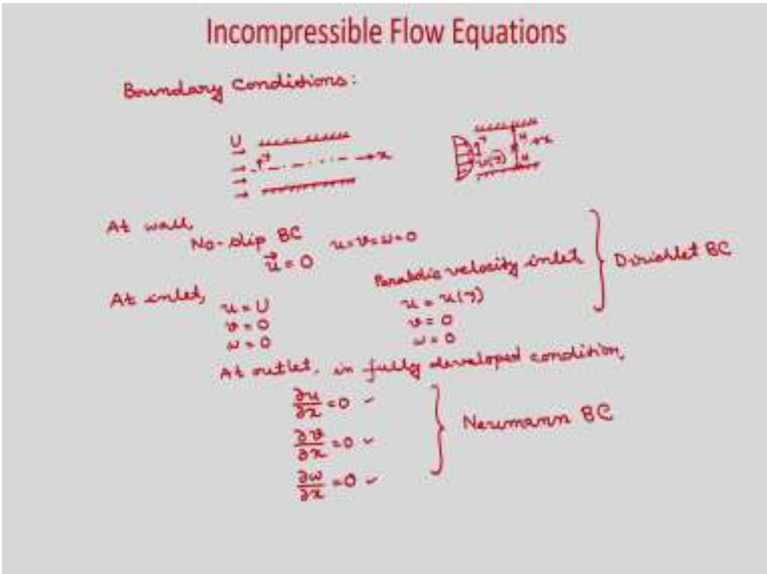
Now, you can write the momentum equation or Navier Stoke's equations in vector form as $\frac{\partial(\rho\vec{u})}{\partial t}+\nabla(\rho\vec{u}\vec{u})=-\nabla p+\nabla(\mu\nabla\vec{u})+\rho b$

So, you can see this is the Navier Stoke's equations. So, in Navier Stoke equation this first term is your temporal term because it is a time bearing term. This is the convective term ok, this is your pressure term because there will be a pressure gradient which is the driving force for the fluid flow and this is the viscous term and if some body force is present that you can incorporate in this term.

So, this Navier Stoke equations if you write for the constant properties, then rho you can take it outside and this mu which is your fluid viscosity dynamic viscosity you can take it outside and rho is the fluid density and P is the pressure. And in non-conservative form and in differential form you can write x component of momentum equation as $\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\rho g_x$.

So, here you can see that this is your temporal term this is your convective term, this is your pressure gradient term, this is your viscous term and this is your body force term. Similarly, you can write y component of momentum equation and z component of momentum equation. So, this we have written for incompressible Newtonian fluid flow.

(Refer Slide Time: 32:30)



So, when you solve these equations you need to have the boundary conditions. So, if we discuss the boundary conditions for fluid flow. So, if you consider flow between parallel plates. So, these are the 2 parallel plates in third direction let us say it is infinite. So, you have fluid flow in axial direction x, this is your y and so obviously if these are walls then at the wall what will be the boundary condition; at wall what will be the boundary condition? So, you can see we are writing the boundary condition for the fluid flow equations. So, when the fluid particle resides on the wall, then it will have the same velocity as the wall.

So, if wall is stationary you will have velocity as 0 and if wall is moving with some particular velocity then that fluid particle will have the same velocity. So, this is known as no slip boundary condition. So, at the wall you have no slip boundary condition ok. So, at the wall the fluid particle which are residing on the wall, so immediate fluid particles will have the same velocity at the wall. So, if wall is stationary then you will have the velocities as 0. So, that means, your x direction velocity u, y direction velocity v and z direction velocity w all will be 0 at the wall and it is known as no slip boundary condition.

At the inlet, generally we define the velocity boundary condition. So, this is kind of Dirichlet boundary condition you have a constant value. So, you can have either uniform velocity inlet like constant velocity inlet. So, at inlet, so you have you can have in this particular case you can write u is equal to let us say U. So, at inlet you can have U where U is constant and v will be 0 and w will be 0. Also you can have some if you have a fully developed boundary condition. So, if you have a fully developed boundary condition, then you can have a parabolic profile. So, at the inlet you can have some parabolic profile.

So, this is your y, this is your x, x is the axial direction and let us say you have a parabolic velocity inlet which is function of y. So, for parabolic velocity inlet you can have u as u function of y ok. So, that you can determine depending on the condition here. So, if it is h and this is your h, then you can write the parabolic velocity boundary condition v will be 0 and w will be 0. At the outlet, most of the time we specify the pressure ok. So, generally p = 0 you put and if you want to write the velocity boundary condition.

So, generally we say that it has reached fully developed condition and at the outlet if it is a fully developed condition, then we can write

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial v}{\partial x} = 0, \ \frac{\partial w}{\partial x} = 0$$

because x is the axial direction and the velocity gradient in the axial directions are 0 if it is at outlet in fully developed condition.

So, obviously, if it is a channel flow ok, so at the outlet you will get the fully developed condition and you can use these boundary conditions for velocity. So, you can see that at wall we have no slip boundary condition where we are specifying the velocities 0;
$u = v = w = 0$ and this is known as Dirichlet type boundary condition and at outlet you can see that we are specifying the velocity gradient.

So, this is kind of Neumann boundary condition and at inlet we are specifying the velocity either it is a uniform velocity inlet u or a parabolic velocity inlet which u is function of y. So, these 2 also we are specifying the velocity. So, this is also known as Dirichlet type boundary conditions. So, these 2 are Dirichlet boundary conditions. So, that means, the value of the velocity is specified at the boundary and this is your gradient specified. So, it is known as Neumann type boundary condition.

So, today we started with the definition of convection, then we have discussed about why we are interested to study the convective heat transfer, then we discuss the basic laws of 3 modes of heat transfer.

At first, we discussed about Fourier's laws of heat conduction and we defined heat transfer rate and heat flux, then we discussed about the Newton's laws of cooling and from there we defined the heat transfer coefficient and we have shown some typical values of heat transfer coefficient in different flow situations. Then we discussed about non-dimensional number, Nusselt number and finally, we have shown that Nusselt number is the non-dimensional temperature gradient at the wall.

Then we discussed about the Stefan Boltzmann law and we have shown the net radiation exchange between a surface and the surrounding and from there we defined the radiation heat transfer coefficient and for combined convection and irradiation we have defined the heat flux. Finally, we discussed about the fluid flow equations, continuity equation and Navier Stokes equations in vector form as well as in differential form.

Thank you.

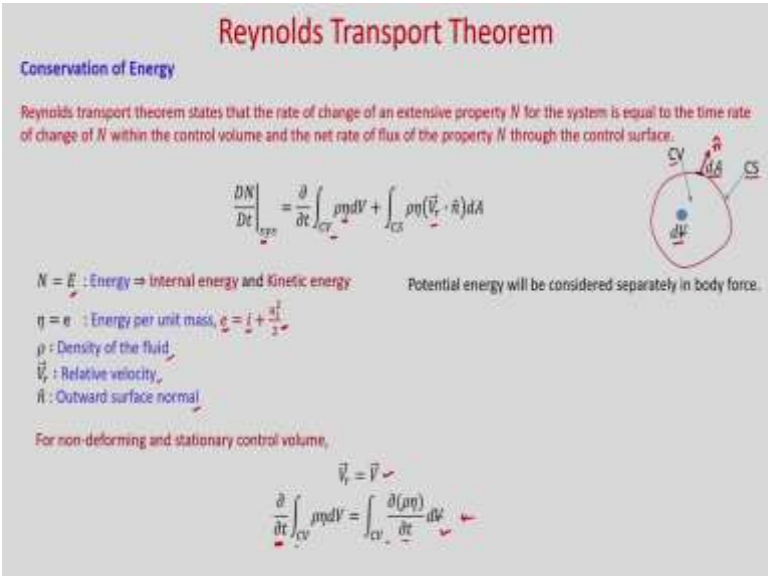
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 02
Preliminary Concepts
Lecture - 03
Derivation of energy equation

Hello everyone. So, you know that there are three conservation laws; conservation of mass, conservation of momentum and conservation of energy, all these three conservation laws must be satisfied at a point in a moving fluid. So, today we will derive the equation for conservation of energy which is known as energy, equation starting from the Reynolds Transport Theorem.

Already you have learnt Reynolds transport theorem in fluid mechanics course. So, we will use Reynolds transport theorem and we will conserve the energy and we will derive the energy equation.

(Refer Slide Time: 01:18)



So, what is Reynolds transport theorem? Reynolds transport theorem states that the rate of change of an extensive property N for the system is equal to the time rate of change of

N within the control volume and the net rate of flux of the property N through the control surface. So, you can see we can write $\frac{DN}{Dt}|_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho\eta dV + \int_{CV} \rho\eta(\vec{V_r}.\hat{n})dA$

So, you consider any arbitrary control volume, where this is the control surface and if you consider one elemental volume that we are denoting with dV and elemental surface is dA. So, you can see that this N is your extensive property, here we will consider this N= E which is your energy.

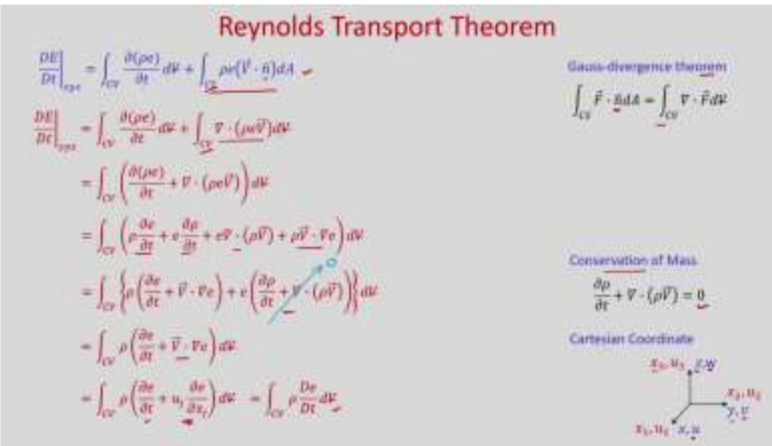
In this case we will consider total energy as summation of internal energy and kinetic energy, we will not consider the potential energy here we will consider this potential energy in the source term separately and all other energy we have neglected.

And this η= e which is your energy per unit mass. So, you can write energy per unit mass =internal energy + kinetic energy . So, now, you can see in this case N = E which is your total energy summation of internal energy plus kinetic energy and η is your energy per unit mass which is $e = i + \frac{u_i^2}{2}$ and; obviously, rho is the density of the fluid and V_r is relative velocity and n is outward surface normal. So, n if you consider here so, always it is outward normal.

So, now, we will assume stationary and non-deforming control volume, in that scenario you can write the relative velocity as V. So, $\vec{V_r} = \vec{V}$ and as your volume is not changing. So, this time derivative you can take inside this integral of this control volume as you are assuming non - deforming and stationary control volume.

So, $\frac{\partial}{\partial t}\int_{CV} \rho\eta dV = \int_{CV} \frac{\partial(\rho\eta)}{\partial t} dV$. So, in this particular case it will be just $\frac{\partial(\rho e)}{\partial t} dV$. So, this is the term we have we can write like this.

(Refer Slide Time: 04:21)



So, now if you put the total energy and energy per unit mass in the Reynolds transport equation then you will get this equation. So, $\frac{DE}{Dt}\Big|_{sys} = \int_{CV} \frac{\partial(\rho e)}{\partial t} dV + \int_{CS} \rho e(\vec{V}.\hat{n}) dA$, where n is the outward surface normal.

Now, what we will do? We will change this surface integral to volume integral using Gauss divergence theorem. So, what is Gauss divergence theorem? You can see if you have a $\int_{CS} \vec{F}.\hat{n}\, dA = \int_{CV} \nabla \vec{F} dV$. So, n is your surface normal you need surface normal.

So, you can write divergence form when you write in the integral inside the integral. So, you can see now if you put it here. So, these term if you use Gauss divergence theorem you can write $\nabla \rho e \vec{V} dV$. So, using Gauss divergence theorem this surface integral we have converted to volume integral.

So, now we will do some numerical algebra. So, you can see just you simplify it. So, you can see you can write, $\frac{\partial(\rho e)}{\partial t} + \nabla \rho e \vec{V}$. So, this derivative you just write as $\rho \frac{\partial e}{\partial t} + e \frac{\partial \rho}{\partial t}$ and this divergence you write $e\nabla(\rho \vec{V}) + \rho \vec{V}.\nabla e$. So, after writing this if you rearrange then this term and this term you write together. So, what you can write, $\rho(\frac{\partial e}{\partial t} + \vec{V}.\nabla e)$ and this term now you write together. So, $e(\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}))$.

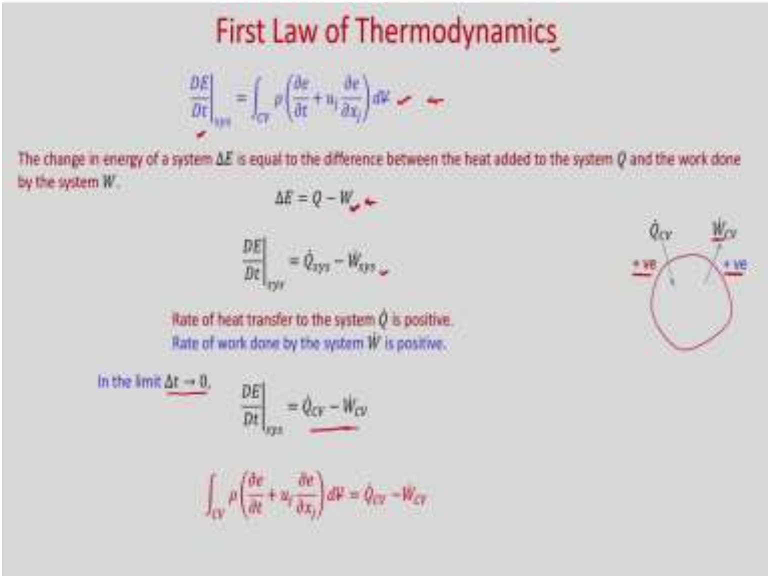
So, now, what it is you can see if you use this Reynolds transport equation and conserve the mass then you will get the continuity equation and this is your $\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{V}) = 0$. So, this is the continuity equation in general, it is applicable for compressible flow as well as incompressible flow. So, if you write this continuity equation. So, it is 0. So, this term will become 0.

So, you can write this as $\rho(\frac{\partial e}{\partial t} + \vec{V}.\nabla e) dV$ and in tensorial form if you write. So, $\vec{V}.\nabla e$ you can write $u_j \frac{\partial e}{\partial x_j}$. So, this term in tensorial form we have written like this or this term together you can write as $\int_{CV} \frac{De}{Dt} dV$. So, $\frac{De}{Dt}$ is material derivative.

So, which contains the temporal term as well as the convection term and in our study we are now considering only the Cartesian coordinate and you can see this is your x direction where velocity is u and y direction velocity v and z direction velocity w and when we will write the in tensorial form. So, you can write that u_1 x_1 is equivalent to u x, similarly y is equivalent to x_2 and v is equivalent to u_2 and similarly u_3 is equivalent to w and x_3 is equivalent to z.

So, now right hand side we have written in this form where left hand side still we need to determine. So, whatever the energy acting on the system that we need to consider.

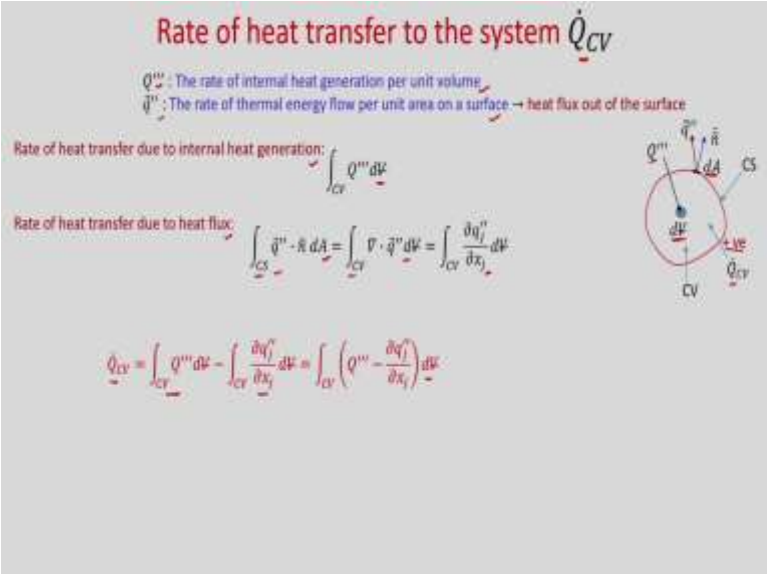
(Refer Slide Time: 08:17)



Now we will use first law of thermodynamics, the change in energy of a system ΔE is equal to the difference between the heat added to the system Q and the work done by the system W. So, from the first law of thermodynamics you can write that $\Delta E = Q - W$. So, that you have already studied in thermodynamics. So, now, if you write in a rate of change sense then you can write this equation as $\frac{DE}{Dt}\Big|_{sys} = \dot{Q}_{sys} - \dot{W}_{sys}$. So, now let us discuss about the sign convention. So, in heat transfer now we will consider that rate of heat transfer to the system is positive and rate of work done by the system is positive. So, you can see this is the control volume, in the control volume if your rate of heat transfer is to the system then we will consider as positive and rate of work done is by the system then we will consider as positive.

So, with this sense just will proceed the derivation and in the limit Δt →0, this system and volume will coincide. So, whatever we have written that $\frac{DE}{Dt}\big|_{sys} = \dot{Q}_{sys} - \dot{W}_{sys} = \dot{Q}_{CV} - \dot{W}_{CV}$. So, now you can see finally, if you put this $\frac{DE}{Dt}\big|_{sys}$ this expression in this equation then you can write $\int_{CV} \rho(\frac{\partial e}{\partial t} + u_j \frac{\partial e}{\partial x_j})dV = \dot{Q}_{CV} - \dot{W}_{CV}$. Now we need to derive the expression for this rate of heat transfer and rate of work done.

(Refer Slide Time: 10:51)



So, first let us consider rate of heat transfer to the system and we will derive the expression for $\dot{Q}_{CV}$. So, you can see there are two types of heating; one is volumetric heating and another is surface heating. So, you can see that if there is a volumetric heat generation then it is a volumetric phenomena and that you consider the rate of internal heat generation per unit volume.

So, if you consider this as a control volume and this is the control surface, in the control volume if you consider one elemental volume dV then this rate of internal heat generation is taking place in this elemental volume dV.

So, if $Q'''dV$ then you will get the rate of heat transfer inside this elemental volume and if you integrate over the whole volume then you will get the total rate of heat transfer due to heat generation. So, you can see. So, rate of heat transfer due to internal heat generation we can write $\int_{CV} Q''' dV$.

Now, there will be surface heating and surface heating will take place due to heat conduction. So, you can see if you consider one elemental area dA on the surface. So, dA and this is your normal unit normal n and your heat conduction is taking place in outward direction. So, it is heat flux $\overrightarrow{q''}$.

So, this q'' is the rate of thermal energy flow per unit area on a surface and it is heat flux out of the surface. Now you see this heat flux is the surface phenomena and it is going out of the surface, but we have considered that heat transfer to the system is positive so, but here your heat flux is going out of the surface. So, there will be a negative sign.

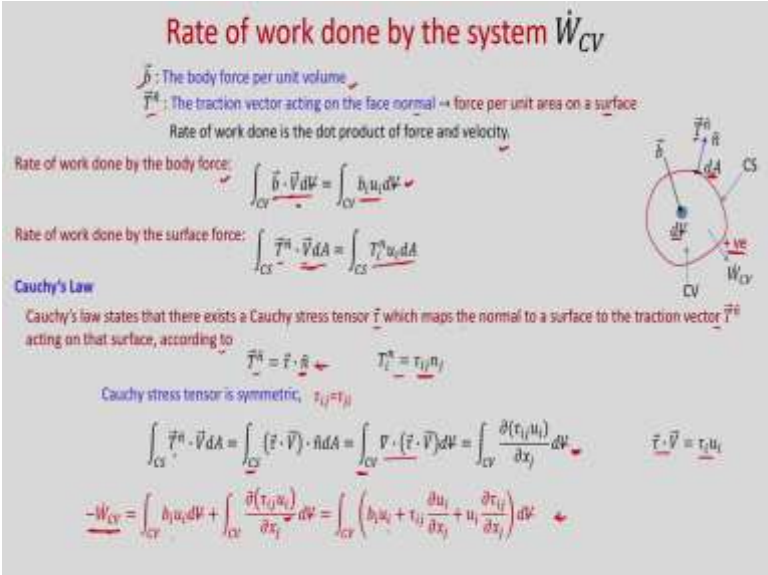
So, if you see rate of heat transfer due to heat flux. So, this q'' is acting on this surface. So, in the normal direction if you take then $\overrightarrow{q''}.\hat{n}\, dA$. So, it is acting on this surface dA, now if we integrate over the whole surface then you will get $\int_{CS} \overrightarrow{q''}.\hat{n}\, dA$.

Now you use Gauss divergence theorem to convert this surface integral to volume integral. So, you can write this $\int_{CV} \nabla.\overrightarrow{q''} dV$ and this if you write in tensorial form then you can write this ∇Q you can write $\frac{\partial q_j''}{\partial x_j}$, but this is negative, because Q dot cv we have taken if it is heat transfer is taking place to the system then it is positive, but it is going out.

So, when we will calculate $\dot{Q}_{CV}$ so, we will write this rate of heat transfer due to internal heat generation and minus because it is going out of the surface. So, this is the minus rate of heat transfer due to heat flux. So, together you can write $\int_{CV} (Q''' - \frac{\partial q_j''}{\partial x_j}) dV$.

So, now, let us consider rate of work done. So, rate of work done is $\dot{W}_{CV}$ and there are two types of forces acting on this fluid element. So, what are the forces, one is your body force which is your volumetric phenomena and you have surface force ok. So, that is your surface phenomena.

(Refer Slide Time: 14:55)



So, you can see the body force per unit volume we are writing as V and the surface force what is acting on this elemental surface dA that we will denote with the traction vector. So, this is denoted as $\vec{T}^{\hat{n}}$ the traction vector acting on the phase normal and it is force per unit area on a surface.

So, these are the forces acting on the volume as well as in the surface we need to calculate the work done right. So, work done is the dot product of this force and the velocity. So, rate of work done is the dot product of force and velocity. So, you can see rate of work done by the body force you can write.

So, V is the body force and you take the dot product of the velocity and if you consider this elemental volume dV. So, the rate of work done by the body force in this elemental volume is $\vec{b}.\vec{V}dV$ and if you integrate over the control volume then you can get the total rate of work done by the body force.

And this you can write in tensorial form as b_i u_i and rate of work done by the surface force. So, this is your normal direction to the elemental surface dA and this is the traction vector acting on this normal. So, you can see that rate of work done by the surface force you can write $\int_{CS} \vec{T}^{\hat{n}}.\vec{V} dA$ and this in tensorial form also you can write $\int_{CS} T_i^n u_i dA$.

So, now this traction vector we will relate with the stress tensor and we will use Cauchy's law. So, Cauchy's law states that there exists a Cauchy stress tensor tau which maps the normal to the surface to the $\vec{T}^{\hat{n}}$ acting on the surface according to this $\vec{T}^{\hat{n}} = \vec{\tau}.\hat{n}$, n is the unit normal outward of the surface and tensorial form you can write $T_i^n = \tau_{ij} n_j$ and Cauchy stress tensor is symmetric. So, you can write $\tau_{ij} = \tau_{ji}$ and now this integral now you consider. So, this $\vec{T}^{\hat{n}}$ you substitute this expression.

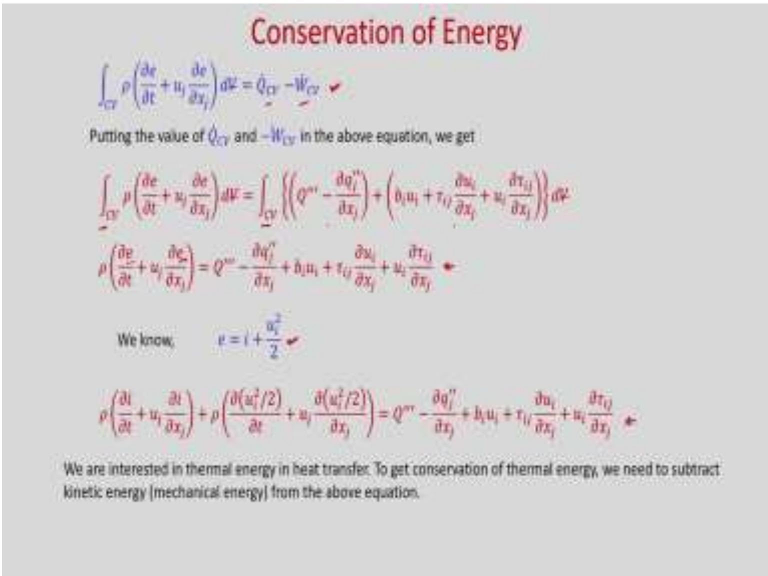
Then you can write $(\vec{\tau}.\overrightarrow{V}).\hat{n}\, dA$, now this surface integral you convert to the volume integral using Gauss divergence theorem and you write $\nabla.(\vec{\tau}.\vec{V})dV$ and in tensorial form we can write $\nabla.(\vec{\tau}.\vec{V})$ as $\frac{\partial(\tau_{ij} u_i)}{\partial x_j}$ because this $\vec{\tau}.\vec{V}$ you can write $\tau_i u_i$ and this a there is a divergence. So, that if you write then you can write $\frac{\partial(\tau_{ij} u_i)}{\partial x_j}$. So, this is the volume integral we have converted you have a rate of work done by the body force and you have rate of work done by the surface force, if you add together then you will get the work done by the force.

So, you can see in thermodynamics we have calculated this work done by the body force and work done by the surface force, but when we considered the sign convention in today's class that work done by the system is positive, but here whatever we have derived these are work done by the force so; that means, there is a change in sign convention.

So, work done by the system we have considered as positive, but here we have calculated work done by the force so; obviously, work done by the system if we consider then there will be a negative sign. So, we have written $-\dot{W}_{CV}$ because this is the sign convention we have taken, because $\dot{W}_{CV}$ work done by the system is positive, but whatever we have considered those are rate of work done by the force.

So, now, if you see. So, if there is a negative sign. So, actually you should write $\dot{W}_{CV}$ is equal to negative of this, but as simplification we are writing $-\dot{W}_{CV} = \int_{CV} b_i u_i dV + \int_{CV} \frac{\partial(\tau_{ij} u_{i)}}{\partial x_j} dV = \int_{CV} (b_i u_i + \tau_{ij} \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial \tau_{ij}}{\partial x_j}) dV$. So, this is your $\dot{W}_{CV}$.

(Refer Slide Time: 20:11)



So, you can see we had this expression, now you substitute this $\dot{Q}_{CV}$ expression in $-\dot{W}_{CV}$ expression here then you can get. So, left hand side will be as it is in the right hand side this is the $\dot{Q}_{CV}$ and this is the $-\dot{W}_{CV}$.

So, we can see both side you have volume integral. So, this is also volume integral this is also volume integral. So, for any arbitrary control volume so, you can write as

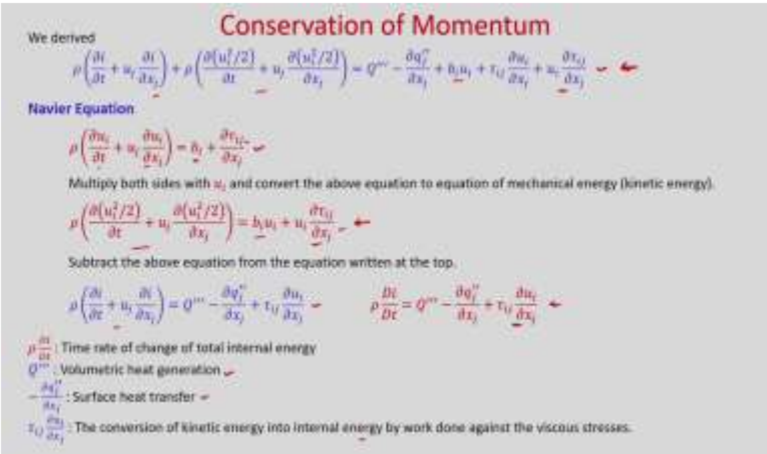
$\rho(\frac{\partial e}{\partial t}+u_j\frac{\partial e}{\partial x_j})=Q''' -\frac{\partial q_j''}{\partial x_j}+b_i u_i+\tau_{ij}\frac{\partial u_i}{\partial x_j}+u_i\frac{\partial \tau_{ij}}{\partial x_j}$. Let us substitute the total energy at summation of internal energy plus kinetic energy ok. So, that is $e=i+\frac{u_i^2}{2}$.

So, if you put it in this expression. So, what you will get, you can see $\rho(\frac{\partial i}{\partial t}+u_j\frac{\partial i}{\partial x_j})+\rho(\frac{\partial(u_i^2/2)}{\partial t}+u_j\frac{\partial(u_i^2/2)}{\partial x_j})$ and write tensor terms will be as it is. Now you see in heat transfer we are interested in your internal energy, but we are having in this expression internal energy plus mechanical energy; that means, your kinetic energy.

So, now we have to subtract somehow this kinetic energy. So, that we can get the equation for conservation of thermal energy; that means, only internal energy will be present. So, to do that, now we will consider the Navier's equation.

So, from the Navier's equation if you multiply with the velocity u $_i$ then you can write the equation for kinetic energy and once we get the equation for kinetic energy and if you subtract that equation from this equation then you will get the equation for conservation of thermal energy.

(Refer Slide Time: 22:34)



So, this is the equation we have derived now Navier's equation, you can see in general we can write like this $\rho(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j})=b_i+\frac{\partial \tau_{ij}}{\partial x_j}$, where b_i is the body force term and τ_{ij} is your stress tensor and this is temporal term and this is your convective term.

So, we have considered Navier equation. So, it is valid in general for compressible and incompressible fluid Newtonian and non - Newtonian fluid flow. So, multiply both sides with u_i and convert the above equation to equation of mechanical energy. So, if you multiply u_i here and if you take in the inside these derivative then you can write $\frac{\partial(u_i^2/2)}{\partial t}$ and here also u_i if you take inside these derivative then you can write $u_j\frac{\partial(u_i^2/2)}{\partial x_j}$ and here $b_i u_i + u_i\frac{\partial\tau_{ij}}{\partial x_j}$.

So, now, subtract these equation from this equation. So, what you will get now you are subtracting the kinetic energy from this equation so that you can get the equation for internal or thermal energy. So, you can see if you subtract.

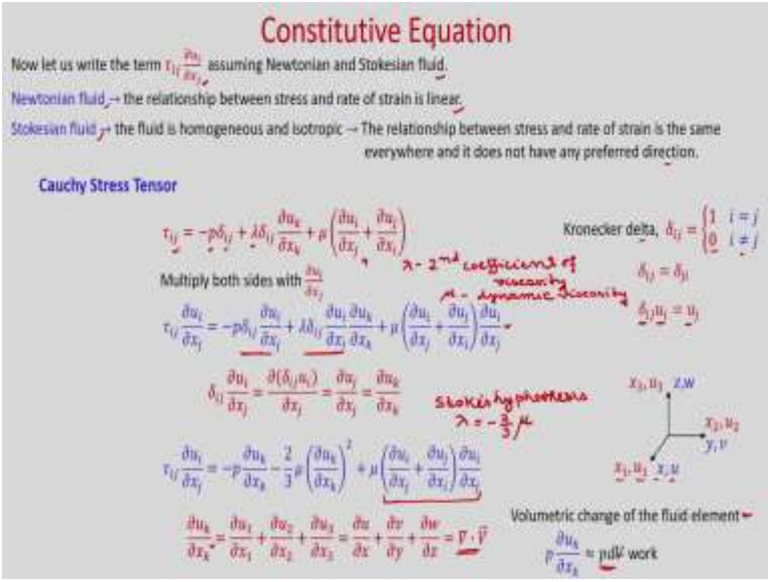
So, this term and this term will get cancel b_i u_i will get cancel. So, you can see you can write $\rho(\frac{\partial i}{\partial t} + u_j\frac{\partial i}{\partial x_j}) = Q''' - \frac{\partial q_j''}{\partial x_j} + \tau_{ij}\frac{\partial u_i}{\partial x_j}$. So, this term also will get cancel with this. So, we will have only $+\tau_{ij}\frac{\partial u_i}{\partial x_j}$. So, this term if you write in terms of material derivative then you can write $p\underbrace{\frac{\partial u_k}{\partial x_k}}^{\nabla.\vec{V}}$ this is the left hand side and right hand side will be same as this equation.

So, now you can see the left hand side you have time rate of change of total internal energy right and in the right hand side the first term Q'''. So, this is your volumetric heat generation, the second term it is due to surface heat transfer and the third term you can see the conversion of kinetic energy into internal energy by work done against the viscous stresses.

Thus so, in the fluid flow will occur and there will be friction between the two fluid layers and that friction will be converted to thermal energy. And due to stresses that you see there will be a generation of internal energy and that is the conversion of kinetic energy into internal energy. Now you can see this is the term we need to expand $\tau_{ij}\dfrac{\partial u_i}{\partial x_j}$.

So, now, first let us see the constitutive equation.

(Refer Slide Time: 25:41)



The constitutive equation gives the expression for the Cauchy's stress tensor. So, you will get the expression for τ_{ij}. So, before that now let us assume that we have Newtonian fluid and Stokesian fluid. So, now, we are assuming that we have the fluid flow for Newtonian and Stokesian fluid. What is Newtonian fluid? The relationship between the stress and the rate of strain is linear and Stokesian fluid the fluid is homogenous and isotropic.

What is homogenous? The relationship between stress and rate of strain is the same everywhere and isotropic means it does not have any preferred direction. So, assuming this Newtonian fluid and Stokesian fluid we can write τ_{ij} as $-p\delta_{ij}$, where p is your thermodynamic pressure, δ_{ij} is your Kronecker delta where δ_{ij} = 1, when i = j and δ_{ij} is =0 when i $\neq$ j.

And $+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}$, where λ is your second coefficient of viscosity $+\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$. So, you can see we have written this τ_{ij} in terms of thermodynamic pressure and the velocity gradient and where μ is your dynamic viscosity.

So, now we need to determine the term $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ right. So, this is the term we want to derive. So $\tau_{ij}\frac{\partial u_i}{\partial x_j}$. So, you multiply $\frac{\partial u_i}{\partial x_j}$ with these terms. So, what you will get, $-p\delta_{ij}\frac{\partial u_i}{\partial x_j}+\lambda\delta_{ij}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_k}$ + with this term you multiply $\frac{\partial u_i}{\partial x_j}$. This you can see that if you operate these δ_{ij} on u_i then you will get u_j.

So, δ_{ij} u_i you will get u_j. So, here now $\delta_{ij}\frac{\partial u_i}{\partial x_j}$. So, this is the term this is the same term we want to write $\delta_{ij}\frac{\partial u_i}{\partial x_j}=\frac{\partial(\delta_{ij}u_i)}{\partial x_j}$. So, I have taken this δ_{ij} here. So, you can see δ_{ij} u_i if you operate this δ_{ij} on u_i it will get u_j. So, $\frac{\partial u_j}{\partial x_j}$ and it is equivalent to write $\frac{\partial u_k}{\partial x_j}$ because both are same j. So, this j we have replaced with k. So, $\frac{\partial u_k}{\partial x_k}$.

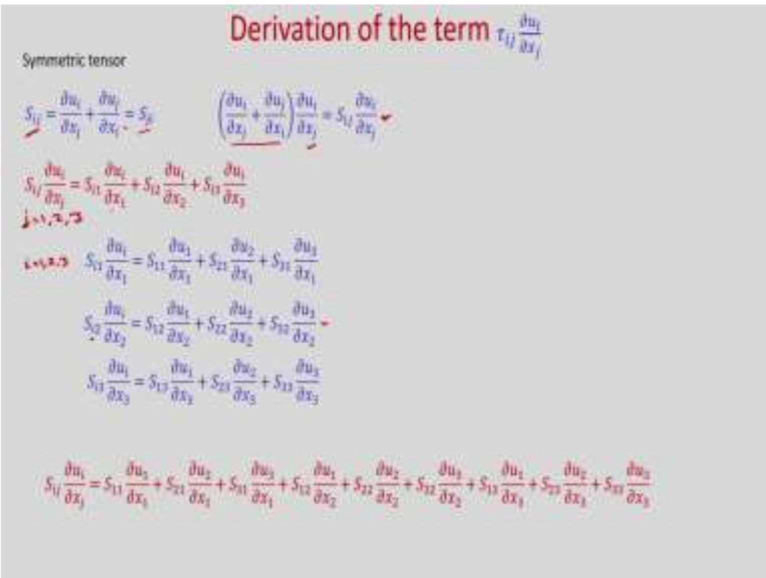
So, you can see this term will become $-p\frac{\partial u_k}{\partial x_k}$ and this term will become $(\frac{\partial u_k}{\partial x_k})^2$. So, it will be $\lambda(\frac{\partial u_k}{\partial x_k})^2$,now we will use Stoke's hypothesis. So, in Stoke's hypothesis we can write the second coefficient of viscosity in terms of the dynamic viscosity as $-\frac{2}{3}\mu$. So, if you put it here.

So, it will be $-\frac{2}{3}\mu(\frac{\partial u_k}{\partial x_k})^2$. So, now, if you expand $\frac{\partial u_k}{\partial x_k}$ what it is, $\frac{\partial u_k}{\partial x_k}$. So, now, k u vary 1 2 3. So, it will be $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}$. So, now, u_1 is equivalent to u. So, you can write $\frac{\partial u}{\partial x}$, x_1 is equivalent to x similarly $+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ at which is nothing, but $\nabla.\vec{V}$.

So, what is $\nabla.\vec{V}$? $\nabla.\vec{V}$ is the volumetric change of the fluid element right, for incompressible fluid $\nabla.\vec{V} = 0$ right, but in general we are writing. So, there will be a volumetric change in the fluid element and that is represented by $\nabla.\vec{V}$ and these p $\nabla.\vec{V}$ is nothing, but the pdV work in thermodynamics.

So, you can see whatever we have written $p\dfrac{\partial u_k}{\partial x_k}$ which is your p $\nabla.\vec{V}$ and that is nothing, but your pdV work in thermodynamics. So, now, let us expand this term.

(Refer Slide Time: 30:52)



So, we will write this term $\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}$ as symmetric tensor S_{ij} and it is equal to S_{ji} because it will remain same. So, it is a symmetric tensor S_{ij} is equal to S_{ji}. So, this term we can write as $S_{ij}\dfrac{\partial u_i}{\partial x_j}$ because this term we have written as S_{ij}. So, $S_{ij}\dfrac{\partial u_i}{\partial x_j}$.

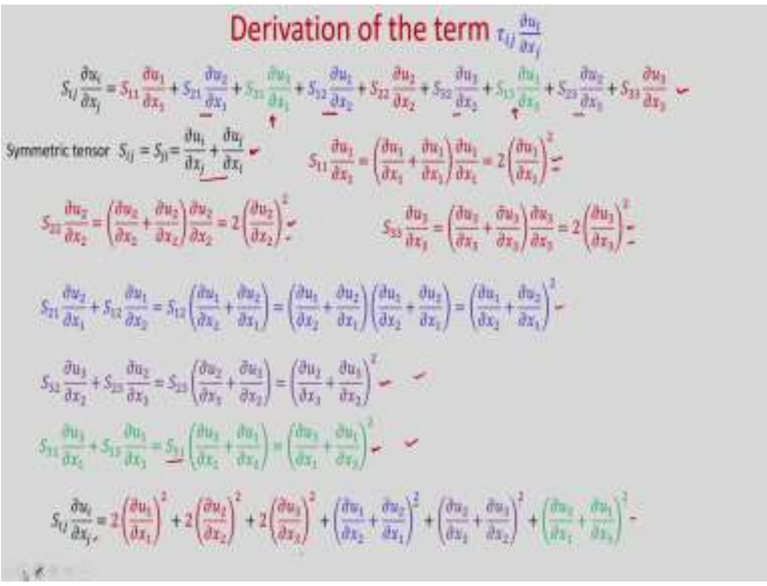
So, now you can see $S_{ij}\dfrac{\partial u_i}{\partial x_j}$, first you vary j =1, 2 and 3. So, if you vary 1, 2, 3 so what will be there? So, it will be $S_{i1}\dfrac{\partial u_i}{\partial x_1} + S_{i2}\dfrac{\partial u_i}{\partial x_2} + S_{i3}\dfrac{\partial u_i}{\partial x_3}$, because j we have varied 1, 2 and 3. Now you vary i = 1, 2 and 3 so, in each term. So, if you take the these term.

So, i you vary 1 2 3. So, it will be $S_{11}\dfrac{\partial u_1}{\partial x_1}+S_{21}\dfrac{\partial u_2}{\partial x_1}+S_{31}\dfrac{\partial u_3}{\partial x_1}$.

Similarly here also you vary i=1,2,3. So, if you put i=2 $S_{12}\dfrac{\partial u_1}{\partial x_2}+S_{22}\dfrac{\partial u_2}{\partial x_2}+S_{32}\dfrac{\partial u_3}{\partial x_2}$ and for i=3, $S_{13}\dfrac{\partial u_1}{\partial x_3}+S_{23}\dfrac{\partial u_2}{\partial x_3}+S_{33}\dfrac{\partial u_3}{\partial x_3}$.

So, now, you see this term you can write as summation of these three. So, there will be nine components. So, we have written just these nine components in the expression of $S_{ij}\dfrac{\partial u_i}{\partial x_j}$. So, now, we have to find what is these S_{11}, S_{21} all these terms and S_{ij} already we have represented as this one.

(Refer Slide Time: 33:05)



So, you can see. So, this is the term we have derived S_{ij} is the expression this one. So, now, you see the red colored terms. So, $S_{11}\dfrac{\partial u_1}{\partial x_1}$. Now S_{11} we have to find from this expression. So, if you put i and j both 1; that means, $\dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_1}{\partial x_1}$ because i and j are 1 and you have $\dfrac{\partial u_1}{\partial x_1}$. So, this will become $2\dfrac{\partial u_1}{\partial x_1}$. So, it will be $2(\dfrac{\partial u_1}{\partial x_1})^2$.

Similarly the other two terms in the red colored you can write $S_{22}\frac{\partial u_2}{\partial x_2}$ as $2(\frac{\partial u_2}{\partial x_2})^2$ and $S_{33}\frac{\partial u_3}{\partial x_3}$ as $2(\frac{\partial u_3}{\partial x_3})^2$.

Now rest other terms let us find. So, you see this term and this term you consider, because you know that it is a symmetric tensor S_{ij} is equal to S_{ji}. So, $S_{21}=S_{12}$. So, these two terms we are considering together and we are writing. So, S_{12} you can take it outside and you can write $\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}$ and S_{12} now i=1, j=2.

So, you put it here i=1, j =2. So, $\frac{\partial u_1}{\partial x_2}$ and j =2. So, $\frac{\partial u_2}{\partial x_1}$. So, now, you can see these two terms are same. So, it will be $(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1})^2$. Similarly you consider rest of the term. So, you can see $S_{32}\frac{\partial u_3}{\partial x_2}$ and $S_{23}\frac{\partial u_2}{\partial x_3}$ you are considering these two terms together because we know S_{32} is equal to S_{23}.

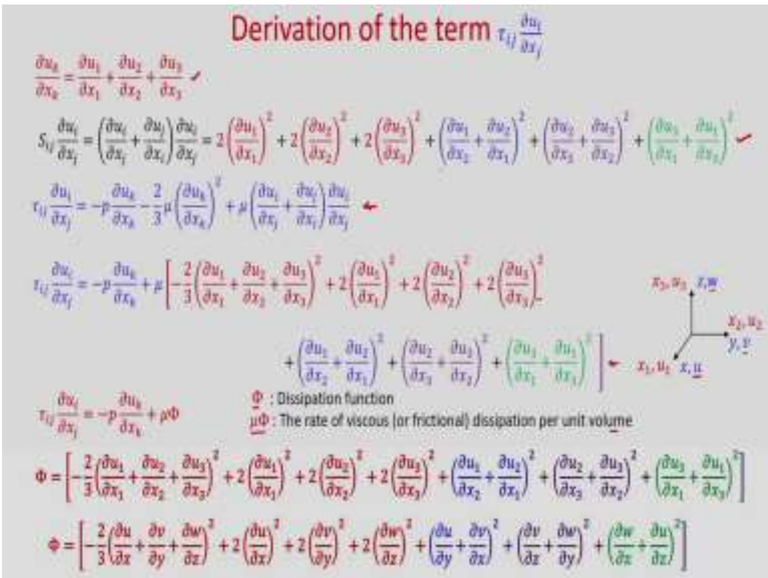
So, if you consider this you can write $(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2})^2$ and similarly $S_{31}\frac{\partial u_3}{\partial x_1}$ and $S_{13}\frac{\partial u_1}{\partial x_3}$. So, S_{13} is equal to S_{31}. So, we can write $S_{31}(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3})$ and this S_{31} you can find here i=3 and j =1. So, you can see $(\frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3})^2$.

So, now, all these terms we have retained in terms of velocity gradients right. So, now, substitute all these terms in $S_{ij}\frac{\partial u_i}{\partial x_j}$. So, it will be you see this is one term, second, third. So, we have written here then we have these terms. So, we have written here. So, now, you look into this expression.

So, what is this? This is the generation of internal energy due to friction right, because you are converting this kinetic energy to thermal energy right. And all these gradients are having square, you see, $(\frac{\partial u_2}{\partial x_2})^2$, $(\frac{\partial u_3}{\partial x_3})^2$ and all these are having square.

So, whether velocity gradient is positive or negative always this term $S_{ij}\dfrac{\partial u_i}{\partial x_j}$ will be positive term, you will get always heating inside the fluid element. So, because due to the friction between two fluid elements it will generate heat and it will be always positive; that means, it will be heat, heating will be there. So, you can see from this expression.

(Refer Slide Time: 36:55)

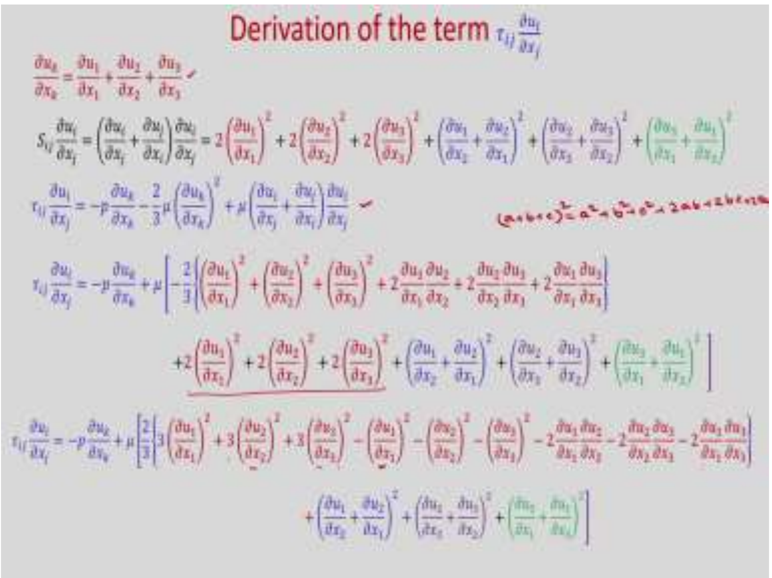


You see $\dfrac{\partial u_k}{\partial x_k}$ already we have written in this expression $S_{ij}\dfrac{\partial u_i}{\partial x_j}$ we have written in this expression and we have $\tau_{ij}\dfrac{\partial u_i}{\partial x_j}$ these expression now you put all these values right. So, $\dfrac{\partial u_k}{\partial x_k}$ this expression and this expression you write from here and you can write $\tau_{ij}\dfrac{\partial u_i}{\partial x_j}$ as this term + μ I am taking outside $-\dfrac{2}{3}$ then $\dfrac{\partial u_k}{\partial x_k}$ is this one. So, it is whole square plus this terms plus these three terms.

So, you can see we have written $\tau_{ij}\dfrac{\partial u_i}{\partial x_j}$ in terms of velocity gradients. So, now, we can see this you can write μ into all these terms you can write as Φ. So, this Φ is known as dissipation function and μΦ is the rate of viscous dissipation per unit volume. So, you can see this expression you can write as $\tau_{ij}\dfrac{\partial u_i}{\partial x_j} = -p\dfrac{\partial u_k}{\partial x_k} + \mu\phi$.

So, Φ is the dissipation function and μΦ is the rate of viscous dissipation. So, you can see now Φ will be this term. So, this we have written and in terms of u v w and x y z if you write then you will get this expression. And if you consider incompressible fluid flow then you can see this will be your 0 because this is your continuity equation $\nabla.\vec{V}$.

(Refer Slide Time: 39:01)



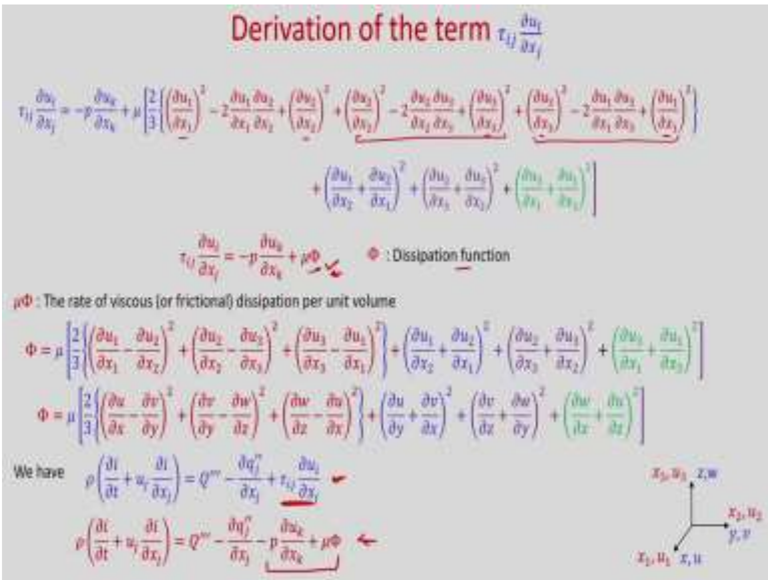
In other way also you can express this dissipation function. So, you can see now this is the expression we have, you take μ outside and $-\dfrac{2}{3}$ we have taken common. Now if you express these $(\dfrac{\partial u_k}{\partial x_k})^2$; that means, this is square.

So, we can write $(\dfrac{\partial u_1}{\partial x_1})^2 + (\dfrac{\partial u_2}{\partial x_2})^2 + (\dfrac{\partial u_3}{\partial x_3})^2 + 2ab + 2bc + 2ca$ right $(a+b+c)^2$. So, this we have written. So, it is $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

So, this expression we have written and we have this term. So, we have written here and now we are taking $\dfrac{2}{3}$ outside. So, this terms we are writing here. So, as $\dfrac{2}{3}$ we have taken outside. So, it will be $3(\frac{\partial u_1}{\partial x_1})^2 + 3(\frac{\partial u_2}{\partial x_2})^2 + 3(\frac{\partial u_3}{\partial x_3})^2$ and $\frac{2}{3}$ is there and negative of these term you have to write. So, negative of this term we have written like this.

So, now, let us rearrange this. So, we can see you have $3(\frac{\partial u_1}{\partial x_1})^2$ and here 1 if you subtract then you will get $2(\frac{\partial u_1}{\partial x_1})^2$. Similarly for this term we will get 2 factor 2 as a coefficient.

(Refer Slide Time: 40:47)



So, now you rearrange it like this. So, you have one you have written here, another you have written here, here you have written, one here, another so here one, here another. So, these you have written then $-2\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}$ is there. So, now, what is this? It is a $(-b)^2$ right. So, if you write in this form. So, it will become. So, $(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2})^2$.

Similarly, these term together you can write $(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3})^2$ and these three terms together you can write $+3(\frac{\partial u_3}{\partial x_3} - \frac{\partial u_1}{\partial x_1})^2$ and all these 3 terms will be there. So, all these terms we have written in terms of dissipation function and this $\tau_{ij}\frac{\partial u_i}{\partial x_j} = -p\frac{\partial u_k}{\partial x_k} + \mu\phi$, where phi is the dissipation function and μΦ is the rate of viscous dissipation per unit volume.

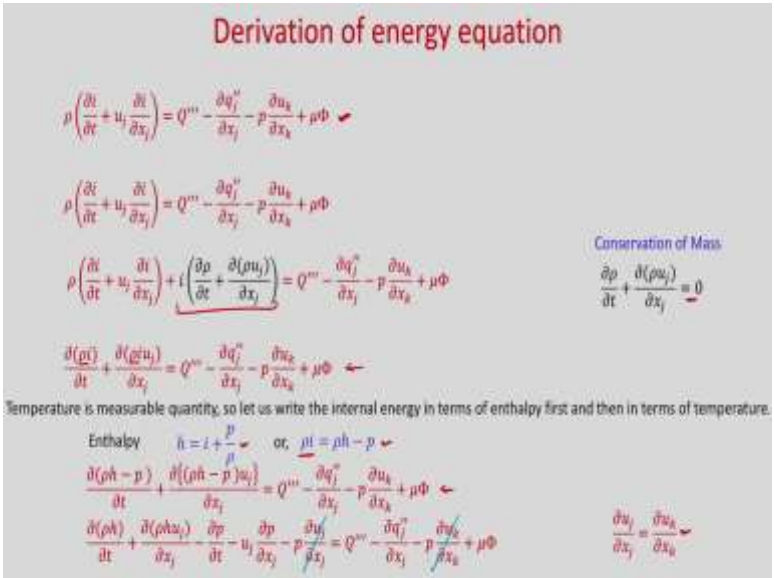
So, Φ now this full expression you can write in terms of velocity u v w and coordinate x y z if you write then it will be

$$[\frac{2}{3}\{(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y})^2+(\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z})^2+(\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x})^2\}+(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})^2+(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})^2+(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})^2].$$

So, this is your dissipation function. So, now, we had this one. So, now, we have derived this term as this. So, if you substitute in this equation. So, you can get this equation. So, this is the two terms whatever we have derived and phi is the dissipation function, it is a big expression in terms of velocity gradient.

So, we have considered Cartesian coordinates. So, this is the expression for dissipation function in Cartesian coordinate, but if you consider cylindrical coordinate or spherical coordinate then this expression for this dissipation function will be different.

(Refer Slide Time: 43:02)



So, we have already derived this now what will do, we will just add this term $i(\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j})$. So, if you add this term it is actually conservation of mass $\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j}$=0 in general, it is applicable for both compressible and incompressible flow.

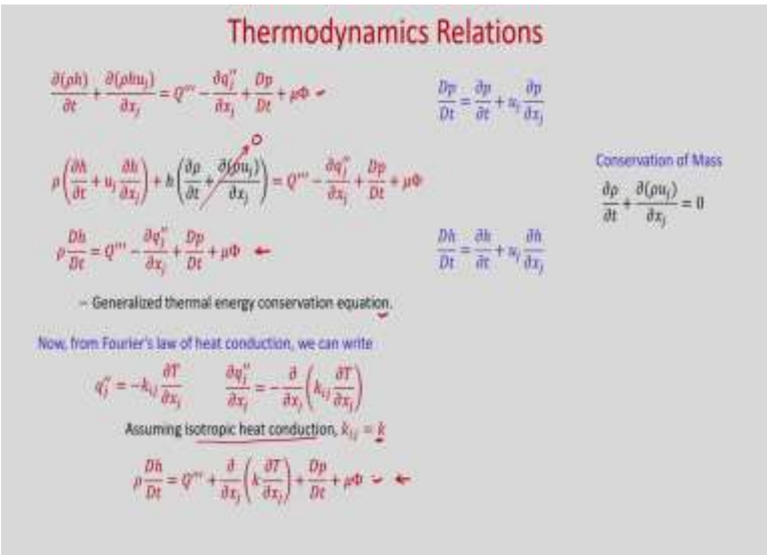
So, if you add this term then together if you write. So, you can write in a conservative form, because this $\rho\frac{\partial i}{\partial t}+i\frac{\partial \rho}{\partial t}$ you can write as $\frac{\partial(\rho i)}{\partial t}$ and $\rho u_j\frac{\partial i}{\partial x_j}+i\frac{\partial(\rho u_j)}{\partial x_j}$ you can write as $\frac{\partial(\rho i u_j)}{\partial x_j}$.

So, now, this is the equation we have derived in terms of internal energy, but now we need to write in terms of some measurable quantity. So, that is temperature. So, first we will write this equation in terms of enthalpy, then we will write in terms of temperature.

So, you know that enthalpy from thermodynamics, you can know that enthalpy, $h=i+\frac{p}{\rho}$, where p is the thermodynamic pressure and ρ is the fluid density. So, you can write $\rho i=\rho h-p$. Now this ρi you substitute here.

So, if you substitute here you will get this equation, $\frac{\partial(\rho h)}{\partial t}+\frac{\partial(\rho h u_j)}{\partial x_j}-\frac{\partial \rho}{\partial t}-p\frac{\partial u_j}{\partial x_j}=Q^{'''}-\frac{\partial q_j^{''}}{\partial x_j}-p\frac{\partial u_k}{\partial x_k}+\mu\phi$.

(Refer Slide Time: 45:27)



So, now you can write. So, this is in conservative form we have written $\frac{\partial(\rho h)}{\partial t}+\frac{\partial(\rho h u_j)}{\partial x_j}=Q^{'''}-\frac{\partial q_j^{''}}{\partial x_j}+\frac{Dp}{Dt}+\mu\phi$ and now if you write these two terms. So,

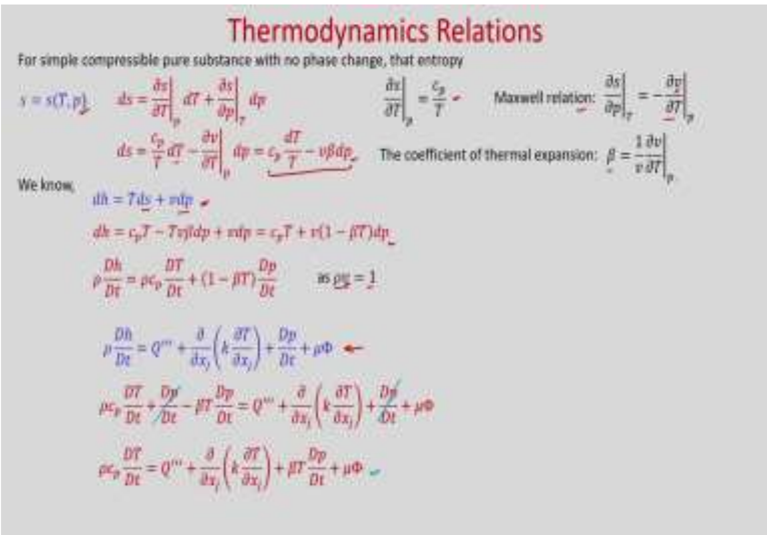
$\rho \frac{\partial h}{\partial t} + h \frac{\partial \rho}{\partial t}$ and here if you write $\rho u_j \frac{\partial h}{\partial x_j} + h \frac{\partial (\rho u_j)}{\partial x_j}$ then you can write in terms of non conservative form.

And here you can see this is your conservation of mass $\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j}$ = 0. So, this is your 0 right. So, we have written this term. So, this is your generalized thermal energy conservation equation we have written in terms of enthalpy.

So, this enthalpy now we have to write in terms of temperature and we have this heat flux $q^{"}$ and that we need to write in terms of temperature. So, first let us write this heat flux using Fourier's law of heat conduction. So, from Fourier's law of heat conduction you know $q_j^{"} = -k_{ij} \frac{\partial T}{\partial x_j}$ and if you write $\frac{\partial q_j^{"}}{\partial x_j} = -\frac{\partial}{\partial x_j}(k_{ij} \frac{\partial T}{\partial x_j})$.

So, now, you assume isotropic heat conduction so; that means, this thermal conductivity will be independent of the directions. So, k_{ij}=k we can write. So, if you write k then in this expression if you put then you can write $\rho \frac{Dh}{Dt} + Q^{'''} + \frac{\partial}{\partial x_j}(k \frac{\partial T}{\partial x_j}) + \frac{D\rho}{Dt} + \mu\phi$.

(Refer Slide Time: 47:30)



So, now let us write the enthalpy in terms of temperature. So, for that we will use some thermodynamic relations which we have already studied in your thermodynamics course, for simple compressible pure substance with no phase change entropy you can write $s = s(T, p)$.

So, we can write s as a function of two variable. So, we can write $ds = \frac{\partial s}{\partial T}|_p dT + \frac{\partial s}{\partial p}|_T dp$. So, now, you can see that you know $\frac{\partial s}{\partial T}|_p = \frac{c_p}{T}$. So, this is your specific heat at constant pressure.

And from Maxwell relations you can see $\frac{\partial s}{\partial p}|_T = -\frac{\partial v}{\partial T}|_p$ where v is your specific volume.

So, if you put these $\frac{\partial s}{\partial T}|_p$.

So, $\frac{c_p}{T} dT$ and $\frac{\partial s}{\partial p}$ you put this expression. So, $-\frac{\partial v}{\partial T}|_p dp$ and now you use the coefficient of thermal expansion expression. So, β you know that $\frac{1}{v}\frac{\partial v}{\partial T}|_p$. So, you can see $\frac{\partial v}{\partial T}|_p$ you can write vβ and this expression $ds = c_p \frac{dT}{T} - v\beta dp$.

Now again you can write the enthalpy, . So, now, $dh = c_p T - Tv\beta dp + vdp$. So, it will be c_pT and v if you take outside. So, (1- βT)dp. So, now, if you write in terms of material derivative so, $\rho \frac{Dh}{Dt} = \rho c_p \frac{DT}{Dt} + (1-\beta T)\frac{Dp}{Dt}$.

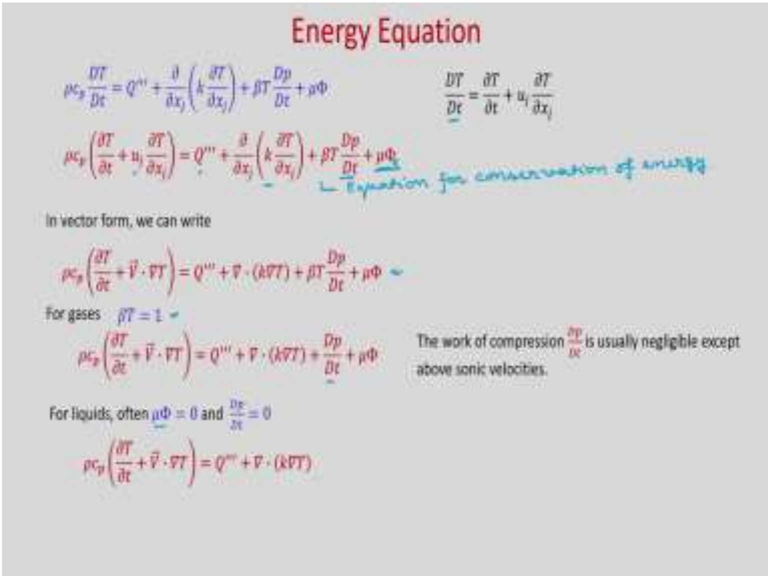
So, this is your material derivative $\frac{Dp}{Dt}$ and ρv=1 because v is specific volume. So, ρ v will be 1. So, this I have put 1. So, now, we can see this $\rho \frac{Dh}{Dt}$ now if you put in this expression then you can write. So, $\rho \frac{Dh}{Dt} = Q''' + \frac{\partial}{\partial x_j}(k\frac{\partial T}{\partial x_j}) + \frac{Dp}{Dt} + \mu\phi$, this we have already derived. So, if you put this expression in the left hand side then you can get these.

Now you see in left hand side you have $\frac{Dp}{Dt}$, right hand side also you have $\frac{Dp}{Dt}$. So, these you can cancel. So, if you now simplify it you will get

$$\rho c_p \frac{DT}{Dt} = Q''' + \frac{\partial}{\partial x_j}(k\frac{\partial T}{\partial x_j}) + \beta T \frac{Dp}{Dt} + \mu\phi .$$

So, now you can see that we have written the energy equation in terms of temperature and left hand side we have written in terms of material derivative $\frac{DT}{Dt}$.

(Refer Slide Time: 51:12)



So, now if you write this material derivative of T as $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}$ then you can see $\rho c_p (\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}) = Q''' + \frac{\partial}{\partial x_j}(k\frac{\partial T}{\partial x_j}) + \beta T \frac{Dp}{Dt} + \mu\phi$. So, this is the equation for conservation of energy.

So, you see left hand side term, ρ is density, c_p is the specific heat this is the temporal variation of this temperature $\frac{\partial T}{\partial t}$ and this is the convective term $u_j \frac{\partial T}{\partial x_j}$, because your these scalar temperature is convicted by this velocity u_j right and right hand side this term is internal heat generation per unit volume and this is the term you can see this is coming from your heat conduction.

So, in the energy equation this is the term while deriving this equation you have seen it is coming from the heat conduction right, but in general if k is constant you can take it outside and you can write $\frac{\partial T}{\partial x_j}$ whole you can write $(\frac{\partial T}{\partial x_j})^2$.

So, this is the term which is coming from the Fourier's law of heat conduction right and substituting that in the expression we have got this term which is known as diffusion term, it is due to the heat conduction right plus beta T$\frac{Dp}{Dt}$. So, $\frac{Dp}{Dt}$ which is your material derivative of p. So, these term having the significance at high velocity. So, above the sonic velocity if you have the flow then this term will have significant contribution.

And μΦ, what is μΦ? μΦ is the viscous dissipation rate right per unit volume. So, and phi is the dissipation function that we have written in terms of velocity gradient.

So, you can see when μ is having very high value then this term will have the significance and if you have a high velocity then due to high velocity there will be contribution from the velocity gradient in the dissipation function Φ then also this term will have some significance. So, you can see this is for high viscosity or high velocity you can consider this term and this term you can consider when you have very high velocity above sonic velocity.

So, in vector form this equation now if you write $\rho c_p(\frac{\partial T}{\partial t}+\vec{V}.\nabla T)=Q^{'''}+\nabla.(k\nabla T)+\beta T\frac{Dp}{Dt}+\mu\phi$ and these if you assume that your thermal conductivity is constant then you can take it outside otherwise you keep it inside. So, most of the gases you know for gases you know that $\beta=\frac{1}{T}$; that means, $\beta T=1$.

So, if you put $\beta T=1$ then this expression here it will be just $\frac{Dp}{Dt}$ and this the work of compression $\frac{Dp}{Dt}$ is usually negligible except above sonic velocities and for liquids open you will get $\mu\Phi = 0$ and $\frac{Dp}{Dt} = 0$. Then you can express this equation in the right hand side only two terms will be there $Q''' + \nabla.(k\nabla T)$, but if there is a significant contribution from μΦ then you can consider in this equation.

So, today we have started with the Reynolds transport equation to derive the equation for conservation of energy. So, for a control volume we have written the total energy as internal energy plus kinetic energy because potential energy we consider in the body force term from first law of thermodynamics we have written $\frac{DE}{Dt}\Big|_{sys} = \dot{Q}_{CV} - \dot{W}_{CV}$, where we are taking that rate of heat transfer to the system $\dot{Q}_{CV}$ is positive and rate of work done by the system $\dot{W}_{CV}$ is positive.

Then we have expressed this $\dot{Q}_{CV}$ and $\dot{W}_{CV}$ considering the volumetric volume heating as well as surface heating, for the rate of heat transfer calculation and for the rate of work done calculation we have considered two different forces body force as well as surface force. Then we have expressed these terms and finally, we have got one term $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ which is your the term which is contributing from the friction because you have a shear stress between the two fluid elements and these frictional force is converting to internal energy.

Then that we have expressed in terms of velocity gradients and finally, these total energy we have written in terms of internal energy plus kinetic energy. So, to get the equation for thermal energy we have subtracted the kinetic energy considering the Navier's equation. Then we have written the energy equation in terms of internal energy and later we have converted this internal energy in terms of enthalpy then in terms of a measurable quantity temperature.

And using some thermodynamics relations we have written the equation for this internal energy in terms of temperature which is your equation for conservation of energy. We have also discussed that the viscous dissipation term μΦ is significant when you have high viscosity fluid or you have high velocities.

And $\frac{Dp}{Dt}$ is significant above the sonic velocities and finally, we have written this equation in terms of vectorial form for gases assuming $\beta = \frac{1}{T}$ and finally, we have written for liquids where most of the time we consider negligible viscous heating and $\frac{Dp}{Dt}$ as 0.

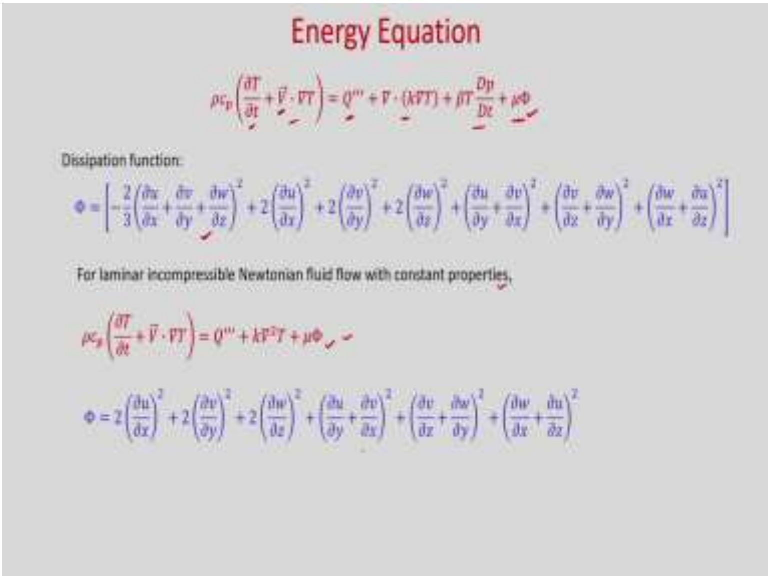
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 02
Preliminary Concepts
Lecture - 04
Derivation of boundary layer equations

Hello everyone. So, in last class, we derived the energy equation. Now, in today's class, first we will just summarize what we have done in last class, and we will write the governing equations in cylindrical and spherical coordinates, then we will simplify these equations for Cartesian coordinate for the boundary layer flows.

(Refer Slide Time: 00:54)

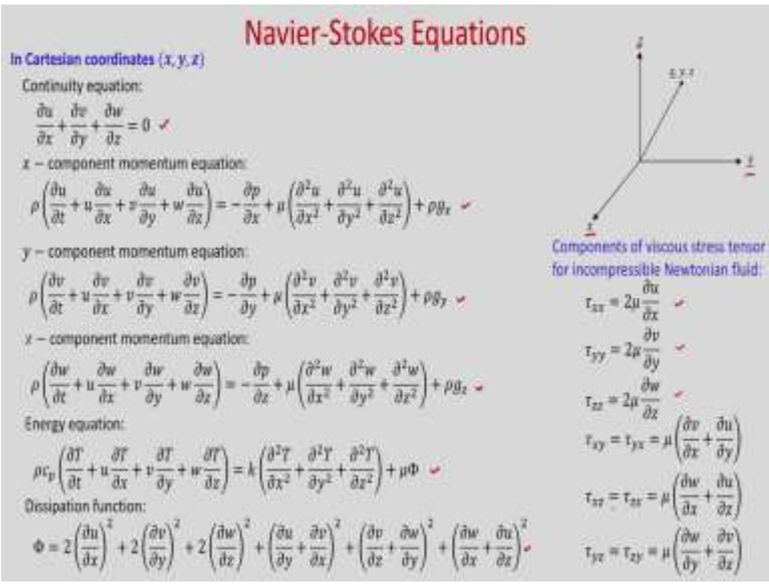


So, we can see we have derived this equation in general, and Φ is the dissipation function, and $\mu\Phi$ is the dissipation term, here this is the temporal term, this is the inertia term where temperature is convected due to the velocity v, this is the heat generation per unit volume, and this is the diffusion term.

And you can see for laminar incompressible Newtonian fluid flow with constant properties, you can write these equations, because constant properties so k you can take it outside.

And you can see for incompressible flow velocity will be low, so you can see these term you can neglect, and in general you can write this equation with this assumptions, where dissipation function you can see this is the term you can neglect because for incompressible flow divergence of v will be 0; that means, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. So, dropping this term, you can write the dissipation function as this.

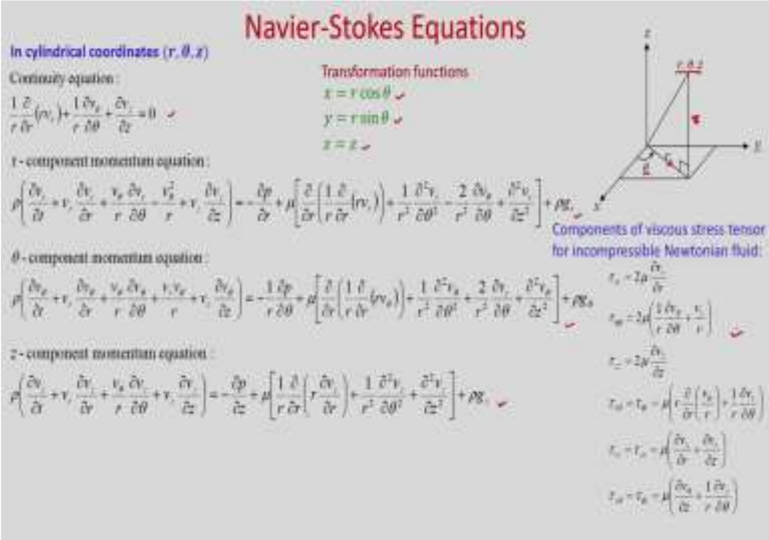
(Refer Slide Time: 02:05)



So, in Cartesian coordinate now let us write the governing equations for laminar incompressible Newtonian fluid flow with constant properties. So, this is the coordinate system Cartesian coordinate x, y, z. So, this is the continuity equation. This is the x component of momentum equation, this is the y component of momentum equation, and this is the z component of momentum equation, where g x, g y, g z are the gravitational acceleration in x, y, z direction respectively. And this is the energy equation, and this is the dissipation function.

And for these equations you can see you can find the components of viscous stress tensor like this. So, there will be six components, because this stress tensor is symmetric. So, $\tau_{xy}=\tau_{yx}$, so obviously, we will have total six components in the stress tensor.

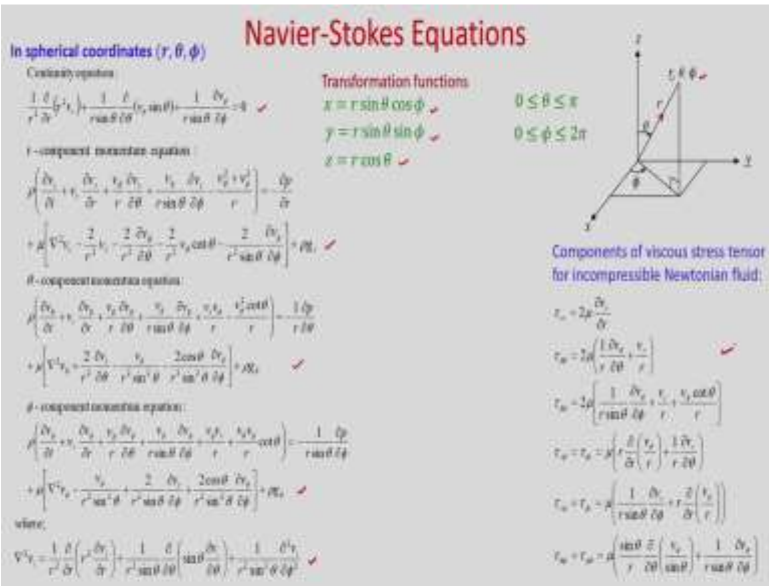
(Refer Slide Time: 03:05)



Now, if you use some suitable transformation function, then you can convert these equations in Cartesian coordinate to the equations in cylindrical coordinate. So, if you use this as a cylindrical coordinate, so you can see this is the r, and this is the θ, and this is the z.

Then for r, θ, z coordinate in cylindrical coordinate, if you use the transformation function as x = r cosθ, y = r sinθ, and z =z, then you can write the continuity equation as these; r component of equation as this; θ component of equation as this; and z component of momentum equation as this. And corresponding viscous stress tensor will be this.

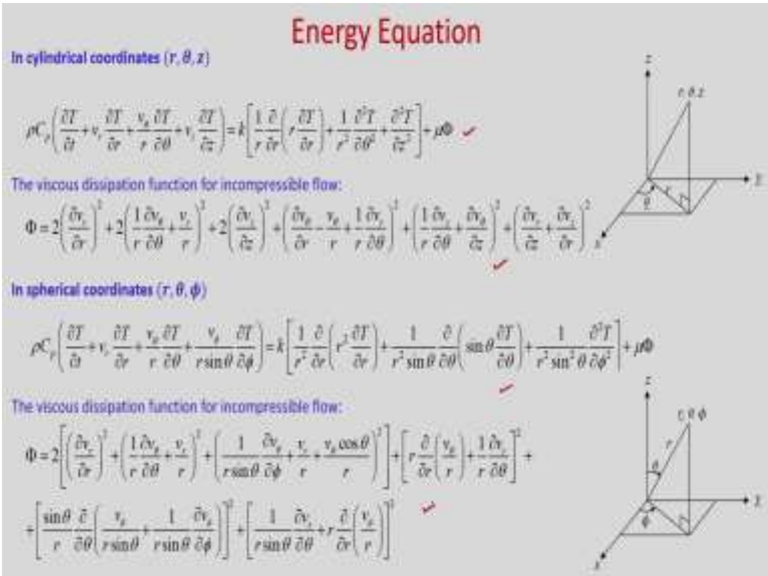
(Refer Slide Time: 03:58)



And in spherical coordinate, so we are considering r, θ, Φ, so this is the Φ. So, Φ obviously, you can see it will vary 0 to 2 π; this is the θ, it will be 0 to π; and this is the r. So, if you use this transformation functions $x = r \sin\theta \cos\Phi$; $y = r \sin\theta \sin\Phi$; and $z = r \cos\theta$. Then you can write the continuity equation in spherical coordinate like these.

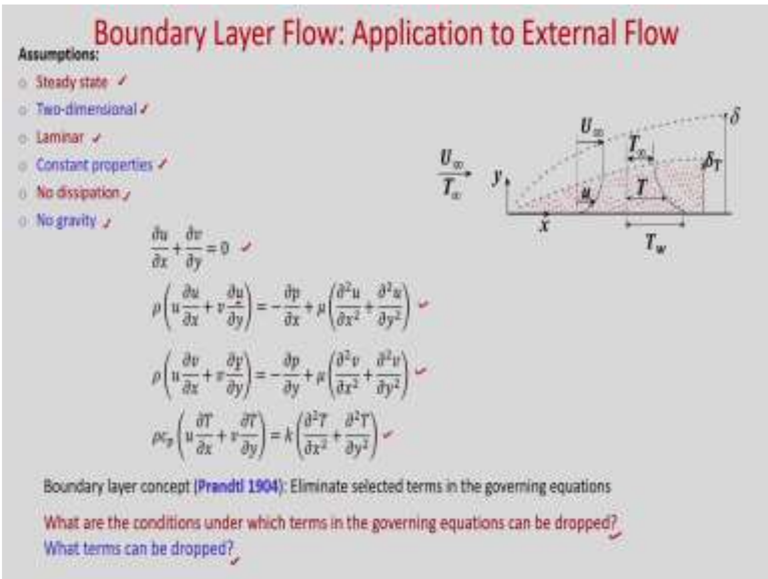
This is the r component of momentum equation; this is the θ component of momentum equation; and this is the Φ component of momentum equation, where $\nabla^2 . v_i$ is given by this expression. And corresponding components of viscous stress tensor can be written like this.

(Refer Slide Time: 04:48)



And energy equation now in cylindrical and spherical coordinate you can write like this. And this is the viscous dissipation function. And similarly in spherical coordinate this is the energy equation, and this is the viscous dissipation function.

(Refer Slide Time: 05:05)



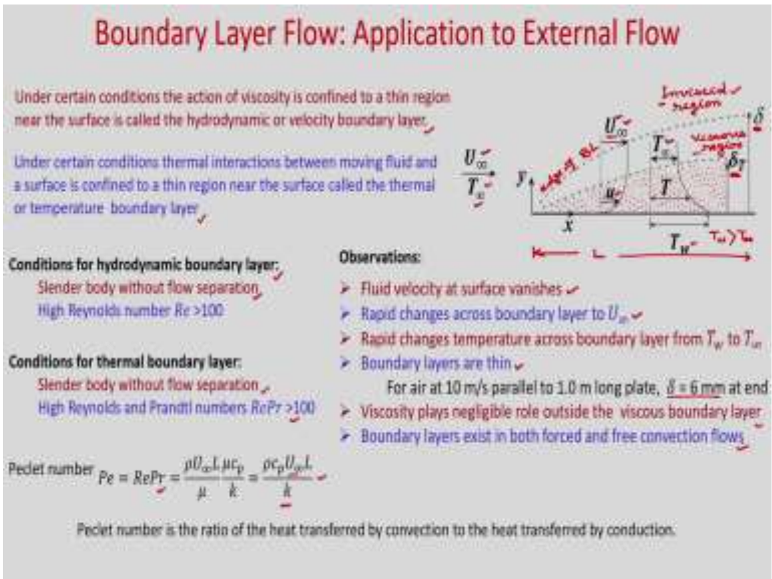
Now, we will discuss about the boundary layer flow. We will consider Cartesian coordinate and we will consider external flows. First we will assume that it is a steady, two-dimensional and laminar flow with constant properties and neglecting dissipation and gravity.

So, invoking these assumptions, you can write the governing equations as this. This is the continuity equation it is a two-dimensional flow. So, this is the continuity equation. This is the u_x component of momentum equation, and this is the y component of momentum equation, and this is the energy equation. And obviously, you can see u is the velocity in x direction; v is the velocity in y direction.

So, in today's class, we will use boundary layer concept, and we will see that if you can drop some term from these equations for boundary layer flows. So, scientist Prandtl actually used scaling analysis, and showed that few terms in the governing equation can be dropped because those terms are very small compared to the others.

So, we will ask these questions now. What are the conditions under which terms in the governing equations can be dropped, and what terms can be dropped? So, these questions now we will answer one by one.

(Refer Slide Time: 06:44)



First let us understand what is boundary layer. Let us consider that you have a external flow over a flat plate. If you see that there will be some region close to the wall, where viscous effect is significant, and some other region away from the surface you will find that there is no effect of viscosity.

So, you can see here you have a flat plate let us say of length L, and you have a uniform flow U_∞, and temperature T_∞. This velocity is known as free stream velocity and this temperature is known as free stream temperature; x is measured in the axial direction, and y is measured perpendicular to the plate.

Now, if you see that the velocity boundary layer, so there is some region where you have effect of viscosity, and this is known as viscous region. And some region away from the surface, there will be no effect of viscosity and that is known as inviscid region.

And if you see the velocity profile obviously to invoke the no slip condition at the wall velocity will be 0. So, the fluid flow which is residing on the top of this flat plate, the velocity will be 0. And gradually this velocity will increase from 0 to U_∞, where U_∞ is the free stream velocity.

Now, we can see that there is a some region which is known as boundary layer; inside that you have viscous region, and outside that you will have inviscid region. So, if you consider that this is the edge of the boundary layer, then you can see this distance from the normal distance from the flat plate is known as hydrodynamic boundary layer thickness that is denoted by δ.

δ is the normal distance from the plate at which these velocity U becomes almost 99 % of U_∞, so that is known as hydro dynamic boundary layer thickness and this fictitious line where at every distance, we have the hydro dynamic boundary layer thickness, so that is known as edge of boundary layer.

So, the inside of this edge of boundary layer you can see there is a viscous region, and outside it is inviscid region. So, you can see under certain conditions the action of viscosity is confined to a thin region near the surface is called the hydrodynamic or velocity boundary layer.

Now, if you consider the thermal part, so you can see you have free stream temperature T_∞ and let us say wall temperature is T_w, and $T_w > T_\infty$. Then you can see that there will be some region where thermal effect will be there, and temperature gradient will exist. And this temperature will vary inside this layer from T_w to T_∞.

And outside this region you can see it will maintain the free stream temperature T_∞. So, the normal distance from the plate up to which you have the effect of this temperature gradient, so that is known as thermal boundary layer thickness and denoted as δ_T.

So, you can see that these fictitious line inside, there is a temperature gradient, but outside you have free stream temperature T_∞. So, this is the edge of thermal boundary layer. So, you can see under certain conditions, thermal interactions between moving fluid and a surface is confined to a thin region near the surface called the thermal or temperature boundary layer.

So, here we have to consider two important assumptions, one is that there is no flow separation. Then under that condition we can derive the boundary layer equations and you have a slender body that means thickness of the body is much, much smaller than the length of the body.

So, you can see conditions for hydro dynamic boundary layer, we have slender body without flow separation, and we have to consider high Reynolds number flows. And Reynolds number should be >100. Conditions for thermal boundary layer, slender body without flow separation and high Reynolds and Prandtl numbers flow that means, the Peclet number which is the product of Reynolds number and Prandtl number should be >100.

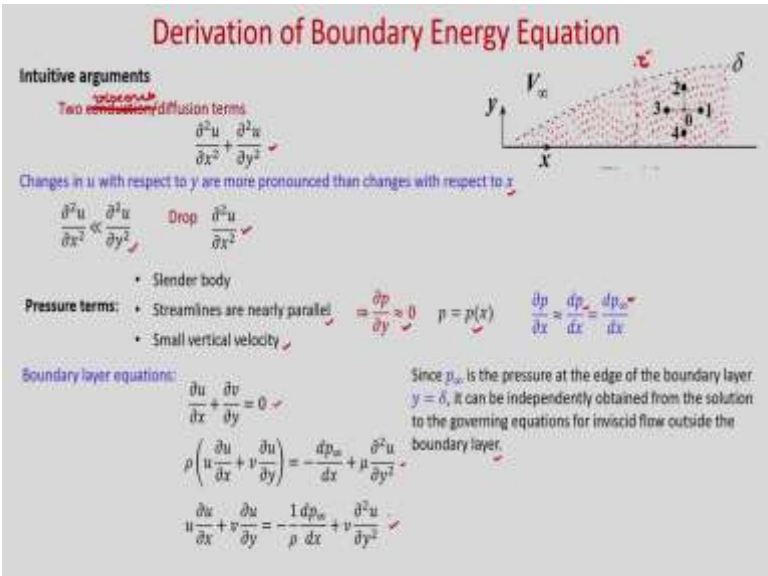
And you can see Peclet number you can write in this expression. So, you can see that Peclet number is the ratio of the heat transferred by the convection to the heat transferred by conduction. So, let us list down the observations of these boundary layer flow. Fluid velocity at surface vanishes rapid changes across boundary layer to U_∞.

So, from 0 to U_∞, these changes occurring inside the boundary layer. Rapid change temperature across boundary layer from T_w to T_∞. So we can see it is changing from T_w to T_∞ inside the thermal boundary layer.

Another observation is that boundary layers are thin. From the experiment it is seen that for air at 10 m/s parallel to 1 m long plate this boundary layer thickness will be of the order of 6 mm at the end; at x equal to 1 m, you will get 6 mm.

So, you can see that you have the length of the plate as 1 m and the boundary layer thickness is 6 mm, so it is very very small compared to the length of the plate. So, you can see boundary layers are very thin. Viscosity plays negligible role outside the viscous boundary layer which is your inviscid region; and boundary layer exist in both forced and free convection flows.

(Refer Slide Time: 14:17)



Now, we will use two different approaches to see that which term we can drop from the governing equations. So, first let us use the intuitive arguments. So, you can see that in the right hand side of the momentum equations, we have viscous terms. Let us see that if you can drop one term. So, there are two viscous terms, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. So, this is not conduction. This is diffusion or viscous. So, these are the two terms.

Now, let us consider one small insect which is sitting inside the boundary layer, and it is not disturbing the flow. And let us say that it is flying at this position 0. Now, it is experiencing a high velocity and it wants to shift to a lower velocity region. So, by intuitive arguments what can you say that where it will go 1, 2, 3 or 4 positions.

So, you can see obviously that it will travel to position 4, because it will have less velocity right, because if it is travels from 0 to 4, then it will have less velocity. And if it is ultimately goes to the surface, then it will not feel any velocity.

So, you can see that changes in u with respect to y are more pronounced than changes with respect to x. So, it does not answer that the change of these gradient $\frac{\partial u}{\partial x}$ is which one is smaller than the other. Now, you see that the insect is just one step away from the surface. So, you can see that if it goes to the surface, then obviously, the velocity will become 0, and the changes in the velocity gradient will be more.

If you consider that insect is at the edge of the boundary layer and if it goes away from the surface, then you will find that there will be not much variation in the velocity or velocity gradient. And if you consider in the axial direction, if it moves then obviously there will be not much change in the velocity gradient. So, that means, that your velocity gradient with respect to y is much higher than the velocity gradient with respect to x.

So, considering that you can see that $\frac{\partial^2 u}{\partial x^2}$ will be much less than $\frac{\partial^2 u}{\partial y^2}$, and you can drop the term from the momentum equation. So, now, what about the pressure terms. So, if you consider slender body, so obviously, streamlines are nearly parallel and you will have very small particle velocity.

So, if you have very small vertical velocity and if you consider the y momentum equations, then all the inertia terms and viscous terms will become 0. So, you can consider that the pressure gradient $\frac{\partial p}{\partial y}$ will be almost 0 very small.

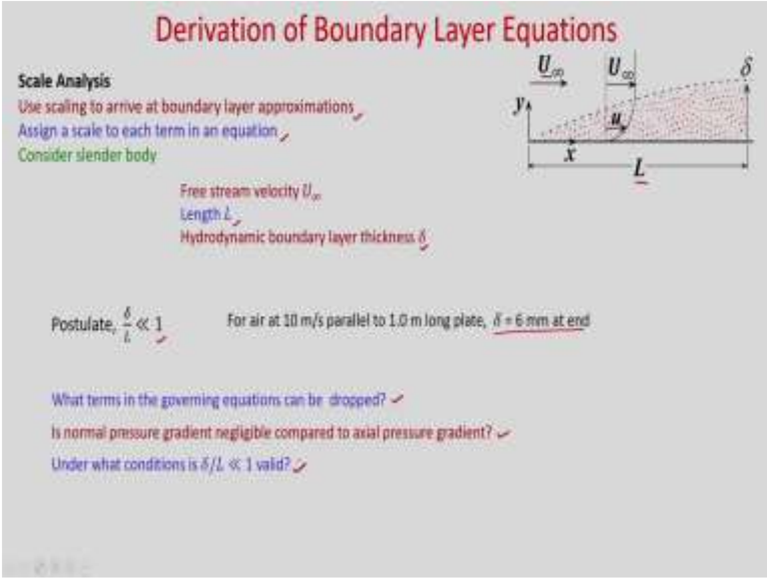
So, obviously, $\frac{\partial p}{\partial y}$ will be close to 0. So, you can write p is only function of x, and $\frac{\partial p}{\partial x}$ you can write as $\frac{dp}{dx}$. And as pressure gradient along the y direction almost 0, whatever pressure is there outside the boundary layer p_∞ so that will also be impressed inside the boundary layer because there is no change in the pressure in y direction.

So, whatever pressure is there p_∞ here so inside also at this x location will be same pressure everywhere. So, you can see that you can write $\frac{dp}{dx} = \frac{dp_\infty}{dx}$. And since p_∞ is the pressure at the edge of the boundary layer that means at $y=\delta$, it can be independently obtained from the solution to the governing equation for inviscid flow outside the boundary layer.

So, you can see that in the momentum equation, then we can write $\frac{dp}{dx} = \frac{dp_\infty}{dx}$. So, we have these boundary layer equations, this is the continuity equation. And $\frac{dp}{dx}$ we have written $\frac{dp_\infty}{dx}$, and dropping the term $\frac{\partial^2 u}{\partial x^2}$ this is the term we have written. So, now you can see that if you divide by ρ, then you can write in right hand side $-\frac{1}{\rho}\frac{dp_\infty}{dx}$ and $\frac{\mu}{\rho}$ is your kinematic viscosity. So, $\nu\frac{\partial^2 u}{\partial y^2}$.

Now, let use another approach which is your mathematical approach that is your scale analysis. So, here we will use the order of magnitude analysis. And we will see the order of magnitude of each term in the governing equations and which term is having less order of magnitude compared to the other, we will drop from the governing equations.

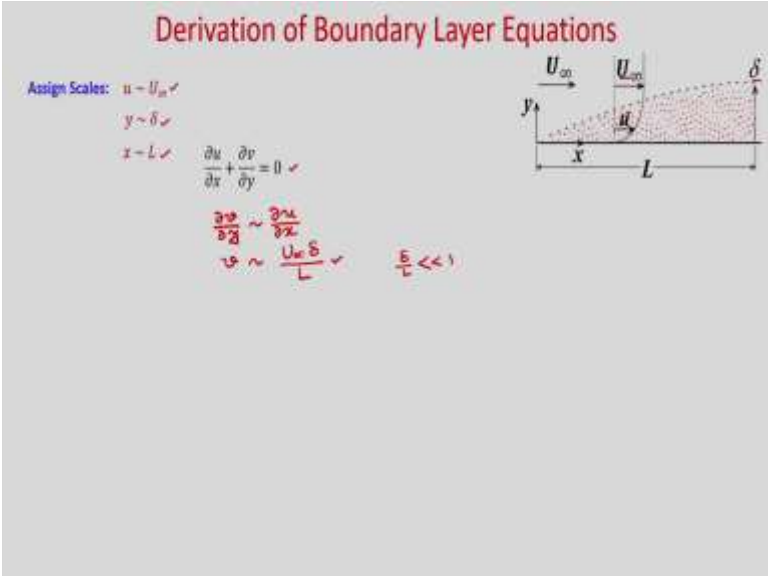
(Refer Slide Time: 20:03)



So, use scaling to arrive at boundary layer approximations, assign a scale to each term in the equation. So, obviously, you have free stream velocity U_∞, length of the plate as L, and hydrodynamic boundary layer thickness δ. Now, we will postulate that $\frac{\delta}{L}$ is much, much smaller than 1, because we have already seen that for air at 10 m/s parallel to 1 m long plate δ is of the order of 6 mm at the end. So, obviously, we can assume that $\frac{\delta}{L} << 1$.

Now, let us answer these questions. What terms in the governing equation can be dropped? Is normal pressure gradient negligible compared to the axial pressure gradient? And under what conditions is $\frac{\delta}{L} << 1$ valid? So, we will answer these questions one by one.

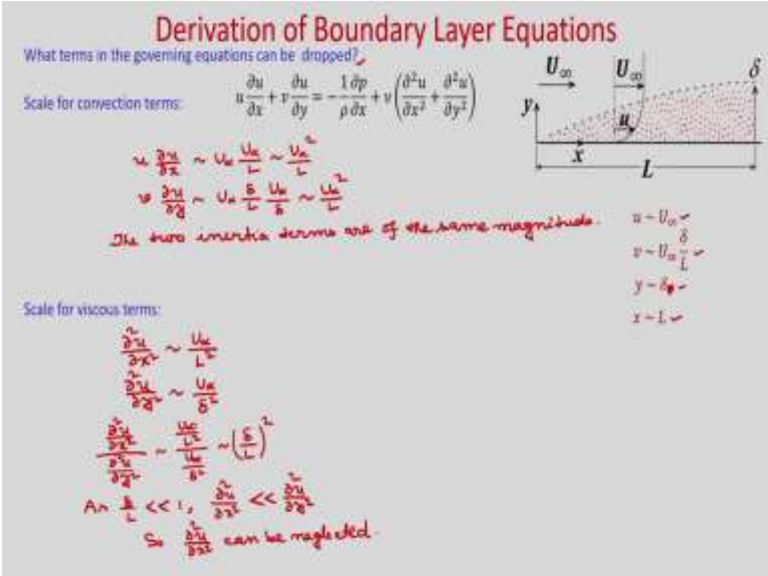
(Refer Slide Time: 21:00)



So, first assign the scales. So, velocity scale you can see the free stream velocity is U_∞. So, we can assign the scale for these velocity u as U_∞. We have hydro dynamic boundary layer thickness δ. So, in y direction, so obviously, y we can write as order of δ. And length of the plate is L, so the order of x, we can write as L. Now, let us consider the continuity equation, and find what is the order of velocity v. So, this is the continuity equation. So, you can see $\frac{\partial v}{\partial y}$, you can write as order of $\frac{\partial u}{\partial x}$.

So, you can write the order of v as this is the $\frac{\partial u}{\partial x}$, what is the order of u? It is U_∞. What is the order of δ_y, this is δ and divided by x. So, you can see the order of velocity v is $\frac{U_\infty \delta}{L}$. So, here you can see we have already assume that $\frac{\delta}{L}$ is much smaller than 1. So, we can see v will be much smaller than U_∞. So, the velocity in y direction v is very small compared to U_∞. And we got the scale for v velocity as this.

(Refer Slide Time: 22:37)



Now, let us answer this question what terms in the governing equations can be dropped. So, we have already assigned the scales, and we have found the scale for velocity v. So, you can see this is the u scale, this is the v scale, this is the y scale which is δ, and this is the x scale L.

Now, let us consider the convection terms. So, $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$. So, first term is $u\frac{\partial u}{\partial x}$. So, what is the order? u is U_∞, this is your U_∞ and this is L, so that means, $\frac{U_\infty^2}{L}$. Now, consider the other convection term $v\frac{\partial u}{\partial y}$.

So, what is the order of v you see this is your $U_\infty\frac{\delta}{L}$, and u is U_∞ and y is δ. So, you see this is $\frac{U_\infty^2}{L}$. So, you can see both are of same order. So, you cannot drop any terms. So, the two inertia terms are of the same magnitude. So, we cannot drop any term in the convection terms.

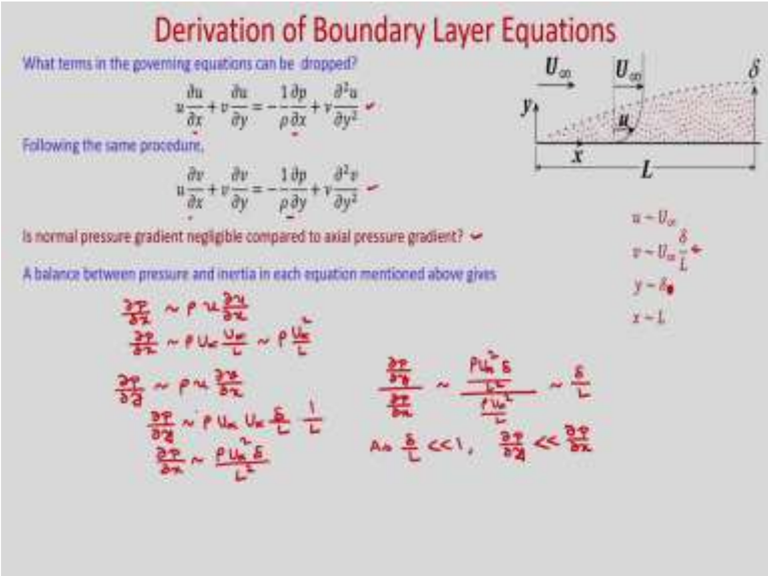
Now, let us consider the viscous terms. So, we have $\frac{\partial^2 u}{\partial x^2}$. So, you can see this is the order of $\frac{U_\infty}{L^2}$, and you have $\frac{\partial^2 u}{\partial y^2}$ is order of $\frac{U_\infty}{\delta^2}$. So, now you can see that we can write $\frac{\frac{\partial^2 u}{\partial x^2}}{\frac{\partial^2 u}{\partial y^2}}$.

So, this ratio will be order of $\dfrac{\frac{U_\infty}{L^2}}{\frac{U_\infty}{\delta^2}}$. So, you can see you can write these as order of $(\dfrac{\delta}{L})^2$.

So, we have already assume that $\dfrac{\delta}{L}$ is much smaller than 1.

So, from here you can see that $\dfrac{\partial^2 u}{\partial x^2}$ is much smaller than $\dfrac{\partial^2 u}{\partial y^2}$. So, as $\dfrac{\delta}{L}$ << 1. So, $\dfrac{\partial^2 u}{\partial x^2} << \dfrac{\partial^2 u}{\partial y^2}$. So, $\dfrac{\partial^2 u}{\partial x^2}$ can be neglected from the viscous terms.

(Refer Slide Time: 25:43)



So, you can see that if $\dfrac{\delta}{L}$ << 1, then you can drop the term $\dfrac{\partial^2 u}{\partial x^2}$. So, dropping the $\dfrac{\partial^2 u}{\partial x^2}$ term, you can write the x component momentum equation as this. And following the same procedure you can drop $\dfrac{\partial^2 v}{\partial y^2}$ from the y momentum equation and you can write like this.

Now, the next question is that is normal pressure gradient negligible compared to the pressure gradient? So, next question is that is normal pressure gradient negligible compared to axial pressure gradient? So, now, you can see that in the u momentum equation we have the term $\frac{\partial p}{\partial x}$, and in the v momentum equation we have $\frac{\partial p}{\partial y}$. Now, let us consider these each pressure gradient term with corresponding inertia term.

So, in the u momentum equation compare these term with the inertia term and v momentum equation compare this pressure gradient term with the inertia term, because both terms are of the same order. So, we have already seen. So, you can take any one term. So, we can see that $\frac{\partial p}{\partial x}$ will be order of $\rho u \frac{\partial u}{\partial x}$. So, you can see $\frac{\partial p}{\partial x}$ will be ρU_∞, this is U_∞ divided by x length is L. So, this is $\rho \frac{U_\infty^2}{L}$.

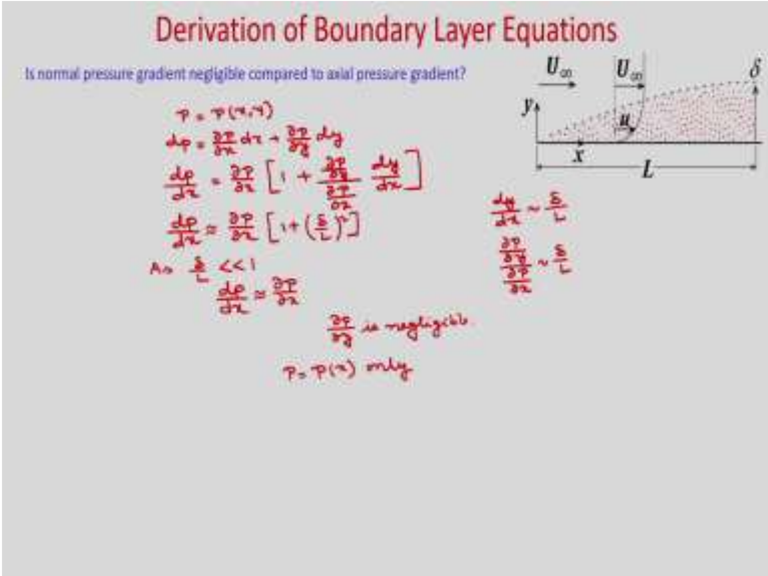
Now, if you see the pressure gradient term in the y component of momentum equation, so you can write $\frac{\partial p}{\partial y}$ will be order of $\rho u \frac{\partial v}{\partial x}$. So, $\frac{\partial p}{\partial y}$ will be ρU_∞, and v is the order of $U_\infty \frac{\delta}{L}$, so $U_\infty \frac{\delta}{L}$ and $\frac{1}{L}$. So, $\frac{\partial p}{\partial y}$ you can write order of $\frac{\rho U_\infty^2 \delta}{L^2}$.

So, now see the ratio $\frac{\frac{\partial p}{\partial y}}{\frac{\partial p}{\partial x}}$. So, this is the order of so $\frac{\partial p}{\partial y}$ is this one $\frac{\rho U_\infty^2 \delta}{L^2}$ and divided by $\frac{\partial p}{\partial x}$. So, it is $\frac{\rho U_\infty^2}{L}$. So, you can see this will be order of $\frac{\delta}{L}$.

So, from here you can see that $\frac{\delta}{L}$, we have assumed as much smaller than 1, so obviously, your $\frac{\partial p}{\partial y}$ will be much smaller than $\frac{\partial p}{\partial x}$. So, you can see as $\frac{\delta}{L}$ is much smaller than 1.

So, obviously, $\frac{\partial p}{\partial y} << \frac{\partial p}{\partial x}$. So, you can see that $\frac{\partial p}{\partial y}$ will be order of 0, because it is very small. You can see that we have done the balance between the pressure gradient and the inertia term you can also balance the pressure gradient with the viscous term and you will get the same result.

(Refer Slide Time: 29:26)



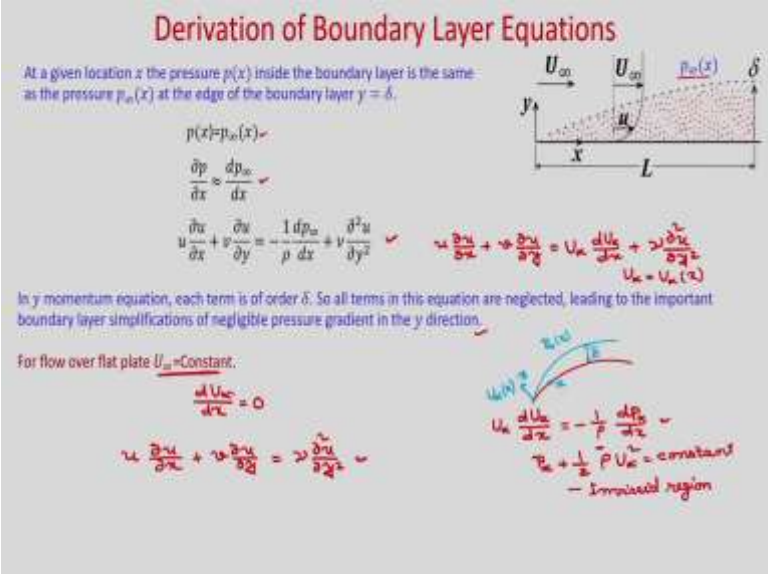
So, now, you can see for the two-dimensional flow p is function of x and y. So, for 2D flow we know that p is function of x and y. So, you can write. Now, you can write $\frac{\partial p}{\partial x}$.

So, we have divided by d x. So, you can write $\frac{\partial p}{\partial x}$.

And if you take common then you will get $1+\frac{\frac{\partial p}{\partial y}}{\frac{\partial p}{\partial x}}\frac{dy}{dx}$. So, now, you can see what is the order of $\frac{dy}{dx}$; $\frac{dy}{dx}$ is order of y is δ, and x is L. And $\frac{\partial p}{\partial y}$, and the $\frac{\frac{\partial p}{\partial y}}{\frac{\partial p}{\partial x}}$ that is also we have derived in the last slide as $\frac{\delta}{L}$.

So, you can see $\frac{dp}{dx} \approx \frac{\partial p}{\partial x}[1+(\frac{\delta}{L})^2]$, because $\frac{dy}{dx}$ is order of $\frac{\delta}{L}$ and $\frac{\partial p}{\partial y}$. So, it will be $\frac{\delta}{L}$. So, you can write this as order of this. So, as $\frac{\delta}{L}$ is much smaller than 1, so you can write $\frac{dp}{dx} \approx \frac{\partial p}{\partial x}$. So, now you can see that in this as $\frac{\partial p}{\partial y}$ is negligible. So, $p = p(x)$.

(Refer Slide Time: 31:27)



So, now, at a given location x the pressure $p(x)$ inside the boundary layer is the same as the pressure p_∞ at the edge of the boundary layer y = δ. So, if you can see that outside the velocity boundary layer, if you have a pressure gradient p_∞ which is function of x, then $p(x) = p_\infty(x)$, and $\frac{dp}{dx} \approx \frac{dp_\infty}{dx}$.

And in the u momentum equation, you can see that it will become $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$. So, this is the boundary layer equation.

What about the v momentum equations? So, we have seen that v is much smaller than U_∞ right, and it is having very small value. And from there we have derived the $\frac{\partial p}{\partial y}$ also is very small value. So, in the y momentum equation, if you see all the terms will be order of δ.

So, you can neglect the y momentum equation. So, in y momentum equation, each term is of order δ. So, all terms in this equation are neglected leading to the important boundary layer simplifications of negligible pressure gradient in the y direction.

So, now we can see that this equation we have derived in general, where p may be function of x, and as $\frac{\partial p}{\partial y} = 0$. So, obviously the pressure at the outside the boundary layer will be also function of x. So, you can see that p_∞ (x), so that can be impressed inside the boundary layer and we will have at a certain x location it will have the same pressure.

Now, this pressure gradient if you consider for a let us say you have a in general curved surface, and you can have the boundary layer like these where this is the boundary layer thickness delta, and x is measured along the surface, and y is measured perpendicular to the surface. In this particular case, actually you will get that outside pressure p_∞ (x). And for that reason your velocity free stream velocity ∞ will be function of x.

So, you can see that for this particular case from this equation if you apply outside the boundary layer, then what will happen? So, obviously, you can see that from here you can write U_∞ and $\frac{\partial u}{\partial x}$ so it will be u(x), so you can write $\frac{dU_\infty}{dx}$.

And v is very small negligible, so you can drop this term as $-\frac{1}{\rho}\frac{dp_\infty}{dx}$. And obviously, outside the boundary layer u is function of x only, so $\frac{\partial^2 u}{\partial y^2}$ will be 0. So, you will get this equation. So, $U_\infty \frac{dU_\infty}{dx} = -\frac{1}{\rho}\frac{dp_\infty}{dx}$.

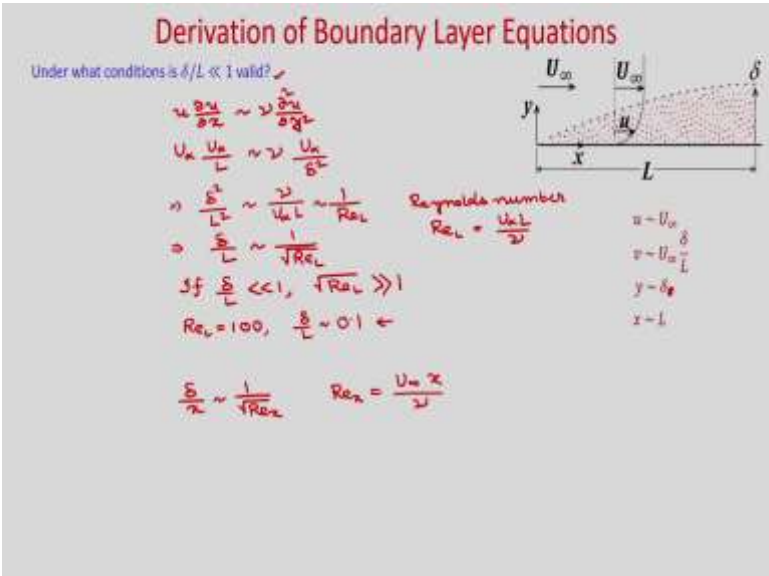
So, if you write it here then you can get, so you can write it here $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$, where U_∞(x). And if you integrate this equation what you will get? If you integrate this situation you can see you will get $p_\infty + \frac{1}{2}\rho U_\infty^2 =$ constant. And it is valid in the inviscid region outside the boundary layer.

Now, if you consider a special case that flow over flat plate. So, in case of flow over flat plate, this U_∞ will be constant; it is not function of x. So, for flow over flat plate, U_∞ is constant. So, here you can see that if U_∞ is constant, then $\frac{dU_\infty}{dx}$ = 0, because U_∞ is constant.

So, $\frac{dU_\infty}{dx}$ will be 0. So, if you have a flow over flat plate where U_∞ is constant, then you can write your boundary layer equation as $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$, so that means, your $\frac{dp_\infty}{dx}$ =0.

So, you can write this equation as a special case for flow over flat plate. But if it is a curved surface like wedge or flow over a cylinder, then you have a curved surface, then in that case your U_∞ is function of x, and you have to consider the pressure gradient term $\frac{dp_\infty}{dx}$ in the momentum equation.

(Refer Slide Time: 37:19)



Now, the last question is that under what conditions is $\frac{\delta}{L}$ much small than 1 valid? So, let us answer this question. So, what will do now? Now, you can see in the left hand side the inertia terms are of the same order. So, we will take one inertia term, and we will compare it with the viscous term because we have only one viscous term right $\nu\frac{\partial^2 u}{\partial y^2}$. So, this we will compare.

So, you can see $u\frac{\partial u}{\partial x}$ one inertia term we will compare with the viscous term $\frac{\partial^2 u}{\partial y^2}$. So, what is the scale? So, $U_\infty \frac{U_\infty}{L} \sim \nu \frac{U_\infty}{\delta^2}$.

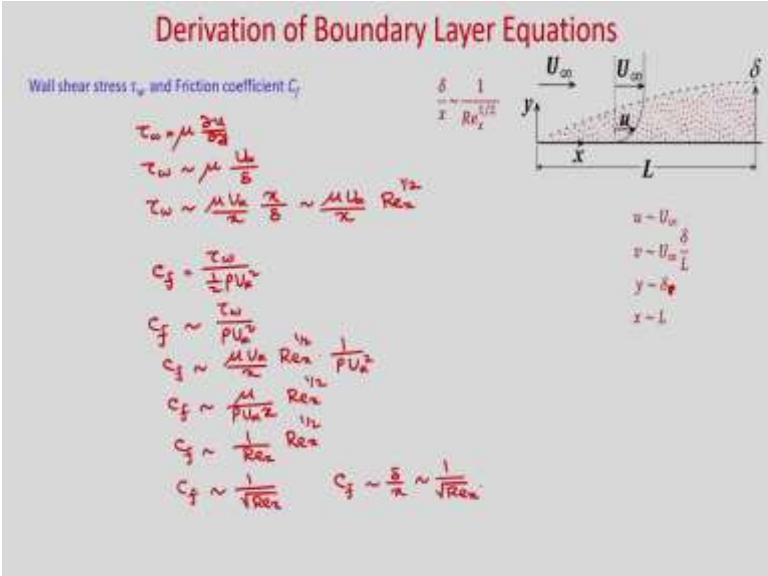
So, if you take delta square in the left hand side, if you divide both side by L, then you will get L^2, then you can see here you will get $\frac{\nu}{U_\infty L}$. And we can define Reynolds number based on free stream velocity ∞ and plate length L, then we can write $R_{e_L} = \frac{U_\infty L}{\nu}$. So, you can see from here it will be $\frac{1}{R_{e_L}}$, and $\frac{\delta}{L} \sim \frac{1}{\sqrt{R_{e_L}}}$.

Now, we will answer this. So, if $\frac{\delta}{L}$ delta by L is much smaller than 1, so what will be this term. So, if $\frac{\delta}{L}$ is much smaller than 1 then, $\frac{1}{\sqrt{R_{e_L}}}$ will be much greater than 1. So, you can see that it is a high Reynolds number flow. So, if you have high Reynolds number flow, but not in the range of turbulent, then you will get the hydro dynamic boundary layer thickness is very small compared to the length of the plate.

So, if you consider Reynolds number 100, then let us see that what is the $\frac{\delta}{L}$ order. So, if consider R_{e_L} as 100, then you can see from here $\frac{\delta}{L}$ will be order of, so it will be 100, so $\sqrt{100} = 10$. So, it will be 0.1.

So, you can see that it is very small. So, generally we say that these equations are valid when Reynolds number is greater than 100. Now, for any length x, this expression you can write as $\frac{\delta}{x} \sim \frac{1}{\sqrt{R_{e_x}}}$, where the Reynolds number $R_{e_x} = \frac{U_\infty x}{\nu}$.

(Refer Slide Time: 40:26)



Now let us find what is the wall shear stress and the friction coefficient. So, what is the order of these terms? So, now, we know that for two-dimensional flow, wall shear stress $\tau_w = \mu \dfrac{\partial u}{\partial y}$.

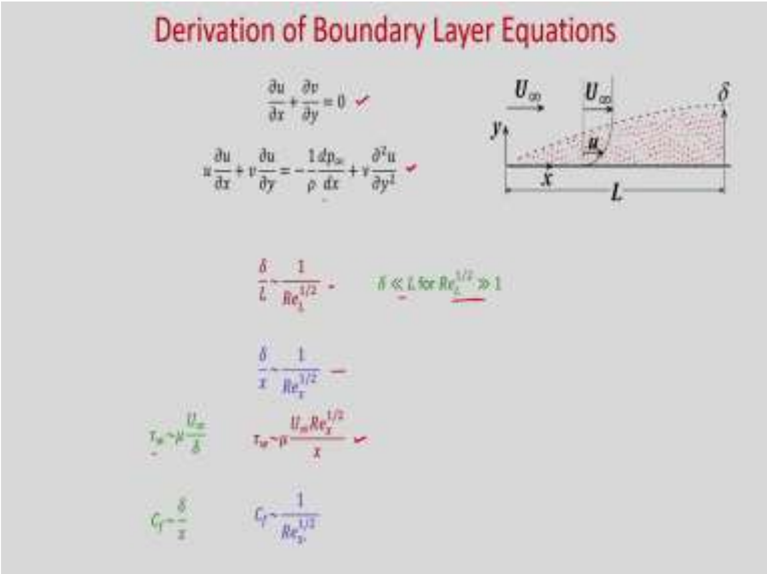
So, for 2D flow we know $\tau_w = \mu \dfrac{\partial u}{\partial y}$. So, you can see that $\tau_w \sim \mu \dfrac{U_\infty}{\delta}$. And we know that $\dfrac{\delta}{x} \sim \dfrac{1}{\sqrt{R_{e_x}}}$. So, $\tau_w \sim \mu \dfrac{U_\infty}{x}\dfrac{x}{\delta} \sim \mu \dfrac{U_\infty}{x} R_{e_x}^{1/2}$.

And we know that friction coefficient $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$. So, you can see that friction coefficient C_f will be generally $\dfrac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$. So, if you see $C_f \sim \dfrac{\tau_w}{\rho U_\infty^2}$, we will not consider half because we are doing the order of magnitude analysis.

So, you can see that $C_f \sim \dfrac{\mu U_\infty}{x} R_{e_x}^{1/2} \dfrac{1}{\rho U_\infty^2}$. So, if you see from here that if you $C_f \sim \dfrac{\mu}{\rho U_\infty x} R_{e_x}^{1/2}$. And C f will be $C_f \sim \dfrac{1}{R_{e_x}} R_{e_x}^{1/2}$.

So, if you see that $C_f \sim \frac{1}{\sqrt{R_{e_x}}}$. So, if $\frac{1}{\sqrt{R_{e_x}}}$ and $\frac{\delta}{x}$ is also $\frac{1}{\sqrt{R_{e_x}}}$. So, obviously, you can see $C_f \sim \frac{\delta}{x} \sim \frac{1}{\sqrt{R_{e_x}}}$.

(Refer Slide Time: 42:43)



So, let us summarize what we have studied in today's class. So, first we have just written the governing equations in Cartesian, cylindrical and spherical coordinates. Then we considered external flow, and we have discussed about boundary layer.

So, we have seen that near to the wall, we have a small region where you have an effect of viscous region. So, that is known as viscous region then outside this and away from the surface there is a region that is known as inviscid region where there is no effect of viscosity. Then we used intuitive arguments and scale analysis, and we have derived the boundary layer equations.

So, you can see this is the continuity equation and this is the momentum equation. We have derived expressing the term p as p_∞ because from y momentum equation we have seen that $\frac{\partial p}{\partial y} \sim 0$.

So, this is the equation we have seen. And as a special case also we have discussed that if you have a flow over flat plate, then U_∞ is constant and there will be no axial pressure gradient. So, there will be $\frac{\partial p_\infty}{\partial x} = 0$. So, this term you can drop for flow over flat plate.

Then we have derived this relation $\frac{\delta}{L} \sim \frac{1}{R_{e_L}^{1/2}}$. And if $\frac{\delta}{L}$ is much smaller than 1, obviously, it has to be a high Reynolds number flow. And for any x we have written $\frac{\delta}{x}$ as this expression and from here we have written the shear stress $\tau_w \sim \mu \frac{U_\infty}{\delta}$, and we got this expression. And then we have seen that your friction coefficient $C_f \sim \frac{\delta}{x}$ and $C_f \sim \frac{1}{R_{e_x}^{1/2}}$.

Thank you.

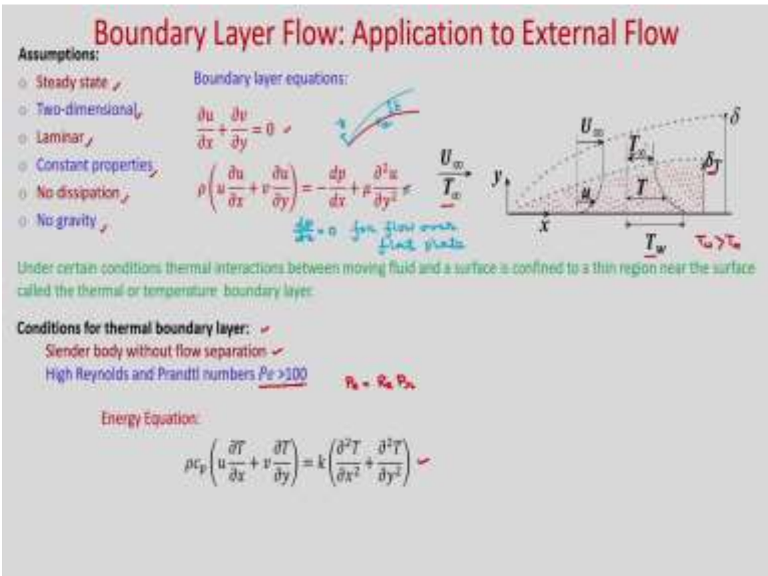
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 02
Preliminary Concepts
Lecture - 05
Derivation of boundary layer energy equation

Hello everyone. So, in last class, we discussed about the velocity boundary layer and we have seen that near to the wall, there is a effect of viscosity and this is a viscous region and away from the wall, there is a region where there is no effect of viscosity and that is known as inviscid region and we have discussed about the edge of the boundary layer and boundary layer thickness.

Today, we will discuss about thermal boundary layer. We can see there is a region near to the wall, there will be an effect of this temperature gradient, but away from the surface, it will be at the free stream temperature T ∞.

(Refer Slide Time: 01:18)



So, we will start with the assumptions. So, we will consider steady state, two-dimensional and laminar flow with constant properties and neglecting dissipation and gravity.

So, in last class, you can see we have derived these boundary layer equations so, this is your continuity equation and this is the momentum equation in general. So, if you have any curved surface and over it if there is a velocity boundary layer then obviously, you can see normal to this surface if you measure the distance from the surface to the edge of this boundary layer so, that is known as boundary layer thickness and there will be a change in the pressure along the x direction. So, along the x direction and perpendicular to this surface this is your y.

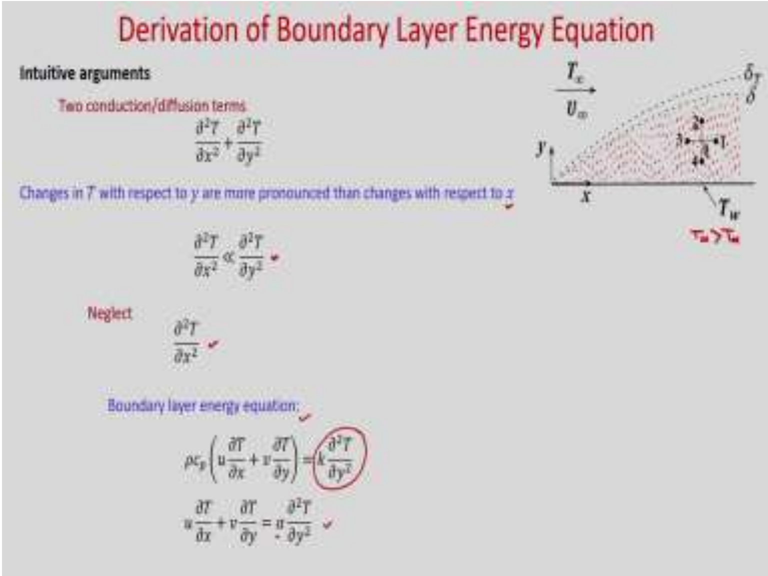
So, in general, we have derived this equation, but as a special case, if you consider flow over flat plate, then this $\frac{dp}{dx}$ becomes 0 for flow over flat plate. Now, let us discuss about the thermal boundary layer. Let us consider that the wall is maintained at a higher temperature than the free stream temperature. In this condition, you can see that you have a wall temperature T_w and free stream temperature is T_∞. Obviously, we have considered $T_w > T_\infty$. So, wall is hot and obviously, your $T_\infty < T_w$.

There will be some region where you can see that there will be change of temperature and at a certain distance from the plate; you will find that this temperature will become equal to free stream temperature T_∞. So, that distance is known as thermal boundary layer thickness δ_T. So, we measure normal distance from the surface.

So, obviously, you can see that under certain conditions thermal interactions between moving fluid and a surface is confined to a thin region near the surface called the thermal or temperature boundary layer and the distance from the normal from the wall the distance at which this temperature becomes almost 99 % of the free stream velocity normal to the surface is known as thermal boundary layer thickness.

So, now we have energy equation, this is your energy equation. Now, under certain conditions can we drop some term from this equation when we consider external flows which we consider as the boundary layer flows. So, in this case also we consider that these are the conditions for thermal boundary layer slender body without flow separation and it is a high Reynolds and Prandtl numbers flows and we know that Peclet number is product of Reynolds number and Prandtl number and generally if it is > 100, then thermal boundary layer is significant.

(Refer Slide Time: 05:30)



So, like earlier class, now we will consider both approaches intuitive approach as well as scaling approach. First, let us discuss about intuitive approach. Let us consider that one insect is there at position 0 and it finds very hot at this position so, it wants to move to some position 1, 2, 3, 4 and we considered that $T_w > T_\infty$.

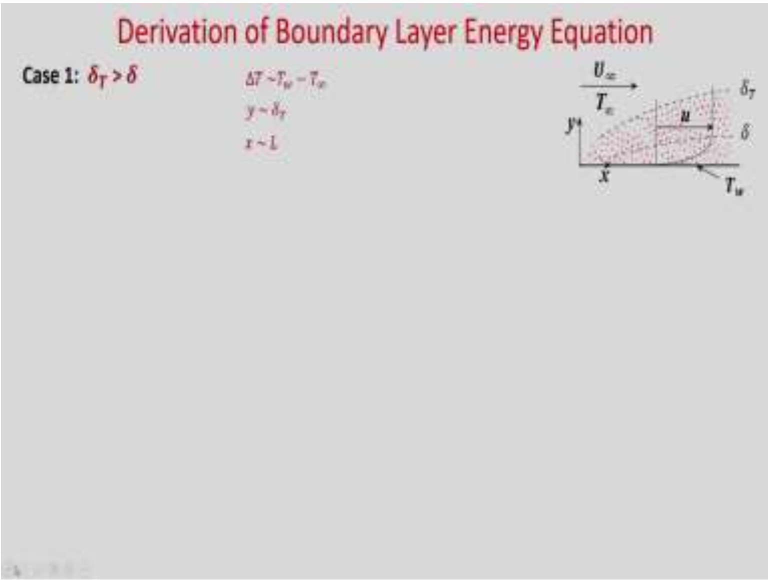
So, under this condition, this insect if it is feeling very hot at this position where it should move? So, you can see that if it moves to 1 position or 3 position obviously, it will not find that much change in the temperature, but if it moves to position 2, then obviously, it will get some relief.

So, from intuitive approach, you can see that obviously, if it is go away from the surface to position 2, then it will feel somewhat cooler than earlier position. Now, let us consider that the insect is near to the wall surface. Let us consider that the insect is near to the wall and it wants to go to a some position where it will feel some significance really, then obviously, you can see that it will go away from the surface because the temperature change is more in that direction, but if it travels in the axial direction, then obviously, there will be not much change in the temperature.

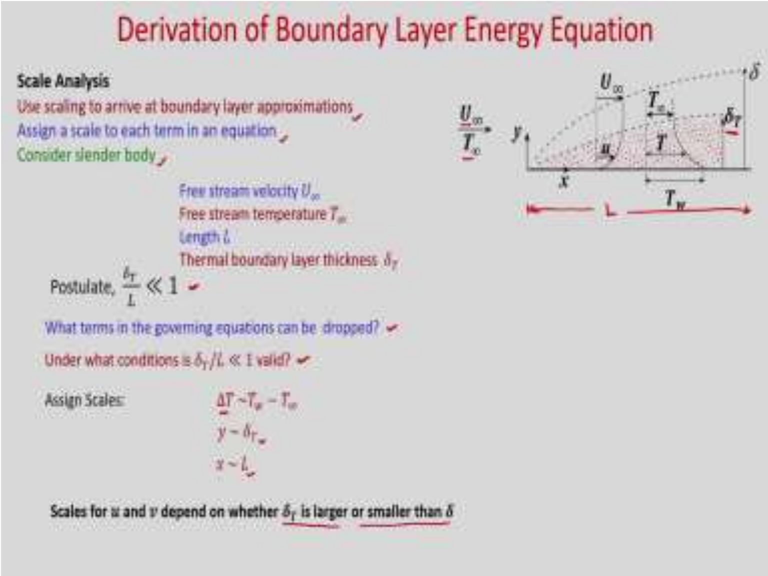
So, now we are considering about the change of temperature right so that means, gradient. So obviously, from the intuitive approach, you can see that your normal direction temperature gradient is higher than the axial temperature gradient. So, from there you can see that $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$. So, changes in T with respect to y are more pronounced than changes with respect to x. So, under this condition, you can neglect $\frac{\partial^2 T}{\partial x^2}$. So, you neglect this term.

So, if you see now the energy equation, we can write as considering only one term in the right-hand side which is your diffusion term. So, this is known as boundary layer energy equation and if you divide by ρ C_p, then you can write $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial x} = \frac{k}{\rho C_p}$ that is your thermal diffusivity $\alpha \frac{\partial^2 T}{\partial y^2}$. So, we can see that we have dropped one term $\frac{\partial^2 T}{\partial x^2}$ because it is much smaller than the $\frac{\partial^2 T}{\partial y^2}$.

(Refer Slide Time: 08:34)



(Refer Slide Time: 08:38)

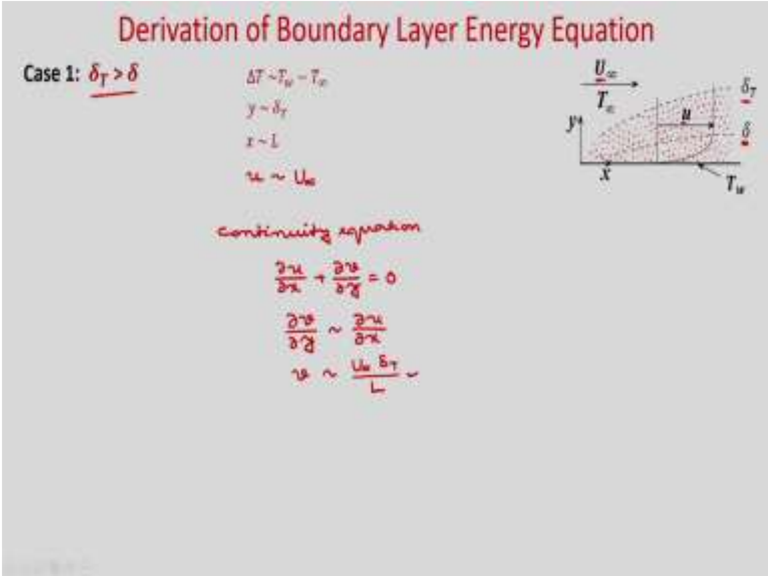


Now, let us consider the scaling approach. In scaling approach, we will examine the order of magnitude of each term and we will see that if some term can be dropped. So, use scaling to arrive at boundary layer approximation. Assign a scale to each term in an equation and obviously, we are considering slender body. So, you know that we have a free stream velocity U_∞. Free stream temperature T_∞ and the plate length is L and we have thermal boundary layer thickness δ_T.

Now, let us postulate that $\frac{\delta_T}{L} \ll 1$. So, with that assumption, let us go ahead and we will see that under what condition this is valid. So, now, let us see what terms in the governing equations can be dropped and under what conditions is $\frac{\delta_T}{L} \ll 1$ valid.

So, now let us assign the scales. We will assign the scales temperature difference $\Delta T \sim T_w - T_\infty$, $y \sim \delta_T$ thermal boundary layer thickness and x in axial direction it is order of the plate length L. In this particular case, now the scale of u will depend on the value of Prandtl number and obviously, your scale of v will also depend on the value of Prandtl number. So, we can see scales for u and v depend on whether δ_T is larger or smaller than δ.

(Refer Slide Time: 10:27)



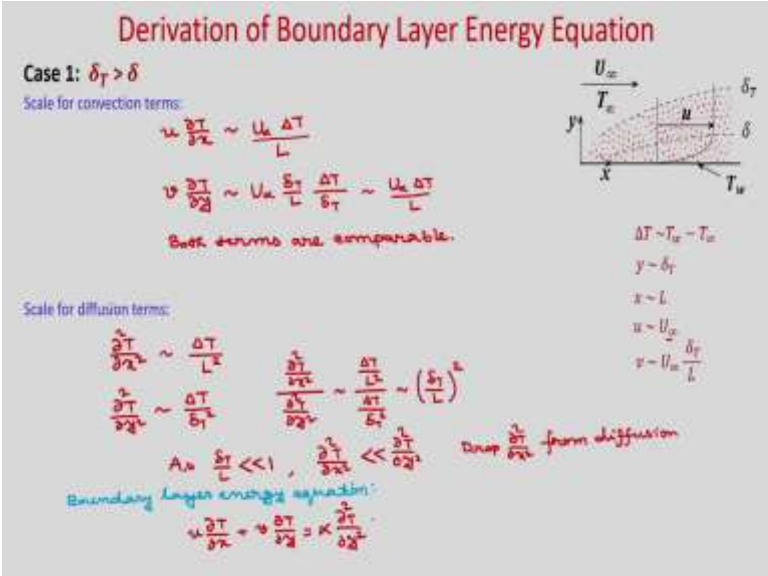
So, first let us consider that thermal boundary layer thickness is higher than the velocity boundary layer thickness. So, you see here. So, this is your thermal boundary layer thickness and it is higher than the velocity boundary layer thickness δ. If you see the velocity distribution obviously, the velocity will vary up to the velocity boundary layer thickness δ which is known as also hydro dynamic boundary layer thickness and after that it will have the same velocity U_∞ right.

So, the scale of u we can consider in this particular case as the order of U_∞ which is your free stream velocity. When $\delta_T > \delta$, we will consider the scale of velocity u as order of free stream velocity U_∞ because you can see that at δ_T , we have the free stream velocity U_∞.

Now, we will use continuity equation to find the order of velocity v. So, we know the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ so obviously, you can write $\frac{\partial v}{\partial y} \sim \frac{\partial u}{\partial x}$. So, the v will be order of so, what is the order of u? It will be U_∞, order of $y \sim \delta_T$ and $x \sim L$. So, you can see $v \sim \frac{U_\infty \delta_T}{L}$.

Now, let us examine the convection terms. We have two convection terms and let us see that what is the order of magnitude of each term.

(Refer Slide Time: 12:31)



So, if you see the scale of convection term so, we have $u\dfrac{\partial T}{\partial x}$ this is the first term and what is the order of this? So, $u \sim U_\infty$, $\delta_T \sim \Delta T$ so, that we will write $\dfrac{\delta_T}{\delta_x}$ so, $x \sim L$.

So, $\dfrac{U_\infty \Delta T}{L}$.

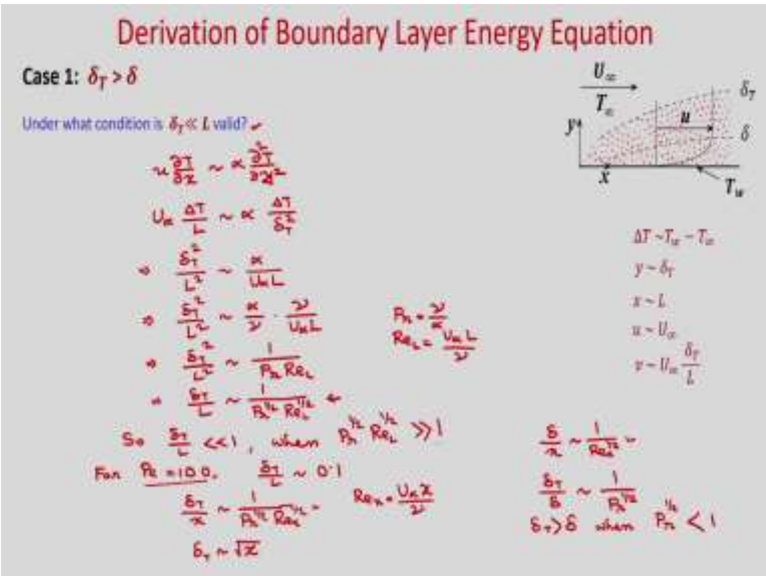
Similarly, the other convection term $v\dfrac{\partial T}{\partial y}$. So, this is the order of v so, v is $\dfrac{U_\infty \delta_T}{L}$ and $\delta_T \sim \Delta T$ which is $T_w - T_\infty$ and your $y \sim \delta_T$. So, you can see these δ_T δ_T will cancel out so, it will have $\dfrac{U_\infty \Delta T}{L}$. So, you can see both the terms are having the same order. So, we cannot drop any term in the convection. So, both terms are comparable. So, you cannot drop any term from this convection.

Now, let us consider the diffusion terms. So, first term is $\frac{\partial^2 T}{\partial x^2}$. So, what is the order of this? So, you can see it is $\frac{\Delta T}{L^2}$ and the next term is $\frac{\partial^2 T}{\partial y^2}$. So, what is the order? It is $\frac{\Delta T}{\delta_T^2}$.

So, now, let us see. So, we know that $\dfrac{\frac{\partial^2 T}{\partial x^2}}{\frac{\partial^2 T}{\partial y^2}} \sim \dfrac{\frac{\Delta T}{L^2}}{\frac{\Delta T}{\delta_T^2}} \sim (\frac{\delta_T}{L})^2$.

Now, we have already postulated that $\frac{\delta_T}{L} \ll 1$. So, what does it mean? From here, you can see that $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$. So, as $\frac{\delta_T}{L} \ll 1$ so obviously, you can see $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$. So, drop $\frac{\partial^2 T}{\partial x^2}$ from diffusion. So, if you drop these term, then obviously, essentially you will get the boundary layer energy equation. So, boundary layer energy equation you can write $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}$ both are comparable so, we did not drop any term is equal to now you have thermal diffusivity $\alpha\frac{\partial^2 T}{\partial y^2}$ dropping the term $\frac{\partial^2 T}{\partial x^2}$.

(Refer Slide Time: 16:16)



So, now, let us see that under what condition is δ_T << L valid. So, for that we will take one convection term, one diffusion term and obviously, as both terms are there in the boundary layer energy equation so, both will be comparable. So, you can see that the convection term $u\frac{\partial T}{\partial x}$, we can compare with the diffusion term $\alpha\frac{\partial^2 T}{\partial y^2}$. So, now put the scale. So, what is the scale of u? It is $U_\infty \frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_T^2}$.

So, if you see, now we can write δ_T^2 this side and we will write L^2 so, we have divided by L in the left-hand side so, in the right-hand side, we will write $\frac{\alpha}{U_\infty L}$,we have divided by L in the left-hand side so, we will also divide in the right-hand side. So, we can write $\frac{\delta_T^2}{L^2} \sim \frac{\alpha}{\nu}\frac{\nu}{U_\infty L}$, you see.

What is $\frac{\nu}{\alpha}$? $\frac{\nu}{\alpha}$ is nothing but Prandtl number and Reynolds number is $\frac{U_\infty L}{\nu}$. So, Prandtl number is $\frac{\nu}{\alpha}$ and Reynolds number based on the characteristic length L in this particular case, it is plate length is $\frac{U_\infty L}{\nu}$ where ν is your kinematic viscosity. So, you can see, we can write $\frac{\delta_T^2}{L^2} \sim \frac{1}{P_r R_{e_L}}$ based on L.

So, you can see that $\frac{\delta_T}{L} \sim \frac{1}{P_r^{1/2} R_{e_L}^{1/2}}$. So, from here, you can see when this $\frac{\delta_T}{L}$<<1. So, you can see here. So, $\frac{\delta_T}{L}$<< 1 when you can see your $P_r^{1/2} R_{e_L}^{1/2}$ >> 1. So, from this you can see .

So, $\frac{\delta_T}{L}$, we have already postulated that it is << 1 when your $P_r^{1/2} R_{e_L}^{1/2}$ >> 1. So, that means, this is your Peclet number. So, $P_e^{1/2}$ >> 1; that means, Peclet number also will be >>1.

So, in this case, if you consider let us say Peclet number is 100. So, if you consider Peclet number 100, then what will be your $\frac{\delta_T}{L}$? So, if it is Peclet number 100 so, you can see for Peclet number is 100 so, $\frac{\delta_T}{L}$ from here what will be that value? So, it will be $\sqrt{100}$. So, it will be 10, $\frac{1}{10}$ so, it will be 0.1. So, you can see that Peclet number is 100 $\frac{\delta_T}{L}$ will be 0.1. So, it is very small. So, generally we say that for thermal boundary layer to exist Peclet number should be greater than 100.

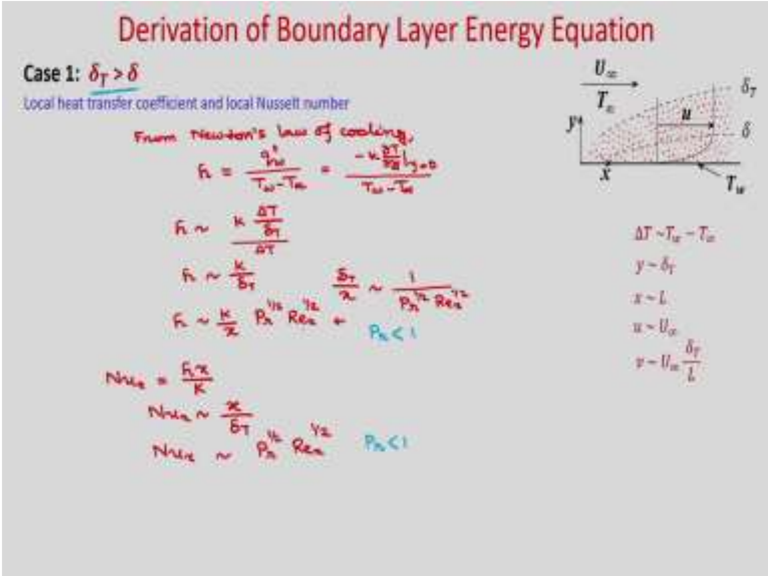
Now, you can see that we have written $\frac{\delta_T}{L}$ as this expression. So, for any x we can write $\frac{\delta_T}{x} \sim \frac{1}{P_r^{1/2} R_{e_x}^{1/2}}$ so that means, R_{e_x} we are defining as $\frac{U_\infty x}{\nu}$. So, for at any x or $\frac{\delta_T}{x}$, we can write $\frac{1}{P_r^{1/2} R_{e_x}^{1/2}}$.

So, from this expression, you can see how your thermal boundary layer thickness varies with x. So, you can see from this expression, it varies with root x and velocity boundary layer also varies with $\sqrt{x}$. So, you can see from this expression that δ_T varies with $\sqrt{x}$. You can see that we have already found the hydrodynamic boundary layer thickness $\frac{\delta}{x} \sim \frac{1}{R_{e_x}^{1/2}}$. So, $\frac{\delta}{x}$ is this expression and $\frac{\delta_T}{x}$ is having this expression. So, you can see $\frac{\delta_T}{\delta}$ what you can write? So, $\frac{\delta_T}{\delta}$ so, this you can write as $\frac{1}{P_r^{1/2}}$.

So, now, you can see $\frac{\delta_T}{\delta} \sim \frac{1}{P_r^{1/2}}$. So, what does it mean that $\delta_T > \delta$ when your $P_r^{1/2} < 1$.

So, you can see that when you have low Prandtl number fluids, then your thermal boundary layer thickness will be higher than the hydro dynamic boundary layer thickness. So, generally you see that liquid metals are having low Prandtl number values. So, for those cases, you can have the thermal boundary layer thickness, you can have higher than the hydro dynamic boundary layer thickness.

(Refer Slide Time: 22:51)



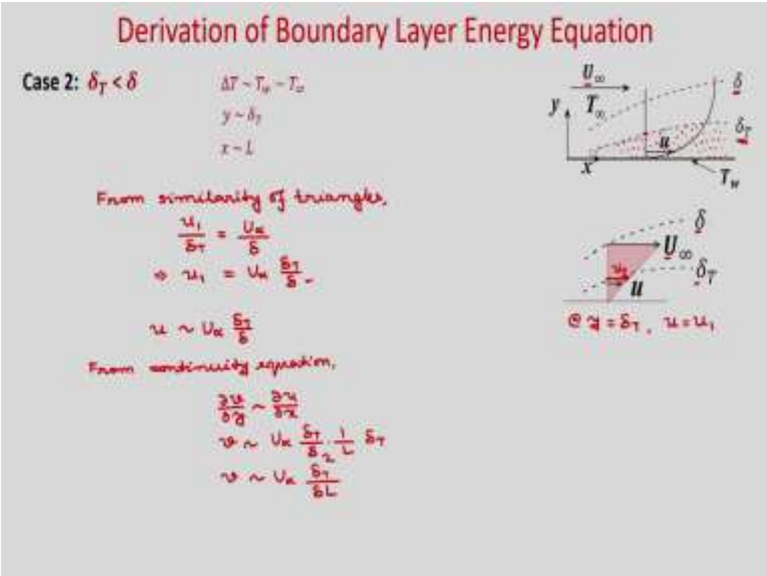
Now, let us discuss about the local heat transfer coefficient and local Nusselt number and what is the order we will find. So, you know local heat transfer coefficient definition from Newton's law of cooling, you can write; from Newton's law of cooling you can write $h = \dfrac{q_w^{"}}{T_w - T_\infty}$ and what is your heat flux? Heat flux is nothing but $-\dfrac{k\dfrac{\partial T}{\partial y}\Big|_{y=0}}{T_w - T_\infty}$.

Now, let us see what is the order of h. So, now, $h \sim \dfrac{k\dfrac{\Delta T}{\delta_T}}{\Delta T}$. So, that means, you can see $h \sim \dfrac{k}{\delta_T}$ and we know what is the value of $\dfrac{\delta_T}{x}$ so, from there, you can write $\dfrac{\delta_T}{x} \sim \dfrac{1}{P_r^{1/2} R_{e_x}^{1/2}}$ so, from here, you can see that $h \sim \dfrac{k}{x} P_r^{1/2} R_{e_x}^{1/2}$. So, this is the local heat transfer coefficient order and if you consider local Nusselt number, then the local Nusselt number $N_{u_x} = \dfrac{hx}{k}$.

So, now you can see $h \sim \dfrac{k}{\delta_T}$. So, Nusselt number you can see will be order of h if it is order of $\dfrac{k}{\delta_T}$ so, it will be $\dfrac{x}{\delta_T}$ and $\dfrac{x}{\delta_T}$ from here, you can see $N_{u_x} \sim P_r^{1/2} R_{e_x}^{1/2}$ and from here also you can see $N_{u_x} = \dfrac{hx}{k}$ so, it is $P_r^{1/2} R_{e_x}^{1/2}$ and you can see that these expressions are valid when you have $\delta_T > \delta$ and when it is valid? When you have low Prandtl number fluids; that means, Prandtl number < 1.

So, these expression is valid when you have low Prandtl number fluids; that means, Prandtl number < 1 because we have taken the case when $\delta_T > \delta$ so obviously, it is valid when Prandtl number < 1 and under this condition your h and Nusselt number x will be order of these expressions.

(Refer Slide Time: 26:22)



Now, let us consider when thermal boundary layer thickness is less than the hydrodynamic boundary layer thickness. So, you consider here your thermal boundary layer thickness is δ_T and this is less than your hydrodynamic boundary layer thickness δ.

So, you have free stream velocity U_∞. So, these U_∞ obviously, you can see that when your velocity will develop from 0 to ∞ and at the edge of this velocity boundary layer, it will become free stream velocity U_∞ right. So, you can see that inside the thermal boundary layer thickness at the edge of thermal boundary layer, the velocity will be lower than the free stream velocity because it is not same as U_∞ right. So, at this position, you can see your velocity will be lower than the free stream velocity U_∞ and that will be the scale for u in case of $\delta_T < \delta$.

So, in this case, we will have one assumptions. We will assume that your velocity is varying inside the thermal boundary layer linearly. So, we can see. So, in this case, we are considering that this is your free stream velocity $U_\infty \big|_{y=\delta}$; at y = δ. So, at y = δ_T; let us say you have velocity u = u_1 some scale at the edge of thermal boundary layer. So, let us say this is your some scale u_1.

Now, we are assuming that your velocity is varying linearly as you have very small velocity boundary layer thickness, if you assume that it is varying linearly, then whatever expression we will derive you will find that there is not much error. So, in that condition, now you find with the similar triangle what is the velocity at the edge of thermal boundary layer assuming the linear velocity profile.

So, you can see from similarity of triangles so, there are two triangles. So, this is one triangle, this is one triangle. So, you can see that $\frac{u_1}{\delta_T} = \frac{U_\infty}{\delta}$ because that y = δ you have U_∞. So, $u_1 = U_\infty \frac{\delta_T}{\delta}$.

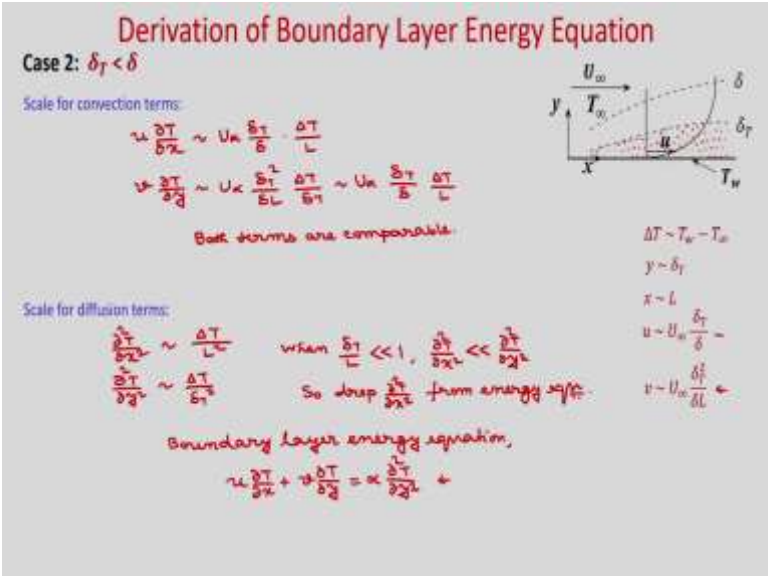
So, now when $\delta_T < \delta$, we will consider the scale for velocity u inside the thermal boundary layer as u_1. So, we will consider $u_1 \sim U_\infty \frac{\delta_T}{\delta}$. Here, we cannot consider u as U_∞ because U_∞ is becoming at y = δ, but we are considering thermal boundary layer thickness so, y = δ_T. So, inside this, we are assuming that u will be less than the free stream velocity U_∞ and the order of velocity will be $U_\infty \frac{\delta_T}{\delta}$.

So, we can see when we consider $\delta_T > \delta$ obviously, outside the velocity boundary layer thickness you have U_∞ everywhere. So, you can we took the scale for u as U_∞, but in this particular case, we cannot take because at the edge of velocity boundary layer, you have U_∞, but now $\delta_T < \delta$. So, at the edge of thermal boundary layer, we do not have the velocity scale as U_∞. So, for that reason we considered the velocity scale in this particular case u as u_1 and this u_1 we have found from similarity of triangles as $U_\infty \frac{\delta_T}{\delta}$.

So, now from continuity equation, you can find the scale for v from continuity equation. So, from continuity equation, you can see that $\frac{\partial v}{\partial y} \sim \frac{\partial u}{\partial x}$. So, we can see $v \sim U_\infty \frac{\delta_T}{\delta} \frac{1}{L} \delta_T$.

So, you can see $v \sim U_\infty \frac{\delta_T^2}{\delta L}$.

(Refer Slide Time: 31:26)



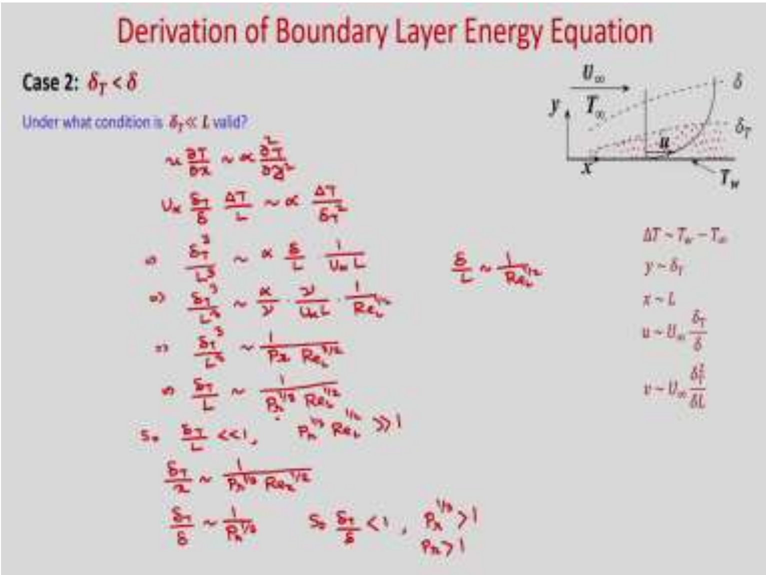
Now, with these let us see that if we can drop some term in the energy equation. So, first we will consider the convection terms. So, it is $u\frac{\partial T}{\partial x}$. So, it is order of u is scale of $U_\infty \frac{\delta_T}{\delta}$. So, $U_\infty \frac{\delta_T}{\delta}$, δ_T is δ, x is L. So, and $v\frac{\partial T}{\partial y}$ as order of so, $v\frac{\partial T}{\partial y} \sim U_\infty \frac{\delta_T^2}{\delta L}\frac{\Delta T}{\delta_T} \sim U_\infty \frac{\delta_T}{\delta}\frac{\Delta T}{L}$.

So, you can see this is also $U_\infty \frac{\delta_T}{\delta}$ because here δ_T^2 and here δ_T is there so, it will become δ_T and $\frac{\delta_T}{L}$. So, we can see both are having the same order. So, you cannot drop any term from the convection. Both terms are comparable. So, you cannot drop any term.

Now, let us consider the conduction terms or diffusion terms, So, $\frac{\partial^2 T}{\partial x^2}$ so obviously, you can see it will be $\frac{\Delta T}{L^2}$ and $\frac{\partial^2 T}{\partial y^2} \sim \frac{\Delta T}{\delta_T^2}$ and with similar argument, you can see when is $\frac{\delta_T}{L} << 1$, $\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2}$. So, drop $\frac{\partial^2 T}{\partial y^2}$ from energy equation.

So, if you drop this term from the energy equation, you can write boundary layer energy equation as $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$. So, this is the same expression when we derived for $\delta_T > \delta$. Now, let us see under what condition is $\frac{\delta_T}{L} << 1$, valid.

(Refer Slide Time: 34:05)



So, again we will compare the diffusion term with the convection term. So, you can see that $u\frac{\partial T}{\partial x} \sim \alpha\frac{\partial^2 T}{\partial y^2}$,now put the scale. So, $U_\infty \frac{\delta_T}{\delta}\frac{\Delta T}{L} \sim \alpha\frac{\Delta T}{\delta_T^2}$. So, here you can see this δ_T^2 if you take in the left-hand side, then it will be δ_T^3 and divide by L^2 both side. So, here L is there so, it will become L^3 will be order of α.

Then, we have $\frac{\delta}{L}$, L^2 we have divided right in right-hand side so, $\frac{\delta}{L}$ will be there and will be $\frac{1}{U_\infty L}$ and δ_T δ_T will get cancelled. So, you can see it will be $\frac{\delta_T^3}{L^3} \sim \frac{\alpha}{\nu} \frac{\nu}{U_\infty L} \frac{1}{R_{e_L}^{1/2}}$. So, we can write $\frac{\delta}{L} \sim \frac{1}{R_{e_L}^{1/2}}$. So, you can see $\frac{\nu}{U_\infty L}$ it is $\frac{1}{R_{e_L}}$ and $\frac{\nu}{\alpha}$ as Prandtl number.

So, you can see $\frac{\delta_T^3}{L^3} \sim \frac{1}{P_r} \frac{1}{R_{e_L}^{3/2}}$. So, you can see $\frac{\delta_T}{L} \sim \frac{1}{P_r^{1/3} R_{e_L}^{1/2}}$.

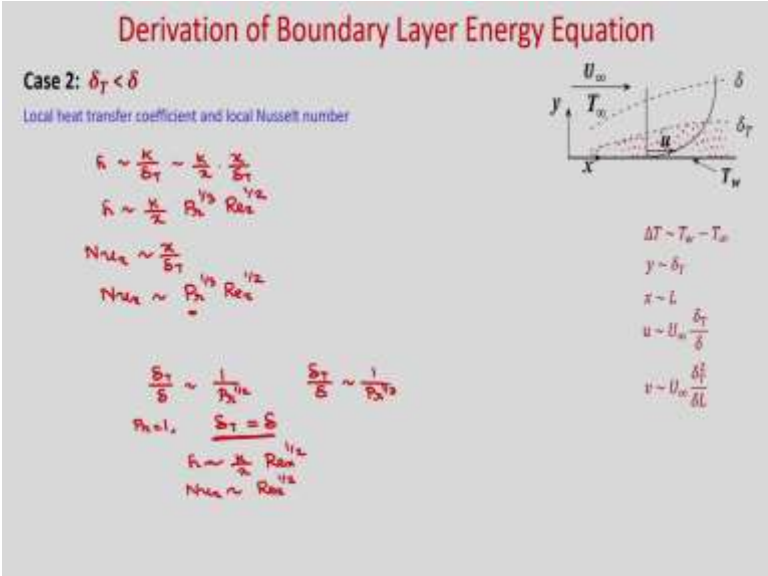
So, if you observe the expression of this and the term which we derive for $\delta_T > \delta$, you can see Prandtl number, $P_r^{1/3}$ and in case of $\delta_T > \delta$, it is Prandtl number $P_r^{1/2}$.

So, you can see so, $\frac{\delta_T}{L} << 1$ when you have $P_r^{1/3} R_{e_L}^{1/2} >> 1$. So, generally, you can see that if Peclet number >>1, then $\delta_T <<$ L.

So, now for any x also you can write $\frac{\delta_T}{x} \sim \frac{1}{P_r^{1/3} R_{e_x}^{1/2}}$ and $\frac{\delta_T}{\delta} \sim \frac{1}{P_r^{1/3}}$. So, now, from this expression you see when you consider $\delta_T < \delta$ that means, your thermal boundary layer thickness is less than the hydrodynamic boundary layer thickness when it will happen? When your $P_r^{1/3} > 1$ that means, it is high Prandtl number fluids.

So, you can see that. So, $\frac{\delta_T}{\delta}$ will be so, we have consider $\frac{\delta_T}{\delta} < 1$ so obviously, your $P_r^{1/3} > 1$. So, if you consider fluid like oils, it will have high Prandtl number. So, for that the hydrodynamic boundary layer thickness will be greater than the thermal boundary layer thickness that means, $\delta > \delta_T$ for oils and you can see that $\frac{\delta_T}{\delta} \sim \frac{1}{P_r^{1/3}}$.

(Refer Slide Time: 38:52)



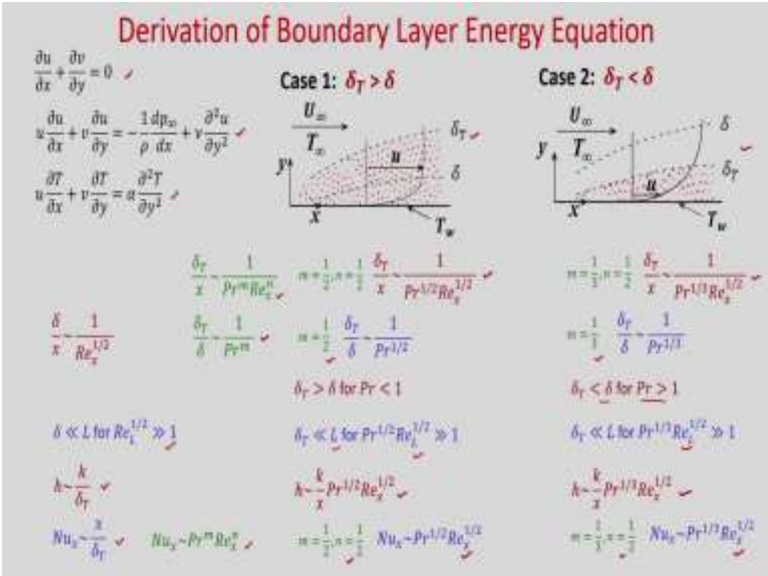
Now, under this condition, let us consider what is the local heat transfer coefficient and local Nusselt number. So, again $h \sim \frac{k}{\delta_T}$. So, we have already derived it. So that means, you can see it will be $\frac{k}{x}\frac{x}{\delta_T}$ and $\frac{\delta_T}{x}$ already we have found. So, it will be $\frac{k}{x}\frac{x}{\delta_T}$ is just $P_r^{1/3} R_{e_x}^{1/2}$ and local Nusselt number $N_{u_x} \sim \frac{x}{\delta_T}$ and so, $N_{u_x} \sim P_r^{1/3} R_{e_x}^{1/2}$.

So, you can see the difference when $\delta_T > \delta$ is in the power of Prandtl number. So, if Prandtl number is 1 so, most of the gases will have the Prandtl number as 1 say let say air, air Prandtl number is almost 0.71 right.

So, it is of the order of 1 and so, for most of the gases, you have Prandtl number is order of 1. So, like air you have Prandtl number as 0.71. So, if Prandtl number is 1 so, we can see for both the cases, we have derived $\frac{\delta_T}{\delta} \sim \frac{1}{P_r^{1/2}}$ and another case we have consider $\frac{\delta_T}{\delta} \sim \frac{1}{P_r^{1/3}}$.

So, if you consider Prandtl number = 1, then obviously, $\frac{\delta_T}{\delta}$ so, because your thermal boundary layer thickness will be equal to hydrodynamic boundary layer thickness and in that particular case, you can see your $h \sim \frac{k}{x} R_{e_x}^{1/2}$ and $N_{u_x} \sim R_{e_x}^{1/2}$.

(Refer Slide Time: 41:06)



So, let us summarize what we have done today. So, we started with the energy equation and using order of magnitude analysis, we have derived the boundary layer energy equation and then, we considered two different cases: first case is when thermal boundary layer is larger than the hydrodynamic boundary layer and next we considered that hydrodynamic boundary layer is greater than thermal boundary layer.

So, we can see. So, finally, we have derived these boundary layer equations, this is the continuity equation, this is the u momentum equation where we have dropped the term $\frac{\partial^2 u}{\partial x^2}$ and this is the energy equation dropping the term $\frac{\partial^2 T}{\partial x^2}$.

You can see here that we have considered two different cases. In this particular case, we have considered $\delta_T > \delta$ and in this case, we have considered $\delta_T < \delta$. So, from here we have derived $\frac{\delta_T}{x} \sim \frac{1}{P_r^m R_{e_x}^n}$ where m = 1/2 and n = 1/2 when $\delta_T > \delta$ so, from here, you will get this expression and when you consider $\delta_T < \delta$, then you can put m = 1/3 and n =1/2, then you will get this expression.

From the velocity boundary layer equation also we have derived $\frac{\delta_T}{x} \sim \frac{1}{R_{e_x}^{1/2}}$. So, if you now divide the thermal boundary layer to the hydrodynamic boundary layer, then you can find $\frac{\delta_T}{\delta} \sim \frac{1}{P_r^m}$. So obviously, you can see m =1/2 when $\delta_T > \delta$ and m =1/3 when you have $\delta_T < \delta$ and from here, you can see δ_T will be δ for Prandtl number <1 that means, low Prandtl number fluids and $\delta_T < \delta$ for high Prandtl number fluids. Also, we have seen that $\delta \ll l$ when Reynolds number is very high and $\delta_T \ll L$ when Peclet number will be very high.

Then, we considered the local heat transfer coefficient and local Nusselt number and you can see that we have derived each of the order of $\frac{k}{\delta_T}$ and this is the expression for two different cases and $N_{u_x} \sim \frac{x}{\delta_T}$ you can see from here you can write $N_{u_x} \sim P_r^m R_{e_x}^n$. So obviously, m =1/2, n =1/2 when $\delta_T > \delta$ and you will get this expression and for $\delta_T < \delta$, you will get m =1/3 and n =1/2 and you will get this expression.

So, today we will just stop here and in the next class, we will start with the solution of these energy equation analytically and also using your approximate solution that means, integral approach.

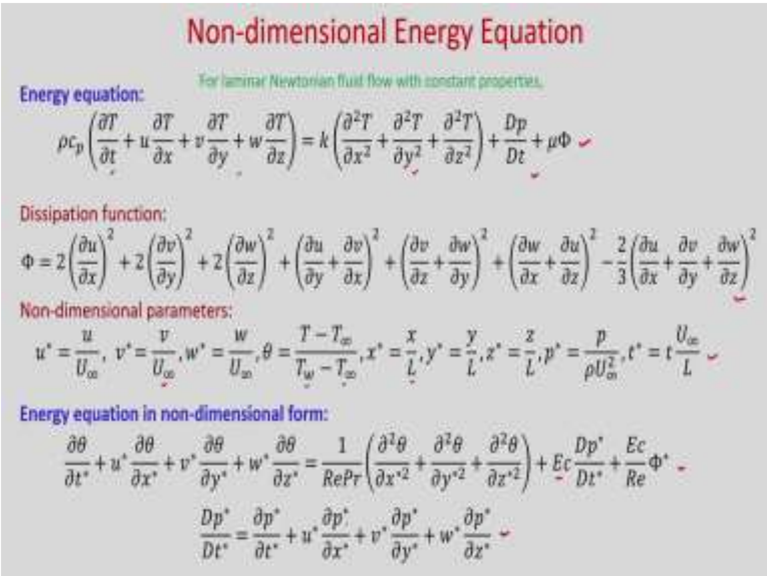
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 03
Convective Heat Transfer in External Flows – I
Lecture – 06
Blasius solution: similarity method

Hello everyone. So, in today's class first we will derive the energy equation in non-dimensional form, then we will solve the governing equation for flow over flat plate using similarity method. This solution is known as Blasius solution. Blasius first solved this equation, using similarity method and derived the ordinary differential equation which you can solve using some numerical technique.

(Refer Slide Time: 01:02)



So, first let us see what we derived, the energy equation in dimensional form. So, if you see for laminar Newtonian fluid flow with constant properties, this is the energy equation we derived right. So, this is the temporal term, this is the convective term, this is the diffusion term and as it is constant properties so, the thermal conductivity we can take it outside and this is $\frac{Dp}{Dt}+\mu\phi$, where Φ is the dissipation function.

So, dissipation function in general is given in this form. If it is an incompressible flow; then obviously, $\nabla V = 0$. So, last term will become 0. Using this non dimensional parameters now you please convert this energy equation in non-dimensional form. So, what we will use? We will use some reference velocity U_∞ and reference length L.

So, these are some characteristic length L and characteristic velocity U_∞. For external flows generally, we take the free stream velocity as reference velocity that is your U_∞ and for any geometry we find the characteristic length and that length we take as reference length L. So, you can see using this reference velocity we have non-dimensionalized the velocity u* as $\frac{u}{U_\infty}$.

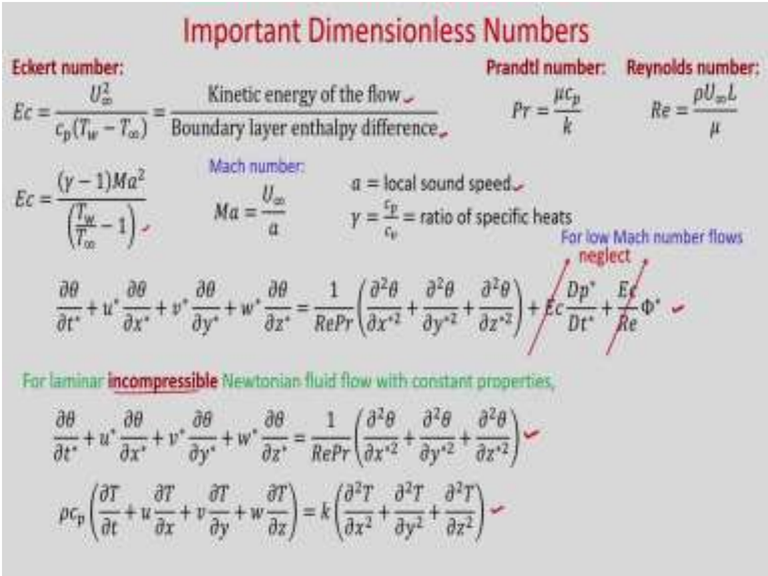
Similarly, $v^* = \frac{v}{U_\infty}$, $w^* = \frac{w}{U_\infty}$, the temperature non dimensional temperature we have defined as $\theta = \frac{T - T_\infty}{T_w - T_\infty}$, where T_w is the wall temperature and T_∞ is the free stream temperature.

Similarly, the coordinate, we have non-dimensionalized using the characteristic length L. So, $x^* = \frac{x}{L}$, $y^* = \frac{y}{L}$, $z^* = \frac{z}{L}$, and the pressure, pressure we have used $p^* = \frac{p}{\rho U_\infty^2}$ and the time $t^* = t\frac{U_\infty}{L}$. So, these are some non-dimensional parameters used to convert this dimensional form of the energy equation to non-dimensional form of the energy equation.

So, as homework you just do it and finally, you will get these equations. So, you can see we have written in non-dimensional form. So, this is the convective term. In diffusion term, these non-dimensional numbers comes one by Reynolds number into Prandtl number.

In this term, one new non dimensional number comes that is your Eckert number and with the dissipation function you will get $\frac{E_c}{R_e}$. And obviously, you know how the $\frac{Dp^*}{Dt^*}$ is defined. So, this is the material derivative. So, now let us define the non-dimensional numbers.

(Refer Slide Time: 04:06)



So, this conversion from dimensional form to non-dimensional form using these non-dimensional parameters, you can do as a homework. And now, you can see the definition of these non-dimensional numbers, which we encountered in non-dimensional form of the energy equation. So, first non-dimensional number is Eckert number. So, what is Eckert number? Eckert number $E_c = \frac{U_\infty^2}{c_p(T_w - T_\infty)}$.

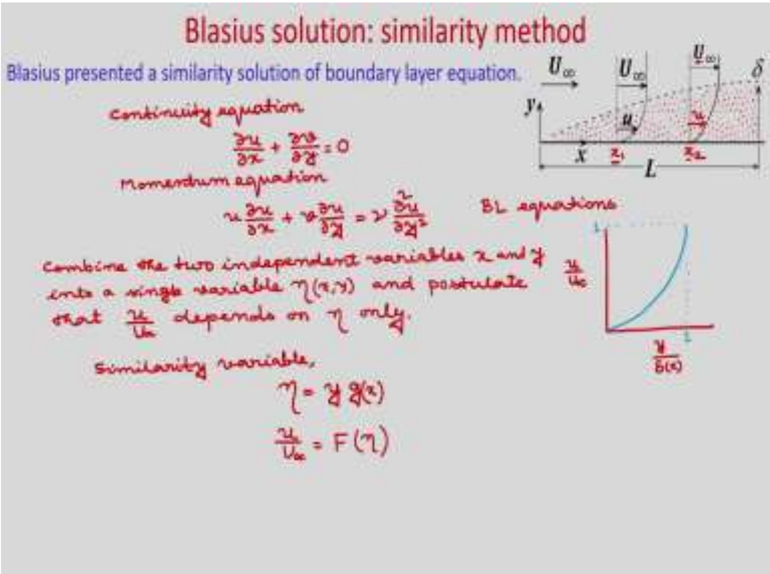
So, that means, it is the temperature difference. So, you can see that in the numerator you have kinetic energy of the flow and in denominator you have boundary layer enthalpy difference. So, you can see Eckert number is the ratio of kinetic energy of the flow to the boundary layer enthalpy difference. And Prandtl number you know, Prandtl number $P_r = \frac{\mu c_p}{k}$ and Reynolds number, $R_e = \frac{\rho U_\infty L}{\mu}$, where rho is the fluid density, U_∞ is the reference velocity, L is the reference length and mu is the dynamic viscosity of the fluid.

So, these two non-dimensional numbers you are familiar with, but this is the new number Eckert number which we can write in this form. So, with some mathematical algebra, you can write Eckert number, $E_c = \frac{(\gamma -)Ma^2}{(\frac{T_w}{T_\infty} - 1)}$.

So, you know what is Mach number? Mach number is the ratio of fluid velocity to the local sound velocity. So, that means, Mach number you can defined, $Ma = \frac{U_\infty}{a}$, where a is the local sound speed and $\gamma = \frac{c_p}{c_v}$. So, you can see in this equation that, if Mach number is very low then Eckert number also will become very low. So, here you can see whatever energy equation we have written in non-dimensional form. So, if Mach number is very low then these last two terms you can neglect, because Eckert number will be very-very low.

So, you can see these two terms in the energy equation you can neglect. And, for laminar incompressible flow generally Mach number is very low and for that you can drop these two terms and you can write the energy equation in this form. So, it is in non-dimensional form and if you write in dimensional form, the energy equation for incompressible fluid flow, then this will be your energy equation.

(Refer Slide Time: 07:03)



Now, let us consider flow over flat plate. So, this is the simplest solution we can do for flow over flat plate, because your pressure gradient is 0. So, first let us write the governing equation. We have already derived the boundary layer equations for flow over flat plate. So, continuity equation; so, what is continuity equation? $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

So, while deriving the boundary layer equation for flow over flat plate what are the assumptions we took? We have assumed that it is a steady flow, two dimensional flow, and incompressible Newtonian fluid flow. So, for that you can write the continuity equation like this and momentum equation you can write $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$.

So, this is the boundary layer equations right, these are boundary layer equations. Now, you can see these equations are non-linear, because the momentum equation you can see these terms are non-linear. So, it is difficult to solve analytically. Blasius first time solved these equations using similarity method.

So, when can you use similarity method? First you see the velocity distribution for flow over flat plate. So, if you see this is the length of the plate L, x is the axial direction, y is measured from the flat plate, U_∞ is the free stream velocity and if you see at any location x_1, so this will be your velocity profile. So, velocity from 0, it will develop and at the edge of the boundary layer.

So, this is the edge of the boundary layer it will become free stream velocity U_∞. Similarly, if you see at location x_2; so, here also the velocity will vary; velocity will vary u from 0 to U_∞, because U_∞ is the free stream velocity. So, at the free stream velocity U_∞ is constant for flow over flat plate. So, you can see in these two locations that, velocity profile looks similar except it is with a scaling factor.

So, if you see at x_2 location, whatever velocity profile you can see if you scale it down, it will be similar to the velocity profile at x_1 location. So, we can see that as this velocity profile looks like similar with a scaling factor so we can use the similarity method to solve these governing equations. So, Blasius first observed that, if these two independent variables x, y; if you convert it into a single variable η, where η is function of x and y then, with the scaling factor all the velocity profile at different locations will fall in a same curve.

So, here if you see that if you plot the velocity u by U_∞; so, u by U_∞, so you can see that you will be 0 at y = 0 and at the edge of boundary layer you have U_∞, so it will become 1. And, if you plot this velocity distribution with $\frac{y}{\delta}$, δ is the boundary layer thickness.

So, at any location δ is the distance from the flat plate to the edge of bond layer, where the velocity becomes close to free stream velocity U_∞. And, at different location if you plot this u by U_∞ versus $\frac{y}{\delta(x)}$, because δ will vary with x.

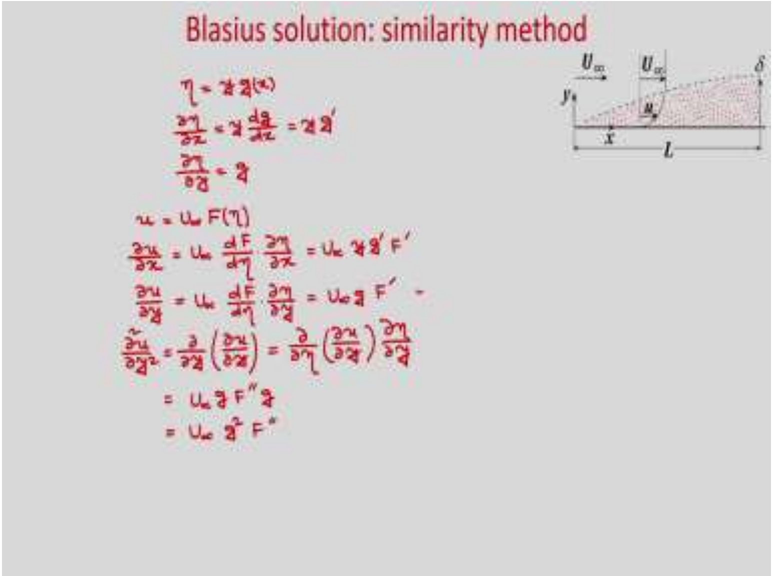
So, δ is function of x, then he observed that it will fall in the same curve. So, it will look like. So, you can see this is the velocity 1 and this will be also 1, because at $y = \delta$, this will become 1. So, it will fall in the same curve at different location, along the axial direction at different x location if you plot the velocity in this curve $\frac{u}{U_\infty}$ versus $\frac{y}{\delta(x)}$, it will fall in the same curve.

So, you can see that velocity profiles are similar. So, that is why we can use the similarity approach to solve these governing equations. Now, you can see that we have defined $\frac{y}{\delta}$, δ is function of x; so obviously, we can see that we can define one variable η combining the two independent variables x and y such that, it will become y into g(x). So, combine the two independent variables; two independent variables x and y into a single variable η, which is function of x and y and postulate that $\frac{u}{U_\infty}$ depends on η only.

So, now this η will be function of x and y. We do not know; what is the function, but it will be function of y and x. So, we will write, because we can see from here that $\delta(x)$. So, it will be function of x, but that is unknown. So, let us define the similarity variable η; similarity variable $\eta = yg(x)$, we do not know what is the function; so, which is function of x.

So, now what we can do? So, now we can see that these independent variables x and y we have combined into a single variable η such that, $\frac{u}{U_\infty}$ is function of eta only; some function, we do not know the function, but function of η only. Now, we have the governing equations we know u with some function of F. So, we will just substitute all these in these equations and we will derive the Blasius equation.

(Refer Slide Time: 15:31)



So, we have seen that η=yg(x). So, we can write $\frac{\partial \eta}{\partial x} = y\frac{dg}{dx}$. So, you can write ordinary derivative; that means, yg'. So, g' represent the derivative of g with respect to x. Similarly, $\frac{\partial \eta}{\partial y} = g$. Now, we know the velocity profile $u = U_\infty F(\eta)$. And, we will see the derivative of u with respect to x and y.

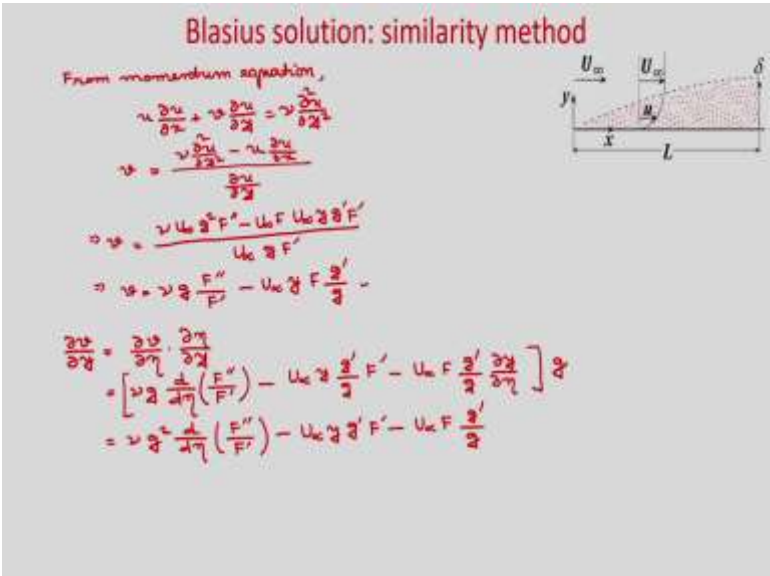
So, u = U_∞, where U_∞ is the free stream velocity into F. F is function of η only. So, it is function of η only and g is function of x only. So, first let us write $\frac{\partial u}{\partial x} = U_\infty \frac{dF}{d\eta}\frac{\partial \eta}{\partial x}$. So, $\frac{\partial \eta}{\partial x} = yg'$ and $\frac{dF}{d\eta} = F'$. So, you can write $U_\infty yg'F'$.

Similarly, you take the derivative of u with respect to y. So, $\frac{\partial u}{\partial y}$. So, it will be $U_\infty \frac{dF}{d\eta}\frac{\partial \eta}{\partial y}$. So, $\frac{\partial \eta}{\partial y} = g$. So, you can write $U_\infty gF'$. Now, take the derivative of $\frac{\partial u}{\partial y}$ with respect to y. So, we can write $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial y})$.

So, what you can write; $\frac{\partial u}{\partial y}$ we know right. So, we can write $\frac{\partial}{\partial \eta}(\frac{\partial u}{\partial y})\frac{\partial \eta}{\partial y}$. So, this you can write. So, now you see, this is the $\frac{\partial u}{\partial y}$. So, you write the derivative of it with respect to η. So, U_∞ is constant, g is function of x only; so, it will be F'' and $\frac{\partial \eta}{\partial y} = g$. So, you can write $U_\infty g F''$ and $\frac{\partial \eta}{\partial y} = g$.

So, you can write it as $U_\infty g^2 F''$. So, $\frac{\partial^2 u}{\partial y^2} = U_\infty g^2 F''$. Now, let us substitute this in the momentum equation.

(Refer Slide Time: 18:48)



So, from momentum equation, you can write $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$. So, here you can see, here we have already found $u\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$, but v is unknown.

So, let us find v. So, $v = \dfrac{\nu\frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$. So, you can write v is equal to; so, $\nu\frac{\partial^2 u}{\partial y^2}$.

So, we have already derived $U_\infty g^2 F''-u$, u =U∞ F right and $\frac{\partial u}{\partial x}$ we have derived as $U_\infty y g' F'$ and divided by $\frac{\partial u}{\partial y}$.

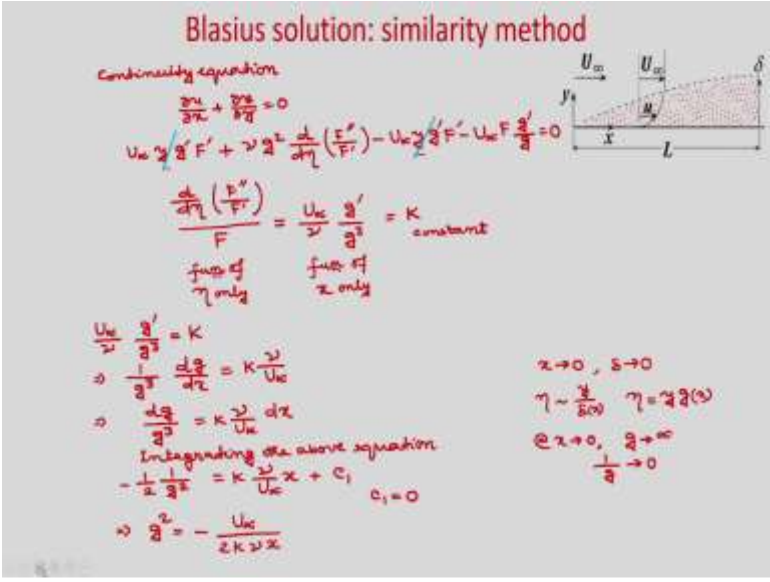
So, this is your $U_\infty g F'$. So, from here you can see that, you can write v is equal to. So, you see U∞ U∞ will cancel out, one g will get cancelled, so you will get $\nu g \frac{F''}{F'}$ and the second term one U∞ will get cancelled and $F' F'$ will get cancelled, so you will get $U_\infty y F \frac{g'}{g}$.

So, we have found the v. Now, if you see the continuity equation so, in the continuity equation we have $\frac{\partial v}{\partial y}$. So, let us find $\frac{\partial v}{\partial y}$ from here. So, $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}$. So, now you see $\frac{\partial v}{\partial \eta}$ from here you just find. So, νg. So, now we can write $\frac{d}{d\eta}(\frac{F''}{F'})$.

So, this is your first term and minus now, $\frac{\partial}{\partial \eta}$ of this so it will be so, you can see; so, there are two terms y and F. So, g is function of x only. So, you can write $U_\infty y \frac{g'}{g} F'$. So, we have taken the; we have taken the derivative of F minus. So, $U_\infty F \frac{g'}{g}\frac{\partial y}{\partial \eta}$ and we have $\frac{\partial \eta}{\partial y}$. So, what is $\frac{\partial \eta}{\partial y}$? $\frac{\partial \eta}{\partial y}$ is nothing but, g and here $\frac{\partial \eta}{\partial y}$ is there. In the denominator if you see it will be $\frac{\partial \eta}{\partial y}$ and that will be also g.

So, now you multiply g and write it. So, it will be $\nu g^2 \frac{d}{d\eta}(\frac{F''}{F'})$; so, these g g will get cancelled. So, it will be $U_\infty y g' F'$ minus you can see this, $\frac{\partial \eta}{\partial y}$ one g will be there and one g will get cancelled here. So, you can write $U_\infty F \frac{g'}{g}$. So, this is your $\frac{\partial v}{\partial y}$. Now, let us use the continuity equation. So, you can write the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

(Refer Slide Time: 23:32)



So, this is the continuity equation. Now, we have already found $\frac{\partial u}{\partial x}$ and also we have found $\frac{\partial v}{\partial y}$. So, you can see $\frac{\partial u}{\partial x}$ whatever we have found $\frac{\partial u}{\partial x}$ is $U_\infty y g' F'$ and $\frac{\partial v}{\partial y}$ is this expression. So, let us put in the continuity equation.

So, if you put it here so, $U_\infty y g' F' + \nu g^2 \frac{d}{d\eta}\left(\frac{F''}{F'}\right) - U_\infty y g' F' - U_\infty F \frac{g'}{g} = 0$.

So, if you can see that this term and this term is same with a minus sign so, you can cancel it. And, now you notice, you see g is function of x only and F is function of η only. So, you rearrange such a way and separate the variables such that, that in the left hand side will be function of η only and right hand side will be function of x only. So, if you write and rearrange then, you can write $\frac{\frac{d}{d\eta}\left(\frac{F''}{F'}\right)}{F}$.

So, this will be divided by F and if you take this in the right hand side, it will become positive and you can write as $\frac{U_\infty}{\nu}\frac{g'}{g^2}$; so, it you are dividing so it will be g^3. Now, you can see in this equation we have separated the variables left hand side is function of eta only and right hand side is function of x only and these are equal. So obviously, it will be equal to some constant right.

So, you can write that equal to some constant K. So, it will be some constant K, because it is function of η only; function of η only left hand side and this term is function of x only.

So obviously, because U_∞ and ν are constant for the flow over flat plate and g is function of x only. So, it will be equal to some constant K and that we need to find. Now, if you see; so, if you consider this term equal to K, then you can rearrange it as $\frac{U_\infty}{\nu}\frac{g'}{g^3} = K$. So, you can see you will be able to integrate it $\frac{1}{g^3}\frac{dg}{dx} = K\frac{\nu}{U_\infty}$. If you rearrange it again so you can write $\frac{dg}{g^3} = K\frac{\nu}{U_\infty}dx$.

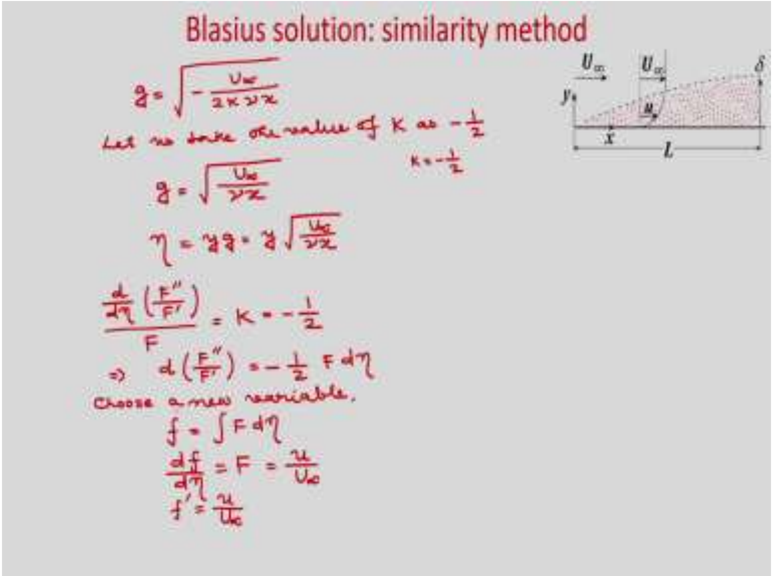
So, now you can see that, this equation now you will be able to integrate. So, integrating the above equation. So, you can see you can write $-\frac{1}{2}\frac{1}{g^2} = K\frac{\nu}{U_\infty}x + c_1$. So, this integration constant c_1 what will be the value? Now, you see physically, if you see this figure as x →0, your boundary layer thickness δ→0.

So, you can see x → 0, at the leading edge of the plate your boundary layer thickness will be tending to 0. It is not actually 0; although, we write that x = 0 ,δ = 0, but at the edge at the leading edge of the flat plate delta will not be exactly 0.

So, that is why we are writing x →0, δ →0 and δ is function of x and we have seen that, $\eta \sim \frac{y}{\delta(x)}$, because we have already seen that, the plot for the similar velocity so it will be $\frac{y}{\delta}$ and that we have written $\eta = yg(x)$, because δ is function of x and g is function of x. So, you can see as x →0, δ → 0; obviously, at x →0, g →∞; that means, $\frac{1}{g}$ → 0.

So, from here you can see that, x →0, $\frac{1}{g}$ → 0; so obviously, the integration constant c_1=0. So, now if you see from here you can write $g^2 = -\frac{U_\infty}{2K\nu x}$.

(Refer Slide Time: 30:16)



So, you can write $g = \sqrt{-\frac{U_\infty}{2K\nu x}}$. So, now you can see in this expression, K is constant right. So, for simple calculation we can take any value of K.

So, let us take the value of K as $-\frac{1}{2}$. So, that this minus will go away, because minus minus will become positive and here 2 is there and if you take half then, these 2 2 will get cancel and you will get a simple expression. So, let us take the value of K which is constant as $-\frac{1}{2}$. So, if you put K = $= -\frac{1}{2}$, then you will get $g = \sqrt{\frac{U_\infty}{\nu x}}$. In some book, you will find this $\eta = \frac{U_\infty}{2\nu x}$.

So, depending on how they have taken this constant it will vary, but in our calculation we are assuming the constant as $-\frac{1}{2}$, so that we will have the simple expression. So, g now we can write as $\sqrt{\frac{U_\infty}{\nu x}}$. So, now you can write η, $\eta = yg$. So, you can see that it will be $y\sqrt{\frac{U_\infty}{\nu x}}$. So, this is the similarity variable. Now, we got the expression of g which is a function of x as $\sqrt{\frac{U_\infty}{\nu x}}$. So, that we have found.

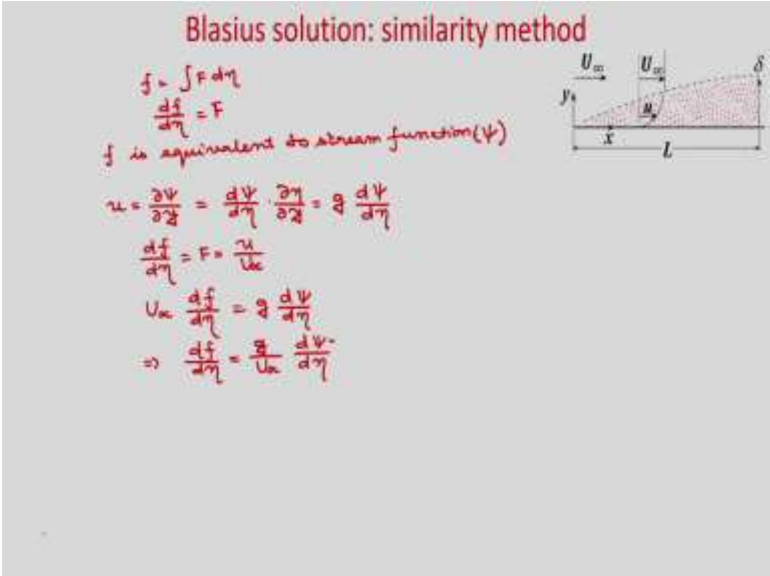
Now, let us take the other part. So, this part $\frac{\frac{d}{d\eta}(\frac{F''}{F'})}{F} = K$. And now, K we have already assumed $-\frac{1}{2}$. So, this will be equal to $-\frac{1}{2}$.

So, it will be $-\frac{1}{2}$. So, if you see it will be $d(\frac{F''}{F'}) = -\frac{1}{2}Fd\eta$. So, you can see in this equation still F is unknown; but F is function of η. So, now let us take that you can integrate it. So, if you integrate this equation, then you will get $\int Fd\eta$ and now, as F is unknown it will be difficult. So, let us assume or choose a new variable $f = \int Fd\eta$.

So, we are choosing $f = \int Fd\eta$ such that, $\frac{df}{d\eta} = F$. So, you can see that, we are choosing $f = \int Fd\eta$ as a new variable F such that, you will get $\frac{df}{d\eta} = F$. So, here you can see in this integration when we are writing it; so obviously, in here this it includes the constant so that, when we are integrating these so obviously, there will be integration constant and if you take the derivative. So, this constant will become 0; so, it will be $\frac{df}{d\eta} = F$.

So, now we can see, what is $\frac{df}{d\eta}$. So, we have written $F = \frac{u}{U_\infty}$. So, $\frac{df}{d\eta}$ which you can represent as $f' = \frac{u}{U_\infty}$. Now, what is the physical significance of f. So, we have chosen a new variable f. So, what is the physical significance of it? Let us see.

(Refer Slide Time: 35:17)



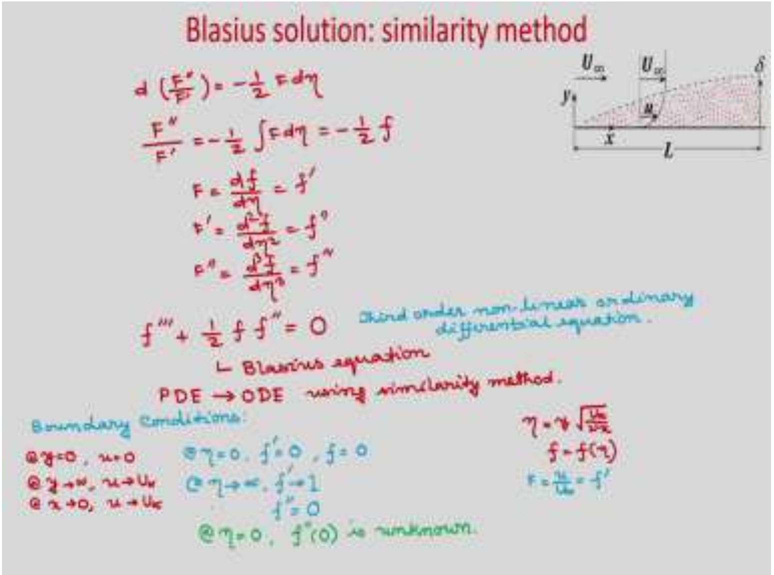
So, we have defined $f = \int F d\eta$ such that, $\dfrac{df}{d\eta} = F$. So, this F, whatever F we are defining this is having the physical significance. So, f is equivalent to stream function. So, whatever new variable we have defined f, it is having the physical significance because it is equivalent to stream function. So, let us see, so how do you define the velocity u? u you define if you have Ψ is the stream function then how you define u, $u = \dfrac{d\psi}{dy}$.

So, $u = \dfrac{d\psi}{dy}$. Now, you can write $\dfrac{d\psi}{dy}\dfrac{d\eta}{dy} = g\dfrac{d\psi}{d\eta}$. And, in earlier slide you can see we have seen that, $\dfrac{df}{d\eta} = F = \dfrac{u}{U_\infty}$.

So, you can see this u, you can write as $U_\infty \dfrac{df}{d\eta} = g\dfrac{d\psi}{d\eta}$ so that means, you can write $\dfrac{df}{d\eta} = \dfrac{g}{U_\infty}\dfrac{d\psi}{d\eta}$. So, you can see f is equivalent to stream function with a factor g by U_∞, but the physical significance of f is that, it is equivalent to stream function.

So, at the wall in this plate, you can assume the value of f as any constant, because it is a you can assume flat plate has a stream line and along a streamline your stream function will be constant and any value we can take. So, for simplification we will take the value of stream function or F as 0 on the flat plate. So, now, what we are getting from this expression. So, you can see now $f = \int F d\eta$.

(Refer Slide Time: 38:16)



So, you can write $d(\frac{F''}{F'}) = -\frac{1}{2} F d\eta$. So, if you integrate it. So, you will get $\frac{F''}{F'} = -\frac{1}{2}\int F d\eta$ such that, it will be $-\frac{1}{2} f$ and integration constant I have told that it is already included in the f. So, you can write $-\frac{1}{2} f$. And, from here you can see F we have defined $F = \frac{df}{d\eta}$.

So, $F' = \frac{d^2 f}{d\eta^2}$. So, it will be f'' and $F'' = \frac{d^3 f}{d\eta^3}$; that means, f'''. So, now if you put it here and rearrange what you are going to get? $f''' + \frac{1}{2} f f'' = 0$, which is known as Blasius equation.

So, now you can see we started with the partial differential equations, because continuity equation and momentum equation which are boundary layer equations. So, those equations were partial differential equations. Using the similarity method we converted this partial differential equation to ordinary differential equation, because this you can see, this is your ordinary differential equation, because they are ordinary derivative right $\frac{df}{d\eta}, \frac{d^2 f}{d\eta^2}$. So, this is ordinary derivative.

So, you have converted PDE to ODE using similarity method. Now, you see about this equation ordinary differential equation; so obviously, you can see it is a non-linear equation, because you have ff'' and it is a third order ordinary differential equation because you have f'''. So, you can see this is a third order non-linear ordinary differential equation. And, here f is function of η only that is why it is ordinary derivative.

Now, how to solve this equation. So, Blasius used a power series to solve this equation, but in this course we will not elaborate that, but you can use some numerical approach to solve this ordinary differential equation, because you know different numerical techniques right. So, you can use Rungekutta method to solve this equation with some initial and boundary conditions.

So, in this case; obviously, it is a boundary value problem, you can see from the boundary conditions which are known and which are not known. So, first let us write the boundary condition boundary conditions. So obviously, you see that at y = 0 at the wall, you have u = 0 and Ψ = constant at $y \rightarrow \infty$; that means, $y \rightarrow \delta$ right.

So, you have $u \rightarrow U_\infty$, because it will approach free stream velocity. And, at $x \rightarrow 0$, where at the leading edge of the flat plate here also your $u \rightarrow U_\infty$. Now, you convert the boundary condition in terms of f.

So, at y = 0. Now, you see what is the expression of η? So, you see η we have already written $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$ and f is function of η only. Now, you can see here. So, at y = 0; obviously, η = 0. So, at η = 0.

So, u = 0; means you can see u, how we have defined $f = \frac{u}{U_\infty}$. And f is nothing but, f'.

So, from here you can see at η = 0 u = 0; that means, $f' = 0$ right. Now, if you see in terms of stream function, because we have already shown that, f is equivalent to stream function. So, you can consider the flat plate as a stream line and along a stream line stream function will be constant and that constant value we can choose 0 for convenience.

So, we can write f = 0. Now, if you see these two equations, now at $y \to \infty$ right, y_δ means ∞ also you can write means away from the wall. So, you can write at $y \to \infty$ or $x \to 0$ in both the cases $\eta \to \infty$ right, because $y \to \infty$ then, $\eta \to \infty$ and $x \to 0$ that also $\eta \to \infty$. So, what will be your value? So, you can see $u \to U_\infty$; so that means, f' will become 1.

So, $f' \to 1$. Now, you can see $f' \to 1$ in this case. And, what is the physical significance of f''. So, in the f'', so from here you can see. So, at $\eta \to \infty$ what will be the velocity gradient? So, at the edge of the boundary layer your and outside the boundary layer you have free stream velocity U_∞. So obviously, you can see the velocity gradient will be 0 and what will be the velocity gradient you can see f'' will represent the velocity gradient.

So, $f'' = 0$, ok. So, at the edge of boundary layer and outside the boundary layer you can see that, $f'' = 0$, because velocity gradient will become 0. So, now we can see from this boundary condition we have converted the boundary condition in terms of f and η.

So, you can see $\eta = 0$, your stream function is 0, we have just assumed 0 value, $f' = 0$ and at $\eta \to \infty$, $f' \to 1$ and velocity gradient. So, f'' will represent the velocity gradient.

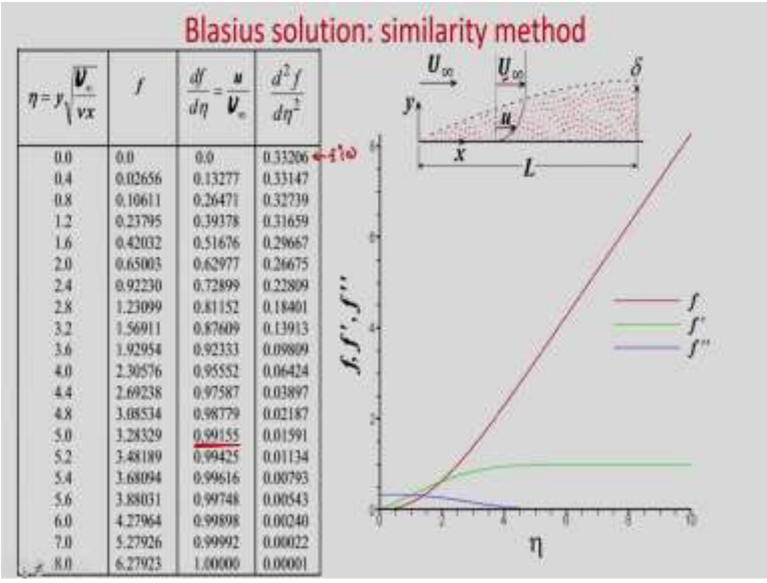
So, $f'' = 0$. Now, you can see that this ordinary differential equation you can solve using some numerical technique. So, you can refer some book, you can use Rungekutta method to solve this ordinary differential equation with this condition. So, you can see here, your the value of f'' at $\eta = 0$ is unknown. So, at $\eta = 0$, $f''(0)$ is unknown.

So, you solve such a way that, you assume the value of $f''(0)$ at $\eta = 0$, because to solve this equation you need it and satisfy that at $\eta \to \infty$, ∞ you cannot have. So, as far distance for η high value f'' will become 0, because velocity will become 0.

So, you can see f'' f values are known at $\eta = 0$, but at $\eta = 0$, f'' is unknown. So, you assume the value of $f''(0)$ and such a way that, you satisfy f'' at $\eta \to \infty$ means far away from the wall it will become 0.

So, it is known as shooting technique, because you have to assume and check whether it is satisfying that condition or not if it does not satisfy you change the value of $f''(0)$. So, in this way you solve these equations you solve these ordinary differential equation using Rungekutta method and shooting technique and satisfy f'' at $\eta \to \infty$ as 0, assuming the value of f'' at $\eta = 0$.

(Refer Slide Time: 48:23)



$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{V_\infty}$	$\frac{d^2 f}{d\eta^2}$
0.0	0.0	0.0	0.33206 ← f''(0)
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
6.0	4.27964	0.99898	0.00240
7.0	5.27926	0.99992	0.00022
8.0	6.27923	1.00000	0.00001

If you solve this then, you will get these values. You can see that if you assume f'' value at $\eta = 0$ as 0.33206, then you can see at $\eta \to \infty$ at a higher value it is becoming 0.

So, this is the value η. So, this will be U_∞. So, you can see this is the η value. So, 0 is at wall and as η increases; that means, you are going away from wall. So, and you can see the value of f, f is equivalent to stream function; $\frac{df}{d\eta}$ is U_∞. So, this is your U_∞ and $\frac{d^2 f}{d\eta^2}$ it is the velocity gradient. So, that you are assuming this value such that, your at $\eta \to \infty$ it will become 0. So, if you plot it. So, you can see the plot here.

So, this is the plot of f. So, we have assumed f = 0. So, it is satisfying and away from wall means, η at higher value; it is increasing. f' is the representation of the velocity. So, now you can see f'. So, f' at $\eta = 0$, it is 0, but at $\eta \to \infty$; obviously, outside the boundary layer it will become constant value right, 1. So, you can see this is almost 1.

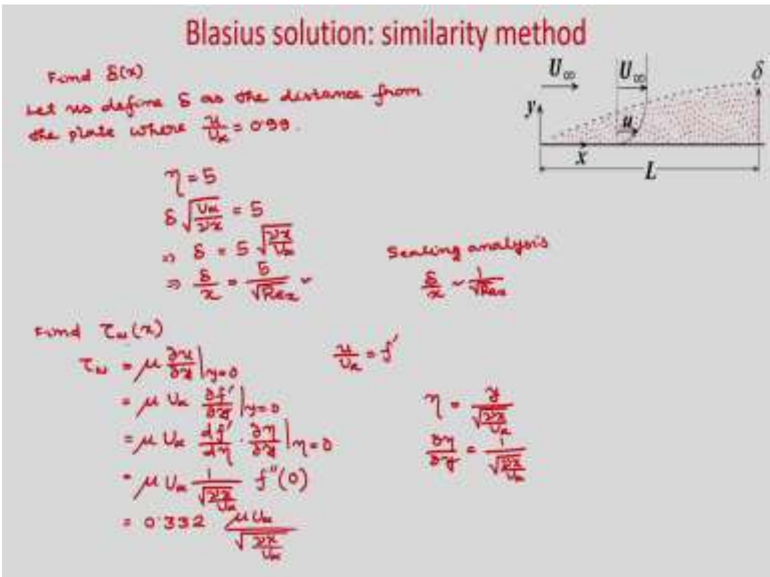
So, this is your u = U_∞ right; so that means, $\frac{u}{\infty} = 1$ and you can see around $\eta = 5$ it is becoming almost constant, because it is outside the boundary layer and it is maintaining the free stream velocity U_∞. And, f'' is the representation of velocity gradient.

So obviously, you can see velocity gradient will be higher at the wall, because wall shear stress will be there. And, as you go away from the wall your velocity gradient will be will decrease and outside the boundary layer; obviously, it will become 0 and you can see you have higher value at $\eta = 0$ as η increases your velocity gradient decreases and at the edge of the boundary layer almost, it is becoming 0.

So, you can see almost at $\eta = 5$, your velocity is becoming almost 99 % of U_∞ and that will define the hydrodynamic boundary layer thickness. So, δ will define where u becomes almost 99 % of U_∞. So, that you can see, it is becoming 1 and velocity gradient also is becoming 0.

So, you can tabulate it, you can solve this ordinary differential equation using Rungekutta method and plot this table as well as this figure, ok. Now, from the solution Blasius solution at $\eta = 5$ your velocity is becoming almost 99 % of U_∞. So, you can see from this table. So, at $\eta = 5$, you can see it is becoming almost 99 % of the U_∞.

(Refer Slide Time: 51:46)



So, let us find what is $\delta(x)$. So, let us define δ as the distance from the plate, where $\frac{u}{U_\infty} = 0.99$. So obviously, you can see. So, around η = 5, it is becoming 99 %. So, at y = δ; obviously, it is becoming this. So, η we know. So, at place of y you put $\delta\frac{U_\infty}{\nu x}$. So, this is the expression of η we are writing is equal to 5 and you can write $\delta = 5\sqrt{\frac{\nu x}{U_\infty}}$ and if you rearrange it so, you can see $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$.

So, now you can see from the Blasius solution, we got the value of $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$ and if you recall that when we use scaling analysis, we have shown that delta by x is order of 1 by root R e x. So, from scaling analysis $\frac{\delta}{x} \sim \frac{1}{\sqrt{R_{e_x}}}$. So, now you can see that $\frac{\delta}{x} \sim \frac{1}{\sqrt{R_{e_x}}}$ and you can see from the Blasius solution. In some book, you can see that this value 5 whatever we have taken, it can be taken as 5.2 or 4.9, because from this table you can see this is also 0.99.

So, in different books you can see that, these value of η is taken as 5.2 or 4.9 kind of thing, but we have chosen just 5. So, $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$. Now, let us find what is the wall shear stress? Right. So, find wall shear stress, which is function of x. So, now, $\tau_w = \mu\frac{\partial u}{\partial y}|_{y=0}$ right, at the wall y = 0 we are finding τ_w and $\frac{u}{U_\infty}$ we know that it is f'.

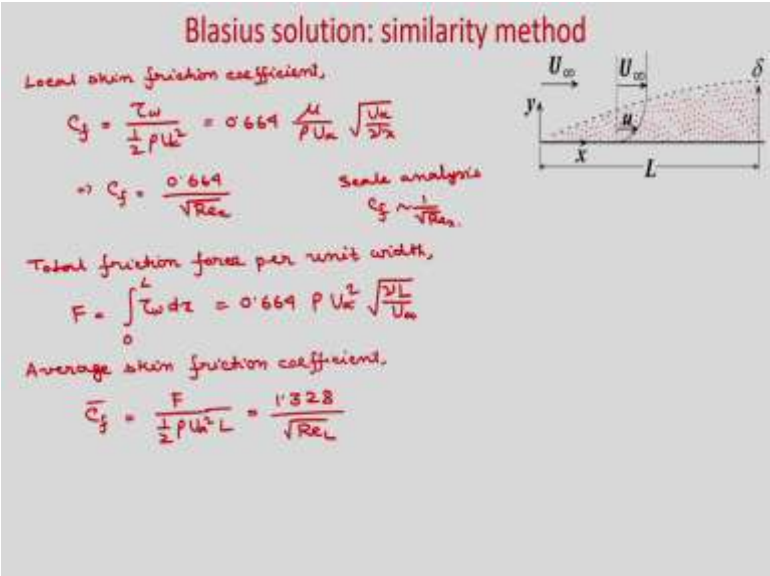
So, you can see now you can write μ $\tau_w = \mu U_\infty \frac{\partial f'}{\partial y}|_{y=0}$. Now, this you write in terms of η derivative right. So, you can write $\tau_w = \mu U_\infty \frac{df'}{dy}\frac{\partial \eta}{\partial y}|_{\eta=0}$; so, you can see $\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}$. So,

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{\frac{\nu x}{U_\infty}}}.$$

So, if you put these values. So, you can see this will be $\tau_w = \mu U_\infty \frac{1}{\sqrt{\frac{\nu x}{U_\infty}}} f''(0)$. So, these value f'', what is the physical significance of f''? We have shown that it is a velocity gradient and velocity gradient now you know from the Blasius solution, what is the value at η = 0.

So, if you go back to this table, you can see this is the value of f'' at η = 0. So, if you put this value here and rearrange you will get 0.332; 0 6 I am not writing. So, it will be $\tau_w = 0.332 \frac{\mu U_\infty}{\sqrt{\frac{\nu x}{U_\infty}}}$. So, this is your wall shear stress.

(Refer Slide Time: 56:06)



Now, let us write local skin friction coefficient. So, this you can define as $C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$.

So, if you put that you will get $0.664 \frac{\mu}{\rho U_\infty} \sqrt{\frac{U_\infty}{\nu x}}$ and if you rearrange it you will get $C_f = \frac{0.664}{\sqrt{R_{e_x}}}$.

So, if you remember that using scaling analysis also we have found $C_f \sim \frac{1}{\sqrt{R_{e_x}}}$. So, from scale analysis already we have shown that, $C_f \sim \frac{1}{\sqrt{R_{e_x}}}$. Now, let us find what is the total friction force per unit width? So, total friction force per unit width of the plate.

So, $F = \int_0^L \tau_w dx$, because this is the length of the plate. So, if you do it you will get $0.664 \rho U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}}$. Now, average skin friction coefficient now you can write; average skin friction coefficient $\overline{C_f} = \frac{F}{\frac{1}{2}\rho U_\infty^2 L}$.

So, that is L .1. So, $L = \frac{1.328}{\sqrt{R_{e_L}}}$. So, in today's class, we have seen that we have started with the boundary layer equation for flow over flat plate. We used similarity method, proposed by Blasius and we converted the partial differential equations to ordinary differential equation using similarity method.

So, similarity variable we have used as η, which is function of two independent variables x and y and we have used $\frac{u}{U_\infty}$ as capital F which is function of η only. And, we have converted these boundary equations to Blasius equation which is your third order non-linear ordinary differential equation.

Then, we have written down the boundary conditions and you know that, the value of F'', f'' which is the representation of velocity gradient is unknown at the wall. So, using some numerical technique like Rungekutta method and shooting technique you can assume the value of f'' at η = 0 and satisfy that f'' at η →∞ will become 0, because at the near to the edge of the boundary layer and outside the boundary layer your velocity gradient will become 0 for flow over flat plate.

Then, we have tabulated the values of f f'' with η and we have also shown the plot of f f'' and f'', where f is the representation of stream function f' is representation of the velocity and f'' is the representation of velocity gradient.

Then, we have found the hydrodynamic boundary layer thickness δ from the Blasius solution and we have shown that $\frac{\delta}{x} = \frac{5}{\sqrt{R_{e_x}}}$ and also we have found the wall shear stress and from wall shear stress we have found the local skin friction coefficient and average skin friction coefficient.

Thank you.

Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

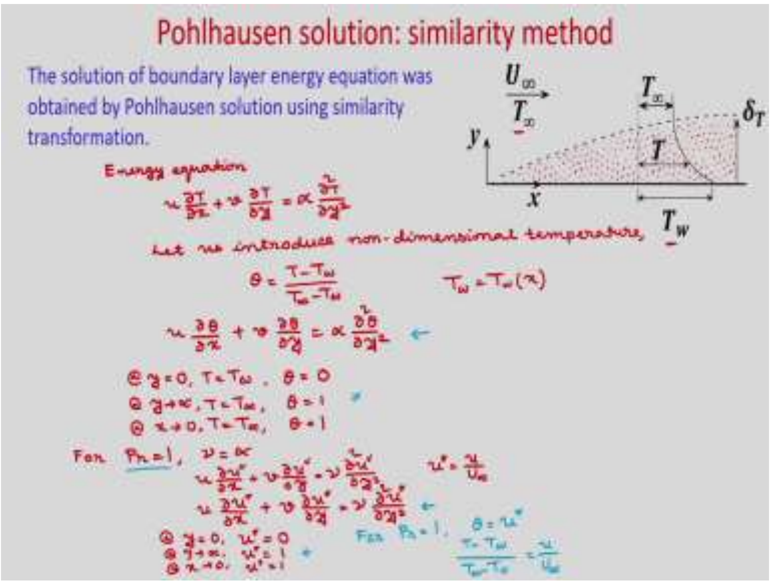
Module – 03
Convective Heat Transfer in External Flows - I
Lecture – 07
Pohlhausen solution: similarity method

Hello everyone. So, today we will consider thermal boundary layer over a flat plate and we will find the temperature distribution using Similarity Method and this solution is known as Pohlhausen solution.

We will consider two-dimensional, steady, laminar flow with constant properties. The free stream temperature T_∞ is constant and the wall temperature T_w in general will consider that it varies with x which is the axial direction, later we will consider a special case where we will assume T_w as constant.

In last class, we have already solved the velocity distribution using similarity method. As we are assuming that properties are constant so, velocity distribution is independent of temperature distribution. Here, we are considering low speed flow which is incompressible flow. So, we can neglect viscous dissipation effect.

(Refer Slide Time: 01:45)



First let us write down the energy equation and let us introduce non-dimensional temperature, $\theta = \frac{T - T_w}{T_\infty - T_w}$. So, energy equation for steady laminar flow we can write $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$. So, this is the thermal boundary layer equation right.

So, we are neglecting the viscous dissipation effect and now let us introduce non-dimensional temperature; non-dimensional temperature $\theta = \frac{T - T_w}{T_\infty - T_w}$. So, here T_∞ is the free stream velocity and that is constant, but T_W which is your wall temperature in general, we will consider T_W as function of x and as a special case, later we will consider that T_W is constant and you can see if $T_W > T_\infty$ so, temperature distribution will look like this because at the wall, we have maximum temperature and free stream temperature as $y \rightarrow \infty$.

So, with this now, if I write the energy equation, then you can write $u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$.

So, now, let us write the boundary conditions. You can see at the wall where y = 0, you have $T = T_W$ and $y \rightarrow \infty$ means away from the boundary, you have free stream temperature T_∞.

You can see that as $x \rightarrow 0$ at the leading edge of the flat plate, we have free stream temperature T_∞. So, at y = 0, you have $T = T_W$ so, θ will be 0 at or $y \rightarrow \infty$ so obviously, it is away from the wall you can see at the edge of the bond layer, you have T_∞ so, $T = T_\infty$ and you can write θ = 1 and at $x \rightarrow 0$, you have $T = T_\infty$ so, θ will be 1.

Now, we have already solved the velocity distribution using similarity method. Can we have some similarity with the momentum equation and the energy equation when ν =α. So, when ν is when will be the ν = α? When Prandtl number, P_r= 1.

So, as a special case let us say for Prandtl number= 1 your ν you can write is equal to α and the momentum equation whatever we have so, you can write down $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \alpha\frac{\partial^2 u}{\partial y^2}$,and this u you write in terms of non-dimensional velocity $u* = \frac{u}{U_\infty}$ then, you can see you can write $u\frac{\partial u*}{\partial x} + v\frac{\partial u*}{\partial y} = \alpha\frac{\partial^2 u*}{\partial y^2}$.

And what are the boundary conditions for this equation at y = 0, u* = 0 no slip boundary condition, at y $\rightarrow\infty$, you have u*= 1 and at x $\rightarrow$0, you have u* = 1.

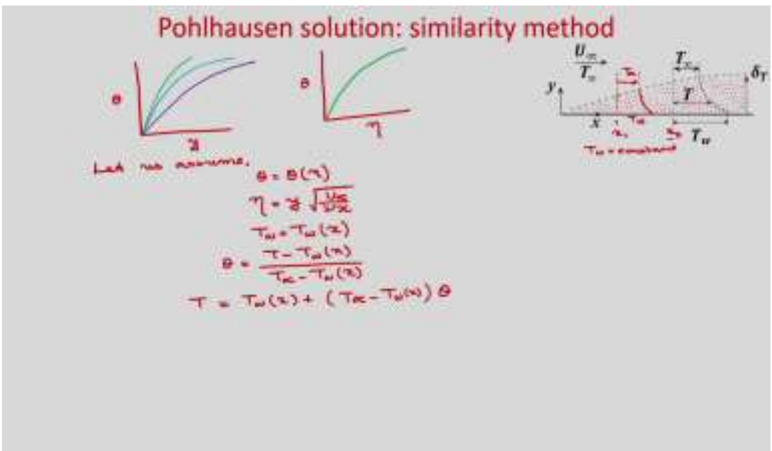
Now, you see these equations. This is the energy equation where θ is the non-dimensional temperature with these boundary conditions and this is the momentum equation where u* is the non-dimensional velocity and these are the boundary conditions. So, if you compare these two for Prandtl number= 1 so, these are the same equation because if you replace θ = u*, then you can see the boundary conditions are same and $\nu = \alpha$. So, both equations are same.

So, you can write for Prandtl number =1, θ can be replaced with u* that means, you can write θ = u*; that means, $\frac{T - T_w}{T_\infty - T_w} = \frac{u}{U_\infty}$. So, you can see already we have solved the velocity distribution using similarity method and we could convert the partial differential equation to ordinary differential equation using similarity approach.

In this case, whatever we have shown now that for Prandtl number =1 as a special case, the governing equations, energy equation as well as the momentum equation are same, and θ = u*. So obviously, we can have the similarity solution at least for Prandtl number=1 for the temperature distribution.

So, we will use the similarity method to solve these energy equation and we will find the temperature distribution and if we can convert for other Prandtl number for Prandtl number $\neq$ 1, this partial differential equation to ordinary differential equation; that means, similarity solution exist for Prandtl number $\neq$1.

(Refer Slide Time: 08:53)



So, you can see that the temperature distribution at any location say let us say x_2 and another location if you see here so, it will have free stream temperature T_∞ and you have let us say T_w so, this is your T_∞, this is your T_w and this is location x_1.

So, you can see that the temperature distribution if you see here and here for let us say T_w is equal to constant; you can see that these are similar profiles. So, if you can scale down the temperature distribution at location x_2 with a scaling factor, then you can get the same temperature distribution at x_1. So that means, so, whatever temperature distribution say we are getting here θ versus y at different x location.

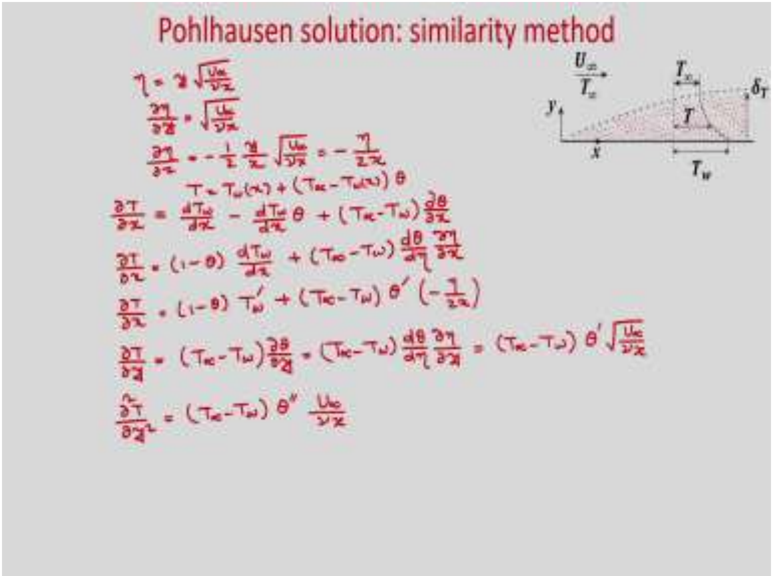
So, this will be you can see that at y =0, you have θ = 0 and y →∞, it will be 1. So, you will have another profile. So, you can see these are the temperature profile at different x locations. Now, if you use this similarity variable, then you can see that if you plot θ versus η, then it will all the temperature profile will fall in same curve. So, it will fall in same curve.

So, let us start with whatever similarity variable η we have derived in earlier class and we will assume that θ is function of η only because here you can see it might fall in the same profile and using similarity method, if we can convert this partial differential equation to ordinary differential equation; that means, your similarity solution exists. If you see the energy equation, in the energy equation u and v are known from the solution of velocity distribution; that means, this is linear equation because velocity profiles are known, only you need to find the temperature distribution.

So, let us assume that θ is function of η only and η already we have found in last class as $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$. So, this is the similarity variable. So, we have written in terms of two independent variables x and y and properties ν is constant and U_∞ is the free stream velocity that is also constant.

Also, let us assume in general that T_w is a function of x. So, $\theta = \frac{T - T_w(x)}{T_\infty - T_w(x)}$ and now $T = T_w(x) + (T_\infty - T_w(x))\theta$. So, let us write the energy equation the derivatives in terms of these with respect to the derivative with respect to η.

(Refer Slide Time: 13:03)



So, $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$. So, $\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}$ and $\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{y}{x}\sqrt{\frac{U_\infty}{\nu x}} = -\frac{\eta}{2x}$.

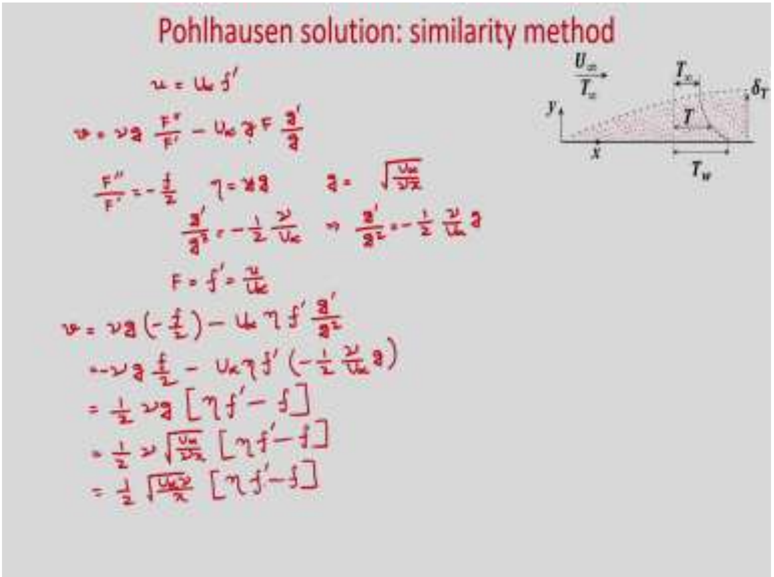
Now, let us find the derivative of temperature. So, we have $\frac{\partial T}{\partial x}$. So, it will be. So, $T = T_w(x) + (T_\infty - T_w(x))\theta$. So, $\frac{\partial T}{\partial x}$, you can see. So, this will be total derivative; $\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \frac{dT_w}{dx}\theta + (T_\infty - T_w)\frac{\partial \theta}{\partial x}$.

So, now $\frac{\partial T}{\partial x}$ you can write as so, you can see this you can $\frac{\partial T}{\partial x} = (1-\theta)\frac{dT_w}{dx} + (T_\infty - T_w)\frac{d\theta}{d\eta}\frac{\partial \eta}{\partial x}$. So, now, $\frac{\partial T}{\partial x} = (1-\theta)T_w^{'} + (T_\infty - T_w)\theta'(-\frac{\eta}{2x})$.

Now, similarly you find the derivative of T with respect to y. So, $\frac{\partial T}{\partial y}$ you can write now you see T_w is function of x. So, its derivative with respect to y will be 0 so, you will get $\frac{\partial T}{\partial y} = (T_\infty - T_w)\frac{\partial \theta}{\partial y} = (T_\infty - T_w)\frac{\partial \theta}{\partial \eta}\frac{\partial \eta}{\partial y}$.

So, you can see, you can write $\frac{\partial T}{\partial y} = (T_\infty - T_w)\theta'\sqrt{\frac{U_\infty}{\nu x}}$ and $\frac{\partial^2 T}{\partial y^2} = (T_\infty - T_w)\theta''\frac{U_\infty}{\nu x}$.

(Refer Slide Time: 16:51)



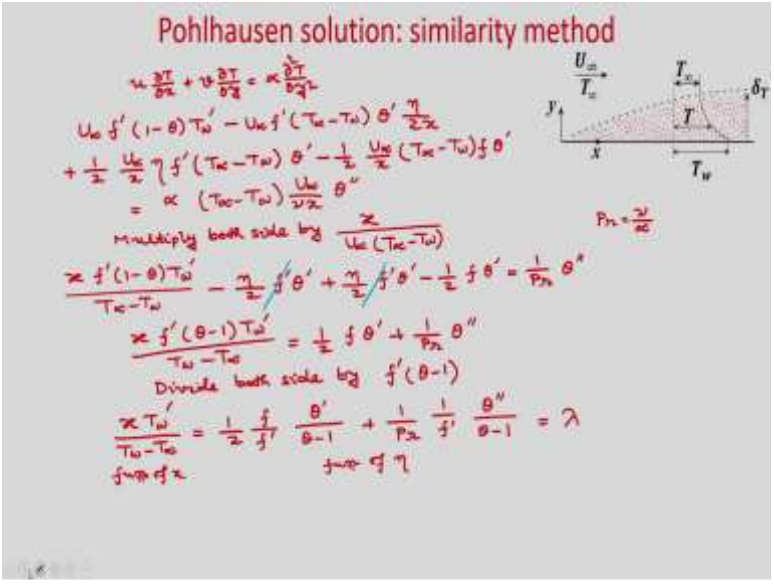
Now, we have also velocities u and v. So, that let us find. So, u obviously, you know that $u = U_\infty f'$ and in last class, we have already derived the velocity v in terms of F first let us write that and then we will write in terms of F. So, v if you see in last class, we have written $v = \nu g \dfrac{F''}{F'} - U_\infty y F \dfrac{g'}{g}$.

And if after separation of variables if you see that we have written $\dfrac{F''}{F'} = -\dfrac{f}{2}$. H already you know so, that we have written $\eta = yg$, here y we can write $\dfrac{\eta}{g}$ and g we have written $g = \sqrt{\dfrac{U_\infty}{\nu x}}$ and $\dfrac{g'}{g^3}$, if you see the equation where we have separated the variables that we have written $\dfrac{g'}{g^3} = -\dfrac{1}{2}\dfrac{\nu}{U_\infty}$. So, from here we can write $\dfrac{g'}{g^2} = -\dfrac{1}{2}\dfrac{\nu}{U_\infty} g$ and F obviously, you can see $F = f' = \dfrac{u}{U_\infty}$.

So, now, if you substitute all these here, you will get velocity v. So, it will be $v = \nu g(-\dfrac{f}{2}) - U_\infty \eta f' \dfrac{g'}{g^2}$. So, $v = -\nu g \dfrac{f}{2} - U_\infty \eta f'(-\dfrac{1}{2}\dfrac{\nu}{U_\infty} g)$.

So, if you take half ν g outside and it will be $v=\frac{1}{2}\nu g[\eta f'-f]$. So, it will be $v=\frac{1}{2}\nu\sqrt{\frac{U_\infty}{\nu x}}[\eta f'-f]$. So, this ν if you take inside the root, then you will get $v=\frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}[\eta f'-f]$.

(Refer Slide Time: 19:57)



So, let us put all these values in the energy equation. So, energy equation is $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$. So, now, one by one let us put. So, you have u, $U_\infty f'(1-\theta)T_w'$ and another term will be there. So, you can see this term we have written and this term you can write now. So, it will be $-U_\infty f'(T_\infty - T_w)\theta'\frac{\eta}{2x}$.

Now, we have $v\frac{\partial T}{\partial y}$ so, if you write it

$$\frac{1}{2}\frac{U_\infty}{x}\eta f'(T_\infty - T_w)\theta' - \frac{1}{2}\frac{U_\infty}{x}(T_\infty - T_w)f\theta' = \alpha(T_\infty - T_w)\frac{U_\infty}{\nu x}\theta''.$$

So, now, you rearrange this equation. So, if you rearrange you will get so, first multiply both side by $\frac{x}{U_\infty(T_\infty - T_w)}$. So, you can see in the first term. So, U_∞, U_∞ will get cancelled so, you can write $\frac{xf'(1-\theta)T_w'}{(T_\infty - T_w)}$. The next term you can see U_∞ will get cancelled $T_\infty - T_w$ also will get cancelled and this x, x. So, you will get finally, $\frac{\eta}{2} f'\theta'$.

Then, here also $\frac{U_\infty}{x}(T_\infty - T_w)$ is there so, you will get $\frac{\eta}{2} f'\theta'$. Then, the next term you can see here you will get only $-\frac{1}{2} f\theta'$ and here you can see $\frac{\alpha}{\nu}$ that means, it is $\frac{1}{P_r}$ because Prandtl number is $\frac{\nu}{\alpha}$ right, Prandtl number is $\frac{\nu}{\alpha}$ moving from diffusivity to thermal diffusivity. So, you can write $\frac{1}{P_r}$ and θ''.

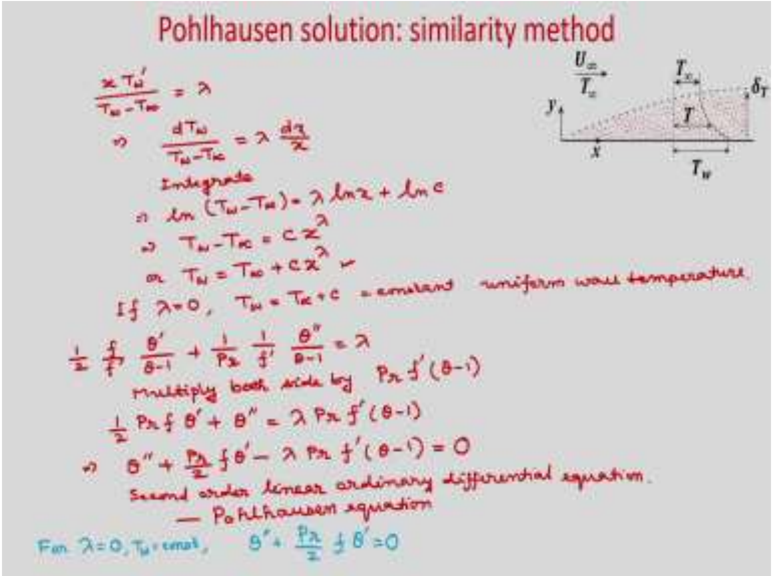
So, here you can see this is $T_\infty - T_w$, it is $1-\theta$ we will write here $\frac{xf'(\theta-1)T_w'}{T_w - T_\infty}$ and you have minus we taken right hand side. So, it will be $\frac{1}{2} f\theta' + \frac{1}{P_r}\theta''$.

Now, divide both sides; divide both side by $f'(\theta-1)$. So, if you rearrange it, you will get $\frac{xT_w'}{T_w - T_\infty} = \frac{1}{2}\frac{f}{f'}\frac{\theta'}{\theta-1} + \frac{1}{P_r}\frac{1}{f'}\frac{\theta''}{\theta-1}$.

So, you see we have separated the variables. If you see the left-hand side, it is function of x only and right hand side it is function of η only because AP is function of θ and θ is also a function of η. So, we have separated the variables.

So, you can see this is T_w. So, T_w is a function of x only so, all these terms are function of x so, this is function of x and this right-hand side terms are function of η only. So, as left-hand side equal to right-hand side and left-hand side is function of x and right-hand side function of η so, it will be equal to some constant and that constant let us say that it is λ. So, this will be equal to λ.

(Refer Slide Time: 25:17)



We have separated the variable so; you can write it as $\frac{xT_w^{'}}{T_w - T_\infty} = \lambda$. So, you can write dT_w so, $T_w^{'}$ is $\frac{dT_w}{dx}$. So, we will write $\frac{dT_w}{T_w - T_\infty} = \lambda \frac{dx}{x}$.

So, now, integrate. So, you will get $\ln(T_w - T_\infty) = \lambda \ln x + \ln c$. So, you can write $T_w - T_\infty = Cx^\lambda$ or $T_w = T_\infty + Cx^\lambda$. We have assumed that wall temperature T_w is function of x and it varies in this way.

As a special case, you can see that if λ = 0; if λ = 0, then you will get $T_w = T_\infty + C$. So, T_w and T_∞ is constant, and c is constant so, this is equal to constant. So, you can see that it is a case of uniform wall temperature. So, this is a special case. But in general, we have derived and T_w varies as $T_\infty + Cx^\lambda$.

So, now let us consider the other terms which is a function of η only. So, if you consider that term, then you will get $\frac{1}{2}\frac{f}{f'}\frac{\theta'}{\theta' - 1} + \frac{1}{P_r}\frac{1}{f'}\frac{\theta''}{\theta' - 1} = \lambda$. Now multiply both side by $P_r f'(\theta - 1)$. So, what you will get? You will get $\frac{1}{2} P_r f'\theta + \theta'' = \lambda P_r f'(\theta - 1)$. So, you will get $\theta'' + \frac{P_r}{2} f'\theta - \lambda P_r f'(\theta - 1) = 0$.

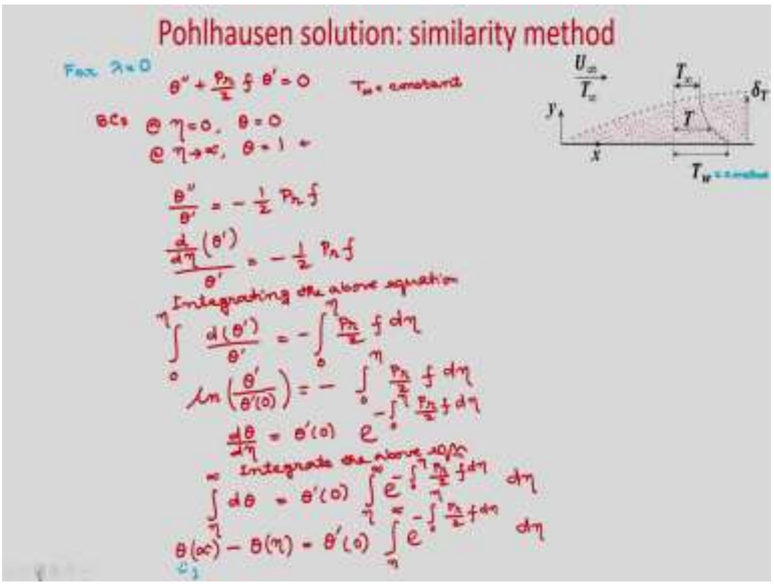
So, you see that this equation is ordinary differential equation. So, we started with the partial differential equation and we used similarity transformation and transferred this partial differential equation to ordinary differential equation for Prandtl number $\neq 1$ as well because it is a function of Prandtl number.

So that means, similarity solutions for this temperature distribution exist and we could get this second order linear ordinary differential equation. You can see this is second order and as f is known f'f and $f'$are known from the velocity distribution, so obviously, this is linear equation. So, you can see this is second order linear ordinary differential equation and this equation is known as Pohlhausen equation.

And you can see as a special case for uniform wall temperature you can put $\lambda = 0$. So, for λ=0 which is a special case T_w is constant for that you can write the Pohlhausen equation as $\theta'' + \frac{P_r}{2} f'\theta = 0$. So, you can see here, this is your linear equation because f is known from the velocity distribution and this is the second order linear ordinary differential equation.

So, this equation now you can solve using some numerical technique and find the temperature distribution and once you get the temperature distribution, you will be able to calculate the heat flux and from there you can calculate the heat transfer coefficient and Nusselt number.

(Refer Slide Time: 30:39)



Now, let us consider the special case when the flat plate is maintained at constant uniform temperature so; that means, T_w is constant and whatever Pohlhausen equation we got, now let us find the solution of this ordinary differential equation applying the boundary conditions.

So, for λ =0, here now T_w is constant we can write $\theta''+\frac{P_r}{2}f'\theta=0$ where T_w is constant. So, this is a special case for λ=0.

Now, what are the boundary conditions? At η=0 because y = 0 you will get η =0 and it is your T_w so that means, T_w - T_w = 0 so; that means, θ will be 0 and η → ∞ when y →∞ and x →0 these two boundary conditions are merged into 1 and for η →∞ you can write θ= 1.

So, now, you can integrate this equation so, you can write $\frac{\theta''}{\theta'}=-\frac{1}{2}P_r f$ so, what is θ''?

You can write $\frac{d}{d\eta}$ right; $\frac{\frac{d}{d\eta}(\theta')}{\theta'}=-\frac{1}{2}P_r f$.

So, integrating the above equation so, what we will get? So, $\int_0^\eta \frac{d(\theta')}{\theta'}=-\int_0^\eta \frac{P_r}{2}fd\eta$. So, if you integrate what you will get here? $\ln(\frac{(\theta')}{\theta'(0)})=-\int_0^\eta \frac{P_r}{2}fd\eta$.

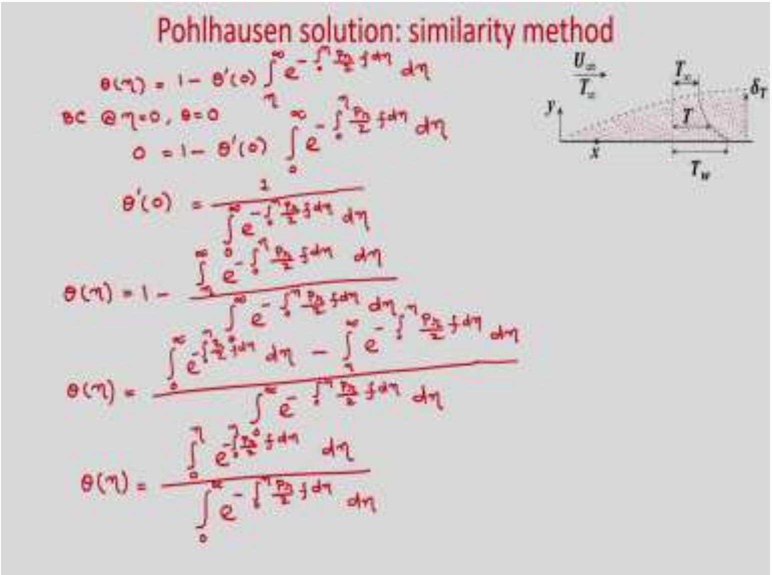
So, unless you know the velocity distribution f so, f' and f, f is the stream function equivalent to a stream function, you will not be able to integrate so, you are keeping it in terms of integral form. So, if you write this so, you can write $\frac{d\theta}{d\eta}=\theta'(0)e^{-\int_0^\eta \frac{P_r}{2}fd\eta}$.

So, integrate this above equation; integrate the above equation. So, we will integrate $\int_\eta^\infty d\theta=\theta'(0)\int_\eta^\infty e^{-\int_0^\eta \frac{P_r}{2}fd\eta}d\eta$.

Here, we will put the limit from η to ∞; η to ∞. So, if you put it, then what you will get?

θ at η = ∞ minus θ at η = $\theta'(0)$ and this is your η to ∞, $\int_{\eta}^{\infty} e^{-\int_0^{\eta}\frac{P_r}{2}fd\eta} d\eta$. So, now, you see θ at η →∞ we have the boundary condition θ = 1. So, you can put this is as 1. So, these value is 1.

(Refer Slide Time: 35:35)



So, now, θ (η) we express in terms of other terms. So, you can write $\theta(\eta) = 1-\theta'(0)\int_{\eta}^{\infty} e^{-\int_0^{\eta}\frac{P_r}{2}fd\eta} d\eta$.

So, now, we are left with another boundary condition at η =0, θ = 0. So, now, let us put that boundary condition. So, another boundary condition is there at η = 0, θ = 0. So, if you put that value so, $0 = 1-\theta'(0)\int_{0}^{\infty} e^{-\int_0^{\eta}\frac{P_r}{2}fd\eta} d\eta$.

Now, we can find the value of $\theta'(0)$. So, $\theta'(0) = \dfrac{1}{\int_{0}^{\infty} e^{-\int_0^{\eta}\frac{P_r}{2}fd\eta} d\eta}$. So, once $\theta'(0)$ is known so, now, we can find θ(η).

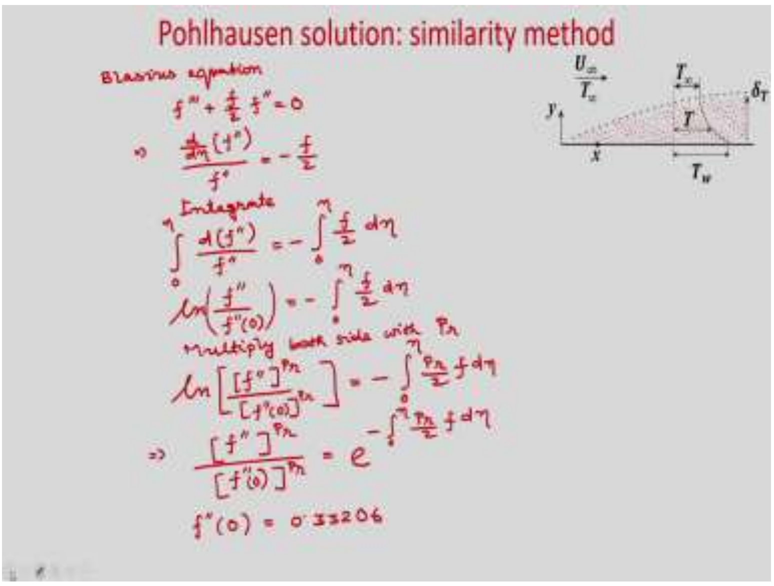
So, $\theta(\eta) = 1 - \dfrac{\int_{\eta}^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$. So, this is the temperature distribution. So, if you know the function f, then you will be able to integrate and you will be able to find the temperature distribution using some numerical technique you can solve this equation.

So, in this equation now you can see that if you write $\theta(\eta) = \dfrac{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta - \int_{\eta}^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$.

So, you can see another dη will be here another dη will be here.

So, now you can see this integrant is same in both the integral, but the limits are different 0 to ∞ and η to ∞. So, if you subtract that so obviously, you can get 0 to η. So, θ(η) $\theta(\eta) = \dfrac{\int_0^{\eta} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}{\int_0^{\infty} e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta} d\eta}$.

(Refer Slide Time: 39:47)



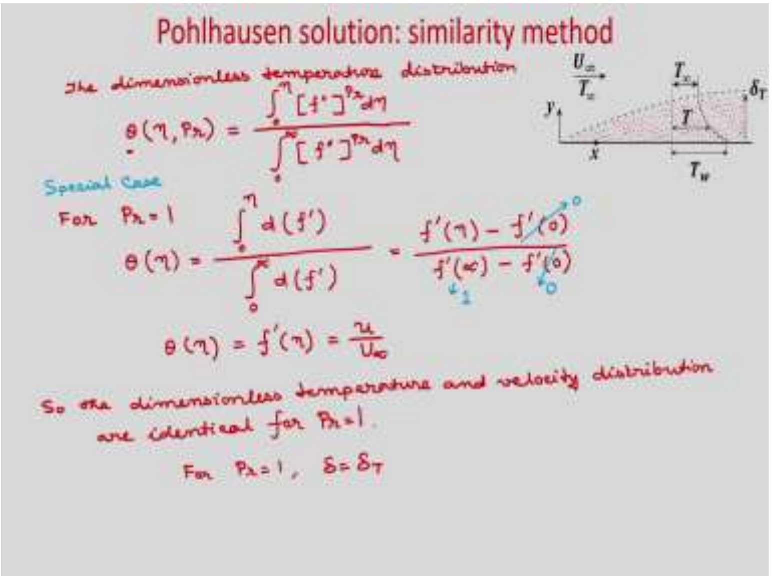
So, this expression also we can write in terms of f''. So, we will start with the Blasius equation. So, whatever we have derived in the last class so, you can see Blasius equation that is your $f''' + \frac{f}{2} f'' = 0$. So, you can write $\frac{\frac{d}{d\eta}(f'')}{f''} = -\frac{f}{2}$. So, integrate this equation so, you will get $\int_0^\eta \frac{d(f'')}{f''} = -\int_0^\eta \frac{f}{2} d\eta$.

So, now, you can see that you will get $\ln(\frac{f''}{f''(0)}) = -\int_0^\eta \frac{f}{2} d\eta$. Now multiply both sides with Prandtl numbers; multiply both side with Prandtl number. So, if you see here if you multiply Prandtl number so, you can write this as $\ln(\frac{[f'']^{P_r}}{[f''(0)]^{P_r}}) = -\int_0^\eta \frac{f}{2} d\eta$.

So, from here, you can write $\frac{[f'']^{P_r}}{[f''(0)]^{P_r}} = e^{-\int_0^\eta \frac{P_r}{2} f d\eta}$.

So, if you see in the last expression whatever you have $e^{-\int_0^\eta \frac{P_r}{2} f d\eta}$ so, that expression we have written in terms of f''. So, $f''(0)$ you know right because $f''(0)$ we have found from the velocity distribution so, this is the velocity gradient at η = 0 and that value is 0.33206.

(Refer Slide Time: 42:29)



So, now, if you put this in this expression; in this expression so, what you will get? You will get so, the dimensionless temperature; dimensionless temperature distribution you will get $\theta(\eta, P_r) = \dfrac{\int_0^{\eta} [f'']^{P_r} d\eta}{\int_0^{\infty} [f'']^{P_r} d\eta}$.

So, you can see here in this expression so obviously, $[f''(0)]^{P_r}$ is there and so, whatever expression we have got here $e^{-\int_0^{\eta} \frac{P_r}{2} f d\eta}$ so, that expression you put in the expression here in this expression and you just rearrange it finally, you can write the temperature distribution as the dimensionless temperature distribution.

So, after rearrangement you will get θ which is function of η and Prandtl number is equal to $\dfrac{\int_0^{\eta} [f'']^{P_r} d\eta}{\int_0^{\infty} [f'']^{P_r} d\eta}$. So, after doing some rearrangement, you will get this as the final temperature distribution and you can see if you know the value of f'' from the velocity distribution, you will be able to calculate the temperature distribution. So, this equation you can solve numerically.

So, now, let us see that as a special case whatever we started with that for Prandtl number =1 what is the temperature distribution is it same as the velocity distribution let us see. So, for Prandtl number =1 this is a special case. So, for Prandtl number= 1 you can see what you can write θ(η) for Prandtl number =1 is equal to 0 to η.

So, Prandtl number =1. So, you can write $\theta(\eta) = \dfrac{\int_0^{\eta} d(f')}{\int_0^{\infty} d(f')} = \dfrac{f'(\eta) - f'(0)}{f'(\infty) - f'(0)}$.

So, now what is f prime? f' is the velocity $\dfrac{u}{U_\infty}$ and at η = 0 $f'(0)$. So, you can see this is your 0 and this is also 0 and what is $f'(\eta) \to \infty$? So, that is your 1 because u →U_∞. So, this is your 1. So, you can see that from this expression that θ (η) = $f'(\eta)$ and what is f prime? It is nothing, but $\frac{u}{U_\infty}$.

So, the dimensionless temperature and velocity distribution are identical for Prandtl number = 1 that means for Prandtl number = 1, you have $\delta = \delta_T$. So, it will be the non-dimensional temperature distribution will be same as non-dimensional velocity distribution.

So, in today's class, we have started with the energy equation and we defined one non-dimensional temperature $\theta = \frac{T - T_w}{T_w - T_\infty}$. We have assumed in the beginning that T_w is function of x and we have derived the Pohlhausen equation in general which is your second order linear ordinary differential equation and we have shown that using similarity variable approach, we could convert the PDE to ODE for any Prandtl number.

For a special case, λ = 0, the wall temperature becomes constant and for that we have found the solution for temperature distribution and this temperature distribution we have expressed in terms of your velocity gradient f'' once you know the velocity distribution from the solution of Blasius equation you will be able to find the temperature distribution.

So, to you can solve this equation using some numerical technique and at last we have shown that for Prandtl number =1, the temperature distribution and velocity distribution are identical and you can write that $\theta = \frac{u}{U_\infty}$

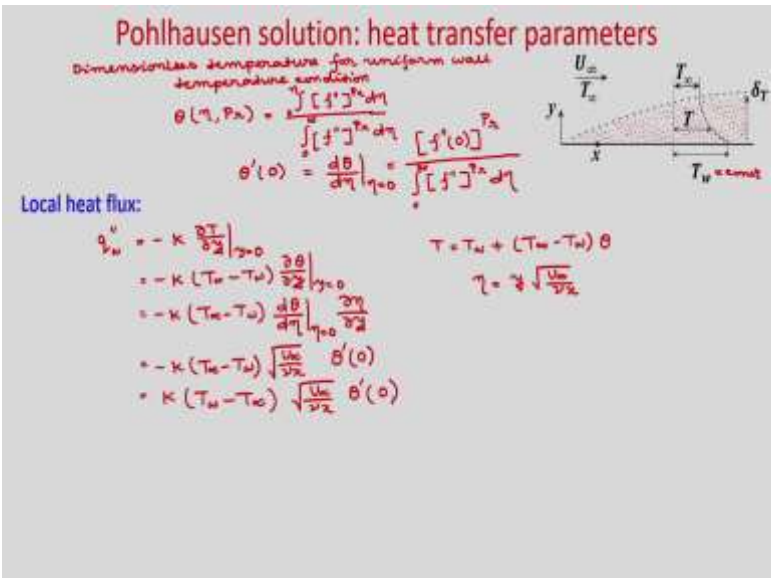
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 03
Convective Heat Transfer in External Flows - I
Lecture – 08
Pohlhausen Solution: Heat Transfer Parameters

Hello, everyone. So, in the last lecture, we have derived the Pohlhausen equation starting from the energy equation. Today, we will find the different Heat Transfer Parameters like heat flux, heat transfer coefficient, Nusselt number, from the solution of Pohlhausen equation.

(Refer Slide Time: 00:53)



So, you can see that dimensionless temperature we have derived in last class, temperature for uniform wall temperature condition; that means, T_w is constant θ which is your dimensionless temperature which is function of η and Prandtl number, $\theta(\eta, \Pr) = \dfrac{\int_0^{\eta} [f'']^{\Pr} d\eta}{\int_0^{\infty} [f'']^{\Pr} d\eta}$.

And the derivative of θ we have also derived and that η = 0, the expression is, $\theta'(0) = \frac{d\theta}{d\eta}|_{\eta=0} = \frac{[f''(0)]^{\text{Pr}}}{\int_0^\infty [f'']^{\text{Pr}} d\eta}$.

So, once you know the temperature distribution from the Pohlhausen solution then you will be able to calculate the local heat flux as well as local heat transfer coefficient and local Nusselt number and you need the value of $\frac{d\theta}{d\eta}|_{\eta=0}$; that means, the temperature gradient at the wall to find the heat flux. So, first let us find what is the local heat flux.

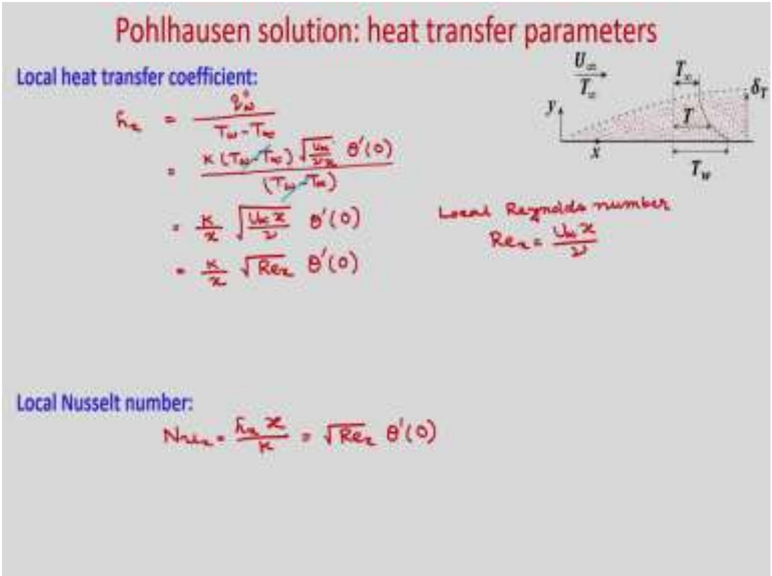
So, you know by definition, $q_w^{"} = -K\frac{\partial T}{\partial y}|_{y=0}$. So, at this wall the temperature gradient $\frac{\partial T}{\partial y}$ and that is coming from Fourier's law of heat conduction. And we have taken the dimensionless temperature such a way that $T = T_w + (T_\infty - T_w)\,\theta$. And the similarity, variable $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$.

So, now, you can see that this we can write - K$\frac{\partial T}{\partial y}$. So, T_w is constant in this case. So, you can write $-K(T_\infty - T_w)\frac{\partial\theta}{\partial y}|_{y=0} = -K(T_\infty - T_w)\frac{\partial\theta}{\partial\eta}|_{\eta=0}\frac{\partial\eta}{\partial y}$

So, $\frac{\partial\eta}{\partial y}$ you can see it is $\sqrt{\frac{U_\infty}{\nu x}}$. So, you can write $-K(T_\infty - T_w)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$. So, this you can write this minus you can take it inside. So, you can write, $K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$.

So, we have found the local heat flux. Now, we will calculate the local heat transfer coefficient. So, local heat transfer coefficient you know from the equation from Newton's law of cooling you will be able to calculate.

(Refer Slide Time: 04:59)



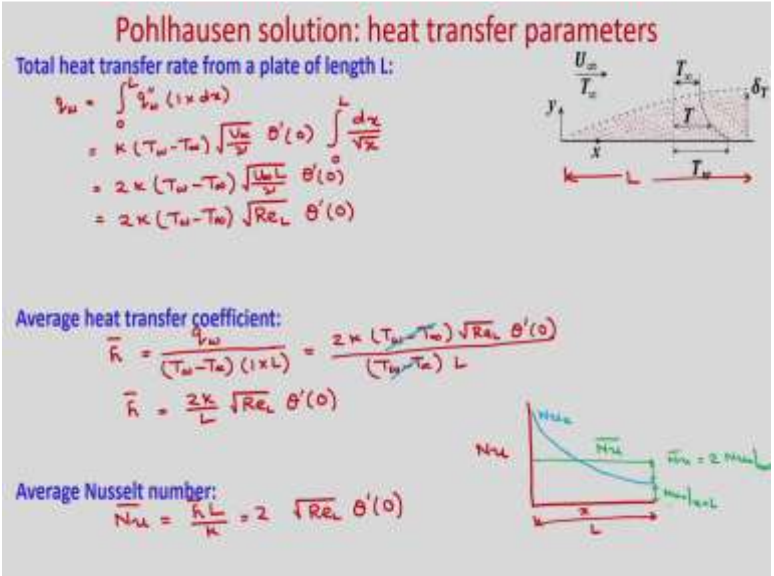
So, local heat transfer coefficient $h_x = \frac{q_w''}{T_w - T_\infty}$ and this we are writing from the Newtons law of cooling. So, q_w'', just we have calculated. So, this is your, $\frac{K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)}{(T_w - T_\infty)}$.

So, this you can cancel so you will get we will just write $\frac{K}{x}$ and we will multiply in the numerator x and we will take inside this inside this root. So, it will be x^2. So, you will get $\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$. What is $\frac{U_\infty}{\nu x}$?

So, you know the Reynolds number definition right. So, this will be local Reynolds number at any location x. So, local Reynolds number R_{e_x}, you can write $\frac{U_\infty}{\nu x}$. So, this now you can write $\frac{K}{x}\sqrt{R_{e_x}}\theta'(0)$.

Now, once you know the local heat transfer coefficient you can calculate the local Nusselt number because you know the definition of local heat transfer as $\frac{hx}{K}$. So, local Nusselt number is definition by $\frac{h_x x}{K}$. So, you can see this x $\frac{x}{K}$ if you take this side. So, this will be left with $\sqrt{R_{e_x}}\theta'(0)$. So, once you can find the derivative $\theta'(0)$, then you will be able to find the value of local heat transfer coefficient and local Nusselt number.

(Refer Slide Time: 07:35)



Now, let us calculate the total heat transfer from the plate of length L and we will calculate it for unit width. So, total heat transfer rate from a plate of length L you can calculate q_w. So, just local heat flux you have calculated into area. So, per unit width so, it will be, $q_w = \int_0^L q_w''(1dx)$. L is the length of the plate.

So, if you put the expression of q_w''. So, the constant you can take it outside the integral. So, it will be $K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu}}\theta'(0)\int_0^L \frac{dx}{\sqrt{x}}$. So, if you integrate it you will get twice $K(T_w - T_\infty)$.

And if you put the limit L then you will get $\sqrt{L}$ in the numerator. So, that we will take $2K(T_w - T_\infty)\sqrt{\frac{U_\infty L}{\nu}}\theta'(0)$. So, from here you can see $2K(T_w - T_\infty)$ and what it is? It is Reynolds number right $\frac{U_\infty L}{\nu}$ so, based on the plate length. So, $\sqrt{R_{e_L}}\theta'(0)$. Once you know the total heat transfer rate you will be able to calculate the average heat transfer coefficient.

So, average heat transfer coefficient we will calculate $\bar{h} = \frac{q_w}{(T_w - T_\infty)(1.L)}$, right this is Newton's law of cooling. So, per unit to it so, 1xL. So, if you put the value so, you will get $\frac{2K(T_w - T_\infty)\sqrt{R_{e_L}}\theta'(0)}{(T_w - T_\infty)L}$.

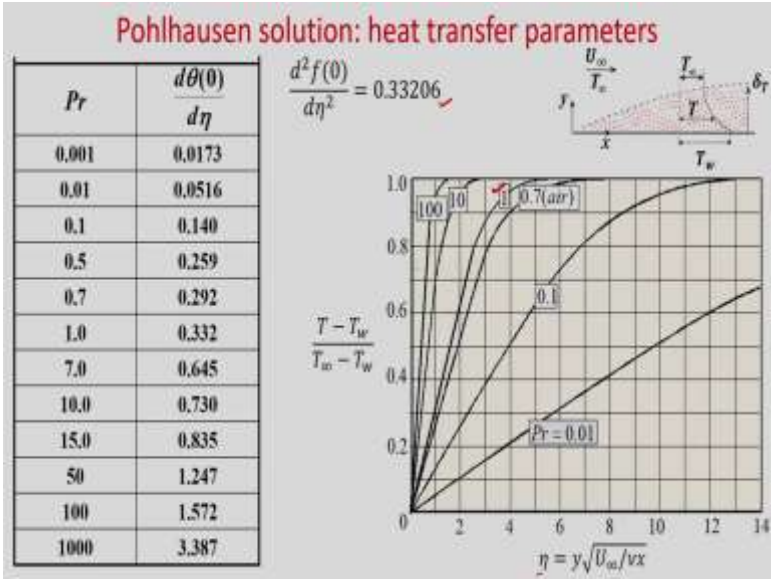
So, this you cancel. So, you will get, $\bar{h} = \frac{2K}{L}\sqrt{R_{e_L}}\theta'(0)$. From here average Nusselt number you will calculate. So, average Nusselt number $\overline{N_u} = \frac{\bar{h}L}{K}$. So, from this expression you can see it will be $2\sqrt{R_{e_L}}\theta'(0)$.

So, all these expression we have written in terms of $\theta'(0)$. So, $\theta'(0)$ is still unknown because you need to find it from the temperature distribution. Here you notice the local heat transfer coefficient and average heat transfer coefficient. So, you can see that your average heat transfer coefficient is double of the local heat transfer coefficient at x = L. And similarly, average heat transfer coefficient is twice of the local Nusselt number at x= L.

So, if you see, let us say this is the Nusselt number distribution with length L. So, this is the plate length L. So, how it varies? So, let us say your local Nusselt number varies like this. So, if this is the local Nusselt number variation, then your average Nusselt number will be twice at x =L. So, this will be the average Nusselt number it is average Nusselt number and you can see it is value is double of this.

So, what is this value? So, this is your Nusselt number x at x = L. And this $\overline{N_u} = 2N_{u_x}|_{x=0}$. So, this is your double, so this is the same distance. So, it will be the same value, so obviously, it will be the $\overline{N_u} = 2N_{u_x}|_{x=0}$.

(Refer Slide Time: 12:37)

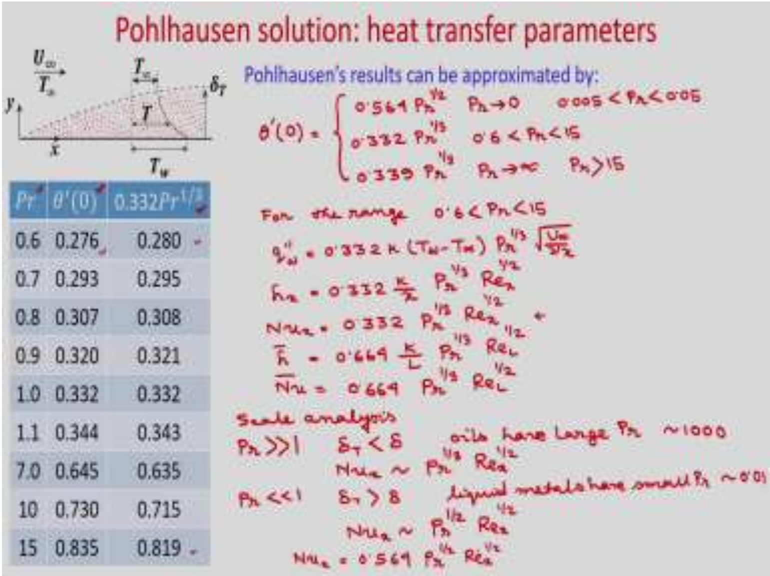


So, in this slide we are showing the temperature gradient at η = 0 for different Prandtl number. So, this has been evaluated numerically for a range of Prandtl numbers by Pohlhausen. That the numerical solutions and $\frac{d^2 f}{d\eta^2}$ = 0, already we have found from the Blasius solution.

And if you see the temperature distribution θ which is $\frac{T-T_w}{T_\infty - T_w}$ versus η. So obviously, at different x location all the temperature profile falls in the same curve, but it varies for different Prandtl numbers.

So, you can see Prandtl number = 1. So, this is the case where Prandtl number = 1. So, in this profile temperature profile will be same as the velocity profile, that we have already discussed. And for other Prandtl number you can see how it varies with η. So, at different Prandtl numbers.

(Refer Slide Time: 13:43)



Pr	$\theta'(0)$	$0.332Pr^{1/3}$
0.6	0.276	0.280
0.7	0.293	0.295
0.8	0.307	0.308
0.9	0.320	0.321
1.0	0.332	0.332
1.1	0.344	0.343
7.0	0.645	0.635
10	0.730	0.715
15	0.835	0.819

So, now, whatever results we have shown Pohlhausen's results now can be approximated by for different Prandtl number, this is the numerical value of temperature gradient at η = 0. So, this actually is approximated by Pohlhausen as $0.332\,\mathrm{Pr}^{1/3}$. So, you can see if you see 0.6. So, if you put 0.6 Prandtl number here so $0.6^{1/3}\,X\,0.332 = 0.280$.

And for different Prandtl number you can see that almost this is comparable. So, this is the numerical solution and this is approximate, $0.332\,\mathrm{Pr}^{1/3}$. So, 0.6 to 15 we have shown here and in this range so, $\theta'(0) = 0.332\,\mathrm{Pr}^{1/3}$.

So, you can approximate the first derivative of dimensionless temperature at η = 0 as $0.564\,\mathrm{Pr}^{1/2}$. For other range I am writing where Prandtl number →0. Generally, it will be valid in the range of 0.005 < Pr < 0.005. Then in this range 0.6 to 15 this is the approximation.

So, $0.332\,\mathrm{Pr}^{1/3}$ in the range of 0.6 and 15 and it will be $0.339\,\mathrm{Pr}^{1/3}$ for high Prandtl number. So, generally it is Prandtl number greater than 15. So, now, you got some approximate value of $\theta'(0)$ in terms of Prandtl number. So, from the Pohlhausen numerical solutions that is approximated in the power of Prandtl number.

So, for the range 0.6 and 15 you can write $q_w^"$ because we have already found the expression. So, if we put the value of $\theta"(0) = 0.332\,\mathrm{Pr}^{1/3}$. So, you can write as, $q_w^" = 0.332K(T_w - T_\infty)\,\mathrm{Pr}^{1/3}\sqrt{\frac{U_\infty}{\nu x}}$.

Local heat transfer coefficient you can write $h_x = 0.332\frac{K}{x}\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}$. So, local Nusselt number you can write $Nu_x = 0.332\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}$. And average heat transfer coefficient it will be twice of h_x at x = L. So, it will be $\bar{h} = 0.664\frac{K}{L}\mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2}$. And, average Nusselt number will be $\overline{Nu} = 0.664\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2}$.

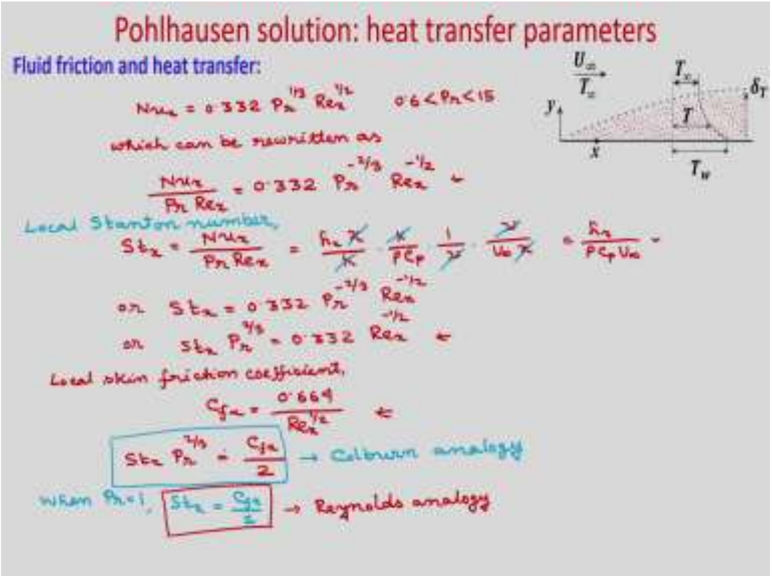
You can see that already we have seen this scale of this Nusselt number and the heat transfer coefficient using that scale analysis. So, if you recall using scale analysis we have written for high Prandtl number fluids, high Prandtl number of fluids; that means, your thermal boundary layer thickness will be less than hydro dynamic boundary layer thickness and, generally oils have large Prandtl number of the order of 1000.

So, here if you recall we have written $Nu_x \sim \mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}$. So, you can see here same order with a value 0.332. Similarly, for low Prandtl number fluids $\delta_T > \delta$ generally, liquid metals have small Prandtl number of the order of 0.01.

So, if you recall, we have found $Nu_x \sim \mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}$. So, you can see the power of Prandtl number for high Prandtl number fluids the power is 1/3 and low Prandtl number of fluids the power of Prandtl number is 1/2. So, that we have already found from the scale analysis.

And from this numerical solution you can see that for low Prandtl number fluids, Prandtl number →0 your $\theta'(0) = 0.564\,\mathrm{Pr}^{1/2}$. So, if you write the local Nusselt number for low Prandtl number fluids then, $Nu_x = 0.564\,\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2}$.

(Refer Slide Time: 20:01)



Now, we will discuss about the fluid friction and heat transfer relation. So, you can see that we have found the local Nusselt number $Nu_x = 0.332\,\text{Pr}^{1/3}\,\text{Re}_x^{1/2}$ in the range of Prandtl number 0.6 and 15. Now, this we can rewrite as, which can be rewritten as $\frac{Nu_x}{\text{Pr}\,\text{Re}_x}$. So, if you divide it and you will get in the right-hand side as $0.332\,\text{Pr}^{-2/3}\,\text{Re}_x^{-1/2}$.

So, you can see left hand side is the dimensionless group right. So, this is called local Stanton number. So, the left-hand side it is Stanton number. So, this is $\frac{Nu_x}{\text{Pr}\,\text{Re}_x}$. So, this is known as local Stanton number. So, from this relation you can see that $Nu_x = \frac{h_x x}{K}$.

Prandtl number is $\frac{\nu}{\alpha}$. So, you can write $\alpha = \frac{K}{\rho C_p}$ and $\frac{1}{\nu}$ and Reynolds number. So, $\frac{U_\infty x}{\nu}$; so from here you can rearrange it. So, you can write it as $\frac{h_x}{\rho C_p U_\infty}$. So, you can see the local Stanton number is given by this relation $\frac{h_x}{\rho C_p U_\infty}$.

So, now you can write $St_x = 0.332\,\text{Pr}^{-2/3}\,\text{Re}_x^{-1/2}$ or you can write $St_x\,\text{Pr}^{2/3} = 0.332\,\text{Re}_x^{-1/2}$.

From the Blasius solution, we have found the local skin friction coefficient. So, if you write down the expression for local skin friction coefficient which is known as friction coefficient because for flow over flat plate the friction is the dominant drag due to sheer stress.

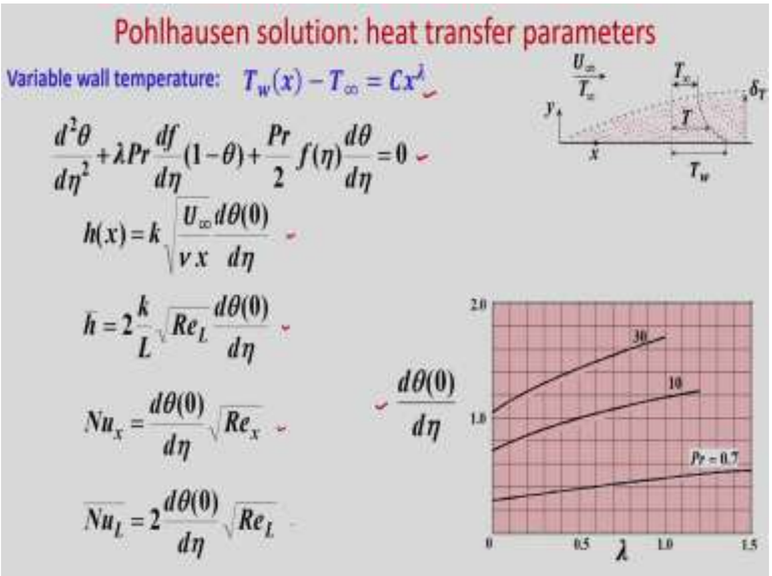
So, you can write $C_{f_x} = \frac{0.664}{Re_x^{1/2}}$. So, you can see from this expression and this expression you can write $St_x \Pr^{2/3} = \frac{C_{f_x}}{2}$. So, this relation is known as Colburn analogy. So, this relation is known as Colburn analogy.

So, this is between the fluid friction and heat transfer for laminar flow on a flat plate you can write this expression. So, what is the advantage of using this analogy because, if you know the friction coefficient then you will be able to calculate the heat transfer right from this expression.

Now, when Prandtl number = 1; so this expression will become $St_x = \frac{C_{f_x}}{2}$ and this expression is known as Reynolds analogy. So, you can see you can calculate local heat transfer coefficient, when local friction coefficient is known on a flat plate under the conditions in which no heat transfer is involved.

So, now, let us consider the variable wall temperature what we started during the derivation and we have shown from the analysis that T_w varies as $T_w = T_\infty + Cx^\lambda$ C x to the power λ where C is the constant.

(Refer Slide Time: 25:49)



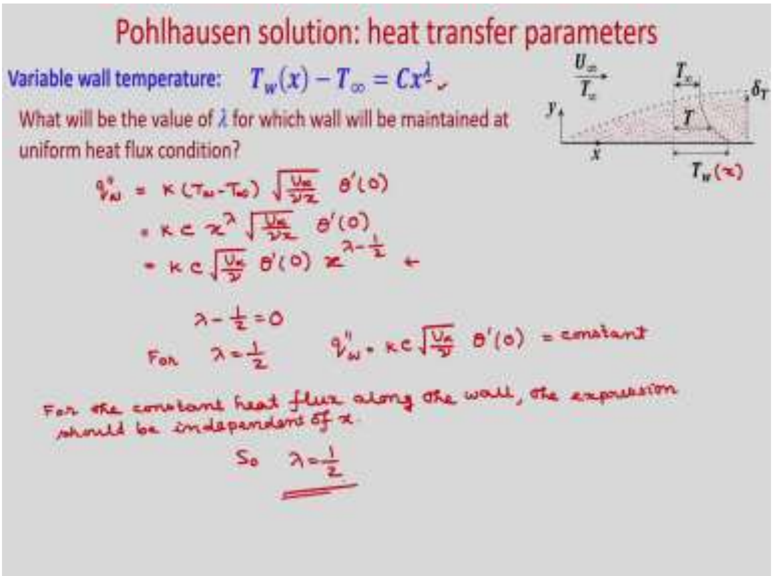
So, you can see variable wall temperature $T_w(x) - T_\infty = Cx^\lambda$, and this is the Pohlhausen equation right. So, this is the ordinary differential equation. So, for a special case we put λ = 0, where wall temperature become constant; so we drop this term.

So, from this equation if you numerically solve then if you can calculate the nondimensional temperature derivative at η = 0; that means, at the wall for different value of λ then for different Prandtl number this is the variation. So, you can see this is the dimensionless temperature gradient variation at the wall, for different value of λ, and at different Prandtl number. So, you know that λ = 0, these denotes for flow over flat plate with uniform wall temperature case.

So, by numerical techniques if you can solve then you can plot this and once you know this value then the same expression for local heat transfer coefficient, average heat transfer coefficient, local Nusselt number, and average Nusselt number you will be able to calculate.

Now, when you consider variable wall temperature, then can you find the value of λ for which the wall will be maintained that constant heat flux condition.

(Refer Slide Time: 27:25)



So, this is your variable wall temperature T_w is function of x. Now, what will be the value of λ, for which wall will be maintained at uniform heat flux condition? So, we have to find what is the value of λ? So, the heat flux local heat flux you can write, $q_w^{"} = K(T_w - T_\infty)\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$.

So, this expression just in previous slide we have derived now, you can see $T_w - T_\infty$, so you have T_W function of x so, $T_w - T_\infty$ you can put Cx^λ. So, you can put $KCx^\lambda\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$.

So, now, you can write it as $KC\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)x^{\lambda-\frac{1}{2}}$ because here in the denominator you have $x^{1/2}$. So, here x^λ so, $x^{\lambda-\frac{1}{2}}$; so this is the expression for local heat flux right. So, now, to maintain at uniform wall heat flux condition it should be independent of x right, then only the wall will be maintained at constant heat flux condition.

So, you can see here K is the thermal conductivity that is constant, C is the integration constant, U_∞ free steam velocity is constant. This is your fluid kinematic viscosity that is also constant; $\theta'(0)$ which is your temperature gradient at η = 0 so, that having some numerical value. So, that is also constant, but it varies with $x^{\lambda-\frac{1}{2}}$.

So, these terms would be 1 right. Then only it will be independent of x. When it will be 1? When $\lambda - \frac{1}{2}$ will be 0; so that means, $\lambda - \frac{1}{2}$ will be 0 then it will be x^0; that means, 1. So, it will become independent of x. So, λ = 1/2.

So, for λ =1/2 what will be $q_w^{"}$. So, you can see this will be 1. So, it will be $KC\sqrt{\frac{U_\infty}{\nu x}}\theta'(0)$ which is constant. So, for the constant heat flux along the wall, the expression should be independent of x; so λ will be 1/2.

So, in today's class we have found the local heat transfer coefficient, average heat transfer coefficient, local Nusselt number, average Nusselt number, in terms of temperature gradient at η=0. Then later we have shown the numerical solution of Pohlhausen equation, and there we have tabulated the value of $\theta'(0)$ for different Prandtl number. Then Pohlhausen approximated this temperature gradient at the wall with

Prandtl number relation and we have shown that for the range of Prandtl number between 0.6 and 15.

The Nusselt number varies with $\mathrm{Pr}^{1/3}$ and $\mathrm{Re}^{1/2}$. The same we have shown earlier from the scale analysis as well. Later we have defined the Stanton number and from there we have shown the Colburn analogy as well as the Reynolds analogy. So, this can be used to find the heat transfer coefficient if you know the friction coefficient.

Then, for variable temperature boundary condition we have shown the numerical solution and later we have found the value of λ for which your flat plate wall will be maintained at uniform wall heat flux condition and the value of λ is 1/2.

Thank you.

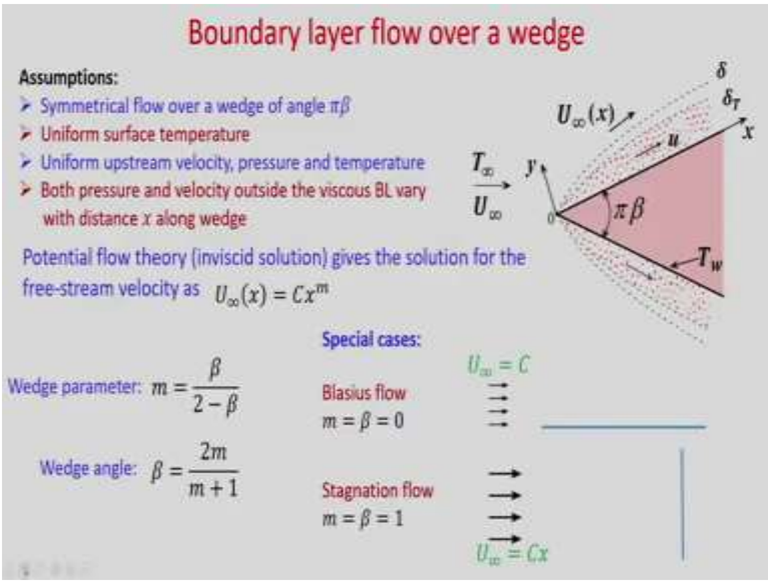
Fundamentals of Convective Heat Transfer

Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 03
Convective Heat Transfer in External Flows – I
Lecture - 09
Falkner-Skan equation: Boundary layer flow over a wedge

Hello everyone. So, till now we have considered flow over flat plate, with constant or variable temperature boundary condition. In those cases, pressure gradient was 0; right, $\frac{dp}{dx}$ is 0 for flow over flat plate because your free stream velocity U_∞ was constant. Today, we will consider Boundary layer flow over a wedge, where U_∞ is function of x. So, for potential flow you know outside the boundary layer velocity will vary as Cx^m, where m is wedge parameter.

(Refer Slide Time: 01:15)



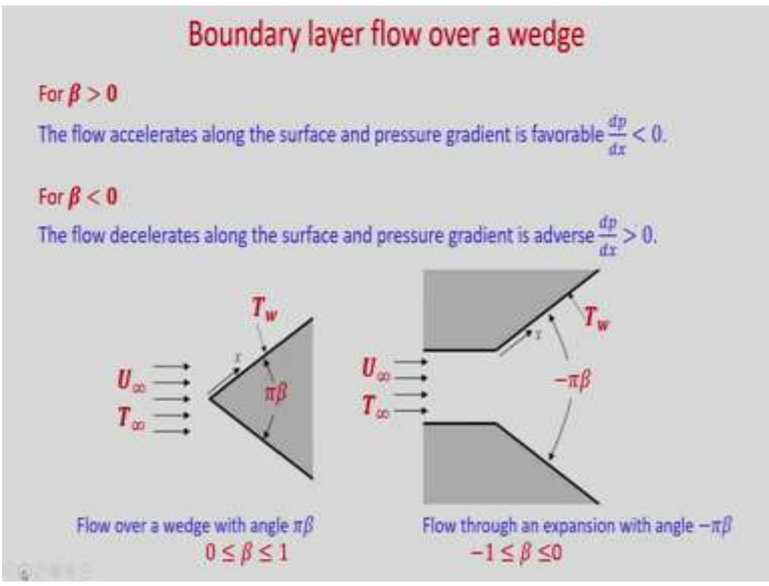
So, let us consider laminar boundary layer flow over a wedge. This is the case. So, you can see this is the wedge, with the wedge angle π β and your free stream temperature is T_∞ and free stream velocity is U_∞, x is measured along the surface and y is normal to the surface. So, obviously, you can see when fluid will flow over this wedge, it will be accelerating; the fluid velocity will accelerate.

So, obviously U_∞ will be function of x. So, these are the assumptions we will consider symmetrical flow over a wedge of angle πβ. Uniform surface temperature; so, we will consider T_w as wall temperature as constant. Uniform upstream velocity, pressure and temperature. And, both pressure and velocity outside the viscous boundary layer vary with distance x along wedge. So, potential flow theory gives the solution for the free stream velocity as $U_\infty(x) = Cx^m$, where m is the wedge parameter and C is constant.

The relation between these wedge parameter m and wedge angle β is given here. So, you can see wedge parameter m, you can write as $m = \frac{\beta}{2-\beta}$ and wedge angle you can write in terms of the wedge parameter $\beta = \frac{2m}{m+1}$. Now, you can see that as a special case, if you put β = 0 m will become 0. And, what will be the flow situation? If β becomes 0, so it will be flow over flat plate.

So, you see if this wedge angle β, if you put 0, then obviously, it will become flow over flat plate. And, if β is 0; that means, m will be 0 and if m is 0, so U_∞ will be constant. So, it will be a Blasius flow over a flat plate. And, if β = 1, so that will means it will become π, so it will be a vertical plate. So, if flow occurs then it is known as stagnation flow and for β = 1, m will be 1 and if you put m = 1 here, you can see $U_\infty = Cx$, so it is a stagnation flow.

(Refer Slide Time: 04:14)



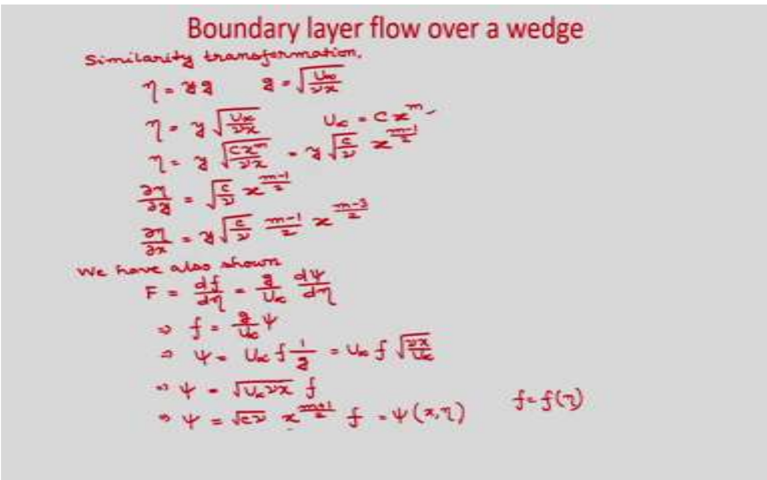
So, here you can see two different situations for $\beta > 0$. So, in this case you can consider that, flow over a wedge with angle $\pi\ \beta$ and β varies between 0 and 1, then the flow accelerates along the surface and pressure gradient is favourable. Because $\frac{dp}{dx} < 0$, so pressure gradient will be favourable and flow will accelerate.

For $\beta < 0$, so if you consider flow through an expansion with angle $-\pi\ \beta$. So, β will vary between -1 and 0, so you can consider this expansion. So, it is kind of diffuser. So, fluid is entering with velocity U_∞, once it is comes here, so you can see it is kind of diverging. So, the wall temperature is T_w and x is measured along this wedge surface. And, in this particular case, you can see the flow decelerates along the surface and pressure gradient is adverse that means, $\frac{dp}{dx} > 0$.

And, in this situation, it may happen that there will be a flow separation, and if flow separates boundary layer flow theory will not be valid. So, we will see that, at which wedge angle β flow separates? And, how do you know that flow separation has happened? When you will see the shear stress τ_w or the velocity gradient will become 0, so that time you will know that the, at that point, flow separates. So, after that boundary layer flow theory will not be valid.

So, let us consider laminar boundary layer flow over a wedge and we will use the similarity transformation technique similar to what we have done for solving the boundary layer equation for a flow over flat plate. We will use that, similarity variable as well as the velocity distribution.

(Refer Slide Time: 06:44)



So, you can see what we derived in similarity transformation approach, similarity transformation. So, we have used the similarity variable $\eta = yg$ and g is function of x, and $g = \sqrt{\frac{U_\infty}{\nu x}}$.

So, you can see from our earlier lecture that, we have derived g which is function of x as, $g = \sqrt{\frac{U_\infty}{\nu x}}$. So, you can write $\eta = y\sqrt{\frac{U_\infty}{\nu x}}$. And, what is the U_∞? U_∞ is your free stream velocity and we have considered it as $U_\infty = Cx^m$, where m is the wedge parameter.

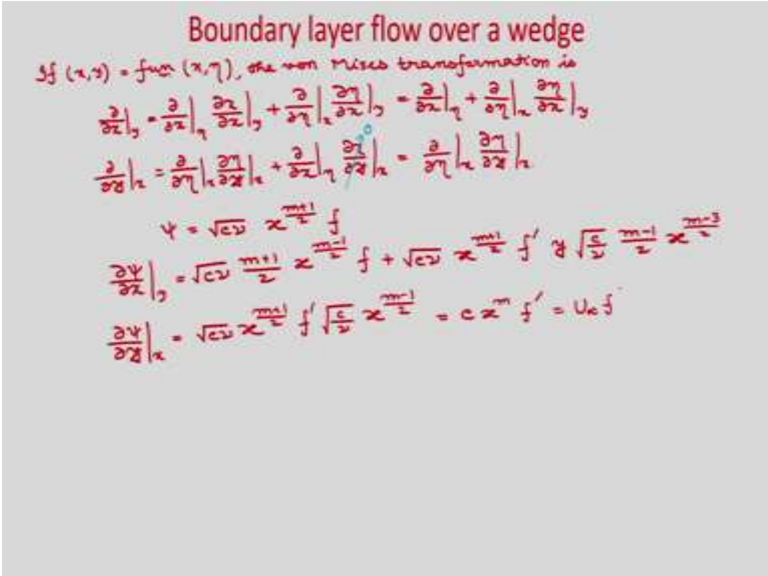
So, here if you put Cx^m then what you will get η? η you will get as $\eta = y\sqrt{\frac{Cx^m}{\nu x}}$ and you can write as $y\sqrt{\frac{C}{\nu}}x^{\frac{m-1}{2}}$. Now, let us take the derivative of η with respect to y as well as x. So, what you will get? $\frac{\partial \eta}{\partial x} = \sqrt{\frac{C}{\nu}}x^{\frac{m-1}{2}}$ and $\frac{\partial \eta}{\partial x} = y\sqrt{\frac{C}{\nu}}\frac{m-1}{2}x^{\frac{m-3}{2}}$.

So, now, let us write down the expression for f', whatever we have derived in earlier lecture. So, we have also shown $F = \frac{df}{d\eta} = \frac{g}{U_\infty}\frac{d\psi}{d\eta}$. So, where Ψ is the stream function and we have already told that f is having the physical significance of stream function.

So, from here you can write $f = \frac{g}{U_\infty}\psi$ equivalent to that. And, you can write $\psi = U_\infty f\frac{1}{g}$. And, what is g? $g = \sqrt{\frac{U_\infty}{\nu x}}$. So, you can write $\psi = U_\infty f\sqrt{\frac{\nu x}{U_\infty}}$. So, from here you can write $\psi = \sqrt{U_\infty \nu x}\, f$ and $U_\infty = Cx^m$.

So, if you put it here, so you will get $\psi = \sqrt{C\nu}\,x^{\frac{m+1}{2}}f$. So, you can see Ψ(x,η) because f is function of η we know, right. And, also you can see Ψ is function of x. So, it will be Ψ(x,η).

(Refer Slide Time: 11:04)



So, now let us write the von Mises transformation. So, what is von Mises transformation? So, if from any book you can see this von Mises transformation, if (x,y) = func (x,η), the von Mises transformation you can write as, $\frac{\partial}{\partial x}\Big|_y = \frac{\partial}{\partial x}\Big|_\eta \frac{\partial x}{\partial x}\Big|_y + \frac{\partial}{\partial \eta}\Big|_x \frac{\partial \eta}{\partial x}\Big|_y = \frac{\partial}{\partial x}\Big|_\eta + \frac{\partial}{\partial \eta}\Big|_x \frac{\partial \eta}{\partial x}\Big|_y$.

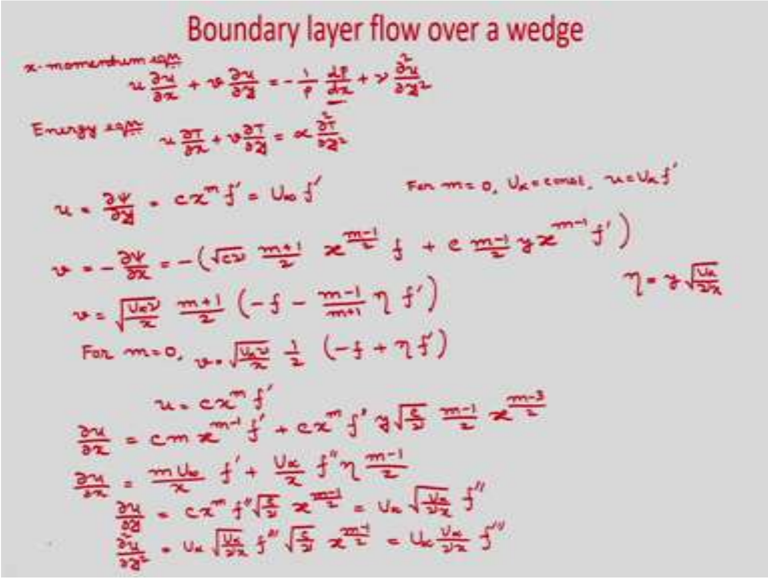
Similarly, you can write $\frac{\partial}{\partial y}\Big|_x = \frac{\partial}{\partial \eta}\Big|_x \frac{\partial \eta}{\partial y}\Big|_x + \frac{\partial}{\partial x}\Big|_\eta \frac{\partial x}{\partial y}\Big|_x = \frac{\partial}{\partial \eta}\Big|_x \frac{\partial \eta}{\partial y}\Big|_x$.

So, from here now, $\psi = \sqrt{C\nu}\, x^{\frac{m+1}{2}} f$.

So, from here you can write $\frac{\partial \psi}{\partial x}\Big|_y = \sqrt{C\nu}\,\frac{m+1}{2} x^{\frac{m-1}{2}} f + \sqrt{C\nu}\, x^{\frac{m+1}{2}} f'\, y\sqrt{\frac{C}{\nu}}\,\frac{m-1}{2} x^{\frac{m-3}{2}}$

Similarly, $\frac{\partial \psi}{\partial y}\Big|_x = \sqrt{C\nu}\, x^{\frac{m+1}{2}} f' \sqrt{\frac{C}{\nu}} x^{\frac{m-1}{2}}$. So, what you can write from here? So, you can see it will be $Cx^m f'$. And, it is nothing but, $U_\infty f'$. So, now, let us write the governing equations for flow over wedge.

(Refer Slide Time: 15:16)



So, we will have the assumptions steady laminar flow, so obviously, you can write the momentum equation as $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$. Now, in this case you will have pressure gradient. So, it will be $-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$. So, this is your x momentum equation.

And, what is the energy equation? Energy equation will remain same, neglecting the viscous distribution you can write $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$. So, these are boundary layer equations, right. So, in this case as pressure gradient is not 0, so we have written $-\frac{1}{\rho}\frac{dp}{dx}$.

Now, we have found the stream function and its gradient with respect to x and y. So, you will be able to find the value of u, v and its gradient. So, first let us write u. So, $u=\frac{\partial \psi}{\partial y}$. So, already we have found it. So, it will be just $Cx^m f'$ and it is same as whatever we have written for the Blasius solution. So, for flow over flat plate we have already defined $f'=\frac{u}{U_\infty}$. So, from here you can write the x direction velocity $u=U_\infty f'$. And, for m = 0 obviously, flow over flat plate, U_∞ is constant and $u=U_\infty f'$ we have already shown.

Now, the velocity v you can write as $v = -\frac{\partial \psi}{\partial x}$, and that we have found already. So, if you see it will be $-(\sqrt{C\nu}\,\frac{m+1}{2}x^{\frac{m-1}{2}}f + C\frac{m-1}{2}yx^{m-1}f\,')$. So, you can see from the previous expression from here. So, that we have rearranged and we have written in this form.

So, if you rearrange it, you can see you can write, $v = \sqrt{\frac{U_\infty \nu}{x}}\,\frac{m+1}{2}(-f - \frac{m-1}{m+1}\eta f\,')$

And, if you see this is the similar expression this in the inside the bracket. So, if you put m = 0; m = 0, then what will be v? So, v will be m = 0. So, it will be, $v = \sqrt{\frac{U_\infty \nu}{x}}\,\frac{1}{2}(-f + \eta f\,')$. So, this is the expression already we have seen for flow over flat plate.

Now, u, v we have found. Now, we let us find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. So, you can see from here u you have written $Cx^m f\,'$, so $\frac{\partial u}{\partial x}$ you can write $Cmx^{m-1}f\,'$. Again, we are using the von Mises transformation. You refer in last slide and use that expression $y\sqrt{\frac{C}{\nu}}\,\frac{m-1}{2}x^{\frac{m-3}{2}}$. So, $\frac{\partial u}{\partial x}$ expression you can write as, now again Cx^m will write as U_∞. Hence,

$$\frac{\partial u}{\partial x} = Cmx^{m-1}f\,' + Cx^m f\,''\, y\sqrt{\frac{C}{\nu}}\,\frac{m-1}{2}x^{\frac{m-3}{2}}$$

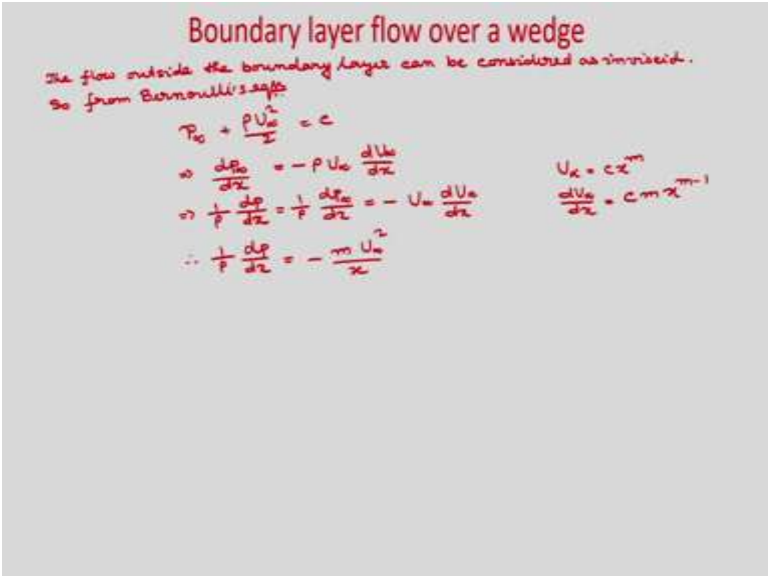
So, it will be $\frac{\partial u}{\partial x} = \frac{mU_\infty}{x}f\,' + \frac{U_\infty}{x}f\,''\eta\frac{m-1}{2}$.

And $\frac{\partial u}{\partial y} = Cx^m f\,''\sqrt{\frac{C}{\nu}}x^{\frac{m-1}{2}} = U_\infty\sqrt{\frac{U_\infty}{\nu x}}f\,''$.

And, $\frac{\partial^2 u}{\partial y^2} = U_\infty\sqrt{\frac{U_\infty \nu}{x}}f\,'''\sqrt{\frac{C}{\nu}}x^{\frac{m-1}{2}} = U_\infty\frac{U_\infty}{\nu x}f\,'''$. Now, you see we have written the expression for u, $\frac{\partial u}{\partial x}$, v, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$.

Now, we need to find the pressure gradient. So, for pressure gradient, so you can see for boundary layer flow, $\frac{\partial p}{\partial y}$=0, right. So, whatever pressure will be there outside the boundary layer that will be impressed inside the boundary layer. So, outside you can use Bernoulli's equation and find, what is the temperature gradient.

(Refer Slide Time: 22:46)

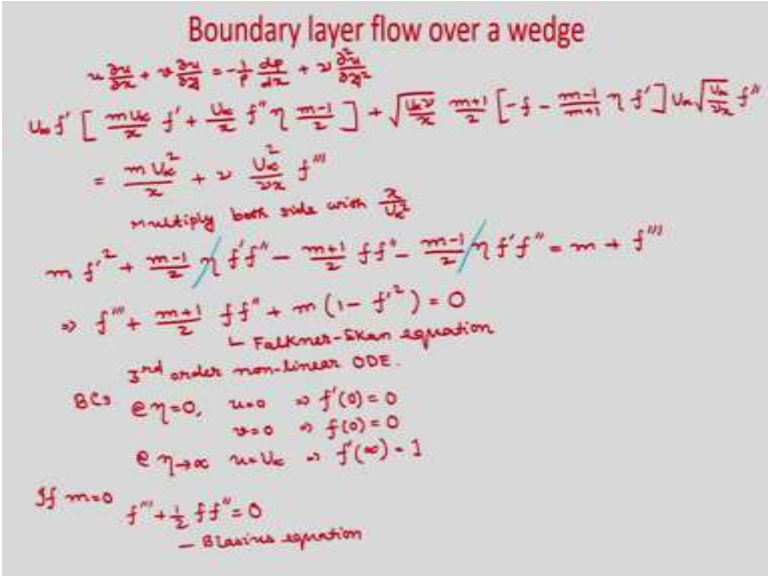


So, you can see the flow outside the boundary layer can be considered as inviscid. So, from Bernoulli's equation you can write $p_\infty + \frac{\rho U_\infty^2}{2} = C$ or you can write $\frac{dp_\infty}{dx} = -\rho U_\infty \frac{dU_\infty}{dx}$. So, you can write $\frac{1}{\rho}\frac{dp}{dx}$.

So, inside the boundary layer whatever pressure gradient will be there, that will be equal to the outside pressure gradient. So, that will be $\frac{1}{\rho}\frac{dp_\infty}{dx}$ which will be $-U_\infty \frac{dU_\infty}{dx}$.

And, we know $U_\infty = Cx^m$. So, $\frac{dU_\infty}{dx} = Cmx^{m-1}$. So, from here you can see $\frac{1}{\rho}\frac{dp}{dx}$ you can write as $-\frac{mU_\infty^2}{x}$. So, now, the expressions for u, v, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{dp}{dx}$ you put in the x momentum equation.

(Refer Slide Time: 25:15)



We have x momentum equation $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$. So, all these expressions you put in this x momentum equation. So, we will get, $U_\infty f'[\frac{mU_\infty}{x}f'+\frac{U_\infty}{x}f''\eta\frac{m-1}{2}]+\sqrt{\frac{U_\infty \nu}{x}}\frac{m+1}{2}[-f-\frac{m-1}{m+1}\eta f']U_\infty\sqrt{\frac{U_\infty}{\nu x}}f''$.

Then, we have $-\frac{1}{\rho}\frac{dp}{dx}$, you can write as $\frac{mU_\infty^2}{x}$. And, $\frac{\partial^2 u}{\partial y^2}$. So, you can write, $\nu\frac{U_\infty^2}{\nu x}f'''$. So, this is the expression you simplify it, multiply both side by $\frac{x}{U_\infty^2}$. So, multiply both side with $\frac{x}{U_\infty^2}$ and simplify it.

So, if you do it, you will get, $mf'^2+\frac{m-1}{2}\eta f' f''-\frac{m+1}{2}ff''-\frac{m-1}{2}\eta f' f''$ and that will be just $m+f'''$.

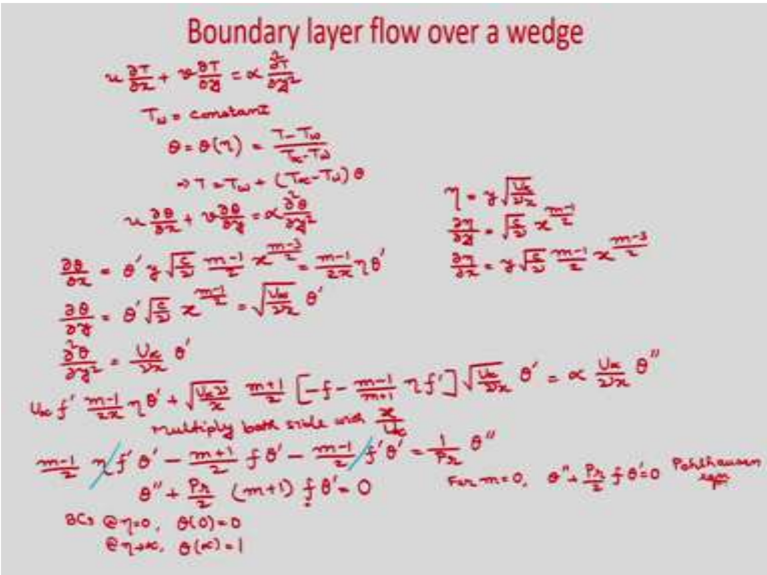
So, you can see here, this terms will get cancelled, this term and this term and you will get, $f'''+\frac{m+1}{2}ff''+m(1-f'^2)=0$. So, you can see we started with partial differential equation, then after using the similarity transformation we could transfer the PD to ordinary differential equation. So, this equation is known as Falkner-Skan equation.

So, you can see this is a third order ODE and non-linear, third order non-linear ODE. And, what are the boundary conditions? Boundary conditions you have, at $\eta = 0$, ok, from $u = 0$; you can write $f'(0)= 0$ and from $v = 0$ you have the expression of v. So, from there you can write $f'(0)= 0$. It is kind of a stream function where you are assuming the value of stream function on the wall as 0. And, at $\eta \rightarrow\infty$, so $u = U_\infty$. So, from here you can write $f'(\infty) = 1$.

If you put m =0 then, you will get flow over flat plate and you will get the Blasius equation back. So, you let us see whether we get it or not. So, if m = 0, then you can see from this equation m = 0, so this is the last term will become 0 and here it will be 1/2. So, it will be, $f''' + \frac{1}{2} ff'' = 0$; which is your Blasius equation.

Now, we have solved for the velocity profile because, these third order non-linear ordinary differential equation if you solve using numerical technique, then you will get the velocity profile for flow over a wedge. Now, let us consider energy equation as a special case we will consider surface or the wall as a uniform temperature, so temperature will remain constant; with that assumption let us find what will be the equation to find the temperature distribution.

(Refer Slide Time: 31:16)



So, if you see we have written the energy equation as $u\dfrac{\partial T}{\partial x}+v\dfrac{\partial T}{\partial y}=\alpha\dfrac{\partial^2 u}{\partial y^2}$. So, now, already we have the transformation details. Here, we are assuming T_w = constant. And, we are assuming θ is function of η only, $\theta=\theta(\eta)=\dfrac{T-T_W}{T_\infty-T_w}$. So, from here you can see, $T=T_w+(T_\infty-T_w)\theta$.

So, the equation you can see you can write if you put here, so you will get $u\dfrac{\partial\theta}{\partial x}+v\dfrac{\partial\theta}{\partial y}=\alpha\dfrac{\partial^2\theta}{\partial y^2}$. And, θ is function of η only and we have the transformation variable η, right. It is $\eta=y\sqrt{\dfrac{U_\infty}{\nu x}}$. So, let us find the value of $\dfrac{\partial T}{\partial x}$, $\dfrac{\partial T}{\partial y}$ and $\dfrac{\partial^2 T}{\partial y^2}$ and already we have the value for u and v.

So, $\dfrac{\partial\theta}{\partial x}$ now you find. So, $\eta=y\sqrt{\dfrac{U_\infty}{\nu x}}$. And, $\dfrac{\partial\eta}{\partial y}=\sqrt{\dfrac{C}{\nu}}x^{\frac{m-1}{2}}$ and $\dfrac{\partial\eta}{\partial x}=y\sqrt{\dfrac{C}{\nu}}\dfrac{m-1}{2}x^{\frac{m-3}{2}}$. And again, we will use the von Mises transformation and we will find $\dfrac{\partial\theta}{\partial x}$.

In this particular case, θ is function of η only, so easily you can find the derivative. So, you can write $\dfrac{\partial\theta}{\partial x}=\theta' y\sqrt{\dfrac{C}{\nu}}\dfrac{m-1}{2}x^{\frac{m-3}{2}}$. So, this you can write as $\dfrac{m-1}{2x}\eta\theta'$.

And, $\dfrac{\partial\theta}{\partial y}$ if you write it will be $\dfrac{\partial\theta}{\partial y}=\theta'\sqrt{\dfrac{C}{\nu}}x^{\frac{m-1}{2}}$. So, you can write $\sqrt{\dfrac{U_\infty}{\nu x}}\theta'$. And, $\dfrac{\partial^2\theta}{\partial y^2}=\dfrac{U_\infty}{\nu x}\theta'$.

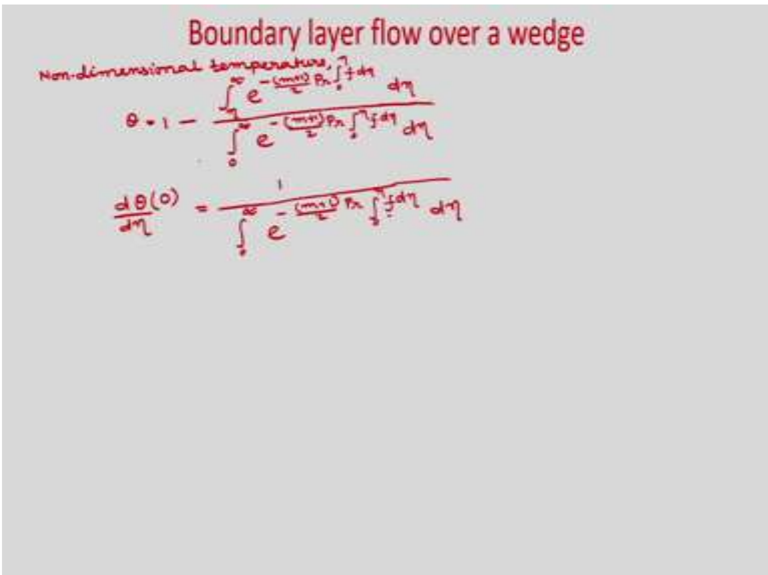
So, if you now substitute this with the value of u v here. So, what you will get? $U_\infty f'\dfrac{m-1}{2x}\eta\theta'+\sqrt{\dfrac{U_\infty\nu}{x}}\dfrac{m+1}{2}[-f-\dfrac{m-1}{m+1}\eta f']\sqrt{\dfrac{U_\infty}{\nu x}}\theta'=\alpha\dfrac{U_\infty}{\nu x}\theta''$, multiply both side with $\dfrac{x}{U_\infty}$ and rearrange it.

So, you will get as $\frac{m-1}{2}\eta f'\theta' - \frac{m+1}{2} f\theta' - \frac{m-1}{2}\eta f'\theta' = \frac{\alpha}{\nu}\theta''$ that means, it will be, $\frac{1}{\Pr}\theta''$. So, this term this term will get cancelled, so you will get finally, $\theta'' + \frac{\Pr}{2}(m+1) f\theta' = 0$. So, you can see this is the ODE and this is linear equation, because already you know the value of f, right from the velocity distribution. So, this is known. So, from the Falkner-Skan solution you will be knowing the velocity profile. So, from there you can calculate the f.

And, if you put for m = 0 as a special case flow over flat plate, then you can see m = 0; that means, it will be $\theta'' + \frac{\Pr}{2} f\theta' = 0$. This is your Pohlhausen equation we have already derived for flow over flat plate. And, what are the boundary conditions? So, boundary conditions at $\eta = 0$ $T = T_w$. So, if it is $T = T_w$, then $\theta = 0$. And, at $\eta \to \infty$, so T will be T_∞, so it will be 1.

And, you can see this is the second order ODE. Using some numerical technique you can solve this equation, you can find the temperature distribution and if you know that, already we have derived the expression for temperature non-dimensional temperature for flow over flat plate, similarly, you can do the analysis here and you can find the temperature distribution as non-dimensional temperature.

(Refer Slide Time: 37:42)

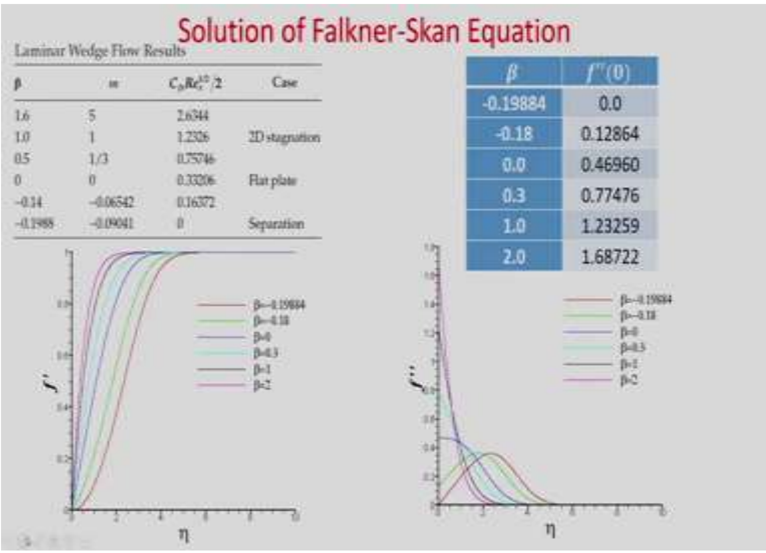


$$\theta = 1 - \frac{\int_{\eta}^{\infty} e^{-\frac{(m+1)}{2}\Pr\int_0^{\eta} f d\eta} d\eta}{\int_0^{\infty} e^{-\frac{(m+1)}{2}\Pr\int_0^{\eta} f d\eta} d\eta}.$$

And, if you remember, we have written the expression for $\frac{d\theta}{d\eta}$ it is required to find the Nusselt number or heat transfer coefficient. So, that at η = 0, right; $\frac{d\theta(0)}{d\eta} = \frac{1}{\int_0^{\infty} e^{-\frac{(m+1)}{2}\Pr\int_0^{\eta} f d\eta} d\eta}$. So, once you know the f from the velocity distribution you will be able to find, what is $\frac{d\theta}{d\eta}$? And, using some numerical technique you can find the value.

So, if you solve this Falkner-Skan equation whatever we have derived. So, you will get the velocity profile. And, once you know the velocity profile from the energy equation whatever we have derived ordinary differential equation you will be able to find the temperature distribution. And, once you know the temperature distribution you will be able to find $\frac{d\theta}{d\eta}$ = 0 that means, the temperature gradient at the wall and you will be able to find, the local heat transfer coefficient, local Nusselt number, average heat transfer coefficient and average Nusselt number.

(Refer Slide Time: 40:08)



So, if you see laminar wedge flow, so for different β and corresponding m value is shown here and if you see $C_{fx} \frac{\mathrm{Re}_x^{1/2}}{2}$, this value if you see here then, you can see for β=0, m = 0. So, what is your $f''(0)$? That means, your velocity gradient at the wall. So, that is 0.33206. So, that you have already found, right. So, this is the flat plate case.

And, if you see the case, where this is your $f''(0)$; that means, your velocity gradient at the wall. So, velocity gradient at the wall become 0. What does it mean? Shear stress is 0. And, if it become shear stress as 0; then obviously, the fluid particle will just float, and that is the point of flow separation. And, flow separation occurs at that point. So, you corresponding β value you can see an m value you can see where flow separates.

So, after that your boundary layer theory is not valid, because the important assumptions, one of the important assumptions we have taken while deriving the boundary layer equation is that flow does not separate. So, here flow is separating at this β value. So, this is the separation point.

Now, for different β value, so this is the similar case. And, at this point you can see that $f''(0)$ becomes 0 for this particular β, and if you see the f' versus η, what is f'? $f' = \frac{u}{U_\infty}$. So, $\frac{u}{U_\infty}$ and η, so you can see what is the velocity profile at different value of β.

So, this is the case, where flow separates. And, if you plot the f'' versus η then you will be able to see, so for this particular case where flow separates you can see, the value of f'' is 0. And, raised other β values, you can see you have a positive value that means, shear stress is present at the wall, but at β value of -0.19884 your velocity gradient becomes 0 and flow separates.

(Refer Slide Time: 42:50)

Boundary layer flow over a wedge

	$Nu_x / Re_x^{1/2} = \frac{d\theta(0)}{d\eta}$				
m	$Pr = 0.7$	0.8	1.0	5.0	10.0
−0.0753	0.242	2.53	0.272	0.457	0.570
0	0.292	0.307	0.332	0.585	0.730
0.111	0.331	0.348	0.378	0.669	0.851
0.333	0.384	0.403	0.440	0.792	1.013
1.0	0.496	0.523	0.570	1.043	1.344
4.0	0.813	0.858	0.938	1.736	2.236

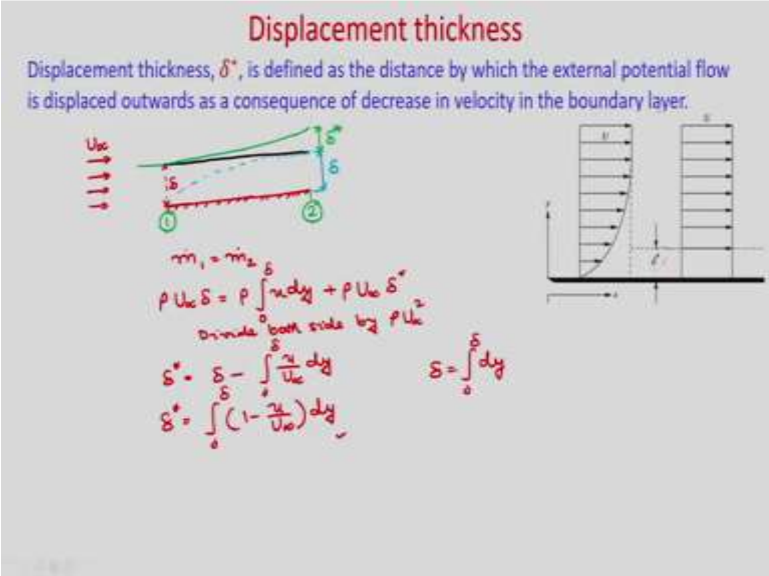
$$h_x = \frac{k}{x}\sqrt{Re_x}\frac{d\theta(0)}{d\eta} \qquad \bar{h} = \frac{2k}{L}\sqrt{Re_L}\frac{d\theta(0)}{d\eta}$$

$$Nu_x = \sqrt{Re_x}\frac{d\theta(0)}{d\eta} \qquad \overline{Nu} = 2\sqrt{Re_L}\frac{d\theta(0)}{d\eta}$$

Now, if you see the $\frac{d\theta(0)}{d\eta}$; that means, $\frac{Nu_x}{\text{Re}_x^{1/2}}$. So, for different values of m and these for different Prandtl number 0.7, 0.8, 1, 5 and 10, these are the values of $\frac{d\theta(0)}{d\eta}$ at different m.

So, now once you know the value of $\frac{d\theta}{d\eta}$ you will be able to calculate the local heat transfer coefficient because, $h_x = \frac{k}{x}\sqrt{\text{Re}_x}\frac{d\theta(0)}{d\eta}$. So, for at any Prandtl number and any m value you will be able to find local heat transfer coefficient, local Nusselt number, average heat transfer coefficient and average Nusselt number. From this table you can see. So, from the solution of the Falkner-Skan equation and the energy equation the value of $\frac{d\theta(0)}{d\eta}$, at η = 0, it is diluted.

(Refer Slide Time: 43:56)



Now, let us define displacement thickness, momentum thickness and shape factor for boundary layer flow. So, first let us discuss displacement thickness. Consider just free stream velocity U_∞. So, if you consider this case then obviously, you can see your stream line will be all parallel to each other, right.

So, if you consider a stream line, so it will be parallel to each other all stream line. Now, if you bring one flat plate here. So, what will happen? So, due to the presence of flat plate there will be formation of boundary layer, over this flat plate, so it will. So, this is the boundary layer. So, this, at this location, this is your boundary layer thickness δ.

Now, you can see that in the presence of flat plate, this stream line will be no longer flat, ok, because it will deflect. Why? Because velocity gradient will be there and due to the velocity gradient to have the same mass flux at each location these stream line will deflect. So, your new stream line will be like this. So, it was earlier flat sorry a straight line, but there will be deflection due to the presence of this flat plate.

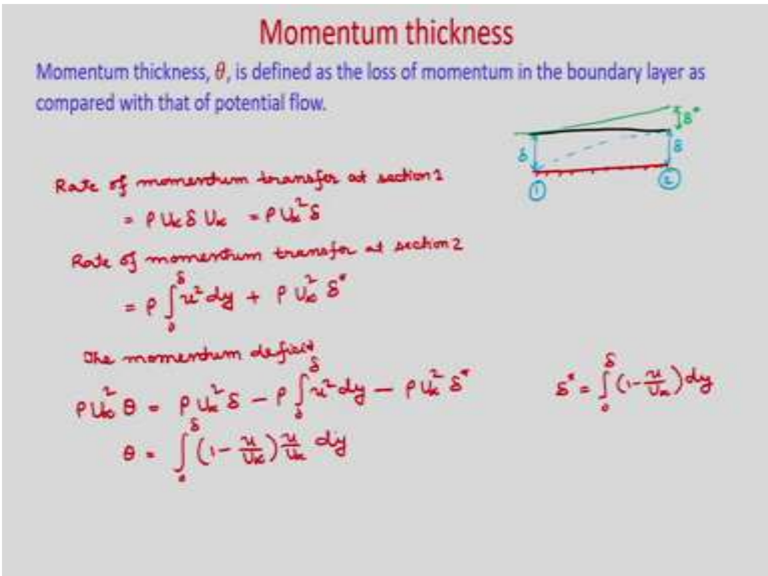
Because, you can see at this location 1 and at this location 2, you should have same mass flow rate and as it is a stream line there will be no flow across the streamline. So, mass flow rate at 1 should be equal to mass flow rate 2. And hence, as velocity gradient will be there these stream line will deflect and these deflection is known as displacement thickness δ*.

So, now, let us find, what is the δ*? So, from here you can see that at section 1 what is mass flow rate, will be equal to mass flow rate at section 2. So, at section 1 it will be just $\rho U_\infty \delta$ because here also, it is same thickness δ. And, at section 2 there will be velocity variation here.

So, it will be $\rho\int_0^\delta u dy + \rho U_\infty \delta^*$. Now, divide both side by ρU_∞ and rearrange. So, you can find $\delta^* = \delta - \int_0^\delta \frac{u}{U_\infty} dy$. Now, this δ we can write $\delta = \int_0^\delta dy$, so δ* if you put it here and you can write $\delta^* = \int_0^\delta (1 - \frac{u}{U_\infty}) dy$.

So, this is the mathematical expression of displacement thickness. And, also you can define in this way say you have a boundary layer and this is the velocity distribution, and if you have a free stream velocity then, how much distance you have to shift this wall to maintain the same mass flux. So, that distance is known as also δ*. So, you can see here displacement thickness δ* is defined as the distance by which the external potential flow is displaced outwards as a consequence of decrease in velocity in the boundary layer.

(Refer Slide Time: 48:20)



Now, we will talk about the momentum thickness. So, momentum thickness θ is defined as the loss of momentum in the boundary layer as compared with that of potential flow.

So, let us consider the same figure here. So, this is your flat plate and this is your stream line, and there will be formation of boundary layer, and this is your δ, this is also δ, section 1 and this is your section 2, but there is a deflection in a boundary layer and so this will be your δ*, right. This is your displacement thickness.

So, you can see that, if you see the rate of momentum at section 1 and section 2, it will not be same although, mass flow rates are same. So, there will be deficit in momentum at section 2. So, this deficit is known as momentum thickness. So, mathematical if you see here. So, what is the rate of momentum transfer at section 1, at section 1? So, what is that? That will be your $\rho U_\infty \delta U_\infty$. So, that is your momentum.

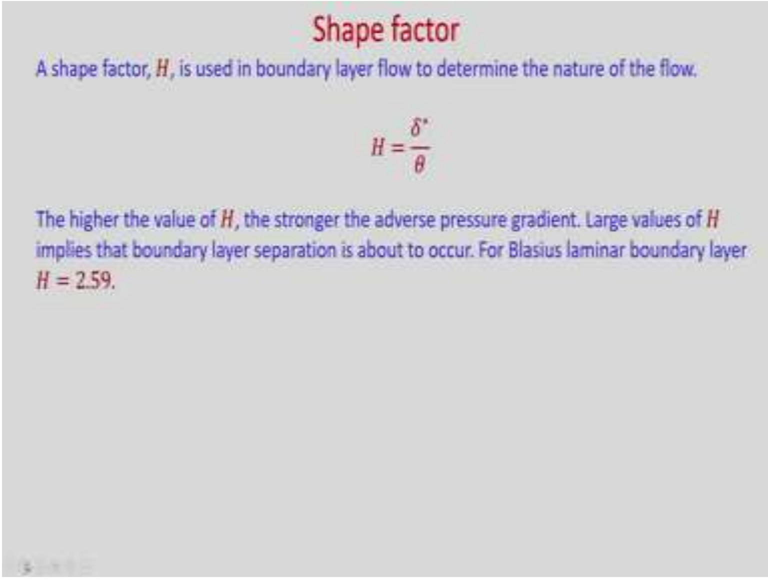
So, $\rho U_\infty \delta U_\infty$. So, that will be $\rho U_\infty^2 \delta$. And, rate of momentum transfer at section 2. So, at section 2 you can see; so, it will be $\rho \int_0^\delta u^2 dy$. So, this is your momentum transfer in this section and in outside it will be $+\rho U_\infty^2 \delta *$.

So, now, you see there will be a deficit. And, this deficit momentum deficit you can find the momentum deficit. So, this the difference you can write as, $\rho U_\infty^2 \theta = \rho U_\infty^2 \delta - \rho \int_0^\delta u^2 dy - \rho U_\infty^2 \delta *$.

And, you know the value of δ* we have found in last slide. So, it is $\delta * = \int_0^\delta (1 - \frac{u}{U_\infty}) dy$.

So, you put these values in the δ*, you divide by ρU_∞^2, rearrange this, you will get $\theta = \int_0^\delta (1 - \frac{u}{U_\infty}) \frac{u}{U_\infty} dy$.

(Refer Slide Time: 51:53)



And shape factor, so a shape factor H is used in boundary layer flow to determine the nature of the flow. So, H is defined as ratio of displacement thickness to the momentum thickness. So, $H = \frac{\delta *}{\theta}$. And, you can see it is known as shape factor, because it solely depends on the shape of the velocity. So, the higher the value of H; the stronger the adverse pressure gradient and large values of H implies that boundary layer separation is about to occur. And, for Blasius laminar boundary layer flow you can find H = 2.59.

So, in today's class we considered laminar boundary layer flow over a wedge and we defined the wedge angle and the wedge parameter and velocity varies as Cx^m as a potential flow. And then, we define the stream function and the similarity variable η and from there we found the velocity u and v from the stream function gradient.

And later, we from the momentum equation we substituted this value of u, v and velocity gradients and we derived the Falkner-Skan equation. And, you can see the Falkner-Skan equation is the third order non-linear ordinary differential equation, so using any numerical technique you can solve this ordinary differential equation.

Then, we considered energy equation keeping the T_w as constant. And, again we define the non-dimensional temperature θ which is function of η only, and we converted this partial differential equation to ordinary differential equation using similarity transformation and that is second ordered ODE and Linear.

And, from there we express the non-dimensional temperature and also $\frac{d\theta(0)}{d\eta}$. Then, we have shown the numerical solution for this Falkner-Skan equation, and we have shown that flow separates at a particular value of β and after that your laminar boundary layer theory will not be valid. At last we defined the displacement thickness as well as the momentum thickness and shape factor.

Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 04
Convective Heat Transfer in External Flows - II
Lecture – 10
Momentum integral equation for flat plate boundary layer

Hello everyone, till now we have solved the boundary layer equation using similarity method, where you convert the partial differential equation to ordinary differential equation and you can easily solve ordinary differential equations with given boundary conditions. Today we learn one new method, which is an approximate method known as integral method.

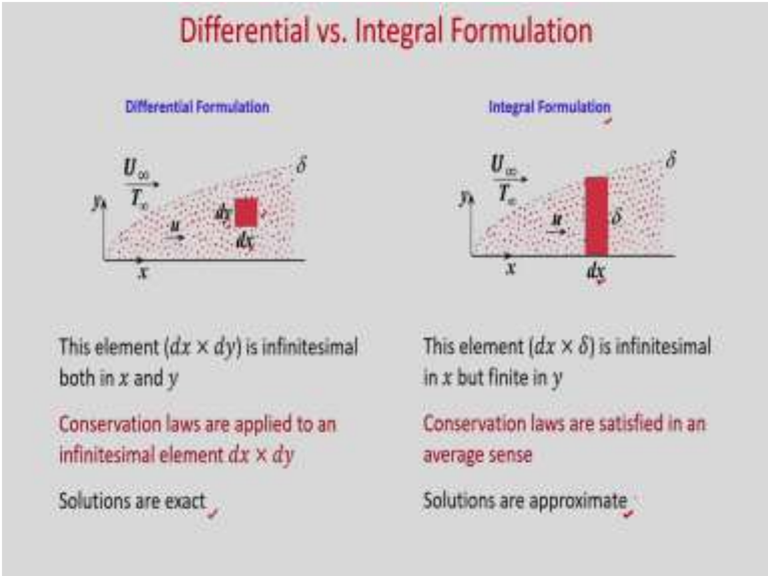
(Refer Slide Time: 01:07)

There are many situations where it is desirable to obtain the approximate analytical solutions. When can we have this approximate analytical solution? When exact solution is not available or cannot be easily obtained and when solutions are too complex implicit or require numerical integration. So, the advantage of these approximate solutions is; the integral method is simple and it can deal with complicated factors.

The integral method is used extensively in fluid flow heat transfer and mass transfer. The mathematical simplifications are there in approximate solutions because number of independent variables are reduced.

When you consider 2 dimensional situation, you can see that using integral method you can convert the partial differential equation to ordinary differential equation. So, you can see that there is reduction in order of differential equation.

(Refer Slide Time: 02:05)

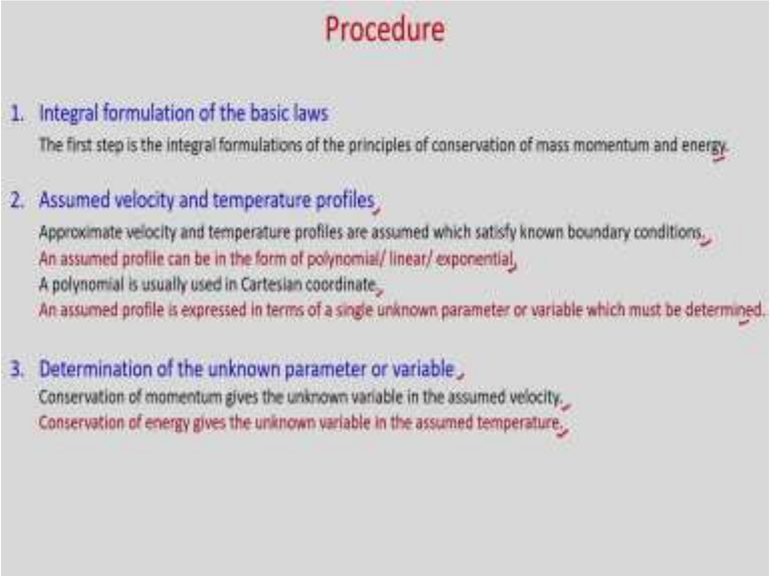


So in differential formulation we know that we consider one infinitesimal element of length dx and dy and the conservations laws are applied to this infinitesimal element. So, the solutions are exact whereas, in integral formulation the element is infinitesimal in x, but finite in y.

In this particular case when you consider boundary layer equations you can see for a flow over flat plate δ is the boundary layer thickness and your infinitesimal element is dx. So, the element is (dx X δ). So; obviously, we apply the conservation laws in an average sense and hence, solutions are approximate in integral formulation.

Now, let us discuss what is the procedure we will follow when we use this approximate method or integral solution.

(Refer Slide Time: 03:10)



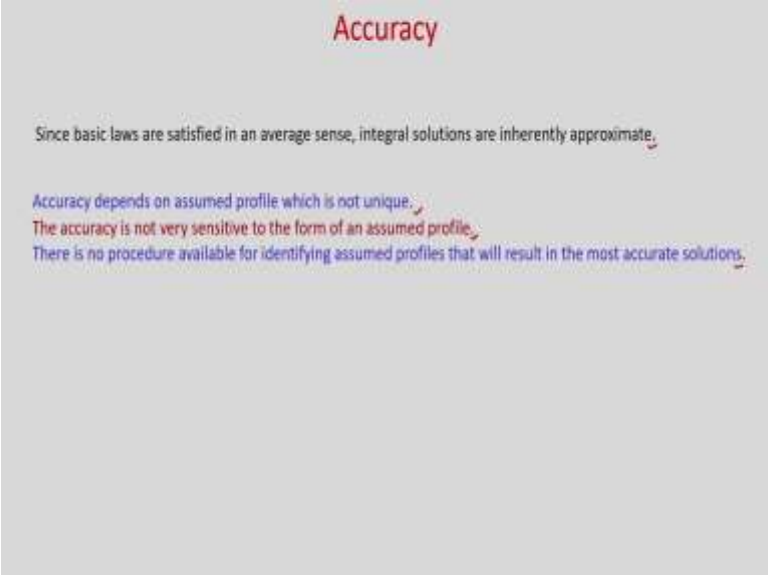
First, we have this integral formulation of the basic laws. The first step is the integral formulation of the principles of conservation of mass, momentum and energy. So, first you see, what are the governing equations for that fluid flow and heat transfer phenomena, then you write it in integral form.

Next, you assume the velocity and temperature profiles. So, approximate velocity and temperature profiles are assumed, which satisfy known boundary conditions. An assumed profile can be in the form of exponential. A polynomial is usually used in Cartesian coordinate. An assumed profile is expressed in terms of a single unknown parameter or variable which must be determined.

So, in this boundary layer equation for flow over flat plate, we will see that this unknown variable is your boundary layer thickness δ. And finally, you determine the unknown parameter or variable.

So, conservation of momentum gives the unknown variable in the assumed profile and conservation of energy gives the unknown variable in the assumed temperature. So, if you follow these 3 steps, then you can use this integral approach to solve the boundary layer equations and we can find the velocity and temperature profile, as well as we can find the heat transfer coefficient and the Nusselt number.

(Refer Slide Time: 04:47)

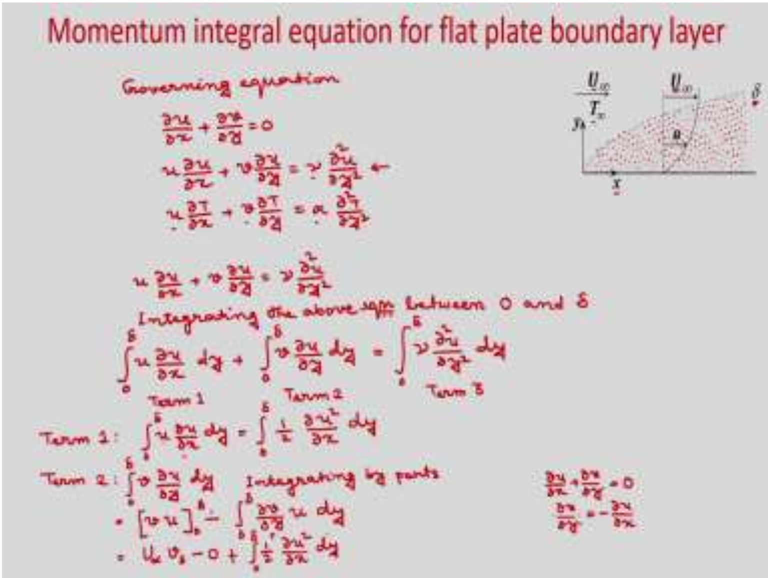


So, what is the accuracy of this integral approach? Since basic laws are satisfied in an average sense, integral solutions are inherently approximate. So, accuracy depends on assumed profile which is not unique, because you can use third order polynomial or second order polynomial. So accordingly, you will get the velocity profile or the temperature profile.

The accuracy is not very sensitive to the form of an assumed profile. So you can see there will be little variation when you use different degree of polynomial in the final solution of temperature profile or velocity profile or the boundary layer thickness. There is no procedure available for identifying assumed profiles that will result in the most accurate solutions. We do not know the optimum temperature or velocity profile for which you will get the results which is closer to the exact solutions.

Today, we will consider only fluid flow, because you know that in convective heat transfer we need to solve the fluid flow equation as well as the energy equation. In today's class, we will solve the fluid flow equation using the integral method and we will find what is the boundary layer thickness, then we will find what is the shear stress acting on the flat plate and then we will find the coefficient of friction.

(Refer Slide Time: 06:20)



So now, let us consider flow over a heated flat plate. First, we will solve the fluid flow equation, then we will solve the energy equation for two different boundary conditions; with constant wall temperature and with constant heat flux.

So you can see, this is your heated flat plate; this is the x direction; perpendicular to the plate is y direction; your free stream velocity U_∞; and free stream temperature T_∞; and this is your age of the boundary layer. And this is the boundary layer thickness δ; obviously, it is hydrodynamic boundary layer thickness and the velocity profile will vary like this where U is function of x and y and here U_∞ is the free stream velocity.

So, let us write the governing equations. First is continuity equation; that is your $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. So; obviously, you are considering a 2 dimensional case, then you have boundary layer equation for flat plate, you know, $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$.

And; obviously, energy equation you can write as $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$. So; obviously, you can see ν is your kinematic viscosity and α is your thermal diffusivity.

So now you can see, once to solve the energy equation we need to know the velocity profile, because in the energy equation you have the velocities. So first, let us solve this equation for the case flow over flat plate using this integral approach.

So, already we discussed the first approach is that you have to integrate the governing equation. So, first we are considering the momentum equation, which is your $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$ right.

So now, let us integrate this equation in the boundary layer because 0 to δ. So, as you are using integral approach; obviously, in y direction we are using δ and in x direction it is infinitesimal distance dx. So, if you integrate it; integrating the above equation between 0 and δ, where δ is your hydrodynamic boundary layer thickness.

So you can see, you can write $\int_0^\delta u\frac{\partial u}{\partial x}dy+\int_0^\delta v\frac{\partial u}{\partial y}dy=\int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy$. Now, we will consider this each term separately and we will integrate step by step. So, first let us give the term as 1, this is the first term; the second term is this one in the left hand side and in the right hand side let us name as term 3.

So, term 1; let us write as $\int_0^\delta u\frac{\partial u}{\partial x}dy$. We will take this u inside this derivatives. So, if you put it inside the derivative then you can write $\int_0^\delta \frac{1}{2}\frac{\partial u^2}{\partial x}dy$. So, you can see; so, if you take the derivative; obviously, you will get 2 u and you will get back this term. Now term 2; so, this is the term 2. So, this is your $\int_0^\delta v\frac{\partial u}{\partial y}dy$.

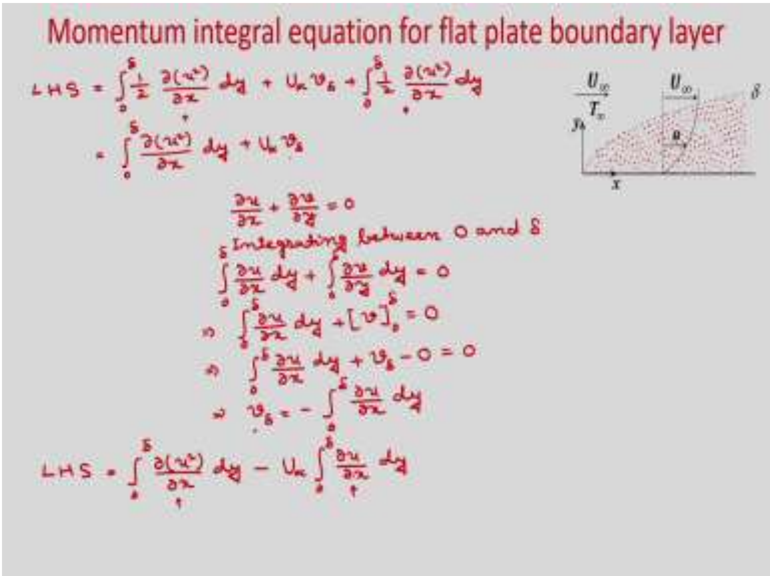
So now, we will use integration by parts. And, you because there are 2 variables, v and $\frac{\partial u}{\partial y}$. So use integration by parts. So, if you write it, so integrating by parts what you will get? So, $[vu]_0^\delta-\int_0^\delta \frac{\partial v}{\partial y}udy$.

So now, let us see the continuity equation. So, what is your continuity equation? Continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. So, this $\frac{\partial v}{\partial y}$ term, you can write as $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$.

So you see, this we put the limit. So if you put y = δ, so at y =δ, what is the velocity u? u will be U_∞, because that is the age of the boundary. So you will have U_∞. So, you can write $U_\infty v_\delta$, this is unknown and if you put the lower limit so velocity is at 0. So, it will be 0 only.

So, if you put $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$ and if you multiply with u and this u if you take inside this derivative then you can write $\int_0^\delta \frac{1}{2}\frac{\partial u^2}{\partial x}dy$.

(Refer Slide Time: 13:31)



So, the left hand side, let us write all the terms. So, in the left hand side we have all the terms as $\int_0^\delta \frac{1}{2}\frac{\partial (u^2)}{\partial x}dy+U_\infty v_\delta+\int_0^\delta \frac{1}{2}\frac{\partial (u^2)}{\partial x}dy$.

Now, in this equation you can see v_δ is unknown right. So, what will do now, we will again use the continuity equation and we will integrate between 0 and δ. So, these two terms if you put together, it is 1/2 and 1/2, so it will be 1. So it will be just $\int_0^\delta \frac{\partial(u^2)}{\partial x}dy + U_\infty v_\delta$. And, this we need to determine, v_δ.

So, let us consider the continuity equation. So continuity equation is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. And, integrating between 0 and δ; within the boundary layer thickness.

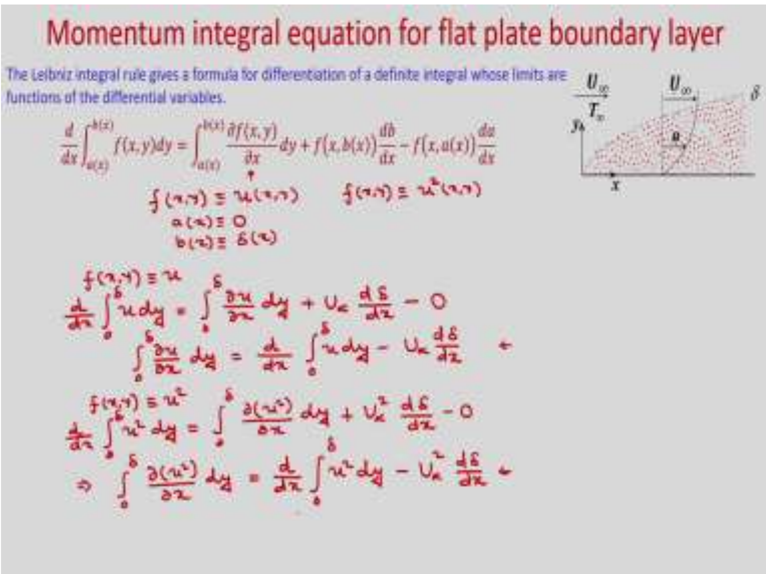
So, we can see it will be $\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta \frac{\partial u}{\partial y}dy = 0$. So now you can see this term will remain as it is, $\int_0^\delta \frac{\partial u}{\partial x}dy$ at this term. So, it will be integral 0 to δ dv. So it will be just $[v]_0^\delta = 0$.

So you can see, you can write as $\int_0^\delta \frac{\partial u}{\partial x}dy + v_\delta - 0 = 0$. So you can find the $v_\delta = -\int_0^\delta \frac{\partial u}{\partial x}dy$.

So this v_δ now you put here. So now, left hand side you can see; you can write as $\int_0^\delta \frac{\partial(u^2)}{\partial x}dy - U_\infty \int_0^\delta \frac{\partial u}{\partial x}dy$.

So you can see, in the first term, so in the first term you have, $\frac{\partial(u^2)}{\partial x}dy$. So, this now; we will use the Leibniz theorem. So that we can integrate this term which is having derivative and you can take this derivative outside the integral.

(Refer Slide Time: 17:01)



So, you can see the Leibniz integral rule gives a formula for differentiation of a definite integral whose limits are functions of the differential variable. So, if you have, $\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)dy = \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x} + f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx}$. So, a and b are the limits and function of x.

So now, you see in our left hand side the first term, so u^2 you can take as f right. So you can see, that this f (x,y) so this you can take as u and this limits; obviously, you can see a x. So this is your lower limit, so it is 0.

And, the upper limit in our case, it is δ and which is function of x, and; obviously, u is function of x and y. So now, you can see that in the left hand side we have this term. So, this we want to write this derivative with respect to a, we can take outside the integral using this Leibniz theorem.

So you can see now, so you can see we have this term as well as this term; $\frac{\partial u}{\partial x}$. So, f will be one time u and another time will be u^2 and we will take the derivative outside the integral and, another time f (x,y) will take u^2.

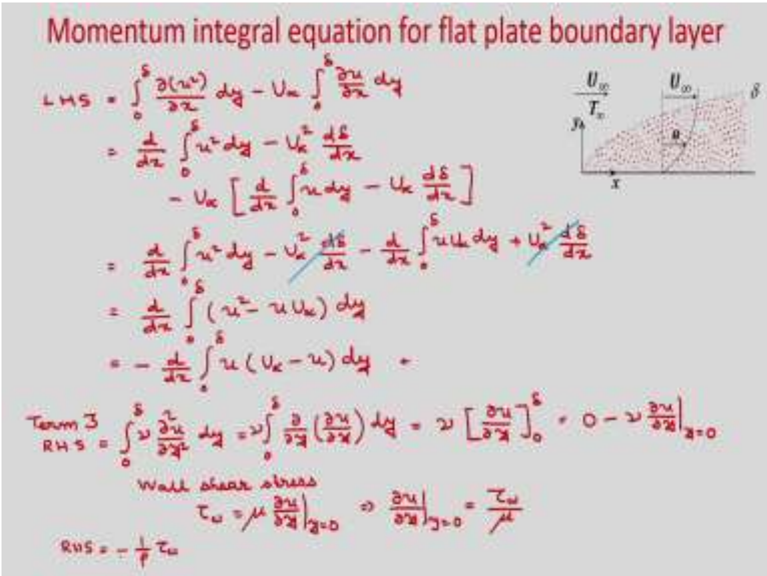
So now you can see, if you put this f (x,y) as u then you can write $\frac{d}{dx}\int_0^\delta u dy = \int_0^\delta \frac{\partial u}{\partial x} dy$. now, f at x = b. So now, u at y = δ. So, you have to see that f at y = b, so in this case u at y = δ.

So, what is that? That is your free stream velocity u. So, you can write plus U_∞ and what is b? b is your δ. So, $\frac{d\delta}{dx}$ and minus, so this is your at y = 0. So that is your 0 right. So a is also 0. So this term will become 0.

So, you can see, you can write $\int_0^\delta \frac{\partial u}{\partial x} dy = \frac{d}{dx}\int_0^\delta u dy - U_\infty \frac{d\delta}{dx}$. And, if you use f (x,y) = u^2, then what you can? Write $\frac{d}{dx}\int_0^\delta u^2 dy = \int_0^\delta \frac{\partial (u^2)}{\partial x} dy + U_\infty^2 \frac{d\delta}{dx} - 0$.

So, you can see, you can write $\int_0^\delta \frac{\partial(u^2)}{\partial x} dy = \frac{d}{dx}\int_0^\delta u^2 dy - U_\infty^2 \frac{d\delta}{dx}$. So you notice these two equations. So now, inside the integral we had partial derivative, but using this Leibniz integral rule you can see you have used ordinary derivative $\frac{d}{dx}$ and $\frac{d\delta}{dx}$, similarly for this equations.

(Refer Slide Time: 22:10)



Now, you put this expression in the left hand side terms. So, we can write finally, so left hand side we had $\int_0^\delta \frac{\partial(u^2)}{\partial x} dy - U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy$. So now, this using Leibniz rule whatever you have got, so that you just write it.

So, this we have written, $\frac{d}{dx}\int_0^\delta u^2 dy - U_\infty^2 \frac{d\delta}{dx} - U_\infty [\frac{d}{dx}\int_0^\delta u dy - U_\infty \frac{d\delta}{dx}]$.

So, from Leibniz integral rule whatever we got, the partial differentiation inside the integral that we have written in terms of the ordinary derivative outside the integral; so that we have just put the terms in this expression. So, now, if you rearrange you can see; so, it will be, $\frac{d}{dx}\int_0^\delta u^2 dy - U_\infty^2 \frac{d\delta}{dx} - \frac{d}{dx}\int_0^\delta u U_\infty dy + U_\infty^2 \frac{d\delta}{dx}$

.

So, you cancel these two terms. So, you will get $\frac{d}{dx}\int_0^\delta (u^2 - uU_\infty)dy$. And you can write $-\frac{d}{dx}\int_0^\delta u(U_\infty - u)dy$. Now, let us consider the third term, which was in right hand side of the boundary layer equation. So, if you see the third term. So that is in the right hand side. So you have $\int_0^\delta \nu \frac{\partial^2 u}{\partial y^2}dy$.

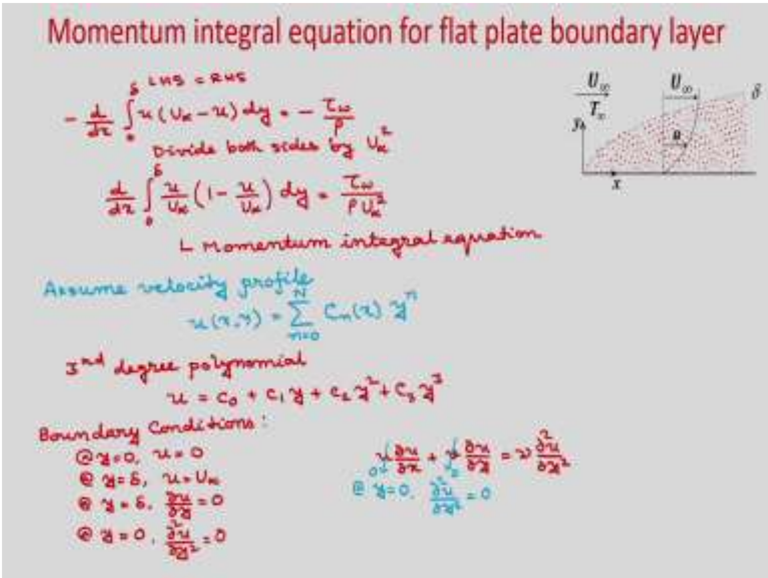
So, what you can write this? 0 to δ; ν we can take it outside because this is your fluid property and constant. So, you can write $\nu\int_0^\delta \frac{\partial}{\partial y}(\frac{\partial u}{\partial y})dy$. So you can see. So this del y del y will get cancelled, so you can write this as $\nu[\frac{\partial u}{\partial y}]_0^\delta$. Now you see, that at y= δ which is your edge of the boundary layer.

So, what is the velocity gradient? $\frac{\partial u}{\partial y}$,because at the edge of the boundary layer and outside the age of the boundary layer you have a free stream velocity U_∞. So, velocity gradient becomes 0. So that means, $\frac{\partial u}{\partial y}$ will be 0 at y = δ. So you can see, at y = δ it will be 0 and $-\nu\frac{\partial u}{\partial y}|_{y=0}$ at wall will have some value right. So that is y = 0.

Now, this term we can represent in terms of shear stress; wall shear stress right. In terms of wall shear stress, τ_w. So, you can write wall shear stress. So, $\tau_w = \mu\frac{\partial u}{\partial y}|_{y=0}$. So you can see, you can write $\frac{\partial u}{\partial y}|_{y=0} = \frac{\tau_w}{\mu}$.

So now you see, the left hand side term is this one; which has one negative sign and right hand side this is the term and if you put the value of $\frac{\partial u}{\partial y}|_{y=0}$ you will get as minus, so ν is your, what is ν? So $\nu = \frac{\mu}{\rho}$. So you will get $\frac{1}{\rho}\tau_w$.

(Refer Slide Time: 28:16)



So you can see, you have minus. So, now if you write left hand side is equal to right hand side, so what you are going to get? So, if you put left hand side is equal to right hand side, so we will get $-\frac{d}{dx}\int_0^\delta u(U_\infty - u)dy = -\frac{\tau_w}{\rho}$. So, you divide both side by U_∞^2.

So, and this minus and right hand side minus we will cancel, so you can write $\frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy = \frac{\tau_w}{\rho U_\infty^2}$. So this equation is known as momentum integral equation.

So, we started with the momentum equation. We integrated between 0 and δ and finally, we have arrived in this expression. So, you can see this is known as momentum integral equation and you can see that this is your ordinary derivative $\frac{d}{dx}$. Now, we need to assume the velocity profile. Now, we need to go to the second step.

So, we need to approximate the velocity profile. Now, let us assume the velocity profile. So, assume velocity profile. So, we will use polynomial expression. So, $u(x,y) = \sum_{n=0}^{N} C_n(x)y^n$. So today we will consider 3rd degree polynomial. So if you consider 3rd degree polynomial, then you can express this velocity profile as 3rd degree polynomial.

You will get $u = c_o + c_1 y + c_2 y^2 + c_3 y^3$. So, considering 3rd degree polynomial we got 4 coefficients. So we need to determine c_0 ,c_1 ,c_2 and c_3. So how many boundary condition do you require to find this 4 coefficients? Obviously, you need 4 boundary conditions.

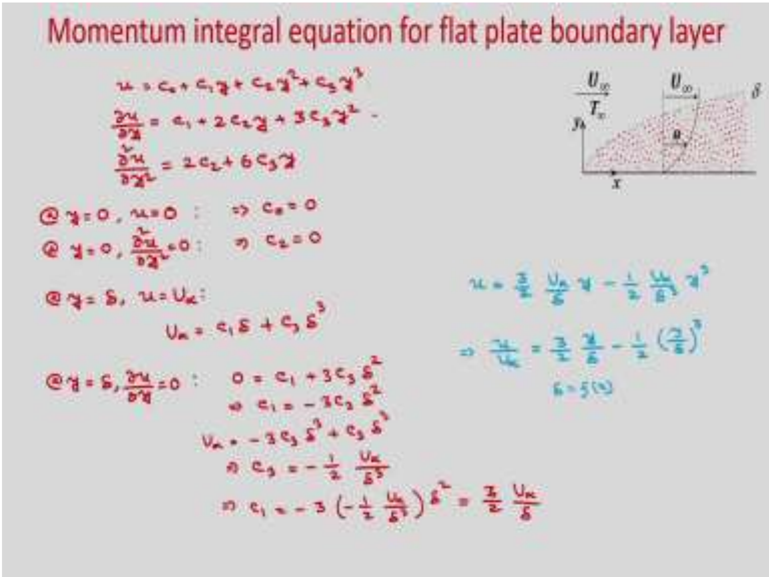
So, now you can see, easily you can find 2 boundary conditions at y = 0, no slip boundary condition. So u = 0 and y = δ, you have u = ∞. Another boundary condition at y = δ, you can easily find that velocity gradient is 0, so; that means, that y = δ, $\frac{\partial u}{\partial y}$ =0. So this 3 boundary conditions you got easily.

So we need another boundary conditions. Now the fourth boundary condition will derive, satisfying the Navier-Stokes equation or satisfying the boundary layer equation at the wall. So at y = 0, let us see whatever boundary layer equation you have, what expression you get and from there we will get the boundary condition that is why it is known as derived boundary condition.

So let us write the boundary conditions. So, at y = 0 you have u = 0 at y = δ, you have u = ∞ that is your free stream velocity, at y = δ, your velocity gradient is 0. So $\frac{\partial u}{\partial y}$ = 0 and at y = 0. Now, we will see the boundary layer equation is $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$.

So this is your boundary layer equation. Now, you put at y = 0. What happens? So we will derive this boundary condition. So at y = 0, u is 0; at y = 0, v is 0. So we can see, left hand side 2 terms are 0. So, we will get $\frac{\partial^2 u}{\partial y^2}$= 0. So, this is your derived boundary condition. So, $\frac{\partial^2 u}{\partial y^2}$= 0. So now, you have 4 boundary conditions, so find these 4 coefficients.

(Refer Slide Time: 34:26)



So, we have $u = c_o + c_1 y + c_2 y^2 + c_3 y^3$. So $\frac{\partial u}{\partial y}$, if you take the derivative with respect to y.

So you find the, $\frac{\partial u}{\partial y} = c_1 + 2c_2 y + 3c_3 y^2$ and $\frac{\partial^2 u}{\partial y^2} = 2c_2 + 6c_3 y$.

Now, apply those boundary conditions. So, at y = 0, you have u = 0. So, if you see this equation at y = 0. So these last 3 terms will become 0 and left hand side u = 0, so that will give c_0 is 0.

Then at y = 0, you have $\frac{\partial^2 u}{\partial y^2}$=0. So, if you put it here, y = 0 left hand side this is 0, so it will give c_2 = 0. And now at y = δ, you have u = U_∞ right. So, if you put it in this expression, so you will get $U_\infty = c_1 \delta + c_3 \delta^3$.

And at y = δ, you have $\frac{\partial u}{\partial y}$ = 0. So, if you put here, so left hand side is $0 = c_1 + 3c_3 \delta^2$. So, you can see, you can find $c_1 = -3c_3 \delta^2$. And if you put it here, so you will get $U_\infty = -3c_3 \delta^3 + c_3 \delta^3$.

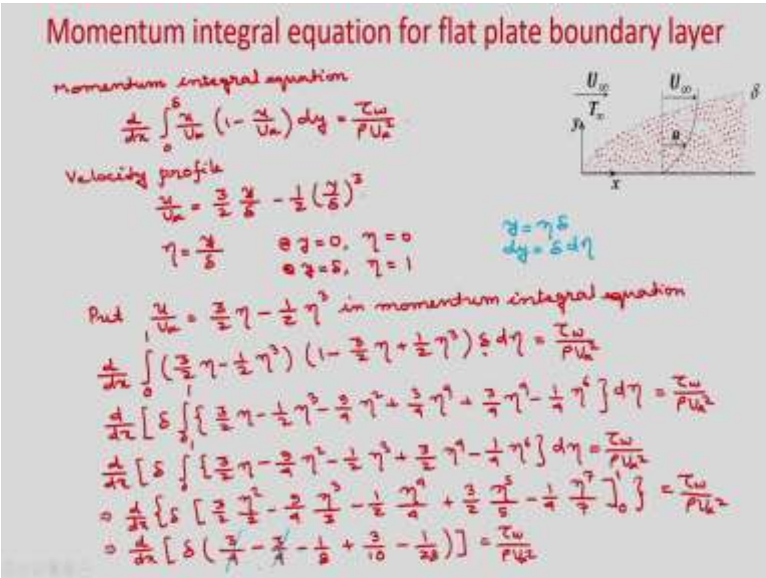
So, what you will get? So, you will get you see here,- $2c_3 \delta^3$. So, $c_3 = -\frac{1}{2}\frac{U_\infty}{\delta^3}$.

And now, $c_1 = -3(-\frac{1}{2}\frac{U_\infty}{\delta^3})\delta^2$. So what you will get? So, it will get plus so it will be $\frac{3}{2}\frac{U_\infty}{\delta}$.

So now, you can write the velocity profile as $u = \frac{3}{2}\frac{U_\infty}{\delta}y - \frac{1}{2}\frac{U_\infty}{\delta^3}y^3$. So, you can see your velocity profile. You can write $\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}(\frac{y}{\delta})^3$. So you can see, that assumed velocity is in terms of a single unknown variables δ and $\delta = f(x)$.

So, now, we will go to the next step. So, what is that step? Determination of this unknown variable. So, we have express the velocity profile in terms of hydrodynamic boundary layer thickness δ, which is function of x. So now this unknown variable δ, we need to find. So that will determine.

(Refer Slide Time: 39:11)



So, for this we will use this momentum integral equation. So, we have momentum integral equation, which we have already derived. This is $\frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy = \frac{\tau_w}{\rho U_\infty^2}$.

And we have velocity profile; $\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}(\frac{y}{\delta})^3$.

So now, let us put this velocity profile in this momentum integral equation and find the unknown variable δ. So, we will just use $\eta = \frac{y}{\delta}$. So you can see, at y = 0 η = 0 and at y=δ you will have η =1. So, this limit if you put it here, so all this expression you put, so put $\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$ in momentum integral equation.

So, what we will get? So, we will get $\frac{d}{dx}\int_0^1 (\frac{3}{2}\eta - \frac{1}{2}\eta^3)(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3)\delta d\eta = \frac{\tau_w}{\rho U_\infty^2}$.

So you can see here, your δ is function of x only right. So, this δ you can take it outside this integral, because this you are integrating with respect to η which is your y. So; obviously, you can take it outside the integrals.

So, you can write, $\frac{d}{dx}[\delta\int_0^1 \{\frac{3}{2}\eta - \frac{1}{2}\eta^3 - \frac{9}{4}\eta^2 + \frac{3}{4}\eta^4 + \frac{3}{4}\eta^4 - \frac{1}{4}\eta^6\}]d\eta = \frac{\tau_w}{\rho U_\infty^2}$.

So if you rearrange it, what you will get?

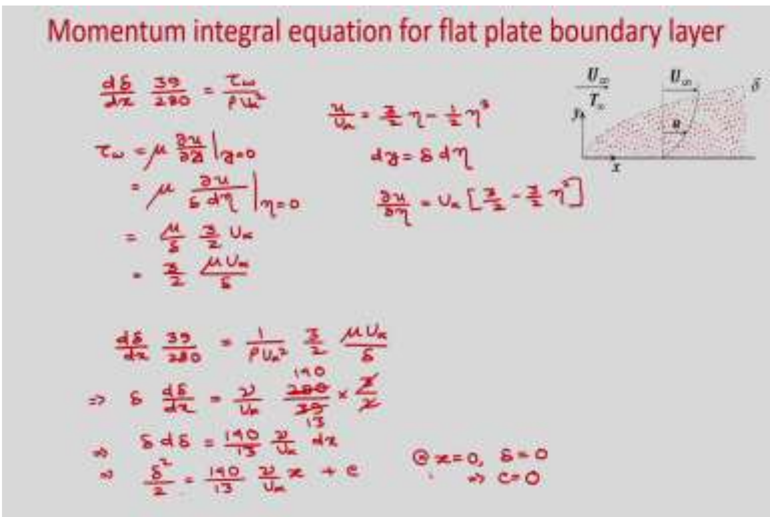
$\frac{d}{dx}[\delta\int_0^1 \{\frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6\}]d\eta = \frac{\tau_w}{\rho U_\infty^2}$.

So if you integrate now, so you will get

$\frac{d}{dx}\{\delta[\frac{3}{2}\frac{\eta^2}{2} - \frac{9}{4}\frac{\eta^3}{3} - \frac{1}{2}\frac{\eta^4}{4} + \frac{3}{2}\frac{\eta^5}{5} - \frac{1}{4}\frac{\eta^7}{7}]_0^1\} = \frac{\tau_w}{\rho U_\infty^2}$.

So now, at η = 0 anyway all these terms will become 0, at η = 1 you just put the value, then what you will get? $\frac{d}{dx}[\delta(\frac{3}{4} - \frac{3}{4} - \frac{1}{8} + \frac{3}{10} - \frac{1}{28})] = \frac{\tau_w}{\rho U_\infty^2}$. So these 3/4 and 3/4 you can cancel.

(Refer Slide Time: 45:39)



So, you can get, so now if you rearrange this and you can get finally, in the left hand side as $\frac{d\delta}{dx}\frac{39}{280}$ and in the right hand side it is $\frac{\tau_w}{\rho U_\infty^2}$. So, now let us find what is the wall shear stress, because wall shear stress you can express in terms of the velocity gradient at y = 0 and velocity profile already you have derived. So you will be able to find what is the shear stress.

So, you can write $\tau_w = \mu \frac{\partial u}{\partial x}|_{y=0}$ and you can see you can have velocity profile $\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$. And also we have dy = δ dη right, because $\eta = \frac{y}{\delta}$. So, you can write $\mu \frac{\partial u}{\delta d\eta}|_{\eta=0}$.

Then you can write, $\frac{\partial u}{\partial \eta} = U_\infty[\frac{3}{2} - \frac{3}{2}\eta^2]$; so, you can see $\frac{\partial u}{\partial \eta}|_{\eta=0}$. So you will get $\frac{\mu}{\delta}\frac{3}{2}U_\infty$.

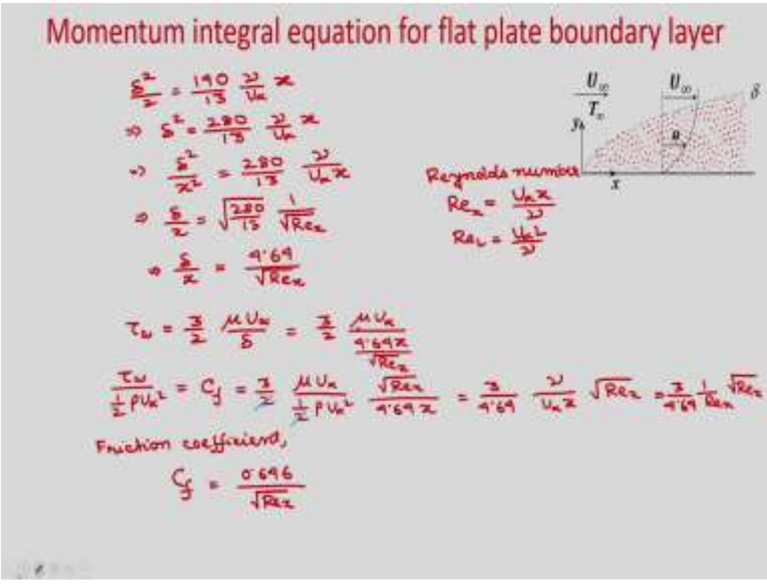
So, finally $\tau_w = \frac{3}{2}\frac{\mu U_\infty}{\delta}$.

So, now, if you put τ_w here, so you will get so you can write $\frac{d\delta}{dx}\frac{39}{280} = \frac{1}{\rho U_\infty^2}\frac{3}{2}\frac{\mu U_\infty}{\delta}$.

So you can see, you can write $\delta\frac{d\delta}{dx} = \frac{\nu}{U_\infty}\frac{280}{39}\frac{3}{2}$. So, this will be $\delta d\delta = \frac{140}{13}\frac{\nu}{U_\infty}dx$.

So if you integrate this equation, so you will be able to find the unknown variable δ. And you know that x = 0, you have the hydrodynamic boundary layer thickness as 0 so; that means, δ =0. So, if you integrate it so you will get $\frac{\delta^2}{2} = \frac{140}{13}\frac{\nu}{U_\infty}x + c$. So now, you put at x = 0, δ = 0 right so; that means, it will give c = 0 ok. So if your c = 0.

(Refer Slide Time: 49:23)



So, you can write $\frac{\delta^2}{2} = \frac{140}{13} \frac{\nu}{U_\infty} x$. So if you rearrange, you will get $\delta^2 = \frac{280}{13} \frac{\nu}{U_\infty} x$ or you can write $\frac{\delta^2}{x^2} = \frac{280}{13} \frac{\nu}{U_\infty x}$.

So now you see, in the last term you have $\frac{\nu}{U_\infty x}$. So that you can express in terms of Reynolds number. So, if you define the Reynolds number. $\mathrm{Re}_x = \frac{U_\infty x}{\nu}$, which is x is your axial direction. So, at x = l for the plate length l you can write $\mathrm{Re}_L = \frac{U_\infty L}{\nu}$. So, if you put it here, so we can see you can write $\frac{\delta}{x} = \sqrt{\frac{280}{13}} \frac{1}{\sqrt{\mathrm{Re}_x}}$. Or you can write $\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}}$.

So now, we have found the unknown variable; the hydrodynamic boundary layer thickness δ. So now, once δ is known now you will be able to calculate the velocity profile ok, because velocity profile we have expressed in terms of δ. So, now, δ you have found, so you will be able to find what is the velocity profile using the integral method.

Now, let us find the coefficient of friction. So, we have found the shear stress $\tau_w = \frac{3}{2}\frac{\mu U_\infty}{\delta}$. So now you can put the value of δ here. So, what you will get? $\frac{3}{2}\frac{\mu U_\infty}{\delta}$ value you can see here. So, δ you can write as, $\delta = \frac{4.64x}{\sqrt{\text{Re}_x}}$.

So if you put this, so you can see you can write as; so now, you can write $\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = C_f = \frac{3}{2}\frac{\mu U_\infty}{\frac{1}{2}\rho U_\infty^2}\frac{\sqrt{\text{Re}_x}}{4.64x}$. So, you can see here, these 2 and 2 you can cancel.

Now, you will get $\frac{3}{4.64}$.

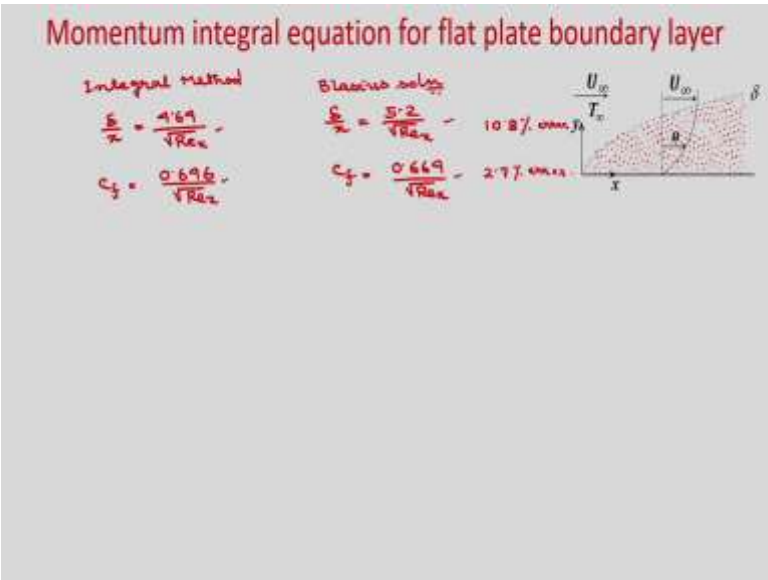
And here you see, it is; so it will be just $\frac{3}{4.64}\frac{\nu}{U_\infty x}\sqrt{\text{Re}_x}$.

So the friction coefficient you will get as $C_f = \frac{0.646}{\sqrt{\text{Re}_x}}$. So today, we considered the flow over flat plate and used approximate solution method to find the boundary layer thickness as well as the friction factor using assumed velocity profile.

So, in integral approach there are three steps. First step is that you have to integrate the governing equation, then you assume the velocity profile and next you find the unknown variable.

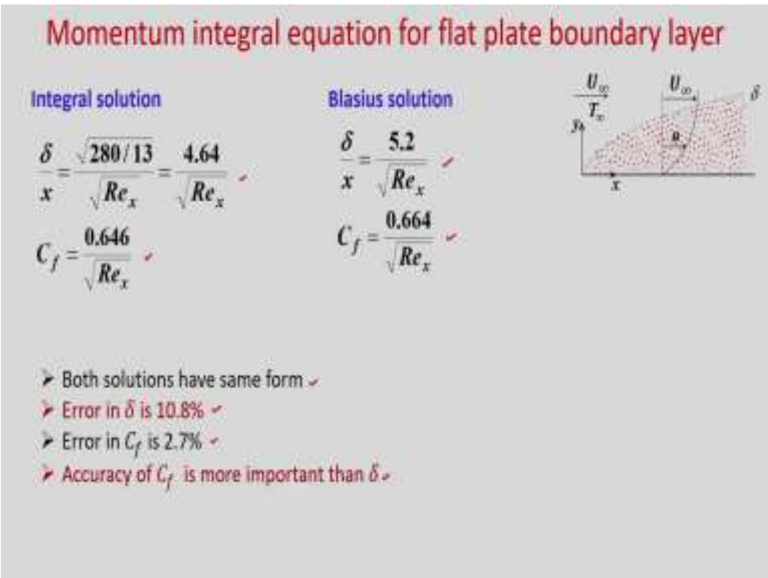
So, using the integral approach we have found $\frac{\delta}{x}$ as well as the friction coefficient C_f in terms of the Reynolds number.

(Refer Slide Time: 54:20)



And finally, you can see that if you compare the results you got from the integral method with the exact solution, so there is; if you see this is your almost 10.8 % error and if you compare the C_f it is 2.7 % error.

(Refer Slide Time: 54:44)



So you can see; finally, we have derived the hydrodynamic boundary layer thickness using integral solution $\frac{\delta}{x} = \frac{4.64}{\sqrt{\text{Re}_x}}$. Similarly, coefficient of friction we have derived

$C_f = \frac{0.646}{\sqrt{\mathrm{Re}_x}}$. So you can see these are approximate solution, because we have used some approximate velocity profile.

Whereas, you have the exact solution from Blasius solution; you can see $\frac{\delta}{x} = \frac{5.2}{\sqrt{\mathrm{Re}_x}}$ and $C_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$. So, you can see both solutions have same form.

If you compare δ by x with this integral solution and Blasius solution, you will get error in δ as 10.8 % and error in C_f is 2.7 %. Obviously, error in C_f is less than the error in δ and accuracy of C_f is more important than δ in design point of view.

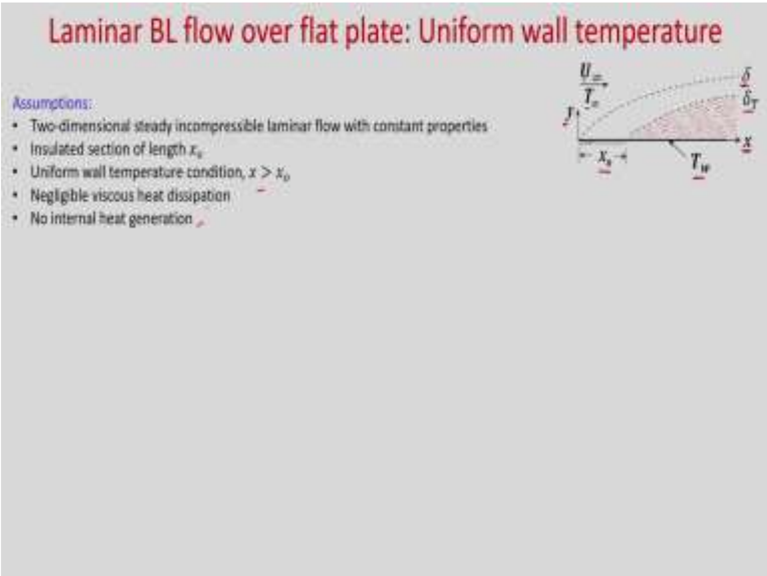
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 04
Convective Heat Transfer in External Flows - II
Lecture - 11
Laminar BL flow over flat plate: Uniform wall temperature

Hello everyone. So, today we will consider Laminar Boundary Layer Flow Over a Flat Plate with Uniform Wall Temperature Condition. In last class, we derived the momentum integral equation, and we derived the hardening boundary layer thickness, and the coefficient of friction. In today’s lecture, we will consider the heat transfer where the boundary is maintained at uniform wall temperature, and we wish to determine the heat transfer coefficient, and the Nusselt number.

(Refer Slide Time: 01:10)



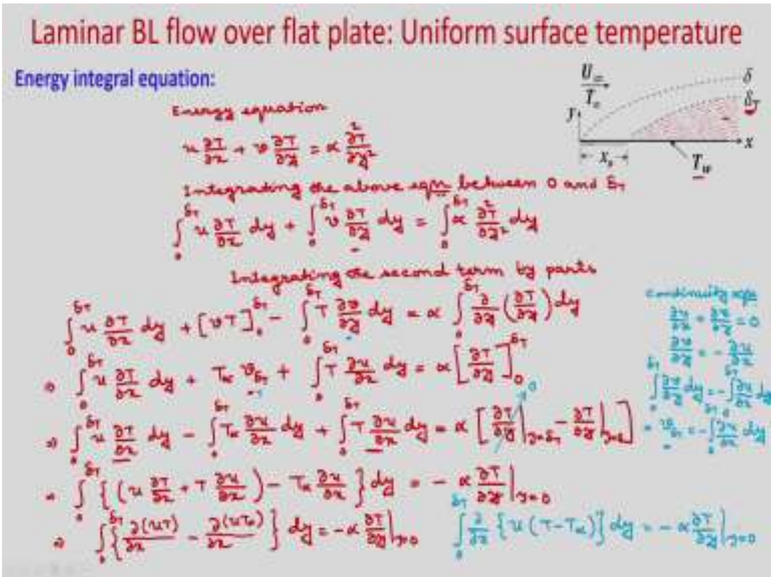
So, let us consider this flat plate x is in the in this direction, and y is perpendicular to the plate. You can see that up to $x = x_0$ it is insulated that means; there is no heat transfer from this surface. So, the free stream velocity is ∞ and free stream temperature T_∞.

So, in this region your wall temperature will be at T_∞. But from $x = x_0$ this wall is maintained at uniform wall temperature T_w. So, obviously, your thermal boundary layer thickness will start developing from x = x_0. And hardening boundary layer thickness obviously, will start developing from x = 0.

So, these are the assumptions we will consider, two-dimensional steady incompressible laminar flow with constant properties, insulated section of length x_0, uniform wall temperature condition T_w for x > x_0. We will neglect viscous heat dissipation and there will be no internal heat generation.

In today's lecture, first we will derive energy integral equation then we will put the temperature distribution and the velocity distribution in the energy integral equation and we will derive the expression for thermal boundary layer thickness, and then we will derive the local heat transfer coefficient and local Nusselt number.

(Refer Slide Time: 02:57)



So, first let us derive the energy integral equation. So, we have the energy equation, you know $u\dfrac{\partial T}{\partial x}+v\dfrac{\partial T}{\partial y}=\alpha\dfrac{\partial^2 T}{\partial y^2}$. So, let us consider today this flow over flat plate where from x =0 to x = x_0 it is unheated region, after that from x > x_0 you can see the flat plate is maintained at uniform wall temperature T_w.

So, velocity boundary layer thickness will start growing from x = 0. However, as it is insulated up to x = x_0 then your thermal boundary layer thickness will start developing from x = x_0.

So, today we will integrate this energy equation in this thermal boundary layer. So, we will integrate from y = 0 to δ_T, where δ_T is your thermal boundary layer thickness. So, integrating the above equation between 0 and δ_T, where δ_T is your thermal boundary layer thickness. So, you can see $\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + \int_0^{\delta_T} v\frac{\partial T}{\partial y}dy = \int_0^{\delta_T} \alpha\frac{\partial^2 T}{\partial y^2}dy$.

So, first the second term you consider in the left hand side and integrate using integration by parts. So, you will see integrating the second term by parts. So, first term you keep it as it is. So you will get, $\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + [vT]_0^{\delta_T} - \int_0^{\delta_T} T\frac{\partial v}{\partial y}dy = \alpha\int_0^{\delta_T} \frac{\partial}{\partial y}(\frac{\partial T}{\partial y})dy$.

So, you can see we will get $\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy$ plus you see at y =0. At y = 0, what is the velocity? Obviously, it is 0. So, you will get 0 v X T. And at y = δ_T, what is the temperature? It is free stream temperature, so free stream temperature is T_∞. And what is the velocity?

Velocity is v δ_T. So, you will get T_∞ v at y = δ_T that we need to find from the continuity equation and these $\frac{\partial v}{\partial y}$. So, what we will do now, you have continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. So, this $\frac{\partial v}{\partial y}$ we can write $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$. So, that we can substitute.

So, you can write $\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + T_\infty v_{\delta_T} + \int_0^{\delta_T} T\frac{\partial u}{\partial x}dy = \alpha[\frac{\partial T}{\partial y}]_0^{\delta_T}$. So, now we need to find v_{δ_T}. So, to find the v_{δ_T}, we will use this continuity equation and we will integrate this continuity equation in the thermal boundary layer. So, if you see from this expression if you integrate it $\int_0^{\delta_T} \frac{\partial v}{\partial y}dy = -\int_0^{\delta_T} \frac{\partial u}{\partial x}dy$.

Now, you see at, so it will be v, right v at the limit 0 and δ_T. So, at y = 0, v = 0, so you will get $v_{\delta_T} = --\int_0^{\delta_T} \frac{\partial u}{\partial x}dy$. So, these you substitute it here. So, what you will get now?

$$\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy - \int_0^{\delta_T} T_\infty\frac{\partial u}{\partial x}dy + \int_0^{\delta_T} T\frac{\partial u}{\partial x}dy = \alpha[\frac{\partial T}{\partial y}|_{y=\delta_T} - \frac{\partial T}{\partial y}|_{y=0}].$$

So, you can see, so $\frac{\partial T}{\partial y}$ at y = δ_T is 0. So, $\alpha[\frac{\partial T}{\partial y}|_{y=\delta_T} - \frac{\partial T}{\partial y}|_{y=0}]$. So, you can see this term will be 0 because at the edge of the thermal boundary layer this is 0.

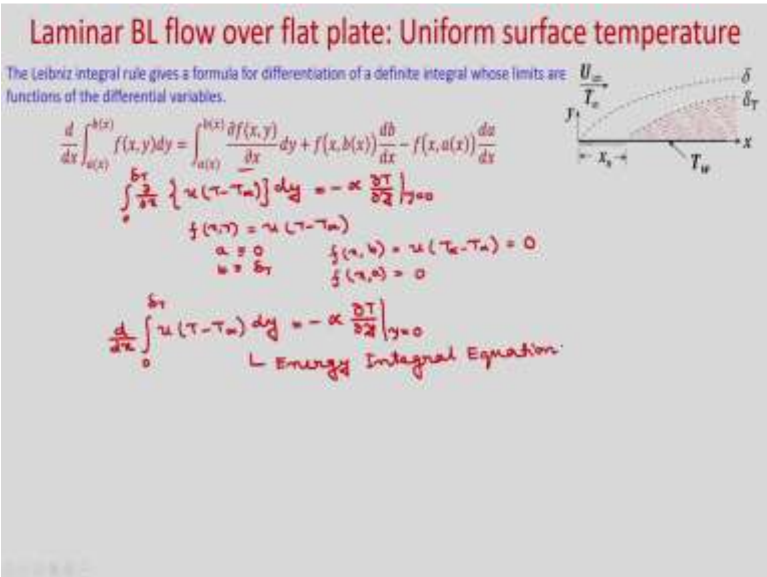
So, now you can write this equation $\int_0^{\delta_T} \{(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}) - T_\infty \frac{\partial u}{\partial x}\}dy = -\alpha \frac{\partial T}{\partial y}|_{y=0}$.

So, what is this term? These two terms together you can write $\int_0^{\delta_T} \{\frac{\partial (uT)}{\partial x} - \frac{\partial (uT_\infty)}{\partial x}\}dy = -\alpha \frac{\partial T}{\partial y}|_{y=0}$.

So, if you take this $\frac{\partial}{\partial x}$ common, then you will get integral $\int_0^{\delta_T} \frac{\partial}{\partial x}\{u(T - T_\infty)\}dy = -\alpha \frac{\partial T}{\partial y}|_{y=0}$.

So, now you can see that you have the derivative $\frac{\partial}{\partial x}$ inside the integral. So, now, using Leibniz rule we will take these $\frac{\partial}{\partial x}$ outside and we will write in terms of ordinary derivative.

(Refer Slide Time: 12:40)



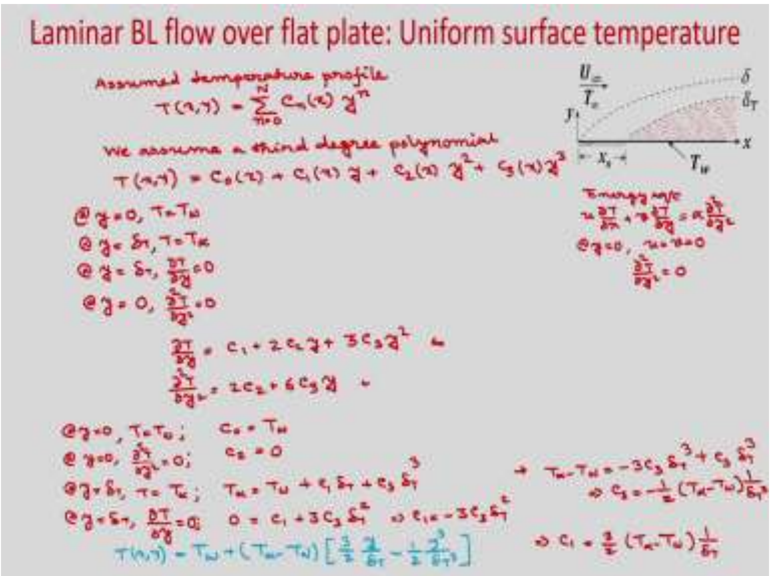
So, now, let us see what is our integral equation we have derived. So, this is your integral $\int_0^{\delta_T} \frac{\partial}{\partial x}\{u(T - T_\infty)\}dy = -\alpha \frac{\partial T}{\partial y}|_{y=0}$. So, if you compare these two terms then you will get $f = u(T - T_\infty)$. Your lower limit a, equivalent to here 0 and upper limit is equivalent to δ_T.

So, if you find f (x,b). So, this term, so f (x,b), so at y = δ_T. At y = δ_T if you find what is the value of f? So, it will be u. What is T? T will be T_∞. So, T_∞ - T_∞, so it will be 0. And what is f(x,a)? At y = a. Means y = 0. At y = 0, what is the velocity?

So, velocity is 0. So, this will be 0. So, these two terms will contribute 0. So, we can write directly these using this Leibniz integral rule we can write $\frac{d}{dx}\int_0^{\delta_T} u(T-T_\infty)dy = -\alpha \frac{\partial T}{\partial y}|_{y=0}$. So, this is known as energy integral equation.

So, we have completed the first step. What is the first step? That you integrate the governing equation and get the integral equation. So, we have got the energy integral equation. So, next step is to find the temperature distribution. So, we will assume polynomial, and in this case, we will consider third degree polynomial and with proper boundary conditions we will find the assume temperature profile. Once we find the assume temperature profile, then we will put that in the energy integral equation.

(Refer Slide Time: 15:06)



So, we will assume the temperature profile as, so $T(x,y) = \sum_{n=0}^{N} C_n(x) y^n$.

Now, we will consider third degree polynomial. So, we will take up to n = 3. So, we assume a third degree polynomial. So, you will get $T(x,y) = C_0 + C_1 y + C_2 y^2 + C_3 y^3$. So, how many unknown coefficients are there? C_0, C_1, C_2, and C_3. So, how many boundary conditions do we need?

We need 4 boundary conditions. Two boundary conditions you can easily find at y = 0 you have uniform wall temperature $T = T_w$, at $y = \delta_T$ you have $T = T_\infty$ which is your free stream temperature, and $y = \delta_T$ again your temperature gradient is 0. And another boundary condition will derive from the energy equation satisfying it at boundary.

So, you can see at y = 0, you have $T = T_w$, at $y = \delta_T$ you have $T = T_\infty$, at $y = \delta_T$ temperature gradient is also 0, so $\frac{\partial T}{\partial y} = 0$. Now, let us see at y = 0 if we satisfy the energy equation. So, what is your energy equation? Energy equation is $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$. So, at y = 0, u and v are 0. So, this left hand side terms will be 0. So, you can see it will be $\frac{\partial^2 T}{\partial y^2}$=0. So, it is derived boundary condition. So, you can write $\frac{\partial^2 T}{\partial y^2}$=0.

So, now, you can see from here. So, if you find $\frac{\partial T}{\partial y}$. So, what you will get? $\frac{\partial T}{\partial y} = C_1 + 2C_2 y + 3C_3 y^2$ if you write. So, you will get $\frac{\partial^2 T}{\partial y^2} = 2C_2 + 6C_3 y$. So, now you put the boundary conditions. At y = 0, $T = T_w$. So, what you will get? So, if you put it here you will get $C_0 = T_w$. Then, at y = 0, $\frac{\partial^2 T}{\partial y^2}$=0. So, this you can see. So, at y = 0 you have $\frac{\partial^2 T}{\partial y^2}$=0, so you will get C_2 =0.

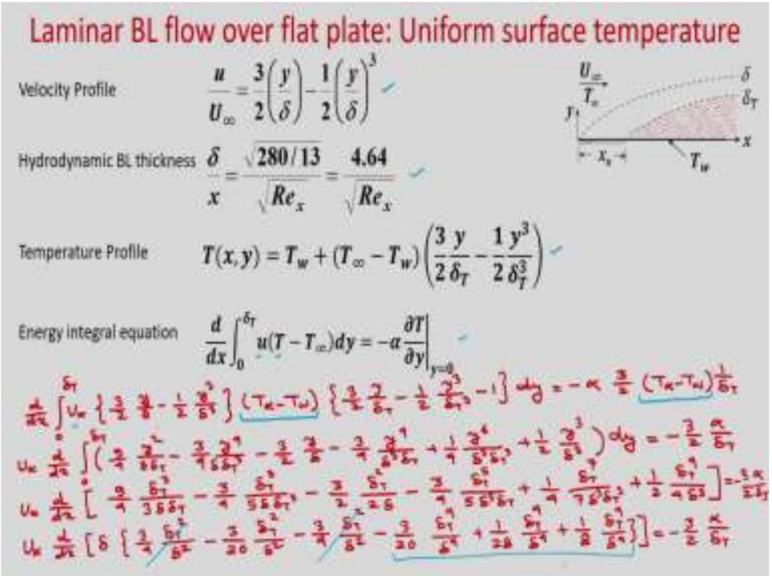
So, $y = \delta_T$, $T = T_\infty$. So, you can see from here this equation you will get $T_\infty = T_w + C_1\delta_T + C_3\delta_T^3$ and $y = \delta_T$ from here you can see $\frac{\partial T}{\partial y}$=0. So, $0 = C_1 + 3C_3\delta_T^2$.

So, you can see from here you will get $C_1 = -3C_3\delta_T^2$. If you put it here then you will get as $T_\infty - T_w = -3C_3\delta_T^3 + C_3\delta_T^3$. So, it will be - 2 $C_3\delta_T^3$. So, you will get $C_3 = -\frac{1}{2}(T_\infty - T_w)\frac{1}{\delta_T}$.

So, you are getting finally, the temperature profile $T(x,y)=T_w+(T_\infty-T_w)[\frac{3}{2}\frac{y}{\delta_T}-\frac{1}{2}\frac{y^3}{\delta_T^3}]$. So, now, we have found the temperature profile, using third degree polynomial.

So, if you see the energy integral equation in the energy integral equation you have velocity U and temperature T. Velocity profile already we have derived in the last class. So, that we need to substitute in the energy integral equation along with this temperature profile and we need to find what is the value of thermal boundary layer thickness.

(Refer Slide Time: 21:19)



So, you can see this is your velocity profile we have derived in the last class. Hydrodynamic boundary layer thickness, this we have already derived in the last class and temperature profile today we have derived this, and this is your energy integral equation. So, in the energy integral equation now you put this u here and this T from this expression and you integrate it and find the thermal boundary layer thickness.

So, if you put it here you can see

$$\frac{d}{dx}\int_0^{\delta_T}U_\infty\{\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\frac{y^3}{\delta^3}\}(T_\infty-T_w)\{\frac{3}{2}\frac{y}{\delta_T}-\frac{1}{2}\frac{y^3}{\delta_T^3}-1\}dy=-\alpha\frac{3}{2}(T_\infty-T_w)\frac{1}{\delta_T}.$$

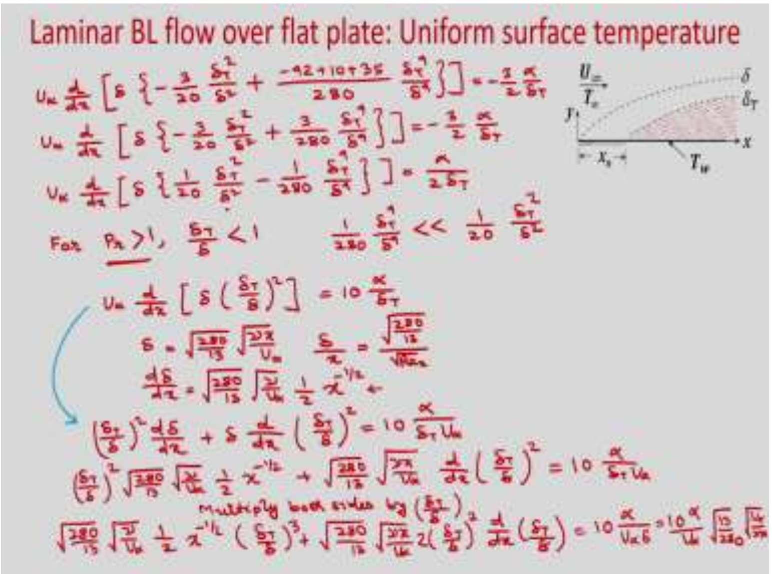
So, now you integrate it. So, this you can see this term T_w -T_∞. And this is constant, so you can take it outside the integral and you can cancel it. So, we will not write this term in the next equation and U_∞ is constant, so you can take it outside. So, now, $U_\infty \frac{d}{dx}\int_0^{\delta_T} (\frac{9}{4}\frac{y^2}{\delta\delta_T} - \frac{3}{4}\frac{y^4}{\delta\delta_T^3} - \frac{3}{2}\frac{y}{\delta} - \frac{3}{4}\frac{y^4}{\delta^3\delta_T} + \frac{1}{4}\frac{y^6}{\delta^3\delta_T^3} + \frac{1}{2}\frac{y^3}{\delta^3})dy = -\frac{3}{2}\frac{\alpha}{\delta_T}$

So, now this we integrate between 0 and δ_T. So, you can see, at y = 0 anyway you every term will become 0. So, what you can do? At y = δ_T you can put, so after taking the integration. So, you can see you will get $U_\infty \frac{d}{dx}[\frac{9}{4}\frac{\delta_T^3}{3\delta\delta_T} - \frac{3}{4}\frac{\delta_T^5}{5\delta\delta_T^3} - \frac{3}{2}\frac{\delta_T^2}{2\delta} - \frac{3}{4}\frac{\delta_T^5}{5\delta^3\delta_T} + \frac{1}{4}\frac{\delta_T^7}{7\delta^3\delta_T^3} + \frac{1}{2}\frac{\delta_T^4}{4\delta^3}] = -\frac{3}{2}\frac{\alpha}{\delta_T}$

So, you can see we will get $U_\infty \frac{d}{dx}$. You take δ outside. So, what you will get? So, you can see it will be $U_\infty \frac{d}{dx}[\delta\{\frac{3}{4}\frac{\delta_T^2}{\delta^2} - \frac{3}{20}\frac{\delta_T^2}{\delta^2} - \frac{3}{4}\frac{\delta_T^2}{\delta^2} - \frac{3}{20}\frac{\delta_T^4}{\delta^4} + \frac{1}{28}\frac{\delta_T^4}{\delta^4} + \frac{1}{8}\frac{\delta_T^4}{\delta^4}] = -\frac{3}{2}\frac{\alpha}{\delta_T}$.

So, you can see here. you have $\frac{3}{4}\frac{\delta_T^2}{\delta^2}$ and here -$\frac{3}{4}\frac{\delta_T^2}{\delta^2}$, so it will cancel. And $\frac{\delta_T^4}{\delta^4}$. So, this we can together write. So, if you write it together, then you can see we will get $U_\infty \frac{d}{dx}[\delta\{-\frac{3}{20}\frac{\delta_T^2}{\delta^2} + \frac{-42+10+35}{280}\frac{\delta_T^4}{\delta^4}\}] = -\frac{3}{2}\frac{\alpha}{\delta_T}$.

(Refer Slide Time: 28:45)



So, you can see these all terms, these 3 terms we have written together here. So, you can see finally, you can write it as $U_\infty \frac{d}{dx}[\delta\{-\frac{3}{20}\frac{\delta_T^2}{\delta^2}+\frac{3}{280}\frac{\delta_T^4}{\delta^4}\}]=-\frac{3}{2}\frac{\alpha}{\delta_T}$.

So, here you can see here 3 is there, 3 is there, 3 is there. So, if you cancel it you can write, $U_\infty \frac{d}{dx}[\delta\{\frac{1}{20}\frac{\delta_T^2}{\delta^2}-\frac{1}{280}\frac{\delta_T^4}{\delta^4}\}]=\frac{\alpha}{2\delta_T}$. So, this is the expression we have derived till now, now we will make one assumptions.

So, let us consider that your $\delta_T < \delta$. So, if $\delta_T < \delta$ that means, Prandtl number > 1. So, for Prandtl number > 1, $\frac{\delta_T}{\delta}<1$. So, we are considering for the cases where Prandtl number > 1.

So, $\frac{\delta_T}{\delta}<1$ and if $\frac{\delta_T}{\delta}<1$, so you can see the last term $\frac{\delta_T^4}{\delta^4}$ will be much much less than this term. So, we can neglect this second term in the left hand side. So, we can write if $\frac{\delta_T}{\delta}<1$ then $\frac{1}{280}\frac{\delta_T^4}{\delta^4}<<\frac{1}{20}\frac{\delta_T^2}{\delta^2}$.

So, what we will do as we are assuming that Prandtl number >1 so obviously, $\delta_T < \delta$ and the second term in the left hand side now we can neglect for simple calculation. So, if you neglect it then it will be easy to integrate. So, you can write it as, $U_\infty \frac{d}{dx}[\delta(\frac{\delta_T}{\delta})^2]=10\frac{\alpha}{\delta_T}$.

So, now from the last class you have derived $\frac{\delta}{x}$ which is your hydrodynamic boundary layer thickness. So, $\frac{\delta}{x}$ we have already derived as $\delta=\sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{U_\infty}}$ or $\frac{\delta}{x}=\frac{\sqrt{\frac{280}{13}}}{\sqrt{\mathrm{Re}_x}}$. And $\frac{d\delta}{dx}=\sqrt{\frac{280}{13}}\sqrt{\frac{\nu}{U_\infty}}\frac{1}{2}x^{-1/2}$, because $x^{1/2}$ is there. So, $\frac{1}{2}x^{-1/2}$. So, this we need it in this calculation.

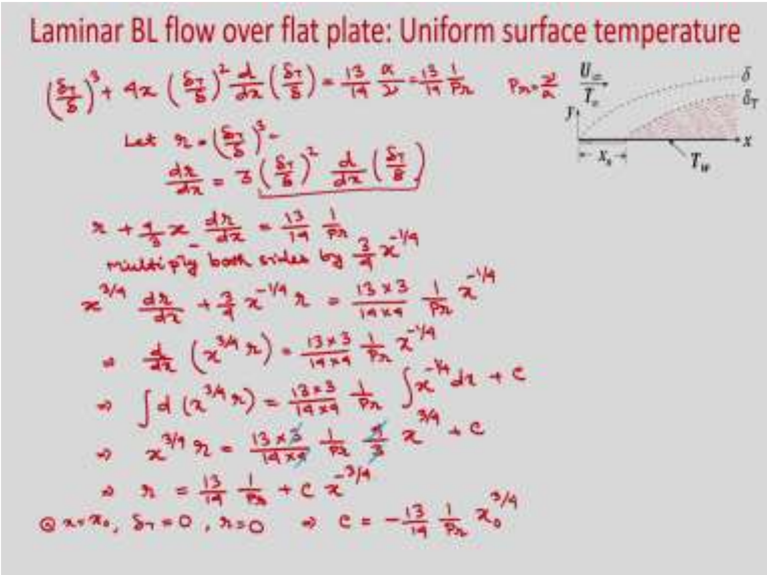
So, from here now you derive U_∞ take in the right hand side, so you can see that it will be $(\frac{\delta_T}{\delta})^2\frac{d\delta}{dx}+\delta\frac{d}{dx}(\frac{\delta_T}{\delta})^2=10\frac{\alpha}{\delta_T U_\infty}$. So, now $\frac{d\delta}{dx}$ we have already found here, so you put it here. So, if you put it here, so you will get

$(\frac{\delta_T}{\delta})^2\sqrt{\frac{280}{13}}\sqrt{\frac{\nu}{U_\infty}}\frac{1}{2}x^{-1/2}+\sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{U_\infty}}\frac{d}{dx}(\frac{\delta_T}{\delta})^2=10\frac{\alpha}{\delta_T U_\infty}$.

So, now multiply both side, multiply both sides by $\frac{\delta_T}{\delta}$. So, $\frac{\delta_T}{\delta}$ we are multiplying in both sides. So, what you will get? So, here you will get $\sqrt{\frac{280}{13}}\sqrt{\frac{\nu}{U_\infty}}\frac{1}{2}x^{-1/2}(\frac{\delta_T}{\delta})^3+\sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{U_\infty}}2(\frac{\delta_T}{\delta})^2\frac{d}{dx}(\frac{\delta_T}{\delta})=10\frac{\alpha}{U_\infty \delta}$.

So, now you can see $10\frac{\alpha}{U_\infty}$. And what is δ? So, δ is this one. So, you can see it will be $\frac{1}{\delta}$, right.

(Refer Slide Time: 36:11)



If you simplify it, what you will get? You will get $(\frac{\delta_T}{\delta})^3+4x(\frac{\delta_T}{\delta})^2\frac{d}{dx}(\frac{\delta_T}{\delta})=\frac{13}{14}\frac{\alpha}{\nu}$, so you can write $\frac{13}{14}\frac{1}{\Pr}$. So, after simplification you will get this expression.

So, now, what we will do we will just put $r=(\frac{\delta_T}{\delta})^3$. So, what will be, $\frac{dr}{dx}$? $\frac{dr}{dx}=3(\frac{\delta_T}{\delta})^2\frac{d}{dx}(\frac{\delta_T}{\delta})$. So, if you put it here you can see you will get, so it will be $r+\frac{4}{3}x\frac{dr}{dx}=\frac{13}{14}\frac{1}{\Pr}$.

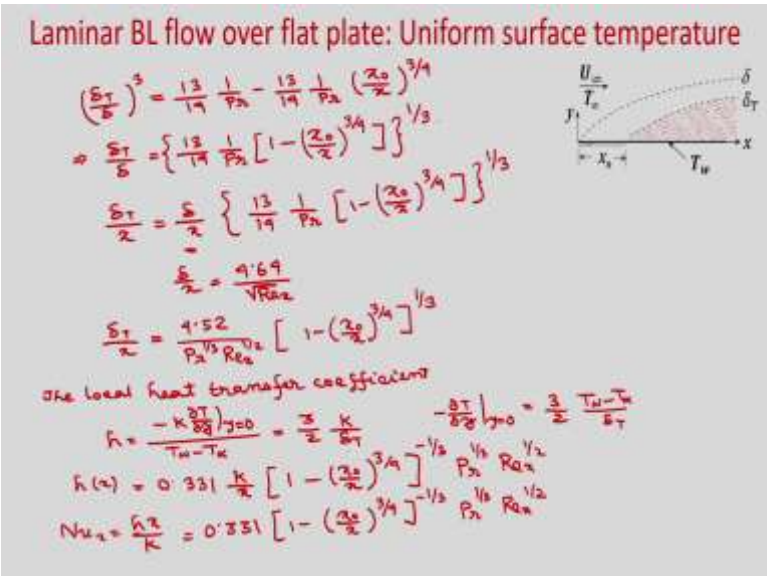
So, multiply both sides by $\frac{3}{4}x^{-1/4}$. So, if you put it here and let us write this as a first term. So, you will get $x^{3/4}\frac{dr}{dx}+\frac{3}{4}x^{-1/4}r=\frac{13\times 3}{14\times 4}\frac{1}{\Pr}x^{-1/4}$.

So, you can see these two terms together you can write $\frac{d}{dx}(x^{3/4}r)=\frac{13\times 3}{14\times 4}\frac{1}{\Pr}x^{-1/4}$.

Now, if you integrate it, $\int d(x^{3/4}r)=\frac{13\times 3}{14\times 4}\frac{1}{\Pr}\int x^{-1/4}dx+C$. So, if you integrate it, it will get $x^{3/4}r=\frac{13\times 3}{14\times 4}\frac{1}{\Pr}\frac{4}{3}x^{3/4}+C$. So, now these 3, these 3, these 4, these 4, will cancel. So, you will get $r=\frac{13}{14}\frac{1}{\Pr}+Cx^{-3/4}$.

So, now, let us put what is the value of δ_T at x = x_0. At x = x_0 you have δ_T as 0, right. So, at x=x_0 you have δ_T = 0. And x = x_0 you can see here δ_T is 0 that means, r = 0 because we are starting your thermal boundary layer thickness is starting from x = x_0, where the value of δ_T is 0. And if δ_T is 0 then r will become 0. So, that will give $C=-\frac{13}{14}\frac{1}{\Pr}x_0^{3/4}$.

(Refer Slide Time: 41:07)

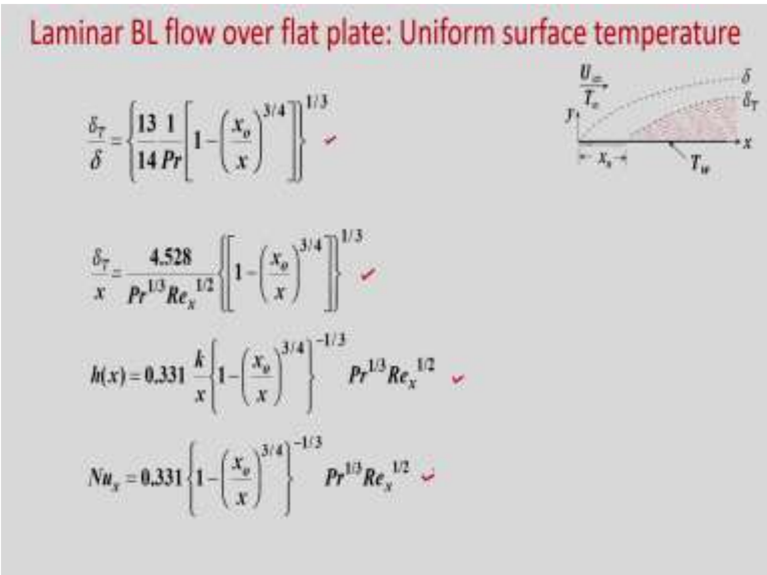


So, if you put it in this expression what you will get? $(\frac{\delta_T}{\delta})^3 = \frac{13}{14}\frac{1}{\Pr} - \frac{13}{14}\frac{1}{\Pr}(\frac{x_0}{x})^{3/4}$. So, that means, you can see $\frac{\delta_T}{\delta} = \{\frac{13}{14}\frac{1}{\Pr}[1-(\frac{x_0}{x})^{3/4}]\}^{1/3}$.

So, $\frac{\delta_T}{\delta}$ now we have found. And we know what is the value of δ, right. So, $\frac{\delta_T}{x} = \frac{\delta}{x}\{\frac{13}{14}\frac{1}{\Pr}[1-(\frac{x_0}{x})^{3/4}]\}^{1/3}$. So, $\frac{\delta}{x}$ we know, right. So, $\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}}$. So, if you put it here, so you will get $\frac{\delta}{x} = \frac{4.52}{\Pr^{1/3}\mathrm{Re}_x^{1/2}}[1-(\frac{x_0}{x})^{3/4}]^{1/3}$. So, once you know the thermal boundary layer thickness you will be able to find what is the heat transfer coefficient, right.

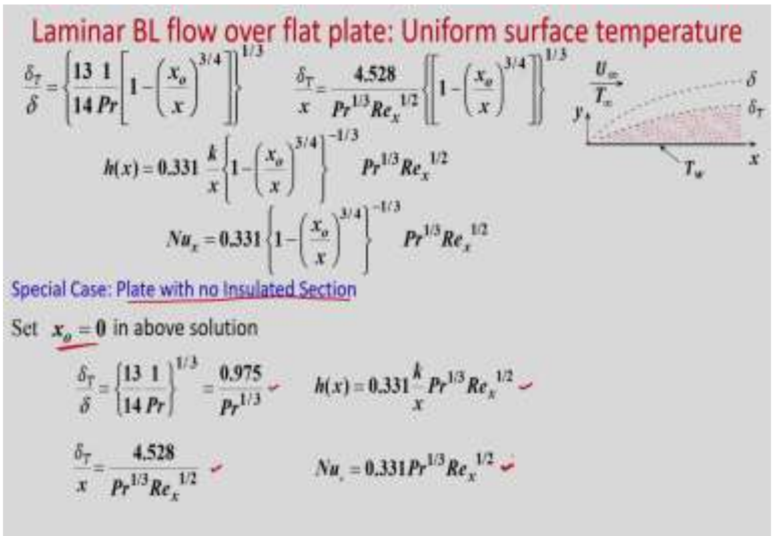
So, you can see, so now, the local heat transfer coefficient. So, $h = \frac{-K\frac{\partial T}{\partial y}|_{y=0}}{T_w - T_\infty}$. So, if you put the value you will get $\frac{3}{2}\frac{K}{\delta_T}$. So, because $-\frac{\partial T}{\partial y}|_{y=0} = \frac{3}{2}\frac{T_w - T_\infty}{\delta_T}$. So, if you put this, so now, you know the value of δ_T, right. If you put the value you will get $h(x) = 0.331\frac{K}{x}[1-(\frac{x_0}{x})^{3/4}]^{-1/3}\Pr^{1/3}\mathrm{Re}_x^{1/2}$. So, now $Nu_x = \frac{hx}{K}$. So, $\frac{hx}{K}$ if you put it here, so you will get $Nu_x = 0.331[1-(\frac{x_0}{x})^{3/4}]^{-1/3}\Pr^{1/3}\mathrm{Re}_x^{1/2}$.

(Refer Slide Time: 44:36)



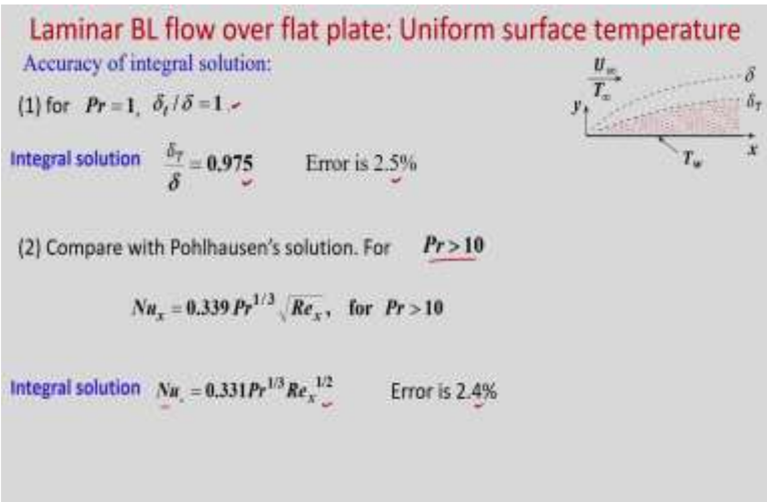
So, you can see that in today's class now starting from the energy equation we have found the thermal boundary layer thickness $\frac{\delta_T}{\delta}$ as this one. Then, putting the value of $\frac{\delta}{x}$ we have found $\frac{\delta_T}{x}$ as this. Then, we have found h(x) which is your local heat transfer coefficient, then local Nusselt number.

(Refer Slide Time: 45:07)



Now, let us consider a special case, plate with no insulated section. So, there is no insulated section so that means, $x_0 = 0$. If you put $x_0 = 0$ then your thermal boundary layer and hydrodynamic boundary layer will start going from $x = 0$. So, if $x_0 = 0$ if you put in the above equation, so you will get $\frac{\delta_T}{\delta}=\frac{0.975}{\mathrm{Pr}^{1/3}}$ and $\frac{\delta_T}{x}=\frac{4.528}{\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}$ and local heat transfer coefficient, and local Nusselt number.

(Refer Slide Time: 45:50)

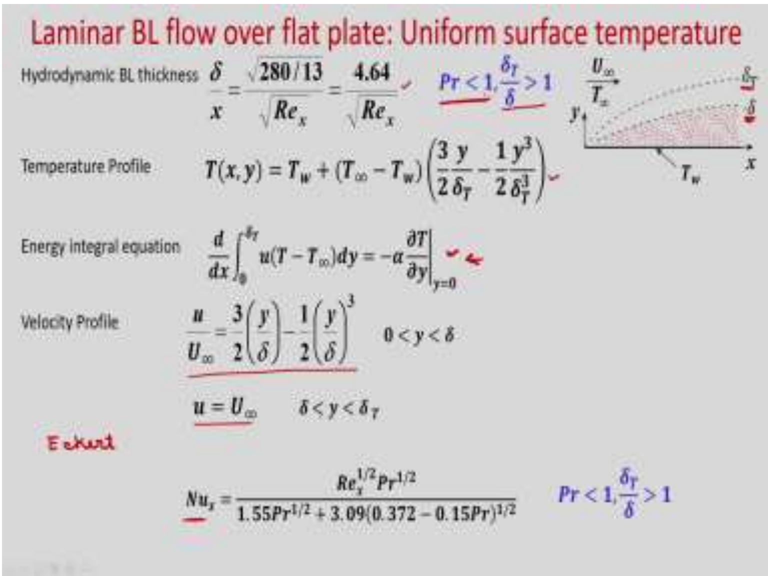


So, now what is the accuracy of this integral solution? If you see for Prandtl number = 1, $\frac{\delta_T}{\delta} = 1$. So, that is the exact solution. But from the integral solution $\frac{\delta_T}{\delta}$, for Prandtl number=1 we got $\frac{\delta_T}{\delta} = 0.975$. You can see if you put it in this expression Prandtl number =1, then $\frac{\delta_T}{\delta} = 0.975$. So, you can see that error is 2.5% using integral solution.

And, if you see the exact solution from the Pohlhausen solution for Prandtl number >10, $Nu_x = 0.339 \Pr^{1/3} \sqrt{\mathrm{Re}_x}$, but in today's class we have found from the integral solution $Nu_x = 0.331 \Pr^{1/3} \sqrt{\mathrm{Re}_x}$. So, you can if you compare it you can see error is 2.4 %.

So, you can see we have assumed that Prandtl number > 1 that means, your $\delta_T < \delta$ and we have found the expression power heat transfer coefficient, Nusselt number and thermal boundary layer thickness. If Prandtl number < 1, where $\delta_T > \delta$ then you have a rigorous derivation. So, that we will not derive, but just I will give the expression, final expression for the Nusselt number.

(Refer Slide Time: 47:19)



So, we can see in this particular case we are considering Prandtl number < 1. So, $\frac{\delta_T}{\delta} > 1$. So, that means, δ_T is higher than the δ. So, in this particular case, you can see $\frac{\delta}{x}$ already we have found, this is your T(x, y) this is your energy integral equation.

So, in the energy integral equation, you can see 0 to δ. 0 to δ you have this velocity distribution, but δ to δ_T here you will have $u = \infty$. So, if you put it these two expression in this energy integral equation then you can find $Nu_x = \frac{\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/2}}{1.55\,\mathrm{Pr}^{1/2} + 3.09(0.372 - 0.15\,\mathrm{Pr})^{1/2}}$. So, this solution actually scientists Eckert found analytically for Prandtl number <1.

So, in today's class, we considered the laminar boundary layer flow over a flat plate and we considered uniform wall temperature T_w from $x = x_0$, and from $x = 0$ to $x = x_0$ it was adiabatic. So, your thermal boundary layer thickness starts developing from $x = x_0$.

And, first we derived the energy integral equation, then assume the third degree polynomial we derived the temperature profile, and we put the velocity profile and temperature profile in the energy integral equation. Then, we found the thermal boundary layer thickness δ_T, and then we found the local heat transfer coefficient and local Nusselt number and putting $x_0 = 0$ we found a special case where there is no heat unheated region. And finally, for Prandtl number < 1 we have discussed that how we can find the Nusselt number expression.

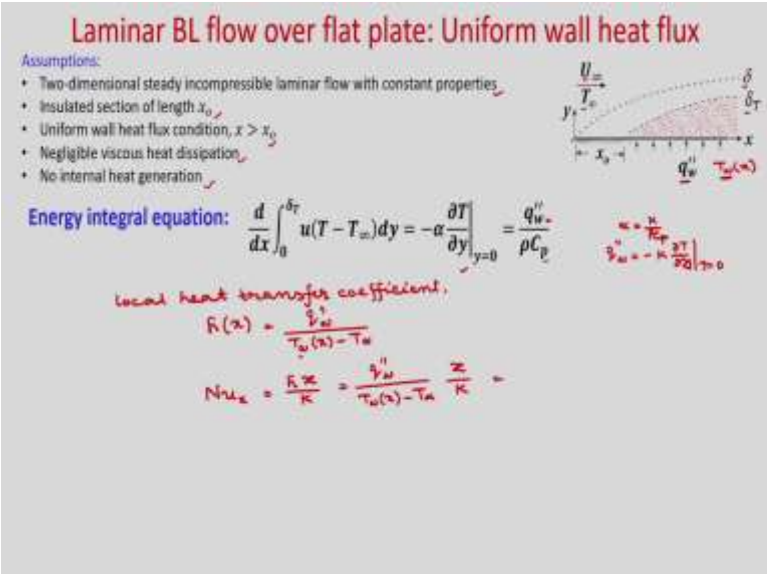
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 04
Convective Heat Transfer in External Flows – II
Lecture – 12
Laminar BL flow over flat plate: Uniform wall heat flux

Hello everyone. So, today we will consider Boundary Layer flow over a flat plate with Uniform wall heat flux condition. So, in last lecture we considered uniform wall temperature condition, but today we will consider uniform heat flux boundary condition. We wish to determine the wall temperature T_w, as a function of x and the local Nusselt number.

(Refer Slide Time: 00:56)



So, let us consider this flat plate, y is measured perpendicular to the flat plate, your free stream velocity is U_∞ and temperature is T_∞. Up to $x = x_0$ it is insulated, so it will be maintained at temperature T_∞ as there will be no heat transfer. From $x = x_0$, you can see this plate is maintained at uniform wall heat flux q''_w.

So, your thermal boundary layer thickness will start developing from $x = x_0$ and hydrodynamic boundary layer thickness will start developing from x = 0. So, these are the assumptions; two dimensional steady incompressible laminar flow with constant properties, insulated section of length x_0, uniform wall heat flux condition $x > x_0$, negligible viscous heat dissipation and no internal heat generation.

So, in last class already we have derived the energy integral equation. So, that we will use. And, we will first find the temperature distribution using third degree polynomial. And, we will put the velocity distribution and temperature distribution in the energy integral equation and we will find the expression for thermal boundary layer thickness.

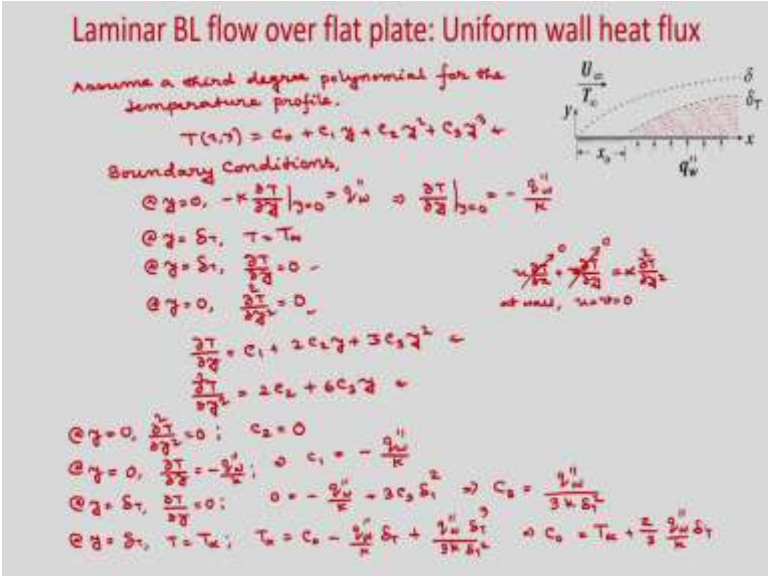
And, then we will find the wall temperature distribution as well as local Nusselt number. So, we can see this is the energy integral equation already we have derived. This right hand side $-\alpha \frac{\partial T}{\partial y}\Big|_{y=0}$ this we can write $\frac{q_w''}{\rho C_p}$, because $\alpha = \frac{K}{\rho C_p}$ and $q_w'' = -K\frac{\partial T}{\partial y}\Big|_{y=0}$.

Hence, this right hand side can be written as $\frac{q_w''}{\rho C_p}$. You know that in this particular case q_w'' is constant and ρC_p are the properties and that also are constant. So, right hand side is a constant term.

However, you can see here temperature T_w, wall temperature T_w will be function of x; because along the x, your T_w will increase. So, as q_w'' is constant, you can find the local heat transfer coefficient $h(x) = \frac{q_w''}{T_w(x) - T_\infty}$. So, you can see T_∞ is your free stream temperature and T_w is function of x. And, q_w'' is constant.

So, this is from Newton's law of cooling we have written. Now, Nusselt number $Nu_x = \frac{hx}{K}$. So, you can see, you can write $Nu_x = \frac{q_w''}{T_w(x) - T_\infty}\frac{x}{K}$. So, this is the expression for Nusselt number. Now, first let us consider a third degree polynomial for temperature distribution and we will find the coefficient using the boundary conditions.

(Refer Slide Time: 04:40)



So, assume a third degree polynomial for the temperature profile. So, T which is function of (x, y) we can write $T(x,y) = C_0 + C_1 y + C_2 y^2 + C_3 y^3$. So, these coefficients Cc$_0$, C_1, C_2, C_3 are function of x. So, now, we have boundary conditions at y = 0; y = 0, you can see your heat flux is given. So, you can write $-K\frac{\partial T}{\partial y}|_{y=0} = q_w^{"}$.

So, you can write $\frac{\partial T}{\partial y}|_{y=0} = -\frac{q_w^{"}}{K}$, and at y = δ_T you have temperature T_∞ and also a temperature gradient $\frac{\partial T}{\partial y}$ = 0. So, at y = δ_T, you have T = T_∞ and also you have at y = δ_T, $\frac{\partial T}{\partial y}$ = 0; because, that is the free stream temperature, so there will be no gradient. And, another boundary condition we will derive from the energy equation satisfying it at the wall.

So, at y = 0, you can write $\frac{\partial^2 T}{\partial y^2} = 0$. So, you remember in last class we have done $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$. So, this is your boundary layer energy equation. So, at wall you have u = v = 0. So, if u and v are 0; so obviously, left hand side terms will be 0.

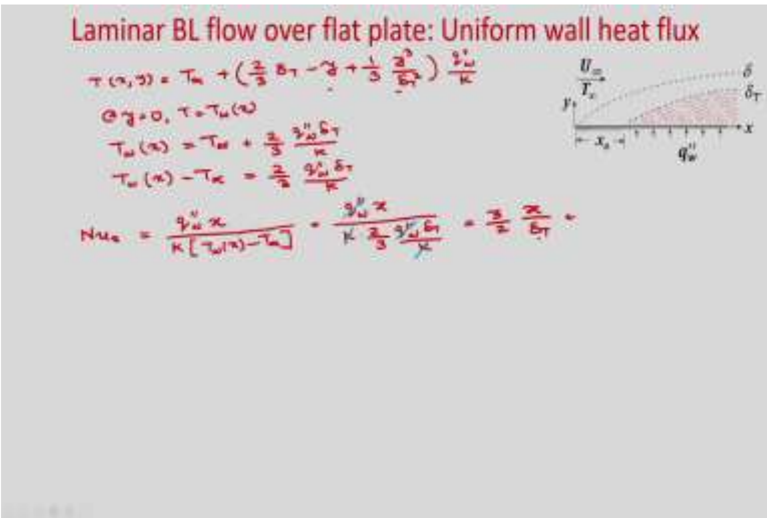
So, $\frac{\partial^2 T}{\partial y^2} = 0$. So, now, you have four boundary conditions and four coefficients. So, those four are known coefficients you can find satisfying these boundary conditions. So, $\frac{\partial T}{\partial y} = C_1 + 2C_2 y + 3C_3 y^2$; $\frac{\partial^2 T}{\partial y^2} = 2C_2 + 6C_3 y$.

So, we can see at y =0, you have $\frac{\partial^2 T}{\partial y^2} = 0$. So, if you satisfy this from this equation, you can see C_2 = 0. Then, at y = 0, you have $\frac{\partial T}{\partial y} = -\frac{q_w^"}{K}$. So, this is your $\frac{\partial T}{\partial y}$. So, at y = 0 if you satisfy, so last two terms will become 0. So, that will give $C_1 = -\frac{q_w^"}{K}$. And, now you see at y = δ_T, you have $\frac{\partial T}{\partial y}$ = 0.

So, if it is 0, so you can see from this equation, you will get $0 = -\frac{q_w^"}{K} + 3C_3 \delta_T^2$. So, that means, from here you will get $C_3 = \frac{q_w^"}{3K\delta_T^2}$. Now, another boundary condition you apply at y = δ_T is, you have T = T_∞.

So, from this equation you can see, you can write $T_\infty = C_0 - \frac{q_w^"}{K}\delta_T + \frac{q_w^" \delta_T^3}{3K\delta_T^2}$. So, you can see this will be $C_0 = T_\infty + \frac{2}{3}\frac{q_w^"}{K}\delta_T$.

(Refer Slide Time: 10:24)



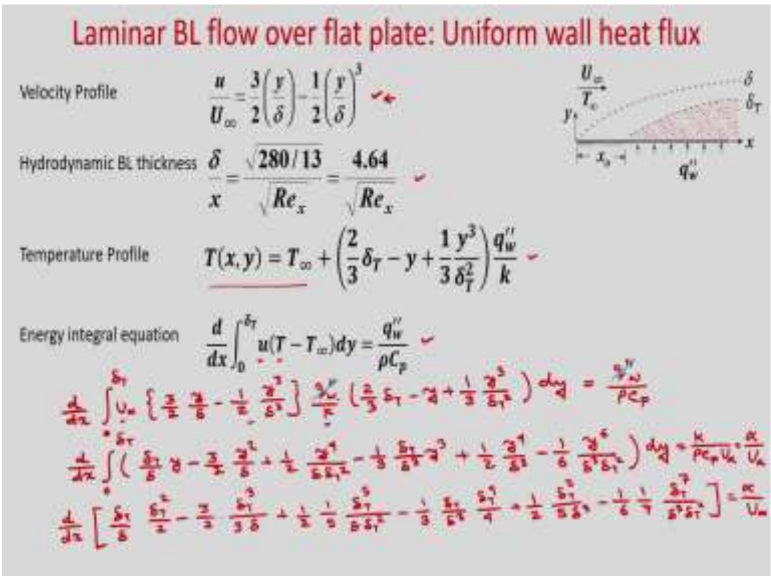
So, we have found four coefficients, now you put it in the temperature expression; then, your final temperature distribution will be for these boundary conditions, $T(x,y) = T_\infty + (\frac{2}{3}\delta_T - y + \frac{1}{3}\frac{y^3}{\delta_T^2})\frac{q_w^"}{K}$. So, this is the general temperature distribution. Now, at the wall if you want to find what is the temperature variation, then you put y = 0.

So, at y = 0, T = T_w(x). So, if you put y = 0; so these two terms will become 0, so you can write $T_w(x) = T_\infty + \frac{2}{3}\frac{q_w^"\delta_T}{K}$. And, also you can write $T(x) - T_\infty = \frac{2}{3}\frac{q_w^"\delta_T}{K}$. So, now, you can put it in the Nusselt number distribution whatever we have found.

So, Nusselt number we have written, $Nu_x = \frac{q_w^" x}{K[T(x) - T_\infty]}$. So, if you put $T(x) - T_\infty$ this expression, so what you will get; $\frac{q_w^" x}{K\frac{2}{3}\frac{q_w^"\delta_T}{K}}$. So, finally, you can write this $Nu_x = \frac{3}{2}\frac{x}{\delta_T}$.

So, you can see in this expression Nusselt number now you can find, once you find the thermal boundary layer thickness δ_T. So, once we find $\frac{\delta_T}{x}$, then you will be able to find the local Nusselt number. So, what we will do now? We know the velocity profile, we know the temperature profile, we have the energy integral equation; so, you put it in the energy integral equation, then you will be able to find the thermal boundary layer thickness δ_T.

(Refer Slide Time: 13:10)



So, we can see we have the velocity profile already we have derived using third degree polynomial and from that solution, hydrodynamic boundary layer thickness we have found, $\frac{\delta_T}{x}$ as this. In today's class we have found the temperature distribution for the given boundary conditions and this is the energy integral equation. So, now, you put the value of u here and value of T here; so you will be able to find, what is the thermal boundary layer thickness?

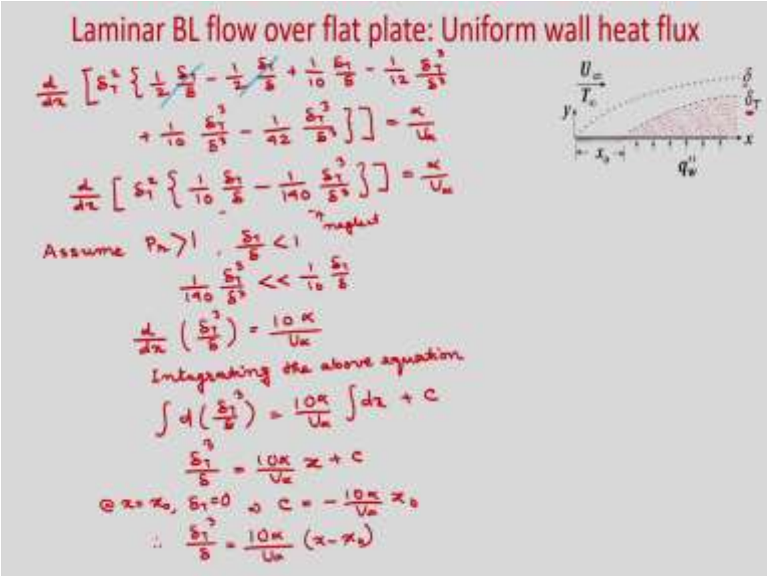
So, you can write $\frac{d}{dx}\int_0^{\delta_T} U_\infty\{\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\frac{y^3}{\delta^3}\}\frac{q_w^{"}}{K}(\frac{2}{3}\delta_T-y+\frac{1}{3}\frac{y^3}{\delta_T^2})dy=\frac{q_w^{"}}{\rho C_p}$.

So, in this expression you can see $q_w^{"}$ is constant. So, these $q_w^{"}$ you can take it outside the integral and you can cancel, right. So, this $q_w^{"}$ can cancel and this k you can take in the right hand side. So, in the next step you can see, we can write $\frac{d}{dx}\int_0^{\delta_T}(\frac{\delta_T}{\delta}y-\frac{3}{2}\frac{y^2}{\delta}+\frac{1}{2}\frac{y^4}{\delta\delta_T^2}-\frac{1}{3}\frac{\delta_T}{\delta^3}y^3+\frac{1}{2}\frac{y^4}{\delta^3}-\frac{1}{6}\frac{y^6}{\delta^3\delta_T^2})dy=\frac{K}{\rho C_p U_\infty}$.

Because, these are constant, so you can take it outside the integral and you take in the right hand side. So, you can write $\frac{K}{\rho C_p U_\infty}$. And, $\frac{K}{\rho C_p}=\alpha$ right, thermal diffusivity; so, $\frac{\alpha}{U_\infty}$. So, now, you integrate it. So, if you integrate it. So, you can see we can find. So, at y =0, this will become 0 anyway and y =δ_T, we will put after the integration.

So, we can write, $\frac{d}{dx}[\frac{\delta_T}{\delta}\frac{\delta_T^2}{2}-\frac{3}{2}\frac{\delta_T^3}{3\delta}+\frac{1}{2}\frac{1}{5}\frac{\delta_T^5}{\delta\delta_T^2}-\frac{1}{3}\frac{\delta_T}{\delta^3}\frac{\delta_T^4}{4}+\frac{1}{2}\frac{\delta_T^5}{5\delta^3}-\frac{1}{6}\frac{1}{7}\frac{\delta_T^7}{\delta^3\delta_T^2}]=\frac{\alpha}{U_\infty}$. So, now you simplify it, you cancel some terms.

(Refer Slide Time: 18:27)



So, you can see here these 3, 3 will get cancel and you can write finally, $\frac{d}{dx}[\delta_T^2\{\frac{1}{2}\frac{\delta_T}{\delta}-\frac{1}{2}\frac{\delta_T}{\delta}+\frac{1}{10}\frac{\delta_T}{\delta}-\frac{1}{12}\frac{\delta_T^3}{\delta^3}+\frac{1}{10}\frac{\delta_T^3}{\delta^3}-\frac{1}{42}\frac{\delta_T^3}{\delta^3}\}]=\frac{\alpha}{U_\infty}$. So, you can see these first two terms. So, you will it will get cancel.

So, $\frac{d}{dx}[\delta_T^2\{\frac{1}{10}\frac{\delta_T}{\delta}-\frac{1}{140}\frac{\delta_T^3}{\delta^3}\}]=\frac{\alpha}{U_\infty}$. So, now, we will assume that Prandtl number is > 1. So, if Prandtl number > 1, then you know $\delta_T < \delta$. And, from this expression, now we will neglect the second term in the left hand side. So, you can see we are assuming, assume Prandtl number > 1 and for Prandtl number > 1, you know $\frac{\delta_T}{\delta} < 1$.

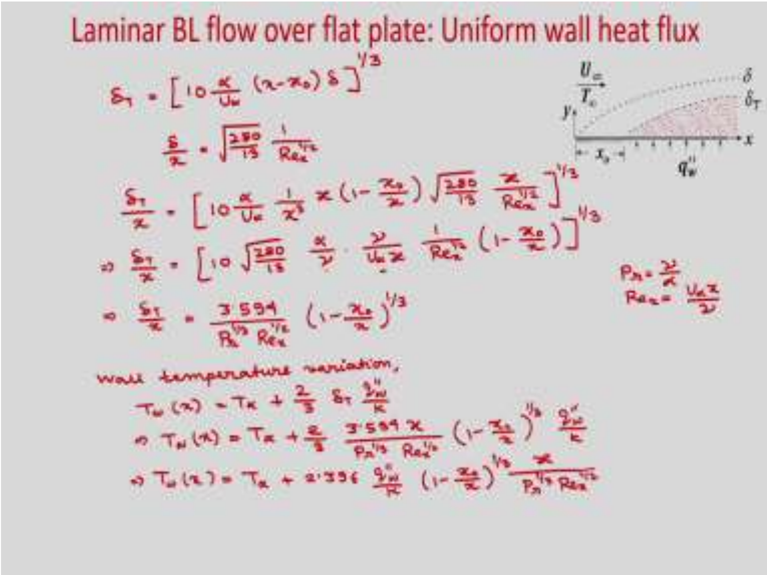
So, in this particular case, you can see that your thermal boundary layer thickness will be less than the hydrodynamic boundary layer thickness. So, if it is so, if you compare these two terms; then you can see $\frac{1}{140}\frac{\delta_T^3}{\delta^3} << \frac{1}{10}\frac{\delta_T}{\delta}$.

So, neglect this term, this term you neglect. So, you can write $\frac{d}{dx}(\frac{\delta_T^3}{\delta^3})=\frac{10\alpha}{U_\infty}$. Let us integrate this. So, this is ordinary differential equation. So, you can integrate it and you know that at x = x_0, you have thermal boundary layer thickness δ_T as 0.

So, if you integrate it, so you will get integrating the above equation $\int d(\frac{\delta_T^3}{\delta^3}) = \frac{10\alpha}{U_\infty}\int dx + C$. So, you will get $\frac{\delta_T^3}{\delta} = \frac{10\alpha}{U_\infty}x + C$. And, we know at x=$x_0$, you have $\delta_T = 0$.

So, your thermal boundary layer thickness starts from x=x_0; so here at x=x_0, $\delta_T = 0$. So, from here you can see, $C = -\frac{10\alpha}{U_\infty}x_0$. Hence, you can see that, $\frac{\delta_T^3}{\delta} = \frac{10\alpha}{U_\infty}(x - x_0)$.

(Refer Slide Time: 22:55)



So, you can write $\delta_T = [10\frac{\alpha}{U_\infty}(x - x_0)\delta]^{1/3}$. Now, let us put the expression for hydrodynamic boundary layer thickness δ that we have already found from solving the momentum integral equation. So, you can write $\frac{\delta}{x} = \sqrt{\frac{280}{13}}\frac{1}{\mathrm{Re}_x^{1/2}}$. So, this is the expression we have.

So, now, you can see, you can write from this expression $\frac{\delta_T}{x}$; so we are dividing by x. So, if you divide the right hand side by x and if you take inside this power; so you will get x^3, right. So, you can write $\frac{\delta_T}{x} = [10\frac{\alpha}{U_\infty}\frac{1}{x^3}x(1 - \frac{x_0}{x})\sqrt{\frac{280}{13}}\frac{x}{\mathrm{Re}_x^{1/2}}]^{1/3}$.

So, you can see, you can find $\frac{\delta_T}{x}$ as so, here you have x^2 and so, if you rearrange it; so you can see, you will get, $\frac{\delta_T}{x} = [10\sqrt{\frac{280}{13}\frac{\alpha}{\nu}\frac{\nu}{U_\infty x}\frac{1}{\mathrm{Re}_x^{1/2}}}(1-\frac{x_0}{x})]^{1/3}$. So, you can write $\frac{\delta_T}{x}$. So, you can see here; what is this expression? This is here $\frac{1}{\sqrt{\mathrm{Re}_x}}$.

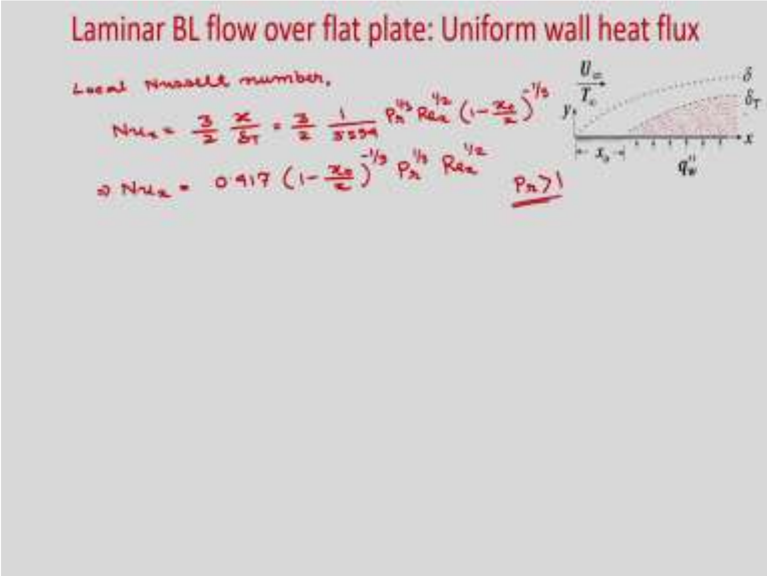
So, $\frac{1}{\sqrt{\mathrm{Re}_x}}$ and here $\mathrm{Re}_x^{1/2}$; so it will be 3/2. So, it will be 3/2 and here $\frac{\nu}{\alpha} = \mathrm{Pr}$. So, you have $Pr = \frac{\nu}{\alpha}$. So, and $\mathrm{Re}_x = \frac{U_\infty x}{\nu}$. So, you will get here $\frac{1}{\mathrm{Pr}}$, here you will get $\frac{1}{\mathrm{Re}_x}$ and $\mathrm{Re}_x^{1/2}$ you have.

So, it will be 3 /2, and outside this bracket if you take, then it will become $\sqrt{\mathrm{Re}_x}$. So, you can see, you can write $\frac{\delta_T}{x} = \frac{3.594}{\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}(1-\frac{x_0}{x})^{1/3}$. So, after simplification, we have now derive the expression for $\frac{\delta_T}{x}$.

Now, once you know $\frac{\delta_T}{x}$, now you will be able to find, what is the temperature distribution and what is the Nusselt number? So, if you put this $\frac{\delta_T}{x}$, then you can get your wall temperature distribution as; wall temperature variation as $T_w(x) = T_\infty + \frac{2}{3}\delta_T\frac{q_w''}{K}$. So, if you put this expression, then you will get, $T_w(x) = T_\infty + \frac{2}{3}\frac{3.594x}{\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}(1-\frac{x_0}{x})^{1/3}\frac{q_w''}{K}$.

So, hence you will get $T_w(x) = T_\infty + 2.396\frac{q_w''}{K}(1-\frac{x_0}{x})^{1/3}\frac{x}{\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}$. So, this is the wall temperature variation. So, you can see from this expression that it is function of x, right. Now let us find, what is that local Nusselt number? Already, we have written local Nusselt number in terms of the thermal boundary layer thickness.

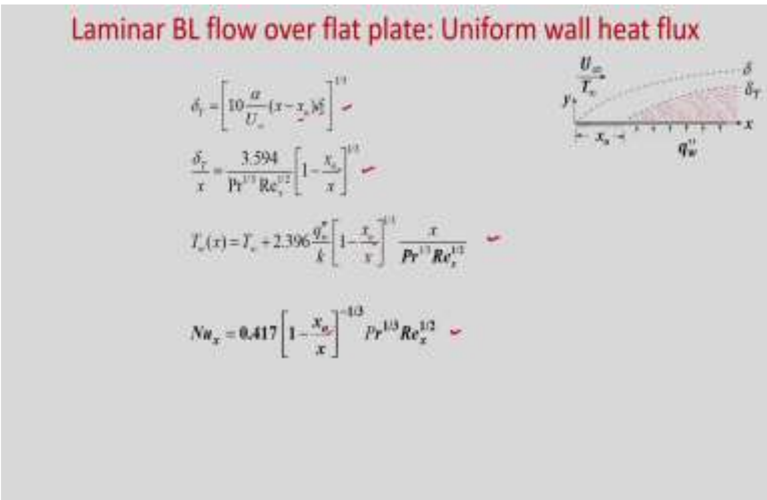
(Refer Slide Time: 28:38)



So, if you remember, we have already derived this expression local Nusselt number $Nu_x = \frac{3}{2}\frac{x}{\delta_T}$. And now, we know the expression of $\frac{\delta_T}{x}$. So, we can write $\frac{3}{2}\frac{1}{3.594}\Pr^{1/3}\mathrm{Re}_x^{1/2}(1-\frac{x_0}{x})^{-1/3}$. So, if you rearrange, you will get Nusselt number as, $Nu_x = 0.417(1-\frac{x_0}{x})^{-1/3}\Pr^{1/3}\mathrm{Re}_x^{1/2}$.

So, this is the Nusselt number expression we have found for Prandtl number > 1 using the approximate method; because we have approximated the velocity profile as well as the temperature profile. So, this is valid for Prandtl number >1, because we have assumed $\delta_T < \delta$.

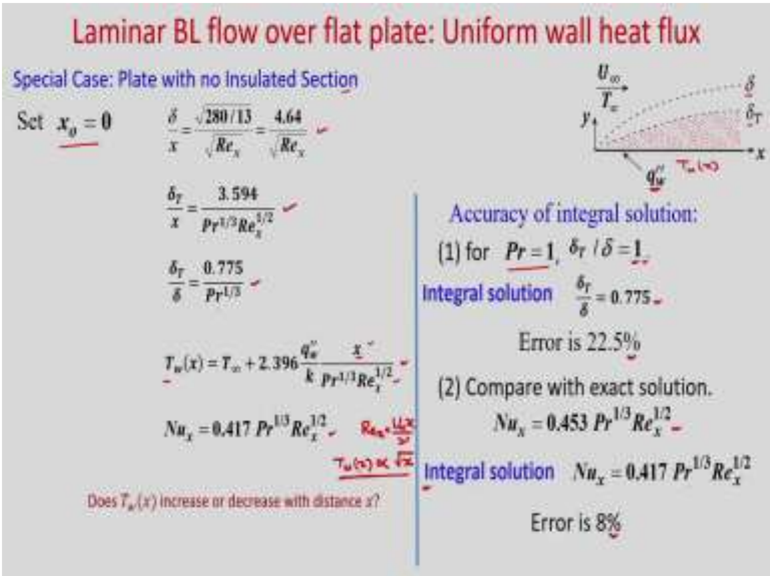
(Refer Slide Time: 30:04)



So, you can see, we have finally derived in today's class this δ_T in terms of hydrodynamic boundary layer thickness δ; then putting the value of δ, we have found $\frac{\delta_T}{x}$. And, you can see it is also function of Prandtl number and Reynolds number. And, putting this expression in the wall temperature variation, we found this is the wall temperature variation and then, we have found the local Nusselt number as this.

Now, let us consider a special situation when there is no insulated region; so that means $x_0 = 0$. So, in this expression you can see, if you put $x_0 = 0$; then, you will get the expression for thermal boundary layer thickness, wall temperature variation and local Nusselt number for the unheated region as 0.

(Refer Slide Time: 31:10)



So, you can see in this particular case $x_0 = 0$; so, thermal boundary layer thickness and hydrodynamic boundary layer thickness starts developing from x = 0. So, in for the special case, in earlier expression if you put $x_0 = 0$, where you have plate with no insulated section; then, we have already found $\frac{\delta}{x}$, then this is your $\frac{\delta_T}{x}$ putting x_0=0and $\frac{\delta_T}{\delta}$.

If you can see that $\frac{\delta_T}{\delta}$ if you put; then, you will get as $\frac{0.775}{\Pr^{1/3}}$. And, wall temperature variation you can see here you will get as $T_w(x) = T_\infty + 2.396\frac{q_w''}{K}(1-\frac{x_0}{x})^{1/3}\frac{x}{\Pr^{1/3}\mathrm{Re}_x^{1/2}}$.

So, you can see it varies with x. And, $Nu_x = 0.417(1-\frac{x_0}{x})^{-1/3}\Pr^{1/3}\mathrm{Re}_x^{1/2}$. In this expression you can see, does $T_w(x)$ increase or decrease with distance x? You can see that, you have here Re_x,one x is there and also here x is there; so you can see that your wall temperature will increase along x. So, if you although in this particular case your; you have this plate with uniform wall heat flux; but T_w which is function of x will increase along x.

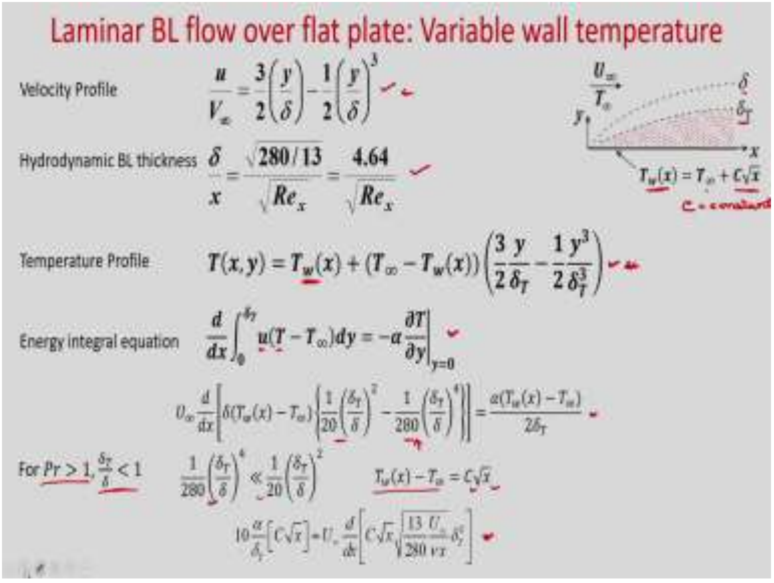
Now, let us see, what is the accuracy compared to the exact solution? Because, in this particular case, we have used approximate method where we have approximated the velocity profile as well as the temperature profile as third degree polynomial; so, we have found what is the thermal boundary layer thickness as well as the Nusselt number. Now, let us compare this with the exact solution. So, you can see for Prandtl number= 1 exact solution $\frac{\delta_T}{\delta}$ should be 1; because $\delta_T = \delta$ for Prandtl number= 1. But, from the integral solution you can see, for Prandtl number =1 , $\frac{\delta_T}{\delta}$ = 0.775. So, error is much in predicting the thermal boundary layer thickness, it is 22.5 %.

Now, if you compare the Nusselt number with the exact solution. So, this is the exact solution, you can see $Nu_x = 0.453\Pr^{1/3}\mathrm{Re}_x^{1/2}$. So, this is your follows a solution with unheated region. So, you can see this is the expression; but from the approximate solution, we have found $0.417\Pr^{1/3}\mathrm{Re}_x^{1/2}$.

So, you can see error is almost 8 %, but it is Nusselt number is predicting well right; but here δ_T is having much difference with the exact solution. So, in this particular expression you can see, your $\mathrm{Re}_x = \frac{U_\infty x}{\nu}$. So, you have; so that means your, in the denominator you have $\sqrt{x}$ and this is your x, so that means T_w varies with $\sqrt{x}$, you can see.

So, this is in the numerator we have x and in the denominator you have $\sqrt{x}$. So, in the $\frac{x}{\sqrt{x}} = \sqrt{x}$. So, T_w varies $\sqrt{x}$. Now, if we assume a variable temperature profile in the flat plate; then, can we get back the same expression of Nusselt number whatever we have got assuming the constant wall heat flux boundary condition. So, let us see that.

(Refer Slide Time: 35:12)



So, now we are considering laminar boundary layer flow over flat plate with variable wall temperature. So, you can see that your wall temperature varies with $\sqrt{x}$. So, we have taken this flat plate where temperature varies as $T_\infty + C\sqrt{x}$, where C is your constant.

And, in last slide we have seen that, generally for constant wall heat flux condition T_w varies as $\sqrt{x}$. So, we have taken $T_x(x) = T_\infty + C\sqrt{x}$. So, you have free stream temperature T_∞ and Prandtl number > 1, so that $\delta_T < \delta$.

So, for this expression if you use the third degree polynomial for velocity profile; so, we have already derived this, $\frac{\delta}{x}$ we have derived this, temperature profile. Now, with these boundary conditions if you see that we have, in the last class we have used uniform wall temperature boundary condition and for that, we have found the temperature profile. So, same temperature profile we can put it, where T_w is function of x.

So, we can see this is the same expression what we have derived already for uniform wall temperature boundary condition and this is the T (x, y); but here T_w is function of x.

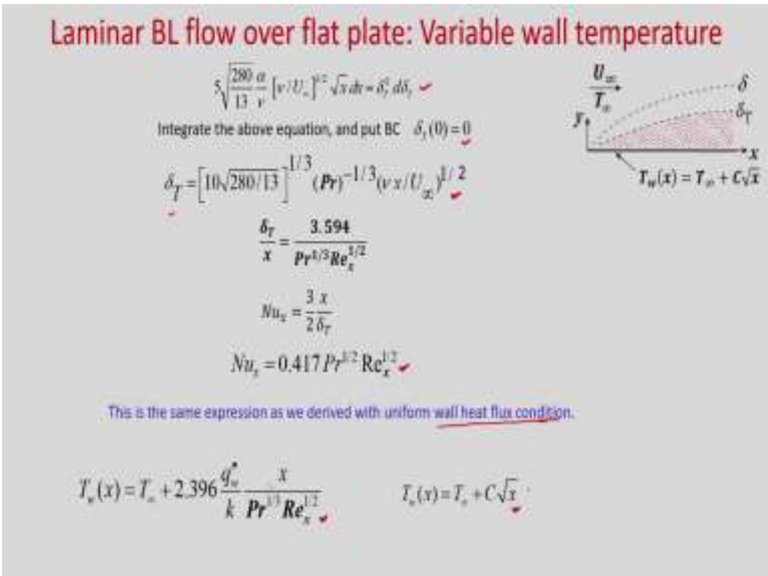
So, $T(x,y) = T_w(x) + (T_\infty - T_w(x))(\frac{3}{2}\frac{y}{\delta_T} - \frac{1}{2}\frac{y^3}{\delta_T^3})$. And, this is the energy integral equation, right. So, now, in this expression you put u and T.

So, already this we have derived and here already we have derived this; but here T_w is function of x, because your wall temperature varies like this. So, if you put it and you will get U_∞ is constant. So, we have taken outside $\frac{d}{dx}$. So, you will get from this you can see, it will be $\frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)dy = -\alpha \frac{\partial T}{\partial y}|_{y=0}$.

So, you will get, $U_\infty \frac{d}{dx}[\delta(T_w(x) - T_\infty)\{\frac{1}{20}(\frac{\delta_T}{\delta})^2 - \frac{1}{280}(\frac{\delta_T}{\delta})^4\}] = \frac{\alpha(T_w(x) - T_\infty)}{2\delta_T}$.

So, if you see these two terms and we have used Prandtl number > 1; so that means $\frac{\delta_T}{\delta}$<1. So, in this particular case you can see, you can neglect this term; because, this term will be much much less than the this term. And, $T_x(x) = T_\infty + C\sqrt{x}$. So, these if you put it here, you can see we will get this expression. And, it is easy to integrate, because you can see here you can put the expression for $T_x(x) - T_\infty = C\sqrt{x}$.

(Refer Slide Time: 38:18)



And, after simplification you will get this, and integrate this above equation and put the boundary condition that at x = 0, you have δ_T = 0. So, you will get a δ_T like this expression. And, if you rearrange this you will get $\frac{\delta_T}{x} = \frac{3.594}{\Pr^{1/3} \mathrm{Re}_x^{1/2}}$ and $Nu_x = \frac{3}{2}\frac{x}{\delta_T}$ and Nusselt number x you will get this. And, you can see that this is the same expression as we derived for uniform wall heat flux condition. And, you can see the temperature profile whatever we have got it from the uniform wall heat flux condition. So, here we can see, if you take from Re_x this $\sqrt{x}$ outside; then, you will get $\frac{x}{\sqrt{x}}$ and it will be $\sqrt{x}$ and all other terms are constant, because $q_w^{"}$ is constant, k is constant, Prandtl number is constant and here free properties and velocity are constant.

So, all these will be constant. So, you can write $T_x(x) - T_\infty = C\sqrt{x}$. So, you can see that, keeping the flat plate at uniform wall heat flux condition or keeping the flat plate as variable wall temperature where wall temperature varies as $\sqrt{x}$, both will give the same result; because, you have seen that Nusselt number expression and these $\frac{\delta_T}{x}$ expressions are same in both the cases. So, in today's lecture, we considered laminar flow over a flat plate with uniform wall heat flux boundary condition.

So, $q_w^{"}$ is constant on the flat plate; however, you have T_w which is your wall temperature varies with x. We considered initially up to x = x_0 as a unheated region, and from x > x_0, it is maintained at a uniform wall heat flux boundary condition.

Then, we found the temperature profile using third degree polynomial; applying four boundary conditions, we found the four coefficients. And finally, these velocity profile as well as the temperature profile, we put it in the energy integral equation. And, integrating that equation we got finally the expression for $\frac{\delta_T}{x}$, which is your δ_T is your thermal boundary layer thickness.

Once you got the expression for $\frac{\delta_T}{x}$; then, we found the wall temperature variation T_w and local Nusselt number Nu_x. And, putting the x_0 = 0; that means there is no unheated region, then we found the as a special condition what are the expression for δ_T as well as the wall temperature and Nusselt number. Next, we have considered variable wall temperature boundary conditions.

So, we have taken the wall temperature variation $T_x(x) = T_\infty + C\sqrt{x}$. And, putting that wall temperature condition and using third degree polynomial of velocity profile and temperature profile, we have found the same thermal boundary layer thickness as well as same Nusselt number. So, we have seen that both conditions are same; however, if you maintain the variable wall temperature $T_x(x) = T_\infty + C\sqrt{x}$, it is equivalent to maintaining the flat plate as uniform wall heat flux condition.

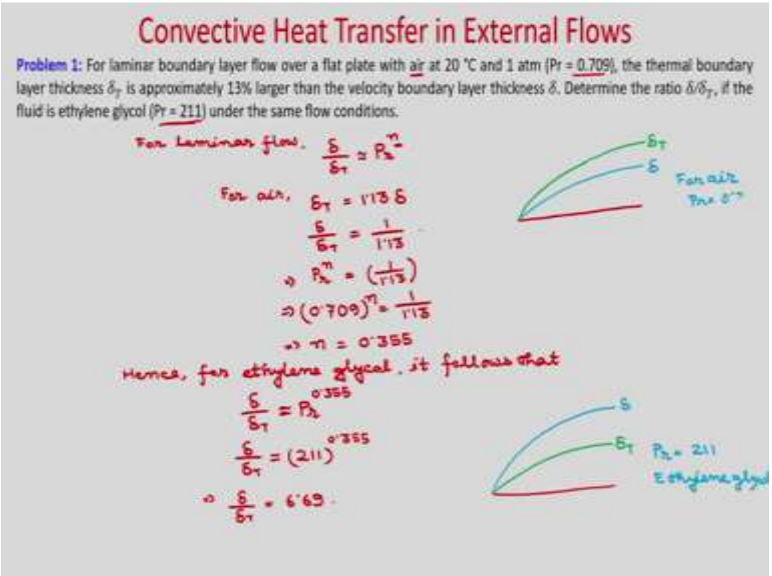
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 04
Convective Heat Transfer in External Flows – II
Lecture – 13
Solution of example problems

Hello, everyone. So, today we will solve few example problems of Convective Heat Transfer in External Flows. So, we have already covered few lectures in last two modules related to convective heat transfer in external flows and let us apply those knowledge to solve some numerical problems.

(Refer Slide Time: 00:57)



So, let us discuss the first problem;

<u>Problem 1</u>: For laminar boundary layer flow over a flat plate with air at 20^0C and 1 atm, Prandtl number = 0.709 the thermal boundary layer thickness δ_T is approximately 13 % larger than the velocity boundary layer thickness δ. Determine the ratio δ / δ_T if the fluid is ethylene glycol Prandtl number = 211 under the same flow conditions. So, you can see for a given condition δ / δ_T is given.

Now, we need to find the δ_T for ethylene glycol where Prandtl number is given. So, we know that for external flows if the flow is laminar then $\frac{\delta}{\delta_T} = \mathrm{Pr}^n$. So, this n for air we know it is 1/3, but for this particular case we need to determine from the given condition. So, we know that for laminar flow $\frac{\delta}{\delta_T} \approx \mathrm{Pr}^n$.

So, here you can see the first fluid is air and for air it is given that your δ_T is approximately 13 % larger than δ. So, you can write $\delta_T = 1.13\delta$ from the given condition. So, you can see that $\frac{\delta}{\delta_T} = \frac{1}{1.13}$. Now, you have the same flow conditions. So, under this now for air we have to find; what is the n?

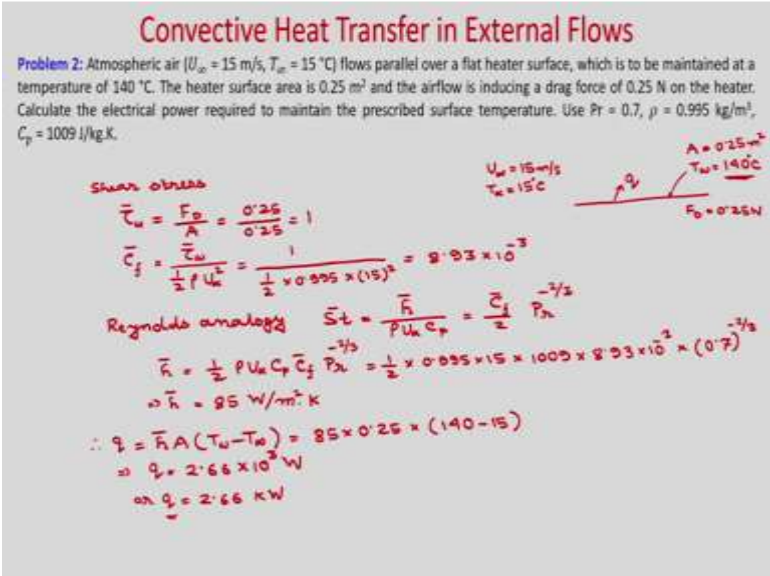
So, you can see here $\frac{\delta}{\delta_T} = \frac{1}{1.13}$. So, if we put it here so, we will get $\mathrm{Pr}^n = \frac{1}{1.13}$. So, now, you can find Prandtl number is given as 0.709. So, you can see $(0.709)^n = \frac{1}{1.13}$. So, from here you find n; n = 0.355. So, for ethylene glycol it follows that $\frac{\delta}{\delta_T} \approx \mathrm{Pr}^n$; n = 0.355 because you have the same flow conditions, so n will be same.

So, now, Prandtl number is different. In this case Prandtl number is 211. So, if you put that. So, you will find $\frac{\delta}{\delta_T} = (211)^{.355}$ so hence you will get $\frac{\delta}{\delta_T} = 6.69$. So, you just notice here that for air $\frac{\delta}{\delta_T} < 1$. You can see here it will be less than 1, but in case of ethylene glycol $\frac{\delta}{\delta_T} > 1$.

So, if you plot the hydrodynamic boundary layer and thermal boundary layer, then you will see for air if it is your boundary layer hydrodynamic boundary layer then your thermal boundary layer will be higher than the hydrodynamic boundary layer. So, this is your δ, this is your δ_T, it is for air and if you consider ethylene glycol then if it is flat plate.

Then in this particular case this will be your thermal boundary layer and hydrodynamic boundary layer will be higher because Prandtl number >1. So, Prandtl number in this case it is 211; you have ethylene glycol and this is your Prandtl number 0.709.

(Refer Slide Time: 06:32)



So, let us take the second problem

<u>Problem 2</u>: Atmospheric air U_∞ =15 m/s, T_∞ =15^{0}C flows parallel over a flat heater surface which is to be maintained at a temperature of 140^{0}C. The heated surface area is 0.25 m^2 and the air flow is inducing a drag force of 0.25N on the heater. Calculate the electrical power required to maintain the prescribed surface temperature. Use Prandtl number =0.7, ρ = 0.995 kg/m^3 and C_p = 1009 J/kgK.

So, in this particular case you can see your surface temperature is to be maintained at 140^{0}C. So, if you consider the flat plate so, the drag force acting on this flat plate is 0.25N, it is given and your surface temperature T_w = 140^{0}C.

And area of this plate is given as 0.25 m^2 and your U_∞ = 15 m/s which is your free stream velocity. Free stream temperature T_∞ = 15^{0}C and you need to find what is the electrical power required; q, to maintain the surface temperature as 140^{0}C.

So, we know the drag force acting on the plate. So, we will be able to calculate the shear stress acting on the surface. So, shear stress so, this is your average shear stress $\tau_w = \frac{F_D}{A} = \frac{0.25}{0.25} = 1$. So, if it is a average shear stress so, just I will denote with $\bar{\tau}_w$.

Similarly, the coefficient of drag, $\overline{C}_f = \frac{\bar{\tau}_w}{\frac{1}{2}\rho U_\infty^2}$. So, it is due to shear and it will be, $\frac{1}{\frac{1}{2} X\, 0.995 X (15)^2}$. So, if you calculate it you will get as $8.93X10^{-3}$.

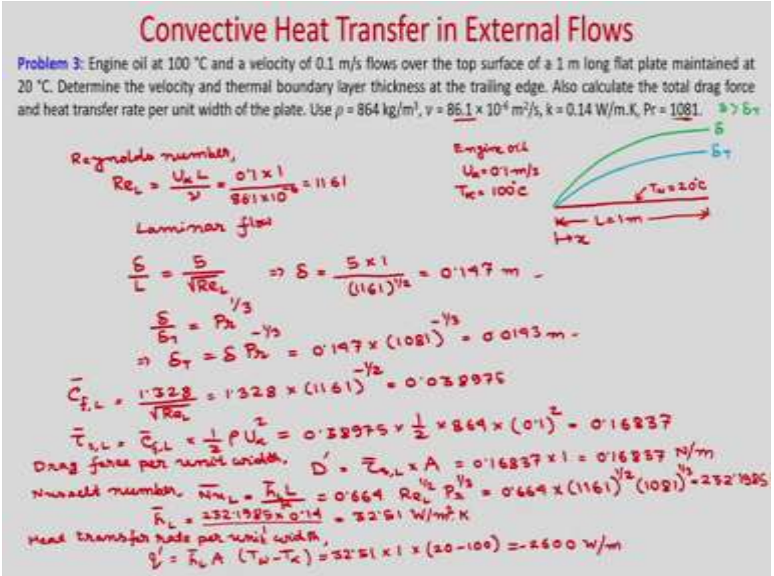
So, now you know C_f, so, we will calculate the heat transfer coefficient using Reynolds analogy. So, Prandtl number you can see it is of the order of 1. So, you can use this Reynolds analogy to calculate the average heat transfer coefficient. So, you know that $\overline{St} = \frac{\bar{h}}{\rho U_\infty C_p}$ and these you can relate with $\frac{\overline{C_f}}{2} \Pr^{-2/3}$.

So, from here you can calculate the average heat transfer coefficient, $\bar{h} = \frac{1}{2}\rho U_\infty C_p \overline{C}_f \Pr^{-2/3}$. So, that will be $\frac{1}{2} X\, 0.995 X 15 X 1009 X 8.93 X 10^{-3} X (0.7)^{-2/3}$. So, if you calculate you will get the average heat transfer coefficient, $\bar{h} = 85 W/m^2K$.

So, now, we have calculated average heat transfer coefficient, you know the total heat transfer area. So, you will be able to calculate the heat transfer rate. So, that is your using Newton's law of cooling you can write as $\bar{h}A(T_w - T_\infty)$. So, $q = \bar{h}A(T_w - T_\infty)$, where it will be $85 X\, 0.25 X (140 - 15)$. So, if you calculate it so, you will get $q = 2.66 X 10^3 W$ or $q = 2.66 kW$.

So, this is the electrical power required to maintain the surface temperature at 140^0C.

(Refer Slide Time: 12:25)



So, let us discuss the 3rd problem.

Problem 3; Engine oil at 100^{0}C and a velocity of 0.1 m/s flows over a top surface of a 1m long flat plate maintained at 20^{0}C. Determine the velocity and temperature boundary layer thickness at the trailing edge also calculate the total drag force and heat transfer rate per unit width of the plate; the properties are given here.

So, you have one flat plate; so, where U_∞ your free stream velocity is 0.1 m/s, T_∞=100^{0}C. So, this engine oil is flowing over this flat plate. Now, the length of the plate is given L is 1 m, x is measured from this leading edge, the T_w =20^{0}C.

Now, we need to determine the thermal boundary layer thickness δ_T and your hydrodynamic boundary layer thickness δ. So, we can see Prandtl number =1081, so, it is >> 1; obviously, $\delta > \delta_T$. So, first let us check whether flow is laminar or not. So, we will calculate the Reynolds number from the given data.

So, Reynolds number, Re based on the length L is $\mathrm{Re}_L = \frac{U_\infty L}{\nu}$ which is $\frac{0.1X1}{86.1X10^{-6}}$. So, if you calculate it you will get as 1161. So, we can see it is a laminar flow. So, now, you know for flow over flat plate $\frac{\delta}{L} = \frac{5}{\sqrt{\mathrm{Re}_L}}$.

So, from this you can calculate the hydrodynamic boundary layer thickness $\delta = \frac{5X1}{(1161)^{1/2}}$.

So, if you calculate this you will get 0.147m and we know $\frac{\delta}{\delta_T} = \Pr^{1/3}$. So, $\delta_T = \delta \Pr^{-1/3}$.

So, $0.147 \times (1081)^{-1/3}$. If you calculate you will get 0.0143m.

So, δ and δ_T now we are found, now we need to find the total drag force and heat transfer rate per unit width of the plate. So, now, $\overline{C}_{f,L} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$. So, this is the expression we have already derived for flow over flat plate. So, we can see it will be $1.328 \times (1161)^{-1/2}$. So, this is your Reynolds number to the power -1/2 . So, you will get 0.038975.

So, now, you will be able to calculate the shear stress. So, shear stress will be just $\overline{\tau}_{s.L} = \overline{C}_{f,L} \times \frac{1}{2} \rho U_\infty^2$. So, you will get as $0.38975 \times \frac{1}{2} \times 864 \times (0.1)^2$. So, if you calculate it you will get it as 0.16837.

So, now drag force per unit width you will be able to calculate as drag force per unit width. So, D unit width means perpendicular to this board so, in that direction. So, that is per unit width we are calculating and we are just prime we are telling per unit width as. So, it will be the $D' = \overline{\tau}_{s.L} \times A$. So, what will be your area? So, per unit width you are calculating. So, it is $0.16837 \times A$.

So, area is length into width per unit width we are calculating, so, into 1. So, we are considering only top surface of the plate because it is written that flows over the top surface ok. So, it will be $0.16837 \times A$; area we are considering on the top side so, it is $\mathrm{L} \times 1$. So, it is 1. So, you will get as 0.16837 N/m.

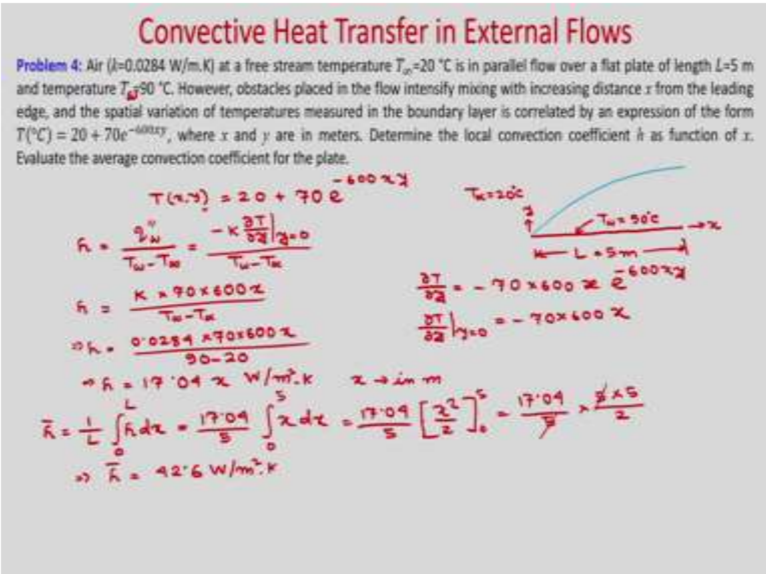
Now, we need to calculate the total heat transfer rate. So, Nusselt number you know relation for the Nusselt number average Nusselt number you can write as $\overline{Nu}_L = \frac{\overline{h}_L L}{K} = 0.664 \mathrm{Re}_L^{1/2} \Pr^{1/3}$. So, this relation we are using because you know that Prandtl number >> 1, 1081.

So, it will be $\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}$. So, from here you just calculate the average heat transfer coefficient. So, it will be, $0.664\times(1161)^{1/2}(1081)^{1/3}=232.1985$. So, heat transfer coefficient you will be able to calculate as $\frac{232.1985\times0.14}{L}$; L is 1 ok. So, this will give as 32.51 W/m^2 K.

So, now, heat transfer rate per unit width you will be able to calculate heat transfer rate per unit width. So, q' I am writing because per unit width we are writing. So, it will be $q'=\bar{h}_L A(T_w-T_\infty)$. So, it will be $32.51\times$A. So, we are considering only top surface one side of the plate we are considering and so, your area will be only 1 as we are considering per unit width in to T_w-T_∞.

So, it will be $32.51\times1\times(20-100)$. So, if you calculate it you will get approximately as 2600 W/m. So, what is the significance of this negative sign because we are calculating q from the surface to the fluid because we have T_w-T_∞, but q is coming as negative; that means, heat transfer is taking place from the fluid to the wall fluid to the surface.

(Refer Slide Time: 21:56)



Now, let us discuss about the next problem.

<u>Problem 4</u>: Air thermal conductivity as 0.0284 W/m^2 K at a free stream temperature T_∞=20^{0}C is in parallel flow over a flat plate of length 5m and temperature T_w at 90^{0}C.

However, obstacles placed in the flow to intensify mixing with increasing distance x from the leading edge, and the spatial variation of temperatures measured in the boundary layer is correlated by the expression of the form $T(^0C) = 20 + 70e^{-600xy}$, where x and y are in meters. Determine the local convection coefficient h as a function of x and evaluate the average heat transfer coefficient for the plate.

So, in this particular case the temperature variation with the coordinate x and y is given. So, from there you will be able to calculate the temperature gradient at the wall. Once you know the temperature gradient at the wall you will be able to calculate; what is the heat flux from the surface and then you will be able to calculate the local heat transfer coefficient. Once you know the local heat transfer coefficient you will be able to calculate the average heat transfer coefficient for the plate.

So, $T(x, y) = 20 + 70e^{-600xy}$, where x and y are in meter. So, if you consider a flat plate L is 5m, your free stream temperature $T_\infty = 20^0$C, your surface temperature $T_w = 90^0$C and y is measured perpendicular to the plate and x along the plate, and your boundary layer thickness will be like this.

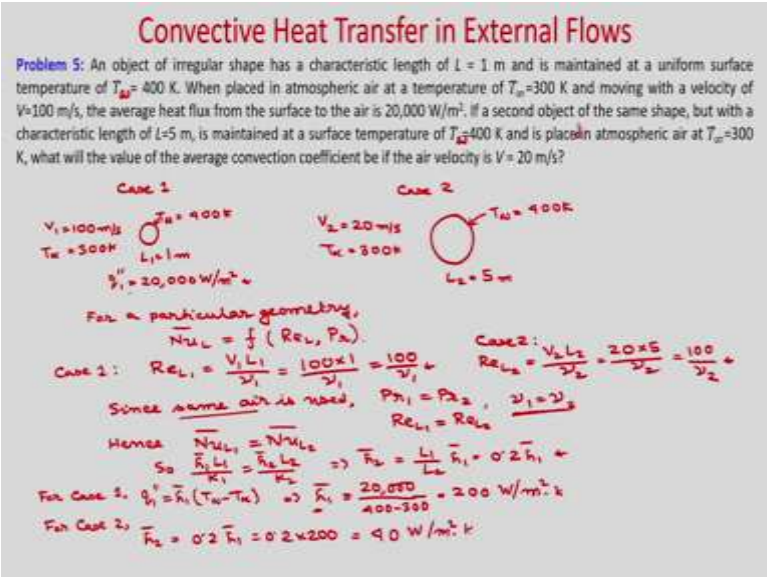
Now, you first calculate the local heat transfer coefficient. So, $h = \frac{q_w^{"}}{T_w - T_\infty}$ this is your local heat transfer coefficient, $\frac{-K\frac{\partial T}{\partial y}|_{y=0}}{T_w - T_\infty}$. So, the temperature is given here. So, $\frac{\partial T}{\partial y} = -70 \times 600xe^{-600xy}$.

So, now, you see at $\frac{\partial T}{\partial y}|_{y=0}$. So, at y = 0, what will be your $\frac{\partial T}{\partial y} = -70 \times 600x$. You will be able to calculate the local heat transfer coefficient h as, $h = \frac{K \times 70 \times 600x}{T_w - T_\infty}$. So, h will be $\frac{0.0284 \times 70 \times 600x}{90 - 20}$. So, hence you will be able to calculate h = 17.04x W/m^{2}K, x in meter.

Now, you know the local heat transfer coefficient now if we integrate over the flat plate and divided by the length of the plate will give you the average heat transfer coefficient. So, average heat transfer coefficient $\bar{h} = \frac{1}{L}\int_0^L h dx$. So, we can see $\bar{h} = \frac{17.04}{5}\int_0^5 x dx$.

So, it will be $\frac{17.04}{5}[\frac{x^2}{2}]_0^5$. So, it will be $\frac{17.04}{5} X \frac{5X5}{2}$; so these 5, 5 will get cancel. So, you can calculate the average heat transfer coefficient $\bar{h} = 42.6 W/m^2K$.

(Refer Slide Time: 27:29)



Now, let us take the next problem.

Problem 5: An object of irregular shape has a characteristic length of 1m and is maintained at a uniform surface temperature 400 K. When placed in atmospheric air at a temperature of T_∞ =300 Kelvin and moving with a velocity of 100 m/s, the average heat flux from the surface to the air is 20000 W/m^2.

If a second object of the same shape, but with a characteristic length of 5m, is maintained at a surface temperature 400 K and is placed in atmospheric air at T_∞ =300 K, what will be the value of average convection coefficient be if the air velocity is 20 m/s?

So, you have two irregular shape. So, if you see if those are geometrically similar then you will be able to equate the non-dimensional number Reynolds number and Nusselt number. So, let us take this irregular shape let us say case 1. So, you have let us say one shape like this whose characteristic length L_1 is 1 m, T_w =400 K, your v_1=100 m/s, T_∞=300 K and $q_1^{''} = 20000 W/m^2$.

Now, if you consider case 2 you have second object of the same shape. So, it is of the same shape, but characteristic length is different. So, in this case your characteristic length is 5m, wall temperature is 400 K, your velocity is 20 m/s and T_∞ =300 K. So, for a particular geometry you know that the Nusselt number is function of Reynolds number and Prandtl number. So, average Nusselt number is function of Reynolds number and Prandtl number.

So, for case 1 what is the Reynolds number? It will be $\mathrm{Re}_{L_1} = \frac{V_1 L_1}{\nu_1} = \frac{100\times 1}{\nu_1} = \frac{100}{\nu_1}$. So, for case 2, what is the Reynolds number? Reynolds number based on the characteristic length L_2 it will be $\mathrm{Re}_{L_2} = \frac{V_2 L_2}{\nu_2}$. So, $\frac{20\times 5}{\nu_2} = \frac{100}{\nu_2}$.

So, now, you can see thus since same air is used so, you will have $\mathrm{Pr}_1 = \mathrm{Pr}_2$. So, you will have $\nu_1 = \nu_2$. So, you can see that $\mathrm{Pr}_1 = \mathrm{Pr}_2$ because same air is used.

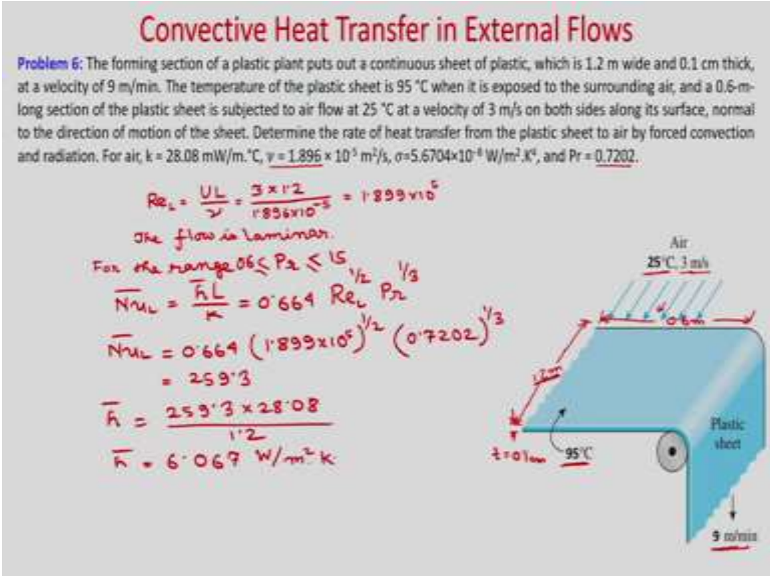
Similarly, your for the same air your kinematic viscosity will be same. So, you can write $\mathrm{Re}_{L_1} = \mathrm{Re}_{L_2}$ from this expression. So, now, as Nusselt number is function of Reynolds number and Prandtl number, but Prandtl number and Reynolds number are same for both the cases. So, Nusselt number also will be same.

So, hence $\overline{Nu}_{L_1} = \overline{Nu}_{L_2}$. So, you can write $\frac{\bar{h}_1 L_1}{K_1} = \frac{\bar{h}_2 L_2}{K_2}$. So, from here you can see you can write $\bar{h}_2 = \frac{L_1}{L_2}\bar{h}_1$ and $\frac{L_1}{L_2} = \frac{1}{5}$. So, $0.2\bar{h}_1$.

So, for case 1 the heat flux $q_1^{''}$ is given. So, you can write $q_1^{''} = \bar{h}_1(T_w - T_\infty)$. So, you can write $\bar{h}_1 = \frac{q_1^{''}}{(T_w - T_\infty)} = \frac{20000}{400-300}$. So, it will be just 200 W/m^{2}K. So, once you know $\bar{h}_1$.

So, now you will be able to calculate the $\bar{h}_2$. So, for case 2, now you will be able to calculate your $0.2\bar{h}_1$. So, 0.2×200 so, it will be 40 W/m^{2}K.

(Refer Slide Time: 33:50)



Let us consider now problem 6.

Problem 6: The forming section of a plastic plant puts out a continuous sheet of plastic, which is 1.2m wide and 0.1cm thick at a velocity of 9 m/min. The temperature of the plastic sheet is 95^0C where it is exposed to a surrounding air and a 0.6m long section of the plastic sheet is subjected to air flow at 25^0C at a velocity of 3m/s on both sides along its surface, normal to the direction of motion of the sheet.

Determine the rate of heat transfer from the plastic sheet to air by forced convection and radiation. For air, thermal conductivity, kinematic viscosity, Stefan Boltzmann constant and Prandtl number are given. So, you can see this is the continuous sheet of plastic so, the surface is maintained at 95^0C and air is flowing perpendicular to this direction of this sheet.

So, this is your 3m/s and air velocity and air velocity is 3m/s and temperature is 25^0C. And, the plastic sheet is having this velocity 9 m/min and you can see the thickness of this plastic sheet is 0.1cm then 1.2 m wide. So, this is your 1.2 m wide and you it is having 0.6 m long section. So, let us say that this is your 0.6m long section.

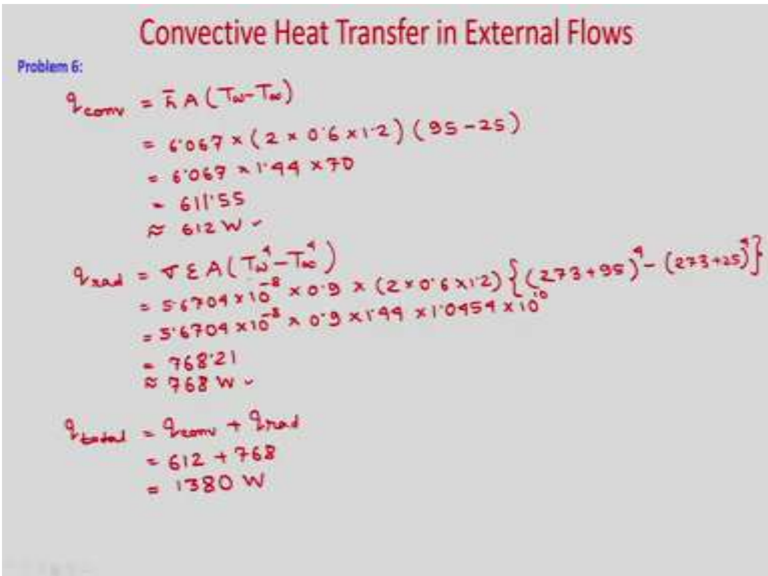
So, now, first let us calculate the Reynolds number and see whether the flow is laminar or turbulent. So, Reynolds number based on the width because flow is taking place in this direction. So, in this direction you can see if you consider this sheet as a flat plate then length of the flat plate will be 1.2 m. So, $\mathrm{Re}_L = \frac{UL}{\nu}$, where ν is the air velocity.

So, this is $\frac{3X1.2}{1.896X10^{-5}}$; so if you calculate the Reynolds number it will come as $1.899X10^5$ and you can see the flow is laminar because you know that if it is less than $5X10^5$ then the flow is laminar.

So, now, we can use the Nusselt number relation for the range of $0.6 \le \mathrm{Pr} \le 15$ and you know here Prandtl number is 0.72. So, obviously, we can use $\overline{Nu_L} = \frac{\bar{h}L}{K} = 0.664\,\mathrm{Re}_L^{1/2}\,\mathrm{Pr}^{1/3}$. So, from here you will be able to calculate the heat transfer coefficient.

So, if you calculate the Nusselt number, $\overline{Nu_L} = 0.664(1.899X10^5)^{1/2}(0.7202)^{1/3}$. So, if you calculate it you will get as 259.3. So, $\bar{h} = \frac{259.3X28.08}{1.2}$. So, h bar will be 6.067 W/m^{2}K.

(Refer Slide Time: 38:57)



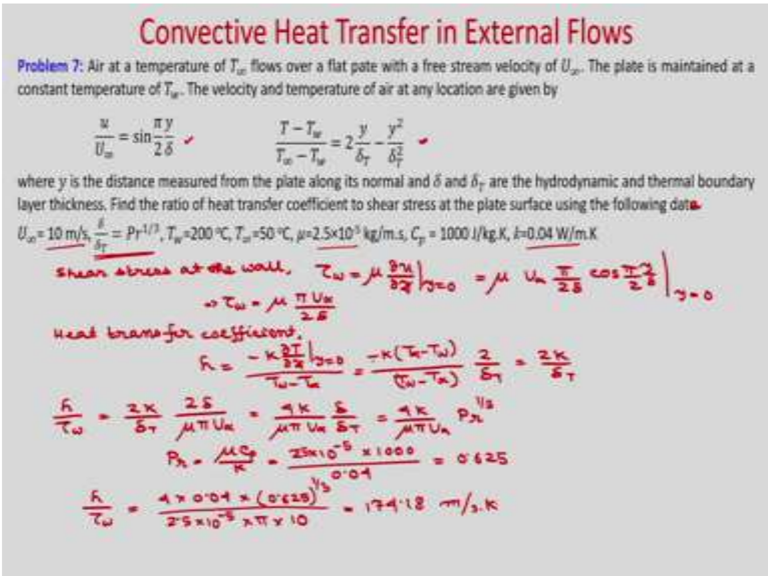
Now, let us calculate the heat transfer rate due to convection heat transfer rate due to convection you know it is $q_{conv} = \bar{h}A(T_w - T_\infty)$. So, what will be your area? So, $\bar{h}$ already we have calculated it is $6.067 \times (2 \times 0.6 \times 1.2)(95-25)$.

So, if you calculate it will come $6.067\times\ 1.4\ 4\ \times 70$. So, it will come as 611.55. So, you can write it as 612W. Now, let us calculate the heat transfer due to now let us calculate the heat transfer rate due to radiation. So, you know you can, $q_{rad} = \sigma\varepsilon A(T_w^4 - T_\infty^4)$. So, in this case this T you need to take in Kelvin because in radiation we know that the temperature you have to use in Kelvin.

So, $5.6704\times10^{-8}\times0.9\times(2\times0.6\times1.2)\{(273+95)^4-(273+25)^4\}$. So, if you calculate it will come as $5.6704\times10^{-8}\times0.9\times1.44\times1.0454\times10^{10}$. So, it will come around 768.21. So, let us take as 768W.

So, you see this is heat transfer rate due to convection; this is heat transfer rate due to radiation. So, the total heat transfer rate will be $q_{total} = q_{conv} + q_{rad}$. So, this is your 612+768. So, it will be 1380W.

(Refer Slide Time: 42:27)



So, now let us discuss about the last problem:

Problem 7: Air at a temperature of T_∞ flows over a flat plate with a stream velocity of U_∞. The plate is maintained at a constant temperature of T_w. The velocity and temperature of air at any location are given by this is $\frac{u}{U_\infty}$.

And, this is your $\frac{T-T_w}{T_\infty - T_w}$, where y is the distance measured from the plate along its normal and δ and δ_T are the hydrodynamic and thermal boundary layer thickness. Find the ratio of heat transfer coefficient to shear stress at the plate surface using the following data.

So, you can see in this particular case your velocity distribution and temperature distribution are given. So, now, we have to calculate the ratio of heat transfer coefficient to the shear stress. So, as you know velocity distribution you will be able to calculate the shear stress and temperature distribution you know. So, you calculate the heat transfer coefficient and make the ratio.

Shear stress at the wall $\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0}$. So, this is your $\frac{u}{U_\infty}$. So, this if you see $\mu \frac{\partial u}{\partial y}$. So, it will be $\mu U_\infty \frac{\pi}{2\delta} \cos \frac{\pi}{2} \frac{y}{\delta}\Big|_{y=0}$. So, if you calculate this τ_w you will get as $\mu \frac{\pi U_\infty}{2\delta}$.

Now, heat transfer coefficient you can calculate as $h = \frac{-K \frac{\partial T}{\partial y}\Big|_{y=0}}{T_w - T_\infty}$. So, this is the temperature distribution if you calculate $\frac{\partial u}{\partial y}\Big|_{y=0}$, then you will get $\frac{-K(T_\infty - T_w)}{(T_w - T_\infty)}$ and at y=0. If you put then after taking the derivative with respect to y you will get $\frac{2}{\delta_T}$ and you see this is T_∞ - T_w if you write in terms of T_w -T_∞ then it will become plus and it will get cancel. So, you will get $\frac{2K}{\delta_T}$.

So, now, we need to calculate the ratio of $\frac{h}{\tau_w}$. So, $\frac{h}{\tau_w} = \frac{2K}{\delta_T} \frac{2\delta}{\mu \pi U_\infty}$. So, you will be able to see this as $\frac{4K}{\mu \pi U_\infty} \frac{\delta}{\delta_T}$ and $\frac{\delta}{\delta_T}$ you see it is given as $\Pr^{1/3}$. So, you can write as $\frac{4K}{\mu \pi U_\infty} \Pr^{1/3}$.

So, for this particular case Prandtl number you can calculate $\Pr = \frac{\mu C_p}{K}$. So, $\frac{2.5 \times 10^{-5} \times 1000}{0.04}$. So, if you see this if you calculate you will get 0.625. So, $\frac{h}{\tau_w} = \frac{4 \times 0.04 \times (0.625)^{1/3}}{2.5 \times 10^{-5} \times \pi \times 10}$.

So, if you calculate it you will get as 174.18 m/sK. So, now, we have found the ratio of heat transfer coefficient to the shear stress at the wall.

Thank you.

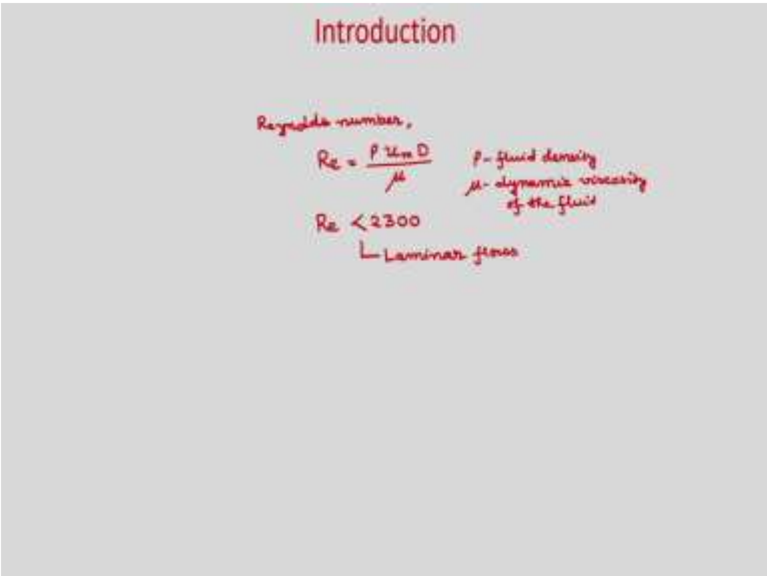
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 05
Convection in Internal Flows – I
Lecture – 14
Hydrodynamic and thermal regions

Hello, everyone. So, in today's lecture we will start Convective Heat Transfer in Internal Flows. So, we will consider internal flows through channels such as ducts, pipes, and parallel plates. You can find the application in heat exchangers.

While considering these internal flows first we will assume the laminar flows. In general for this internal flows through a pipe or circular tube the Reynolds number based on average velocity and the diameter if it is less than 2300, then we consider it as laminar flows.

(Refer Slide Time: 01:11)



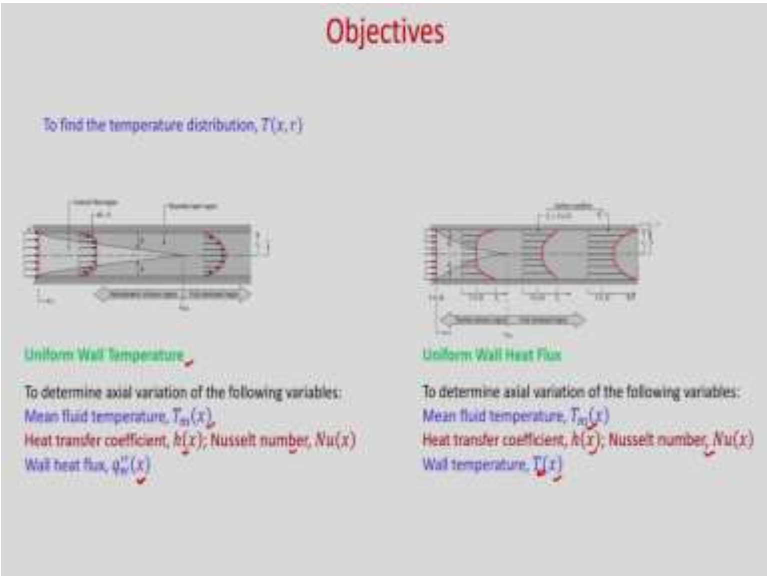
So, Reynolds number you know that it is a ratio of inertia force and the viscous force. So, these Reynolds number we consider as $\mathrm{Re} = \dfrac{\rho u_m D}{\mu}$. So, ρ is the fluid density and μ is the dynamic viscosity of the fluid. So, if Reynolds number < 2300 then we consider it as laminar flows through pipe.

So, in this study for internal flows we will consider two different types of boundary conditions: one is uniform wall temperature where we will assume the temperature to be constant on the walls and secondly, we will consider uniform wall heat flux where we will consider heat flux to be constant on the walls. If heat flux is constant on the walls then wall temperature will vary in axial direction.

Similarly, if we consider uniform wall temperature boundary condition, then we will see that your heat flux will vary along the axial direction. In addition, we will study also the entry region and a developing region and fully developed region. So, you know that in entry region the boundary layer thickness will grow along x and it will merge at the central centre region.

When the entrance region extends to the from the inlet to the section where these boundary layer thickness merges at the central line and the fully developed region starts from there.

(Refer Slide Time: 03:46)

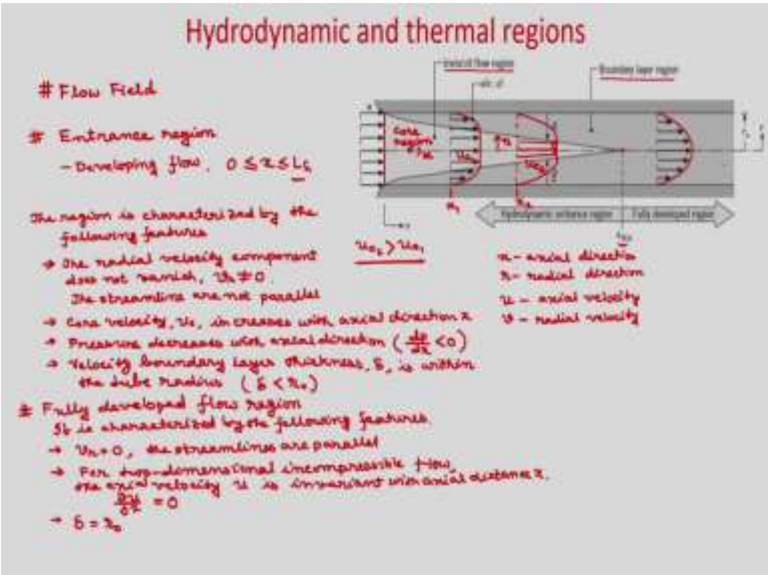


So, what is the objective to study these convective heat transfer in internal flows? Obviously, the first thing is to find the temperature distribution inside the domain. So, once you know the temperature distribution inside the domain then based on the thermal boundary condition on the wall we will have these objectives.

So, you can see that if you have uniform wall temperature, then our objective is to determine the axial variation of mean fluid temperature $T_m(x)$ which is the which is also known as bulk mean temperature, then heat transfer coefficient h_x your heat transfer coefficient will vary along the axial direction and hence we will calculate the Nusselt number. And, as it is uniform wall temperature we want to find the wall heat flux. So, this is the wall heat flux which is function of x.

Similarly, when we will consider the uniform wall heat flux, then we want to determine the axial variation of mean fluid temperature, axial variation of heat transfer coefficient and hence we will calculate the Nusselt number and also we want to calculate the axial variation of wall temperature. So, this is your T_w. So, wall temperature T_w as I told that for uniform wall heat flux condition, your wall temperature will vary along the axial direction.

(Refer Slide Time: 05:30)



So, now let us discuss more about the entrance region and fully developed region. So, as we discussed in general we will consider at inlet you have uniform velocity u_i and uniform temperature T_i, where i denotes the inlet. Now, in the entrance region as we discuss that your velocity boundary layer will grow and it will merge in the central region. So, up to this point from the entrance it is known entrance region.

First let us consider the flow field only. So, you can see that you have a uniform velocity inlet at x = 0, in general x we will consider the axial direction. So, in the central region, so, this is your x. So, this is your central line. So, that is the x direction and from central region now we will consider r which is your radial direction and we will consider u as axial velocity and v as radial velocity in two-dimension case or axis symmetric case.

So, now we can see that when velocity will enter in the duct. So, your velocity boundary layer will start growing. So, then it will merge at the central line. So, up to this point it is known as entrance region or hydrodynamic entrance region as we are considering only the flow field. So, obviously, you can see this is your velocity boundary layer thickness δ and this δ is gradually increases and it merges at the central line and this region is known as entrance region.

So, you can see how the velocity profile will look like. So, obviously, at the inside the boundary layer so, velocity will gradually increase from the wall and it will go to the core region. So, this is known as core region and this is your viscous region and obviously, this is inviscid flow region. Which is your core region, inviscid flow region? And this is your boundary layer region which is your viscous region and at core region you can see you will have uniform velocity.

And, if you consider two different sections say let us say this is your x_1 and if you consider another section x_2 then you can see that your core velocity will increase. So, if this is your core velocity so, this is if your core velocity is constant if it c at section 1 and if it is core velocity u_c at section 2 then $u_{c_2} > u_{c_1}$ in the developing region or entrance region.

So, why it is so? So, you can see that your velocity boundary layer is gradually increasing. So, to have the same mass flow rate at section 1 and section 2 you should have more velocity at the core region at section 2. So, for that reason your core velocity at section 2 is higher than the core velocity at section 1.

So, in this case these developing region this length we will consider as L_h. So, first is your entrance region. So, this is your developing flow because you can see your boundary layer gradually increases and it is $0 \le x \le L_h$. So, whatever here it is mentioned x fully developed h. So, that length we are telling that it is your entrance length which is your L_h.

So, obviously, now the region this entrance region is characterized by the following features ok. So, first thing is that the. So, you can see that in the developing region your v velocity which is your radial direction velocity is not 0. So, obviously, for that reason your streamline will not be parallel in the developing region. So, the radial velocity component radial velocity component does not vanish so; that means, your $v_r \neq 0$ and hence the stream lines are not parallel.

The second observation already we discussed that core velocity u_c increases with axial direction; with axial direction and you can see that obviously, pressure decreases with axial direction pressure decreases with axial direction.

So, your $\frac{\partial u}{\partial x} < 0$ and most importantly your velocity boundary layer thickness; boundary layer thickness your δ is within the tube radius within the tube radius. So, δ will be less than r_0 in the developing region.

So, now, once you can see that this boundary layer reaches to the central line and merges then there will be no further growth of this hydrodynamic boundary layer. So, if there is no growth then obviously, your boundary layer thickness will be r_0 in the fully developed region.

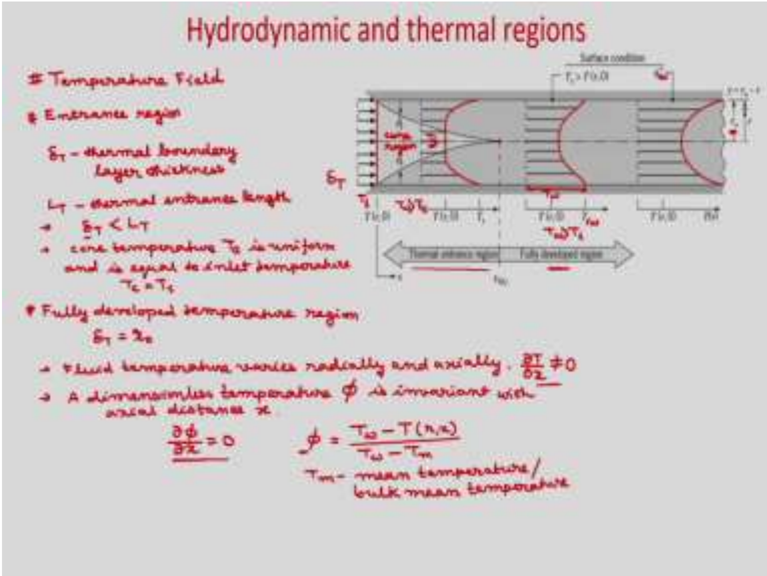
And, you can see that as there will be no development of these hydrodynamic boundary layer your axial velocity u will not vary with x. So, that means, axial velocity distribution will remain constant in the fully developed region and hence $\frac{\partial u}{\partial x} = 0$ and this is the condition for fully developed flow.

So, now, we are discussing about the fully developed flow region. So, it is characterized by the following features. So, first thing is that $v_r = 0$. So, radial velocity will be 0 hence the stream lines are parallel stream lines are parallel.

Then, for two-dimensional incompressible flow the axial velocity u is invariant with axial distance x ok. Hence you can write $\frac{\partial u}{\partial x} = 0$. So, you can see in this region in fully developed region so, velocity becomes parabolic and as your boundary layer thickness merges here, so, your δ in fully developed region will be r_0. So, in fully developed region your δ always will be r_0.

And, at different section if you see in the axial direction these velocity profile will not change. Hence your $\frac{\partial u}{\partial x} = 0$ in fully developed region.

(Refer Slide Time: 16:53)



So, now let us consider the temperature field. So, similarly when your fluid enters at the inlet you have a uniform temperature T_i. Now, when it will come into contact with the wall which may have uniform wall temperature or uniform wall heat flux boundary condition, your thermal boundary layer thickness will grow and merge at central region and up to that region it is known as entrance region and it is known as the distance from the x = 0, it is known as thermal entrance length and after that you can see that thermal boundary layer thickness will not grow further.

So, δ_T which is your thermal boundary layer thickness will be equal to r_0 in the fully developed region. However, in this case your temperature distribution or temperature field will keep on changing with the axial direction in the fully developed region.

Unlike in your hydrodynamic case, where u does not vary in the axial direction in this case as always there will be heat transfer from the wall to the fluid your temperature profile will keep on changing. So, you cannot write in this case for two-dimensional incompressible flow $\frac{\partial T}{\partial x} = 0$, it is not true.

So, in this case entrance region, so, what is happening? You have uniform wall temperature T_i at inlet. So, gradually your thermal boundary layer thickness δ_T generally we consider thermal boundary layer thickness δ_T will start growing and it will merge at the central region, and these distance is known as thermal entrance length and this is your thermal entrance region.

After that this region is known as fully developed region. And, in fully developed region now your temperature will keep on changing in the axial direction. So, in the entrance region; so, δ_T is your thermal boundary layer thickness and when you have L_T if it is your thermal entrance length, then in the entrance region $\delta_T < L_T$. So, this is one condition and also you have core temperature.

Now, you can see this is your core region; it is your core region. So, the heat transfer is taking place from the surface to the fluid if $T_w > T_i$, then obviously, from the wall to the fluid there will be heat transfer and inside this thermal boundary layer thickness there will be variation of temperature. But, at the core region there will be the fluid will remain at temperature T_i. So, this is actually at temperature T_i.

So, there is no effect of the wall temperature in the core region in the thermal entrance region and if you consider two different sections then you will see that your T_i will remain same at two different cross section because your T_i will remain same in the core region. So, core temperature T_c is uniform and is equal to inlet temperature.

So, your T_c will be T_i in the core region and temperature or thermal entrance length or thermal boundary layer thickness will be less than L_T and if you consider a fully developed region so, obviously, now in this case your δ_T always will be r_0 because there will be no further growth of this boundary layer thickness.

So, if your r_0 is the tube radius then δ_T will be always r_0 and in this case your fluid temperature varies radially and axially. And, in this case $\frac{\partial T}{\partial x} \neq 0$ you can see it will keep on changing.

And, here we will define later one dimensionless temperature phi is invariant with axial distance x. And, in this case you can write $\frac{\partial \phi}{\partial x} = 0$ and $\phi = \frac{T_w - T(r,x)}{T_w - T_m}$.

So, it is the ratio of $\frac{T_w - T(r,x)}{T_w - T_m}$ where T m is the mean temperature or sometime it is known as bulk mean temperature or bulk temperature. So, in this case you need to calculate at that cross section what is the bulk mean temperature, we will learn how to calculate this balk mean temperature later.

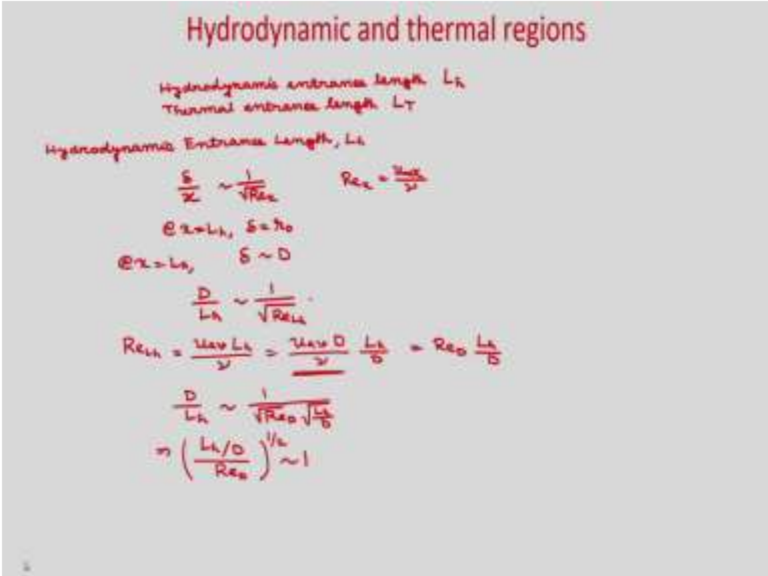
So, these balk mean temperature we can see that the difference $T_w - T(r,x)$ varies similar way as $T_w - T_m$, so that your $\frac{\partial \phi}{\partial x} = 0$. And, this is the condition is valid for both the boundary conditions uniform wall temperature and uniform wall heat flux. And, in this case you can see that if you have uniform wall temperature T_w; so, this is your T_w.

So, these T_w if it is greater than your T than the T_i, then gradually your heat transfer will take place from the wall to the core fluid and from the surface to the fluid and you can see that your temperature distribution will look like this. Because this is the temperature T_w and there is a temperature and in the core region as $T_i < T_w$. So, it will have less temperature than the T_w.

And, if you have uniform wall heat flux which is $q_w^{"}$, then your temperature distribution may look like this. So, this is the temperature distribution in fully developed region and please remember that in this particular case $\frac{\partial T}{\partial x} \neq 0$ here $\frac{\partial \phi}{\partial x} = 0$ where Φ is the non-dimensional temperature or dimensionless temperature as defined $\phi = \frac{T_w - T(r,x)}{T_w - T_m}$.

So, as we are telling $T_w - T(r,x)$ varies in similar way as $T_w - T_m$, hence your Φ is invariant with axial direction and you can write $\frac{\partial \phi}{\partial x} = 0$ in this particular case.

(Refer Slide Time: 27:02)



So, now let us see how to find this hydrodynamic entrance length and thermal entrance length. Already we discussed that hydrodynamic entrance length we have denoted as L_h and thermal entrance length we have denoted as L_T. So, first we will use the scale analysis and we will see what is the order of these hydrodynamic and thermal entrance length.

So, we know from the external flows that so, first let us consider hydrodynamic entrance length; hydrodynamic entrance length. So, L_h and we know from the external flows that $\frac{\delta}{x} \sim \frac{1}{\sqrt{\mathrm{Re}_x}}$ where R e we have defined $\mathrm{Re}_x = \frac{u_{avg} x}{\nu}$.

So, now, at the point where these velocity boundary layer merges then x at $x = L_h$ you have your $\delta = r_0$. Or so, you can write that the order of δ is at $x = L_h$ order of δ we can right diameter of the tube.

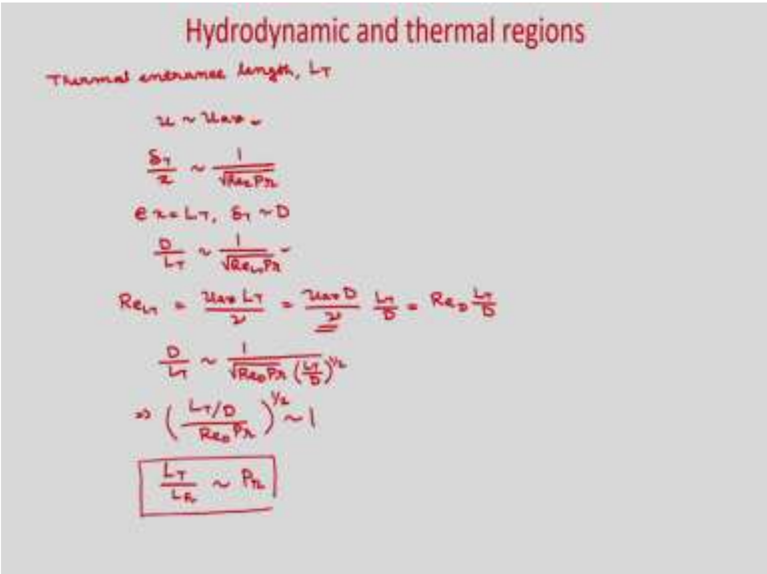
So, now we are doing the scale analysis order of magnitude we are seeing. So, the thermal so, the hydrodynamic boundary layer thickness you can write $\delta \sim D$. So, in that case now you can write $\frac{D}{L_h} \sim \frac{1}{\sqrt{\mathrm{Re}_{L_h}}}$. So, now, Re_{L_h} what we can write? It is just

$\mathrm{Re}_{L_h} = \frac{u_{avg} L_h}{\nu}$.

So, now we will convert it the Reynolds number based on the diameter. So, $\mathrm{Re}_{L_h} = \frac{u_{avg} D}{\nu} \frac{L_h}{D}$. So, you can see that you can write Re_D. So, this quantity is $\mathrm{Re}_{L_h} = \mathrm{Re}_D \frac{L_h}{D}$. So, now, if you substitute it here so, what you will get $\frac{D}{L_h} \sim \frac{1}{\sqrt{\mathrm{Re}_D}\sqrt{\frac{L_h}{D}}}$.

So, now you can write this $\left(\frac{L_h/D}{\mathrm{Re}_D}\right)^{1/2} \sim 1$. So, where L_h is the hydrodynamic entrance length, D is the tube diameter and Reynolds number now based on diameter we have defined. So, $\left(\frac{L_h/D}{\mathrm{Re}_D}\right)^{1/2} \sim 1$.

(Refer Slide Time: 31:19)



Similarly, now consider thermal entrance length thermal entrance length which we have denoted as L_T. So, in this particular case you can see that whether Prandtl number is > 1 or < 1 both cases the thermal boundary layer will merge at the core.

So, in this particular case we will consider that velocity u is order of u_∞ or u_{avg}. So, in this particular case the velocity will take the scale of u_{avg}, irrespective of the low Prandtl number fluids or high Prandtl number fluids because both will margin ultimately at the central region.

So, in this case now u will consider as u_{avg}. So, now, $\frac{\delta_T}{x} \sim \frac{1}{\sqrt{\mathrm{Re}_x \mathrm{Pr}}}$. So, that we have already shown for the external flows because it is a developing region. So, in the developing region you can see that it is kind of external flows and δ_T which is your thermal boundary layer thickness is increasing with x.

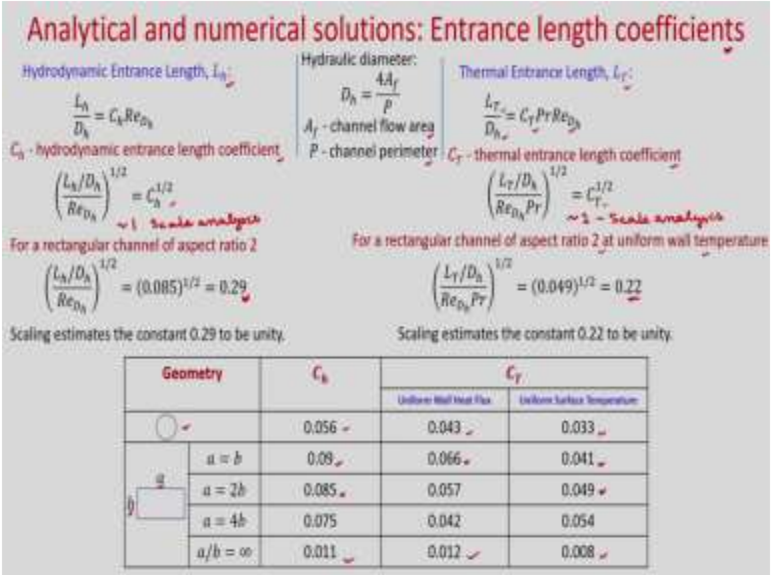
And, as we are considering the velocity scale as the average velocity irrespective of high Prandtl number fluids or low Prandtl number fluids. So, $\frac{\delta_T}{x} \sim \frac{1}{\sqrt{\mathrm{Re}_x \mathrm{Pr}}}$. Now, you can consider that x = L_T your $\delta_T \sim D$ because that will be your the thermal boundary layer thickness will be order of diameter. So, you can write $\frac{D}{L_T} \sim \frac{1}{\sqrt{\mathrm{Re}_{L_T} \mathrm{Pr}}}$.

So, now, you convert this $\mathrm{Re}_{L_T} = \frac{u_{avg} L_T}{\nu}$. So, you can convert in terms of diameter. So, it will be $\mathrm{Re}_{L_T} = \frac{u_{avg} D}{\nu} \frac{L_T}{D}$. So, this you can right now $\mathrm{Re}_{L_T} = \mathrm{Re}_D \frac{L_T}{D}$. So, if you put it in this equation then you will get $\frac{D}{L_T} \sim \frac{1}{\sqrt{\mathrm{Re}_D \mathrm{Pr}} (\frac{L_T}{D})^{1/2}}$.

So, this if you see then you can write $(\frac{L_T/D}{\mathrm{Re}_D \mathrm{Pr}})^{1/2} \sim 1$. So, for hydrodynamic case you have seen $(\frac{L_h/D}{\mathrm{Re}_D})^{1/2} \sim 1$. And, in this case as you are considering the thermal flows, so, it will be $(\frac{L_T/D}{\mathrm{Re}_D \mathrm{Pr}})^{1/2} \sim 1$.

And, now you can see the ratio of this hydrodynamic entrance length and the thermal entrance length will be so, $\frac{L_T}{L_h}$. So, if you divide you can see $\frac{L_T}{L_h} \sim \mathrm{Pr}$. So, using scale analysis we have found that $\left(\frac{L_h / D_h}{\mathrm{Re}_{D_h}}\right)^{1/2} \sim 1$. So, now, after solving numerically or analytically you can find this entrance length.

(Refer Slide Time: 35:56)



Geometry		C_h	C_T	
			Uniform Wall Heat Flux	Uniform Surface Temperature
○		0.056	0.043	0.033
b ▭ a	$a = b$	0.09	0.066	0.041
	$a = 2b$	0.085	0.057	0.049
	$a = 4b$	0.075	0.042	0.054
	$a/b = \infty$	0.011	0.012	0.008

So, now we will define the entrance length coefficient you can see here entrance length coefficient. So, this is C_h is known as hydrodynamic entrance length coefficient and it is given as $\frac{L_h}{D_h} = C_h \mathrm{Re}_{D_h}$.

And, you know that hydraulic diameter we define as $D_h = \frac{4A_f}{P}$, where A_f is your channel flow area and P is the channel perimeter and for circular tube you know that diameter of this pipe will be your hydraulic diameter.

Now, for thermal entrance length L_T we have seen that it is function of Prandtl number and Reynolds number. So, $\frac{L_T}{D_h} = C_T \Pr \mathrm{Re}_{D_h}$, and this C_T is known as thermal entrance length coefficient.

So, you can see that these we can rewrite in this form $\left(\frac{L_T / D_h}{\mathrm{Re}_{D_h}}\right)^{1/2} = C_h^{1/2}$. And, this also you can write $\left(\frac{L_T / D_h}{\mathrm{Re}_{D_h}\,\mathrm{Pr}}\right)^{1/2} = C_T^{1/2}$. And, these using scale analysis we have shown that both are order of one.

So, after doing analytical or numerical solution this entrance length coefficients are found for different geometry. So, you see this table. So, if it is circular tube then your hydrodynamic entrance length co-efficient C_h = 0.056 and C_T which is your thermal entrance length coefficient it depends on the boundary conditions. So, for uniform wall heat flux C_T = 0.043 and for uniform surface temperature it is 0.033.

Now, if you consider a rectangular duct where sides are a b as shown in this figure. So, if a = b, then it is a square cross section channel; for that C_h=0.09 and C_T for uniform wall heat flux it is 0.066 and C_T for uniform surface temperature it is 0.041. And, similarly for a = 2b, a = 4b, and $\frac{a}{b} = \infty$; that means, it is almost infinite parallel plates, so, for that also C_h and C_T values are shown.

So, these are obtained from the analytical and or numerical solutions. So, now, let us see that what we derived using scale analysis so, how it matches with this analytical or numerical results. So, we have already shown that this is order of 1 using scale analysis we have already seen right. So, if you consider a rectangular channel of aspect ratio 2 then your C_h=0.085 and these to the other half is 0.29. So, you can see that scaling estimates the constant 0.29 to be unity.

Similarly, if you consider for a rectangular channel of aspect ratio 2 at a uniform wall temperature, then for this you can see it is 0.049 and $C_T^{1/2}$ it will give 0.22. So, we have seen that using scale analysis it is order of 1, $\left(\frac{L_T / D_h}{\mathrm{Re}_{D_h}\,\mathrm{Pr}}\right)^{1/2} \sim 1$. So, obviously, you can see that scaling estimates the constant 0.22 to be unity.

So, in today's lecture we have first discussed about the objectives to study the convection in internal flows, then we have discussed about the hydrodynamic entrance length and fully developed region. We have seen that in fully developed region the velocity does not vary in the axial direction, so, $\frac{\partial u}{\partial x} = 0$.

And, when we considered the thermal or temperature field then we have seen that in the core region your temperature will remain same as the inlet temperature in the thermal entrance region, but in a fully developed region your temperature will change gradually in the axial direction. So, $\frac{\partial T}{\partial x} \neq 0$ in this particular case.

So, we defined one dimensionless temperature $\phi = \frac{T_w - T(r,x)}{T_w - T_m}$. We discussed that $T_w - T(r,x)$ varies in similar way as $T_w - T_m$, where T_m is the mean temperature. Hence these non-dimensional temperature Φ does not vary in the axial direction. So, that means, your $\frac{\partial \phi}{\partial x} = 0$.

Then we used the scale analysis and we have seen the hydrodynamic entrance length $(\frac{L_h/D}{\text{Re}})^{1/2} \sim 1$, where Reynolds number is defined based on the diameter. And, when we considered the thermal entrance length in that particular case your $(\frac{L_T/D}{\text{Re}_D \Pr})^{1/2} \sim 1$. And, we have also seen the ratio of $\frac{L_T}{L_h} \sim \Pr$.

So, in next class, we will see that how we can find these thermal and hydrodynamic entrance length using analytical and numerical approach.

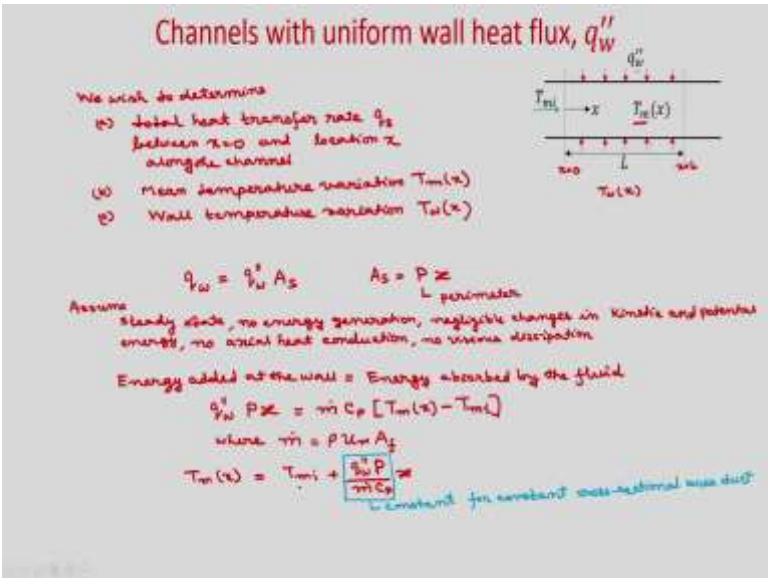
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 05
Convection in Internal Flows - I
Lecture - 02
Energy balance in channel flow

Hello everyone. So, in today's lecture, we will consider different thermal boundary conditions and we will try to find the mean temperature variation as well as for different boundary conditions we will find either the variation of heat flux or variation of temperature on the walls.

(Refer Slide Time: 00:58)



First, we will consider channels with uniform heat flux. So, you can see so, we have a channel where on the channel, we have uniform wall heat flux boundary condition so, q''_w we have given. Now, we consider that a section where x is measured from here where x = 0 and up to x = L.

So, this is the length. So, obviously, you can see at this section, if you considered the temperature mean temperature so, there will be inlet mean temperature will be T_{mi} and at any section x, you can find what is the mean temperature T_m. So, as we considered here uniform wall heat flux, then your temperature at the wall T_w will also vary in axial direction. So, T_w will be function of x.

So, here you can see as heat is added on the surface so, the fluid flowing in the channel, it will take the heat from the wall. So, here we wish to determine; we wish to determine first thing is that total heat transfer rate; total heat transfer rate q_s between x = 0 and location x along the channel.

Then, we want to find the mean temperature variation this mean temperature is also known as bulk mean temperature mean temperature variation so, which will be function of x and also in this case, we will see the wall temperature variation T_w which will be function of x.

In this particular thermal condition as heat flux is uniform, then we can calculate the total heat transfer rate from the wall to the fluid. So, the total heat transfer rate q_w will be just whatever heat flux you have given q_w'' into the surface area heat transfer area q_s. q_w'' So, into heat transfer area A_s. So, what is the A_s in this case? So, if P is the perimeter and x is the length, then your heat transfer area A_s will be P into x where P is the perimeter.

So, now from the energy balance, we will find what is the mean temperature variation. So, we are assuming that it is a steady state assume steady state, no energy generation, negligible changes in kinetic and potential energy, no axial heat conduction; no axial heat conduction and no viscous dissipation.

So, you can see that now with this assumptions, if you do the energy balance whatever heat is transferred from the wall to the fluid actually fluid gain the temperature from inlet while going from inlet to the distance x. So, energy added at the wall will be energy absorbed by the fluid. So, this is just simple energy balance.

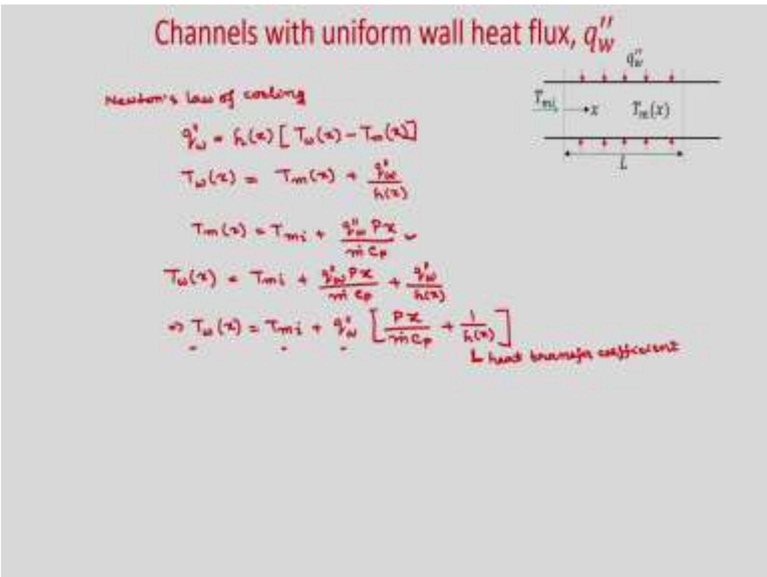
So, now, what is energy added at the wall? That is nothing, but $q_w = q_w'' A_s$. So, $A_s = Px$ area is P into x where P is the perimeter and what is the energy absorbed by the fluid? So, at any section x, $q_w'' Px = \dot{m} C_p [T_m(x) - T_{mi}]$ because this is the temperature difference and where $\dot{m} = \rho u_m A_f$ so, flow area so, that will be for pipe it is πr_0^2 so, this will be flow area A_f.

And now, you can write, $T_m(x) = T_{mi} + \frac{q_s^{"}P}{\dot{m}C_p}x$. So, you can see that this is the expression for bulk mean temperature variation along the x where T_{mi} is the mean temperature at the inlet and q_s or you can write q_w here $q_w^{"}$ which is your heat flux constant.

P is the perimeter that is also constant for constant cross sectional channel, $\dot{m}$ is the mass flow rate and C_p is the specific heat and with x obviously, it will vary so, but you can see that $\dot{m}$. So, you can see that $\frac{q_w^{"}P}{\dot{m}C_p}$. So, this is your constant for this particular case for constant cross sectional; constant cross sectional area duct.

So, now, you can see the temperature is varying from inlet T_{mi} to the any distance x T_m. So, the fluid properties you need to determine at the average temperature. If it is T_{mi} and T_m at any location x, then at average temperature you determine the fluid properties while calculating the variation of this $T_m(x)$.

(Refer Slide Time: 09:36)



Now, we want to calculate the axial variation of wall temperature. So, for that we will do the we will use the Newton's law of cooling. So, from Newton's law of cooling what you can write? So, Newton's law of cooling what you can write?

So, $q_w^{"}$ what is your heat flux at the wall that you can write as, $q_w^{"} = h(x)[T_w(x) - T_m(x)]$.

In this case, when you consider the internal flows, you can see that you cannot take as T_∞ or T inlet as your reference temperature while calculating the heat transfer coefficient or the Nusselt number. Here, we will consider the bulk mean temperature at any cross section.

So, while writing the Newton's law of cooling, the wall heat flux we have written as the temperature difference as $T_w - T_m$. So, for internal flows we will consider $T_w - T_m$. So, you can write the, $T_w(x) = T_m(x) + \frac{q_w^{"}}{h(x)}$ and T_{mx} already we know, $T_m(x) = T_{mi} + \frac{q_w^{"} P}{\dot{m} C_p} x$.

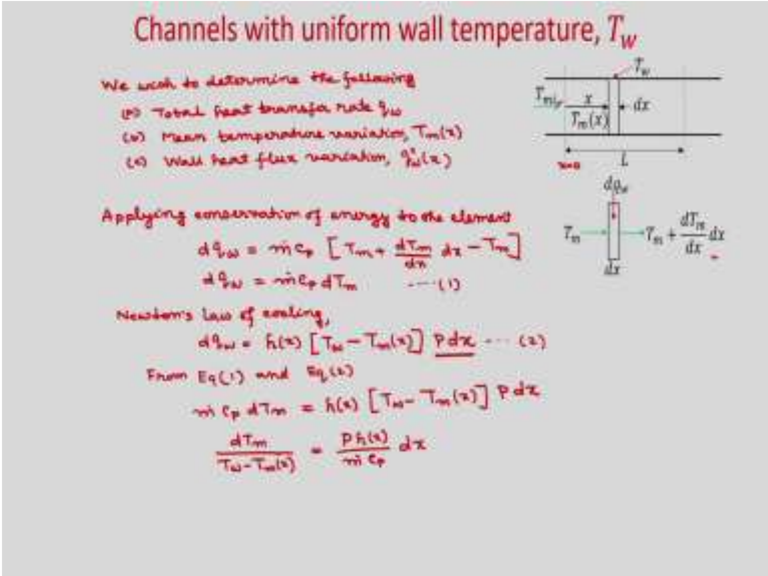
So, you can write in terms of T_{mi} now this T_w (x) so, if you substitute this, you can write $T_w(x) = T_{mi} + \frac{q_w^{"} Px}{\dot{m} C_p} + \frac{q_w^{"}}{h(x)}$. So, you can rewrite it as, $T_w(x) = T_{mi} + q_w^{"}[\frac{Px}{\dot{m} C_p} + \frac{1}{h(x)}]$.

Now, you can see that to calculate the wall temperature variation, you need to know the heat transfer coefficient because h is your heat transfer coefficient. So, you can see from this expression T_{mi} is your inlet mean temperature that will be known, this is your wall heat flux that will be known and it is constant, P is the perimeter, x is the axial direction so, at any location x, you can calculate the T_w, $\dot{m}$ and C_p are also known only unknown is h(x).

So, you can see that while deriving this, we did not take the assumptions whether it is laminar or turbulent or the region is developing or fully developed. So, these expression this $T_m(x)$ and $T_w(x)$ is the expression of $T_w(x)$ and $T_m(x)$ are valid for both laminar and turbulent flows as well as for entrance region as well as fully developed region.

Only thing is that while calculating the wall temperature, you need to calculate the h(x) and h(x) will depend whether it is developing region or fully developed region at the same time whether it is laminar or turbulent flows. So, this h(x) we need to find. So, depending on h(x) if you know h(x), then you can find $T_w(x)$ for constant wall heat flux boundary condition.

(Refer Slide Time: 13:58)



Now, let us consider channels with uniform wall temperature. So, you can see this is the channel at x=0, you have inlet mean temperature T_{mi} the walls are maintained at temperature T_w and T_w is constant.

So, in this case, you can see that heat flux is not constant. So, heat flux will vary in axial direction. So, for that reason, while solving this problem, we cannot consider the full length of the channel as we considered for the uniform heat flux case because in earlier case, the heat transfer rate q_w you could calculate using q''_w into the surface area A_s.

But here as q''_w is not constant for uniform wall temperature condition so obviously, we cannot consider the full length and do the energy balance for that we will consider a small elemental area in the flow region and we will do the energy balance.

In this particular case also, our objective is to find what is the axial variation of mean temperature, axial variation of the heat flux and what is the total heat transfer rate at the wall. So, we wish to determine the following: one is total heat transfer rate q_w, then mean temperature variation; mean temperature variation so, T_m as function of x and in this particular case as T_w is constant so, we want to calculate the wall heat flux variation q''_w as function of x.

So, now, in this region, you consider one small elemental volume at a distance x of distance dx. So, at this inlet of this elemental volume, you can see that we have the inlet mean temperature as T_m; obviously, at a distance dx using the Taylor series expansion what you can write?. So, at a distance dx using the Taylor series expansion you can write this.

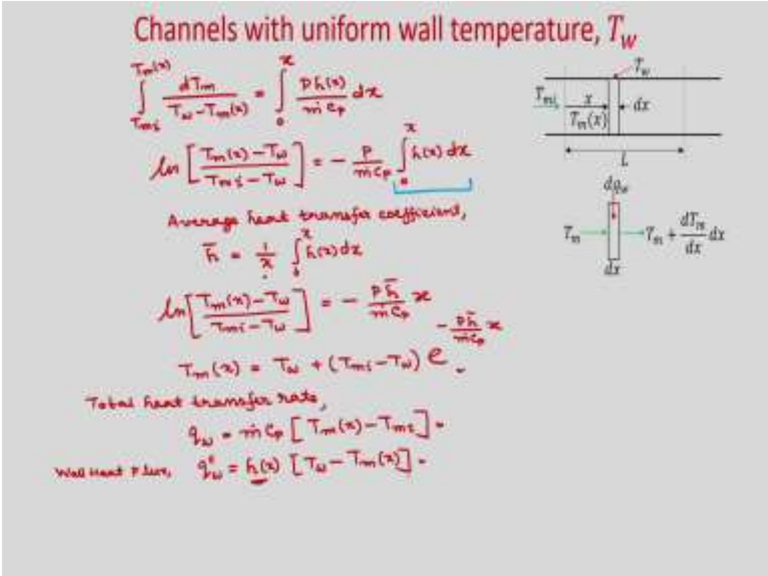
And so, you have the wall temperature constant so, there will be the heat transfer from the wall to the fluid in this elemental volume so, that is your dq_w. So, whatever heat is added here the fluid which is passing through this dx distance that is actually that fluid absorbed this heat. So, you can do the energy balance as.

So, applying conservation of energy to the element what you can write? dq_w so, dq_w is the heat added to the fluid from the surface. So, what is heat absorbed that is your just $\dot{m} C_p \delta_T$. So, δ_T is nothing, but you can see it will be $T_m + \frac{dT_m}{dx} dx - T_m$. So, this is the temperature difference . So, you can write $dq_w = \dot{m} C_p dT_m$.

Now, you apply the Newton's law of cooling. So, Newton's law of cooling you can write dq_w so, that you can write the h which is function of x the temperature $[T_w - T_m(x)]A$, so, area is Pdx. So, this is the P, P is the perimeter and dx is the length. So, your heat transfer area will be Pdx.

So, now, this both you can equate. So, if it is equation number 1 and if it is equation number 2 so, from equation 1 and equation 2, you can write, $\dot{m} C_p dT_m = h(x)[T_w - T_m(x)]Pdx$. So, now, you rearrange it. So, $\frac{dT_m}{T_w - T_m(x)} = \frac{Ph(x)}{\dot{m} C_p} dx$.

(Refer Slide Time: 20:38)



So, now, you integrate it from x =0 to any distance x so, if you do that. So, now, integrate this. So, from at x =0, you have T_{mi} and at any distance x, you have $\int_{T_{mi}}^{T_m(x)} \frac{dT_m}{T_w - T_m(x)} = \int_0^x \frac{Ph(x)}{\dot{m}C_p} dx$.

So, now, you can integrate and put the limits. So, this you can see it will be, $\ln[\frac{T_m(x) - T_w}{T_{mi} - T_w}] = -\frac{P}{\dot{m}C_p} \int_0^x h(x)dx$. So, if you know the variation of heat transfer coefficient in axial direction, then you will be able to integrate this.

So, we can find the average heat transfer coefficient as; average heat transfer coefficient ok. So, that will be $\bar{h} = \frac{1}{x}\int_0^x h(x)dx$. So, now, we will substitute this part as x$\bar{h}$. So, if you do so, you can write $\ln[\frac{T_m(x) - T_w}{T_{mi} - T_w}] = -\frac{P\bar{h}}{\dot{m}C_p} x$.

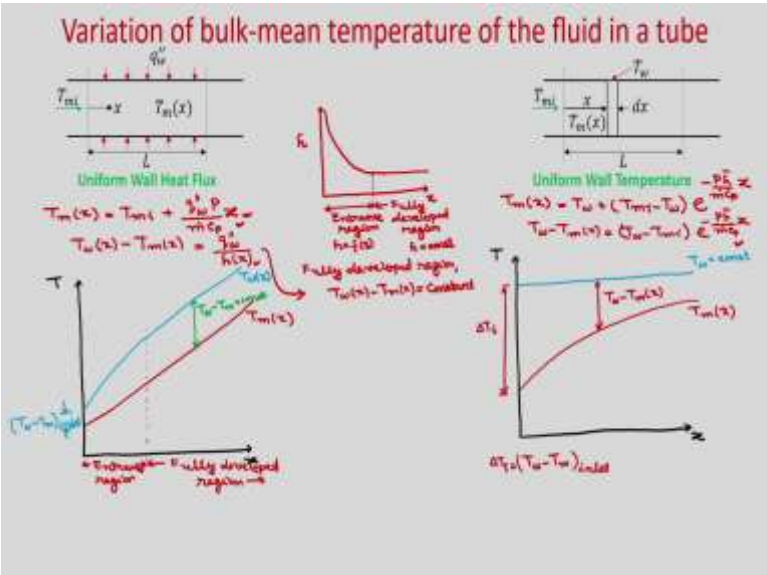
So, you can now find the axial variation of mean temperature as, $T_m(x) = T_w + (T_{mi} - T_w)e^{-\frac{P\bar{h}}{\dot{m}C_p}x}$. So, you can see your mean temperature varies exponentially right.

So, you can see that while deriving this expression, we did not assume whether flow is laminar or turbulent or whether it is entrance region or fully developed region. So, this is valid for all the conditions, but now you have to find what is the heat transfer coefficient. So, while finding the heat transfer coefficient, you need to have these assumption so, whether it is laminar or turbulent or it is entrance region or fully developed region.

So, now let us calculate the total heat transfer rate from the wall. So, we can write total heat transfer rate. So, total heat transparent rate q_w so, what is that we can write $q_w = \dot{m} C_p [T_m(x) - T_{mi}]$. So, you can write and $q''_w = h(x)[T_w - T_m(x)]$.

So, you can see that once you calculate the h(x) so, T_w is known so, you can calculate the mean temperature. Once you know the mean temperature, then you can calculate the total heat transfer rate as well as the heat flux. So, this is your wall heat flux. So, because h(x) you need to find once h(x) is known and T_{mx} is known from this expression, then you will be able to calculate what is the wall heat flux q''_w.

(Refer Slide Time: 26:02)



So, now for both the thermal conditions, we have found what is the axial variation of mean temperature as well as the total heat transfer rate. Now, let us see the variation of this bulk mean temperature of the fluid in the tube. So, you can see for uniform wall heat flux so, what is the variation of T_m?

T_m which is function of x we have written as $T_m(x) = T_{mi} + \frac{q_w^{"} P}{\dot{m} C_p} x$. So, how it is varying?

You can see your T_m is varying linearly because $\frac{q_w^{"} P}{\dot{m} C_p}$ this quantity is constant. So, T_m is varying linearly with x.

When we consider uniform wall heat flux, the T_m variation we have seen, $T_m(x) = T_w + (T_{mi} - T_w) e^{-\frac{P\bar{h}}{\dot{m} C_p} x}$. So, in this case, you can see your mean temperature varies exponentially.

So, now, if we plot. So, this is your temperature axis and this is your axial direction and in this case also this is your temperature axis and this is your axial direction. So, your this is your x and this is your T.

So, now, you can see that in this case, you have uniform wall heat flux right. So, uniform wall heat flux so, you can see that also we have found. So, here you can see that, $T_w(x) - T_m(x) = \frac{q_w^{"}}{h(x)}$

So, now, you can see that in this particular case, h(x) is unknown obviously, your heat transfer coefficient will vary in axial direction, but we will show later that in fully developed region fully developed means both hydrodynamically and thermally, in that region h is not function of x, h is constant.

So, if you plot h with axial direction, then you will see that when the fluid is entering in the channel so, it will have very high heat transfer coefficient in the developing region it will gradually decrease, then once it will become fully developed both hydrodynamically and thermally, then your heat transfer coefficient will become constant and we will show it later that h is constant for a fully developed flow.

So, we plot this h as a function of x. So, in the developing region, it will decrease, then after that it will become constant. So, you can see if this is your entrance region or developing region, then it will gradually decrease and in fully developed region; fully developed region, you can see h is no longer function of x it is constant. So, in fully developed range, h is constant and in this region, h is function of x.

So, you can see that $T_w - T_m = \frac{q_w^{"}}{h(x)}$. So, in fully developed region, h(x) is constant hence, $T_w - T_m$ will be constant.

So, in fully developed region; fully developed region; fully developed region $T_w - T_m$ for uniform wall heat flux boundary condition will become constant. This is also true that fully developed region h is constant for uniform wall temperature. So, h is constant for both the thermal condition uniform wall temperature and uniform wall heat flux for both hydrodynamically and thermally fully developed region.

So, now, you can see here uniform wall heat flux. So, your T_m will vary linearly. You can see from this expression so, you draw this. So, this is linear variation. So, this is your T_m which is function of x it is varying linearly and you can see that $T_w - T_m$ will be constant in the fully developed region.

So, if you see that this is your fully developed region and this is your entrance region, then in this region fully developed region $T_w - T_m$ will be constant. So, you can see that your T_w also will vary linearly. So, it will also vary linearly. So, this is your T_w (x) and you can see that this difference in fully developed region, it is constant $T_w - T_m$ is constant.

However, in your entrance region, there will be variation. So, it may vary like this and this is your the temperature difference $T_w - T_m$ at inlet. So, this is the temperature variation. So, you can see that this T_w will vary linearly in the fully developed region and so, that $T_w - T_m$ will remain constant.

Now, you plot for uniform wall heat flux. So, for uniform wall heat flux so, T_w is constant right. So, you can draw this T_w it is constant. So, this is your T_w is constant and your T_m will vary exponentially so, if you plot it so, there will be variation like this. So, this is your T_m which will vary exponentially with x. So, this is $\Delta T_i = (T_w - T_m)_{inlet}$.

So, you can see that there will be always decrease in the temperature $T_w - T_m$. T_m is function of x, but T_w is constant and that you can see from here that it will how it will vary. So, you can see $T_w - T_m(x) = (T_w - T_{mi})e^{-\frac{P\bar{h}}{\dot{m}C_p}x}$.

So, in today's lecture, we considered two different types of thermal conditions one is uniform wall heat flux and uniform wall temperature and we have just done the energy balance and we have tried to find what is the variation of mean temperature and the total heat transfer rate from the wall in both the cases.

However, in uniform wall temperature case, your heat flux at the wall also will vary so, we have found what is the variation of wall heat flux in axial direction and for uniform wall heat flux, your wall temperature will vary axially and we have found what is the variation of T_w with x.

Then, we have plotted this temperature; mean temperature and the wall temperature and also in this expression of mean temperature, we have seen that your heat transfer coefficient arises.

So, depending on whether the flow is laminar or turbulent or whether it is developing region or fully developed region, you need to find the heat transfer coefficient and also we have told that in fully developed region for both the thermal condition, your h is constant so, for a uniform wall heat flux condition $T_w - T_m$ will be constant. However, in uniform wall temperature boundary condition, $T_w - T_m$ will also vary exponentially.

We have plotted this mean temperature and wall temperature for both the thermal conditions and we have shown that in fully developed region, the temperature difference between wall and the bulk mean temperature will remain constant as your heat transfer coefficient remains constant in fully developed region for both the thermal conditions.

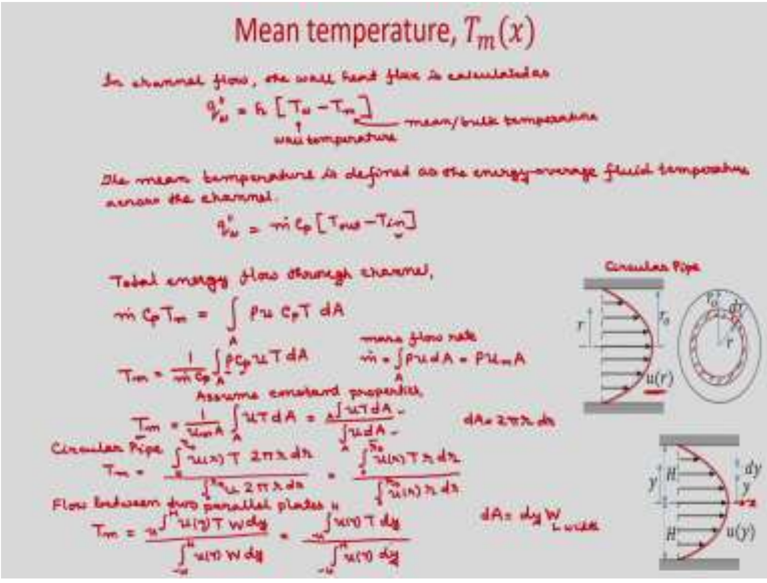
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 05
Convection in Internal Flows - I
Lecture - 16
Determination of heat transfer coefficient

Hello everyone. So, in last lecture we have calculated the mean temperature for two different types of boundary conditions; uniform wall heat flux and uniform wall temperature. Today, we will discuss about the mean temperature first; then we will discuss about the dimensionless temperature in thermally fully developed region, then we will determine the heat transfer coefficient and Nusselt number.

(Refer Slide Time: 01:03)



In external flow, we have seen that you can calculate the wall heat flux as $q_w^{"} = h[T_w - T_m]$ which is your Newton's law of cooling. Here, T_w is the wall temperature and T_∞ is the free stream temperature. But, internal flows you can see that, there is no such free stream temperature.

Hence, we need to find some constant temperature at different axial location. And, we will define the mean temperature or bulk temperature. So, you can see that in channel flow, the wall heat flux is calculated as $q_w^{"} = h[T_w - T_m]$. So, here you can see T_w is the wall temperature and T_m is the mean and sometime it is known as bulk temperature. And, depending on thermal boundary condition, T_w maybe constant or T_w may be function of x and T_m is always function of x.

Now, let us define the mean temperature. The mean temperature is defined as the energy average fluid temperature across the channel. You can see that, we can also calculate the wall heat flux as, $q_w^{"} = \dot{m} C_p [T_{out} - T_{in}]$. So, you can see that there will be variation of temperature at different axial location. So, at outlet and inlet you can have radial variation of temperature; so obviously, this will not be useful if you do not know a constant temperature at the outlet and inlet. So, now, this T_m will give you a proper area weighted average or a mean temperature at any location.

So, you can write that, total energy flow through channel. So, that is you can write $\dot{m} C_p T_m = \int \rho u C_p T dA$. So, if you see that, $T_m = \frac{1}{\dot{m} C_p} \int \rho u C_p T dA$.

Now, what is $\dot{m}$? $\dot{m}$ is your mass flow rate , mass flow rate. So, $\dot{m} = \int \rho u dA$; if you define a mean velocity at that cross section, then $\dot{m} = \rho u_m A$. So, flow cross sectional area, not the heat transfer area; it is a flow cross sectional area.

So, if you put it here and if you assume constant properties, assume constant properties, then this ρC_p you can take it outside and this $\dot{m}$ this ρ you can take it outside. So, you can write $T_m = \frac{1}{u_m A} \int uT dA$; because, ρ C_p you can cancel or you can write $\frac{\int_A uT dA}{\int_A u dA}$.

So, for any channel of different cross section maybe circular cross section or square cross section or flow between two parallel plates; you can calculate the mean temperature using this formula, where you can write $T_m = \frac{\int_A uT dA}{\int_A u dA}$. And, this is your bulk mean temperature T_m.

So, now if you consider a circular pipe; so this is your circular pipe. So, if you consider a fully developed region, then that will be your velocity distribution u r,. So, it is hydrodynamically fully developed ; hydrodynamically fully developed flow; then, your velocity will be only function of r, so u is functional of r.

And, if you see the flow cross sectional area, so that is circular, because it is a circular pipe. And, if you consider a small elemental area at a radial distance r of distance dr. So, this is the elemental flow area you are considering. So, this is your radial distance r and this thickness is dr; then, you can write for circular pipe.

For a fully developed profile u(r) you can write T_m as; so, $T_m = \frac{\int_0^{r_0} u(r)T2\pi rdr}{\int_0^{r_0} u2\pi rdr}$.

So, that is the elemental flow cross sectional area. So, this 2π you can cancel out in the denominator and numerator. So, you can write $\frac{\int_0^{r_0} u(r)Trdr}{\int_0^{r_0} u(r)rdr}$.

Similarly, if you consider flow between two parallel plates. So, in this particular case let us say two infinite parallel plates. So, you can have the central line as shown here. So, this is your axial direction x and this is your y. And, the parallel plates are separated by a distance 2 H. So, when you calculate the flow cross sectional area, so you can see that at a distance y, you take a small elemental area that is of distance dy.

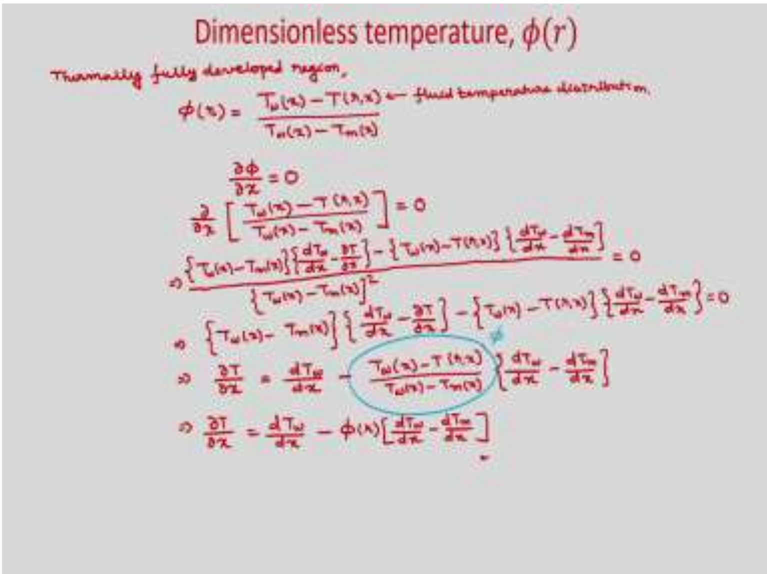
So, in this particular case if you consider flow between two parallel plates; two parallel plates , then what will be your dA? So, dA in this particular case you can see that it will be dy , dy into third direction whatever width you have; so, into w we can write or per unit width also you can calculate. So, dA = dyW, where W is the width, so perpendicular to this board.

So, if you calculate the mean temperature in this particular case, then it will be you can see that your y is varying -H to H. So, u which is function of y, then your cross sectional area is dA = dyW; so, $T_m = \dfrac{\int_{-H}^{H} u(\mathrm{y})TWdy}{\int_{-H}^{H} u(\mathrm{y})Wdy}$. So, w is constant.

So, you can write now $T_m = \dfrac{\int_{-H}^{H} u(\mathrm{y})Tdy}{\int_{-H}^{H} u(\mathrm{y})dy}$. So, depending on the flow cross sectional area, you can consider the elemental flow area dA. And, accordingly you can calculate the mean temperature or it is also known as bulk temperature.

So, now let us discuss about the dimensionless temperature in thermally fully developed region. So, already we have introduced this dimensionless temperature Φ which is $\phi(r) = \dfrac{T_w - T}{T_w - T_m}$. So, in this particular case you can see that T_w-T varies in similar way as T_w-T_m; so that, in axial direction there is no variation of this dimensionless temperature phi in fully developed region. So, Φ is function of r only.

(Refer Slide Time: 13:49)



So, thermally fully developed region, we can introduce this dimensionless temperature $\phi(r) = \frac{T_w(x) - T(r,x)}{T_w(x) - T_m(x)}$. So, you know that T_w is the wall temperature, T_m is the mean temperature and T(r,x) is the fluid temperature distribution. So, this is your fluid temperature distribution. So, this dimensionless temperature expression is valid for both thermal condition constant wall temperature and constant wall heat flux.

So, now, for thermally fully developed condition $\frac{\partial \phi}{\partial x} = 0$; because, Φ is function of r only, so $\frac{\partial \phi}{\partial x} = 0$. So, if it is so, so you can see that $\frac{\partial}{\partial x}[\frac{T_w(x) - T(r,x)}{T_w(x) - T_m(x)}] = 0$. So, if you take the derivative; so, you can write,

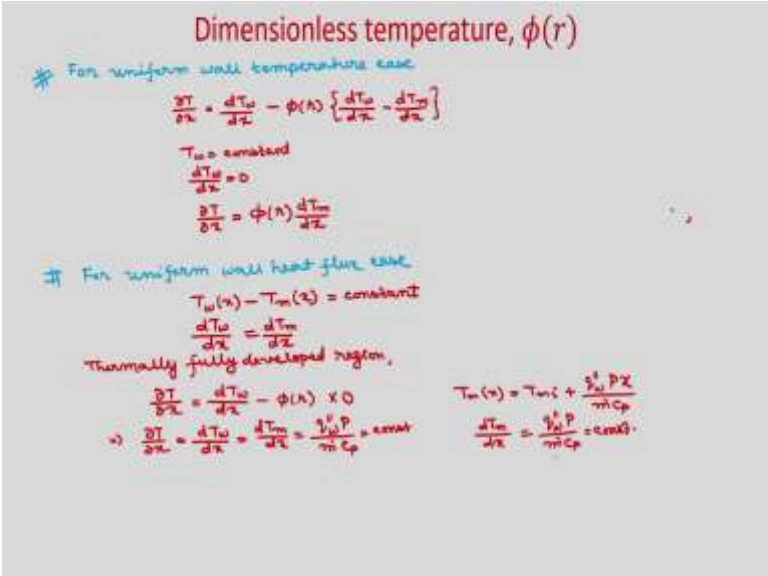
$$\frac{\{T_w(x) - T_m(x)\}\{\frac{dT_w}{dx} - \frac{\partial T}{\partial x}\} - \{T_w(x) - T(r,x)\}\{\frac{dT_w}{dx} - \frac{dT_m}{dx}\}}{[T_w(x) - T_m(x)]^2} = 0.$$

$$\{T_w(x) - T_m(x)\}\{\frac{dT_w}{dx} - \frac{\partial T}{\partial x}\} - \{T_w(x) - T(r,x)\}\{\frac{dT_w}{dx} - \frac{dT_m}{dx}\} = 0$$

So, if you rearrange it, you can write $\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \frac{T_w(x) - T(r,x)}{T_w(x) - T_m(x)}\{\frac{dT_w}{dx} - \frac{dT_m}{dx}\}$. So, you can see that, this quantity is nothing but, Φ. So, you can write it now $\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \phi(r)[\frac{dT_w}{dx} - \frac{dT_m}{dx}]$.

So, this result will be used in analyzing thermally developed flow in channels. So, later we will use this relation, but now let us consider two different boundary conditions. So, now, let us consider two different boundary condition and see the simplification in finding $\frac{\partial T}{\partial x}$.

(Refer Slide Time: 18:37)



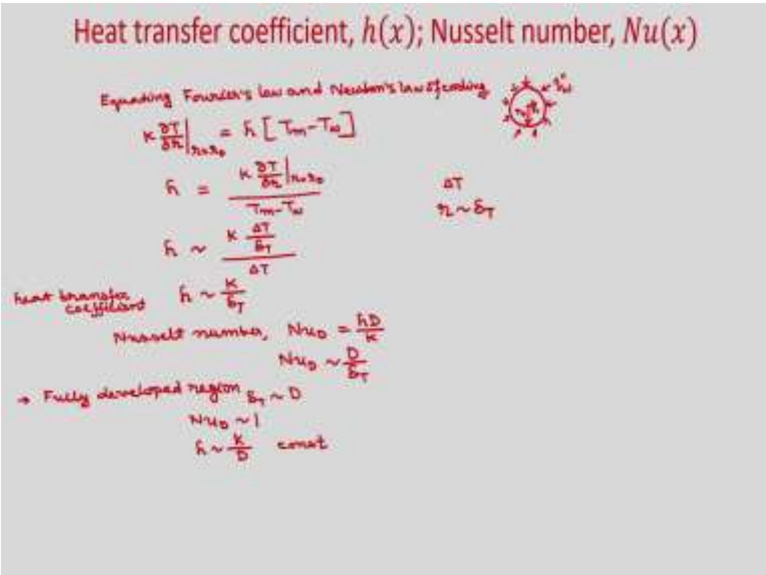
So, for uniform wall temperature case. So, if you consider this, then what you can write? So, our expression is $\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \phi(r)\{\frac{dT_w}{dx} - \frac{dT_m}{dx}\}$. So, in this particular boundary condition T_w = constant. So, $\frac{dT_w}{dx} = 0$. So, you put it in this expression. So, T_w = constant ; hence, $\frac{dT_w}{dx} = 0$. So, you can write $\frac{\partial T}{\partial x} = \phi(r)\frac{dT_m}{dx}$.

So, the axial variation of this temperature profile T, $\frac{\partial T}{\partial x}$ you can express in terms of $\phi(r)\frac{dT_m}{dx}$. Now, if you consider for uniform wall heat flux case, then these expression you can write as; so, for uniform wall heat flux case. So, for uniform wall heat flux case we know that, $T_w(x) - T_m(x) =$ constant. So, that we have shown.

So, this is your constant. So, if it is constant, then you can write $\frac{dT_w}{dx} = \frac{dT_m}{dx}$; because this is constant, so it will be 0. So, $\frac{dT_w}{dx} = \frac{dT_m}{dx}$. So, thermally fully developed region, you can write $\frac{\partial T}{\partial x} = \frac{dT_w}{dx} - \phi(r) X 0$, because you can see $\frac{dT_w}{dx} = \frac{dT_m}{dx}$. So, it will be 0.

So that means, you have $\frac{\partial T}{\partial x} = \frac{dT_w}{dx} = \frac{dT_m}{dx}$. And, already we know that you have $T_m(x) = T_{mi} + \frac{q_w^{"}Px}{\dot{m}C_p}$. So, $\frac{dT_m}{dx} = \frac{q_w^{"}P}{\dot{m}C_p}$. So, now, you can see this quantity is constant, because $\frac{q_w^{"}P}{\dot{m}C_p}$ all are constant. So, $\frac{dT_m}{dx}$ is constant. So, this you can write $\frac{q_w^{"}P}{\dot{m}C_p}$=constant. So, these are the simplification to find the $\frac{\partial T}{\partial x}$ for a thermally fully developed region for both the thermal conditions. So, now, to calculate the heat transfer coefficient, first we will use the scale analysis.

(Refer Slide Time: 22:41)



So, you can see that, if you have let us say a circular pipe of radius r_0 and this is your radial direction. And, if you take the q_w in inward direction. So, $q_w^{"}$; then, you can define whatever heat is conducted that will be conducted. So, from is equating the Fourier's and Newton's law, you can write equating Fourier's law and Newton's law of cooling. What you can write? You can write; so, $K\frac{\partial T}{\partial r}|_{r=r_0}$.

So, at r =r_0. So, it is plus, it is not minus, because you are taking in a negative r direction, because this $K\frac{\partial T}{\partial r}$ in general for Fourier's law heat conduction we write minus; but in this particular case you can see that radial direction is in this outward direction, but $q_w^{"}$ we are considering in the inward direction. So, it will be plus, $K\frac{\partial T}{\partial r}|_{r=r_0} = h[T_m - T_w]$.

So, you can see that $\mathrm{h} = \frac{K\frac{\partial T}{\partial r}|_{r=r_0}}{T_m - T_w}$. So, as earlier we will use the scale analysis. So, the temperature difference we will take the scale of ΔT. So, we will take ΔT and the radius we will take order of the thermal boundary layer thickness δ_T. So, $h \sim \frac{K\frac{\Delta T}{\delta_T}}{\Delta T}$.

So, you can see that $h \sim \frac{K}{\delta_T}$. So, now, if you see the Nusselt number based on the diameter, then you can calculate Nusselt number. So, this is your heat transfer coefficient and Nusselt number now you can calculate as; so, $Nu_D = \frac{hD}{K}$. So, you can see $Nu_D \sim \frac{h}{K}$. So, it will be $Nu_D \sim \frac{D}{\delta_T}$.

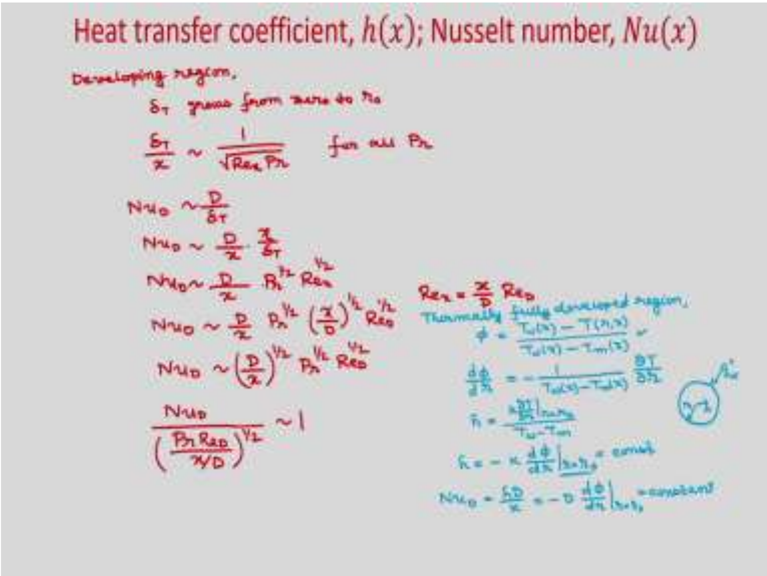
So, now, if you consider entrance region and the fully developed region, then what will be your Nusselt number? So, you can see for fully developed region , we are talking about thermal fully developed region ; thermal fully developed region. So, in this case your δ_T will be order of diameter, because in a fully developed region your $\delta_T \sim D$. So, your $Nu_D \sim 1$. So, you can see in the fully developed region, Nusselt number is constant right. And, heat transfer coefficient you can see it is $h \sim \frac{K}{\delta_T}$.

So, in fully developed region, what will be your h? $h \sim \frac{K}{D}$; because your thermal boundary layer thickness merges at the center line. So, your thermal boundary layer thickness will be just δ_T as D; actual case the thermal boundary layer thickness will be order of r naught which is your radius of the pipe; but, as we are using the scale analysis,

$\delta_T \sim D$. So, you can write $h \sim \dfrac{K}{D}$. So, you see k is the for a constant properties k is constant, D is the diameter is constant. So, it will be also constant in a fully developed region.

So, using scale analysis we are showing that heat transfer coefficient and Nusselt number are constant in a thermally fully developed region. And, later when you will actually calculate the value of Nusselt number, it will be constant. Now, if you consider entrance region. So, in the entrance region your thermal boundary layer thickness will start growing. So, you can take $\dfrac{\delta_T}{x} \sim \dfrac{1}{\sqrt{\text{Re}_x \, \text{Pr}}}$.

(Refer Slide Time: 28:05)



So, for developing region or entrance region; developing region or entrance region; so, what you can write? δ_T grows from 0 to r_0, right. So, your $\dfrac{\delta_T}{x} \sim \dfrac{1}{\sqrt{\text{Re}_x \, \text{Pr}}}$ that we have shown in external flows right, $\dfrac{\delta_T}{x} \sim \dfrac{1}{\sqrt{\text{Re}_x \, \text{Pr}}}$ for all Prandtl number range.

So, now, $Nu_D \sim \dfrac{D}{\delta_T}$. So, you can write $Nu_D \sim \dfrac{D}{x}\dfrac{x}{\delta_T}$. So, Nusselt number D will be; now, $Nu_D \sim \dfrac{D}{x} \text{Pr}^{1/2} \, \text{Re}_x^{1/2}$.

So, now this convert this Reynolds number based on x to Reynolds number based on diameter. So, $\mathrm{Re}_x = \frac{x}{D}\mathrm{Re}_D$,right. So, $Nu_D \sim \frac{D}{x}\mathrm{Pr}^{1/2}(\frac{x}{D})^{1/2}\mathrm{Re}_D^{1/2}$. So, $Nu_D \sim (\frac{D}{x})^{1/2}\mathrm{Pr}^{1/2}\mathrm{Re}_D^{1/2}$.

So, in $\frac{Nu_D}{(\frac{\mathrm{Pr}\,\mathrm{Re}_D}{x/D})^{1/2}} \sim 1$. So, you can see that your Nusselt number will be will depend on Prandtl number and Reynolds number and it is in developing region, the Nusselt number will be $\mathrm{Pr}^{1/2}\mathrm{Re}_D^{1/2}$ into some constant.

So, later we will consider one case, where we will consider a fully developed, hydrodynamically fully developed flow and thermally developing flow where we will calculate the Nusselt number. But, in the other two cases, we will consider both hydrodynamically and thermally fully developed region. So, it is easy to calculate the Nusselt number and you have we have shown now that Nusselt number will be constant value for both the thermal boundary conditions.

Now, if you write this heat transfer coefficient in terms of dimensionless temperature; then, $\phi = \frac{T_w(x)-T(r,x)}{T_w(x)-T_m(x)}$, where T_m is the mean temperature. So, if you take the derivative with respect to r. So, what we will get? $\frac{d\phi}{dr} = -\frac{1}{T_w(x)-T_m(x)}\frac{\partial T}{\partial r}$. So, now, let us calculate the heat transfer coefficient h.

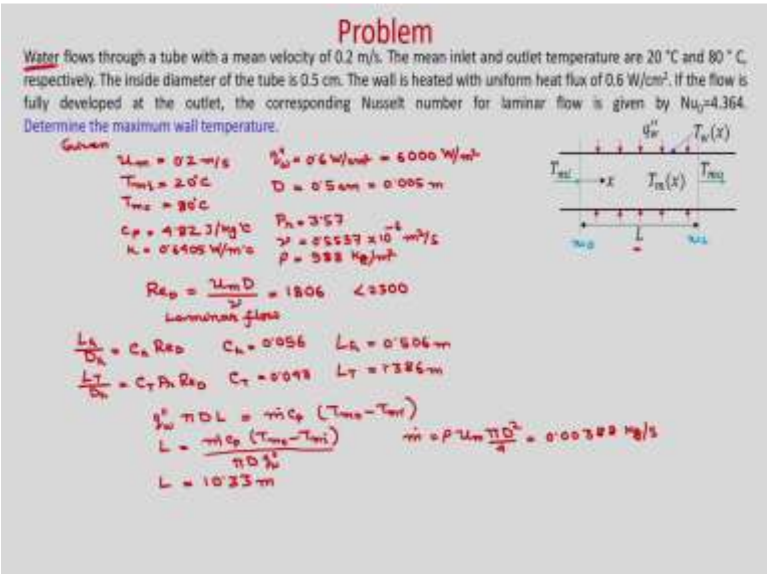
So, you can see that h equating the Fourier's law and the Newton's law, you can write h equal to; so, equating the Fourier's law and the Newton's law of cooling, you can write $\mathrm{h} = \frac{K\frac{\partial T}{\partial r}|_{r=r_0}}{T_w - T_m}$. So, we are considering this as r, tube radius is r_0 and q_w we are considering this. So, now, you can see. So, $\frac{\partial T}{\partial r}$ if you put it here; so, what you will get? So, h is equal to; so, $h = -K\frac{d\phi}{dr}|_{r=r_0}$.

So, and you can see here k is thermal conductivity, right. So, for constant properties r is constant and $\frac{d\phi}{dr}|_{r=r_0}$, at r = r_0; so obviously, this will be also constant. So, for a fully developed region, so you can write that, this is equal to constant.

So, from the scale analysis we have already shown that, your h and Nusselt number will be constant for a thermally fully developed region. And, in this case also you can see k is the thermal conductivity that is constant and $\frac{d\phi}{dr}$, because Φ is function of r only, right. So, $\frac{d\phi}{dr}$ you are calculating at r = r_0; that means, at the tube surface.

So, obviously $\frac{d\phi}{dr}$ will also constant, so h will be constant. And, similarly Nusselt number you can write $Nu_D = \frac{hD}{K}$. So, from here you can see it will be $-D\frac{d\phi}{dr}|_{r=r_0}$. So, you can see here also this is constant. So, for thermally fully developed region you can calculate the $Nu_D = -D\frac{d\phi}{dr}|_{r=r_0}$, where Φ is the dimensionless temperature defined as this and it is true for both the thermal boundary conditions.

(Refer Slide Time: 34:45)



So, now let us solve one example problem; Water flows through a tube with a mean velocity of 0.2 m/s. The mean inlet and outlet temperature are 20^0C and 80^0C respectively. The inside diameter of the tube is 0.5 cm. The wall is heated with uniform heat flux of 0.6 W/cm^2. If the flow is fully developed at the outlet, the corresponding Nusselt number for laminar flow is given by Nu_D = 4.364. Determine the maximum wall temperature.

So, you can see that, first we have to consider whether this is laminar flow or turbulent flow. And, whether it is developing region or fully developed region. If it is fully developed region, then the Nusselt number is given for this particular constant wall heat flux condition. So, you can see schematically. So, this is your q_w right, constant wall heat flux is given and at x =0 you have T_{mi} and at x = L you have T_{mo}. And, in this case T_w is function of x and T_m is also function of x. And, you can see that your mean velocity is given as 0.2 m/s.

Your inlet mean temperature is given as 20^{0}C and outlet mean temperature is given as 80^{0}C, $q_w^"$ =0.6 W/cm^2. So, if you convert it to W/m^2. So, it will be 6000 W/m^2. And, diameter of the pipe is 0.5 cm, so it will be 0.005 m. And, the properties are also provided; so, at mean temperature. So, the properties are given as C_p = 4.82 J/kgK or J/kg^{0}C, thermal conductivity 0.6405 W/m^{0}C, Prandtl number is given as 3.57.

Kinematic viscosity of the fluid is given as 0.5537 X10^{-6} m^2/s, and density is given as 988 kg/m^2. So, the fluid is a water, because it is already given water. So, now, you calculate the Reynolds number. So, Reynolds number you can calculate as $\mathrm{Re}_D = \frac{u_m D}{\nu}$. So, if you put all these values, you will get 1806. So, you can see that the Reynolds number < 2300, obviously this is the laminar flow. Now, you see, whether it is developing or fully developed region. So, we know that $\frac{L_h}{D_h} = C_h \mathrm{Re}_D$.

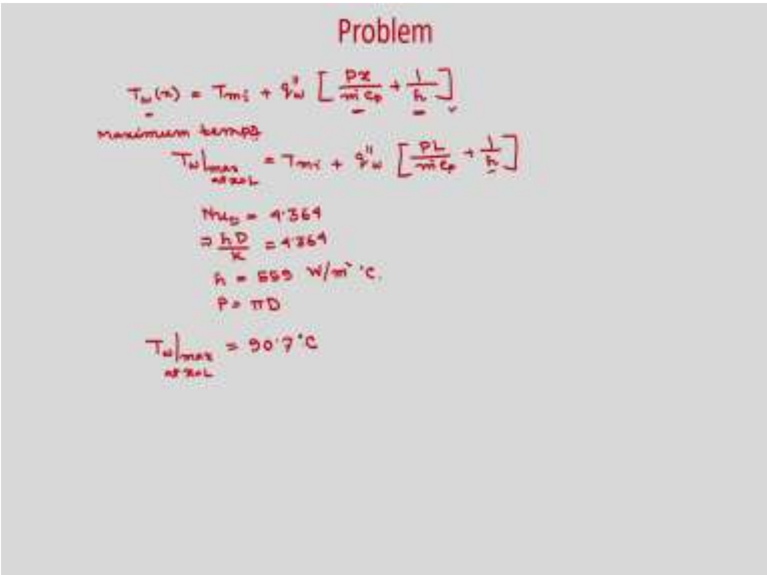
And, from here you can see that C_h for the circular pipe is given as 0.056 from the table we have already shown. So, L_h will be 0.506 m. And, $\frac{L_T}{D_h} = C_T \mathrm{Pr}\,\mathrm{Re}_D$. And, for this circular pipe your C_T =0.043, for the constant wall heat flux boundary condition; so, L_T will be 1.386 meter. So, now, let us calculate the tube length, because this is unknown, right.

So, these L we have to calculate from this equation. So, we can see that $q_w^{''}$ into the heat transfer area. So, that is your, $q_w^{''}\pi DL = \dot{m}C_p(T_{mo} - T_{mi})$. So, your $L = \dfrac{\dot{m}C_p(T_{mo} - T_{mi})}{\pi D q_w^{''}}$.

So, you can see that $\dot{m} = \rho u_m \dfrac{\pi D^2}{4}$, and all the properties are given and q_w also is given. So, T_{mo} and T_{mi} are given. So, you just calculate the length of the pipe. And, this length of pipe is you will get as; $\dot{m}$ if you calculate, it will be 0.00388 kg /s. So, L will be if you put it 10.33 m.

So, now, the tube length is 10.33 m and your entrance length, you can see L_h and L_T it is just 0.506 m hydrodynamic entrance length, and thermal entrance length is 1.386 m. So, it is very very small compared to the length of the pipe. And, you are asked to find the maximum wall temperature at the maximum wall temperature, and maximum wall temperature will occur at the outlet for this particular case.

(Refer Slide Time: 41:35)



So, you can see your T_w (x) for this particular case you know that it is $T_w(x) = T_{mi} + q_w^{''}[\dfrac{Px}{\dot{m}C_p} + \dfrac{1}{h}]$. So, in this particular case as it is a fully developed flow, so h is constant. So, in this expression you can see that, if this term is constant; so as x increases, your T_w also will increase.

So, maximum temperature will occur, maximum temperature you will get, $T_w\big|_{\max@x=L} = T_{mi} + q_w^{"}[\frac{PL}{\dot{m}C_p} + \frac{1}{h}]$. So this heat transfer coefficient for fully developed flow,

Nusselt number is given. So, that is your 4.364. So, this is nothing but, $\frac{hD}{K} = 4.364$. So, if you calculate the h you will get 559 W/m^2 ^{0}C.

So, now this T_w maximum you can now calculate h you know, P L $\dot{m}C_p$. What is P? P is nothing but, $P = \pi D$, $\dot{m}$ already you have calculated, $q_w^{"}$ you know, T_{mi} you know; you put all the values this $T_w\big|_{\max@x=L}$ you will get as 90.7 ^{0}C.

So, in this particular case we have to calculate first, whether it is the flow is laminar or turbulent and where are you are calculating at outlet, whether it is developing region or fully developed region that we have calculated the entrance length for both thermal and hydrodynamic.

So, we have seen that it is very very small compared to the length of the pipes. So, at the outlet it will become anyway fully developed flow and fully developed flow Nusselt number, from Nusselt number you can calculate the heat transfer coefficient. So, in today’s lecture, first we have calculated the mean temperature, ok. So, if you know the velocity profile and the temperature profile, then you can calculate the mean temperature at any cross section.

Then, we have discussed about the dimensionless temperature phi in a thermally fully developed region. And, we have simplified the $\frac{\partial T}{\partial x}$ for two thermal boundary conditions. Then, we have calculated the heat transfer coefficient and the Nusselt number using scale analysis. And, we have shown that in fully developed region h is constant and also Nusselt number constant.

However, in developing region as delta increases with x, so your h and Nusselt number both will increase in axial, both h and Nusselt number will vary in axial direction. Then, also we have calculated the heat transfer coefficient and a Nusselt number in terms of the dimensionless temperature getting $\frac{d\phi}{dr}$. And, here also we have shown for a fully developed for thermally fully developed region, the heat transfer coefficient and Nusselt number are constant.

Thank you.

Fundamentals of Convective Heat Transfer

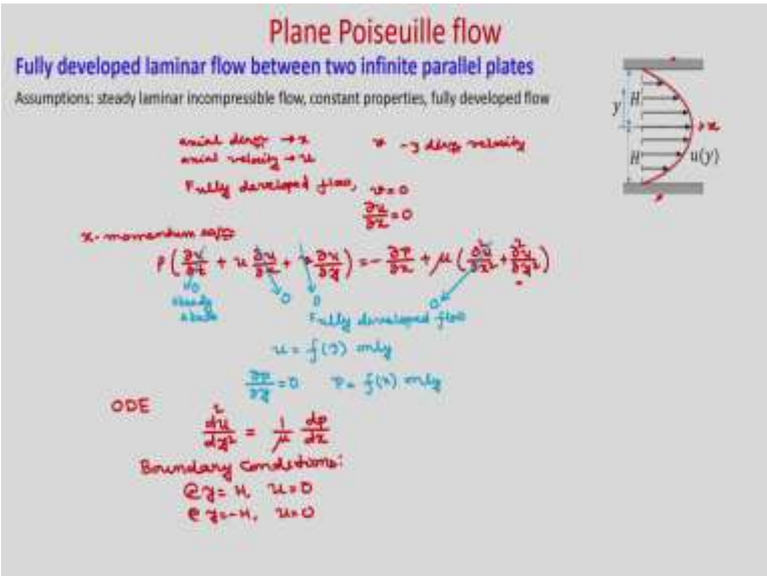
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 05
Convection in Internal Flows - I
Lecture – 17
Velocity profile in fully-developed channel flows

Hello everyone. So, in today's lecture, we will find the fully developed velocity profile in channel flows. So, we will consider three different cases. One is flow between two parallel plates. Then we will consider flow between two parallel plates where one plate is moving with respect to the other. And finally, we will consider flow inside pipe. As you know that to solve the temperature or to solve the energy equation, you need to know the velocity profile.

So, if you have a fully developed condition under which if you want to find the velocity profile, so in today's lecture we will find this fully developed velocity profile for different situations.

(Refer Slide Time: 01:26)



First we will consider plane Poiseuille flow which is known as fully developed laminar flow between two infinite parallel plates you can consider two infinite parallel plates. So, in third direction, as it is infinite you can consider as a two-dimensional flow. And you can have the assumptions of steady, laminar and incompressible flow. Then you can have the assumptions of constant properties, where density and viscosity remain constant, and we are anyway considering the fully developed flow.

You have already seen that in fully developed flow the velocity v in y direction, it is 0 or first let us consider that you have axial direction as x. So, this is your x, and axial velocity as u. And in y direction you have v velocity, and this is your y direction velocity.

So, from fully developed velocity profile, so for fully developed flow condition you can write v = 0 everywhere; and axial velocity is constant in the axial direction, so $\frac{\partial u}{\partial x} = 0$. So, under this assumptions now let us write the x momentum equation.

So, you can see your x momentum equation is $\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial P}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$. So, this is your x momentum equation.

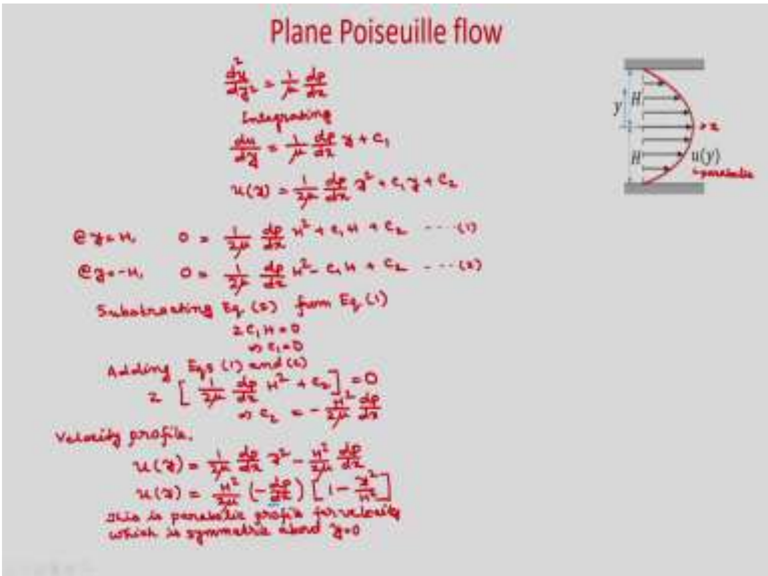
So, as we have assumed that it is a steady state flow, then obviously, $\frac{\partial u}{\partial t} = 0$. So, this is your 0 as it is steady state, $\frac{\partial u}{\partial x} = 0$ fully developed condition; v = 0 fully developed condition; and $\frac{\partial^2 u}{\partial x^2}$=0 as fully developed condition, so fully developed flow. So, we can see that as $\frac{\partial u}{\partial x} = 0$, then u is function of y only.

Similarly, if you write the y momentum equation, then you will find that only $\frac{\partial P}{\partial y} = 0$. So, all terms will get cancelled – the temporal term, convection term, the diffusion term, so all these terms will get cancelled and you will have just all these terms will be 0, and you will have $\frac{\partial P}{\partial y} = 0$. So, you will have from y momentum equation you will get $\frac{\partial P}{\partial y} = 0$, so that means, p is a function of x only.

So, under this situation this x momentum governing equation you can write as, so u is function of y only, so you can write $\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$, so because p is function of x only. So, this is your now you got ordinary differential equation. So, we started with the partial differential equation, but invoking the assumptions and putting some terms 0, you got the ordinary differential equation. Now, with proper boundary condition, you will be able to integrate this ordinary differential equation.

What are the boundary conditions? So, you can see that we have taken axis in the central line as x and y is measured from the central line from here so y. And these two parallel plates are separated by a distance 2 h. So, the boundary conditions you can write. So, at y=h, your u =0 as well as at y = - h ,u = 0, because you have no slip boundary condition on upper plate and the bottom plate.

(Refer Slide Time: 06:46)



So, now let us integrate this equation. So, you have $\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx}$. So, if you integrate it, you will get $\frac{du}{dy} = \frac{1}{\mu}\frac{dP}{dx}y + C_1$. And again if you integrate, then you will get $u(y) = \frac{1}{2\mu}\frac{dP}{dx}y^2 + C_1 y + C_2$. So, C_1,C_2 are integration constant.

Now, you invoke the boundary conditions, you have two boundary conditions because there is as you have two boundary conditions you will be able to find the two constant C_1 and C_2. So, y =H , u = 0. So, you will get $0=\frac{1}{2\mu}\frac{dP}{dx}H^2+C_1H+C_2$ and similarly at y = -H, you have $0=\frac{1}{2\mu}\frac{dP}{dx}H^2-C_1H+C_2$.

So, if you say that this is equation number 1 and this is equation number 2, then if you subtract equation 2 from 1. So, subtracting equation 2 from equation 1 ; so what you will get? So, if you subtract, you will get $2C_1H=0$. That means, $C_1 = 0$ and if you add these two equation adding equations 1 and 2, what you will get?

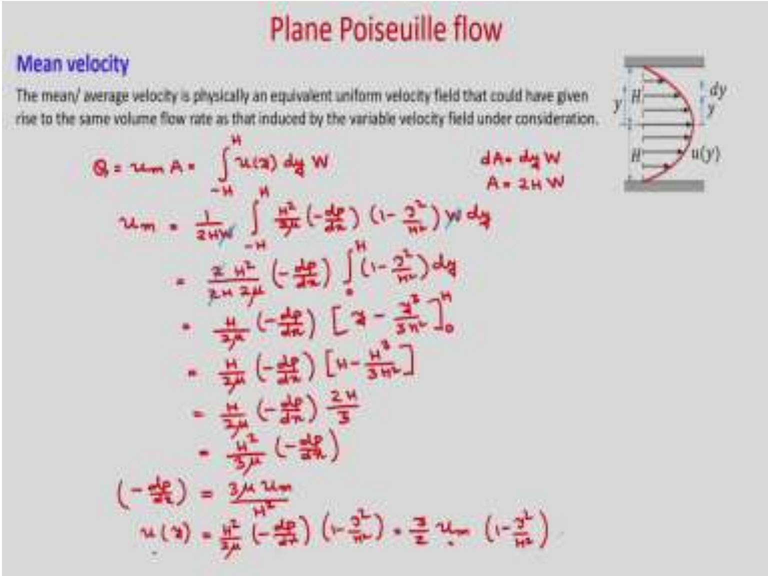
So, this term will get cancelled, this $C_1H - C_1H$. So, this will be 0. So, you will get $2[\frac{1}{2\mu}\frac{dP}{dx}H^2+C_2]=0$, so that means, you will get $C_2=-\frac{H^2}{2\mu}\frac{dP}{dx}$.

So, if you put these values C_1, C_2, so you will get the final velocity profile. So, velocity profile you will get, $u(y)=\frac{1}{2\mu}\frac{dP}{dx}y^2-\frac{H^2}{2\mu}\frac{dP}{dx}$. So, this you can write, $u(y)=\frac{H^2}{2\mu}(-\frac{dP}{dx})[1-\frac{y^2}{H^2}]$.

So, you can see that it is a parabolic profile. So, this is parabolic profile for velocity which is symmetric about y = 0. So, you can see the velocity profile here. So, this is your x direction. So, it is actually symmetric about y = 0. So, you will get at y = 0 if you put, then velocity will be maximum, because there you will get maximum velocity which is your central line velocity and the profile will be parabolic. So, u y is parabolic.

You can see here that we have written this $\frac{dP}{dx}$ as -$\frac{dP}{dx}$. So, you can see that flow will takes place from high pressure region to low pressure region. So, in the positive x-direction or along the axial direction, your pressure will decrease. Hence -$\frac{dP}{dx}$ will be a positive quantity that is why we have written as-$\frac{dP}{dx}$. So, now, we are interested to find what is the mean velocity or average velocity at any cross section.

(Refer Slide Time: 11:51)



So, you can see that mean or average velocity is physically an equivalent uniform velocity field that could have given rise to the same volume flow rate as that induced by the variable velocity field under consideration. So, if you calculate the volume flow rate Q, so you can write $Q = u_m A$, where u_m is the mean velocity.

And that if you see now if you take a small elemental flow area of distance d at a distance y from the central line of distance dy. If you consider an elemental flow area at a distance y from the central line of distance dy, then your this flow area will be just dy into the width of the channel, width means perpendicular to this board whatever distance you have.

So, that if you consider as width, then you can write dA= Wdy. And total flow area will be, so this is your twice h is the distance between two parallel plates. So, it will be $A = 2HW$.

So, now, if you consider this elemental area, so what is flow is happening so that you can integrate from $\int_{-H}^{H} u(y) dyW$, so that is the elemental area dA= Wdy, so that we have written. So, now, your mean velocity will be $u_m = \frac{1}{2HW} \int_{-H}^{H} \frac{H^2}{2\mu}(-\frac{dP}{dx})(1-\frac{y^2}{H^2}) Wdy$ s.

So, now, you will get, $u_m = \frac{2H^2}{2H 2\mu}(-\frac{dP}{dx})\int_0^H (1-\frac{y^2}{H^2})dy$.

So, you can see that we have written this, $u_m = \frac{H}{2\mu}(-\frac{dP}{dx})[y - \frac{y^3}{3H^2}]_0^H$.

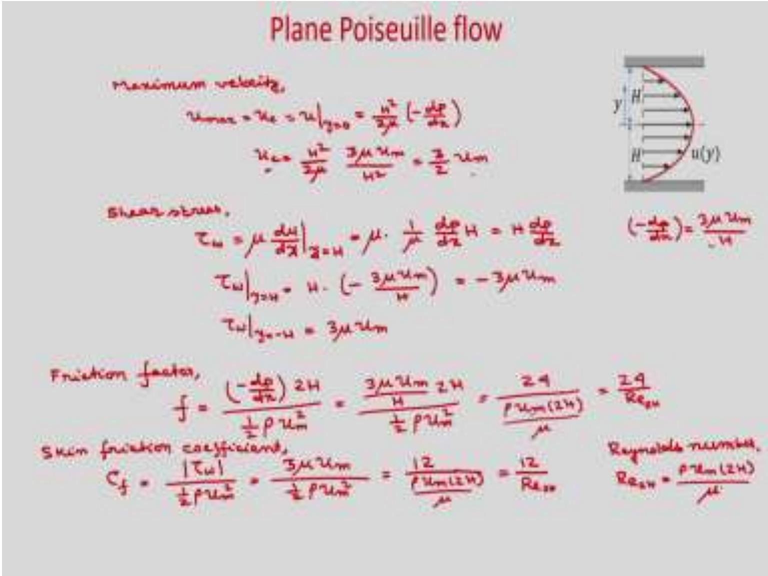
So, if you put H, then we are going to get $u_m = \frac{H}{2\mu}(-\frac{dP}{dx})[H - \frac{H^3}{3H^2}]$. So, this you can see that it will be 2 by 3. So, $u_m = \frac{H}{2\mu}(-\frac{dP}{dx})\frac{2H}{3}$. So, hence you can write $u_m = \frac{H^2}{3\mu}(-\frac{dP}{dx})$. So, now, you can see that you can represent the constant pressure gradient minus dp by dx in terms of the mean velocity.

So, you can write minus $(-\frac{dP}{dx}) = \frac{3\mu u_m}{H^2}$. So, we can see here that your right hand side, you have μ which is your fluid property, dynamic viscosity that is positive quantity, u_m which is your velocity mean velocity that is also constant and positive, and H which is your the distance from the central line so that is also constant. So, obviously, right hand side is constant, that means, $-\frac{dP}{dx}$ is constant.

So, now if you find the velocity profile, so you can write in terms of mean velocity, $u(y) = \frac{H^2}{2\mu}(-\frac{dP}{dx})(1-\frac{y^2}{H^2})$.

So, now, this $-\frac{dP}{dx}$ if you put it here, you are going to get, $u(y) = \frac{3}{2}u_m(1-\frac{y^2}{H^2})$. So, your velocity profile you can write in terms of mean velocity as, $u(y) = \frac{3}{2}u_m(1-\frac{y^2}{H^2})$. And if you find the maximum velocity, then you have to find the velocity at central line which is y =0.

(Refer Slide Time: 17:42)



So, you will get maximum velocity. So, maximum velocity will occur at central line, where u at y =0. So, if you put it, you are going to get $u_{max} = u_c = u|_{y=0} = \frac{H^2}{2\mu}(-\frac{dP}{dx})$. And that if you write $-\frac{dP}{dx}$ in terms of $\frac{3\mu u_m}{H^2}$, then you are going to get , $u_c = \frac{H^2}{2\mu}\frac{3\mu u_m}{H^2}$.

So, it will be just $\frac{3}{2}u_m$. So, you can see that in case of flow fully developed flow inside two infinite parallel plates, your maximum velocity is 1.5 times the average velocity, because here you can see that u_c, which is your maximum velocity is 1.5 times the mean velocity for fully developed flow. Now, if you find what is the shear stress acting on the wall, then you can write shear stress, so that you can write $\tau_w = \mu\frac{du}{dy}$.

So, at any wall you can calculate. So, y = H let us calculate. So, you can write $\tau_w = \mu\frac{du}{dy}|_{y=H}$. So, what is $\frac{du}{dy}$? So, we can see this easily you will get $\tau_w = \mu\frac{1}{\mu}\frac{dP}{dx}H = H\frac{dP}{dx}$, and $(-\frac{dP}{dx}) = \frac{3\mu u_m}{H}$. So, if you put it, then, $\tau_w|_{y=H} = H(-\frac{3\mu u_m}{H}) = -3\mu u_m$.

So, similarly if you calculate, $\tau_w \Big|_{y=-H} = 3\mu u_m$. And you can see that your shear stress inside the fluid, it will vary linearly, now we will define the friction factor. So, this is the non-dimensional representation of the pressure gradient. So, if you calculate non-dimensional pressure population, then that is known as the friction factor.

So, friction factor you can calculate, so this is $f = \dfrac{(-\dfrac{dP}{dx})2H}{\dfrac{1}{2}\rho u_m^2}$. So, this if you find, so what you are going to get? So, $-\dfrac{dP}{dx}$ is this one. So, you can write $f = \dfrac{\dfrac{3\mu u_m}{H}2H}{\dfrac{1}{2}\rho u_m^2}$. So, if you rearrange you will get this as, $\dfrac{24}{\dfrac{\rho u_m(2H)}{\mu}}$.

So, you can see this is your Reynolds number based on the mean velocity and the channel height, then you can write $\dfrac{24}{\mathrm{Re}_{2H}}$. So, friction factor we have represented in the terms of the non-dimensional number Reynolds number based on the channel height. So, friction factor $f = \dfrac{24}{\mathrm{Re}_{2H}}$.

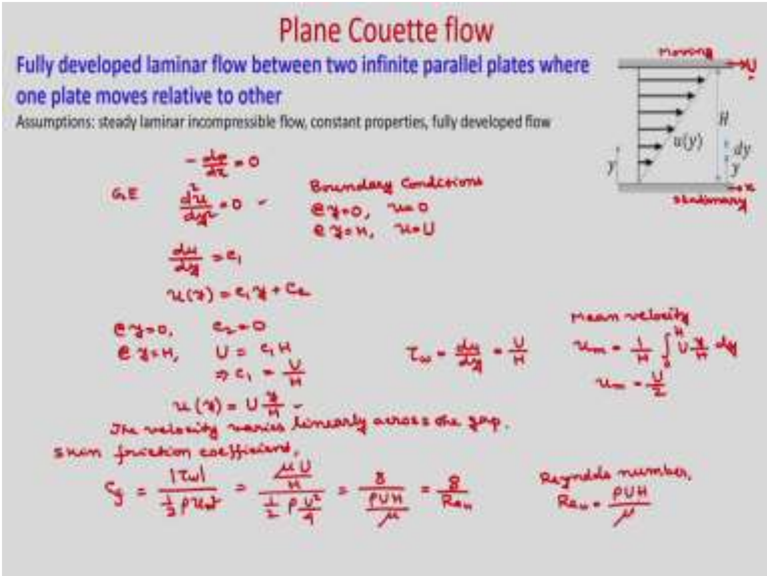
If you calculate the skin friction coefficient, so this is dimensionless representation of the wall shear stress, so skin friction coefficient. So, it is represented as C_f . So, this is $C_f = \dfrac{|\tau_w|}{\dfrac{1}{2}\rho u_m^2} = \dfrac{3\mu u_m}{\dfrac{1}{2}\rho u_m^2}$.

So, you can rearrange it, and you will get $\dfrac{12}{\dfrac{\rho u_m(2H)}{\mu}}$, and you can write $\dfrac{12}{\mathrm{Re}_{2H}}$. So, for the fully developed flow inside two parallel plates, we have calculated the shear stress on the wall.

And, hence we have calculated the skin friction factor which is your dimensionless representation of the wall shear stress as $\frac{12}{\mathrm{Re}_{2H}}$ where Re_{2H} is the Reynolds number based on the channel height.

So, this is your Reynolds number. So, this is your based on 2H. So, Reynolds number we have defined based on mean velocity u_m, and the channel height 2H.

(Refer Slide Time: 23:53)



Next we will consider plane Couette flow. This is the flow between two infinite parallel plates where one plate is moving with respect to the other. So, you can see that these are the two fixed plates, and this upper plate is moving with velocity u in the axial direction ok. So, some constant velocity u it is moving with respect to the bottom wall.

So, this is your x-direction, and y is measured from the bottom wall, and bottom wall is stationary. So, bottom wall is stationary and upper wall is moving with constant velocity u, and y is measured from the bottom wall and the distance between two parallel plates is H.

So, in this scenario now you can see that the flow is taking place. So, here you can see that the flow is taking place due to the shear. So, the upper plate is moving. So, due to the shear there will be velocity inside the channel. So, this is known shear driven flow where your pressure gradient is absent in this particular case, so that is why we have told at as plane Couette flow. So, where you have the external pressure gradient as well along with the shear driven flow, then it will be Couette flow.

But here we are considering plane Couette flow where we are assuming that pressure gradient is 0. So, in this particular case, now we are considering $-\frac{dP}{dx}=0$ and fully developed condition. So, obviously, your governing equation whatever we have derived in earlier case, you can write $\frac{d^2u}{dy^2}=0$, because we have considered fully developed flow. So, your y direction velocity v =0 as well as your $\frac{\partial u}{\partial x}=0$, because there will be no change in the axial velocity in the axial direction. So, your governing equation is $\frac{\partial^2 u}{\partial y^2}=0$.

And what are the boundary conditions? Boundary conditions are at y = 0 bottom wall, u=0 and upper wall at y =H, it is moving with velocity U. So, you integrate this equation. So, you will get $\frac{du}{dy}=C_1$, and $u(y)=C_1y+C_2$. So, C_1 and C_2 are constants. So, you can see that you will get a linear velocity profile.

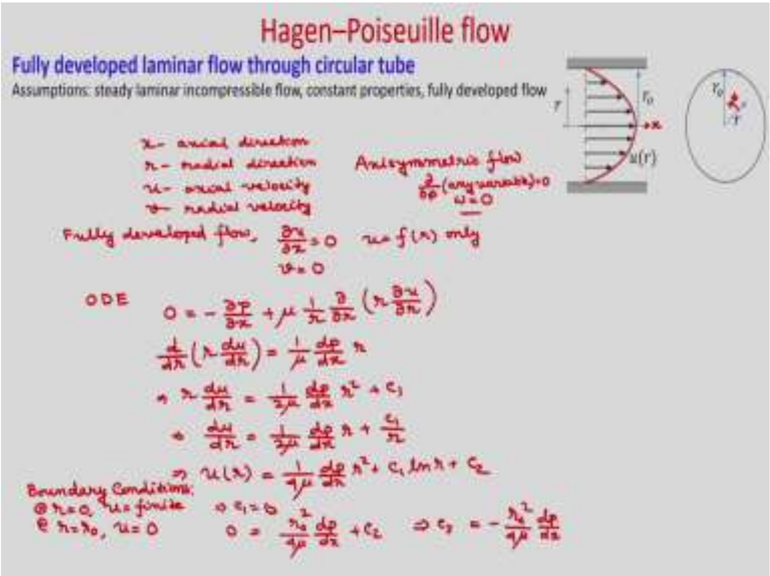
So, now, invoke the boundary conditions at y = 0, u = 0, so u = 0, so we will get C_2 = 0; and at y = H, u = 0, so you will U = u. So, you will get $U=C_1H$; C_2 = 0, so obviously $C_1=\frac{U}{H}$. So, hence you will get the velocity profile $u(y)=U\frac{y}{H}$. And you can see that y = 0, it is 0 velocity at y = H it is U, and it is linearly varying, it is linearly varying because from the velocity profile you can see that it is a velocity varies linearly across the gap.

And if you calculate the shear stress on wall, so you can see that $\frac{du}{dy}$ will be a constant, because here you can see $\tau_w=\frac{du}{dy}$ and $\frac{du}{dy}$ is constant C_1, so that is $\frac{U}{H}$. So, in both the plates, you can find that your wall shear stress $\tau_w=\frac{U}{H}$. So, along inside the flow also the shear stress will be constant. So, now, if you calculate the skin friction coefficient on what will be your mean velocity?

So, if you calculate the mean velocity, mean velocity, so it is $u_m = \frac{1}{H}\int_0^H U\frac{y}{H}dy$. And what is u? $u(y) = U\frac{y}{H}$ dy, so it will be $\frac{y^2}{2}$, hence $u_m = \frac{U}{2}$.

If you calculate the skin friction coefficient, so you will find for this particular case $C_f = \frac{|\tau_w|}{\frac{1}{2}\rho u_m^2}$. So, $\frac{\frac{\mu U}{H}}{\frac{1}{2}\rho\frac{u^2}{4}}$. So, if you rearrange, you will get $\frac{8}{\frac{\rho UH}{\mu}}$, so you will get $\frac{8}{\mathrm{Re}_H}$. So, we are writing this Reynolds number based on your; based on your upper velocity, upper plate velocity u, so that will be $\mathrm{Re}_H = \frac{\rho UH}{\mu}$.

(Refer Slide Time: 30:25)



So, now, we will consider fully developed flow inside pipe. So, this flow is known as Hagen-Poiseuille flow. So, you can see this is your axial direction x, and radial direction is r.

And we are considering axisymmetric flow that means that in circumferential direction in phi direction there will be no variation of any variable, so that is your axisymmetric. So, in case of axisymmetric your u will be function of r only; and as it is fully developed condition then the axial velocity u will be constant or u is function of r only; it is not function of x. So, $\frac{\partial u}{\partial x} = 0$.

So, if you consider that. So, you can see that it is a pipe, and these are your pipe wall, and the pipe radius is r_0. And this is your central line, and r is measured from the center. So, if you write the fully developed condition, so in this particular case x is your axial direction ok, r is your radial direction, u is your axial velocity, and v is your radial velocity ok.

So, as it is a fully developed flow, so $\frac{\partial u}{\partial x} = 0$, because your in axial direction there will be no change in the velocity profile. So, $\frac{\partial u}{\partial x} = 0$. So, u is function of r only. And radial velocity everywhere v = 0. And one most important assumptions that we have taken it is axisymmetric flow.

So, if you considered that your circumferential direction if you miss that, so let us say this is your Φ or θ let us say Φ, then $\frac{\partial}{\partial \phi}(any\, variable) = 0$, and the velocity in Φdirection also 0. So, if you say that velocity w which is in Φ direction, so it will be this it will be 0 in axisymmetric flow.

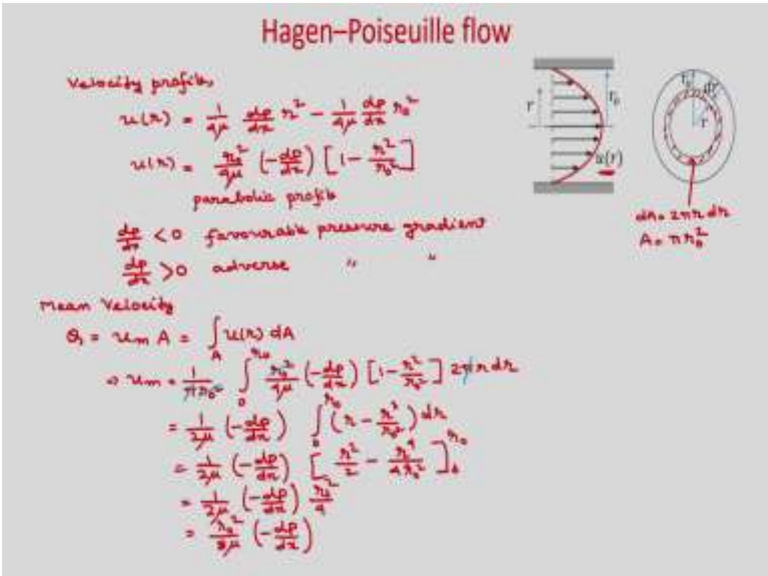
So, if you consider the momentum equation, then you can see that you are invoking all these assumptions, you will get this ordinary differential equation which is your governing equation. You will get $0 = -\frac{\partial P}{\partial x} + \mu \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})$.

Here also you will find that p is function of x only, and you can write $\frac{\partial P}{\partial x} = \frac{dP}{dx}$, and u is function of r only. So, you can write $\frac{d}{dr}(r\frac{du}{dr}) = \frac{1}{\mu}\frac{dP}{dx} r$. So, now, if you integrate it, so you will get $r\frac{du}{dr} = \frac{1}{2\mu}\frac{dP}{dx}r^2 + C_1$. If you divide by r, then $\frac{du}{dr} = \frac{1}{2\mu}\frac{dP}{dx}r + \frac{C_1}{r}$. And again if you integrate, then you will get the velocity profile as $u(r) = \frac{1}{4\mu}\frac{dP}{dx}r^2 + C_1 \ln r + C_2$.

So, now you invoke the boundary condition, what are the boundary conditions? At r = 0, you have u finite; and at r = r_0 which is your wall you have velocity 0, u = 0. So, if you write boundary conditions, boundary conditions, so at r = 0, u is finite;. So, you can see that if u is finite, then C_1 must be 0. So, C_1 is 0. And r = r_0 ,u = 0 because that is your wall, so it will be r_0.

So, you will get $0 = \frac{r_0^2}{4\mu}\frac{dP}{dx} + C_2$. So, $C_2 = -\frac{r_0^2}{4\mu}\frac{dP}{dx}$.

(Refer Slide Time: 36:14)



Hence you can write the velocity profile as so velocity profile you can write now, u is function of r only. And if you put the constant, so you can see that $C_2 = -\frac{r_0^2}{4\mu}\frac{dP}{dx}$. So, you can write as $u(r) = \frac{1}{4\mu}\frac{dP}{dx}r^2 - \frac{1}{4\mu}\frac{dP}{dx}r_0^2$.

So, hence you will get $u(r)=\frac{r_0^2}{4\mu}(-\frac{dP}{dx})[1-\frac{r^2}{r_0^2}]$. So, you can see that this is also your parabolic profile.

So, you can see you will get a parabolic velocity profile, where r = 0 you will get the maximum velocity; and at walls obviously it is 0. So, r = 0, r = r_0, it is 0. So, you will get a velocity profile which is u is function of r, so and you will get a parabolic profile.

And we are writing u as $-\frac{dP}{dx}$ in term, we are writing u(r) in terms of $-\frac{dP}{dx}$ because $-\frac{dP}{dx}$ is positive quantity because in axial direction you have $-\frac{dP}{dx}$ is positive, because your pressure gradient is decreasing in the axial direction. So, generally if $\frac{dP}{dx}$ < 0, so this is known as favourable pressure gradient; and if $\frac{dP}{dx}$>0, then it is known as adverse pressure gradient, because you have pressure gradient as positive and your flow reversal may take place.

So, generally $\frac{dP}{dx}$<0, then it is known as favourable pressure gradient. And if $\frac{dP}{dx}$>0, then it is a adverse pressure gradient. Now, you want to calculate the mean velocity in Hagen-Poiseuille flow. So, you can calculate the mean velocity as, so $\mathrm{Q}=\mathrm{u}_m A=\int_A u(r)dA$. So, in this particular case, you can see that you have a circular cross section.

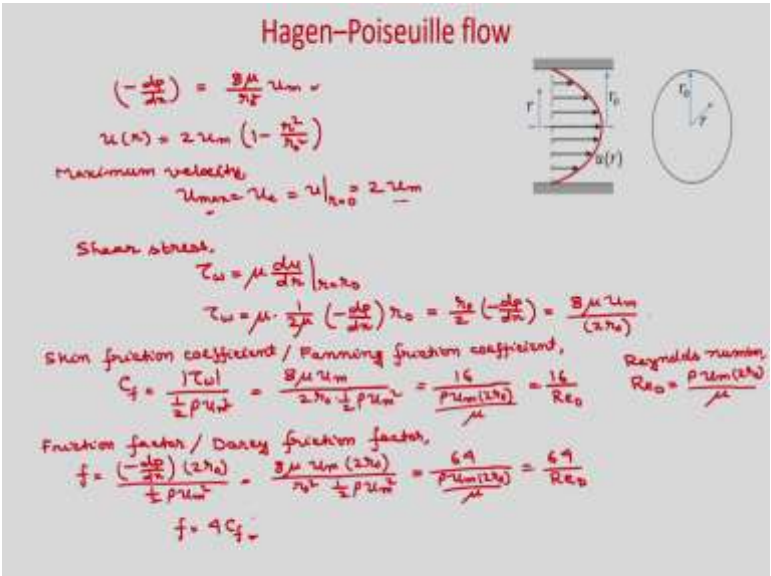
So, at a distance r you take one elemental flow area. So, this is your elemental flow area of distance dr. So, what will be your dA in this particular case? So, this is the elemental flow area. So, $dA=2\pi rdr$. So, this is the area. And total area is $A=\pi r_0^2$.

So, if you put it here, so you will get as $\mathrm{u}_m=\frac{1}{\pi r_0^2}\int_0^{r_0}\frac{r_0^2}{4\mu}(-\frac{dP}{dx})[1-\frac{r^2}{r_0^2}]2\pi rdr$.

So, this π, π you cancel. And you can write it as, so these other r_0^2 also you can cancel because this is constant. So, you can write as $\mathrm{u}_m = \frac{1}{2\mu}(-\frac{dP}{dx})[r - \frac{r^3}{r_0^2}]\mathrm{dr}$.

So, if you integrate it, so you will get $\mathrm{u}_m = \frac{1}{2\mu}(-\frac{dP}{dx})[\frac{r^2}{2} - \frac{r^4}{4r_0^2}]_0^{r_0}$. So, if you put the value, so it will be $\mathrm{u}_m = \frac{1}{2\mu}(-\frac{dP}{dx})\frac{r_0^2}{4}$. So, hence you will get $\mathrm{u}_m = \frac{r_0^2}{8\mu}(-\frac{dP}{dx})$. So, now, you can express $-\frac{dP}{dx}$ in terms of mean velocity.

(Refer Slide Time: 41:59)



So, $-\frac{dP}{dx}$ you can express in terms of mean velocity, so that will be $-\frac{dP}{dx} = \frac{8\mu}{r_0^2}\mathrm{u}_m$. So, now, you can express this velocity profile in terms of the mean velocity. So, you can see that, $\mathrm{u(r)} = 2\,\mathrm{u}_m(1 - \frac{r^2}{r_0^2})$.

Similarly, now if you find the maximum velocity which will occur at the central line where r = 0, then you can write maximum velocity, $\mathrm{u}_{\max} = u_c = u|_{r=0} = 2u_m$. So, in this particular case, you can see that when you consider the flow inside pipe, then your maximum velocity will be twice of the average velocity or mean velocity.

So, this your u_{max}, you can see it is twice into the mean velocity. And when we consider flow between two parallel plates u_{max} was 1.5 times the mean velocity u_m. So, you should remember here.

So, when you consider pipe flow, your maximum velocity is twice the mean velocity; and when you consider flow between parallel plates, then your maximum velocity will be 1.5 times the average velocity. So, now, we want to calculate the shear stress at the wall, that means, at r = r_0 we want to calculate the shear stress. So, $\tau_w = \mu \frac{du}{dr}|_{r=r_0}$.

So, shear stress you can calculate $\tau_w = \mu \frac{du}{dr}|_{r=r_0}$. So, you can see $\tau_w = \mu \frac{1}{2\mu}(-\frac{dP}{dx})r_0$. So, it will be just $\frac{r_0}{2}(-\frac{dP}{dx})$. And $-\frac{dP}{dx} = \frac{8\mu}{r_0^2}\text{u}_m$. So, you will you will see that if you put it, you will get $\tau_w = \frac{8\mu u_m}{(2r_0)}$.

So, now, if you want to calculate the skin friction coefficient, what is skin friction coefficient? It is the dimensionless shear stress, or sometime it is known as fanning friction coefficient. So, you can write $C_f = \frac{|\tau_w|}{\frac{1}{2}\rho u_m^2}$. So, you can see that $C_f = \frac{8\mu u_m}{2r_0 \frac{1}{2}\rho u_m^2}$.

So, if you rearrange it, you can write it as $\frac{16}{\frac{\rho u_m(2r_0)}{\mu}}$. So, you can write $\frac{16}{\text{Re}_D}$. D is the diameter of the pipe, so that is $2r_0$. So, a Reynolds number we have defined here, Reynolds number we have defined here based on the diameter and mean velocity. So, $\text{Re}_D = \frac{\rho u_m(2r_0)}{\mu}$.

Similarly, if you to want to calculate the friction factor which is your non-dimensional pressure gradient, so that you can write it as friction factor for this particular case it is known as Darcy friction factor also. So, you can see that $f = \frac{(-\frac{dP}{dx})(2r_0)}{\frac{1}{2}\rho u_m^2}$. So, $f = \frac{\frac{8\mu u_m}{r_0^2}(2r_0)}{\frac{1}{2}\rho u_m^2}$.

So, if you rearrange it, so you will get $\frac{64}{\frac{\rho u_m (2r_0)}{\mu}}$. So, you will get $\frac{64}{\mathrm{Re}_D}$. So, your friction factor is $\frac{64}{\mathrm{Re}_D}$. So, if you see that your friction factor is 4 times your skin friction coefficient. Friction factor, so this is your 16×4 and $C_f = \frac{16}{\mathrm{Re}_D}$. So, $f=4c_f$.

So, in today's lecture, we have found the fully developed velocity profile for three different cases. First case, we considered as flow between two infinite parallel plates, where your v velocity is 0, and $\frac{\partial u}{\partial x} = 0$ which is your axial velocity does not change in the x direction. So, with that condition, we found the velocity profile and we have found that it is parabolic in nature. And then we can calculated the mean velocity, and then we calculated the skin friction coefficient as well as the friction factor.

Next we considered the plane-Couette flow where we consider the shear driven flow between two parallel plates where bottom plate is stationary and upper plate is moving with velocity u in the positive x direction. And we have seen that velocity varies linearly from bottom plate to top plate.

Next we considered the Hagen-Poiseuille flow; this is the fully developed flow inside pipe. Here we found the velocity profile u as function of r, and that is also parabolic in nature. In this particular case also, we found the mean velocity, then skin friction coefficient and friction factor. And we have found that friction factor is 4 times the skin friction coefficient.

In Hagen-Poiseuille flow, we have seen that your maximum velocity is 2 times the average velocity; and in case of plane-Poiseuille flow your maximum velocity is 1.5

times the average velocity. In fully developed case, we have found the velocity profile in different channels, and we have found the mean velocity. These we will use in the next module when we will find that temperature distribution inside this channel flow.

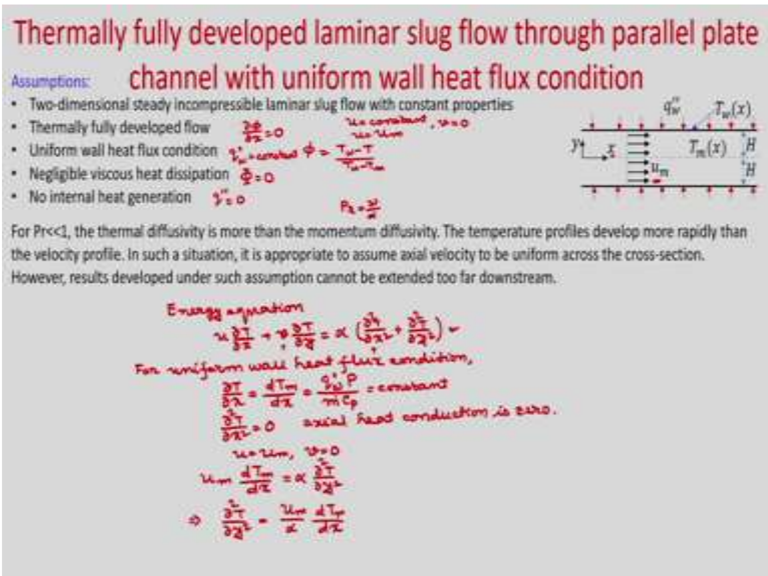
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 06
Convection in Internal Flows – II
Lecture - 18
Thermally fully developed laminar slug flow with uniform wall heat flux condition

So, today we will consider Thermally Fully developed laminar slug flow with uniform wall heat flux condition. So, today we will consider two different types of channel; first we will consider flow through parallel plates channel and next, we will consider flow through circular pipe. As you know that it is thermally fully developed flow; so obviously, the non-dimensional temperature, we have defined as $\frac{T_w - T}{T_w - T_m}$. So, it will not vary in axial direction and we will consider uniform wall heat flux and you know that in this particular case, $\frac{\partial T}{\partial x}$ will be constant.

(Refer Slide Time: 01:17)



So, you can see that for Prandtl number << 1, where Prandtl number is defined as $\Pr = \frac{\nu}{\alpha}$. $\Pr = \frac{\nu}{\alpha}$, where ν is the momentum diffusivity; α is the thermal diffusivity. So, for Prandtl number << 1, the thermal diffusivity is more than the momentum diffusivity. The temperature profiles develop more rapidly than the velocity profile near to the inlet.

In such a situation, it is appropriate to assume axial velocity to be uniform across the cross section. However, results developed under such assumption cannot be extended too far downstream. And, these axial velocity when we considered as uniform that is known as slug flow.

So, the assumptions for today's class, we will consider two-dimensional steady incompressible laminar slug flow with constant properties. So, slug flow means u is constant, u is constant and in this particular case, we will take u =u_m. So, you can see this is the channel with two infinite parallel plates; x is the axial direction, y is measured from the center line, $q_w^{"}$ is the constant heat flux applied to both the walls. These two plates are separated by distance 2H and you have uniform velocity because it is slug flow. Obviously, if it is uniform, then your v velocity will be also 0.

So, v is also 0; v is the velocity in y direction. Thermally fully developed flow, so $\frac{\partial \phi}{\partial x} = 0$, where Φ is the non-dimensional temperature. $\frac{T_w - T}{T_w - T_m}$ uniform wall heat flux condition. So, $q_w^{"}$ is constant. Negligible viscous heat dissipation.

So, Φ = 0 and no internal heat generation. So, $q^{'''} = 0$. So, for this particular case, let us write the energy equation in general for steady two-dimensional situation. So, energy equation, $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2})$, neglecting the viscous dissipation and heat generation.

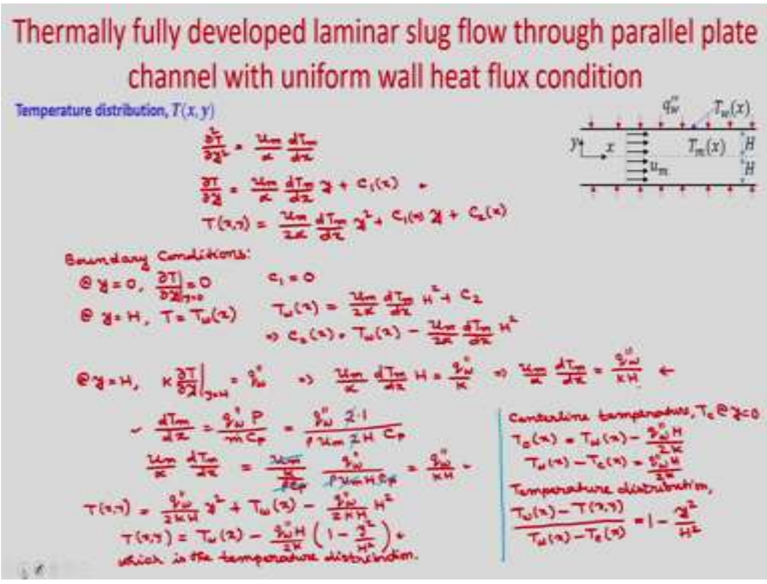
So, now for uniform wall heat flux boundary condition, you know the $\frac{\partial T}{\partial x}$ right. We have already derived. So, for uniform wall heat flux, $\frac{\partial T}{\partial x} = \frac{dT_m}{dx} = \frac{q_w^{"} P}{\dot{m} C_P}$ and you can see that $q_w^{"}$ is constant, P is the perimeter.

If you have a constant cross-sectional channel, then P is also constant, $\dot{m}$ is constant, C_p is constant. So, this will be constant. Hence, $\frac{\partial^2 T}{\partial x^2}$ =0; because, $\frac{\partial T}{\partial x}$ is constant; so, $\frac{\partial^2 T}{\partial x^2}$ =0. So, for this particular case, you can see that your axial heat conduction is 0.

So, for slug flow, you can put u = u_m and b =0. So, u is equal to u_m and v = 0. So, all these if put it in this energy equation, what you will get? So, you will get u_m. So, $\frac{\partial T}{\partial x} = \frac{dT_m}{dx}$; then, v = 0. So, this term will get 0, $\frac{\partial^2 T}{\partial x^2}$ =0. So, you will write $\alpha \frac{\partial^2 T}{\partial y^2}$.

So, you can write it as, $\alpha \frac{\partial^2 T}{\partial y^2} = \frac{u_m}{\alpha} \frac{dT_m}{dx}$.

(Refer Slide Time: 06:26)



So, we have the equation, $\alpha \frac{\partial^2 T}{\partial y^2} = \frac{u_m}{\alpha} \frac{dT_m}{dx}$. So, this is constant. So, what you can do? Now, you can integrate it. So, you can write $\frac{\partial T}{\partial y} = \frac{u_m}{\alpha} \frac{dT_m}{dx} y + C_1(x)$; because,

$$\mathrm{T(x, y)} = \frac{u_m}{2\alpha} \frac{dT_m}{dx} y^2 + C_1(x) y + C_2(x).$$

So, now let us discuss about the boundary conditions. So, at central line, we have a finite temperature and you can see in this particular case about the center line, you have geometrical symmetry, because both walls are at distance H from the center line. At the same time, we have thermal boundary condition which is symmetric because in both walls, we have uniform wall heat flux boundary condition. So, you can see that it is geometrically and thermally symmetric.

So, at y = 0 at the center line, you will have either maximum or minimum temperature. Hence, you can write $\frac{\partial T}{\partial y}|_{y=0} = 0$; which is your center line will be 0. So, one boundary condition is; so, one boundary condition is at y = 0. You have $\frac{\partial T}{\partial y}|_{y=0} = 0$. And, another boundary condition at y = H, you can have temperature $T = T_w(x)$.

So, we are assuming that you have wall temperature T_w which varies in axial direction and that is $T = T_w$ at y = H. So, now if you put at y = 0, $\frac{\partial T}{\partial y} = 0$, then you can see from this equation, this equation you can see that C_1 will be 0 and at y = H, if you put $T = T_w$, then you can see that it will be $T_w(x) = \frac{u_m}{2\alpha}\frac{dT_m}{dx}H^2 + C_2$. So, hence your, $C_2(x) = T_w(x) - \frac{u_m}{2\alpha}\frac{dT_m}{dx}H^2$.

So, now let us simplify this $\frac{u_m}{2\alpha}\frac{dT_m}{dx}H^2$. So, let us apply the boundary condition at $r = r_0$, you have a uniform wall heat flux $q_w^{"}$ =constant. So, obviously, you can write at y = H,. Now, $K\frac{\partial T}{\partial y}|_{y=H}$. So, it is y =0 will be $q_w^{"}$. So, you can see your y is in the positive upward direction, right. So, but $q_w^{"}$ is the negative y direction, what we considered in this diagram.

So, obviously, you are in negative y direction, the heat conduction will be, $K\frac{\partial T}{\partial y}|_{y=H} = q_w^{"}$. So, from here, now we can see $\frac{\partial T}{\partial y}$ from this equation $C_1 = 0$. So, $\frac{\partial T}{\partial y}$ if you put, then you will get K; K you just divide in the right hand side, so you will get $\frac{u_m}{\alpha}\frac{dT_m}{dx}H = \frac{q_w^{"}}{K}$. And hence, you can write $\frac{u_m}{\alpha}\frac{dT_m}{dx} = \frac{q_w^{"}}{KH}$. This we can also similar way, we can derive a straight forward from this expression of $\frac{dT_m}{dx}$, because you know that $\frac{dT_m}{dx}$ for this uniform heat flux boundary condition is $\frac{dT_m}{dx} = \frac{q_w^{"} P}{\dot{m} C_P}$. So, this is constant already you have written. So, now you tell me what is P perimeter?

So, this perimeter is the heaters per area on the wall. So, per unit width if you considered, then it will be 2 or if you w is the width if you considered; then, on the upper wall, it is w; bottom wall, it is w, then 2 w or per unit width if you consider, then 2 into 1. And, what is the flow area? Flow area is the distance between 2 parallel plates is 2 H. So, 2 H into w or per unit width if you consider, then it will be 2 H.

So, you can write $\frac{q_w^{"} 2.1}{\rho u_m 2HC_P}$. So, you can see that you can write $\frac{u_m}{\alpha}\frac{dT_m}{dx} = \frac{u_m}{\frac{K}{\rho C_p}}\frac{q_w^{"}}{\rho u_m H C_P}$ and similarly, from directly this expression also same, we got $\frac{q_w^{"}}{KH}$.

Now, you write the final temperature distribution putting the value of C_2. So, if you write the temperature distribution. So, C_1=0, C_2 is this, so will be, $T(x,y) = \frac{q_w^{"}}{2KH} y^2 + T_w(x) - \frac{q_w^{"}}{2KH} H^2$.

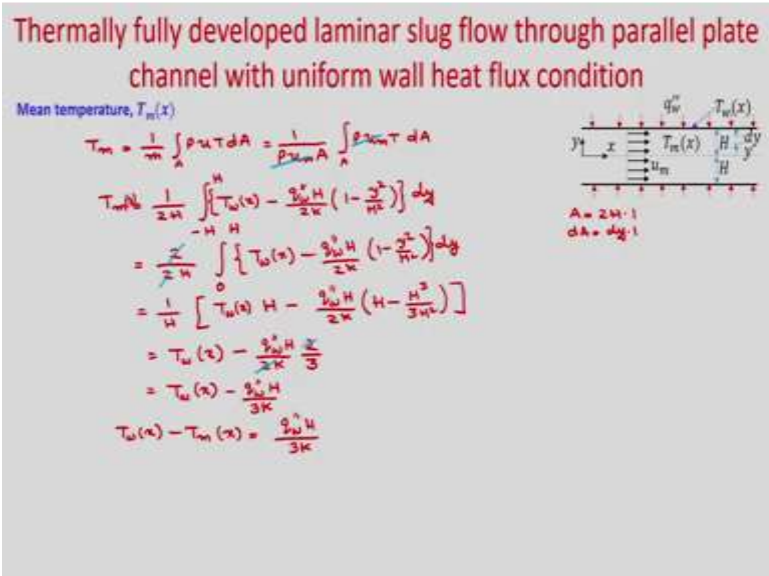
So, now you can see, so if you rearrange it. So, we will rearrange it as, $T(x,y) = T_w(x) - \frac{q_w^{"} H}{2K}(1 - \frac{y^2}{H^2})$.

So, this is the temperature distribution. This temperature distribution also we can write in terms of the centerline temperature. So, what will be centerline temperature? At y = 0, you will get the centerline temperature. So, if it is so, so you will get centerline temperature T_c at y = 0, right. So, y = 0, you will get the centerline temperature.

So, if you put y = 0 in this equation, you can find the center line temperature, $T_c(x) = \mathrm{T}_w(\mathrm{x}) - \frac{q_w^{"} H}{2K}$. So, you can see that if you can write $\mathrm{T}_w(\mathrm{x}) - T_c(x) = \frac{q_w^{"} H}{2K}$. And, if you put it here and you can write the temperature distribution in terms of the central line temperature.

So, it will be $\frac{T_w(x) - T(x,y)}{T_w(x) - T_c(x)} = 1 - \frac{y^2}{H^2}$. So, to calculate the heat transfer coefficient, we need to find the mean temperature. Because, we will define the Nusselt number or the heat transfer coefficient based on the difference between wall temperature and the mean temperature.

(Refer Slide Time: 18:20)



So, let us calculate the mean temperature now. You know that mean temperature, you can calculate as $T_m = \frac{1}{\dot{m}} \int_A \rho u T dA$. So, in this particular case, as you consider the slug flow. So, u_m is constant. So, you can write $\frac{1}{\rho u_m A} \int_A \rho u_m T dA$.

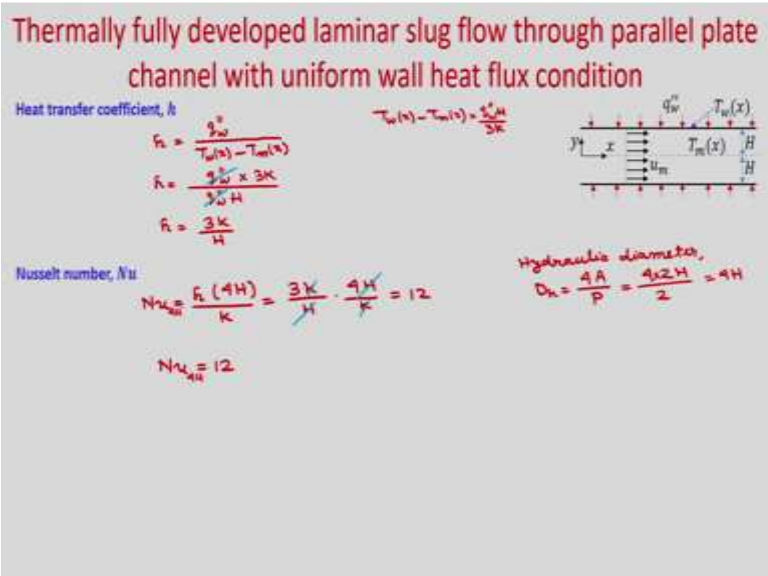
So, what will be the flow area in this particular case? So, particular case per unit width your flow area is 2 H. And, what is your d A? d A = d y. So, d y into 1; per unit width dA=dy. So, dy into 1 and this is also into 1. So, per unit width we are defining, so this ρu_m will cancel out.

So, you can write $T_m(x) = \frac{1}{2H}\int_{-H}^{H}\{T_w(x) - \frac{q_w^{"}H}{2K}(1-\frac{y^2}{H^2})\}dy$. So, now this integral minus H to H will write as 0 to H. So, 2 you can take it outside; so, 2 H.

So, it will be $T_m(x) = \frac{2}{2H}\int_{0}^{H}\{T_w(x) - \frac{q_w^{"}H}{2K}(1-\frac{y^2}{H^2})\}dy$. So, you can write now $T_m(x) = \frac{1}{H}[T_w(x)H - \frac{q_w^{"}H}{2K}(H - \frac{H^3}{3H^2})]$.

So, now if you multiply width 1 /H. So, it will be $T_m(x) = T_w(x) - \frac{q_w^{"}H}{2K}\frac{2}{3}$. So, you will get finally, T_m as T_w (x). So, the difference between the wall temperature and the mean temperature you can write as $T_w(x) - T_m(x) = \frac{q_w^{"}H}{3K}$.

(Refer Slide Time: 22:44)



So, now we can calculate the heat transfer coefficient. So, what is your heat transfer coefficient? Heat transfer coefficient is $h = \frac{q_w^{"}}{T_w(x) - T_m(x)}$ that we got that equating the Fourier’s law and the Newton’s law of cooling.

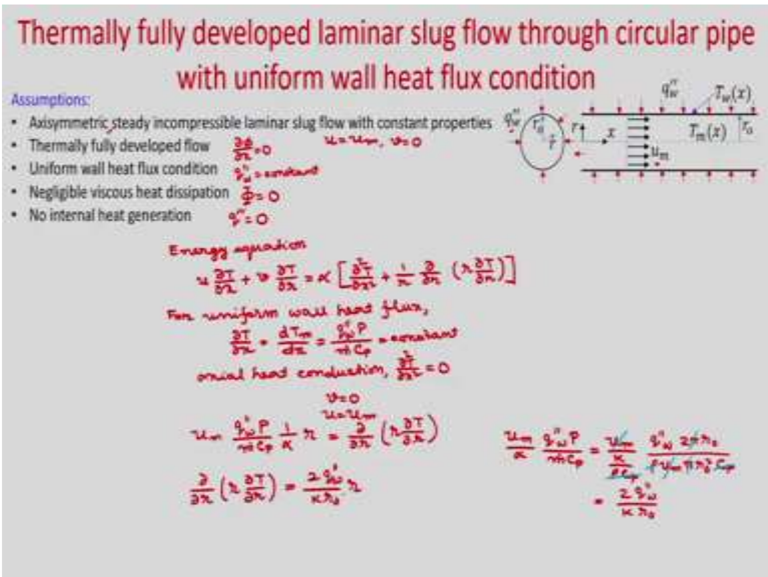
So, you can write $h=\frac{q_w^{"} X 3K}{q_w^{"} H}$. So, you can write H in terms of thermal conductivity of the fluid and the distance from the centerline to the wall as, $h=\frac{3K}{H}$.

So, now you will be able to calculate the Nusselt number. Nusselt number you can calculate based on the hydraulic diameter. So, what is the hydraulic diameter of this particular case? So, you have a two parallel plates separated by distance 2 H. So, your hydraulic diameter in this particular case, so hydraulic diameter will be $D_h=\frac{4A}{P}$.

So, what will be that $\frac{4\times 2H}{2}$. so, it will be 4H. So, we will define the Nusselt number based on the hydraulic diameter that is 4H and the difference between the mean temperature; difference between the wall temperature and mean temperature.

So, Nusselt number now will defined as, $Nu_{4H}=\frac{h(4H)}{K}$. So, hence you can see that it will be 12. So, Nusselt number for these slug flow and thermally fully developed flow inside the parallel plates channel is 12. So, it is based on 4 H, hydraulic diameter based on 4 H,. Next let us consider thermally fully developed laminar slug flow through circular pipe with uniform wall heat flux condition.

(Refer Slide Time: 26:07)



So, the assumptions will take that it is axisymmetric. Axisymmetric means circumferential variation is 0 and velocity is also 0. So, what is axisymmetric? Axisymmetric means the circumferential variation of any quantity is 0 and in that direction, velocity is also 0. And, you can see in this particular case geometrically and thermally because you have uniform wall heat flux boundary condition; so, it is axisymmetric.

So, you can write for axisymmetric steady incompressible laminar slug flow with constant properties so, u = u_m and v = 0. So, you can see x is the axial direction, r is the radial direction and it is measured from the center line; at r =r_0, your heat flux $q_w^{"}$ is constant and you have a slug flow.

So, your velocity profile is constant and that is equal to u_m. So, thermally fully developed flow so, $\frac{\partial \phi}{\partial x} = 0$. Uniform wall heat flux condition, so $q_w^{"}$ = constant. Negligible viscous heat dissipation, so Φ=0 and no internal heat generation, $q^{'''} = 0$.

So, now, let us write the energy equation in cylindrical coordinate for axisymmetric and steady incompressible and laminar flow. So, energy equation you can write in cylindrical coordinate as, $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})]$.

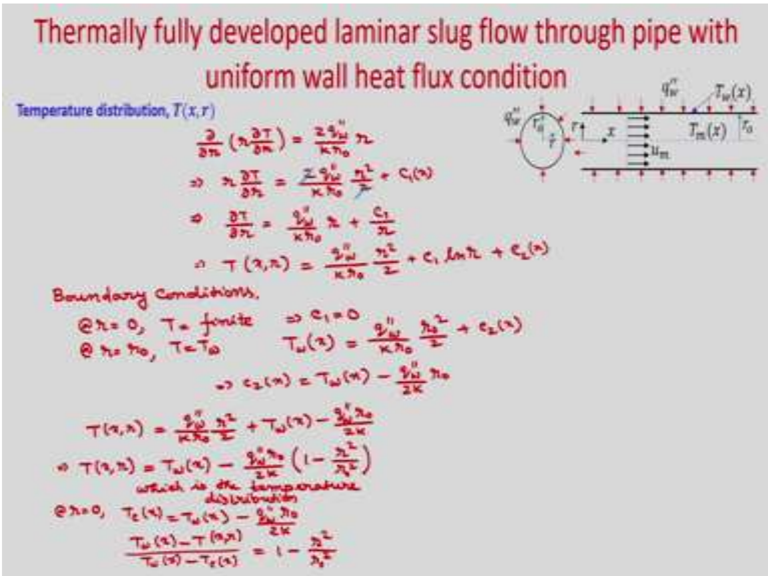
So, now for constant heat flux boundary condition, we can write $\frac{\partial T}{\partial x}$=constant and hence, $\frac{\partial^2 T}{\partial x^2}$ =0. So, for uniform wall heat flux $\frac{\partial T}{\partial x} = \frac{dT_m}{dx} = \frac{q_w^{"} P}{\dot{m} C_P}$ = constant. Hence, your axial heat conduction $\frac{\partial^2 T}{\partial x^2}$ =0. And for slug flow, you can see v = 0. So, if you put these in these energy equation. So, you will get u = u_m. So, you can write, $u_m \frac{q_w^{"} P}{\dot{m} C_P}\frac{1}{\alpha} r = \frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$. So, now in left hand side, you put the values of P perimeter,

$\dot{m}$, mass flow rate and α as $\dfrac{K}{\rho C_p}$. So, if you put it and simplify it what you will get? So, you can see. So, you can write $\dfrac{u_m}{\dfrac{K}{\rho C_p}} \dfrac{q_w^{"} 2\pi r_0}{\rho u_m \pi r_0^2 C_P}$.

So, you will get finally this $u_m \dfrac{q_w^{"} P}{\dot{m} C_P} = \dfrac{2 q_w^{"}}{K r_0}$. So, if you put it here, so you will get $\dfrac{\partial}{\partial r}(r \dfrac{\partial T}{\partial r}) = \dfrac{2 q_w^{"}}{K r_0} r$.

So, now this equation, you can integrate twice and you can find the temperature distribution for thermally fully developed laminar slug flow through circular pipe with applying two boundary conditions at r = 0, you have T is finite and at r = r_0, you have wall temperature T_w right. So, that is why we have started with the simple problem considering the slug flow, because your right hand side is becoming constant. So, it is easy to integrate and find the temperature distribution and the heat transfer coefficient.

(Refer Slide Time: 31:54)



So, now let us find the temperature distribution. So, you have $\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) = \frac{2q_w^{"}}{Kr_0}r$. So, if you integrate it first, then you will get $r\frac{\partial T}{\partial r} = \frac{2q_w^{"}}{Kr_0}\frac{r^2}{2} + C_1(x)$. Then, if you divide by r, then you will get so these 2, 2 will get cancelled. So, you will write $\frac{\partial T}{\partial r} = \frac{q_w^{"}}{Kr_0}r + \frac{C_1}{r}$.

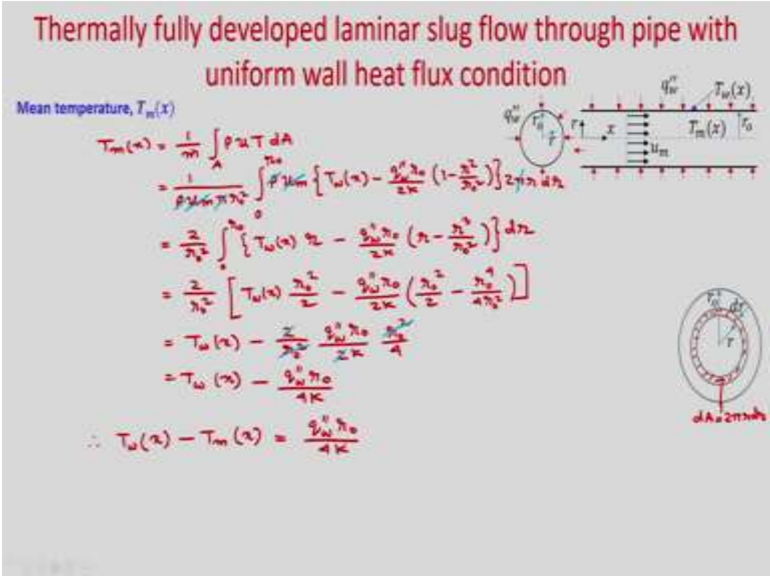
Now, if we integrate again; so, you will get, $T(x,r) = \frac{q_w^{"}}{Kr_0}\frac{r^2}{2} + C_1 \ln r + C_2(x)$. So, what are the boundary conditions? So, one boundary condition is that at r = 0, T is finite. And, another boundary condition, you can write at $r = r_0$, you have a wall temperature T_w.

So, boundary conditions at r = 0, your T is finite. And also, you can see that geometrically and thermal it is symmetric. So, $\frac{\partial T}{\partial r}$=0. So, you can see if r = 0 if you put, so obviously, $C_1 = 0$. So, because this is your 0, so T is finite. So, this cannot be infinite. So, C_1 must be 0. So, C_1=0 and $r = r_0$, you have wall temperature T_w.

So, if you put it that $T_w(x) = \frac{q_w^{"}}{Kr_0}\frac{r_0^2}{2} + C_2(x)$. So, you can see $C_2(x) = T_w(x) - \frac{q_w^{"}}{2K}r_0$. So, now, these constants if you put in the temperature profile and find the temperature distribution. So, you can write $T(x,r) = \frac{q_w^{"}}{Kr_0}\frac{r^2}{2} + T_w(x) - \frac{q_w^{"}r_0}{2K}$. So, these you can write as $T(x,r) = T_w(x) - \frac{q_w^{"}r_0}{2K}(1 - \frac{r^2}{r_0^2})$. So, which is your temperature distribution.

Now, if we write in terms of the centerline temperature, then you put at r = 0, T =T_c. So, at r = 0, your central line temperature will be $T_c(x) = T_w(x) - \frac{q_w^{"}r_0}{2K}$. So, if you put it here. So, if you rearrange, you are going to get this temperature profile $\frac{T_w(x) - T(x,r)}{T_w(x) - T_c(x)} = 1 - \frac{r^2}{r_0^2}$.

(Refer Slide Time: 36:41)



So, now next we need to find the mean temperature to find the heat transfer coefficient. So, you can write $T_m(x) = \frac{1}{\dot{m}}\int_A \rho uTdA$. So, now, what is $\dot{m}$? $\dot{m} = \frac{1}{\rho u_m}$. And, what is the area? Flow area is πr_0^2 and this you can write,

$$T_m(x) = \frac{1}{\rho u_m \pi r_0^2}\int_0^{r_0} \rho u_m \{T_w(x) - \frac{q_w^{"} r_0}{2K}(1-\frac{r^2}{r_0^2})\}2\pi r dr .$$

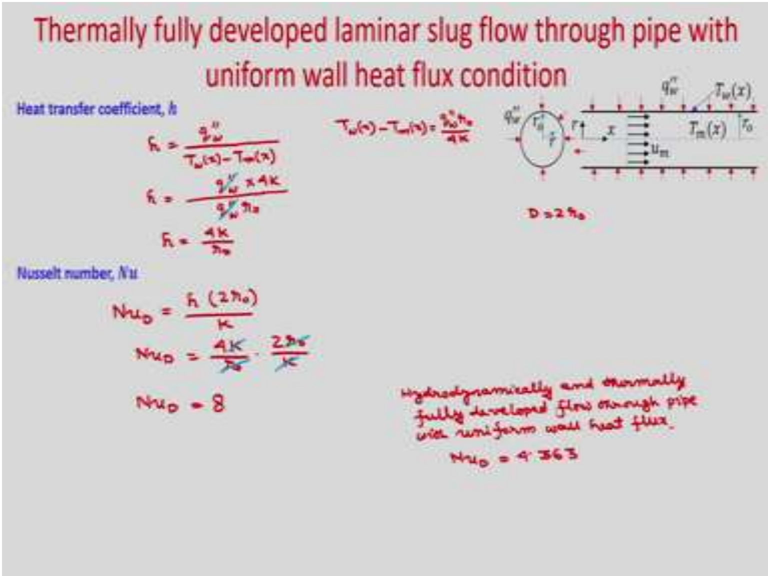
So, that you can write here. So, now, u_m u_m, ρ, ρ, this π, π will get cancelled.

So, and now you integrate it from 0 to r_0. So, what you will get? So, you can write $T_m(x) = \frac{2}{r_0^2}\int_0^{r_0} \{T_w(x)\,\mathrm{r} - \frac{q_w^{"} r_0}{2K}(r-\frac{r^3}{r_0^2})\}dr$.

So, now you integrate it. So, twice you can integrate it $T_m(x) = \frac{2}{r_0^2}[T_w(x)\frac{r_0^2}{2} - \frac{q_w^{"} r_0}{2K}(\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2})]$.

So, if you put the limit, it will be $T_m(x) = T_w(x) - \frac{2}{r_0^2}\frac{q_w^{"} r_0}{2K}\frac{r_0^2}{4}$ .You can write this as $T_m(x) = T_w(x) - \frac{q_w^{"} r_0}{4K}$. So, now, you can write the temperature difference $T_w(x) - T_m(x) = \frac{q_w^{"} r_0}{4K}$. So, now, you find the heat transfer coefficient. Heat transfer coefficient, you can write as $h = \frac{q_w^{"}}{T_w(x) - T_m(x)}$.

(Refer Slide Time: 40:55)



So, heat transfer coefficient, so $h = \frac{q_w^{"}}{T_w(x) - T_m(x)}$. So, in the last slide, we have found $T_w(x) - T_m(x) = \frac{q_w^{"} r_0}{4K}$. So, this h will be then, $h = \frac{q_w^{"} X\, 4K}{q_w^{"} r_0}$.

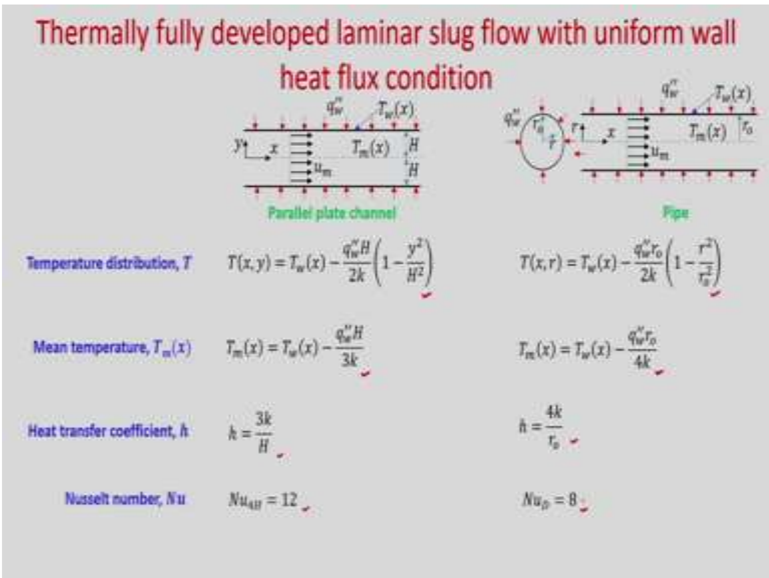
So, this you cancel, then you will get $h = \frac{4K}{r_0}$. Now, we will find the Nusselt number. Based on the hydraulic diameter and for a pipe, hydraulic diameter is just diameter of the pipe. So, it is 2 r_0. So, we will find the Nusselt number based on 2 r_0. So, Nusselt number based on diameter.

So, diameter is just 2 r_0. This is your hydraulic diameter. So, Nusselt number based on diameter, you can write $Nu_D = \frac{h(2r_0)}{K}$. So, Nusselt number based on diameter; $Nu_D = \frac{4K}{r_0}\frac{2r_0}{K}$.

So, you will find Nusselt number based on diameter and the difference between the wall temperature and mean temperature will be 8. Later, we will calculate this thermally fully developed and hydrodynamically fully developed condition with uniform wall heat flux, for this condition for pipe flow.

And Nusselt number, you will find it as, so hydrodynamically and thermally fully developed; thermally fully developed flow through pipe with uniform wall heat flux this will calculate. So, we will calculate later. So, you can calculate Nusselt number based on diameter, it will be 4.363, you will get Nusselt number as 4.363. So, obviously, you can see that as you have uniform velocity obviously, near to the wall you have more velocity. Hence, you will get more heat transfer and that you can see for Nusselt number as 8. And also, you notice that your Nusselt number is constant for this fully developed flow.

(Refer Slide Time: 44:34)



So, let me summarize. So, today, we have considered thermally fully developed laminar slug flow in two different types of channels; one is parallel plate channel and one is circular pipe. And, we considered uniform wall heat flux boundary condition. For both the cases, we calculated the temperature distribution first.

You can see, this is the temperature distribution for parallel plate channel. And for pipe, this is the temperature distribution here T is function of x and r. And in this particular case, T is function of x and y. Then, we calculated the mean temperature in terms of the wall temperature because wall temperature is also varying.

So, you can see for parallel plate channel, this is the mean temperature variation and for pipe flow, this is the mean temperature variation. Once we calculated the mean temperature, then we can calculate the difference between wall temperature and mean temperature.

And, from there we calculate the heat transfer coefficient and this is the heat transfer coefficient for parallel plate channel $\frac{3K}{H}$ and for pipe flow, it is $\frac{4K}{r_0}$. Then, we calculated the Nusselt number based on the hydraulic diameter and the temperature difference between the wall temperature and the mean temperature. And, we have found for this parallel plate channel Nusselt number based on hydraulic diameter as 12 and for pipe flow based on the diameter of the pipe as 8.

And, we have also discussed that if you consider the hydrodynamically fully developed flow then obviously, velocity will vary from u = 0 at the wall to r = 0 maximum velocity. And hence, you will get less heat transfer in fully developed flow. Here, we consider slug flow as your velocity is uniform and constant. Hence, we got higher heat transfer for this pipe flow. So, that is 8, we have shown and in the case of fully developed hydrodynamically fully developed flow it is 4.36.

Thank you.

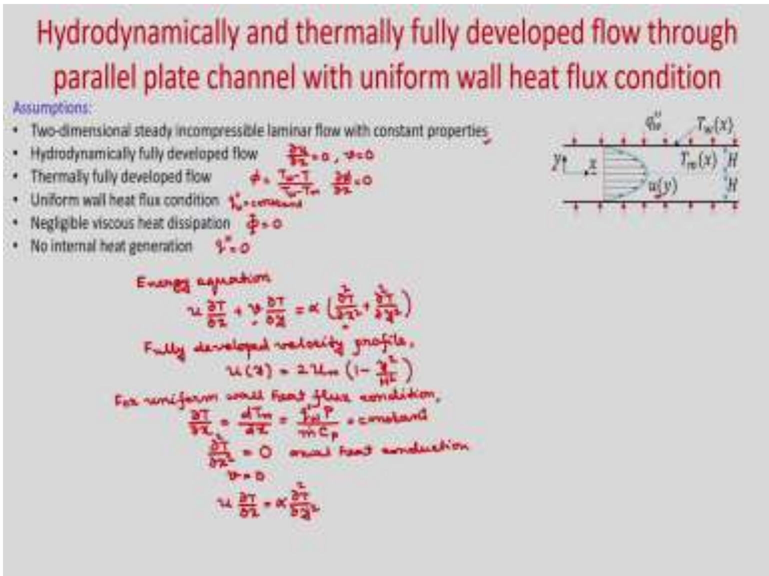
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 06
Convection in Internal Flows -II
Lecture – 19
Hydrodynamically and thermally fully developed flow with uniform wall heat flux condition

Hello everyone. So, in last lecture we considered slug flow where axial velocity remained constant. Today, we will consider Hydrodynamically and thermally fully developed flow with constant or uniform wall heat flux boundary condition.

So, it is the velocity profile we have already derived for a fully developed condition earlier for two different channels; one is parallel plate channel and circular pipe. In today's lecture, we will consider these two types of channel; first we will consider flow through parallel plate channel, then we will consider circular pipe.

(Refer Slide Time: 01:23)



So, first let us consider the assumptions, we will consider two dimensional steady incompressible laminar flow with constant properties. Two dimensional we are considering; that means in the third direction it is in finite. So, there will be no change or no gradient in that direction; so obviously, we will consider for flow through parallel plates channel x as the axial direction and y we will measure from the center line.

You can see this is the two infinite parallel plates channel. We have the velocity profile which is fully developed and u is function of y only; x is the axial direction and y is measured from the central line and uniform wall heat flux is imposed on both walls. For this particular case you can see that, your wall temperature will vary in axial direction. So, T_w is function of x, and these two plates are separated by a distance two H. So, hydro dynamically fully developed flow. So, in this particular case, you know that $\frac{\partial u}{\partial x} = 0$ and v=0.

Thermally fully developed flow, so we have this non dimensional temperature which we defined as $\phi = \frac{T_w - T}{T_w - T_m}$, where T_w is the wall temperature and T_m is the mean temperature.

So, if it is a thermally fully developed flow, then we can write $\frac{\partial \phi}{\partial x} = 0$; because actual variation of this phi will be 0. Uniform wall heat flux condition, so $q_w^{''}$ is constant, and we will have negligible viscous heat dissipation; that means phi will be 0 and no internal heat generation. So, $q^{'''} = 0$.

Considering these assumptions, let us write the energy equation. So, what is your energy equation in for steady laminar flow in two dimensions? So, you can write energy equation. So, $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2})$, neglecting the viscous heat dissipation and internal heat generation.

So, this is your energy equation, we will simplify invoking those assumptions now. First of all let us write the fully developed velocity profile. So, what is the fully developed velocity profile for this particular case? Fully developed velocity profile.

So, in this particular case, you know that $\mathrm{u(y)} = 2\,\mathrm{u}_m(1 - \frac{y^2}{H^2})$. Now, we know for constant heat flux boundary condition, $\frac{\partial T}{\partial x} = \frac{dT_m}{dx}$ and that is also constant.

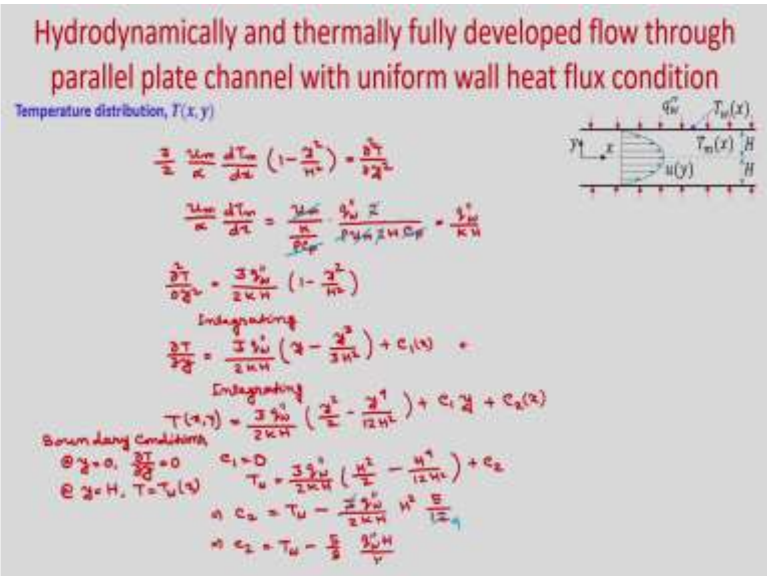
So, for uniform wall heat flux condition, you know that $\frac{\partial T}{\partial x} = \frac{dT_m}{dx}$; we have already derived it in earlier classes. So, $\frac{dT_m}{dx} = \frac{q_w^{"} P}{\dot{m} C_P}$. So, where P is the perimeter, $\dot{m}$ is the mass flow rate.

And you can see, for this particular case $q_w^{"}$ constant; for a constant cross sectional channel P is constant, $\dot{m}$ is constant and C_p is specific heat that is also constant. So, this will be constant.

Now, if you take derivative with respect to x, then you can write $\frac{\partial^2 T}{\partial x^2}$. So, as it is constant it will be 0. So, we can see that in the diffusion term, you can actually write $\frac{\partial^2 T}{\partial x^2}$ =0. So, now, you invoke all the assumptions and write the energy equation.

So, this is your axial heat conduction is 0. So, axial heat conduction, for this particular case it will be 0 and fully developed flow. So, v = 0. So, you can see that v = 0 and this is 0; so your equation now you can write as $u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}$.

(Refer Slide Time: 07:09)



.

Now, putting these values as $\frac{3}{2}\frac{u_m}{\alpha}\frac{dT_m}{dx}(1-\frac{y^2}{H^2})=\frac{\partial^2 T}{\partial y^2}$.

So, now let us find the term left hand side $\frac{u_m}{\alpha}\frac{dT_m}{dx}$. So, what is the value of this? So, u_m is the mean velocity, $\alpha=\frac{K}{\rho C_P};\frac{dT_m}{dx}=\frac{q_w^{"}P}{\dot{m}C_P}$, in this particular case P is your 2 X1 per unit width.

So, if you consider in third direction unit width, then it will be $\frac{u_m}{\frac{K}{\rho C_P}}\frac{q_w^{"}2}{\rho u_m 2HC_P}$. So, now if you simplify it, then ρ, ρ, C_p, C_p will get cancelled; u_m, u_m will get cancel; this 2, 2 will get cancel. So, you can write this as $\frac{q_w^{"}}{KH}$. So, now, you can write, $\frac{\partial^2 T}{\partial y^2}=\frac{3q_w^{"}}{2KH}(1-\frac{y^2}{H^2})$.

So, now this equation we will integrate twice and we will apply the boundary condition; then we will find the temperature distribution inside these parallel plates. So, integrating you will get, $\frac{\partial T}{\partial y}=\frac{3q_w^{"}}{2KH}(y-\frac{y^3}{3H^2})+\mathrm{C}_1(x)$. And again if you integrate, then you will get $\mathrm{T(x,y)}=\frac{3q_w^{"}}{2KH}(\frac{y^2}{2}-\frac{y^4}{12H^2})+\mathrm{C}_1 y+C_2(x)$. Now, we will apply two boundary conditions and find these constant C_1 and C_2. So, one boundary condition is that, at y = 0 which is your central line. So, your temperature gradient with respect to y, $\frac{\partial T}{\partial y}$=0, because the problem is thermally and geometrically symmetric.

So, the maximum or minimum temperature will occur at the central line. So, at y =0, we will put $\frac{\partial T}{\partial y}$=0. And another boundary condition we will take that, at the wall temperature as T_w which is function of x. So, at y = H, you have T_w

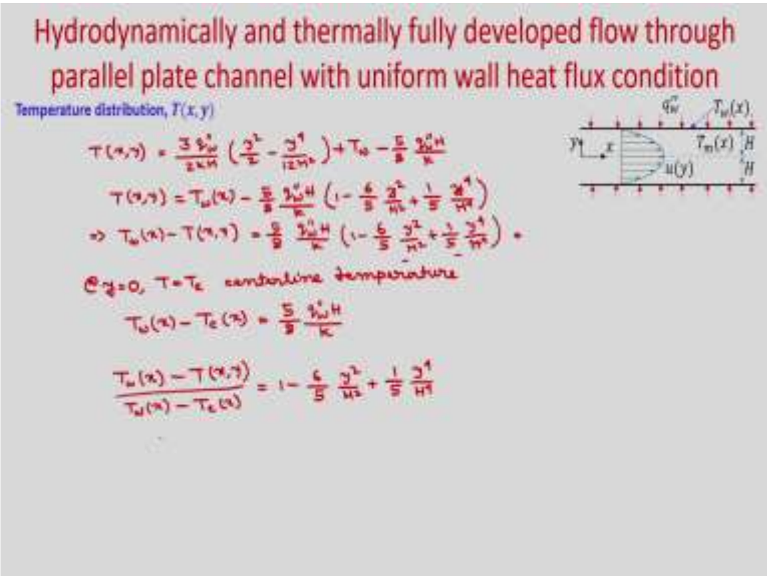
So, boundary conditions. So, at y = 0, we will put $\frac{\partial T}{\partial y}$=0. So, you can see from this equation; if you put y = 0, then this right hand side first term will become 0. So, $\frac{\partial T}{\partial y}$=0.

So, C_1 will be 0. And at y = H, we will put T as T_w which is function of x. So, if you put that, so you will get T_w; here I am not writing function of x, only Tw I am writing here. So, $T_w = \frac{3q_w^{"}}{2KH}(\frac{H^2}{2} - \frac{H^4}{12H^2}) + C_2$.

So, you can see that your, $C_2 = T_w - \frac{3q_w^{"}}{2KH} H^2 \frac{5}{12}$. So, if you rearrange it, so you will get $C_2 = T_w - \frac{5}{8}\frac{q_w^{"}H}{K}$.

.

(Refer Slide Time: 13:05)



.

So, now, let us put this constant in the temperature distribution and find the temperature distribution T(x, y). So, $T(x, y) = \frac{3q_w^{"}}{2KH}(\frac{y^2}{2} - \frac{y^4}{12H^2}) + T_w - \frac{5}{8}\frac{q_w^{"}H}{K}$.

So, if you rearrange it in this form. So, you can write, $T(x, y) = T_w(x) - \frac{5}{8}\frac{q_w^{"}H}{K}(1 - \frac{6}{5}\frac{y^2}{H^2} + \frac{1}{5}\frac{y^4}{H^4})$.

So, now I can also write this as $T_w(x) - T(x, y) = \frac{5}{8}\frac{q_w^{"}H}{K}(1 - \frac{6}{5}\frac{y^2}{H^2} + \frac{1}{5}\frac{y^4}{H^4})$. So, this is the temperature distribution along the y and also it is function of x and y; because T_w is function of x.

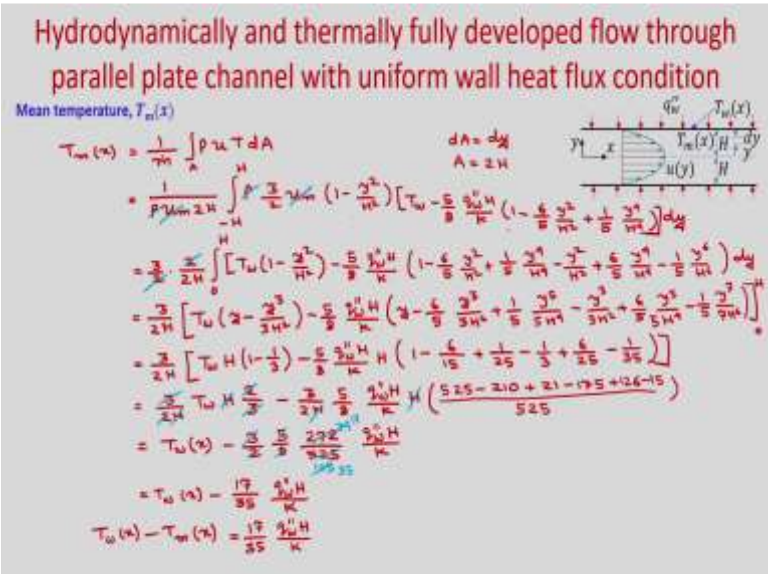
Now at the center y = 0. So, what will be the temperature, centerline temperature T_c? So, it will be either maximum or minimum temperature depending on what is the inlet temperature. So, let us write the centerline temperature as at y = 0, T = T_c which is your centerline temperature. So, you can write $T_w(x) - T_c(x) = \frac{5}{8}\frac{q_w^{"}H}{K}$.

Now, we can just divide this equation with this equation, then what you will get? $\frac{T_w(x) - T(x, y)}{T_w(x) - T_c(x)} = 1 - \frac{6}{5}\frac{y^2}{H^2} + \frac{1}{5}\frac{y^4}{H^4}$.

So, you can see that we have written this temperature distribution in non-dimensional form; because we are dividing with T_w - T_c, where T_c is the centerline temperature and right hand side you can see this is also non-dimensional. So, this is the temperature distribution.

Now, to find the heat transfer coefficient; we need to find, what is the mean temperature? Because we need to define the Nusselt number based on the difference between the wall temperature and the mean temperature. So, first let us find, what is the mean temperature?

(Refer Slide Time: 16:55)



So, mean temperature you can find $T_m(x)=\frac{1}{\dot{m}}\int_A \rho uTdA$. So, if you consider from the centreline, one small elemental flow area of distance dy at a distance y from the centreline.

And in other direction if you take a unit width, then it will be dA = dy X 1and the total flow area will be 2 H X 1. So, this if you write, then you will get $\dot{m}=\rho u_m 2H$. Now, you integrate, $T_m(x)=\frac{1}{\rho u_m 2H}\int_{-H}^{H}\frac{3}{2}u_m\left(1-\frac{y^2}{H^2}\right)\left[T_w-\frac{5}{8}\frac{q_w^{"}H}{K}\left(1-\frac{6}{5}\frac{y^2}{H^2}+\frac{1}{5}\frac{y^4}{H^4}\right)\right]dy$.

So, dA = dy. So, you can see here, you can this u_m, u_m, ρ, ρ you can cancel and you can write this,

$$T_m(x)=\frac{3}{2}\frac{2}{2H}\int_{0}^{H}\left[T_w\left(1-\frac{y^2}{H^2}\right)-\frac{5}{8}\frac{q_w^{"}H}{K}\left(1-\frac{6}{5}\frac{y^2}{H^2}+\frac{1}{5}\frac{y^4}{H^4}-\frac{y^2}{H^2}+\frac{6}{5}\frac{y^4}{H^4}-\frac{1}{5}\frac{y^6}{H^6}\right)\right]dy .$$

So, now it will be easy to integrate. So, if you cancel it 2, 2; so you can write

$$T_m(x)=\frac{3}{2H}\left[T_w\left(y-\frac{y^3}{3H^2}\right)-\frac{5}{8}\frac{q_w^{"}H}{K}\left(y-\frac{6}{5}\frac{y^3}{3H^2}+\frac{1}{5}\frac{y^5}{5H^4}-\frac{y^3}{3H^2}+\frac{6}{5}\frac{y^5}{5H^4}-\frac{1}{5}\frac{y^7}{7H^6}\right)\right]_0^H .$$

So, after integration we have written this, but we need to put the limit from 0 to H. So, if you see y = 0 if you put all terms will become 0; only H you put, then what you will get $T_m(x)=\frac{3}{2H}\left[T_wH\left(1-\frac{1}{3}\right)-\frac{5}{8}\frac{q_w^{"}H}{K}\left(1-\frac{6}{15}+\frac{1}{25}-\frac{1}{3}+\frac{6}{25}-\frac{1}{35}\right)\right]$. So, now, if you see; so you can write, $T_m(x)=\frac{3}{2H}T_wH\frac{2}{3}-\frac{3}{2H}\frac{5}{8}\frac{q_w^{"}H}{K}H\left(\frac{525-210+21-175+126-15}{525}\right)$.

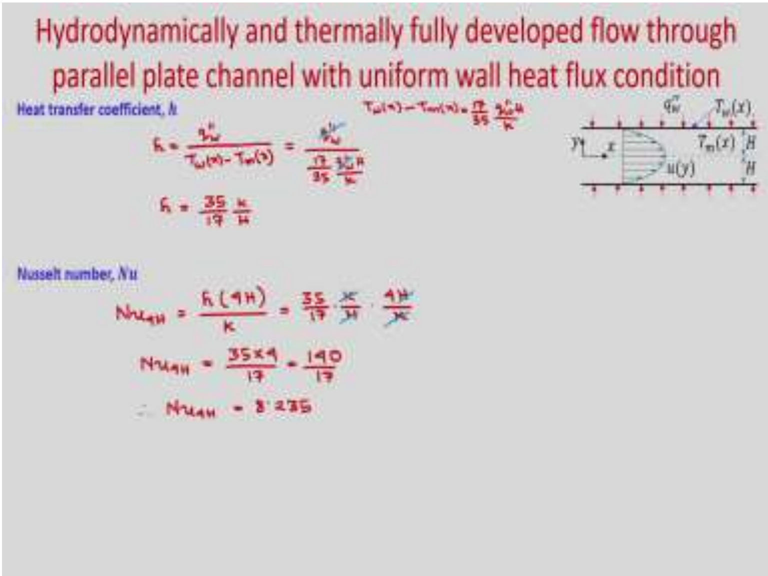
So, you can see this H, H will get cancel; 2, 2; 3, 3 here also H, H will get cancel. So, you can write it as $T_m(x)=T_w(x)-\frac{3}{2}\frac{5}{8}\frac{272}{525}\frac{q_w^{"}H}{K}$.

So, you can see, if you rearrange. So, you will get $T_m(x)=T_w(x)-\frac{17}{35}\frac{q_w^{"}H}{K}$.

So, we have now represented the mean temperature which is function of x in terms of the wall temperature and the heat flux. So, from here now you can actually find the temperature difference between the wall temperature and the mean temperature.

So, you can see, you can write $T_w(x) - T_m(x) = \frac{17}{35}\frac{q_w^{''}H}{K}$. So, now, we need to calculate the heat transfer coefficient.

(Refer Slide Time: 26:25)



So, heat transfer coefficient will be. So, heat transfer coefficient is $h = \frac{q_w^{''}}{T_w(x) - T_m(x)}$. So, in earlier slide we have found what is $T_w(x) - T_m(x)$. So, that is $T_w(x) - T_m(x) = \frac{17}{35}\frac{q_w^{''}H}{K}$.

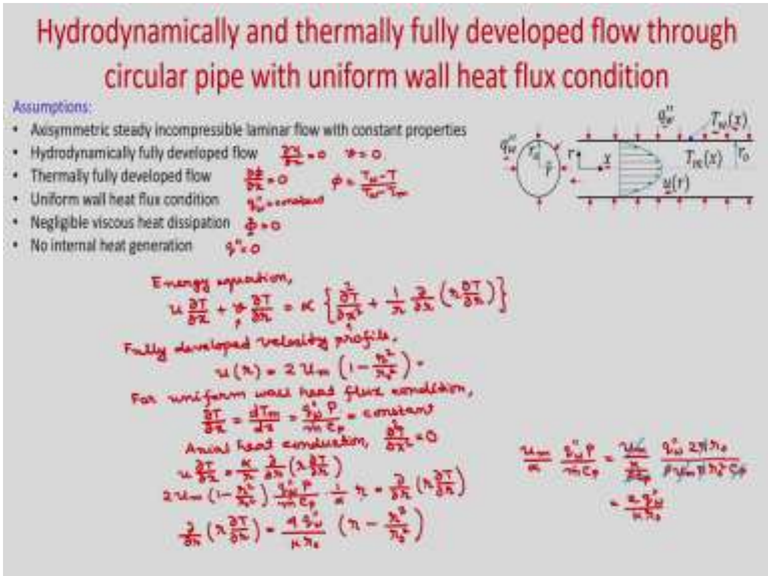
So, you can see if you put it here. So, we will get $h = \frac{q_w^{''}}{\frac{17}{35}\frac{q_w^{''}H}{K}}$.

So, you can write it as $h = \frac{35}{17}\frac{K}{H}$. So, now, Nusselt number; so Nusselt number we will calculate, Nusselt number we will calculate based on the hydraulic diameter and the difference between wall temperature and the mean temperature. So, what is the hydraulic diameter? Already you have calculated for this particular geometry, so that is your 4 H right, so 4 H. So, based on 4 H we will calculate the Nusselt number.

So, Nusselt number will be based on 4 H and the difference between wall temperature and the mean temperature. So, $Nu_{4H} = \frac{h(4H)}{K}$. So, $h = \frac{35}{17}\frac{K}{H}$ and thus $Nu_{4H} = \frac{35}{17}\frac{K}{H}\frac{4H}{K}$. So, you can see H, H will get cancel. So, you will get Nusselt number based on 4 H, $Nu_{4H} = \frac{35 \times 4}{17}$.

So, that will be $Nu_{4H} = \frac{140}{17}$. So, $Nu_{4H} = 8.235$. So, you can see the Nusselt number for hydrodynamically and thermally fully developed flow it is constant and independent of Reynolds number and Prandtl number.

(Refer Slide Time: 28:59)



.

Now, let us consider hydrodynamically and thermally fully developed flow through circular pipe with uniform wall heat flux boundary condition.

So, you can see this is your cross section of the circular pipe q_w'' is constant and r_0 is the radius of this pipe; x is the axial direction, r is measured from the centreline, and we have uniform wall heat flux on the wall and velocity profile is parabolic u(r) and already we have derived it. So, T_w in this particular case also will be varying in axial direction. So, T_w will be function of x. So, you can see the assumptions in this particular case will take axisymmetric steady incompressible laminar flow with constant properties.

So, what is axisymmetric flow? So, you can see that in if circumferential direction velocity is zero and any gradient in that direction is zero; then it is axisymmetric flow. So, you can see in this particular case we have a circular pipe. So, we have a solid wall, and velocity obviously will be zero in circumferential direction and your thermal boundary condition is also uniform over the wall. So, it is geometrically and thermally axisymmetric.

So, we can have the assumption that your temperature will vary in r and x direction. Hydrodynamically fully developed flow, so hydrodynamically fully developed flow means, your $\frac{\partial u}{\partial x} = 0$; and if v is the velocity in radial direction, then v will be also 0. Thermally fully developed flow, so $\frac{\partial \phi}{\partial x} = 0$, where Φ is the non-dimensional temperature; already we have defined as, $\phi = \frac{T_w - T}{T_w - T_m}$.

Uniform wall heat flux condition, so for uniform wall heat flux condition $q_w^{"}$ will be constant. Negligible viscosity heat dissipation, so Φ will be 0; no internal heat generation, $q^{'''}$ =0. So, in cylindrical coordinate now let us write the energy equation.

So, energy equation you can write as assuming the axisymmetric flow, $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left\{\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right\}$.

So, now let us invoke all these assumptions in this energy equation and let us simplify it. First let us write the uniform velocity profile. So, first let us write the fully developed velocity profile. So, fully developed velocity profile for circular pipe, what is that? So, u is function of r only right in this particular case. So, $u(r) = 2u_m\left(1 - \frac{r^2}{r_0^2}\right)$.

And similarly for uniform wall heat flux boundary condition, you can write $\frac{\partial T}{\partial x}$= constant. And hence $\frac{\partial^2 T}{\partial x^2}$ = 0; that means axial heat conduction will be zero in this particular case, or in this thermal boundary condition where $q_w^{"}$ = constant.

So, for constant or uniform for uniform wall heat flux boundary condition, you can write $\frac{\partial T}{\partial x} = \frac{dT_m}{dx} = \frac{q_w'' P}{\dot{m} C_P}$ which is constant; P is the constant, $\dot{m}$ is constant, C_p is constant, so that means this is equal to constant. Hence your axial heat conduction $\frac{\partial^2 T}{\partial x^2}$ = 0. So, and for hydrodynamically fully developed flow v = 0. So, you can see here v = 0 and $\frac{\partial^2 T}{\partial x^2}$ =0.

So, you can write invoking this condition the energy equation as $u \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$. So, now, $\frac{\partial T}{\partial x}$ you can put this value. So, if you put this value; so what you will get?

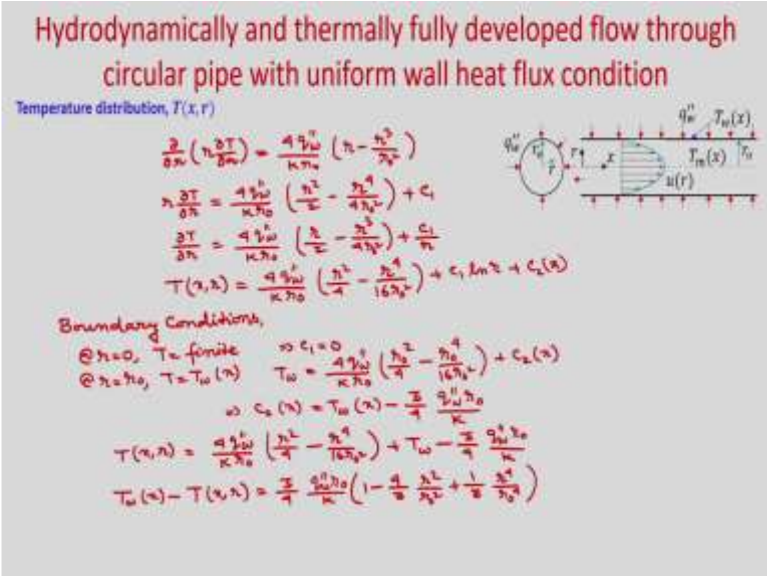
So, you can write u is the velocity profile, you put this one. So, it will be $2u_m \left(1 - \frac{r^2}{r_0^2} \right) \frac{q_w'' P}{\dot{m} C_P} \frac{1}{\alpha} r = \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$.

So, now in the left hand side let us simplify it. So, you can see, you have $\frac{u_m}{\alpha} \frac{q_w'' P}{\dot{m} C_P} = \frac{u_m}{\frac{K}{\rho C_P}} \frac{q_w'' 2\pi r_0}{\rho u_m \pi r_0^2 C_P}$.

So, you can see if you simplify it. So, this ρ C_p , ρ C_P will get cancel, u_m, π π and one r_0. So, you can write this as $\frac{2q_w''}{Kr_0}$. So, this equation now we can write. So, first let us write, $\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) = \frac{4q_w''}{Kr_0}\left(r - \frac{r^3}{r_0^2} \right)$.

So, this equation now if we integrate twice, then you will be able to find the temperature distribution. So, now, this differential equation we will integrate twice and find the temperature distribution inside the pipe.

(Refer Slide Time: 37:13)



.

So, what is your equation finally we got? So, that is your $\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{4q_w^{"}}{Kr_0}\left(r-\frac{r^3}{r_0^2}\right)$.

So, you integrate twice. So, first time if you integrate, then you will get $r\frac{\partial T}{\partial r}=\frac{4q_w^{"}}{Kr_0}\left(\frac{r^2}{2}-\frac{r^4}{4r_0^2}\right)+C_1$. And you divide by r both side, then you will get, $\frac{\partial T}{\partial r}=\frac{4q_w^{"}}{Kr_0}\left(\frac{r}{2}-\frac{r^3}{4r_0^2}\right)+\frac{C_1}{r}$. So, now, another time if you integrate; so you will get, $T(x,r)=\frac{4q_w^{"}}{Kr_0}\left(\frac{r^2}{4}-\frac{r^4}{16r_0^2}\right)+C_1\ln\mathrm{r}+\mathrm{C}_2(x)$.

So, this is the temperature distribution we got with the two integration constant C_1 and C_2. So, we need two boundary conditions; one boundary condition is that at centerline r=0 you have finite temperature, and at $r = r_0$ you have wall temperature T_w. So, boundary conditions if you put, boundary conditions. So, at r = 0 ,T is finite, right.

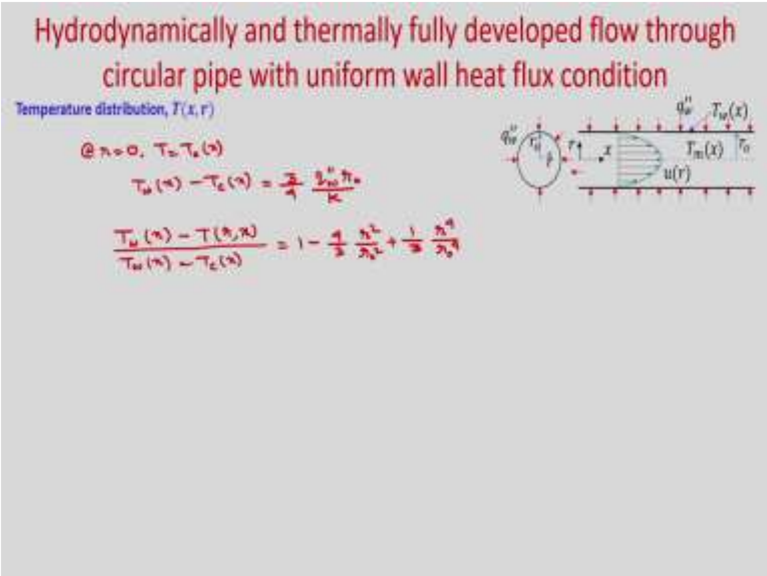
So, if you see in this equation if you put r = 0. So, to have left hand side T finite, C_1 must be 0. So, $C_1 = 0$, and at $r = r_0$, $T = T_w (x)$. If you put it that, so, $\mathrm{T}_w=\frac{4q_w^{"}}{Kr_0}\left(\frac{r_0^2}{4}-\frac{r_0^4}{16r_0^2}\right)+\mathrm{C}_2(x)$.

So, you can find, $C_2(x) = T_w(x) - \frac{3}{4}\frac{q_w^{"} r_0}{K}$.

So, final temperature distribution now we can write, $T(x,r) = \frac{4q_w^{"}}{Kr_0}\left(\frac{r^2}{4} - \frac{r^4}{16r_0^2}\right) + T_w(x) - \frac{3}{4}\frac{q_w^{"} r_0}{K}$.

So, if you rearrange it, you can write it as $T_w(x) - T(x,r) = \frac{3}{4}\frac{q_w^{"} r_0}{K}\left(1 - \frac{4}{3}\frac{r^2}{r_0^2} + \frac{4}{3}\frac{r^4}{r_0^4}\right)$. Now, we will find the centerline temperature putting the r = 0.

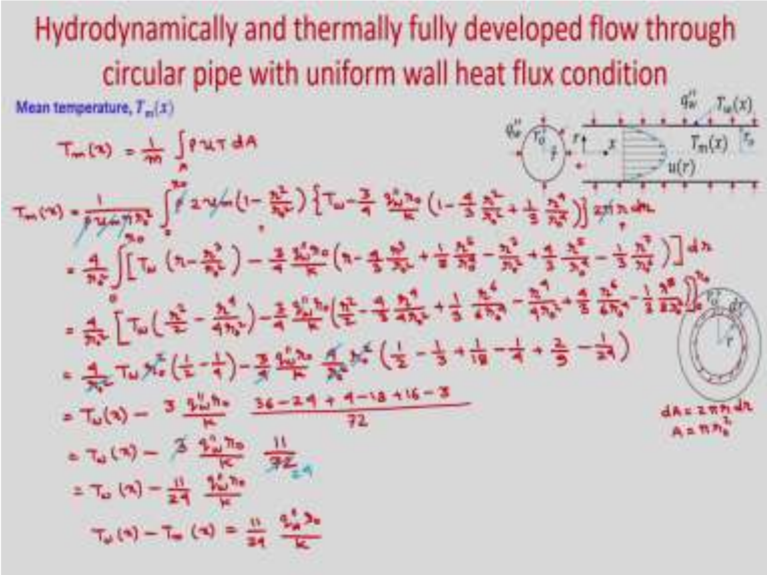
(Refer Slide Time: 41:27)



So, at r = 0, T = T_c which is function of x. So, if you write it. So, you will get $T_w(x) - T(x,r) = \frac{3}{4}\frac{q_w^{"} r_0}{K}$. So, this is your centerline temperature distribution. Now, you can also write the final temperature distribution $\frac{T_w(x) - T(x,r)}{T_w(x) - T_c(x)} = 1 - \frac{4}{3}\frac{r^2}{r_0^2} + \frac{4}{3}\frac{r^4}{r_0^4}$.

So, this is your temperature distribution. So, next we will find the mean temperature to find the Nusselt number.

(Refer Slide Time: 42:41)



So, now let us calculate the mean temperature. So, that you can calculate $T_m(x) = \frac{1}{\dot{m}} \int_A \rho u T dA$. So, if you consider this elemental flow area d A at a distance r of distance dr. So, dA will be your 2πrdr. And what will be the flow area? It will be πr_0^2.

So, if you put it here. So, you will get, you see you will get $T_m(x) = \frac{1}{\rho u_m \pi r_0^2} \int_0^{r_0} \rho 2u_m \left(1 - \frac{r^2}{r_0^2}\right) \left\{ T_w - \frac{3}{4} \frac{q_w'' r_0}{K} \left(1 - \frac{4}{3}\frac{r^2}{r_0^2} + \frac{4}{3}\frac{r^4}{r_0^4}\right) \right\} 2\pi r dr$.

So, these u_m, u_m is constant ρ is constant, π π will get cancel. So, you can write it as, $T_m(x) = \frac{4}{r_0^2} \int_0^{r_0} \left[T_w \left(r - \frac{r^3}{r_0^2} \right) - \frac{3}{4} \frac{q_w'' r_0}{K} \left(r - \frac{4}{3}\frac{r^3}{r_0^2} + \frac{1}{3}\frac{r^5}{r_0^4} - \frac{r^3}{r_0^2} + \frac{4}{3}\frac{r^5}{r_0^4} - \frac{1}{3}\frac{r^7}{r_0^6} \right) \right] dr$. So, now, you integrate it. So, it will be,

$$T_m(x) = \frac{4}{r_0^2} \left[T_w \left(\frac{r^2}{2} - \frac{r^4}{4r_0^2} \right) - \frac{3}{4} \frac{q_w'' r_0}{K} \left(\frac{r^2}{2} - \frac{4}{3}\frac{r^4}{4r_0^2} + \frac{1}{3}\frac{r^6}{6r_0^4} - \frac{r^4}{4r_0^2} + \frac{4}{3}\frac{r^6}{6r_0^4} - \frac{1}{3}\frac{r^8}{8r_0^6} \right) \right]_0^{r_0}.$$

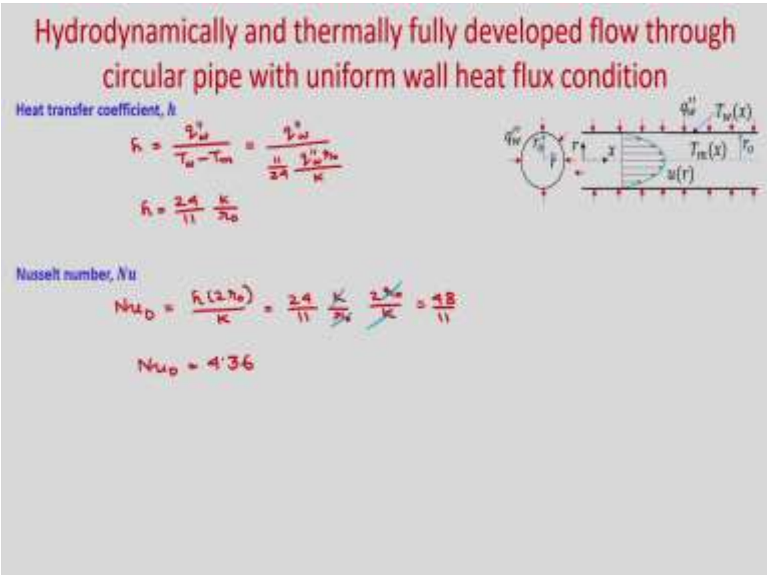
So, if we put r =0, all terms will become 0. So, you put just upper limit r = r_0 and rearrange it. So, what you will get,

$T_m(x) = \frac{4}{r_0^2} T_w r_0^2 \left(\frac{1}{2} - \frac{1}{4}\right) - \frac{3}{4}\frac{q_w^{"} r_0}{K}\frac{4}{r_0^2} r_0^2 \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{18} - \frac{1}{4} + \frac{2}{9} - \frac{1}{24}\right)$. So, you simplify it. So, these r_0^2, r_0^2 will cancel. $\mathrm{T}_m(x) = \mathrm{T}_w(x) - 3\frac{q_w^{"} r_0}{K}\frac{36-24+4-18+16-3}{72}$.

So, we will get $\mathrm{T}_m(x) = \mathrm{T}_w(x) - 3\frac{q_w^{"} r_0}{K}\frac{11}{72}$. So, you can see this will be $\mathrm{T}_m(x) = \mathrm{T}_w(x) - \frac{11}{24}\frac{q_w^{"} r_0}{K}$.

So, you can write the temperature difference $\mathrm{T}_w(x) - \mathrm{T}_m(x) = \frac{11}{24}\frac{q_w^{"} r_0}{K}$. So, now, you are in a position to calculate the heat transfer coefficient right; because you can calculate q double prime w minus the temperature difference between wall temperature and mean temperature.

(Refer Slide Time: 50:29)



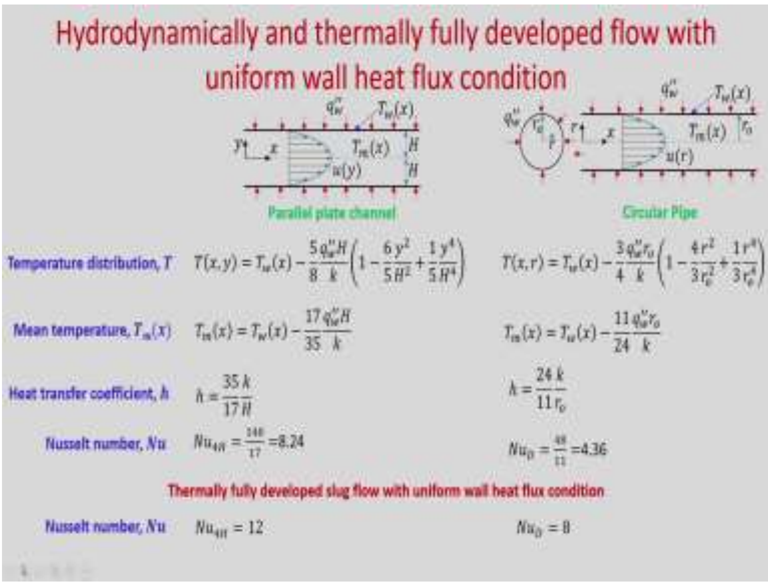
So, we will calculate the heat transfer coefficient, $h = \frac{q_w^{"}}{T_w - T_m}$. And that you can write $h = \frac{q_w^{"}}{\frac{11}{24}\frac{q_w^{"} r_0}{K}}$.

So, $h = \frac{24}{11}\frac{K}{r_0}$. Hence, now, Nusselt number you can calculate based on the hydraulic diameter and in this case hydraulic diameter is D.

So, it will be $Nu_D = \frac{h(2r_0)}{K}$. So, if you put the value of h, it will be $\frac{24}{11}\frac{K}{r_0}\frac{2r_0}{K}$. So, it will be 24×2. So, it will be $\frac{48}{11}$.

So, Nusselt number based on the diameter it is 4.36. So, you can see in this also particular case, your Nusselt number is independent of Reynolds number and Prandtl number and it is constant. So, thermally and hydrodynamically fully developed flow in a circular pipe, your Nusselt number is 4.36 and which is constant.

(Refer Slide Time: 52:05)



So, let us summarize what we have done in today's class. Today we considered two different types of duct; one is parallel plate channel and second is circular pipe.

We considered hydrodynamically and thermally fully developed flow with uniform wall heat flux boundary condition. In both the cases, first we have found the temperature distribution; you can see that we know the fully developed velocity profile put it putting it in the energy equation, we and integrating the equation we got the temperature profile.

And you can see here, the temperature distribution for parallel plates this one and for the circular pipe is this one. Now, we have represented this temperature which is function of x and r in terms of the wall temperature which is function of x.

Then we found the mean temperature right in both the cases. So, you can see for the parallel plate channel, your mean temperature is $T_m(x) = T_w(x) - \frac{17}{35}\frac{q_w^{"}H}{K}$; and whereas in circular pipe, it is $\mathrm{T}_m(x) = \mathrm{T}_w(x) - \frac{11}{24}\frac{q_w^{"}r_0}{K}$.

So, now, we know the temperature difference between wall temperature and the mean temperature. Hence you can calculate the heat transfer coefficient and you can see that for parallel plate channel, heat transfer coefficient is $h = \frac{35}{17}\frac{K}{H}$; in case of circular pipe $h = \frac{24}{11}\frac{K}{r_0}$. Then from heat transfer coefficient we calculated the Nusselt number. And we have seen in both the cases, Nusselt number is constant and independent of Reynolds number and Prandtl number.

So, when you consider thermally and hydrodynamically fully developed flow for uniform wall heat flux boundary condition, Nusselt number for parallel plate channel we calculated as 8.24 and for circular pipe we calculated as 4.36.

In last lecture we calculated the Nusselt number for thermally fully developed slug flow with uniform wall heat flux boundary condition and we found the Nusselt number for parallel plate channel as 12 and for circular pipe as 8 8. So, you can see that when you consider the fully developed velocity profile, then it is 8.24; but when you consider slug flow, where you have the everywhere you have the same velocity u m.

So, in that case you get higher Nusselt number in both the cases; that means when you have fully developed velocity profile, that means your velocity is decreasing towards the wall, so obviously you are getting less heat transfer compared to the slug flow. Because in slug flow, you have the velocity is same near to the wall as near to the centerline. So, velocity is higher, so you are getting higher heat transfer rate. So, that can be found from this expression Nusselt number 12 and Nusselt number 8 for circular pipe.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 06
Convection in Internal Flows - II
Lecture - 20
Hydrodynamically and thermally fully developed flow through parallel plate channel with uniform wall temperature

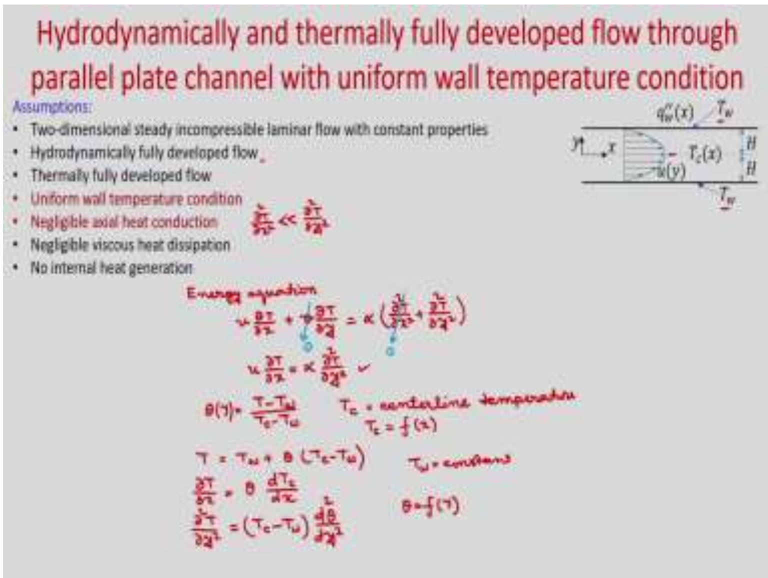
Hello everyone, today we will consider fully developed laminar flow through parallel plate channel with uniform wall temperature. So, you have seen that in earlier classes we considered different cases with uniform wall heat flux boundary condition. In that case, if you remember we have seen that temperature gradient with respect to x; $\frac{\partial T}{\partial x}$ is constant, as you have constant heat flux boundary condition.

Hence, the second derivate of temperature with respect to x, $\frac{\partial^2 T}{\partial x^2} = 0$; that means, axial heat conduction was 0, for the case with uniform wall heat flux boundary condition. But today, we are considering uniform wall temperature boundary condition, hence your axial heat conduction will not be 0.

However, in this particular case we will make a special assumption that your axial heat conduction is very very small compared to your radial heat transfer. So, we can neglect $\frac{\partial^2 T}{\partial x^2}$, as $\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2}$.

So, when we will consider uniform wall temperature boundary condition, we will neglect the axial heat conduction. So, it is the major assumption we are considering in today's class.

(Refer Slide Time: 02:11)



So, you can see in this figure; so, we have parallel plate channel and we have uniform wall temperature boundary condition, x is the axial direction, y is measured from the centre of the channel and these two parallel plates are separated by a distance 2H.

So, we are considering thermally fully developed and hydrodynamically fully developed condition. So, the assumptions are two-dimensional steady incompressible laminar flow; with constant properties, hydrodynamically fully developed flow; so that means, b will be 0 and thermally fully developed flow, uniform wall temperature condition we are considering.

And this is the important assumptions we are making that negligible axial heat conduction; that means, your $\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2}$; so, you can neglect the axial heat conduction. And also we are assuming that negligible discuss it dissipation and no internal heat generation.

So, now with these assumptions we will start with the energy equation and we will invoke these assumptions and make it simplified. So, energy equation is $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$; so this is your energy equation.

Now, we will invoke the assumptions like fully developed condition. So, v = 0 and your axial heat conduction is very very small; so this is also 0. So, you can write the simplified energy equation as, $u\frac{\partial T}{\partial x}=\alpha\frac{\partial^2 T}{\partial y^2}$.

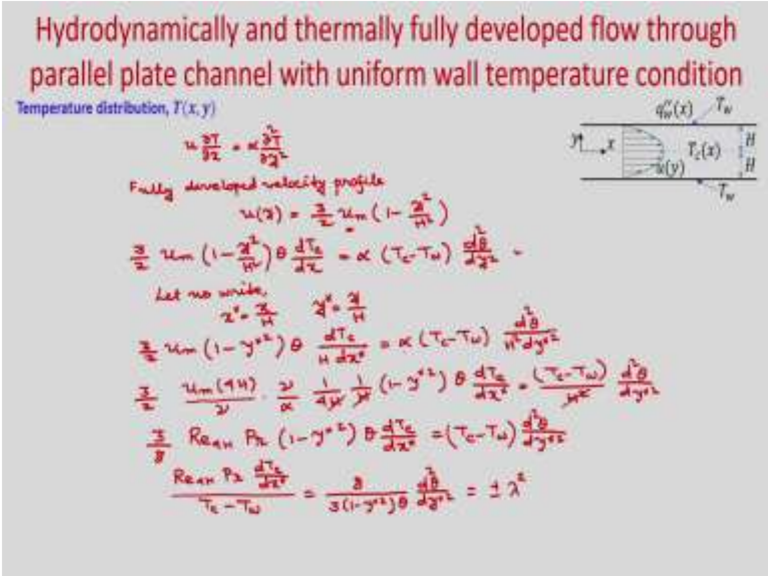
So, now we will consider one non dimensional temperature and we will define with respect to the difference between the temperature of centre line and the wall temperature. So, we will consider one non dimensional temperature $\theta(y)=\frac{T-T_w}{T_c-T_w}$; So, T is your center line temperature here, so T_c obviously, is function of x.

But, if you see the $\frac{T-T_w}{T_c-T_w}$ is no longer function of x, it is only function of y. Like, we defined the non dimensional temperature π with respect to the mean temperature; so similarly here we are defining another non dimensional temperature θ with respect to T_c-T_w.

So, here θ will be function of y only, it will not vary along the axial direction. We are taking this non dimensional temperature so that our calculation will be easier and it will be easy to calculate the Nusselt number. So, now from here you can see that $T=T_w+\theta\left(T_c-T_w\right)$. So, $\frac{\partial T}{\partial x}=\theta\frac{dT_c}{dx}$.

So, here T_W is constant right and hence you can also take the derivative of T with respect to y; $\frac{\partial^2 T}{\partial y^2}$, so you can write as; so now T_c is function of x only and θ is function of y only. So, you can write $\frac{\partial^2 T}{\partial y^2}=\left(T_c-T_w\right)\frac{d^2\theta}{dy^2}$ because θ is function of y and we are taking the second derivative of T with respect to y; so it is $\frac{\partial^2 T}{\partial y^2}=\left(T_c-T_w\right)\frac{d^2\theta}{dy^2}$. So, now you put these values in the energy equation.

(Refer Slide Time: 07:18)



So, what you will get? Our energy equation is $u\dfrac{\partial T}{\partial x} = \alpha \dfrac{\partial^2 T}{\partial y^2}$. Now, if you put these values; you will get and what is your fully developed velocity profile? Fully developed velocity profile for flow through parallel plate channel, it is $u(y) = \dfrac{3}{2}u_m\left(1 - \dfrac{y^2}{H^2}\right)$.

For this configuration this is the mean velocity, fully developed velocity profile. So, for this configuration; this is the fully developed velocity profile and u_m is the mean velocity. So, if you put all these things in this energy equation, you will get $\dfrac{3}{2}u_m\left(1 - \dfrac{y^2}{H^2}\right)\theta \dfrac{dT_c}{dx} = \alpha\left(T_c - T_w\right)\dfrac{d^2\theta}{dy^2}$.

So, now we will also non dimensionalize the x coordinate and y coordinates. So, let us write $x* = \dfrac{x}{H}$ and $y* = \dfrac{y}{H}$. So, if you put it in this equation, so what you are going to get? $\dfrac{3}{2}u_m\left(1 - y*^2\right)\theta \dfrac{dT_c}{Hdx*} = \alpha\left(T_c - T_w\right)\dfrac{d^2\theta}{Hdy*^2}$.

So, if you rearrange it; so you can write as $\dfrac{3}{2}\dfrac{u_m(4H)}{\nu}\dfrac{\nu}{\alpha}\dfrac{1}{4H}\dfrac{1}{H}\left(1 - y*^2\right)\theta \dfrac{dT_c}{dx*} = \dfrac{\left(T_c - T_w\right)}{H^2}\dfrac{d^2\theta}{dy*^2}$.

So, now you see; so you can write $\frac{3}{8}\text{Re}_{4H}\Pr\left(1-y*^2\right)\theta\frac{dT_c}{dx*}=\left(T_c-T_w\right)\frac{d^2\theta}{dy*^2}$.

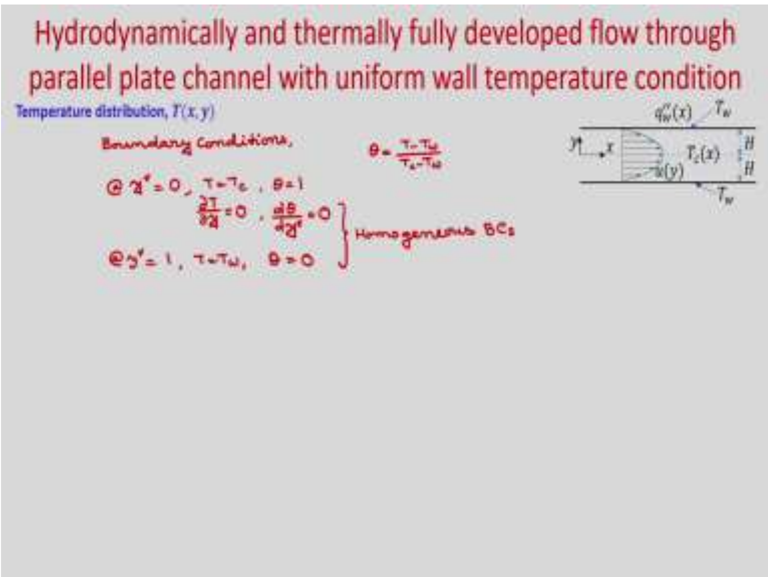
So, now we will separate the variables; so we will put the, which are function of x in the left hand side and which terms are function of y, we will take in the right hand side. So, if you do that then we can write as $\frac{\text{Re}_{4H}\Pr\frac{dT_c}{dx*}}{T_c-T_w}=\frac{8}{3\left(1-y*^2\right)\theta}\frac{d^2\theta}{dy*^2}$.

Now, you see the left hand side; so left hand side your T_c is function of x only, T_w is constant, Reynolds number is constant, Prandtl number is constant and also $\frac{dT_c}{dx*}$ is function of x only.

So, in the left hand side is function of x only; now if you consider the right hand side, so if you see θ ; θ is function of y only and also $\frac{d^2\theta}{dy*^2}$ is function of y only, so right hand side is function of y only. So, we have separated the variables; left hand side is function of x only, right hand side function of y only; so this should be equal to some constant.

So, now what constant we will take, $\pm\lambda^2$. Now, whether it will take plus or minus? So, that we can see that we will use the rules of separation of variables method, before that let us write the boundary conditions.

(Refer Slide Time: 13:12)



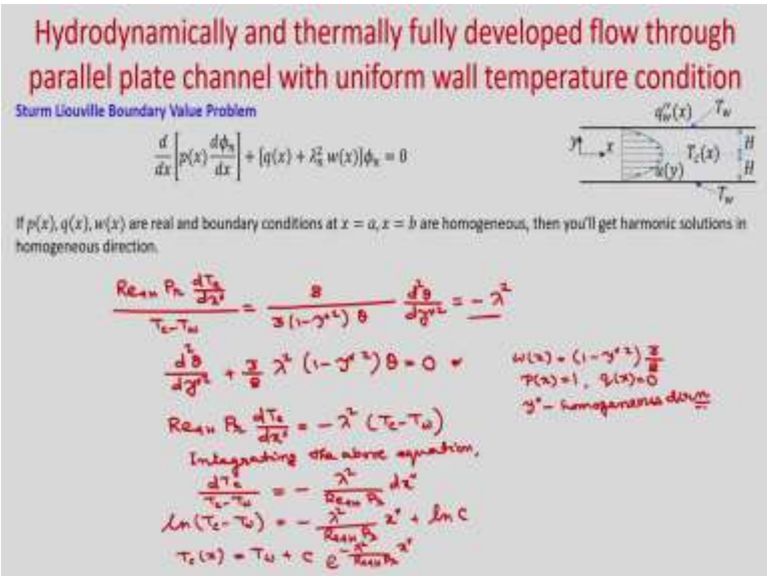
So, what are the boundary conditions? Boundary conditions are; you see, at y* = 0; that means, y is measured from the centreline; so y* = 0 means at the centerline. So, θ we have defined as $\theta(y) = \dfrac{T - T_w}{T_c - T_w}$. So, at y* = 0; obviously, θ = 1 because T = T_c . So, let us write y* = 0, T = T_c; hence your θ = 1.

Again, you see that this problem is geometrically symmetric; as well as thermally symmetric because both the walls are maintained at constant wall temperature T_w and from the centerline both the plates are separated at a distance H; so, it is geometrically symmetric and thermally symmetric. So, maximum or minimum temperature will occur at the centerline; that means you can write $\dfrac{d\theta}{dy^*} = 0$.

So, in this case; you can write $\dfrac{\partial T}{\partial y} = 0$ at y* = 0 or you can write in terms of θ ; $\dfrac{d\theta}{dy^*} = 0$. And at wall which is y* = 1; so obviously, T = T_W; so θ = 0. Now, you see the boundary conditions in y direction.

So, at y* = 0, at the centreline; your $\dfrac{d\theta}{dy^*} = 0$ and at y* = 1, on the wall your θ = 0; that means, you have homogeneous boundary conditions. So, both are homogeneous boundary conditions; so; that means, y is your homogeneous direction.

(Refer Slide Time: 15:33)



So, now let us consider this Sturm Liouville boundary value problem. So, what it is; Sturm Liouville boundary value problem? So, this is your second order differential equation.

So, in this case if p(x), q(x) and w(x) are real and boundary conditions at x =a and x = b are homogeneous, then you will get harmonic solutions in homogeneous directions. And in that direction, so you have to choose the value of λ square such a way that in the homogeneous direction, you get a harmonic solutions.

So, now let us look back the equations; so our equation, if you see we have written $\frac{\mathrm{Re}_{4H}\Pr\frac{dT_c}{dx^*}}{T_c-T_w}=\frac{8}{3\left(1-y^{*2}\right)\theta}\frac{d^2\theta}{dy^{*2}}$; now, whether we will choose $\pm\lambda^2$?

So, you have seen that the boundary conditions in the y direction are homogeneous; so that is the homogeneous direction and we have to get the λ square such a way that in y direction, we get a Sturm Liouville problem.

So that means, if you choose this equal to - λ^2; then what equation you will get? You will get, $\frac{d^2\theta}{dy^{*2}}+\frac{3}{8}\lambda^2\left(1-y^{*2}\right)\theta=0$. And another equation, you will get, $\mathrm{Re}_{4H}\Pr\frac{dT_c}{dx^*}=-\lambda^2\left(T_c-T_w\right)$.

Now, you see the; this equation, so now you can see. So, if we compare with this Sturm Liouville boundary value problem, you can see you have the waiting function, $w(x)=\left(1-y^{*2}\right)\frac{3}{8}$.

And p(x)=1 and q(x) = 0 and y star is homogeneous direction. So, as y* is homogeneous direction and p, q, w are real; so you can have a periodic solution or harmonic solutions in y star direction. So, we have chosen minus λ^2 to get the Sturm Liouville boundary value problem in y* direction .

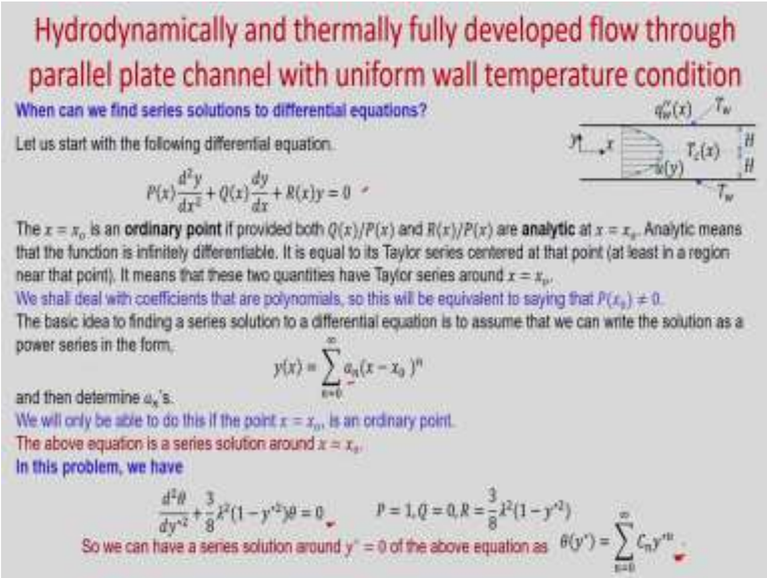
So, now for this equation if you see; so integrate, so what you will get? So, you can see; so you can write $\frac{dT_c}{T_c-T_w}=-\frac{\lambda^2}{\mathrm{Re}_{4H}\Pr}dx^*$.

So, if you see, so you will get $\ln\left(T_c - T_w\right) = -\frac{\lambda^2}{\mathrm{Re}_{4H}\,\mathrm{Pr}} x^* + \ln\mathrm{C}$ and you can write,

$\mathrm{T}_c(x) = T_w + Ce^{-\frac{\lambda^2}{\mathrm{Re}_{4H}\,\mathrm{Pr}} x^*}$.

So, now we have to see the first equation; so now, we have to find the solution of this equation. Now, we want to find a series solution of this equation, now whether we can use the series solution for this particular second order differential equation, let us see.

(Refer Slide Time: 20:39)



So, when can we find series solution to differential equations? So, you let us consider this differential equation, second order differential equation; $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$.

Now, we are just checking or when we can use the series solution, let us see. The x = x_0 is an ordinary point, if provided both $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ are analytic at x = 0. Analytic means that the function is infinitely differentiable, it is equal to its Taylor series centred at that point; at least in a region near that point. It means that these two quantities have Taylor series around x =x_0.

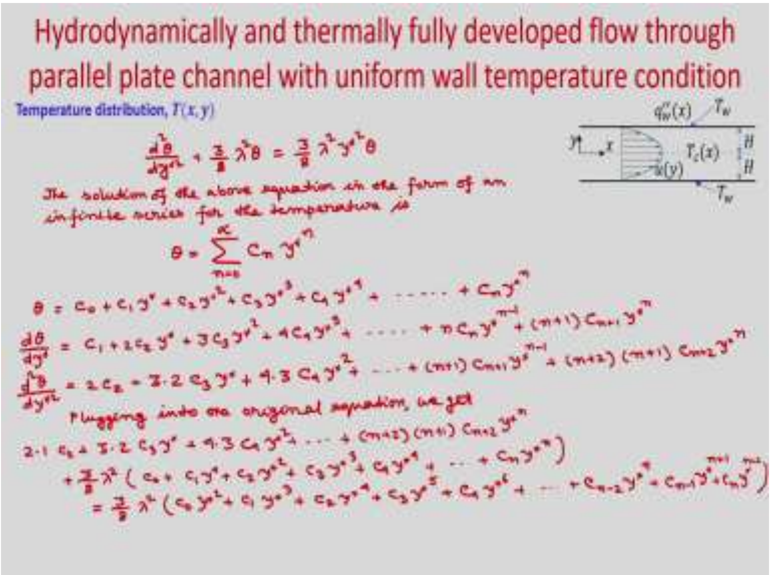
We shall deal with coefficients that are polynomials; so this will be equivalent to saying that $P(x_0) \neq 0$. So, the basic idea to finding a series solution to a differential equation is to assume that we can write the solution as a power series in the form; $y(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$.

We will only be able to do this, if the point x = x_0 is an ordinary point. So, the above equation is a series solution at x = x_0. So, now for our problem let us see; so if you compare our differential equation is this one. So, if you compare with this equation, you see P = 1, Q = 0 and $R=\frac{3}{8}\lambda^2\left(1-y^{*2}\right)$.

So, now you can see that $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ are analytic. So, at x = 0 is an ordinary point; And we can have a series solution around y*=0 of the above equation, as $\theta(y^*)=\sum_{n=0}^{\infty} C_n y^{*n}$.

As y*=0 is an analytic point because you have seen that $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ are analytic. So, we can find the solution of this equation as a series solution and this series solution, we will consider.

(Refer Slide Time: 23:21)



So, our differential equation is $\frac{d^2\theta}{dy*^2}+\frac{3}{8}\lambda^2\theta=\frac{3}{8}\lambda^2 y*^2\,\theta$. So, now the solution of the above equation in the form of an infinite series for the temperature is, $\theta=\sum_{n=0}^{\infty}C_n y*^n$.

So, now let us take the derivative of θ; so θ if we expand,

$$\theta=C_0+C_1y*+C_2\,y*^2+C_3\,y*^3+C_4\,y*^4+....................+C_n\,y*^n\,.$$

Now, if you take the derivative; if you take the derivative with respect to y*, then $\frac{d\theta}{dy*}=C_1+2C_2\,y*+3C_3\,y*^2+4C_4\,y*^3+....................+nC_n\,y*^{n-1}+(n+1)C_{n+1}\,y*^n$.

The next term if you write, so it will be

$$\frac{d^2\theta}{dy*^2}=2C_2+3.2C_3\,y*+4.3C_4\,y*^2+.............+(n+1)C_{n+1}\,y*^{n-1}+(n+2)(n+1)C_{n+2}\,y*^n\,.$$

Now, you plug into the original differential equation. So, if you do that; plugging into the original equation ok, we get so, it is

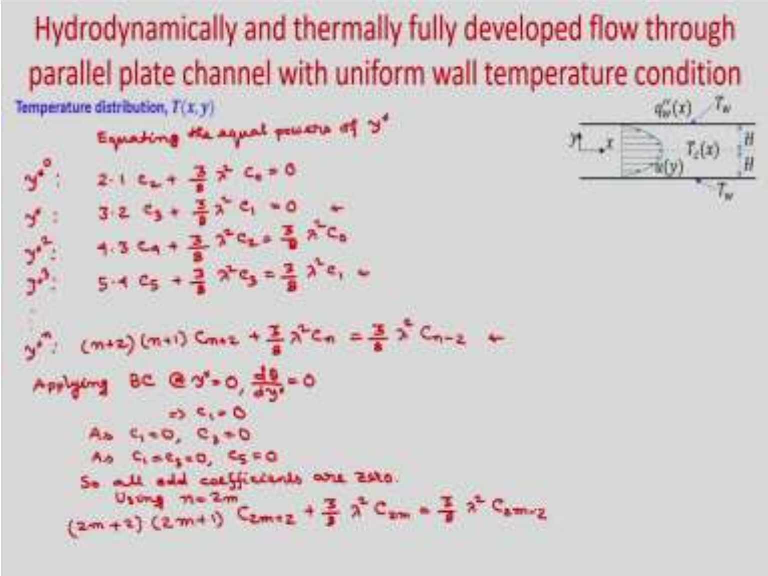
$$2.1C_2+3.2C_3\,y*+4.3C_4\,y*^2+.......+(n+2)(n+1)C_{n+2}\,y*^n+\frac{3}{8}\lambda^2\left(C_0+C_1y*+C_2\,y*^2+C_3\,y*^3+C_4\,y*^4+.....+C_n\,y*^n\right)$$

So, this is your $\frac{d^2\theta}{dy*^2}$. And in the right hand side now,

$$\frac{3}{8}\lambda^2\left(C_0y*^2+C_1y*^3+C_2\,y*^4+C_3\,y*^5+C_4\,y*^6+...........+C_{n-2}\,y*^n+C_{n-1}\,y*^{n+1}+C_n\,y*^{n+2}\right)$$

so, this is the right hand side. Now, what we will do? We will equate the equal power of y*.

(Refer Slide Time: 29:01)



So, equating the equal powers of y*; so if you see $y*^0$, if you see its coefficient; so you can see here, so this is your $y*^0 : 2.1\mathrm{C}_2 + \frac{3}{8}\lambda^2 C_0$ and then this one and right hand side, there is no term; so, it is equal to 0. So, you can write $2.1\mathrm{C}_2 + \frac{3}{8}\lambda^2 C_0 = 0$.

so if you see $y* : 3.2\mathrm{C}_3 + \frac{3}{8}\lambda^2 C_1 = 0$;

Then, $y*^2 : 4.3\mathrm{C}_4 + \frac{3}{8}\lambda^2 C_2$. Then, right hand side $\frac{3}{8}\lambda^2 C_0$.

So, if you write the $y*^2 : 4.3\mathrm{C}_4 + \frac{3}{8}\lambda^2 C_2 = \frac{3}{8}\lambda^2 C_0$. So, similarly you find the other powers ok the coefficient of other powers and equate it.

So, just I am writing here; $y*^3 : 5.4\mathrm{C}_5 + \frac{3}{8}\lambda^2 C_3 = \frac{3}{8}\lambda^2 C_1$. And if you see $y*^n : (n+2)(n+1)\mathrm{C}_{n+2} + \frac{3}{8}\lambda^2 C_n = \frac{3}{8}\lambda^2 C_{n-2}$.

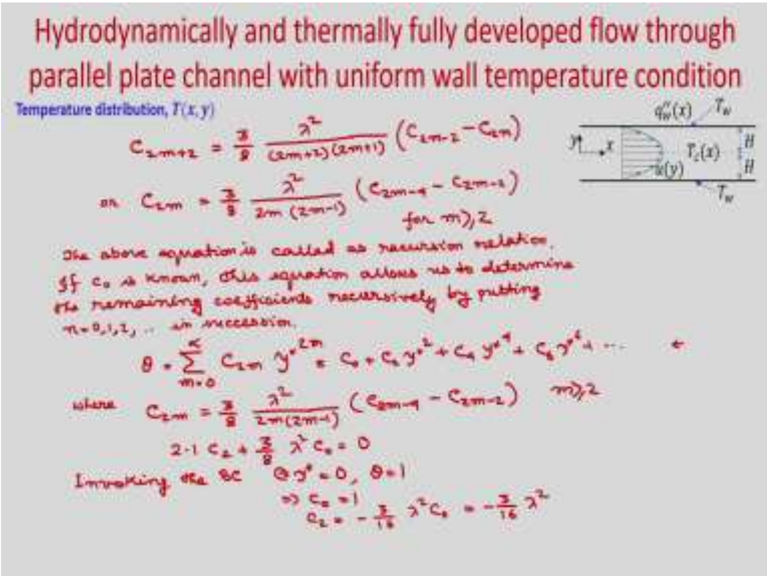
So, now let us apply the boundary conditions. So, applying boundary condition at y* = 0; $\frac{d\theta}{dy*} = 0$. So, if $\frac{d\theta}{dy*} = 0$; you see from this equation, this equation you see; so, if at y* =

0, $\frac{d\theta}{dy*}=0$; that means, C_1 will be 0 because rest all other terms will become 0; so C_1 will be 0.

So, that means, C_1 is 0; now, you see here if C_1 is 0, then from this equation; you see from this equation, if C_1 is 0, C_3 will become 0. Then, if C_1, C_3 is 0; from here you can see C_1, C_3 are 0; so C_5 will be 0. So, you can see all the odd coefficient C will be 0 so; that means, you can write as C_1 = 0, C_3 = 0. As C_1=C_3 =0; C_5 =0; so all odd coefficients are 0.

So, what we can write, we can write using n = 2 m, from here you can see; from this equation you can write $(2m+2)(2m+1)\mathrm{C}_{2m+2}+\frac{3}{8}\lambda^2 C_{2m}=\frac{3}{8}\lambda^2 C_{2m-2}$.

(Refer Slide Time: 34:02)



If you divide it; so you can write $\mathrm{C}_{2m+2}=\frac{3}{8}\frac{\lambda^2}{(2m+2)(2m+1)}(C_{2m-2}-C_{2m})$ or we can write $\mathrm{C}_{2m}=\frac{3}{8}\frac{\lambda^2}{2m(2m-1)}(C_{2m-4}-C_{2m-2})$.

So, we can see this equation is the recursion relation; so it is for m ≥2. So, the above equation is called as recursion relation, if C_0 is known this equation allows us to determine the remaining coefficient recursively by putting n = 0, 1, 2,…..in succession.

So, now let us write the temperature distribution; so your temperature distribution θ , now you can write because you know that odd coefficients are 0, so, $\theta = \sum_{m=0}^{\infty} C_{2m} y^{*2m} = C_0 + C_2 y^{*2} + C_4 y^{*4} + C_6 y^{*6} + ...$.

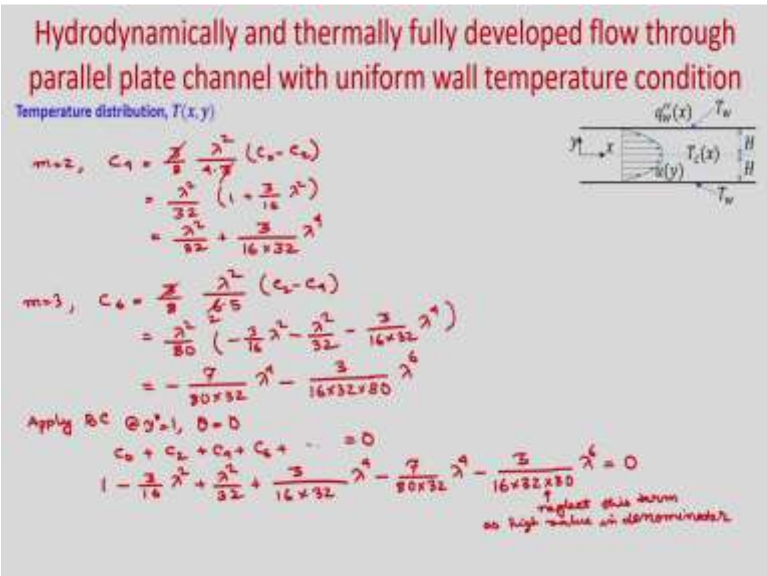
So that means, you can write $C_{2m} = \frac{3}{8}\frac{\lambda^2}{2m(2m-1)}\left(C_{2m-4} - C_{2m-2}\right)$, for m ≥2. And the first equation if you see here, so this is also valid; $2.1C_2 + \frac{3}{8}\lambda^2 C_0 = 0$; so this also we have. So, we also have $2.1C_2 + \frac{3}{8}\lambda^2 C_0 = 0$.

So, if you see; if C_0 is known, C_2 will be known and if C_2 is known then other terms will be known because m; for m ≥, you can use this recursive relation. So, now invoke the other boundary condition; at y*= 0, you have θ = 1.

If y*= 0, θ = 1; so, you can see from this equation; $\theta = C_0 + C_2 y^{*2}$. So, these are; all will become 0, at y*= 0 and θ = 1; so C_0 = 1.

So, if C_0 = 1; then you can write the value of C_2, from this equation. $C_2 = -\frac{3}{16}\lambda^2 C_0$, C_0 = 1; so it will be $C_2 = -\frac{3}{16}\lambda^2$. Now, C_0, C_2 are known. Now, you put this recursive relation and find C_4 and C_6.

(Refer Slide Time: 38:49)



So, you can write C_4, so if you put m = 2; $C_2 = \frac{3}{8}\frac{\lambda^2}{4.3}\left(C_0 - C_2\right)$; C_2 and C_0 you know, so you can write λ^2; so these 3, 3; you can cancel. So, it is $C_4 = \frac{\lambda^2}{32}\left(1 + \frac{3}{16}\lambda^2\right)$.

So, it will be $C_4 = \frac{\lambda^2}{32} + \frac{3}{16\times 32}\lambda^4$. Now, you put m = 3; so, you can find $C_6 = \frac{3}{8}\frac{\lambda^2}{6.5}\left(C_2 - C_4\right)$.

Now, C_4 already we have found here; so, it will be 2; so it will be $\frac{3}{16\times 32}\lambda^4$.

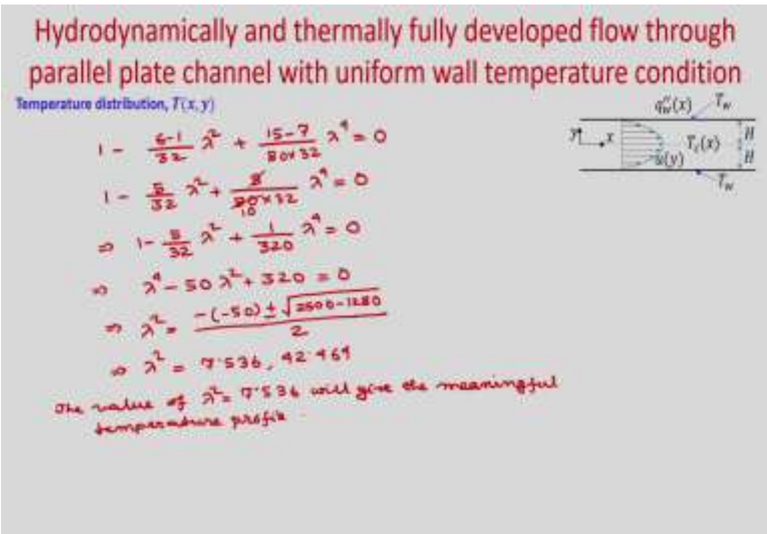
So, if you rearrange it, you will get $C_6 = -\frac{7}{80\times 32}\lambda^4 - \frac{3}{16\times 32\times 80}\lambda^6$. So, now let us apply the another boundary condition at y* = 1, θ =0. Apply boundary condition at y*=1, θ = 0.

So, if you see θ expression; so, θ expression is this and y*=1 if you put, this will become all 1. So, it is just $C_0 + C_2 + C_4 + C_6 + ... = 0$.

Now, we will consider only the first three coefficients, other terms we will neglect. So, you can write, $1 - \frac{3}{16}\lambda^2 + \frac{\lambda^2}{32} + \frac{3}{16\times 32}\lambda^4 - \frac{7}{80\times 32}\lambda^4 - \frac{3}{16\times 32\times 80}\lambda^6 = 0$.

So, you can see the denominator value $16\times 32\times 80$. So, it is a very high value; so if you find, so this will become very small value; so, you neglect this term as you have high value in denominator.

(Refer Slide Time: 42:55)

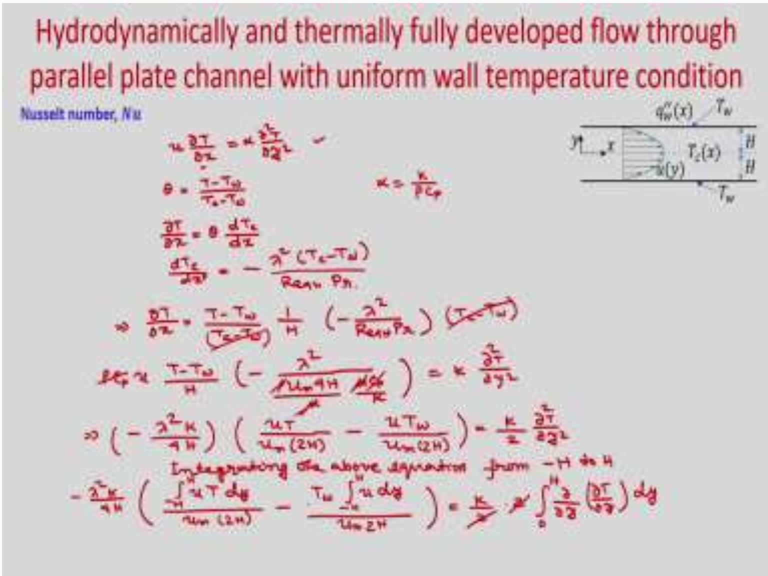


So, if you neglect it; so rest you just rearrange. So, what you will get? $1-\frac{6-1}{32}\lambda^2+\frac{15-7}{80\times 32}\lambda^4=0$. So, $1-\frac{5}{32}\lambda^2+\frac{8}{80\times 32}\lambda^4=0$. So, it will be just, $1-\frac{5}{32}\lambda^2+\frac{1}{320}\lambda^4=0$.

So, you will get a quadratic equation; $\lambda^4-50\lambda^2+320=0$. So, that means $\lambda^2=\frac{-(-50)\pm\sqrt{2500-1280}}{2}$. So that means, we will have $\lambda^2=7.536$; one value and another value will be 42.464.

So, you will see that if you take $\lambda^2=7.536$, it will give physically correct temperature profile, but if you consider $\lambda^2=42.464$, then it will not give a physically correct temperature value. So, you can neglect that; so you just consider the value of $\lambda^2=7.536$. So, the value of $\lambda^2=7.536$ will give the meaningful temperature profile.

(Refer Slide Time: 45:14)



So, now we know the value of λ^2; so if we put in the equation, in the series solution; then you will get the temperature profile. So, in the series solution, if you see that you have found the value of C_0; then C_2, C_4 and C_6; so if you put this, so you will get the temperature profile.

Now, next target is to find the Nusselt number; so we will start from the governing equations. So, it is $u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}$ and if you remember, θ we have defined as $\theta = \frac{T - T_w}{T_c - T_w}$. And we have seen that $\frac{\partial T}{\partial x} = \theta \frac{dT_c}{dx}$ and $\frac{dT_c}{dx*} = -\frac{\lambda^2 \left(T_c - T_w\right)}{\mathrm{Re}_{4H} \mathrm{Pr}}$.

Hence, $\frac{\partial T}{\partial x} = \frac{T - T_w}{T_c - T_w}\frac{1}{H}\left(-\frac{\lambda^2}{\mathrm{Re}_{4H} \mathrm{Pr}}\right)\left(T_c - T_w\right)$, so this is your $\frac{\partial T}{\partial x}$. So, if we put this value in the governing equation and $\alpha = \frac{K}{\rho C_P}$.

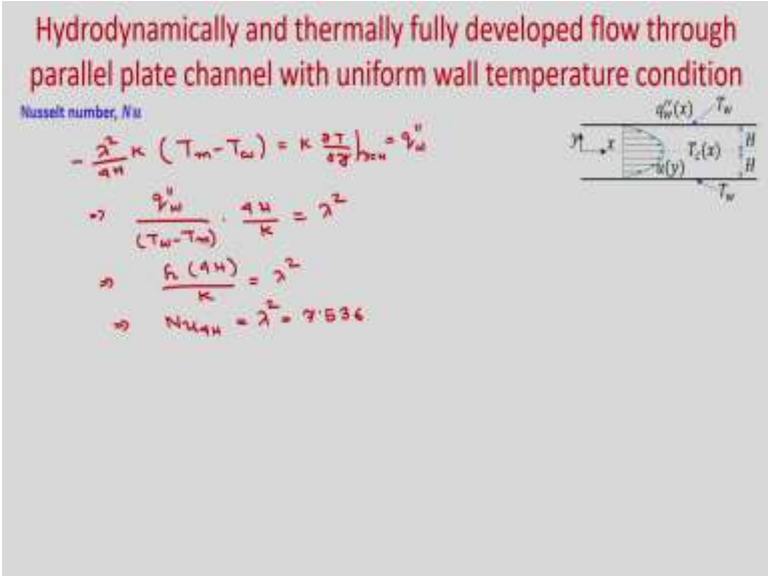
So, now, $\frac{\partial T}{\partial x}$; you put it here and $\alpha = \frac{K}{\rho C_P}$. So, you can see; it will be,

$$\rho C_P u \frac{T - T_w}{H}\left(-\frac{\lambda^2}{\frac{\rho C_P u_m}{\mu}\frac{\mu C_P}{K}}\right) = K\frac{\partial^2 T}{\partial y^2}.$$

So, if you see; so ρ C_p, ρ C_p; you can cancel, then μ, μ will get cancel and now if you write, it will be minus $\left(-\frac{\lambda^2 K}{4H}\right)\left(\frac{uT}{u_m(2H)} - \frac{uT_w}{u_m(2H)}\right) = \frac{K}{2}\frac{\partial^2 T}{\partial y^2}$.

So, now, integrating the above equation from -H to H. So, you see minus $-\frac{\lambda^2 K}{4H}\left(\frac{\int_{-H}^{H} uTdy}{u_m(2H)} - \frac{T_w \int_{-H}^{H} udy}{u_m(2H)}\right) = \frac{K}{2}2\int_0^H \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)dy$. So, this right hand side time you see the integration. So, this $\frac{\partial T}{\partial y}$y at H; at H, so that will give you the wall heat flux and $\frac{\partial T}{\partial y}$; at y = 0, it is centerline temperature gradient and that is equal to 0 because from boundary condition, you have seen.

(Refer Slide Time: 49:53)



So, you can write $-\dfrac{\lambda^2 K}{4H}$. Now, you see this term $\dfrac{\int_{-H}^{H} uTdy}{u_m(2H)}$, so that means, it is your mean temperature right, mean temperature definition; so this is the mean temperature definition. And if you see, this is $\dfrac{\int_{-H}^{H} udy}{u_m(2H)}$, so obviously, this will get cancel; so you will get $T_m - T_w$.

So, you will get $T_m - T_w$; so we have written the definition of mean; bulk mean temperature and because we need to define a Nusselt number based on $T_m - T_w$. So, it will be $K\dfrac{\partial T}{\partial y}|_{y=H}$ and it is nothing, but $K\dfrac{\partial T}{\partial y}|_{y=H} = q_w^{"}$.

So, now you can see $\dfrac{q_w^{"}}{T_m - T_w}\dfrac{4H}{K} = \lambda^2$. So, you can see it is $\dfrac{h(4H)}{K} = \lambda^2$. So that means, so Nusselt number based on hydraulic diameter 4 H $Nu_{4H} = \lambda^2$ and λ^2 value, we have found as 7.536.

So, we have seen that the Nusselt number based on the hydraulic diameter, for this problem the hydraulic diameter is $Nu_{4H} = \lambda^2$ and λ^2; we have already found; so that is your 7.536.

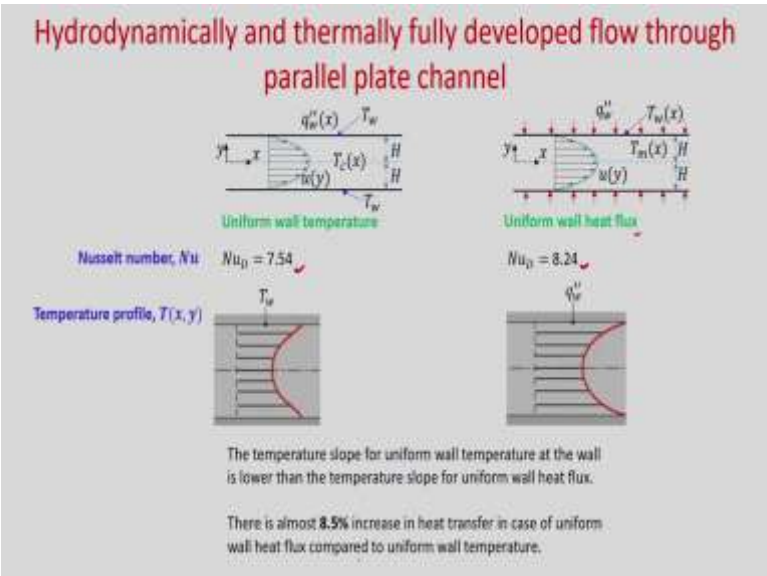
So, let us summarise; so today we considered the fully developed laminar flow through parallel plate channels with constant wall temperature. We considered the non dimensional temperature as $\frac{T - T_w}{T_c - T_w}$ where T_c is the centerline temperature for easy calculation.

From there, we have found the separation of variables we have we; we use the separation of variable method and we have written the Sturm Liouville problem because the boundary conditions are homogeneous in y star direction.

And then, we have found the solution of that governing equation using series solution. And from there we have found the values of λ^2 and then we found the Nusselt number, starting from the governing equation and Nusselt number; we have defined with respect to $T_m - T_w$ where T_m is the mean temperature.

And you can see that for this particular case, the Nusselt number is also constant; independence of Reynolds number and the Prandtl number.

(Refer Slide Time: 53:20)



So, you can see that for uniform wall temperature case where you have that T_w as constant, Nusselt number is 7.54. In earlier lectures, we have found with uniform wall heat flux; for this flow through parallel plate channel, we have found Nusselt number as 8.24. And if you see the temperature profile, for this uniform wall temperature, you can see the slope at the wall is lower than the temperature slope of uniform wall heat flux.

So, if you see the temperature profile here; this slope is higher than this temperature profile, hence your Nusselt number is higher in case of uniform wall heat flux; compared to uniform wall temperature. And there is almost 8.5 % increase in heat transfer, in case of uniform wall heat flux compared to uniform wall temperature.

Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

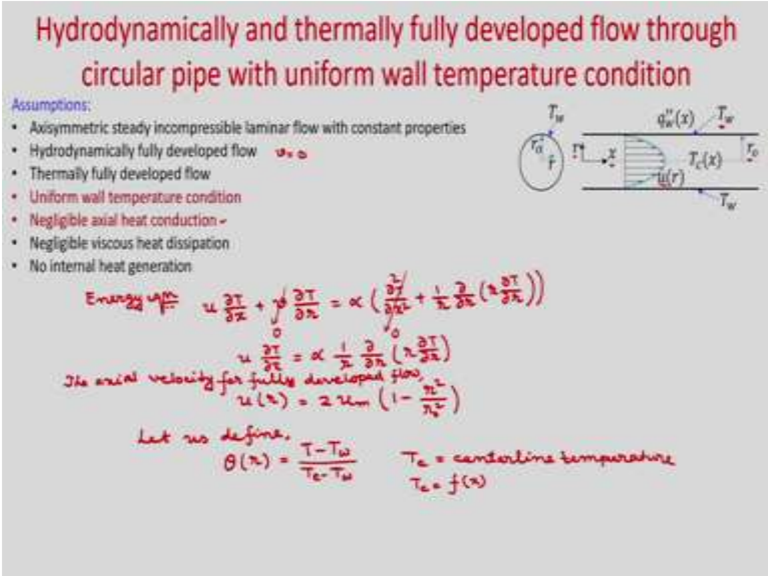
Module - 06
Convection in Internal Flows - II
Lecture - 21
Hydrodynamically and thermally fully developed flow through circular pipe with uniform wall temperature

Hello everyone, today we will consider fully developed laminar flow through circular pipe with uniform wall temperature. For this flow through circular pipe, already we have found the temperature distribution and the Nusselt number with uniform wall heat flux boundary condition. When we will consider uniform wall temperature boundary condition, one important assumptions we have to take that axial heat conduction is negligible compared to the radial heat conduction.

Earlier case when we considered uniform wall heat plus boundary condition, your $\frac{\partial T}{\partial x}$ was constant as your heat flux $q_w^"$ was constant. So, hence the second derivative of T with respect to x, $\frac{\partial^2 T}{\partial x^2}$ becomes automatically 0 for the uniform wall heat flux boundary condition. However, in this particular case when we consider uniform wall temperature boundary condition, we need to assume that your axial heat conduction is negligible compared to the radial heat conduction.

Another assumptions we will take; that it is axisymmetric. What does it mean? It means that in circumferential direction there is no change of any property, that means, $\frac{\partial}{\partial \theta}$ of any quantity is 0. If geometry is symmetric and thermal boundary condition is symmetric, in this particular case we are considering uniform wall temperature boundary condition, hence you have a symmetric thermal boundary condition as well as it is wall circumferential direction there is no change of any quantity. Hence axisymmetric assumption is valid for this particular case.

(Refer Slide Time: 02:42)



So, let us consider fully developed flow inside circular pipe. So, you can see x is the axial direction, r is the radial direction. The circular pipe is having radius r_0, wall is maintained at temperature T_w. So, you can see this is the circular pipe we have considered. So, these are the assumptions axisymmetric steady incompressible laminar flow with constant properties, hydrodynamically fully developed flow.

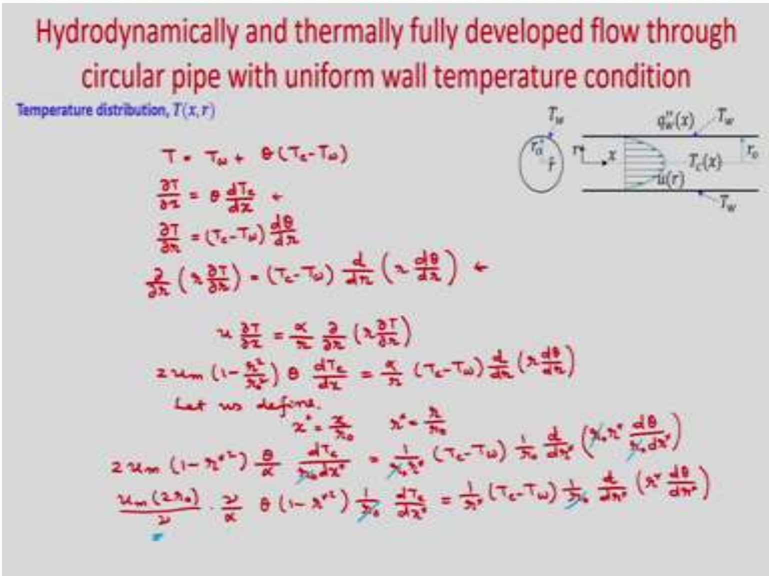
So, the we have considered u is function of r only because it is a fully developed flow, and we can write in terms of mean velocity. Thermally fully developed flow, that means, $\frac{d\phi}{dr} = 0$ and we have considered uniform wall temperature condition. And we are neglecting the axial heat conduction. As well as we have assumed that negligible viscous dissipation, and no internal heat generation.

So, first let us write the governing equation. So, after invoking all these condition, you will be able to write the governing equation as; $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right)$. So, this is your energy equation in general.

So, as it is a fully developed flow hydrodynamically fully developed flow, so v = 0 so. this is 0. And we are neglecting the axial heat conduction, so this is also 0. So, you will get $u\frac{\partial T}{\partial x} = \alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$. And you know the fully developed you know that fully developed velocity profile is, $u(r) = 2u_m \left(1 - \frac{r^2}{r_0^2}\right)$. So, this is your the axial velocity for fully developed flow.

Now, we will define on non-dimensional temperature $\theta = \frac{T - T_w}{T_c - T_w}$ where T_c is your centerline temperature. In this case also, θ will not vary in the axial direction. So, θ is function of r only. So, let us define θ which is function of r only as $\theta(r) = \frac{T - T_w}{T_c - T_w}$, where T_c is your centerline temperature, and T_c is function of x only.

(Refer Slide Time: 06:53)



So, if you now take the derivatives of temperature with respect to x and r, then we can write, so $T = T_w + \theta(T_c - T_w)$. So, you can see T_w is constant. So, θ is function of r only, so you can write $\frac{\partial T}{\partial x} = \theta \frac{dT_c}{dx}$. Similarly, you can write $\frac{\partial T}{\partial r} = (T_c - T_w)\frac{d\theta}{dr}$. And if you write $\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = (T_c - T_w)\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)$.

So, now all these you put in the energy equation. So, you have the energy equation as $u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$. So, u is the axial velocity for fully developed flows. So, now, you put all the values in this energy equation. So, u is the axial velocity for fully developed flow and it can be written as, $2u_m\left(1-\frac{r^2}{r_0^2}\right)\theta\frac{dT_c}{dx}=\frac{\alpha}{r}(T_c-T_w)\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)$.

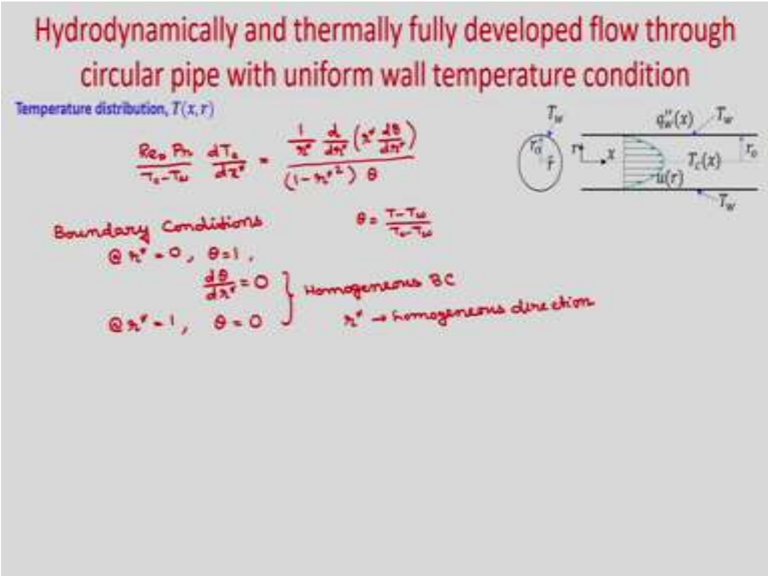
So, now let us define; non-dimensional x coordinate as $x^*=\frac{x}{r_0}$, and radial non-dimensional radial coordinate $r^*=\frac{r}{r_0}$. So, if you put it here, then you will get twice $2u_m\left(1-r^{*2}\right)\frac{\theta}{\alpha}\frac{dT_c}{r_0 dx^*}=\frac{1}{r_0 r^*}(T_c-T_w)\frac{1}{r_0}\frac{d}{dr^*}\left(r_0 r^*\frac{d\theta}{r_0 dr^*}\right)$.

So, if you see this r_0, this r_0, you can cancel, then one r_0 here you can cancel; now, you simplify it. So, you will write,

$$\frac{u_m(2r_0)}{\nu}\frac{\nu}{\alpha}\theta\left(1-r^{*2}\right)\frac{1}{r_0}\frac{dT_c}{dx^*}=\frac{1}{r^*}(T_c-T_w)\frac{1}{r_0}\frac{d}{dr^*}\left(r^*\frac{d\theta}{dr^*}\right).$$

Here you can cancel this r_0, this r_0 and this r_0. So, you see what is this? So, this is your Reynolds number based on diameter 2 r_0, and $\frac{\nu}{\alpha}$ is your Prandtl number.

(Refer Slide Time: 11:53)



So, you can write, $\frac{\mathrm{Re}_D \Pr}{T_c - T_w}\frac{dT_c}{dx*} = \frac{\frac{1}{r*}\frac{d}{dr*}\left(r*\frac{d\theta}{dr*}\right)}{\left(1-r*^2\right)\theta}$. So, after rearranging, you can write in this form. Now, you see the left hand side Reynolds number and Prandtl number are constant, T_c is the centerline temperature which is function of x only, T_w is constant, and $\frac{dT_c}{dx}$ is also function of x, so left hand side is function of x only.

Now, right hand side if you see here all terms are function of r* only. So, now, we have separated the variables. Left hand side is function of x only; right hand side is function of r* only. So, you can see that left hand side is function of x or x*, and right hand side is function of r or r* only. So, we have separated the variables now, these equal to some constant.

So, how we will choose the constant? First let us see the boundary conditions which is the homogeneous direction first let us see. Then accordingly we will choose the sign of this constant such that in homogeneous direction we get the harmonic solution, so that is the rule of using separation of variables method.

So, what are the boundary condition, first let us see. So, you can see that at r *= 0, that means, centerline temperature. How we have defined the θ? θ we have defined as $\theta = \frac{T - T_w}{T_c - T_w}$. So, at r *= 0 that means that the central line, obviously, T = T_c. So, if we put T_c, θ = 1.

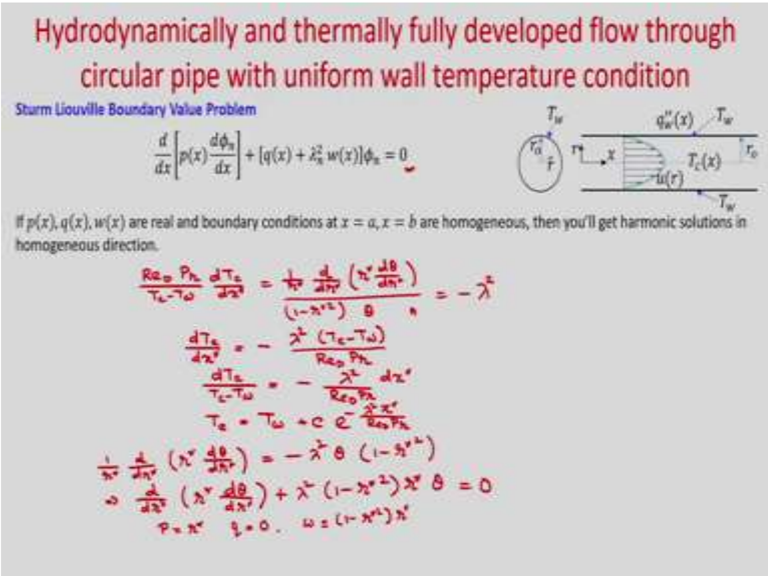
Now, at the same time you can see that the problem is axisymmetric, and it is it is geometrically and thermally symmetric. So, at the center, you will have either maximum of minimum temperature. So, we can write that $\frac{d\theta}{dr*}$ =0. So, although θ = 1, but another you can write at r *= 0, $\frac{d\theta}{dr*}$ =0. And at the wall, what is the boundary condition at r *= 1, so you see T = T_c right, so θ will be 0.

Now, you see at r *= 0, you have $\frac{d\theta}{dr*}$ =0, you have $\frac{d\theta}{dr*}$ =0, and r *= 1 you have θ =0, so that means, these are homogeneous boundary conditions. What is homogeneous boundary condition? If the value of that variable is 0 or its gradient is 0 or combination of these two is 0, so that is known as homogeneous boundary condition.

So, you have in r direction both the boundary conditions are homogeneous; so. it is a homogeneous direction. So, r* is the homogeneous direction. So, you should choose the value of constant, you should choose the sign of the constant such a way that in homogeneous direction you get the harmonic solution. So, this is your homogeneous boundary condition. And r * or r is the homogeneous direction homogeneous direction.

So, now how we will determine that you will have the harmonic solution in the homogeneous direction? So, for that we will use the Sturm Liouville boundary value problem.

(Refer Slide Time: 16:48)



So, you see Sturm Liouville boundary value problem; it is actually given by this second order ordinary differential equation. So, you can see $\frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right]+\left[q(x)+\lambda_n^2 w(x)\right]\phi_n=0$, where w(x) is the weighting function. If p(x), q(x), w(x) are real, and boundary conditions at x = a, and x = b are homogeneous, then you will get harmonic solutions in homogeneous direction.

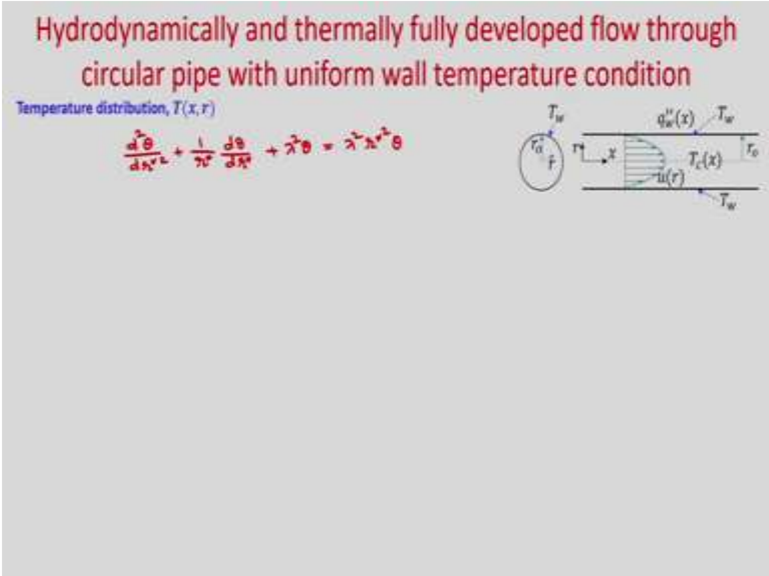
So, now let us see that if we choose the sign of the constant as $-\lambda^2$, then what will happen. So, let us say that this $\frac{\mathrm{Re}_D \Pr}{T_c - T_w} \frac{dT_c}{dx^*} = \frac{\frac{1}{r^*}\frac{d}{dr^*}\left(r^* \frac{d\theta}{dr^*}\right)}{\left(1 - r^{*2}\right)\theta}$ = some constant, and the sign of that constant we are taking $-\lambda^2$. λ^2 we are taking for convenience, it is any constant. So, we are taking the sign of this constant as minus, so that in r* direction we get the harmonic solution.

Now, if you write the equation, so you will get the, if you write first one, so you will get $\frac{dT_c}{dx^*} = -\frac{\lambda^2 (T_c - T_w)}{\mathrm{Re}_D \Pr}$. And you can write $\frac{dT_c}{T_c - T_w} = -\frac{\lambda^2}{\mathrm{Re}_D \Pr} dx^*$. Now, if you integrate it, so you will get $T_c = T_w + Ce^{-\frac{\lambda^2 x^*}{\mathrm{Re}_D \Pr}}$. So, this is the variation of centerline temperature.

So, now if you write for this one, so you will get $\frac{1}{r^*}\frac{d}{dr^*}\left(r^* \frac{d\theta}{dr^*}\right) = -\lambda^2 \theta \left(1 - r^{*2}\right)$. So, if you rearrange, you well get $\frac{d}{dr^*}\left(r^* \frac{d\theta}{dr^*}\right) + \lambda^2 \left(1 - r^{*2}\right) r^* \theta = 0$.

If you compare this equation with the Sturm Liouville boundary value problem, then p=r*, q = 0 and the weighting function w = $\left(1 - r^{*2}\right) r^*$. So, now, you see p, q, r are real and boundary condition in the r* direction both the boundary conditions are homogeneous. So, r * is the homogeneous direction. Hence the solution of this second order differential equation will give harmonic solution.

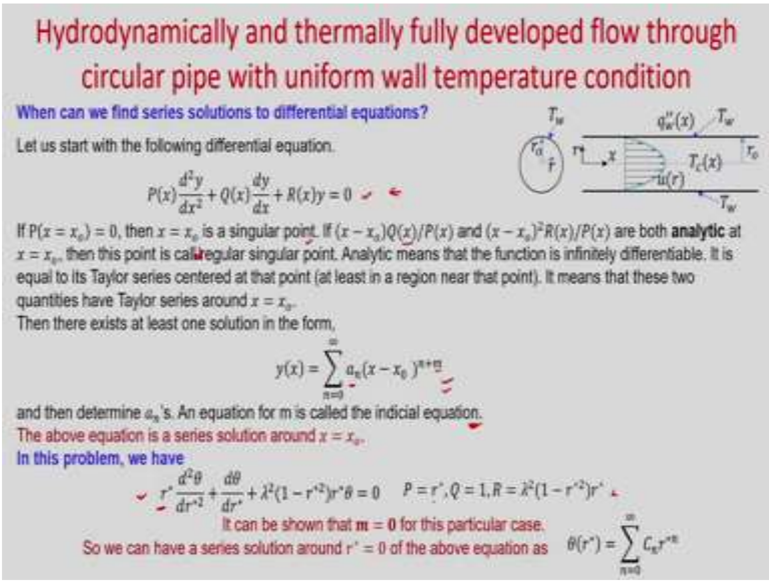
(Refer Slide Time: 20:48)



So, now if you write it, so you will get after rearranging $\frac{d^2\theta}{dr*^2}+\frac{1}{r*}\frac{d\theta}{dr*}+\lambda^2\theta=\lambda^2 r*^2\,\theta$.

So, now we want to seek the solution of this differential equation as a series solution and in last class we have already discussed that whether we can have the series solution of these ordinary differential equation or not for that let us revisit it again.

(Refer Slide Time: 21:35)



So, before going to the solution, let us first see when can we find series solution to differential equations. So, this is the second order ordinary differential equation. In this case, we will see that P at x =x_0, if it is 0, then x =x_0 is a singular point. So, if $\left(x-x_0\right)Q(x)/P(x)$, and $\left(x-x_0\right)^2 R(x)/P(x)$ are both analytic at x = x_0, then this point is called regular singular point.

Analytic means that; the function is infinitely differentiable, so that we already discussed in last class. Now, if we have this, then we can have the solution in the from $y(x)=\sum_{n=0}^{\infty} a_n\left(x-x_0\right)^{n+m}$.

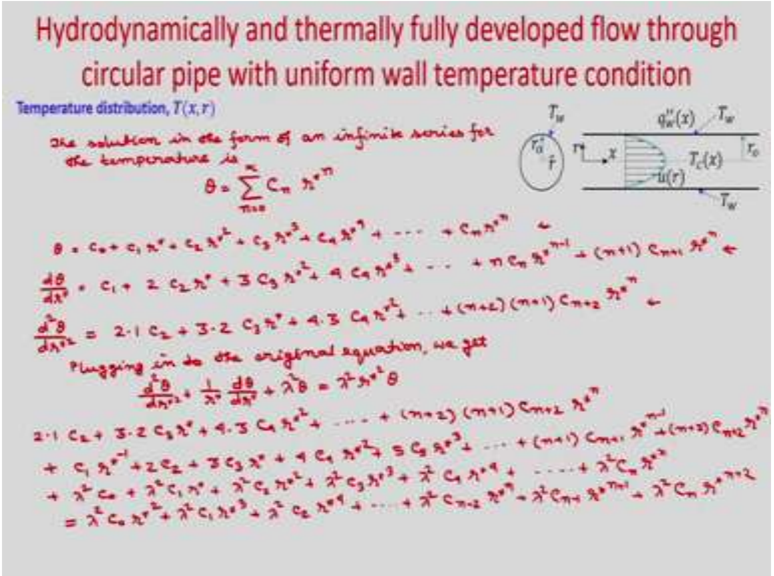
So, here then we have to determine the coefficient a_n. And this a_n, an equation for m is called the indicial equation, so that we also need to find. So, the above equation is just series solution at around x =x_0.

So, for our present problem, you can see that our governing equation whatever we have derived this is the second order differential equation where P = r *, Q = 1, and $R=\lambda^2\left(1-r*^2\right)r*$. So, comparing with this equation just we have found it.

So, if you see that this condition, so for this present case $r*\frac{Q}{P}$, and $r*^2\frac{R}{P}$ are both analytic. And at r = 0 it is singular point. So, you can see that we can write the solution in this form, but we have not written +m, because it can be shown that m = 0 for this particular case.

So, this derivation will be easier because we have already assumed that m = 0, it can be shown actually. So, for this particular case, this is m = 0. So, we can have a series solution around r * = 0 of the above equation is this one. So, now you can see that for this differential equation, your $\frac{Q}{P}$, and $\frac{R}{P}$ are analytic. So, hence you can have the series solution.

(Refer Slide Time: 24:35)



So, if you write the series solution about x about r* = 0, then we can write the solution in the form of an infinite series for the temperature is, so you can write, $\theta = \sum_{n=0}^{\infty} C_n r^{*n}$.

So, first we will find the derivative of θ with respect to r*, then we will put back to the ordinary differential equation, and we will try to find the coefficient equating the similar power. So, we will write $\theta = C_0 + C_1 r^* + C_2 r^{*2} + C_3 r^{*3} + C_4 r^{*4} + ... + C_n r^{*n}$.

So, now you write the derivative of θ with respect to r* as $\frac{d\theta}{dr^*} = C_1 + 2C_2 r^* + 3C_3 r^{*2} + 4C_4 r^{*3} + ... + nC_n r^{*n-1} + (n+1)C_{n+1} r^{*n}$. Then let us find the second derivative of θ. So, you can see that it is $\frac{d^2\theta}{dr^{*2}} = 2.1C_2 + 3.2C_3 r^* + 4.3C_4 r^{*2} + ... + (n+2)(\mathrm{n}+1)C_{n+2} r^{*n}$.

So, now you plug all these into the ordinary differential equation. So, what is our ordinary differential equation? So, plugging into the original equation we get. So, we have the equation $\frac{d^2\theta}{dr^{*2}} + \frac{1}{r^*}\frac{d\theta}{dr^*} + \lambda^2\theta = \lambda^2 r^{*2}\,\theta$. So, this is your equation.

Now, if you put it, so you will get,

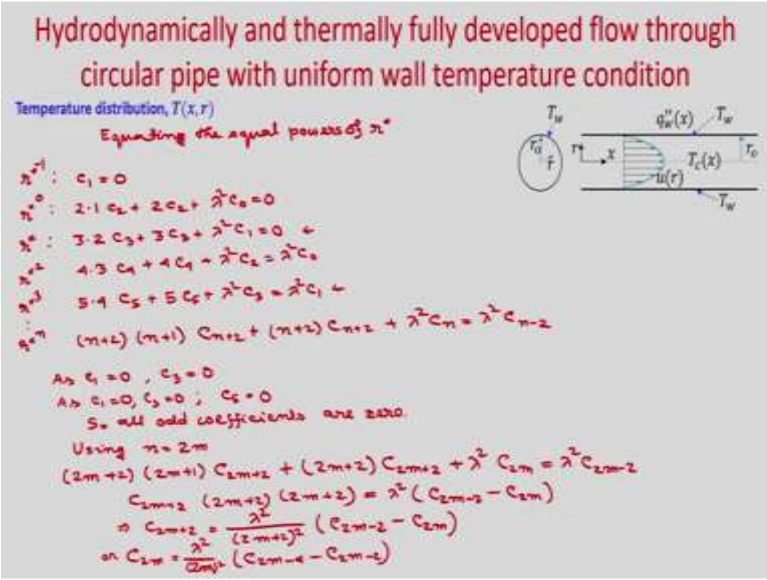
$$2.1C_2 + 3.2C_3 r^* + 4.3C_4 r^{*2} + ... + (n+2)(\mathrm{n}+1)C_{n+2} r^{*n}$$

$$+C_1 r^{*-1} + 2C_2 + 3C_3 r^* + 4C_4 r^{*2} + 5C_4 r^{*3} \ldots.. + (n+1)C_{n+1} r^{*n-1} + (n+2)C_{n+2} r^{*n}$$

$$+\lambda^2 C_0 + \lambda^2 C_1 r^* + \lambda^2 C_2 r^{*2} + \lambda^2 C_3 r^{*3} + \lambda^2 C_4 r^{*4} + \ldots\ldots\ldots + \lambda^2 C_n r^{*n}$$

So, left hand side all we have written, now you write the right hand side. So, right hand side is $\lambda^2 C_0 r^{*2} + \lambda^2 C_1 r^{*3} + \lambda^2 C_2 r^{*4} + \ldots\ldots\ldots + \lambda^2 C_{n-2} r^{*n} + \lambda^2 C_{n-1} r^{*n+1} + \lambda^2 C_n r^{*n+2}$. So, now let us equate the power of r*. So, now, let us equate the equal power of r* and find the coefficient.

(Refer Slide Time: 30:54)



So, equating the equal power powers of r*; so first we will find r^{*-1} that means, $\frac{1}{r^*}$. So, for that what are the if you see left hand side in first line there is no term, the second it is there C_1, and third there is no term and right hand side there is no term; so only C_1 will become 0, so here $C_1 = 0$.

Now, equate the equal powers of r^{*0} that means, in the left hand side you see; so there is no r *, so it is $r^{*0} : 2.1C_2 + 2C_2 + \lambda^2 C_0 = 0$. Now, you equate the power of r *; $r^* : 3.2C_3 + 3C_3 + \lambda^2 C_1 = 0$.

Similarly, if you write for r^{*2}; so you can write $r^{*2} : 4.3C_4 + 4C_4 + \lambda^2 C_2 = \lambda^2 C_0$. So, we have written this one.

Similarly, $r^{*3} : 5.4C_5 + 5C_5 + \lambda^2 C_3 = \lambda^2 C_1$.

And $r^{*n} : (n+2)(\mathrm{n}+1)C_{n+2} + (n+2)C_{n+2} + \lambda^2 C_n = \lambda^2 C_{n-2}$.

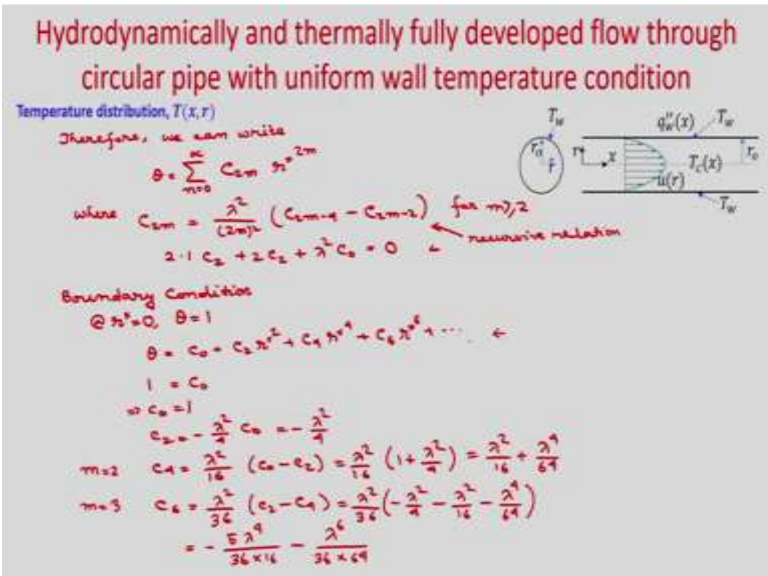
So, now you see the first term C_1= 0 right; if C_1= 0, then from here you can see if C_1= 0, then C_3 will become 0 right. And if C_1,C_3= 0, then from this equation you can see C_5= 0. So, you can see all the odd coefficients are 0; C_1, C_3, C_5, C_7, all odd coefficients will become 0 as C_1=0. So, you can see that as C_1=0 from there you can see C_2,C_3=0, then as C_1=0, C_3=0; then from this equation C_5= 0. So, you can see so all odd coefficients are 0.

So, we can replace n = 2 m , because there will be no odd coefficient only the even coefficients will be there. So, you can write using n = 2 m, we can write twice $(2m+2)(2m+1)C_{m+2} + (2m+2)C_{m+2} + \lambda^2 C_{2m} = \lambda^2 C_{2m-2}$.

So, so now if you rearrange it, so you see $C_{m+2}(2m+2)(2m+2) = \lambda^2 \left(C_{2m-2} - C_{2m}\right)$.

So, you can write $C_{m+2} = \dfrac{\lambda^2}{(2m+2)^2}\left(C_{2m-2} - C_{2m}\right)$; or you can write $C_{2m} = \dfrac{\lambda^2}{(2m)^2}\left(C_{2m-4} - C_{2m-2}\right)$.

(Refer Slide Time: 36:38)



So, now as all odd coefficients are 0, now we can write the solution of θ as $\theta = \sum_{n=0}^{\infty} C_{2m} r^{*2m}$, where we have already found; what is C_{2m} ,

$C_{2m} = \frac{\lambda^2}{(2m)^2}\left(C_{2m-4} - C_{2m-2}\right)$ and this is valid for m ≥ 2. And we have another expression this one , so this we can write $2.1C_2 + 2C_2 + \lambda^2 C_0 = 0$.

So, now let us find the value of the coefficients C_2, C_4, C_6, in terms of C_0. And if we apply the boundary condition, then we will be able to find what is the value of C_0. So, first let us write the boundary condition at r * = 0; θ = 1. So, you can write θ = 1 in the left hand side and right hand side; so you can write first let us expand it, then it will be easier $\theta = C_0 + C_2 r^{*2} + C_4 r^{*4} + C_6 r^{*6}$ ....

So, if you put θ = 1 and r * = 0, so first term will remain and other terms will become 0 that means, $C_0 = 1$. So, now to find the other coefficient C_2, C_4, C_6, in terms of C_0. And you can see that this term this expression $C_{2m} = \frac{\lambda^2}{(2m)^2}\left(C_{2m-4} - C_{2m-2}\right)$, it is a recursive relation.

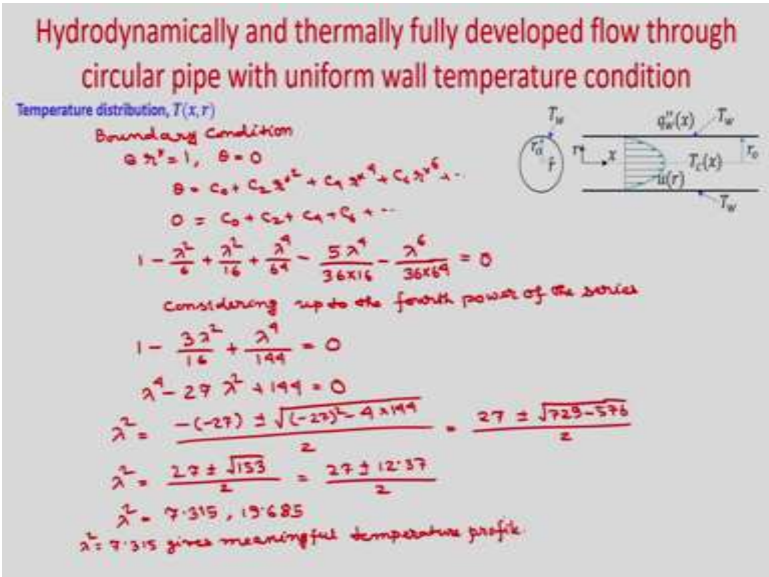
So, if you find C_0, then from there you can find C_2; and if you know C_0 and C_2, then you can find C_4, and C_6, C_8 you can find accordingly. So, from this expression what is the value of C_2? So, you can see $C_2 = -\frac{\lambda^2}{4} C_0 = -\frac{\lambda^2}{4}$.

Then m = 2 if you put in the recursive relation, so this is your recursive relation; so if you put m = 2, then you will get $C_4 = \frac{\lambda^2}{16}(C_0 - C_2)$. Now, C_0 C_2 you know; so if we put the value, $\frac{\lambda^2}{16}(1 + \frac{\lambda^2}{4}) = \frac{\lambda^2}{16} + \frac{\lambda^2}{64}$.

Similarly, if you put m = 3, then $C_6 = \frac{\lambda^2}{36}(C_2 - C_4) = \frac{\lambda^2}{36}(-\frac{\lambda^2}{4} - \frac{\lambda^2}{16} - \frac{\lambda^2}{64})$.

So, if you rearrange it, so you will get $-\frac{5\lambda^4}{36\times16}+\frac{\lambda^6}{36\times64}$. So, if you can find the other constant, then you will be able to find; if you find the other coefficients, then you will be able to find the temperature profile θ, because θ you can write using this expression. So, C_0, C_2, C_4, C_6 already we have found; another coefficients if you find, then you will be able to find the temperature distribution.

(Refer Slide Time: 41:52)



Now, let us find the value of λ^2. Now, apply another boundary condition at r * = 1 means at the wall, θ =0. So, now you know the expression $\theta = C_0 + C_2 r^{*2} + C_4 r^{*4} + C_6 r^{*6}$ .... So, at r * = 1, so θ = 0; so that means $0 = C_0 + C_2 + C_4 + C_6 + ...$.

Now, we know the value of C_0, C_2, C_4, C_6 already we have found let us put it in this expression. Then you can write C_0, C_2, C_4, C_6 in the left hand side, so we can write $1-\frac{\lambda^2}{6}+\frac{\lambda^2}{16}+\frac{\lambda^2}{64}-\frac{5\lambda^4}{36\times16}-\frac{\lambda^6}{36\times64}=0$. So, we have considered up to the fourth power of series.

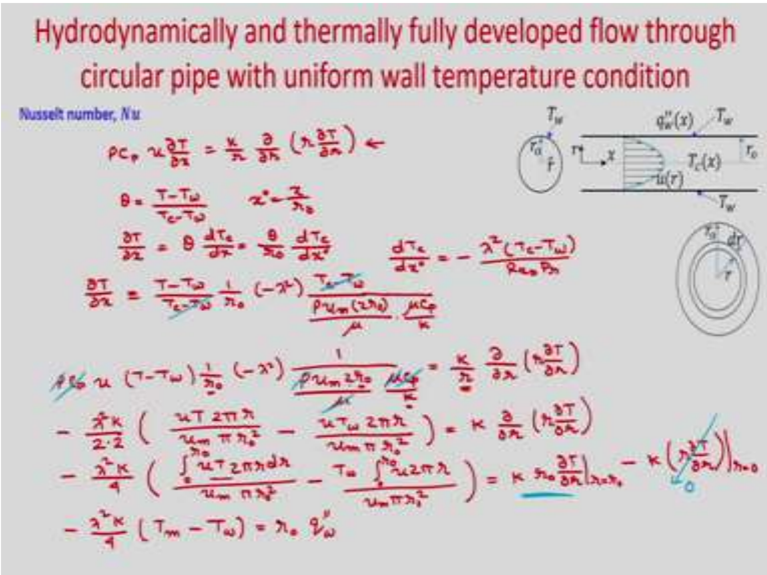
You can see that after that if you see this term $\frac{\lambda^6}{36\times64}$, so the denominator is very high value. So, it will contribute very less here, so and this other term. So, neglecting all the other terms and considering up to the fourth power of series you can write as, $1-\frac{3\lambda^2}{16}+\frac{\lambda^4}{144}=0$. so if you rearrange it, you will get $\lambda^4-27\lambda^2+144=0$.

And λ^2 if you find, so it will be $\lambda^2=\frac{-(-27)\pm\sqrt{(-27)^2-4\times144}}{2}$. So, it will be $\frac{27\pm\sqrt{729-576}}{2}$ and you will get $\frac{27\pm\sqrt{153}}{2}$; so you will get $\frac{27\pm12.37}{2}$, so you will get the value of $\lambda^2=7.315,19.685$.

So, the first value $\lambda^2=7.315$ will give you the physically correct temperature profile, the other one will not give; so you consider the value of $\lambda^2=7.315$. So, $\lambda^2=7.315$ gives meaningful temperature profile, so we are considering only this value .

So, if you now you know the λ^2, you know the values of C_0, C_2, C_4, C_6 and so forth. If you put it in the expression of θ, then you will be find you will be able to find the temperature profile.

(Refer Slide Time: 46:24)



Now, let us find what is the Nusselt number? So, to calculate the Nusselt number we will start from the again governing equation, so it is $\rho C_P u \frac{\partial T}{\partial x} = \frac{K}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$. So, we know $\theta = \frac{T - T_w}{T_c - T_w}$, and also $x^* = \frac{x}{r_0}$, you know $\frac{\partial T}{\partial x} = \theta \frac{dT_c}{dx} = \frac{\theta}{r_0}\frac{dT_c}{dx^*}$.

If you see we have found $\frac{dT_c}{dx^*} = -\frac{\lambda^2 (T_c - T_w)}{\mathrm{Re}_D \Pr}$. So, if you put it here, so you will get $\frac{\partial T}{\partial x} = \frac{T - T_w}{T_c - T_w}\frac{1}{r_0}(-\lambda^2)\frac{T_c - T_w}{\frac{\rho u_m (2r_0)}{\mu}\frac{\mu C_P}{K}}$.

So, you can see this you can cancel $T_c - T_w$, $T_c - T_w$. And you can write in the governing equation if you put it, then you will get, $\rho C_P u (T - T_w)\frac{1}{r_0}(-\lambda^2)\frac{1}{\frac{\rho u_m (2r_0)}{\mu}\frac{\mu C_P}{K}} = \frac{K}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$.

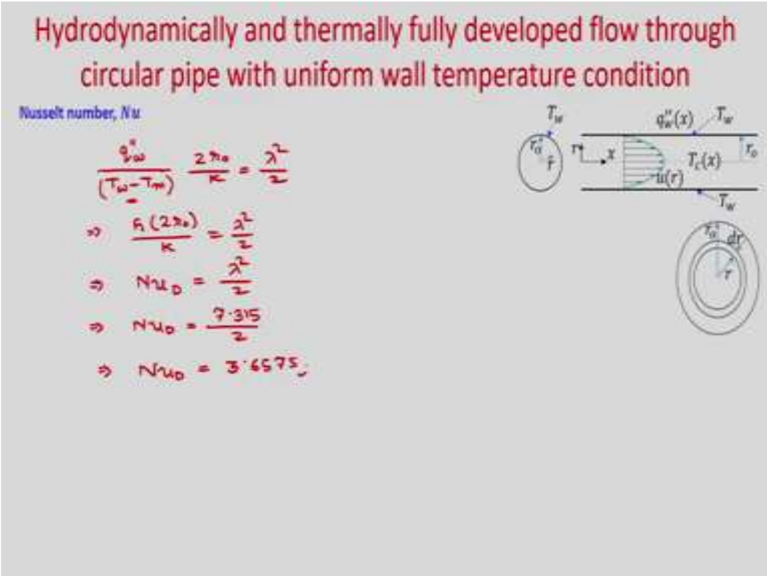
So, you see what are the terms you can cancel, ρC_P, ρC_P; this μ, you can cancel. So, now, if you see you can rearrange it as $-\frac{\lambda^2 K}{2.2}\left(\frac{uT 2\pi r}{u_m \pi r_0^2} - \frac{u T_w 2\pi r}{u_m \pi r_0^2}\right) = K\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$.

So, we have done some rearrangement; now we will integrate both side $-\frac{\lambda^2 K}{4}\left(\frac{\int_0^{r_0} uT 2\pi r dr}{u_m \pi r_0^2} - \frac{T_w \int_0^{r_0} u 2\pi r dr}{u_m \pi r_0^2}\right) = K r_0 \frac{\partial T}{\partial r}\Big|_{r=r_0} - K(r\frac{\partial T}{\partial r})\Big|_{r=0}$.

What is this term? So, this is nothing but the heat flux at the wall right, because $q_w'' = K\frac{\partial T}{\partial r}\Big|_{r=r_0}$ at the wall. So, if you now see this term, what does it mean u T X da you see at a distance r; if you take a small step dr, so whatever the area that is $2\pi r dr$ and that is uTda. So, integral $\frac{\int_0^{r_0} uTda}{u_m \pi r_0^2}$ what is that that is nothing but the definition of mean temperature, so we can write this term as the mean temperature.

Second term $\int_0^{r_0} u 2\pi r dr$, so that is the $u_m \pi r_0^2$ right, so that means this will get cancel. So, essentially you will get $-\frac{\lambda^2 K}{4}(T_m - T_w) = r_0 q_w^{"}$.

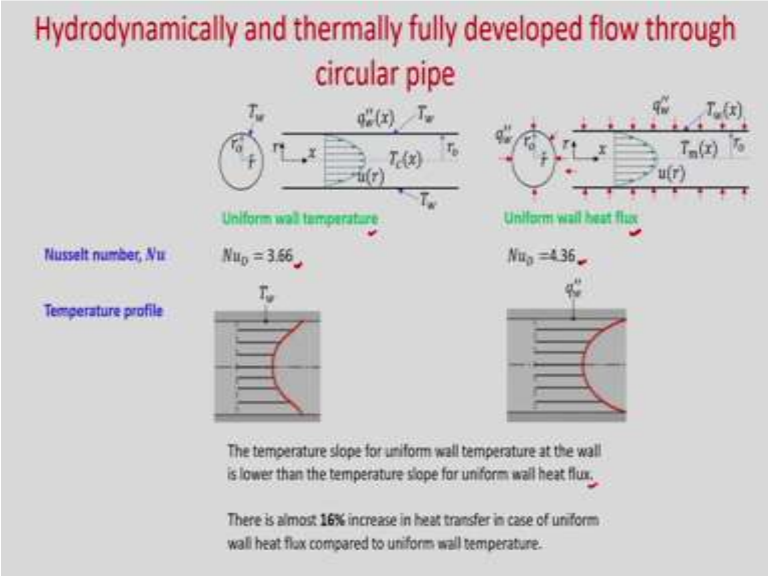
(Refer Slide Time: 53:18)



So, now if you rearrange it, so you can see $\frac{q_w^{"}}{(T_m - T_w)} \frac{2r_0}{K} = \frac{\lambda^2}{2}$ that means, so what is this; this is a local heat transfer coefficient that is $\frac{h(2r_0)}{K} = \frac{\lambda^2}{2}$. What is $\frac{h(2r_0)}{K}$ nothing but local Nusselt number, so $Nu_D = \frac{\lambda^2}{2}$. And we know the value of λ^2 as 7.315 , so it will be, $Nu_D = \frac{7.315}{2}$; so your Nusselt number is 3.6575.

So, you can see for this particular case when we consider fully developed laminar flow through circular pipe, the Nusselt number is independent of Reynolds number and Prandtl number, because it is a constant value. When we consider uniform wall temperature, the value of $Nu_D = 3.6575$.

(Refer Slide Time: 54:36)



Now, let us summarise what we have done today. So, today we considered fully developed laminar flows through circular pipe with uniform wall temperature boundary condition. In this particular case, one important assumptions we have made that is axial heat conduction is negligible compared to the radial heat conduction, and also we have assumed that it is a axisymmetric flow.

Then we have started from the energy equation, then we have separated the variables in terms of x and r and we have compared the homogeneous direction whatever governing equation you are getting, ordinary differential equation with the second order differential equation which is your Sturm-Liouville boundary value problem. And we have shown that that is your in radial direction which is your homogeneous direction, you will get the harmonic solution.

Then we have checked whether we can use the power series solution or not, and considering that we have taken the solution as a power series and we have found the coefficient C_0, C_2, C_4, C_6; and from there we have found the value of lambda square. And then we have found the Nusselt number from the starting from the governing equation, and we have found Nusselt number for this particular case is independent of Reynolds number and Prandtl number and it is constant value.

Now, if you see earlier, we have already found the Nusselt number for this particular case flow through a circular pipe with uniform wall heat flux as 4.36. And when you consider today, uniform wall temperature for this flow through circular pipe the Nusselt number is 3.66. And if you see the temperature profile for uniform wall temperature, if you see the gradient at the wall , the temperature gradient at the wall, and if you see the uniform wall plus case, the temperature gradient at the wall obviously you can see it is much higher than the uniform wall temperature gradient.

So, the temperature slope for uniform wall temperature at the wall is lower than the temperature slope for uniform wall heat flux. And if you see the Nusselt number, there is almost 16 % increase in heat transfer in case of uniform wall heat flux compared to the uniform wall temperature, and it is due to the higher gradient at the wall of this temperature, higher gradient of the temperature at the wall for the case of uniform wall heat flux.

Thank you.

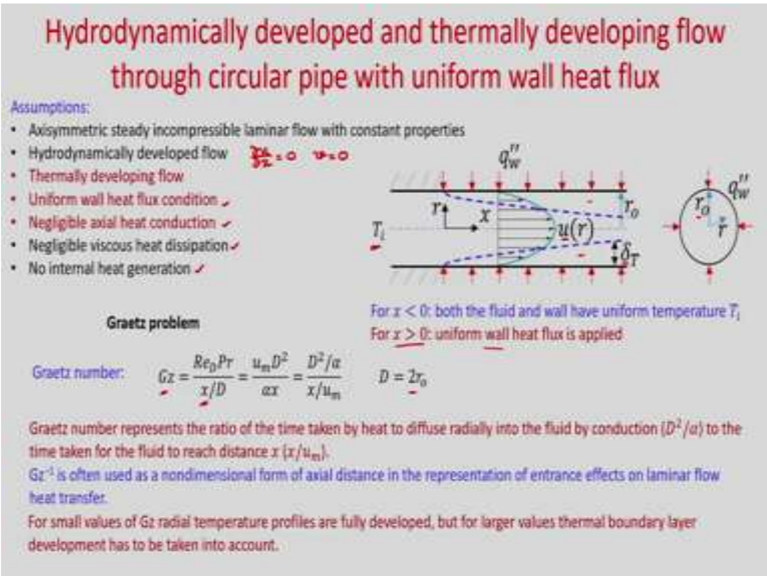
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 07
Convection in Internal Flows - III
Lecture - 22
Hydrodynamically developed and thermally developing flow through circular pipe with uniform wall heat flux

Hello everyone. So, till now we considered Hydrodynamically and thermally fully developed flow. So, in analysis it was easier because the non-dimensional temperature phi whatever we defined it was function of r only, so it does not change in the axial direction.

Today, we will consider hydrodynamically developed, but thermally developing laminar fluid flow through circular pipe with uniform wall heat flux. So, you can see that it is thermally developing flow that means, your temperature will change in axial direction and it will vary only inside the thermal boundary layer. However, in the core region it will remain at the inlet temperature.

(Refer Slide Time: 01:26)



So, let us see. So, this is your circular pipe of radius r 0. We have kept the axis at the center line and at x = 0 we have put in such a way that from there uniform heat flux boundary condition is applied. If x < 0, then both the fluid and wall have uniform temperature T_i. So, after $x \geq 0$, you have uniform wall heat flux, boundary condition and your thermal boundary layer starts growing from x = 0. So, this is your thermal boundary layer thickness.

So, we have considered hydrodynamically fully developed flow. So, you have velocity u which is function of r only and it is fully developed profile parabolic. However, your temperature profile you can see that it will vary inside the thermal boundary layer only. But in the core region it will remain at temperature T_i.

The assumptions for this study are axisymmetric steady incompressible laminar flow with constant properties. You can see that it is geometrically and thermally symmetric and we can have the assumptions of axisymmetric flow. So, there is no variation in of any quantity in circumferential direction. It is hydrodynamically developed flow that means, $\frac{\partial u}{\partial x} = 0$ and radial velocity v = 0. It is thermally developing flow.

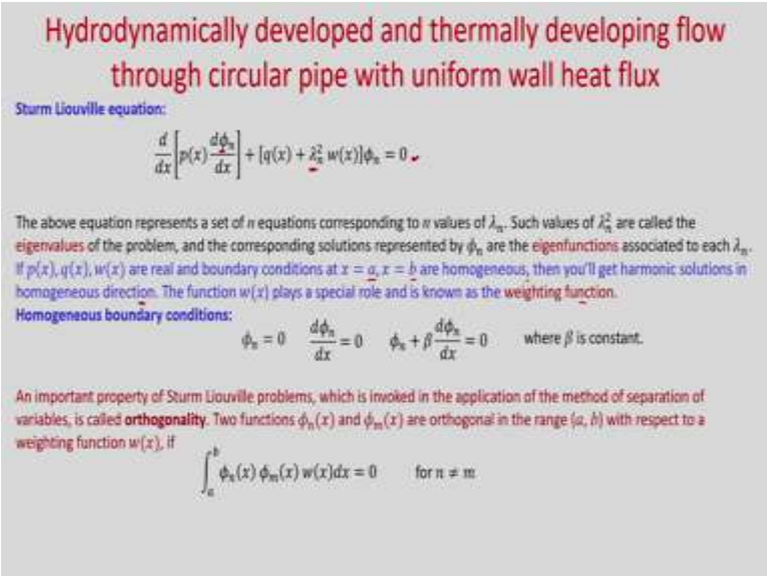
We used uniform wall heat flux condition. In this particular case also, we will neglect the axial heat conduction that means, axial heat conduction is we will assume that axial heat conduction is very very small compared to the radial heat conduction. And we will also assume negligible viscous heat dissipation and no internal heat generation.

So, the problem hydrodynamically developed and thermally developing flow inside a circular pipe is known as Graetz problem. So, we will define the Graetz number as $Gz = \frac{\mathrm{Re}_D \Pr}{x/D}$. So, you can see that this is some inverse of non-dimensional form of the axial distance, x is the axial distance and if you put the variables in Reynolds number and Prandtl number you can write it as $Gz = \frac{u_m D^2}{\alpha x} = \frac{D^2/\alpha}{x/u_m}$. So, you can see that in the numerator and denominator both are having the time scale, where D = 2 r_0.

So, Graetz number represents the ratio of the time taken by heat to diffuse radially into the fluid by conduction to the time taken to the fluid to reach distance x. For small values of Graetz number radial temperature profiles are fully developed. So, you can see that if x is very high then your Graetz number will be small.

So, if x is very high it will be thermally fully developed flow, so Graetz number will be small. So, for small values of Graetz number radial temperature profiles are fully developed, but for larger values of thermal boundary layer development has to be taken into account. So, you can see if x is very small then it will be thermally developing region.

(Refer Slide Time: 05:20)



So, before going to the analysis first let us consider this second order ordinary differential equation. So, this is your second order ordinary differential equation $\frac{d}{dx}\left[p(\text{x})\frac{d\phi_n}{dx}\right] + \left[q(x) + \lambda_n^2 w(x)\right]\phi_n = 0$. So, this is known as Sturm-Liouville equation or Sturm-Liouville boundary value problem.

Here λ_n^2 is known as eigen values and ϕ_n is the solution of this ordinary differential equation, and it is the eigen functions associated with each λ_n. So, if p, q, and w are real and boundary condition at x = a and x = b are homogenous then you will get harmonic solution in homogenous direction. The function w(x) plays a special role and is known as the weighting function.

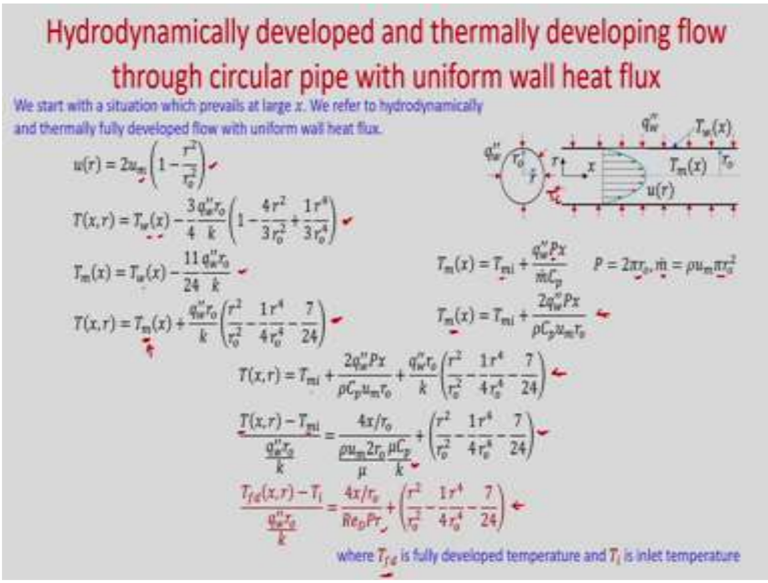
In today's analysis we will use separation of variables method. So, when can we use separation of variables method? If your governing equation is linear and homogenous and in one direction if you have two homogenous boundary conditions and these then you can use the separation of variables method.

And when you will separate the variables then it will be equal to some constant and the sign of constant you need to choose such a way that in homogenous direction you will get harmonic solution. So, we have to choose the λ_n^2 or the constant such a way that in homogenous direction that means, in that direction where you have two homogenous boundary condition, it should give harmonic solution.

So, you can see. So, if you can resemble your governing equation with the Sturm-Liouville equation and p, q, w are real and boundary condition at x = a and x = b are homogenous, then you will get a harmonic solution in the homogenous direction.

One important property of this Sturm-Liouville equation is orthogonality. So, you can see two functions ϕ_n and ϕ_m are orthogonal to each other in the range a, b with respect to weighting function w(x) if $\int_a^b \phi_n(x)\phi_m(x)dx = 0$ for n ≠ m. So, now, for n ≠m these integral will be 0. So, it will be used to find the constant when will use the separation of variables method.

(Refer Slide Time: 08:18)



Now, first let us consider hydrodynamically and thermally fully developed flow through the circular pipe with uniform wall heat flux boundary condition. Already we have studied it. You know the solutions. Just we will represent this solution again here.

So, you can see that fully developed velocity profile is this one, where u_m is the mean velocity or average velocity the temperature T(x,r) which is your fully developed temperature profile $T(x,r) = T_w(x) - \frac{3}{4}\frac{q_w'' r_0}{k}\left(1 - \frac{4}{3}\frac{r^2}{r_0^2} + \frac{1}{3}\frac{r^4}{r_0^4}\right)$. So, this already we have derived the fully developed temperature profile in terms of T_w.

Again, we have derived the mean temperature in terms of wall temperature. So, you can see $T_m(x) = T_w(x) - \frac{11}{24}\frac{q_w'' r_0}{k}$. So, if you substitute T_w in this equation then you will get this equation. So, the temperature profile we have written in terms of mean temperature.

Again, from energy valence we have derived this equation you see $T_m(x) = T_{mi} + \frac{q_w'' P x}{\dot{m} C_p}$.

So, you know that P is the perimeter where in this particular case it is $2\pi r_0$ and $\dot{m}$ is the mass flow rate $\rho u_m \pi r_0^2$, so ρu A. Area is πr_0^2.

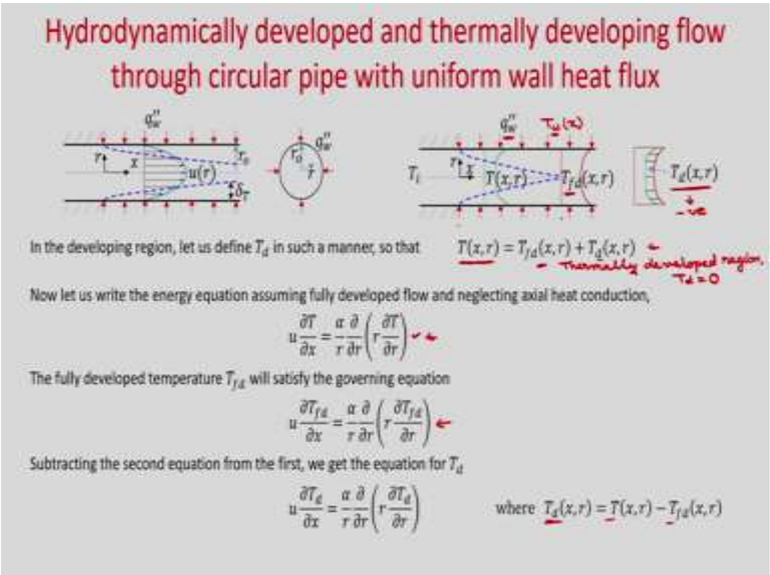
So, these already we have derived from the energy balance, and where T_{mi} is the at inlet you have the mean temperature. So, these P and $\dot{m}$ these value if you put then you can write in this form. Now, these T_m value you put in this equation, see if you put you will get this equation, $T(x,r) = T_{mi} + \frac{2 q_w'' x}{\rho C_p u_m r_0} + \frac{q_w'' r_0}{k}\left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{7}{24}\right)$. So, these T_{mi} if you take in the left hand side and divide by $\frac{q_w'' r_0}{k}$ then you can write in terms of a non-dimensional quantity as, $\frac{T(x,r) - T_{mi}}{\frac{q_w'' r_0}{k}} = \frac{4x/r_0}{\frac{\rho u_m 2 r_0}{\mu}\frac{\mu C_p}{k}} + \left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{7}{24}\right)$.

So, you can see that in this particular case it will become $4x/r_0$ and this after rearrangement you will get Reynolds number into Prandtl number and this is also non-dimensional quantity. So, you can see that we have written this T, what is what is this T? T is your fully developed temperature profile.

So, in a fully developed condition fully developed means hydrodynamically and thermally fully developed. So, $\frac{T_{fd}(x,r)-T_i}{\frac{q_w'' r_0}{k}} = \frac{4x/r_0}{\mathrm{Re}_D \Pr} + \left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{7}{24}\right)$. So, this is the temperature profile, in a fully developed flow, and where T_{fd} we have represented as a fully developed temperature and T_i is the inlet temperature. So, this already we have carried out this analysis earlier. Just we have revisited it.

Now, let us consider thermally developing flow. So, in thermally developing flow we will consider a temperature T_d such a way that your temperature profile at any location whether it is in thermally developing region or fully developed region T will be T_{fd} which is your fully developed temperature profile plus T_d.

(Refer Slide Time: 12:09)



So, you can see here this is our problem. Now, in developing region thermally developing region you can see your thermal boundary layer thickness is growing. So, it is your developing region, thermally developing region. So, the temperature profile it has a T(x), T which is function of x and r. This temperature profile we need to find for thermally developing region. But this T is valid for both in thermally developing region as well as thermally developed region.

And we are defining this temperature as T_{fd} which is your fully developed temperature profile plus some temperature T_d. So, which is we will consider only in the developing region and in fully developed region this T_d will become 0, so in fully developed region thermally fully developed region.

So, in thermally developed region this T_d will be 0. So, we are defining T_d in that way. So, you see we need to find T which is valid in thermally developing region, and thermally developed region these T_{fd} we have already derived which is your temperature profile in thermally developed region.

Now, you can see that this red colored profile is your T_{fd} and this your green colored profile this is your T(x,r). So, obviously, you can see your in axial direction in the core region in a thermally developing region you can see that it will be always T_i because that is the temperature inlet temperature. Only temperature is varying inside the thermally thermal boundary layer.

So, in the core region temperature will remain at T_i. At another location if you consider here also it will be T_i, but once it becomes fully developed region then your core temperature will vary. So, obviously, the temperature profile in the thermally developing region will be lesser than the fully developed region.

This is also true for the wall temperature because at the wall temperature you can see this is a constant wall temperature, this is the constant wall heat flux boundary condition. So, T_w will be function of x and along axial direction your T_w will increase.

So, obviously, when it will come to fully developed region, obviously your T_w will be higher than the T_w at developing region. So, we have represented this green colored temperature profile in the developing region, red colored temperature profile in a fully developed region. So, the difference we are representing with T_d. So, you can see this is actually negative.

This is actually negative, but we are considering T which is your temperature profile at any region whether it is thermally developing region or thermally fully developed region is $T(x,r) = T_{fd}(x,r) + T_d(x,r)$. So, we can see that your T_d will be maximum at x = 0; T_d will be maximum at x = 0. Then, it will start decreasing, decreasing, decreasing; once it becomes thermally fully developed region then your T_d will become 0. So, from the high negative value to 0 it will vary in the developing region.

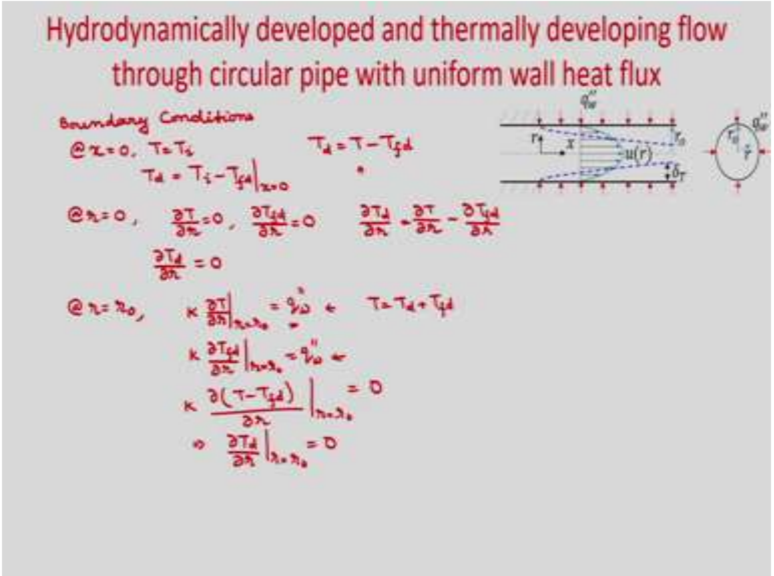
So, you can see that when it is fully developed region T_d will become 0 and T(x, r) will be just T_{fd} . And in developing region T_d will have some negative value, so these negative value will be directed from T_{fd} and you will get the green color this temperature profile.

So, I hope you have understood that how we have defined a temperature in the developing region T_d which is actually negative quantity and these is having a high value at x = 0 at the entrance region and it will decrease along axial direction in the thermally developing zone. After that once it becomes thermally developed region this T_d will become 0 and your temperature profile T will become T_{fd} . So, these T(x,y) we need to find which is valid in both thermally developing region as well as thermally developed region.

So, now let us write the energy equation in booking all the assumptions. So, you can write this is the energy equation $u\frac{\partial T}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$ where ∝ is your thermal diffusivity. So, in fully developed region our temperature profile is T_{fd} . So, obviously, this T_{fd} will satisfy this governing equation, so you can write $u\frac{\partial T_{fd}}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_{fd}}{\partial r}\right)$.

So, now if you subtract this equation from this equations. So, what you will get? You will get $u\frac{\partial T_d}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right)$, so where $T_d(x,r)=T(x,r)-T_{fd}(x,r)$. From this definition you can see your $T_d(x,r)=T(x,r)-T_{fd}(x,r)$. So, you can see that the whatever temperature T_d we have introduced in the thermally developing region it also satisfies the energy equation.

(Refer Slide Time: 18:15)



So, now we need to solve this equation first and we will find the T_d. We know already T_{fd}, so the temperature profile $T(x, y) = T_{fd} + T_d$. Now, m is to find the temperature profile T_d.

So, first let us write the boundary conditions for T_d. So, boundary conditions are at x = 0. So, you have $T = T_i$. And we know $T_d = T - T_{fd}$. We have defined this way. So, if at x = 0, $T = T_i$ we will get $T_d = T - T_{fd}$ at x = 0.

Now, in radial direction the boundary conditions are at r = 0, we can see that your due to the axisymmetric condition as it is thermally and geometrically symmetric you will get maximum or minimum temperature at the central line, where r = 0. So, that means, the temperature gradient with respect to radius will be 0.

So, you will get $\frac{\partial T}{\partial r} = 0$ and in fully developed case also you can write $\frac{\partial T_{fd}}{\partial r} = 0$ because already we have imposed this boundary condition. So, you can see from here you can write $\frac{\partial T_d}{\partial r} = \frac{\partial T}{\partial r} - \frac{\partial T_{fd}}{\partial r}$. So, if $\frac{\partial T}{\partial r} = 0$ and $\frac{\partial T_{fd}}{\partial r} = 0$ then obviously, $\frac{\partial T_d}{\partial r} = 0$, so you can write $\frac{\partial T_d}{\partial r} = 0$ at r = 0.

And at the wall at r = r_0, we have the heat flux boundary condition. So, you can see $q_w^{"}$ we have taken in the negative radial direction. So, what will be your $q_w^{"}$? So, it will be $K\frac{\partial T}{\partial r}$, we are not writing minus sign because $q_w^{"}$ sign is in the negative to radial direction that is why it is positive $K\frac{\partial T}{\partial r}|_{r=r_0} = q_w^{"}$. So, this is your for $\frac{\partial T}{\partial r}$.

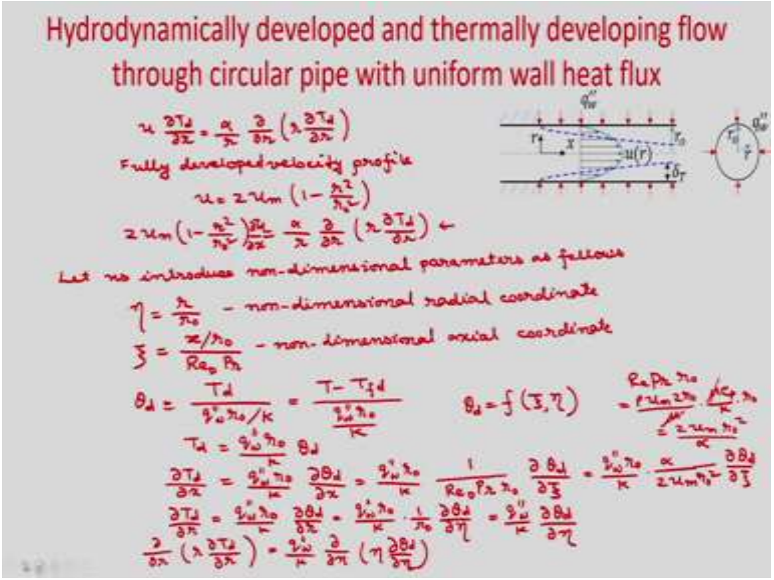
So, now what is T? T = T_d + T_{fd}. And for a fully developed boundary conditions also we have applied this wall heat plus boundary condition. So, you will get $K\frac{\partial T}{\partial r}|_{r=r_0} = q_w^{"}$. So, now, if you subtract, this two, so what you will get?

If you subtract then you will get in the fully developed case where thermally and hydrodynamically fully developed region your temperature boundary condition at r = r_0 will be $K\frac{\partial T_{fd}}{\partial r}|_{r=r_0} = q_w^{"}$ because already we have solved for these boundary condition, right.

So, now, if you subtract these equation from this equation what you will get? $K\frac{\partial (T-T_{fd})}{\partial r}|_{r=r_0} = 0$, right. And what is T -$T_{fd}$? It is nothing, but T_d. So, that means, you will get $\frac{\partial T_d}{\partial r}|_{r=r_0} = 0$.

So, you see in the radial direction at r = and r = r_0 your temperature gradient of T_d =0, so that means, both the boundary conditions are homogenous, that means, it is the homogenous directions. So, when you will use separation of variables method, we have to choose the sign of the constants such a way that you will get the harmonic solution in r direction.

(Refer Slide Time: 23:13)



So, now our governing equation is $u\dfrac{\partial T_d}{\partial x}=\dfrac{\alpha}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T_d}{\partial r}\right)$, because we need to find this value of T_d, right. Because T_{fd} is known if T_d you can find then the temperature profile in general you can write $T_{fd} + T_d$.

In this case, we are using hydrodynamically developed flow, that means, u you know. So, u you can write $u=2u_m\left(1-\dfrac{r^2}{r_0^2}\right)$. So, you can write fully developed velocity profile, you know $u=2u_m\left(1-\dfrac{r^2}{r_0^2}\right)$. So, you put it here, so you will get,

$$2u_m\left(1-\frac{r^2}{r_0^2}\right)\frac{\partial T_d}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right).$$

So, now let us define some non-dimensional parameters. So, we will use the radial direction $\eta=\dfrac{r}{r_0}$. So, this is your non-dimensional radial direction and the non-dimensional axial direction $\zeta=\dfrac{x/r_0}{\mathrm{Re}_D \Pr}$.

So, let us introduce non-dimensional parameters as follows. So, $\eta = \frac{r}{r_0}$, so you can see this is your non-dimensional radial coordinate. Then, $\zeta = \frac{x/r_0}{\mathrm{Re}_D \Pr}$. So, this is your non-dimensional axial coordinate. And the non-dimensional temperature $\theta_d = \frac{T_d}{q_w^{"} r_0 / K}$ which is $\frac{T - T_{fd}}{q_w^{"} r_0 / K}$.

So, now if you put all this values, so you can see you can write $T_d = \frac{q_w^{"} r_0}{K}\theta_d$. So, you can write $\frac{\partial T_d}{\partial x} = \frac{q_w^{"} r_0}{K}\frac{\partial \theta_d}{\partial x}$. So, you see $\theta_d = f(\zeta,\eta)$.

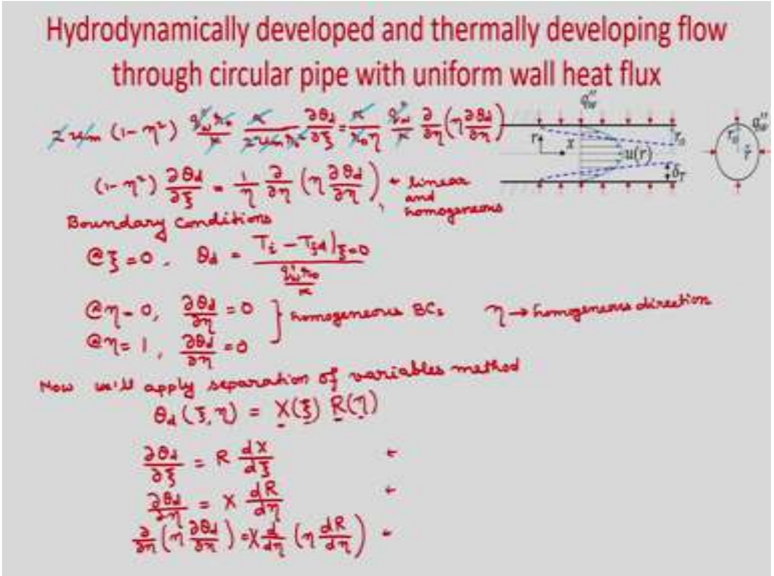
Then, you can write this x coordinate, in terms of ζ. So, you can write $\frac{\partial T_d}{\partial x} = \frac{q_w^{"} r_0}{K}\frac{1}{\mathrm{Re}_D \Pr r_0}\frac{\partial \theta_d}{\partial \zeta}$. So, $\mathrm{Re}_D \Pr r_0 = \frac{\rho u_m 2 r_0}{\mu}\frac{\mu C_p}{K} r_0$.

So, you can see this μ will get cancel. So, you can write $\frac{\partial T_d}{\partial x} = \frac{q_w^{"} r_0}{K}\frac{\alpha}{2 u_m r_0^2}\frac{\partial \theta_d}{\partial \zeta}$.

Similarly, you can write $\frac{\partial T_d}{\partial r} = \frac{q_w^{"} r_0}{K}\frac{\partial \theta_d}{\partial r}$.

So, now, this r will put as r_0 η, so you can write $\frac{\partial T_d}{\partial r} = \frac{q_w^{"} r_0}{K}\frac{1}{r_0}\frac{\partial \theta_d}{\partial \eta}$. So, this r_0 r_0 if you cancel it, then you will get $\frac{\partial T_d}{\partial r} = \frac{q_w^{"}}{K}\frac{\partial \theta_d}{\partial \eta}$. Similarly, if you write $\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right) = \frac{q_w^{"}}{K}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta_d}{\partial \eta}\right)$.

(Refer Slide Time: 29:36)



So, all the derivatives we have found. Now, all these you put it in the governing equation. And write it as, $2u_m\left(1-\eta^2\right)\frac{q_w^{"}r_0}{K}\frac{\alpha}{2u_m r_0^2}\frac{\partial\theta_d}{\partial\zeta}=\frac{\alpha}{r_0\eta}\frac{q_w^{"}}{K}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta_d}{\partial\eta}\right)$. After substituting all the terms.

So, now, you can see here, this α, this α you can cancel u_m, u_m, 2, 2, this r_0 here you will get one and another these r_0. So, you can cancel $q_w^{"}$, $q_w^{"}$,K and K. So, you can write the final equation as $\left(1-\eta^2\right)\frac{\partial\theta_d}{\partial\zeta}=\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta_d}{\partial\eta}\right)$.

So, now, we will use separation of variables method. So, what is separation of variables method? We will write the solution of θ_d as a product of two individual solution x and r, where each solution is function of one coordinate only. So, like we are defining θ_d as product of x and r, where x is function of ζ only and r is function of η only.

So, before going to that let us write the boundary conditions. Boundary conditions at ζ=0 that means, x = 0. So, $\theta_d=\frac{(T_i-T_{fd})\big|_{\zeta=0}}{\frac{q_w^{"}r_0}{K}}$ and at η=0 you have $\frac{\partial\theta_d}{\partial\eta}=0$ and at η=1 because r = r_0. So, η=1 and $\frac{\partial\theta_d}{\partial\eta}=0$.

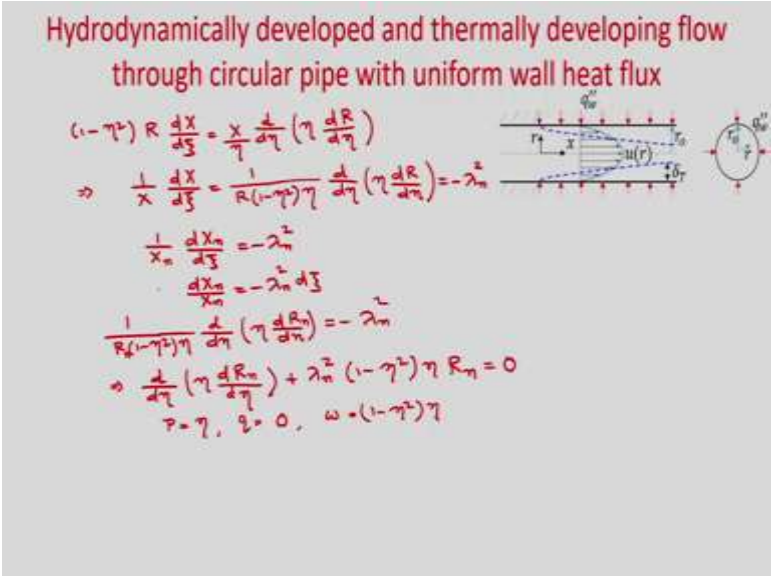
Now, we will apply separation of variables method. So, we will write the solution θ_d which is function of ζ and η as product of two individual solution X which is function of ζ and R which is function of η only. So, you can see you are writing $\theta_d(\zeta,\eta) = X(\zeta)R(\eta)$.

So, when can we use separation of variables method? If the governing equation is linear and homogenous, and in one direction you have two homogenous boundary conditions then you can use separation of variables method. So, you can see our governing equation, these equation is linear and homogenous, and in η direction you have homogenous boundary conditions. That means, η is your homogenous direction. So, η is your homogenous direction.

So, we can use separation of variables (Refer Time: 33:52) and we are writing the solution theta d as product of two individual solution X and R, where X is function of ζ only and R is function of η only. So, now, let us write $\frac{\partial\theta_d}{\partial\zeta} = R\frac{dX}{d\zeta}$. So, now, this is your ordinary derivative were writing because X is function of ζ only.

Similarly, you can write $\frac{\partial\theta_d}{\partial\eta} = X\frac{dR}{d\eta}$. So, similarly if you write , $\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta_d}{\partial\eta}\right) = X\frac{d}{d\eta}\left(\eta\frac{dR}{d\eta}\right)$ and one X will be there. So, all these derivatives you put it in this equation.

(Refer Slide Time: 35:15)



So, what you will get? So, you will get, $\left(1-\eta^2\right) R\frac{dX}{d\zeta} = \frac{X}{\eta}\frac{d}{d\eta}\left(\eta\frac{dR}{d\eta}\right)$. So, you see this equation.

Left hand side X is function of ζ only. That means, the whole term left hand side term is function of ζ only. Right hand side if you see, R is function of η only and, right hand side all the terms are function of η only. So, left hand side is function of ζ, right hand side function of η.

So, this will be equal to some constant. Because left hand side is your function of ζ, right hand side function of η equal to some constant, and that constants sign you have to choose such a way that in η direction you should get the governing equation such a way that you will get the harmonic solution of that governing equation. And obviously, if you can write in R direction the governing equation has Sturm-Liouville equation type, then you will get the harmonic solution in η direction.

So, we will choose the constant as $-\lambda_n^2$, where λ_n^2 is your eigen values. So, you can write $\frac{1}{X_n}\frac{dX_n}{d\zeta} = -\lambda_n^2$. So, you can see its solution will be exponential.

The other term you will get $\frac{1}{R_n\left(1-\eta^2\right)\eta}\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right)=-\lambda_n^2$. So, you can see this will be your $\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right)+\lambda_n^2\left(1-\eta^2\right)\eta R_n=0$.

Now, you see this second order ordinary differential equation. You compare it with the Sturm-Liouville equation. So, you can see your if you compare with the Sturm-Liouville equation, p =η, q will be 0, and the weighting function, $w=\left(1-\eta^2\right)\eta$. And, ok, these are real and in eta direction you have two homogenous boundary conditions so obviously, you will get the harmonic solution in eta direction.

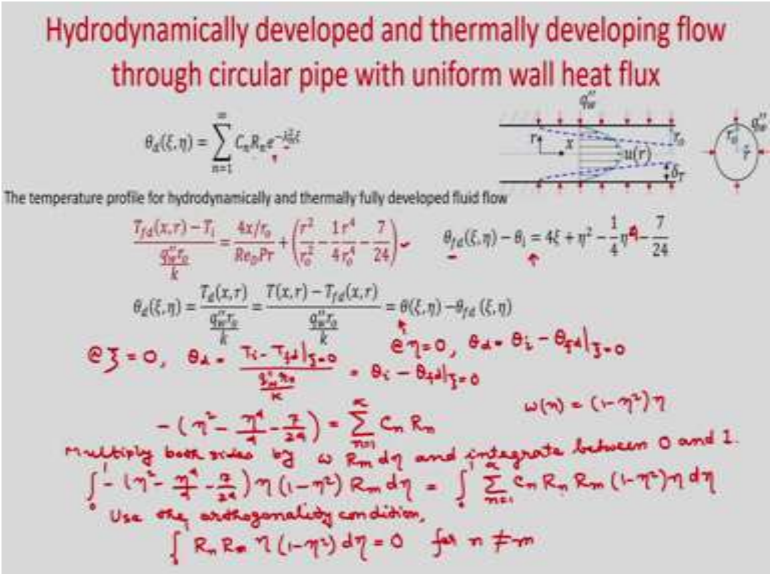
So, if you see this second order differential equation, it is very difficult to solve. So, you can use some numerical technique to solve this second order ordinary differential equation. Once you get the solution which is at the eigen function of this equation R_n, then you can write the product of this two solution one you will get from the X another solution from R then this product of this two solution will be your the temperature profile θ_T. But for different values of eigen values λ_n^2 you will get different solution X_n and different solution R_n.

So, you need to find what is the value of λ_n, eigen values. Numerically, if you solve these equation, with the proper boundary conditions then you will get the eigen functions of these ordinary differential equation R_n, then you can write the solution θ_d, as summation of all the product of solution X and R because for different values of λ_n, you will get different solution X into R. And as it is a linear solution linear governing equation.

So, you can super impose all the solutions for different values of λ_n, and if you super impose, that means you are adding all the solutions. So, it is possible as you have the governing equation linear. Because you have a linear governing equation, so you can super impose all the solutions.

So, now let us write the final solution θ_d as product of X_n R_n, where X_n is the solution where you will get in the exponential form, R_n you need to find solving this second order ordinary differential equation. Then, you can sum it from n is equal to 1 to ∞ because you will get different values of λ_n and you will get the different solutions.

(Refer Slide Time: 42:19)



So, assuming that you would solution of this second order differential equation is R_n which are eigen function and λ_n^2 is eigenvalues then you can write the final solution as. So, you can write the final solution $\theta_d\left(\zeta,\eta\right)=\sum_{n=1}^{\infty}C_nR_ne^{-\lambda_n^2\zeta}$ these are the eigen function of this second order differential equation, and this is the solution, this is the solution from the η direction. So, there are first equation you can see.

So, if you solve this equation. So, you will get X_n is equal to some constant into $\theta_d\left(\zeta,\eta\right)=\sum_{n=1}^{\infty}C_nR_ne^{-\lambda_n^2\zeta}$. So, this product if you write R_n if you find and X_n if you find then you will get the final solution θ_d as product of two individual solutions and as it is linear equation you can super impose all the solution n = 1 to ∞ for different values of λ_n. So, R_n you need to find. So, this is the eigen functions of this second order differential equation.

So, now, we need to apply the boundary conditions, to find these constant C n as well as you need to find λ_n^2 because that is also unknown. So, you can see the temperature profile for hydrodynamically and thermally fully developed fluid flow we have written like this, we have already derived today. So, in non-dimensional form if you write, so $\frac{T_{fd}}{\frac{q_w'' r_0}{k}}$ you can write θ_{fd}, and $\frac{T_i}{\frac{q_w'' r_0}{k}}$ you can write θ.

So, $\theta_{fd}(\zeta,\eta)-\theta_i = 4\zeta+\eta^2-\frac{1}{4}\eta^4-\frac{7}{24}$.

So, now theta d we have defined $\theta_{fd}(\zeta,\eta)=\frac{T_d(x,r)}{\frac{q_w'' r_0}{k}}$ and $T_d(x,r)=T(x,r)-T_{fd}(x,r)$.

$\theta_{fd}(\zeta,\eta)=\frac{T(x,r)-T_{fd}(x,r)}{\frac{q_w'' r_0}{k}}$.That means, $\theta_d(\zeta,\eta)=\theta(\zeta,\eta)-\theta_{fd}(\zeta,\eta)$. So, now, apply the boundary condition at η= 0. At η= 0 we know $\theta_d=\frac{T_i-T_{fd}\big|_{\zeta=0}}{\frac{q_w'' r_0}{k}}$. So, this we already we know. So, we can see that.

So, this is your θ_d and you can see here. So, from these; if you put at η=0 that means, this is your at x = 0, right. So, you can write θ_d as; what is this? This is nothing, but $\theta_d=\theta_i-\theta_{fd}\big|_{\zeta=0}$. So, you can write it as $\theta_d=\theta_i-\theta_{fd}\big|_{\zeta=0}$.

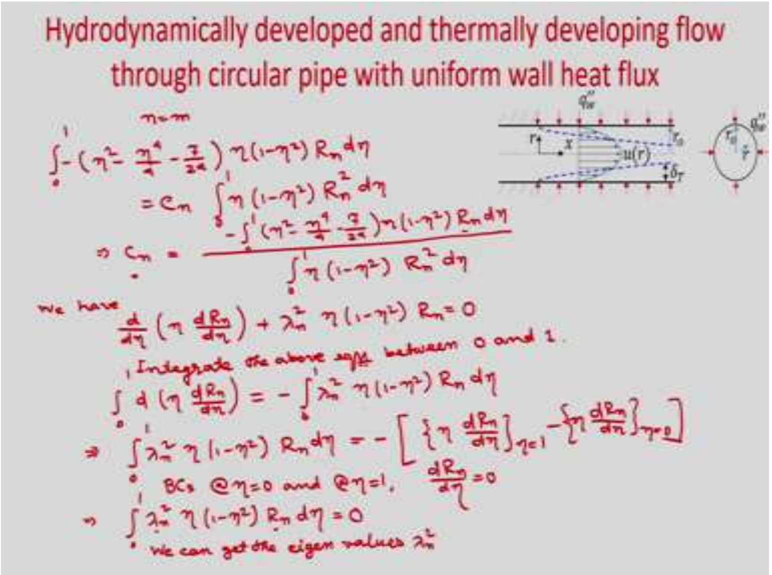
So, you can see from this equation, from this equation if ζ=0 if you put and if you reverse it, so you will get this as, $-\left(\eta^2-\frac{\eta^4}{4}-\frac{7}{24}\right)=\sum_{n=1}^{\infty}C_nR_n$ because at ζ=0, x = 0. So, $e^{-\lambda_n^2\zeta}=1$.

So, now we need to find the constant C_n. Now, we will invoke the orthogonality constant. So, now, we will invoke the orthogonality condition what we discussed in the Sturm-Liouville equation. So, what will do now multiply both side by, weighting function $wR_m d\eta$ and integrate between 0 and 1. So, this w is the weighting function.

What is this? You can see here, $\int_0^1 -\left(\eta^2-\frac{\eta^4}{4}-\frac{7}{24}\right)\eta\left(1-\eta^2\right)R_m d\eta=\int_0^1\sum_{n=1}^{\infty}C_nR_nR_m\left(1-\eta^2\right)\eta d\eta$.

So, if you remember the orthogonality condition for n $\neq$m, $\int_0^1 R_n R_m \eta\left(1-\eta^2\right)d\eta = 0$ for n$\neq$m, C_n is constant. So, C_n you take outside. So, obviously, for n = m this will become 0, so only for n = m only one term will remain.

(Refer Slide Time: 49:08)



So, if you write for n = m if you keep then you can write $\int_0^1 -\left(\eta^2 - \frac{\eta^4}{4} - \frac{7}{24}\right)\eta\left(1-\eta^2\right)R_n d\eta = C_n \int_0^1 \eta\left(1-\eta^2\right)R_n^2 d\eta$; n = m we have put.

So, now, we can find the constant $C_n = \dfrac{\int_0^1 -\left(\eta^2 - \frac{\eta^4}{4} - \frac{7}{24}\right)\eta\left(1-\eta^2\right)R_n d\eta}{\int_0^1 \eta\left(1-\eta^2\right)R_n^2 d\eta}$. So, you can see that, if R_n is known, which is the solution, you will get from the second order differential equation then you will be able to integrate this and you can find the constant C_n.

Now, we need to find the value of λ_n. So, first we will start from the governing equation. So, what is your governing equation? We have $\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right) + \lambda_n^2\eta\left(1-\eta^2\right)R_n = 0$. So, this is the equation we have derived you can see. So, this is the equation.

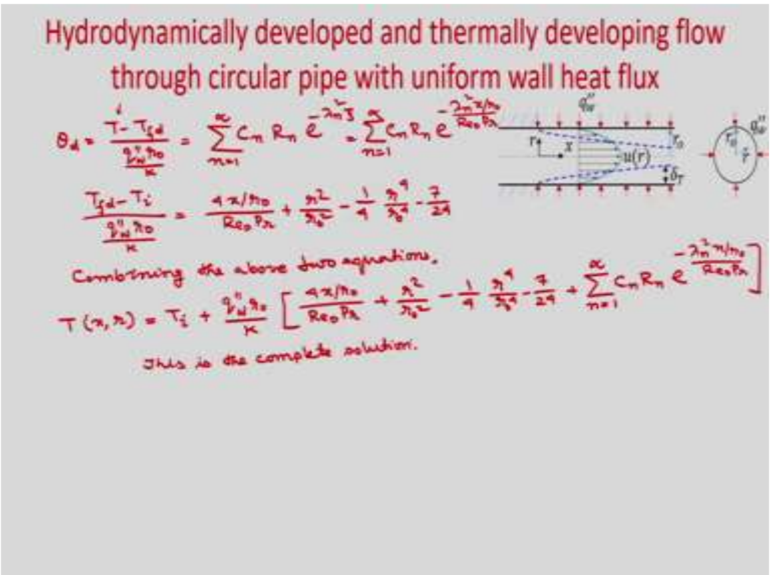
So, now you integrate $\int_0^1 d\left(\eta \frac{dR_n}{d\eta}\right) = -\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta$.

So, this equation you just write in the left hand side, so it will be $\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta = -\left[\left\{\eta \frac{dR_n}{d\eta}\right\}_{\eta=1} - \left\{\eta \frac{dR_n}{d\eta}\right\}_{\eta=0}\right]$.

Now, you recall the boundary conditions. At η=0 and η=1 you have $\frac{\partial \theta_d}{\partial \eta} = 0$. And θ_{fd} is a product of X and R, but R is only function of η. So, your boundary condition at η= 0 ,1, you will get $\frac{dR_n}{d\eta}$ =0. So, that means, boundary condition at η= 0 and at η= 1 you will get $\frac{dR_n}{d\eta}$ =0. So, you can see that these two terms will become 0. So, hence you will get $\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta = 0$.

So, from here we can get the eigen values λ_n^2 because for you can see this is the condition. So, this R_n if you find, so for different λ_n you will get different R_n. So, R_n if you know then from here it should to satisfy these because, to right hand side will be 0, so to satisfy this you can find the value of λ_n^2 from this equation.

(Refer Slide Time: 53:24)



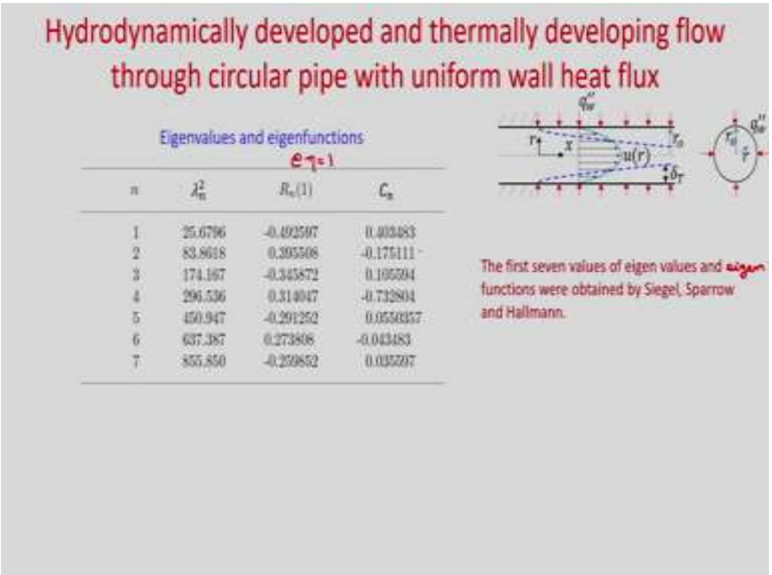
So, now if you put the $\theta_d = \dfrac{T - T_{fd}}{\dfrac{q_w^{"} r_0}{k}} = \sum_{n=1}^{\infty} C_n R_n e^{-\lambda_n^2 \zeta} = \sum_{n=1}^{\infty} C_n R_n e^{-\frac{\lambda_n^2 \, x/r_0}{\mathrm{Re}_D \Pr}}$. So, you see now we are interested to find theta which is the temperature distribution in general.

So, that now you can write $\dfrac{T_{fd} - T_i}{\dfrac{q_w^{"} r_0}{k}} = \dfrac{4\, x/r_0}{\mathrm{Re}_D \Pr} + \dfrac{r^2}{r_0^2} - \dfrac{1}{4}\dfrac{r^4}{r_0^4} - \dfrac{7}{24}$. So, you combining these two you can see.

So, this if you add it then $T(x,r) = T_i + \dfrac{q_w^{"} r_0}{k}\left[\dfrac{4\, x/r_0}{\mathrm{Re}_D \Pr} + \dfrac{r^2}{r_0^2} - \dfrac{1}{4}\dfrac{r^4}{r_0^4} - \dfrac{7}{24} + \sum_{n=1}^{\infty} C_n R_n e^{-\frac{\lambda_n^2 \, x/r_0}{\mathrm{Re}_D \Pr}}\right]$.

So, you can see this is the final temperature distribution T(x, r), which we are interested to find and this T(x, r) is value in both thermally developing region as well as fully developed region and you can write in terms of T_i. So, we can see this is the equation and this is the complete solution.

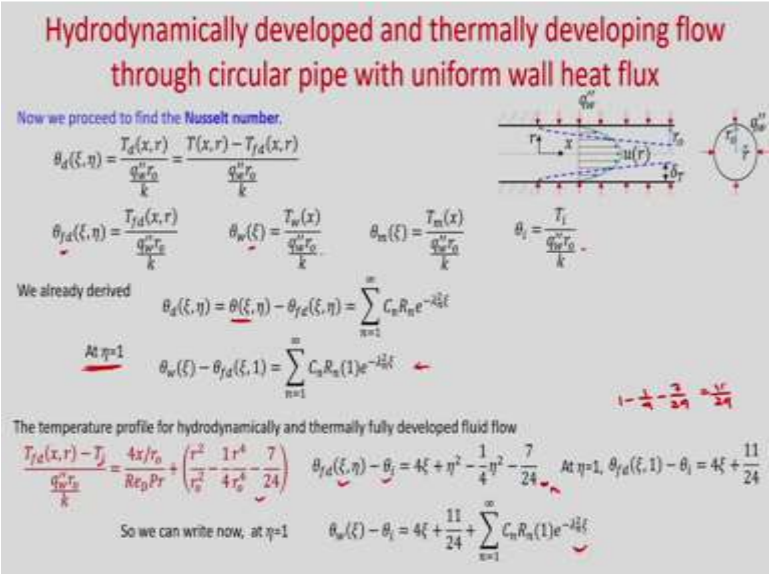
(Refer Slide Time: 56:24)



So, now if you see this eigen functions, you will get different eigenfunction, but at η = 1, $R_n(1)$, this is found by this first 7 values of eigen values and functions these are eigen functions, where obtained by Siegel Sparrow and Hallman. So, you can see from this table.

So, for different values of n, n= 1 to 7 these are the λ_n^2 value and $R_n(1)$ that means, at η=1. So, that at means at the boundary, $R_n(1)$, so we can see this is the first value is negative and if R_n is negative then θ_d will be negative then you can see that it is the negative quantity T_d, whatever we defined. So, actually in the developing region you are calculating the temperature distribution which we actually subtracting this T_d from the T_{fd} part, and this is negative coming. And the constant C_n, you can see these are the C_n.

(Refer Slide Time: 57:29)

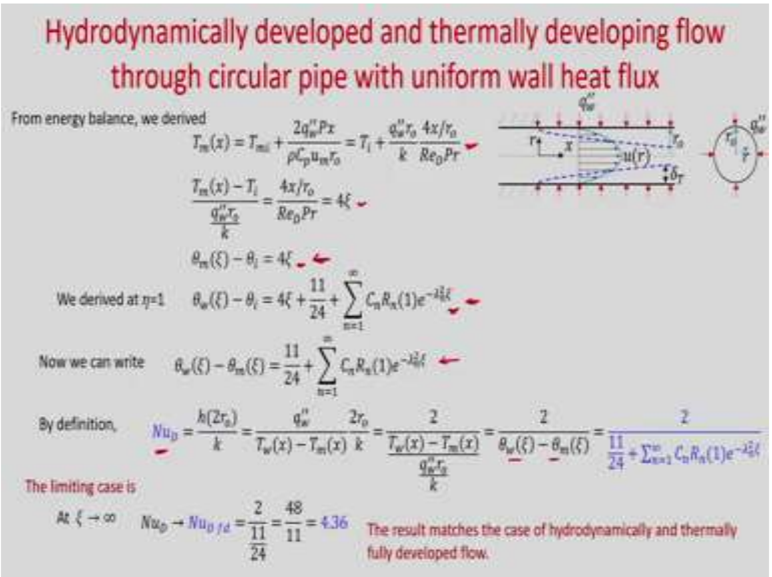


Now, we need to find the Nusselt number. So, you can see that whatever $\theta_d(\zeta,\eta)=\dfrac{T_d(x,r)}{\dfrac{q_w'' r_0}{k}}$. And similarly, fully developed region $\theta_{fd}(\zeta,\eta)=\dfrac{T_{fd}(x,r)}{\dfrac{q_w'' r_0}{k}}$ and mean temperature $\theta_m(\zeta)=\dfrac{T_m(x)}{\dfrac{q_w'' r_0}{k}}$. So, we already derived this θ_d, right. So, this is just $\sum_{n=1}^{\infty} C_n R_n e^{-\lambda_n^2 \zeta}$.

Now, at η=1, at the boundary what is that? θwill be θ_w. So, $\theta_w(\zeta)-\theta_{fd}(\zeta,1)=\sum_{n=1}^{\infty} C_n R_n(1) e^{-\lambda_n^2 \zeta}$ .So, and also we have found the temperature profile for fully developed flow. So, this is the temperature profile.

So, you can write $\theta_{fd}(\zeta,\eta)-\theta_i$ in terms of non-dimensional coordinate, η and ζ. So, at η=1 if you put, so it will get $\frac{T_{fd}-T_i}{\frac{q_w^{"}r_0}{k}}=\frac{4\,{x}/{r_0}}{\mathrm{Re}_D\,\mathrm{Pr}}+\left(\frac{r^2}{r_0^2}-\frac{1}{4}\frac{r^4}{r_0^4}-\frac{7}{24}\right)$. So, this you will get $\theta_{fd}(\zeta,1)-\theta_i=4\zeta+\frac{11}{24}$. So, we can write now at η=1 this θ_w. So, you can see this if you add this two equations these equation and this equation. So, it will be $\theta_w(\zeta)-\theta_i=4\zeta+\frac{11}{24}+\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2\zeta}$.

(Refer Slide Time: 59:29)



Once you get it, so now, you see the from the energy balance T_m we have found like this. So, $\frac{T_m(x)-T_i}{\frac{q_w^{"}r_0}{k}}=\frac{4\,{x}/{r_0}}{\mathrm{Re}_D\,\mathrm{Pr}}=4\zeta$. So, from there you can find. So, in non-dimensional form $\theta_m(\zeta)-\theta_i=4\zeta$.

And we derive that η=1 already in last slide the $\theta_w(\zeta)-\theta_i=4\zeta+\frac{11}{24}+\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2\zeta}$. So, now, we can write, so you can see if you if you subtract the these equation, from this equation subtract this equation from this equation, then you will get, $\theta_w(\zeta)-\theta_m(\zeta)=\frac{11}{24}+\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2\zeta}$.

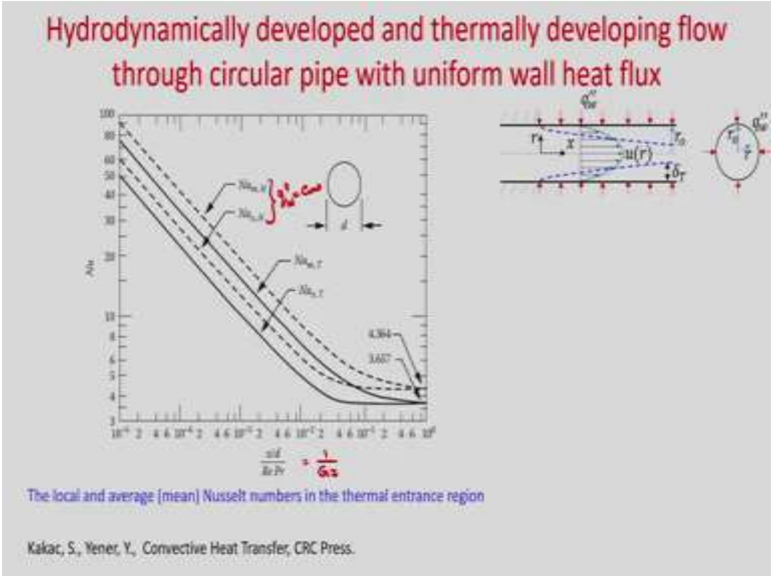
So, now by definition what is the Nusselt number? Thus, $Nu_D=\frac{h(2r_0)}{k}$. So, $\frac{q_w^{"}}{T_w(x)-T_m(x)}\frac{2r_0}{k}$.

Now, we can write $\frac{2}{\frac{T_w(x)-T_m(x)}{\frac{q_w^{"}r_0}{k}}}$ and this is nothing, but $\theta_w(\zeta)-\theta_m(\zeta)$ and these already we have found. So, Nusselt number for this particular case you can see it is $\frac{2}{\frac{11}{24}+\sum_{n=1}^{\infty}C_nR_n(1)e^{-\lambda_n^2\zeta}}$. So, you can see this is the Nusselt number in general. So, these expression is valid for both thermally developing region and fully developed region.

So, let us check whether it is true for the fully developed region or not. So, when you will get the fully developed region when x $\rightarrow\infty$, that means, $\zeta\rightarrow\infty$. So, at $\zeta\rightarrow\infty$, so Nu_D will become fully developed Nusselt number and you can see that as $\zeta\rightarrow\infty$ this term will become 0.

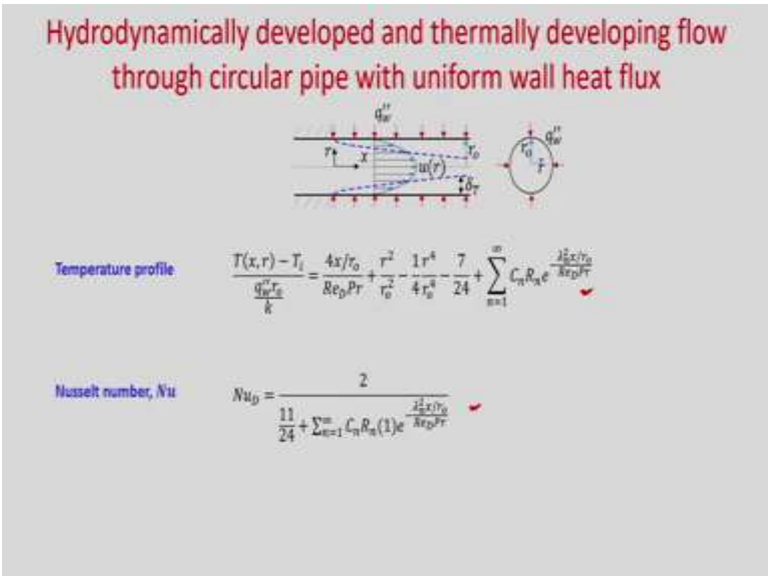
So, this term will become 0, so you will get $Nu_{Dfd}=\frac{2}{\frac{11}{24}}=\frac{48}{11}=4.36$ and these you have already direct, right for hydrodynamically and thermally fully developed region. So, this is true.

(Refer Slide Time: 61:49)



And the temperature profile for this Nusselt number, you can see local and average Nusselt number for thermal intense region. So, this is your $q_w^{"}$ is constant, ok. So, for this case you can see $\frac{x/d}{\text{Re}\,\text{Pr}} = \frac{1}{Gz}$. If you plot, so Nusselt number it will decrease in the developing region and once it becomes fully developed region it will become constant. So, you can see that for high Graetz number you can get the developing region.

(Refer Slide Time: 62:25)



So, let us conclude. So, today we considered hydrodynamically developed and thermally developing flow through circular pipe with uniform wall heat plus boundary condition. And we started with the fully developed boundary condition and we have derived already the temperature profile in that region, and we defined one temperature T_d such a way that the temperature profile $T = T_d + T_{fd}$, and T_d is a quantity negative quantity which you subtract from the fully developed temperature profile.

Then, we used the equation for T_d, and with the boundary condition we got the governing equation and we applied the separation of variables method because the governing equation is linear and homogenous. So, we use the separation of variables method and using separation of variables method we use the orthogonality constant to find the value of C_n.

So, finally, we derived the temperature profile as these which is valid for both in thermal region, thermal developing region and fully developed region and the Nusselt number is these which is also valid for thermally developing region and fully developed region. And you have seen that at $\zeta \to \infty$, that means, in a fully developed region it gives the Nusselt number as $Nu_{Dfd} = \frac{48}{11}$ which you already derived. And it is true for the fully developed region.

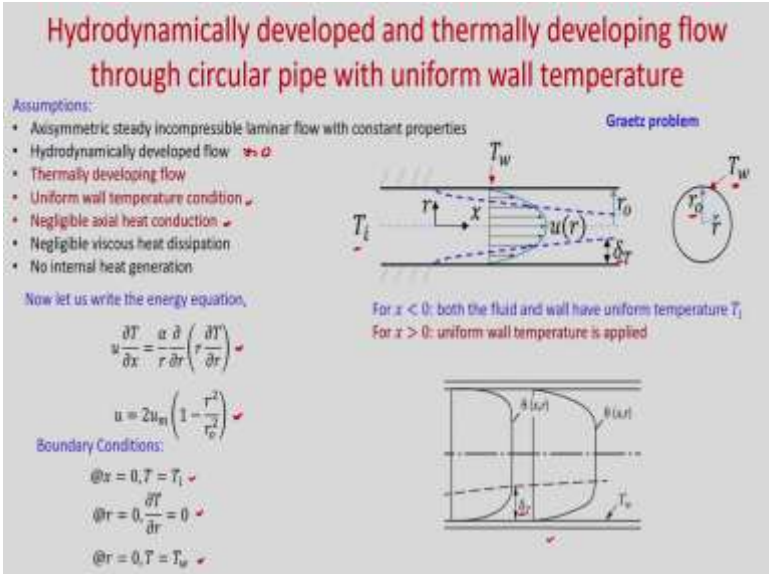
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 07
Convection in Internal Flows – III
Lecture – 23
Hydrodynamically developed and thermally developing flow through circular pipe with uniform wall temperature

Hello, everyone. Today, we will consider Hydrodynamically developed and thermally developing fluid flow through circular pipe with uniform wall temperature. So, this is also known as Graetz problem.

(Refer Slide Time: 00:45)



So, you can see that we are considering hydrodynamically developed and thermally developing; that means, you can see that your thermal boundary layer is growing. This is the thermal boundary layer thickness δ_T and in developing region we need to find what is the temperature distribution.

You can see that your temperature at inlet is T_i and up to x = 0 from the inlet it is adiabatic. So, the wall temperature and free temperature will remain at temperature T_i, but from x = 0 uniform wall temperature is applied at the wall. So, you can see at the wall T_w is applied. So, there will be formation of thermal boundary layer. In this region we are considering fully developed hydrodynamically developed flow; however, you have thermally developing temperature profile.

So, here you can see the radius of the tube is r_0 and r is measured from the central line. So, these are the assumptions already we have discussed only one important assumptions we are taking that it is uniform wall temperature condition and you are neglecting axial heat conduction.

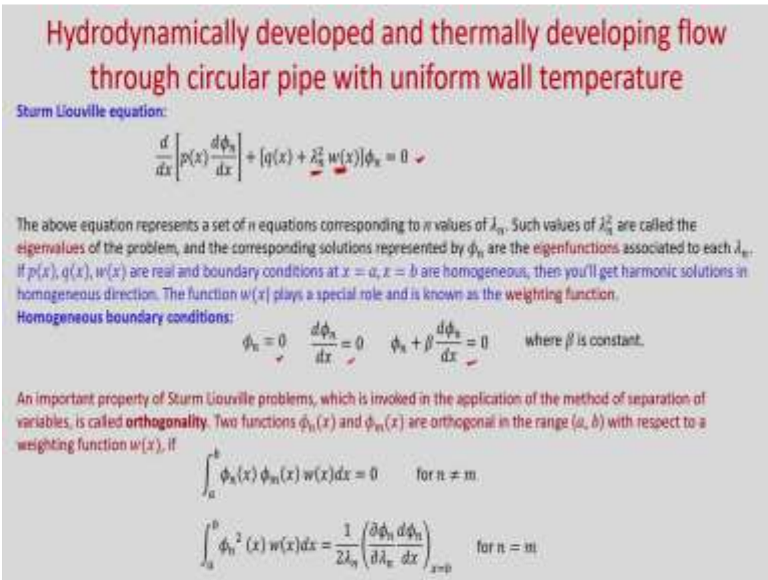
So, with that boundary conditions, you can see that your temperature the energy equation will be $u\frac{\partial T}{\partial x}=\frac{\alpha}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r})$ because we have neglected the axial heat conduction and as it is hydrodynamically developed flow so, v is 0.

So, your one convection term related to $v\frac{\partial T}{\partial y}=0$ and fully developed velocity profile is this one which is parabolic and boundary conditions you can see that x = 0, you have temperature $T = T_i$ at r = 0. So, it is axisymmetric flow. So, $\frac{\partial T}{\partial r}=0$ and at r = 0 we have imposed uniform wall temperature, so, T will be T_w.

In this figure you can see along the axial direction if you go, so there will be change in the temperature profile. The temperature variation will occur only inside the thermal boundary layer and in the core region temperature will remain at temperature T_i. However, when $x \to \infty$; that means, thermally developed flow, then your axial or central line temperature will vary with x.

Now, today we will find the temperature distribution which is valid in both developing region, as well as developed region as well as we will find the Nusselt number.

(Refer Slide Time: 04:01)



Again let us discuss about the Sturm Liouville equation in last class we have already discussed in details. So, this is the second order ordinary differential equation $\frac{d}{dx}\left[p(x)\frac{d\phi_n}{dx}\right] + \left[q(x) + \lambda_n^2 w(x)\right]\phi_n = 0$, where λ_n^2 is the eigen values and ϕ_n is the solution of this ordinary differential equation and these are the eigen functions associated with each λ_n and w(x) is the weighting function.

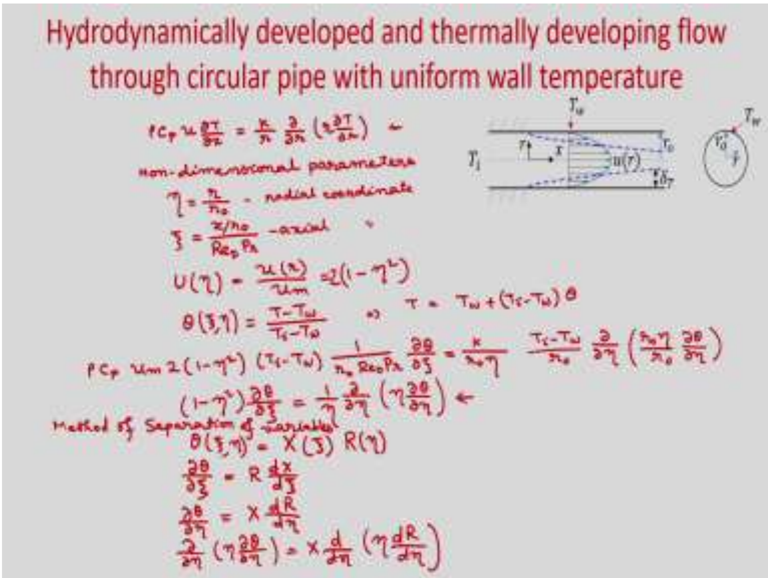
So, it is very important in this particular case when will apply the orthogonality condition it is required. In this analysis also we will use method of separation of variables. So, when can we use the separation of variables method? When the governing equation is linear and homogenous, and in one direction you have two homogenous boundary conditions. And, in homogenous direction it should give the characteristic value problem or harmonic solution in homogenous direction.

So, we can see what is homogenous direction? If the value of that variable is 0 or the gradient of that variable is 0 or combination of this two is 0. So, you can see homogenous boundary condition ϕ_n =0 or $\frac{d\phi_n}{dx}$ =0 or combination of this two where β is the constant is 0. So, in this form if you get the boundary condition then these are the homogenous boundary conditions.

So, we can use separation of variables method if in the homogenous direction if you get characteristic value problem. Now, you see the boundary condition in today's problem we have seen that in r direction it is we can make as homogenous using convenient non-dimensional quantity.

So, one important property of this Sturm Liouville problem is orthogonality. So, we can see the two function ϕ_n and ϕ_m are orthogonal in the range a, b with respect to weighting function w(x) if $\int_a^b \phi_n(x)\phi_m(x)w(x)dx = 0$ for n $\neq$m. And today, we will use another important property where n = m you will get $\int_a^b \phi_n^2(x)w(x)dx = \frac{1}{2\lambda_n}\left(\frac{\partial\phi_n}{\partial\lambda_n}\frac{d\phi_n}{dx}\right)_{x=b}$.

(Refer Slide Time: 06:33)



So, first let us define some suitable non dimensional quantity, so that we get in r direction both boundary conditions as homogenous. So, if you see we have written our energy equation as $\rho C_p u\frac{\partial T}{\partial x} = \frac{K}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$now we are introducing the non-dimensional parameters non-dimensional parameters.

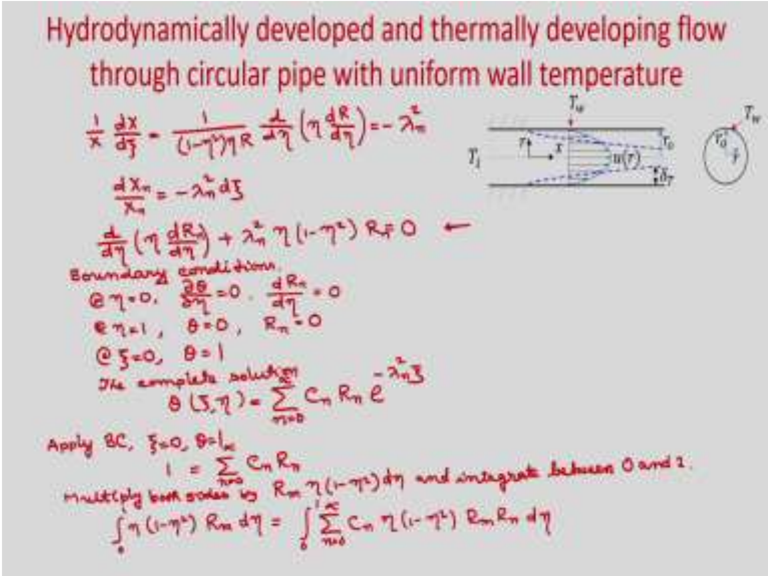
So, $\eta = \frac{r}{r_0}$. So, this is the radial coordinate non-dimensional radial coordinate and we will introduce $\zeta = \frac{x/r_0}{\mathrm{Re}_D \Pr}$. So, this is your axial coordinate. So, we can write the velocity $U(\eta) = \frac{u(r)}{u_m} = 2\left(1-\eta^2\right)$ and we are defining the non-dimensional temperature $\theta\left(\xi,\eta\right) = \frac{T - T_w}{T_i - T_w}$.

Now, you can see using these non-dimensional parameters in r direction at r = r_0 and T=T_w, so, θ will become 0 and at r = 0 , $\frac{d\theta}{d\eta}$ will become 0. So, you will get homogenous direction in r direction or η direction. So, if you put all these in these energy equation what you are going to get? $\rho C_p u_m 2\left(1-\eta^2\right)\left(T_i - T_w\right)\frac{1}{r_0 \mathrm{Re}_D \Pr}\frac{\partial \theta}{\partial \xi} = \frac{K}{r_0\eta}\frac{\left(T_i - T_w\right)}{r_0}\frac{\partial}{\partial \eta}\left(\frac{r_0\eta}{r_0}\frac{\partial \theta}{\partial \eta}\right)$. This we have carried out in last class as well . So, just after rearrangement you can write as $\left(1-\eta^2\right)\frac{\partial \theta}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right)$. So, you carry out this algebra. So, you put the values of Re_D and Pr and you cancel out some parameters, then you will get finally, this equation.

Now, you see this equation is homogenous as well as linear and in r direction, you have two homogenous boundary condition. So, we will be able to use method of separations of variable now we will use separation of variables method where we will find the solution of this equation as theta which is product of two solution x and r, where X is function of ξ only and R is function of η only.

So, we can write θ method of separation of variable . So, $\theta\left(\xi,\eta\right) = X\left(\xi\right)R\left(\eta\right)$, so, now, you can see $\frac{\partial \theta}{\partial \xi} = R\frac{dX}{d\xi}$ you can write $\frac{\partial \theta}{\partial \eta} = X\frac{dR}{d\eta}$ and $\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right) = X\frac{d}{d\eta}\left(\eta\frac{dR}{d\eta}\right)$. So, all these you put in these equation and write.

(Refer Slide Time: 12:03)



Finally, after rearrangement you can write $\frac{1}{X}\frac{dX}{d\xi}=\frac{1}{\left(1-\eta^2\right)\eta R}\frac{d}{d\eta}\left(\eta\frac{dR}{d\eta}\right)$. So, you can see left hand side is function of ξ only and right hand side function of η only. So, this will be equal to some constant and the sign of that constant you have to chose such a way that in r direction which is your homogenous direction you will get characteristic value problem; that means, it is solution will give you harmonic solution.

And, if you can constitute this equation such a way, that in r direction you will get the equation similar to Sturm Liouville equation. So, you will get solution as characteristic value problem. So, we will chose here is equal to $-\lambda_n^2$ because what different values of lambda you will get different solution X and R here λ_n^2 is your eigen values.

So, you can see you can write $\frac{dX_n}{X_n}=-\lambda_n^2 d\xi$. So, the solution you can see this will be exponential right, some constant into $e^{-\lambda_n^2\xi}$. And, the other equation you will get $\frac{d}{d\eta}\left(\eta\frac{dR_n}{d\eta}\right)+\lambda_n^2\eta\left(1-\eta^2\right)R_n=0$.

Now, you compare this equation with the Sturm Liouville equation. So, you can see p =η , q = 0 and weighting function is $\eta\left(1-\eta^2\right)$. So, now, and also in η direction you have two boundary conditions as homogenous. So, boundary conditions if you write so, at η=0 $\frac{\partial \theta}{\partial \eta}$=0.

So, if $\frac{\partial \theta}{\partial \eta}$=0, so, obviously, $\frac{dR}{d\eta}$=0 and at η=1, θ=0; that means, R_n should be 0 at η= 1 and at ξ=0, θ=1 . So, now, you can write the solution right, but you can see that for different values of λ_n you will get a different solution X_n and R_n and as the governing equation is linear, you can super impose all the solutions right.

So, you can sum all the solution for different λ_n and it is possible as it is linear equation. So, we will write the final solution θ as the complete solution $\theta(\xi,\eta)=\sum_{n=0}^{\infty} C_n R_n e^{-\lambda_n^2 \xi}$.

So, now, we need to find what is the value of this constant C_n. So, we will apply the boundary conditions and R_n. What is R_n? R_n is the solution of this second order differential equation and you need to use some numerical technique to find the eigenfunctions of this second order differential equation and you can find the solution R_n.

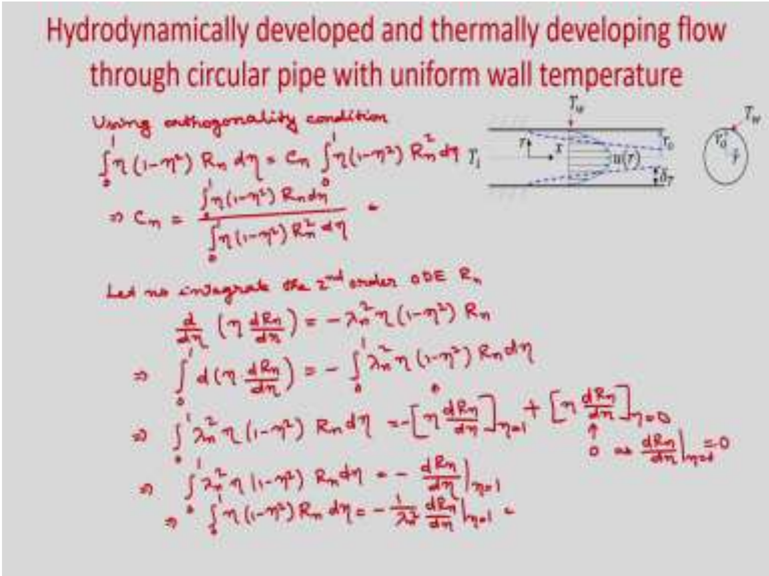
So, this equation you need to use some numerical technique to find the eigen functions R_n and once R_n is known then you can write the complete solution as $C_n R_n e^{-\lambda_n^2 \xi}$. So, this is very important because these equation you need to solve and it is eigenfunction are R_n and in the complete solution this is coming as R_n.

Now, let us apply the boundary condition at ξ=0 , θ=1 . So, if you put it here you will get $1=\sum_{n=0}^{\infty} C_n R_n$. So, what we will do now? So, we will use the orthogonality condition to find this constant C_n.

So, we will multiply both side with R_n into the weighting function; in this case it is $\int_0^1 \eta\left(1-\eta^2\right) R_m d\eta=\int_0^1 \sum_{n=0}^{\infty} C_n \eta\left(1-\eta^2\right) R_m R_n d\eta$.

So, now, you use the orthogonality condition. In the right hand side you can see the summation n =0 to ∞. So, in this particular case you can see if you apply the orthogonality condition for this Sturm Liouville equation, then all the terms will become 0 except n = m . So, for n = to m the integral all will be 0.

(Refer Slide Time: 19:13)



So, if you keep only n = m, then using orthogonality condition so, you can write $\int_0^1 \eta\left(1-\eta^2\right) R_n d\eta = \int_0^1 \sum_{n=0}^{\infty} C_n \eta\left(1-\eta^2\right) R_n^2 d\eta$.

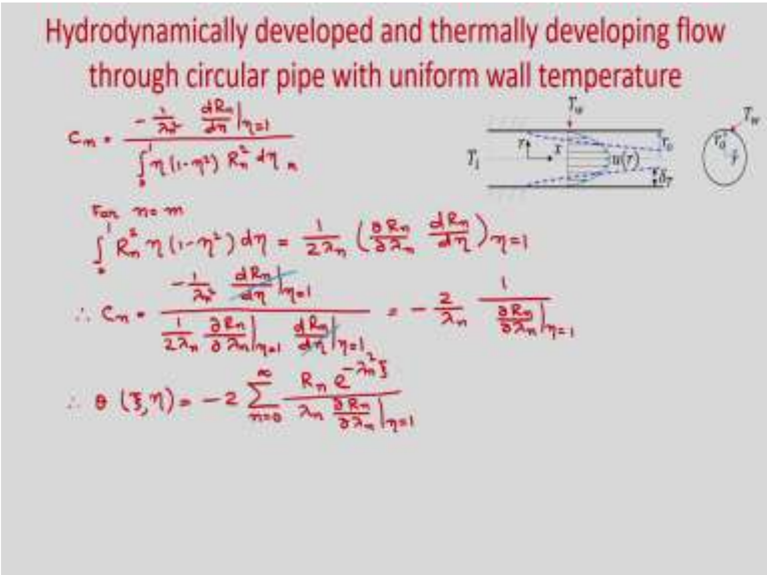
So, now, you can write from here what is the value of constant. $C_n = \dfrac{\int_0^1 \eta\left(1-\eta^2\right) R_n d\eta}{\int_0^1 \eta\left(1-\eta^2\right) R_n^2 d\eta}$.

So, now we will use the second order differential equation of R_n and we will integrate it then we will find. So, first let us integrate the second order ODE of R_n. So, you can see our equation is $\dfrac{d}{d\eta}\left(\eta \dfrac{dR_n}{d\eta}\right) = -\lambda_n^2 \eta\left(1-\eta^2\right) R_n$.

So, now, if you integrate it, so, you will get $\int_0^1 d\left(\eta \dfrac{dR_n}{d\eta}\right) = -\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta$. So, you can see this integral then you can write $\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta = -\left[\eta \dfrac{dR_n}{d\eta}\right]_{\eta=1} + \left[\eta \dfrac{dR_n}{d\eta}\right]_{\eta=0}$.

So, now, we apply the boundary condition at η=0, $\frac{dR_n}{d\eta}$ =0. So, the last term will become 0. So, this is your 0 as $\frac{dR_n}{d\eta}|_{\eta=0}$. So, you will get only $\int_0^1 \lambda_n^2 \eta\left(1-\eta^2\right) R_n d\eta = -\frac{dR_n}{d\eta}|_{\eta=1}$. So, you can see in this constant C_n, so, in the numerator you can replace. So, now, this λ_n^2 you can take it outside. So, you can write $\int_0^1 \eta\left(1-\eta^2\right) R_n d\eta = -\frac{1}{\lambda_n^2}\frac{dR_n}{d\eta}|_{\eta=1}$. So, these value this integral value you can put it in the numerator because this is the integral.

(Refer Slide Time: 23:15)



So, $C_n = \frac{-\frac{1}{\lambda_n^2}\frac{dR_n}{d\eta}|_{\eta=1}}{\int_0^1 \eta\left(1-\eta^2\right) R_n^2 d\eta}$. Now, apply the other important properties of orthogonality at n =m . So, already we have written for the Sturm Liouville equation. So, if you write it for n =m . So, you can write $\int_0^1 R_n^2 \eta\left(1-\eta^2\right) d\eta = \frac{1}{2\lambda_n}\left(\frac{\partial R_n}{\partial \lambda_n}\frac{dR_n}{d\eta}\right)_{\eta=1}$.

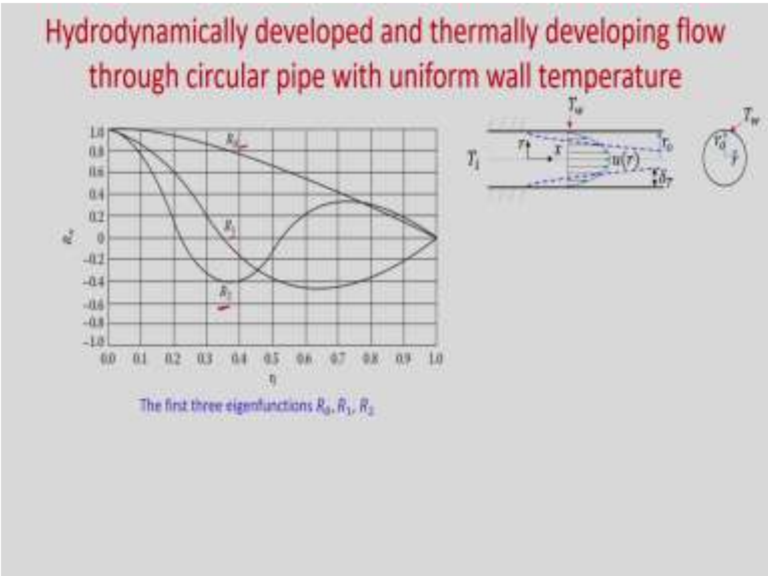
So, if you put in the denominator . So, this you see this is the left hand side this integral.

So, you can write the constant $C_n = \dfrac{-\dfrac{1}{\lambda_n^2}\dfrac{dR_n}{d\eta}\Big|_{\eta=1}}{\dfrac{1}{2\lambda_n}\dfrac{\partial R_n}{\partial \lambda_n}\Big|_{\eta=1}\dfrac{dR_n}{d\eta}\Big|_{\eta=1}}$. So, this and this we will get cancel. So, you will get $-\dfrac{2}{\lambda_n}\dfrac{1}{\dfrac{\partial R_n}{\partial \lambda_n}\Big|_{\eta=1}}$.

So, your temperature distribution, so, you can write C_n you know. So, $\theta(\xi,\eta) = -2\sum_{n=0}^{\infty}\dfrac{R_n e^{-\lambda_n^2\xi}}{\lambda_n \dfrac{\partial R_n}{\partial \lambda_n}\Big|_{\eta=1}}$.

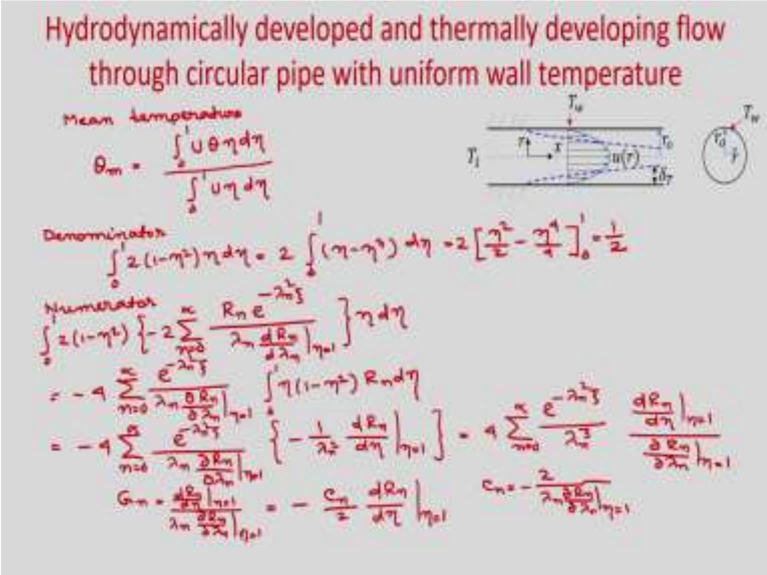
So, if you can find the eigen function of R_n and its derivative with respect to the eigenvalues if you can get at $\eta=1$, then you will be able to find this temperature profile θ.

(Refer Slide Time: 26:05)



So, you can see the first three eigen functions R_0, R_1 and R_2 how it varies; first three eigen functions with η. So, the boundary condition you know . So, at $\eta=1$ R = 0 . So, you can see that here you are getting at $\eta=1$ and R = 0 and this is the R_0 variation, this is the R_1 variation and this is the R_2 variation . So, these are the eigen functions; first three eigen functions R_0, R_1 and R_2.

(Refer Slide Time: 26:41)



Now, we are interested to find the Nusselt number. So, to find the Nusselt number we have to find the difference we need to find the temperature difference between mean temperature and the wall temperature. So, for that first we need to find what is the mean temperature. So, you can see that bulk mean temperature mean temperature $\theta_m = \dfrac{\int_0^1 U\theta\eta d\eta}{\int_0^1 U\eta d\eta}$.

So, now if you write the denominator you integrate first . So, it will be $\int_0^1 2\left(1-\eta^2\right)\eta d\eta = 2\int_0^1 \left(\eta-\eta^3\right)d\eta$ and it will be just $2\left[\dfrac{\eta^2}{2}-\dfrac{\eta^4}{4}\right]_0^1$. So, if you put the limits you will get $\dfrac{1}{2}$.

And, the numerator if you find the integration; so, $\int_0^1 2\left(1-\eta^2\right)\left\{-2\sum_{n=0}^{\infty}\dfrac{R_n e^{-\lambda_n^2\xi}}{\lambda_n \dfrac{dR_n}{d\lambda_n}\Big|_{\eta=1}}\right\}\eta d\eta$ k.

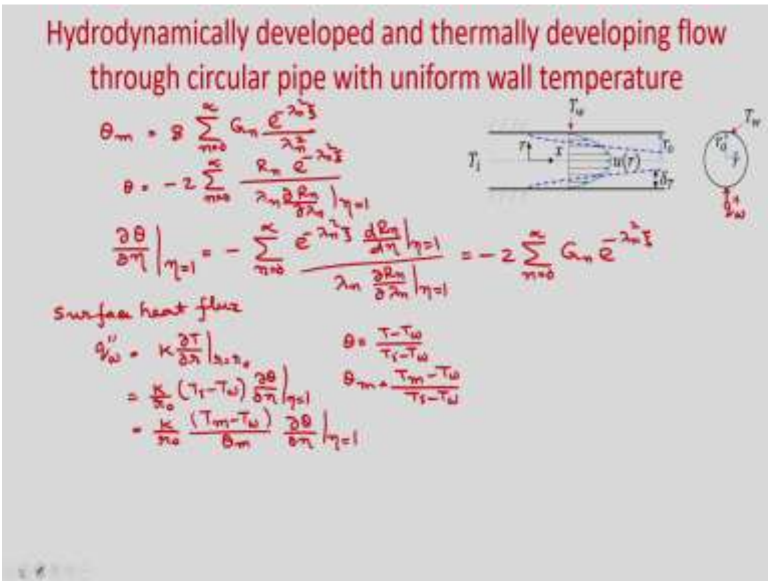
So, if you see if you find this integral, so, these you can see it is not function of η. So, whatever is function of η that you put inside the integral. So, you can write $-4\sum_{n=0}^{\infty}\frac{e^{-\lambda_n^2\xi}}{\lambda_n\frac{dR_n}{d\lambda_n}\big|_{\eta=1}}\int_0^1\eta\left(1-\eta^2\right)R_n d\eta$.

So, you can see that this integral already we have found right. So, this you can write as $-4\sum_{n=0}^{\infty}\frac{e^{-\lambda_n^2\xi}}{\lambda_n\frac{dR_n}{d\lambda_n}\big|_{\eta=1}}\left\{-\frac{1}{\lambda_n^2}\frac{dR_n}{d\lambda_n}\big|_{\eta=1}\right\}$.

So, now if you see, so, you can rearrange it as $4\sum_{n=0}^{\infty}\frac{e^{-\lambda_n^2\xi}}{\lambda_n^3}\frac{\frac{dR_n}{d\eta}\big|_{\eta=1}}{\frac{\partial R_n}{\partial\lambda_n}\big|_{\eta=1}}$. So, now, you can find the θ_m .

So, what we will do? We will now define another constant, $G_n=\frac{\frac{dR_n}{d\eta}\big|_{\eta=1}}{\lambda_n\frac{\partial R_n}{\partial\lambda_n}\big|_{\eta=1}}=-\frac{C_n}{2}\frac{dR_n}{d\eta}\big|_{\eta=1}$. So, you can see the C_n. This is the, $C_n=-\frac{2}{\lambda_n\frac{\partial R_n}{\partial\lambda_n}\big|_{\eta=1}}$.

(Refer Slide Time: 32:09)



So, you will get the final expression for mean temperature $\theta_m = 8\sum_{n=0}^{\infty} G_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n^2}$. So, now, to calculate the Nusselt number we need the temperature gradient at η= 1; that means, at the wall. So, we need the temperature gradient at η=1.

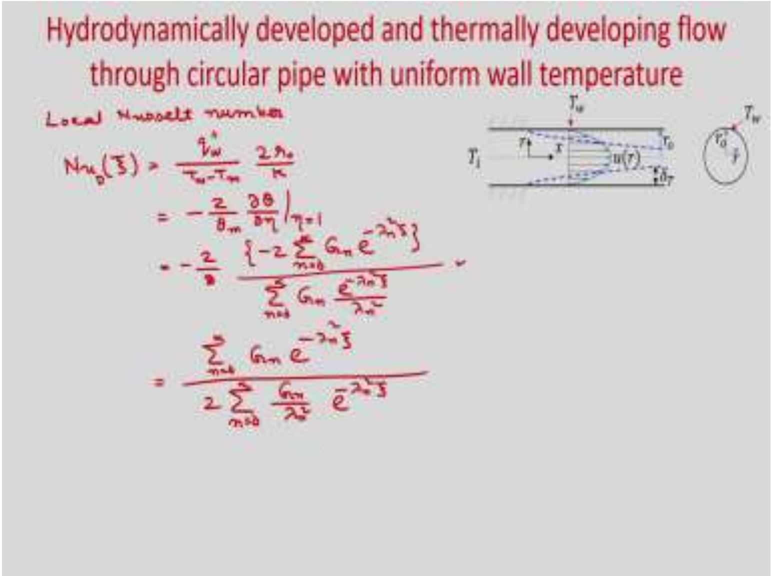
So, we know theta which is $\theta = -2\sum_{n=0}^{\infty} \frac{R_n e^{-\lambda_n^2 \xi}}{\lambda_n \frac{\partial R_n}{\partial \lambda_n}\Big|_{\eta=1}}$.

So, now $\frac{\partial \theta}{\partial \eta}\Big|_{\eta=1} = -\sum_{n=0}^{\infty} \frac{e^{-\lambda_n^2 \xi} \frac{dR_n}{d\eta}\Big|_{\eta=1}}{\lambda_n \frac{\partial R_n}{\partial \lambda_n}\Big|_{\eta=1}}$.

So, this if you write in terms of G, then you can write $-2\sum_{n=0}^{\infty} G_n e^{-\lambda_n^2 \xi}$. We are now in a position to find the Nusselt number. So, first you find what is the surface heat flux. So, surface heat flux $q_w'' = K \frac{\partial T}{\partial r}\Big|_{r=r_0}$. So, what we are telling that q we are taking in the negative direction of r. So, it will become positive.

So, you can see that $\theta = \frac{T - T_w}{T_i - T_w}$ and $\theta_m = \frac{T_m - T_w}{T_i - T_w}$. So, this you can see that it will be $q_w'' = \frac{K}{r_0}(T_i - T_w)\frac{\partial \theta}{\partial \eta}\Big|_{\eta=1}$. And, in terms of mean temperature now if you write, so, it will be $q_w'' = \frac{K}{r_0}\frac{(T_m - T_w)}{\theta_m}\frac{\partial \theta}{\partial \eta}\Big|_{\eta=1}$.

(Refer Slide Time: 35:03)



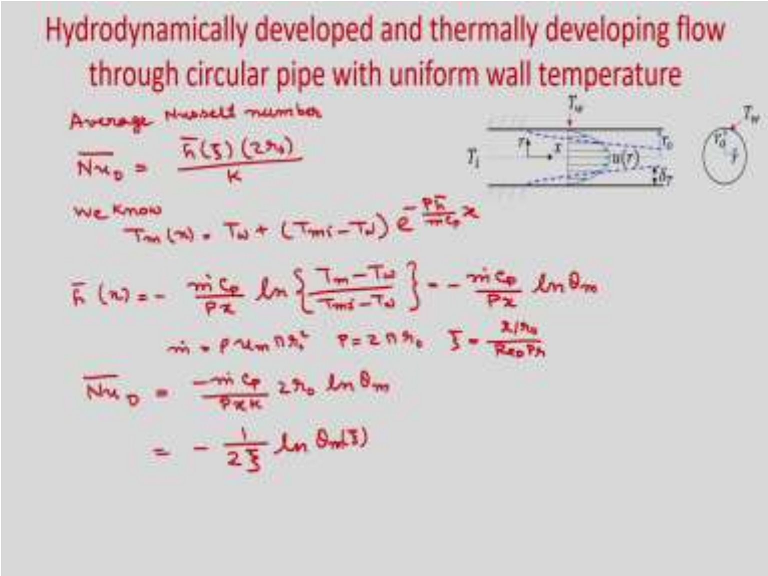
So, now you find the local Nusselt number. So, Nusselt number will be just $Nu_D = \frac{q_w^{"}}{T_w - T_m}\frac{2r_0}{K}$. So, you can see $-\frac{2}{\theta_m}\frac{\partial \theta}{\partial \eta}\Big|_{\eta=1}$.

So, now, $\frac{\partial \theta}{\partial \eta}\Big|_{\eta=1}$ already we have found. So, if you put it. So, you will get, $-\frac{2}{8}\frac{\left\{-2\sum_{n=0}^{\infty} G_n e^{-\lambda_n^2 \xi}\right\}}{\sum_{n=0}^{\infty} G_n \frac{e^{-\lambda_n^2 \xi}}{\lambda_n^2}}$.

So, this you can see that $\frac{\partial \theta}{\partial \eta}$ already we have found. So, if you put the θ_m expression and $\frac{\partial \theta}{\partial \eta}$ then you will get this expression. Finally, you will get this as $\frac{\sum_{n=0}^{\infty} G_n e^{-\lambda_n^2 \xi}}{2\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} e^{-\lambda_n^2 \xi}}$.

So, now, let us find the average Nusselt number. So, we will use the energy balance whatever we have derived earlier.

(Refer Slide Time: 37:01)



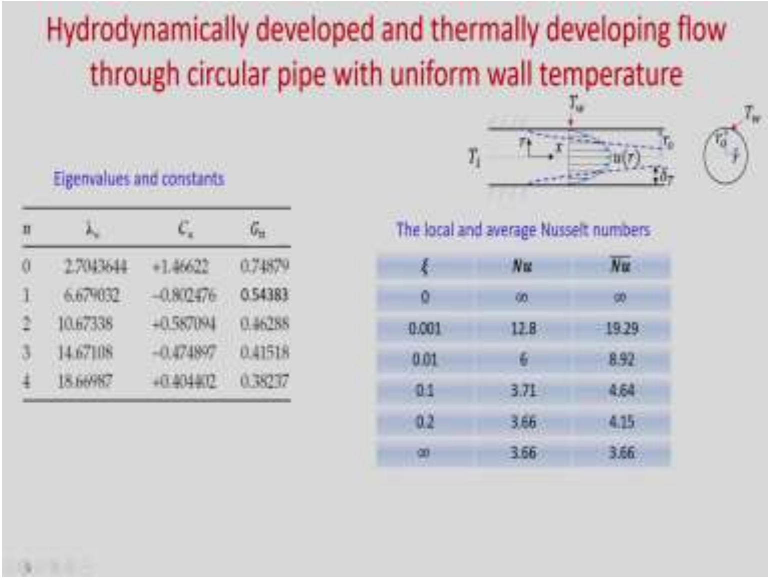
So, average Nusselt number if you calculate . So, you can write $\overline{Nu_D} = \frac{\bar{h}(\xi)(2r_0)}{K}$. So, it will be and from the energy balance we know $T_m(x) = T_w + (T_{mi} - T_w)e^{-\frac{P\bar{h}}{\dot{m}C_p}x}$.

So, from here you can find $\bar{h}(x) = -\frac{\dot{m}C_p}{Px}\ln\left\{\frac{T_m - T_w}{T_{mi} - T_w}\right\}$. So, it will be $-\frac{\dot{m}C_p}{Px}\ln\theta_m$. So, now, $\dot{m} = \rho u_m \pi r_0^2$ and $P = 2\pi r_0$ and $\xi = \frac{x/r_0}{\mathrm{Re}_D \Pr}$.

So, if you rearrange it so, and put it here this Nusselt number you are going to get $\overline{Nu_D} = -\frac{\dot{m}C_p}{Px}2r_0\ln\theta_m$ after rearrangement you will get as $-\frac{1}{2\xi}\ln\theta_m(\xi)$. So, this is the average Nusselt number.

So, now, you can see so, to find this local Nusselt number average Nusselt number and the temperature distribution you need to find the eigenvalues, eigen function and those constants C_n and G_n. Once you find those numerically, then you will be able to find the temperature distribution and the Nusselt numbers.

(Refer Slide Time: 39:33)

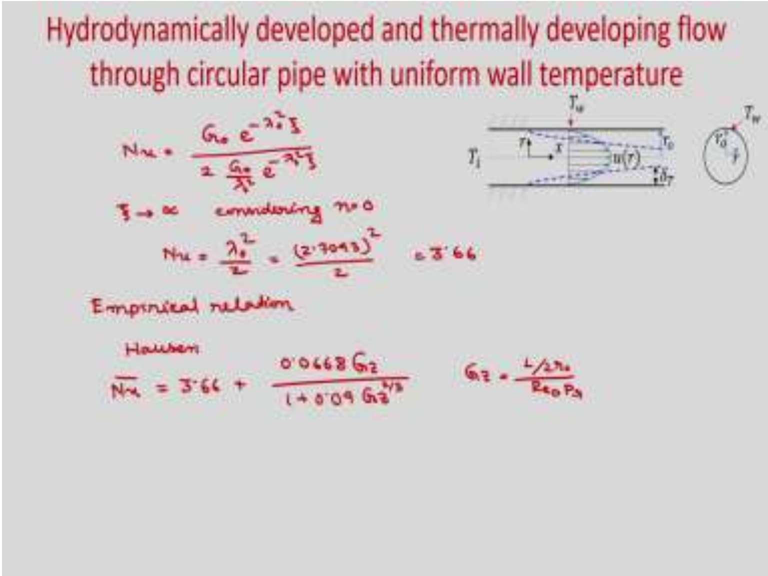


So, you can see for first five eigenvalues and constants from n = 0 to 4 we have written. So, this is the value of λ_n this is the constant C_n and Gn; G_n already we have written in terms of C_n. So, if C_n is known then G_n you will be able to evaluate. And, if you see the local and average Nusselt number so, as x varies. So, for different ξ 0 to ∞, I have written few terms.

Nusselt number, so, at $\xi=0$ it will be almost ∞ and average Nusselt number will be almost ∞ it will be very high value and as it $x \rightarrow \infty$, then it will be fully developed profile. And, if it is a hydrodynamically and thermally fully developed flow you should get the Nusselt number as 3.66 and that will be same as average Nusselt number 3.66.

So, you can see so, whatever expression we have derived for the temperature profile and the Nusselt number it is valid for developing region as well as developed region. So, you can see that if we put at $\xi \rightarrow \infty$ then we are getting back the Nusselt number which is constant and we have found earlier so, that is 3.66. So, you can see from this table.

(Refer Slide Time: 41:03)



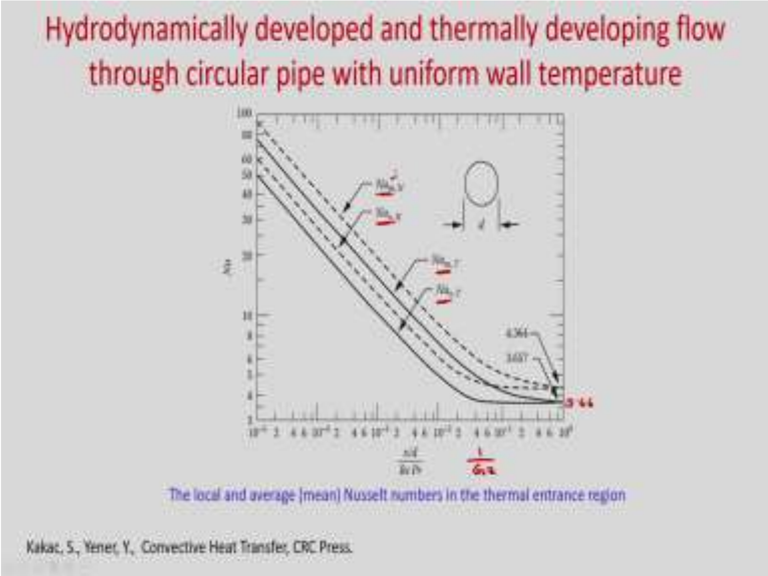
So, now, you see the limiting case. Limiting case at ξ→∞. So, $Nu = \dfrac{G_0 e^{-\lambda_0^2 \xi}}{2\dfrac{G_0}{\lambda_0^2} e^{-\lambda_0^2 \xi}}$. So, now, ξ→∞ considering the first value only for n = 0 neglecting the other terms with n >0, because all those terms are very small because exponentially decaying.

You can see that it will exponentially decaying, so, for n > 0 this will give very small value. So, you can see that $Nu = \dfrac{\lambda_0^2}{2}$ and λ_0^2 from this table if you see this is 2.7043. So, it will be $\dfrac{(2.7043)^2}{2}$; so, it will be 3.66 .

And, another empirical relations given by Hausen empirical relation proposed by Hausen. So, this is, $\overline{Nu} = 3.66 + \dfrac{0.0668 Gz}{1 + 0.04 Gz^{2/3}}$, where $Gz = \dfrac{L/2r_0}{\mathrm{Re}_D \Pr}$.

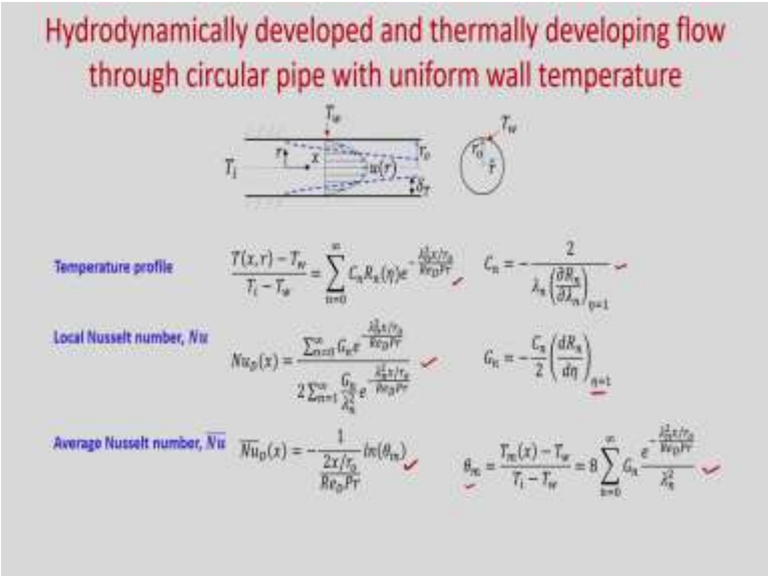
So, you can see the Nusselt number approaches constant value of 3.66 when the tube is sufficiently long because this term will become 0.

(Refer Slide Time: 43:05)



So, we can see the temperature profile. So, Nusselt number is plotted with $\frac{x/d}{\text{Re}_D \text{Pr}}$ which is $\frac{1}{Gz}$. So, for constant temperature, so, this is the average Nusselt number and this is the local Nusselt number. So, we can see local Nu_{NT} is very high value and it is exponentially decaying and it is becoming constant and it is you see it is almost 3.66 and it is for uniform wall heat flux boundary condition. So, local Nusselt number and the average Nusselt number.

(Refer Slide Time: 43:49)



Finally, we got this temperature profile where C_n is given by this expression and the local Nusselt number we have derived like this, where $G_n = -\frac{C_n}{2}\left(\frac{dR_n}{d\eta}\right)_{\eta=1}$ and also we have derived the average Nusselt number. So, this is the expression where $\theta_m = \frac{T_m(x) - T_w}{T_i - T_w}$ with this expression.

So, today we considered hydrodynamically developed and thermally developing fluid flow through circular tube with uniform wall temperature. So, we used separation of variables method with suitable non-dimensional parameters; we defined the $\theta = \frac{T - T_w}{T_i - T_w}$. So, from there we found the eigen functions R_n from the second order differential equation and along x it varies exponentially.

So, with these two solutions product of these two solutions we found the temperature profile θ. Once we know the θ then we can find the Nusselt number and Nusselt number we have defined based on the temperature difference at wall and the mean temperature. And, Nusselt number based on diameter first we have found local Nusselt number, then we have found the average Nusselt number.

And, this is the general expression for Nusselt number when we put x →∞ then obviously, it becomes hydrodynamically and thermally fully developed flow. So, we should get back the constant Nusselt number and we derived earlier it as 3.66. And, today we have shown that if you put x →∞ you are getting back the Nusselt number as 3.66.

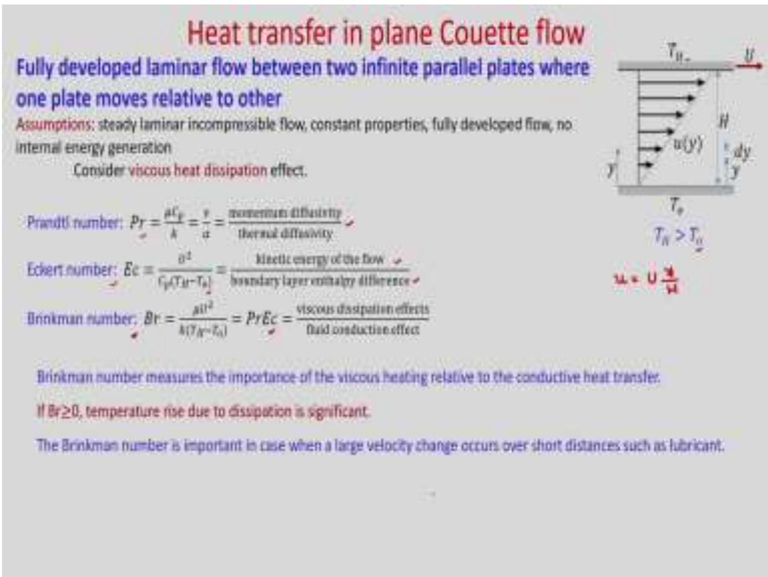
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 07
Convection in Internal Flows - III
Lecture - 24
Heat transfer in plane Couette flow

Hello everyone. Today, we will consider Heat transfer in plane Couette flow. Already we have derived the velocity distribution in plane Couette flow, where upper plate is moving and bottom plate is stationary and you have seen that velocity varies linearly from bottom wall to top wall. So, today, we will assume fully developed laminar flow. There is no internal energy generation. However, we will consider the viscous heat dissipation effect.

(Refer Slide Time: 01:11)



So, let us consider the flow between two parallel plates; the bottom plate here velocity is 0, so it is stationary plate and upper plate, you can see it is moving in the x direction with a constant velocity u. The bottom plate is maintained at temperature T_0 and upper plate is maintained at temperature T_H, where T_H we are considering is greater than T_0 and y is measured from the bottom wall and the distance between two plates is H.

So, you can see that already we have derived the velocity profile u; u is linear and $u = U\frac{y}{H}$. So, before going to that let us revisit the non-dimensional numbers, Prandtl number. What is Prandtl number? Prandtl number is the ratio of momentum diffusivity to thermal diffusivity right.

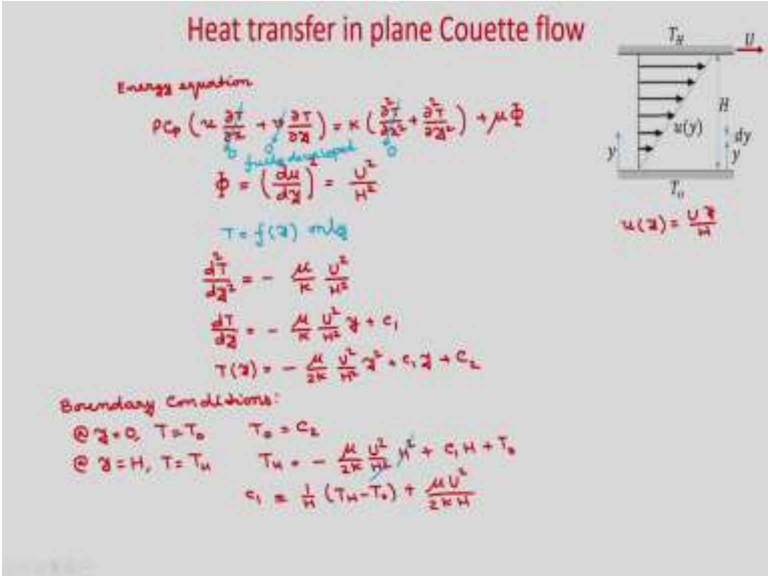
So, you can see that $\Pr = \frac{\mu C_p}{k} = \frac{\nu}{\alpha}$ and it is the ratio of momentum diffusivity to the thermal diffusivity. We also define Eckert number earlier. $Ec = \frac{U^2}{C_p(T_H - T_0)}$.

So, Eckert number is the ratio of kinetic energy of the flow to the boundary layer enthalpy difference. Now, we will define one new non-dimensional number which is Brinkman number. So, Brinkman number is the product of Prandtl number and Eckert number and it is given as $Br = \frac{\mu U^2}{k(T_H - T_0)}$.

So, Brinkman number measures the importance of the viscous dissipation effects to the fluid conductive heat transfer. If Brinkman number ≥ 0, then temperature rise due to dissipation is significant. The Brinkman number is important in case when a large velocity change occurs over a short distance such as lubricant.

So, here you know that we are considering plane Couette flow; that means, it is a shear defined flow, there is no imposed pressure difference right. So, velocity profile is generated due to the shear of the upper plate because upper plate is moving at a constant velocity u. So, let us write the energy equation and invoking the assumptions, let us simplify it.

(Refer Slide Time: 04:07)



So, our energy equation is $\rho C_p\left(u\dfrac{\partial T}{\partial x}+v\dfrac{\partial T}{\partial y}\right)=K\left(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}\right)+\mu\phi$. Here, Φ is viscous dissipation coefficient and it is for this particular case, you can see it will give $\phi=\left(\dfrac{du}{dy}\right)^2$.

So, you have a expression of Φ right. So, if you see all other terms become 0 because your velocity profile u which is function of y only and it is $u(y)=U\dfrac{y}{H}$.

So, in this particular case, you can see the velocity profile is linear. So, it will be just $\dfrac{U^2}{H^2}$ because distance between two parallel plates is H and it is linearly valuing. So, $\dfrac{du}{dy}=\dfrac{U}{H}$. So, $\left(\dfrac{du}{dy}\right)^2=\dfrac{U^2}{H^2}$.

Now, let us simplify it. So, we are telling it is a fully developed flow. So, obviously, your v = 0. In this particular case, as there is no imposed pressure difference, so velocity is generated due to the shears shear force on the exerted on the upper plate. So, your fully developed temperature profile also is constant.

So, in axial direction, there will be no variation of temperature if $\frac{\partial T}{\partial x}$=0. So, it is a fully developed condition. So, in this particular case, your $\frac{\partial T}{\partial x}$=0 because it is fully developed. So, you can see that essentially you can neglect the axial heat conduction. So, $\frac{\partial^2 T}{\partial x^2}$=0 because $\frac{\partial T}{\partial x}$=0 due to fully developed condition; so obviously, $\frac{\partial^2 T}{\partial x^2}$=0. So that means, axial heat conduction is also 0.

So, in this particular case for Couette flow, your temperature is function of y only. So, your fully developed temperature profile, you get as $\frac{\partial T}{\partial x}$=0. So, there is no variation of temperature profile in the axial direction. So, $\frac{\partial T}{\partial x}$=0. Hence, your temperature is only function of y. So, you can write the governing equations as so T is function of y only.

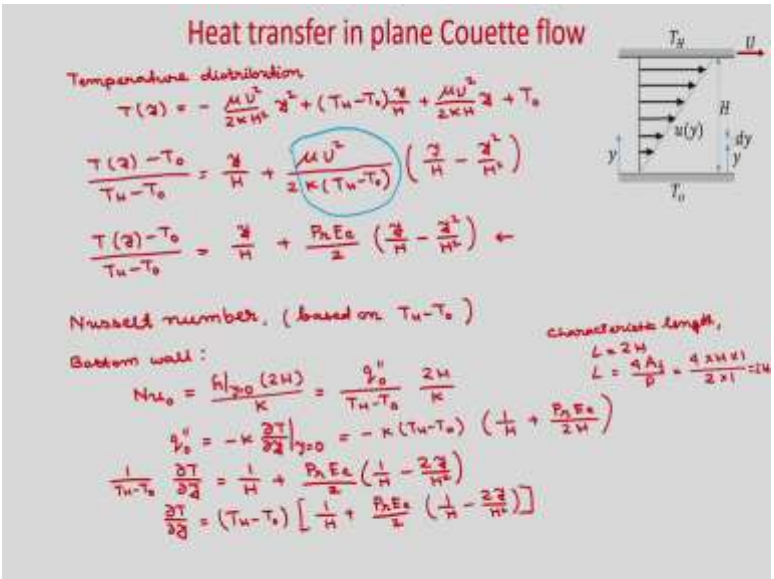
So, your simplified energy equation is $\frac{d^2T}{dy^2} = \frac{\mu}{K}\frac{U^2}{H^2}$. So, this is our governing equation and we have two boundary conditions at y = 0; T =T_0 and at y = H; T = T_H. So, you integrate this equation twice and find the two constants applying the boundary conditions.

So, if you integrate twice, what you will get? $\frac{dT}{dy} = -\frac{\mu}{K}\frac{U^2}{H^2}y + C_1$. If you integrate again, $T(y) = -\frac{\mu}{2K}\frac{U^2}{H^2}y^2 + C_1 y + C_2$.

So, now apply the boundary conditions. So, at y = 0, you have T =T_0. So, if you put that, then T_0 = C_2. Because y = 0, so this right hand side, first two terms will become 0 and at y = H, you have T = T_H.

So, if you put this one, so you will get $T_H = -\frac{\mu}{2K}\frac{U^2}{H^2}H^2 + C_1 H + T_0$. So, you can now write this C_2 = T_0 and $C_1 = \frac{1}{H}(T_H - T_0) + \frac{\mu U^2}{2KH}$. So, now, we have found the two constants. So, let us put it in the temperature profile and find the final temperature distribution.

(Refer Slide Time: 10:09)



So, temperature distribution . So, you can see

$$T(y) = -\frac{\mu}{2K}\frac{U^2}{H^2}y^2 + (T_H - T_0)\frac{y}{H} + \frac{\mu U^2}{2KH}y + T_0.$$

So, if you rearrange it, you can write as $\frac{T(y)-T_0}{T_H - T_0} = \frac{y}{H} + \frac{\mu U^2}{2K(T_H - T_0)}\left(\frac{y}{H} - \frac{y^2}{H^2}\right)$.

So, you can see that this term what is this term? So, if you see, so it is obviously, the product of Prandtl number and Eckert number which is your Brinkman number, already we have defined . So, you can write it $\frac{T(y)-T_0}{T_H - T_0} = \frac{y}{H} + \frac{\mathrm{PrEc}}{2}\left(\frac{y}{H} - \frac{y^2}{H^2}\right)$. So, this is the temperature profile, we will plot it later after finding the Nusselt number.

So, in this particular case, we will calculate the Nusselt number based on the temperature difference T_H -T_0 . Here imposed temperature difference is T_H -T_0 because T_H >T_0 and we will define the Nusselt number based on T_H -T_0.

You can also find the mean temperature T_m and also find the Nusselt number based on the temperature difference T_H –T_H or T_m -T_0, but it is convenient for this particular case to calculate the Nusselt number based on the temperature difference T_H -T_0. So, for this particular case, now find the Nusselt number.

So, Nusselt number we will calculate. So, what is Nusselt number? So, Nusselt number based on temperature difference T_H -T_0, this you should remember because in earlier cases, all we have considered the temperature difference T_w -T_m, where T_m is the mean temperature ok.

But in this particular case, we are calculating the Nusselt number based on T_H -T_0. Obviously, if you calculate the Nusselt number based on that temperature difference T T_H –T_m, then expression will be different.

So, now Nusselt number on bottom wall. So, we can calculate Nu_0. So, it is $q_0^{"}$. So, this is your 0 means at y = 0 divided by or we can write this.

So, under bottom wall, we have Nusselt number at bottom wall that is your we are denoting with 0. So, $Nu_0 = \dfrac{h|_{y=0}(2H)}{K}$.

So, in this particular case, now if you see what is h at y = 0? So, this is $\dfrac{q_0^{"}}{T_H - T_0}\dfrac{2H}{K}$. So, characteristic length, here we are considering as 2 H. So, now what is $q_0^{"}$? So, this is your $q_0^{"} = -K\dfrac{\partial T}{\partial y}|_{y=0}$.

So, you see, we are calculating the heat flux at bottom wall. So, y is perpendicular to the bottom wall. So, $q_0^{"} = -K\dfrac{\partial T}{\partial y}|_{y=0}$. So, you can calculate $-K\dfrac{\partial T}{\partial y}|_{y=0}$. So, you can see from previous expression; $\dfrac{\partial T}{\partial y} = -\dfrac{\mu}{K}\dfrac{U^2}{H^2}y + C_1$.

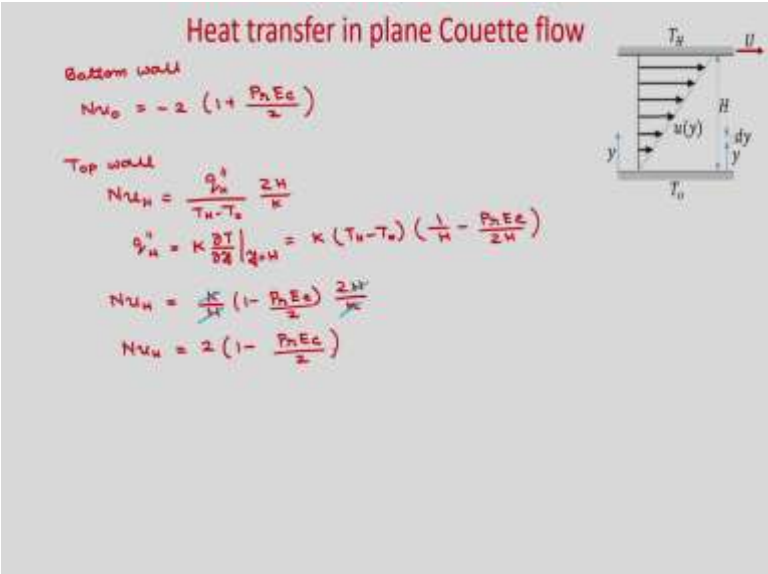
So, from this temperature distribution, first let us calculate the temperature gradient. So, if you take the derivative of this equation with respect to y, you will get $\dfrac{1}{T_H - T_0}\dfrac{\partial T}{\partial y} = \dfrac{1}{H} + \dfrac{\text{PrEc}}{2}\left(\dfrac{1}{H} - \dfrac{2y}{H^2}\right)$. So, this is the temperature gradient. So, $\dfrac{\partial T}{\partial y} = \left(T_H - T_0\right)\left[\dfrac{1}{H} + \dfrac{\text{PrEc}}{2}\left(\dfrac{1}{H} - \dfrac{2y}{H^2}\right)\right]$.

So, you can see $\frac{\partial T}{\partial y}|_{y=0}$, what will be the value? So, y = 0. So, you will get $q_0^{"} = -K\left(T_H - T_0\right)\left(\frac{1}{H} + \frac{\text{PrEc}}{2H}\right)$.

So, now, in this particular case characteristic length twice H, how you did you get? So, you see the characteristic length L, you can find as $L = \frac{4A_f}{P}$. So, in this particular case, you can see flow area is 4 $\times \text{H} \times 1$. So, that is the H is the distance between 2 parallel plates. So, it is your H and in perpendicular direction per unit width, if you consider then 1.

So, this is your flow area divided by the perimeter. What is the perimeter in this particular case? It is 2×1 because in per unit width you are considering right in perpendicular direction. So, you have one on the top plate, one at the bottom plate. So, it will be 2. So, hence, in this particular case, characteristic length becomes 2 H. So, from here we are just calculating the Nusselt number based on the characteristic length 2 H.

(Refer Slide Time: 19:17)



So, now in bottom wall if you calculate the Nusselt number, then you will get, $Nu_0 = -2\left(1 + \frac{\Pr Ec}{2}\right)$. So, if you put all these values and simplify it, you will get this one. Now, you calculate on the top wall. So, Nusselt number at top wall. So, it will be $Nu_H = \frac{q_H^{"}}{T_H - T_0}$. So, this is your heat transfer coefficient H.

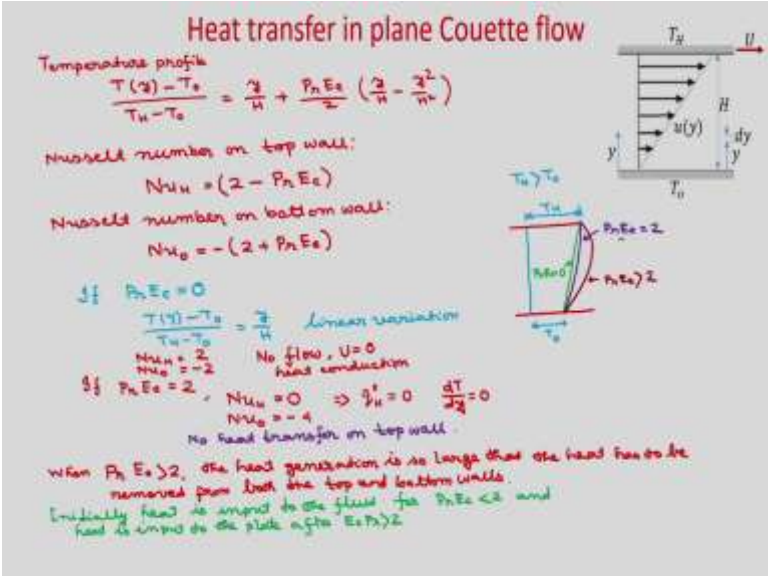
Now, you see in this expression $\frac{\partial T}{\partial y}$, if you want to find the heat flux at top wall. So, it will be $q_H^{"} = K\frac{\partial T}{\partial y}|_{y=H}$ and y = H, if you put it here. So, you are going to get here. So, if you write it, so you will get $q_H^{"}$.

So, it will be now y is in upward direction. Now, when you are calculating the heat flux on the top wall. So, we are calculating it is coming in the negative y direction, so obviously, $q_H^{"} = K\frac{\partial T}{\partial y}|_{y=H}$.

So, it will be $q_H^{"} = K\frac{\partial T}{\partial y}|_{y=H}$. So, now, if you put the value, so it will be $q_H^{"} = K\left(T_H - T_0\right)\left(\frac{1}{H} - \frac{\Pr Ec}{2H}\right)$. So, if it is so, then you can write Nusselt number H as $Nu_H = \frac{K}{H}\left(1 - \frac{\Pr Ec}{2}\right)\frac{2H}{K}$ and it will be $Nu_H = 2\left(1 - \frac{\Pr Ec}{2}\right)$

So, now, we have found the Nusselt number on top wall and bottom wall based on the characteristic length 2 H and the temperature difference T_H - T_0. Now, let us try to plot the temperature profile and see the effect of Brinkman number. That means, the product of Prandtl number and Eckert number.

(Refer Slide Time: 22:51)



So, the temperature distribution we got $\frac{T(y)-T_0}{T_H-T_0}=\frac{y}{H}+\frac{\Pr Ec}{2}\left(\frac{y}{H}-\frac{y^2}{H^2}\right)$.

So, this is your temperature profile and you have Nusselt number on top wall. $Nu_H=\left(2-\Pr Ec\right)$ and Nusselt number on bottom wall, we got $Nu_0=-\left(2+\Pr Ec\right)$.

So, now, let us try to see what happens when your Brinkman number is 0 . So that means, there is no viscous heat dissipation. So, only the temperature at bottom wall is T_0, upper wall is T_H and there is no viscosity. So, you will get a linear profile of temperature distribution.

So, you see this is your top wall, this is your bottom wall separated by distance H. Now, if you see the temperature profile, we have seen that $T_H > T_0$. So, if this is your T_0 on bottom wall, so obviously, T_H will be higher; so, this is let us say T_H on the top wall.

Now, we are seeing the if your Pr × Ec = 0, so if Pr × Ec = 0, so you see what is the temperature profile? So, in this case, you can see this Prandtl number Eckert number will be 0. So, you will get $\frac{T(y)-T_0}{T_H-T_0}=\frac{y}{H}$.

So, you see it is a linear profile right; linear variation of temperature. So, if you plot it ok, so it will be just linear variation. So, this is your Pr $\times$ Ec = 0. Now, you see when Pr$\times$Ec = 2. So, if Pr$\times$Ec = 2, then you see Nusselt number H what you will get?

if Prandtl number Eckert number is 0, then you are getting $Nu_H = 2$ and $Nu_0 = -2$ and what will be the velocity in this particular case, when Eckert number is 0?

Can you tell me, if Pr $\times$ Ec = 0, what will be the velocity profile? Obviously, U = 0; that means, there is no relative motion between 2 plates that means, plates are stationary, U=0. Hence, you are getting only the heat conduction from bottom wall to top wall.

So, that is why you are getting a linear profile of temperature variation. So, in this particular case U is 0; that means, no flow; no flow, U = 0 in this particular case because Pr $\times$ Ec = 0 and Eckert number velocity is there. So, u is 0 and only heat conduction takes place.

But when you are considering Pr $\times$ Ec = 2, then on the top wall Nusselt number is 0. What does it mean? That there is no heat transfer from the top plate right because if Nusselt number is 0, then your q_w'' on top wall is 0 that means, your temperature gradient on the top wall will be 0 .

So that means, here your q_H'' on the top wall is 0 and hence $\frac{dT}{dy}$ will be 0 . So that means, your temperature profile will cut the top wall perpendicularly. So, this will cut perpendicularly ok. So, this is the profile for Pr $\times$ Ec = 0.

Now, if Pr $\times$ Ec >2 , then your viscous heat dissipation effect will dominate and there will be more heat generation and your maximum temperature will occur in between top plate and bottom plate.

So, in this case, you will get no heat transfer on top wall. Now, when Pr $\times$ Ec >2, the heat generation is so large that the heat has to be removed from both the top and bottom walls and what will be the temperature profile in this particular case?

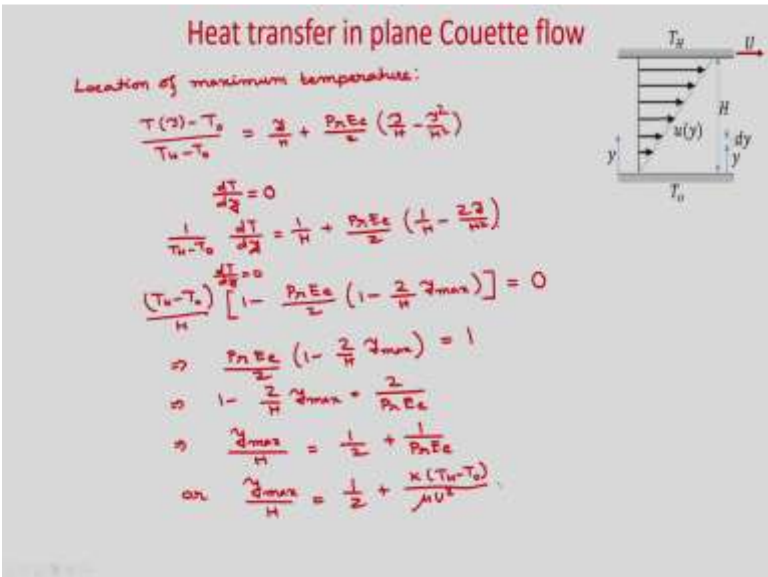
So, your temperature profile will be; so you can see in case of Brinkman number 0 and Brinkman number 2, your maximum temperature occurs on the top wall itself. But when Br > 2, your maximum temperature is occurring in between the domain.

So, this is the case for Pr × Ec >2, this is the plot for Pr × Ec = 2 and in this particular case, Pr × Ec = 0; that means, there is no flow. So, this is the linear profile. Here, your heat transfer is 0 on top wall and for Pr × Ec >2, you will get the maximum temperature inside the domain.

So, if you see so initially heat is input to the fluid, for Pr × Ec < 2 and heat is input to the plate after Ec × Pr >2 because when Ec × Pr >2, then your maximum temperature is occurring inside the domain, so heat transfer will occur from the fluid to the top wall and when you have Pr × Ec < 2, then heat transfer will occur from the top wall to the fluid.

So, when Pr × Ec < 2, heat generation within the fluid is small and therefore, heat stills comes in from water top plate. So, how do we find the location of the maximum temperature? So, simply you just put $\frac{\partial T}{\partial y} = 0$.

(Refer Slide Time: 33:31)



So, your temperature distribution, so location of maximum temperature. So, how we will find? So, let us see the temperature distribution $\frac{T(y)-T_0}{T_H-T_0} = \frac{y}{H} + \frac{\Pr Ec}{2}\left(\frac{y}{H} - \frac{y^2}{H^2}\right)$.

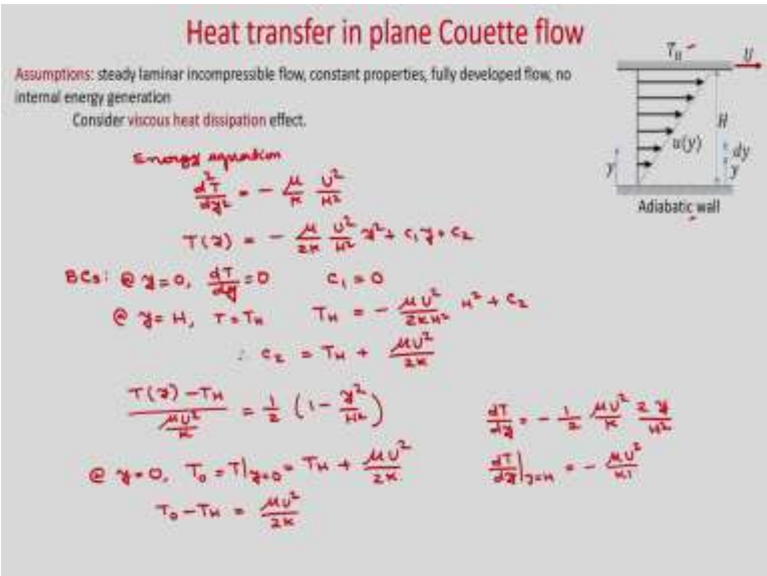
So, you will get so from maximum temperature $\frac{dT}{dy}=0$. If $\frac{dT}{dy}=0$, so that means, $\frac{1}{T_H-T_0}\frac{dT}{dy}=\frac{1}{H}+\frac{\Pr Ec}{2}\left(\frac{1}{H}-\frac{2y}{H^2}\right)$.

So, now if you put $\frac{dT}{dy}=0$, so you will get if you simplify it. So, it will be, $\frac{T_H-T_0}{H}\left[1-\frac{\Pr Ec}{2}\left(1-\frac{2}{H}y_{\max}\right)\right]=0$.

So, if you simplify it, so it will be $\frac{\Pr Ec}{2}\left(1-\frac{2}{H}y_{\max}\right)=1$ or you will get $1-\frac{2}{H}y_{\max}=\frac{2}{\Pr Ec}$ or you will get $\frac{y_{\max}}{H}=\frac{1}{2}+\frac{1}{\Pr Ec}$ or you can write $\frac{y_{\max}}{H}=\frac{1}{2}+\frac{K\left(T_H-T_0\right)}{\mu U^2}$.

So, you see at this location, you will get the maximum temperature inside the domain. So, now let us consider another case, where top plate is having temperature T_H and bottom plate is adiabatic. What does it mean adiabatic? Adiabatic means there will be no heat loss from the bottom wall.

(Refer Slide Time: 36:49)



So, here we are considering T_H on the top wall and adiabatic wall on the bottom. So that means, there will be no heat transfer. So, in this particular case, if you see already we have derived the energy equation and your energy equation is energy equation already we have derived. So, $\frac{d^2T}{dy^2} = -\frac{\mu}{K}\frac{U^2}{H^2}$, we have consider the viscous dissipation effect.

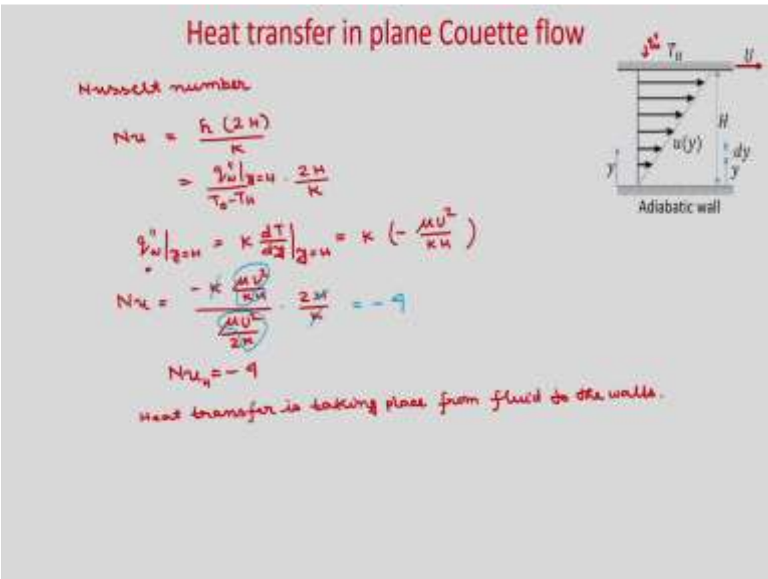
So, if you see, $T(y) = -\frac{\mu}{2K}\frac{U^2}{H^2}y^2 + C_1 y + C_2$. So, now, apply the boundary condition, in this particular case at y = H, you have temperature T_H; but bottom wall is adiabatic. So, heat flux is 0, hence $\frac{dT}{dy}$ will be 0 on the bottom wall.

So, boundary conditions at y = 0 , $\frac{dT}{dy} = 0$. So, if $\frac{dT}{dy} = 0$ on the bottom wall, so you can see that you will get your C_1=0; C_1=0 and at y = H, you will get T as T_H. So, you will get T_H, C_1=0 and you will get, $T_H = -\frac{\mu U^2}{2KH^2}H^2 + C_2$.

So, $C_2 = T_H + \frac{\mu U^2}{2K}$. Now, if you put, then you will get the temperature profile $\frac{T(y) - T_H}{\frac{\mu U^2}{K}} = \frac{1}{2}\left(1 - \frac{y^2}{H^2}\right)$. So, this is a parabolic profile, temperature distribution. Let us find the Nusselt number based on the characteristic length 2H and the temperature difference T_H -T_0. So, you have to find what is the temperature at the bottom wall.

So, if you do. So, so you see at y = 0 , so what will be the temperature ? So, you see if you have a temperature at the bottom wall, if you find it as T_0, then it will be, $T_0 = T\big|_{y=0} = T_H + \frac{\mu U^2}{2K}$. So, you can see that $T_0 - T_H = \frac{\mu U^2}{2K}$. So, now we will find the Nusselt number based on this temperature difference $T_0 - T_H$.

(Refer Slide Time: 40:29)



So, Nusselt number, so we will find Nusselt number based on this temperature difference. So, it will be, $Nu = \frac{h(2H)}{K}$, where $H = \frac{q_w^{''}|_{y=H}}{T_0 - T_H}$.

So, this is the temperature difference, and Nusselt number is Nu=$\frac{q_w^{''}|_{y=H}}{T_0 - T_H}\frac{2H}{K}$, and $q_w^{''}|_{y=H}$ you can find it is $q_w^{''}|_{y=H} = K\frac{dT}{dy}|_{y=H}$. So, what will be that? So, your temperature profile is this one right.

So, you can see you can find $\frac{dT}{dy}$. It will be $-\frac{\mu U^2}{KH}$. So, this is your $\frac{dT}{dy} = -\frac{\mu U^2}{KH}$.

So, it will be just $Nu = \frac{-K\frac{\mu U^2}{KH}}{\frac{\mu U^2}{2K}}\frac{2H}{K}$. So, you can see this simplify it, this H, this H will get cancelled; this $\frac{\mu U^2}{K}$, $\frac{\mu U^2}{K}$ will get cancelled; this K, this K will get cancelled. So, you will get 2. So, it will be -4.

So, Nusselt number you are getting as - 4. So, you can see that bottom wall is adiabatic, so there will be no heat transfer through the bottom wall. However, top wall is maintained at constant wall temperature T_H. Hence, the heat transfer will take place from the fluid to the top wall and temperature of bottom wall anyway it will increase.

As it is adiabatic, so there will be no heat transfer through the bottom wall. However, its temperature will increase. So, the Nusselt number whatever we defined here, you can see the heat flux on the top wall right. So, Nusselt number whatever we have found here is for your top wall.

So, we can write $Nu_H = -4$ and negative sign represents that heat transfer is taking place from the fluid to the top wall because while calculating the heat flux $q_w^{"}\big|_{y=H}$, we took it as negative y direction. So, we assumed that heat transfer is taking place from the top wall to the fluid. However, we have seen that due to the viscous heat dissipation, the temperature of the fluid will increase and the heat transfer will take place from the fluid to the top wall.

So, let us summarize. Today, we considered fully developed laminar flow between 2 parallel plates, where top plate is moving in the positive x direction with a constant velocity U; whereas, your bottom wall is stationary.

We considered two different types of problem, where in first case we considered the temperature on the bottom wall as T_0 and on the top wall as T_H, where $T_H > T_0$ and in other case, we considered bottom wall as adiabatic and top wall is T_H. In both the cases, we consider the viscous heat dissipation effect. First, we found the temperature profile for both the cases, then we found the Nusselt number.

In first case, we considered the Nusselt number for the bottom wall and top wall separately. And this Nusselt number, we have calculated based on the characteristic length 2 H and the temperature difference $T_H - T_0$ and we have seen that if your $Pr \times Ec = 0$, then there is no flow at all.

Hence, your heat conduction will take place from top wall to bottom wall linearly. Next when $Pr \times Ec = 2$, then Nusselt number on the top wall becomes 0; that means, there will be no heat flux on the top wall and $\frac{dT}{dy} = 0$.

In these two cases or in between cases, you can see that when your $Pr \times Ec \leq 2$, your maximum temperature occurs on the top wall and your heat transfer is taking place from the walls to the fluid.

However, when your $Pr \times Ec > 2$, then your viscous heat dissipation effect comes into picture and your temperature becomes higher in the fluid domain than the top wall. Hence, your heat transfer takes place from the fluid region to the top wall.

Next case, we considered the Nusselt number based on the temperature difference between bottom wall and top wall and the characteristic length 2 H. In this particular case, we got the Nusselt number as -4 because we considered that heat transfer is taking place from the top wall to the fluid.

But as it is negative that means, your viscous heat dissipation effect is there and temperature in the fluid zone is higher than the top wall. Hence, your heat transfer is taking place from the fluid to the walls.

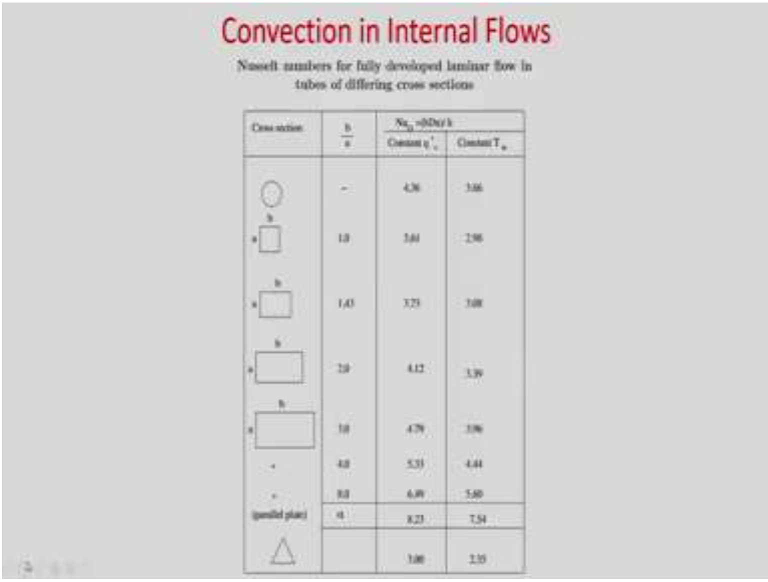
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 07
Convection in Internal Flows – III
Lecture – 25
Solution of example problems

Hello everyone. So, today we will solve few example problems on Convection in Internal Flows.

(Refer Slide Time: 00:39)



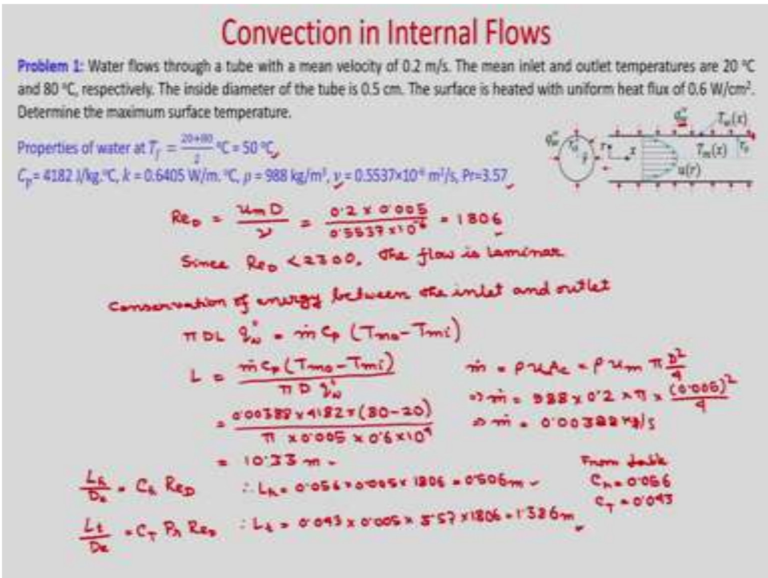
So, first let us summarize that the Nusselt number for thermally and hydro dynamically fully developed flow, you can see for different cross sectional pipe. So, this is your Nusselt numbers for fully developed laminar flow in tubes of differing cross sections. So, you can see here for circular cross section, we have already derived Nusselt number for this particular case.

If you see if it is a constant wall heat flux, then it is 4.36 and if it is constant wall temperature, then 3.66. These we have not derived, but you can see that for a square cross sectional duct. The Nusselt number for constant wall heat flux, it is 3.61 and for constant wall temperature it is 2.98.

Similarly, you see for a rectangular cross sectional channel where $\frac{b}{a}=1.43$. So, b is here a. So, for this particular case for constant wall heat flux boundary condition the Nusselt number is 3.73 and for constant wall temperature 3.08. If you change the $\frac{b}{a}$ ratio, you can see the Nusselt number for two different wall conditions.

Now, if it is a parallel plate so; that means, $\frac{b}{a}$ is infinity. So, in this particular case already we have derived the Nusselt number for constant wall heat flux, it is 8.23 and for constant wall temperature it is 7.54. And, if it is a triangular cross sectional channel then Nusselt number is 3 for constant wall heat flux and 2.35 for constant wall temperature. And you see that the Nusselt number, we have defined based on the hydraulic diameter.

(Refer Slide Time: 02:53)



So, first let us take this problem; Water flows through a tube with a mean velocity of 0.2 m/s. The mean inlet and outlet temperatures are 20^0C and 80 ^{0}C respectively. The inside diameter of the tube is 0.5 cm, the surface is heated with uniform heat flux of 0.6 W/cm^2. Determine the maximum surface temperature. So, you can see it is uniform wall heat flux boundary condition.

So, your wall temperature will vary in axial direction and where it is expected to be maximum at the outlet right; so T_w will be maximum. Now, first for this problem first we have to find what is the Reynolds number. If the Reynolds number is in laminar zone and if it is a fully developed flow for both thermally and hydro dynamically then we know. What is the Nusselt number? So, for this particular case you know that the Nusselt number is 4.364.

So, first let us calculate the Reynolds number and the developing length or entrance length. So, properties of water at film temperature 50^0C. So, it is the average temperature. So, C_p ,k , ρ, ν and Prandtl number are given. So, you calculate first from here the Reynolds number.

So, this is your $Re = \frac{u_m D}{\nu}$. So, u_m is given because it is 0.2 m/s. So, it is $\frac{0.2 \times 0.005}{0.5537 \times 10^{-6}}$. So, if you calculate it will come as 1806 and as Reynolds number < 2300 the flow is laminar.

Now, we need to find the length of the tube that is also unknown and we have to find whether the exit is in developing zone or not. So, we have to find the developing length first so, that if it is a fully developed flow, then we can use Nusselt number = 4.364. How do you calculate the length of the pipe?

So, you use this conservation of energy between the inlet and outlet you can see. So, if q_w'' is at the wall the heat flux is supplied. So, total heat transfer is your $\pi D L q_w'' = \dot{m} C_p \left(T_{mo} - T_{mi}\right)$.

So, length you can calculate as $L = \frac{\dot{m} C_p \left(T_{mo} - T_{mi}\right)}{\pi D q_w''}$. You can see here all the parameters are known except $\dot{m}$. So, $\dot{m}$ you have to calculate. So, the mass flow rate $\dot{m}$ you can calculate as $\dot{m} = \rho u_m A_c$. So, $\dot{m} = \rho u_m \pi \frac{D^2}{4}$.

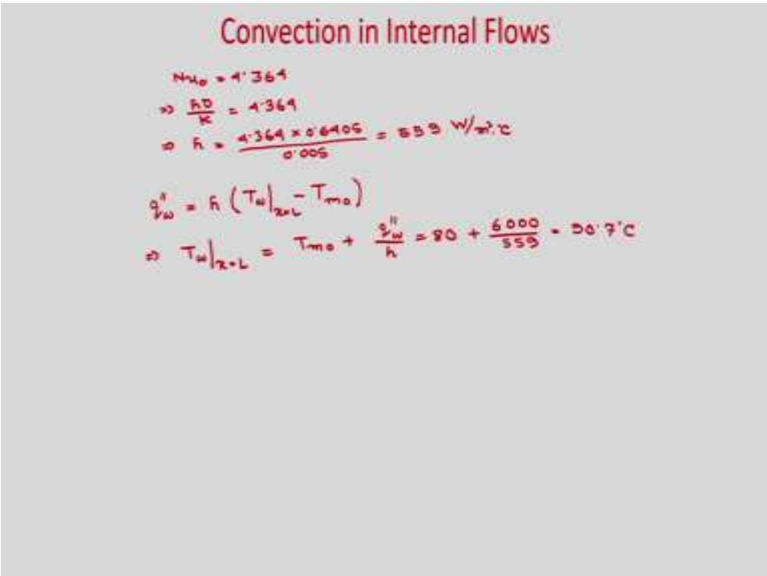
So, $\dot{m} = 988 \times 0.2 \times \pi \times \frac{(0.005)^2}{4}$. So, $\dot{m} = 0.00388 kg/s$. Now, you put it here then you find the length. So, it will be $L = \frac{0.00388 \times 4182 \times (80 - 20)}{\pi \times 0.005 \times 0.6 \times 10^4}$. So, length of the pipe is 10.33 m; so this length we have calculated.

Now, let us calculate the entrance length. For laminar flow in a tube the hydrodynamic and thermal lengths are given by $\frac{L_h}{D_e} = C_h \operatorname{Re}_D$. And for thermal developing length it is $\frac{L_T}{D_e} = C_T \Pr \operatorname{Re}_D$.

So, you know that from table that; C_h this coefficient is 0.056 and C_T is 0.043 for the circular cross sectional pipe. So, if you put it here and Reynolds number is 1806 Prandtl number is 3.57; let us calculate the L_h. So, $L_h = 0.056 \times 0.005 \times 1806$. So, it will be 0.056m and $L_T = 0.043 \times 0.005 \times 3.57 \times 1806$. So, thermal entrance length is 1.386 m.

So, you can see the length of the pipe is 10.33 m. And we want to calculate the maximum surface temperature which will occur at the exit. So, at L = 10.33 m and hydrodynamic entrance length and thermal entrance length are much less than the length of the tube. So, here it will be fully developed. So, you can see it is a laminar flow and it is a fully developed flow.

(Refer Slide Time: 10:42)



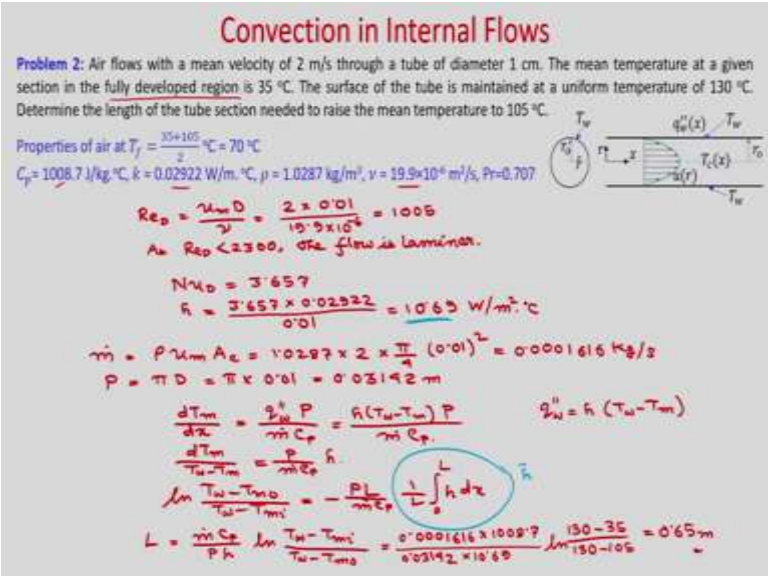
So, Nusselt number for this particular case will be 4.364 .

So, now let us calculate the heat transfer coefficient from here. So, you can see your $\frac{hD}{K}$ is 4.364. So, your; it is average heat transfer coefficient and local heat transfer coefficient as same because, it is a constant value so it is 4.364 thermal conductivity. So, it is $\frac{4.364 \times 0.6405}{0.005}$. So, your heat transfer coefficient is 559 W/m^2.^{0}C.

So, now we have calculate the heat transfer coefficient, but we need to calculate the maximum temperature. So, now, you do the heat flux at the outlet. So, $q_w^{"} = h\left(T_w\big|_{x=L} - T_{mo}\right)$.

So, from here $T_w\big|_{x=L} = T_{mo} + \frac{q_w^{"}}{h}$. So, $80 + \frac{6000}{559}$. So, if you calculate it you will get 90.7^{0}C. So, your maximum temperature at the outlet is 90.7 ^{0}C.

(Refer Slide Time: 12:44)



So, let us take the next problem; Air flows with a mean velocity of 2 m/s through a tube of diameter 1 cm. The mean temperature of a given section in the fully developed region is 35 ^{0}C.

The surface of the tube is maintained at a uniform temperature of 130 ^{0}C. Determine the length of the tube section needed to raise the mean temperature to 105 ^{0}C. So, we can see in this particular case it is a case for uniform wall temperature. And it is a fully developed region already it is told, your mean temperature at a given section it is given as 35 ^{0}C. Now, we have to calculate the length of the tube section, where the mean temperature is to be raised to 105 ^{0}C.

So, first we have to calculate so, first we have to calculate the Reynolds number and we have to check whether it is laminar zone or not. So, if it is a laminar flow and it is a fully developed already told in the problem. So, we can use the Nusselt number for a fully developed condition.

So, Reynolds number first let us calculate. So, $Re = \frac{u_m D}{\nu}$. So, $\frac{2 \times 0.01}{19.9 \times 10^{-6}}$; this is equal to 1005.

So, as Reynolds number < 2300 the flow is laminar. So, the flow is laminar and it is a fully developed region already stated in the problem. So, we can use the Nusselt number based on diameter for this uniform wall temperature is 3.657.

We can calculate the h as $h = \frac{3.657 \times 0.02922}{0.01}$. So, it will be 10.69 W/m^2.^{0}C.

So, now let us calculate the mass flow rate. So, $\dot{m} = \rho u_m A_c$. So, $1.0287 \times 2 \times \pi \times \frac{(0.01)^2}{4}$. So, this if you calculate you will get 0.001616 kg /s. And the P = πD. So, it will be $\pi \times 0.01$=0.03142 m.

Now, from the energy balance analysis we have calculated the $\frac{dT_m}{dx} = \frac{q_w^{"} P}{\dot{m} C_p}$. So, this you know so now, $q_w^{"}$ you can write in terms of heat transfer coefficient so, if you write that.

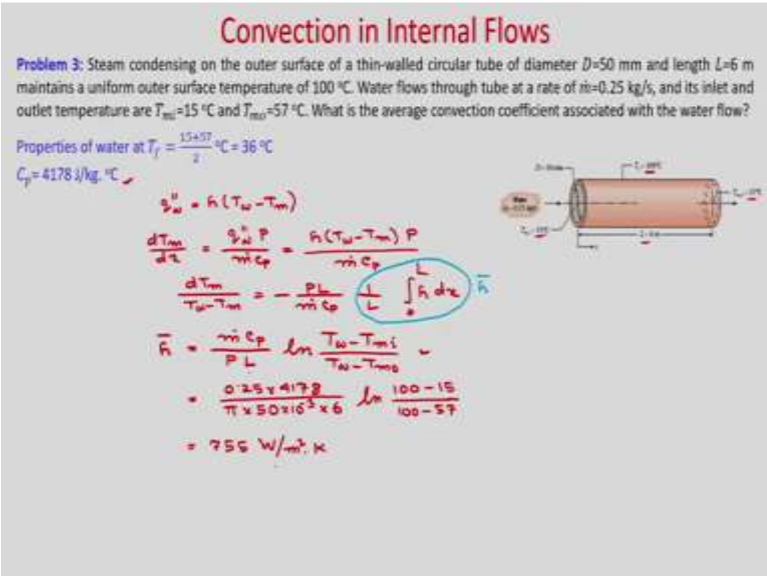
So, what is $q_w^{"}$? $q_w^{"} = h(T_w - T_m)$. So, now if you see that, $\frac{dT_m}{T_w - T_m} = \frac{P}{\dot{m} C_p} h$.

So, if you integrate it from inlet to the outlet. So, you will get $\ln \frac{T_w - T_{mo}}{T_w - T_{mi}} = -\frac{PL}{\dot{m} C_p} \frac{1}{L} \int_0^L h dx$ So, what it is? If you remember this is nothing, but your $\bar{h}$ and in this case it is $\bar{h}$ =h. So, that we have calculated.

So, this value is known so, from here you will be able to calculate the length $L = \frac{\dot{m} C_p}{Ph} \ln \frac{T_w - T_{mi}}{T_w - T_{mo}}$; so this we have reverse because, this minus sign I have taken care here.

So, if you put all the values here. So, you can see it is, $\frac{0.0001616 \times 1008.7}{0.03142 \times 10.69} \ln \frac{130 - 35}{130 - 105}$. So, if you calculate this you will get 0.65 m. So, the length of the tube section needed to raise the mean temperature to 105 ^{0}C is 0.65 m.

(Refer Slide Time: 19:31)



Let us discuss about the next problem; Steam condensing on the outer surface of a thin walled circular tube of diameter D = 50 mm and length L is 6 m, maintains a uniform outer surface temperature of 100 ^{0}C, water flows through tube at a rate of $\dot{m}$ = 0.25 kg /s. And its inlet and outlet temperatures are 15 ^{0}C and 57 ^{0}C respectively. What is the average convection coefficient associated with the water flow?

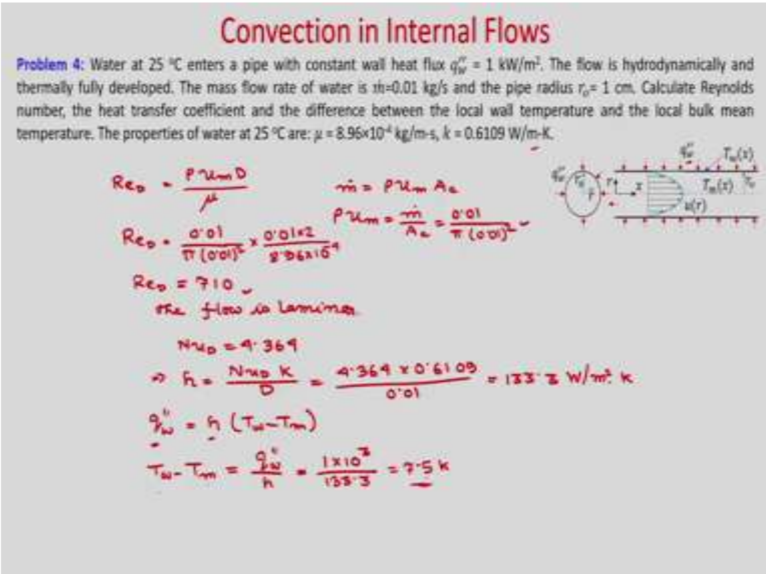
So, we can see this is the tube. So, inner surface is maintained at 100 ^{0}C and water is flowing through, it $\dot{m}$ = 0.25 kg /s. And inlet mean temperature is 15 ^{0}C outlet mean temperature is 57 ^{0}C , L is 6 m. So, now, we have to calculate the average heat transfer coefficient. So, properties of water at T_f = 36 ^{0}C is this one C_p we need for this calculation.

So, now if you see that we know heat flux $q_w^{"} = h(T_w - T_m)$. So, we have calculated just now $\frac{dT_m}{dx} = \frac{q_w^{"}P}{\dot{m}C_p}$. So, this is $\frac{h(T_w - T_m)P}{\dot{m}C_p}$. So, now, $\frac{dT_m}{T_w - T_m} = -\frac{PL}{\dot{m}C_p}\frac{1}{L}\int_0^L h dx$.

So, as earlier problem this represents your average heat transfer coefficient. So, from here you can see your $\bar{h}$, after integration you can write $\bar{h} = \frac{\dot{m}C_p}{PL}\ln\frac{T_w - T_{mi}}{T_w - T_{mo}}$. So, this we have already calculated in the last problem, there you calculate the L in this particular case we are calculating the $\bar{h}$. So, you put all the values.

So, $\dot{m}$ =0.25 kg /s. So, $\frac{0.25 \times 4178}{\pi \times 50 \times 10^{-3} \times 6}\ln\frac{100-15}{100-57}$. So, if you calculate it your $\bar{h}$ you will get 755 W/m^{2}K.

(Refer Slide Time: 23:16)



So, let us discuss about this problem; Water at 25 ^{0}C enters a pipe with constant wall heat flux $q_w^{"}$ =1 kW/m^2. The flow is hydro dynamically and thermally fully developed. The mass flow rate of water is $\dot{m}$ =0.01 kg /s and the pipe radius r_0 is 1cm calculate Reynolds number, the heat transfer coefficient. And the difference between the local wall temperature and the local bulk mean temperature.

The properties of water at 25 ^{0}C are given μ and k . So, this is a case of uniform wall heat flux boundary condition. So, for this particular case first you calculate the Reynolds number . So, it is water. So, rho you know so, your $\mathrm{Re} = \frac{\rho u_m D}{\mu}$. And mass flow rate is given $\dot{m}$.

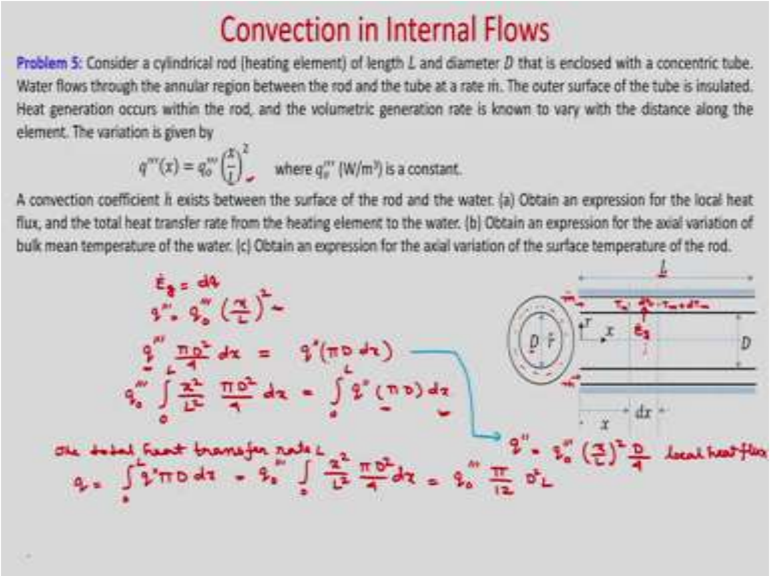
So, you can calculate $\dot{m} = \rho u_m A_c$. So, what is ρu_m ? $\rho u_m = \frac{\dot{m}}{A_c}$ and it is $\frac{0.01}{\pi(0.01)^2}$.

Now, Reynolds number you can calculate as $\mathrm{Re}_D = \frac{0.01}{\pi(0.01)^2} \times \frac{0.01 \times 2}{8.96 \times 10^{-4}}$. So, Reynolds number from here if you calculate you will get 710 . So, this is your; obviously, < 2300. So, the flow is laminar and in fully developed condition Nusselt number for constant wall temperature boundary condition as the flow is laminar and it is a fully developed flow.

So, the Nusselt number for uniform wall heat flux boundary condition is 4.36. So, from here you can see Nusselt number is 4.364. So, from here you can calculate $h = \frac{Nu_D K}{D}$. So, it will be $\frac{4.364 \times 0.6109}{0.01}$. So, if you calculate it you will get 133.3 W/m^2.K Now, what is the difference between the local wall temperature and the local bulk mean temperature?

So, we know $q_w^" = h(T_w - T_m)$ where $q_w^"$ is known h we have found. So, you can calculate $T_w - T_m = \frac{q_w^"}{h}$. So, $q_w^"$ =1 kW/m^2. So, $\frac{1 \times 10^3}{133.3}$; so you will get 7.5 Kelvin. So, you can see the difference between the local wall temperature and local bulk mean temperature is 7.5 K.

(Refer Slide Time: 27:38)



So, now we will consider this problem; Consider a cylindrical rod heating element of length L and diameter D that is enclosed with a concentric tube. Water flows through the annular region between the rod and the tube, at a rate m dot the outer surface of the tube is insulated heat generation occurs within the rod. And the volumetric generation rate is known to vary with a distance along the element. The variation is given by this relation.

So, q''' is the heat generation rate and per unit volume and q_0''' is a constant, a convection coefficient h exist between the surface of the rod and the water. So, obtain an expression for the local heat flux. And the total heat transfer rate from the heating element to the water, b) obtain an expression for the axial variation of the bulk mean temperature of the water.

And c) obtain an expression for the axial variation of surface temperature of the rod. So, you can see the inside tube. So, this diameter D is shown. So, for this is the solid rod and it is the heat generating rod heating element. And for the flow is occurring in the annulus.

So, you can see this is the space through which your flow is fluid is flowing. So obviously, you can see that if a at a distance x if you take a dx length. So, you can see in this volume whatever heat is generated so, a from the surface at a steady state whatever it is leaving the surface that is actually taken by the fluid. So, that energy balance we can do.

So, you can see it will be just $\dot{E}_g$ whatever is heat generated. So, that is the dq taken by the fluid. And if at the inlet the mean temperature is T_m. So, T_m + d T_m. So, there is a increase in the mean temperature d T_m and $\dot{m}$ is the mass flow rate. So, the energy balance you can see that $\dot{E}_g = dq$ from this elemental volume. So, $q''' = q_0'''\left(\frac{x}{L}\right)^2$.

So, you can see if you put it here $\dot{E}_g$. So, $\dot{E}_g$ is q''' into the volume. So, what is that? So, it will be $q''' = \frac{\pi D^2}{4}dx$. So, you can see the surface heat transfer surface is πDdx through which actually, this is going. And what is the heat is generated in the volume? So, this is the volume right.

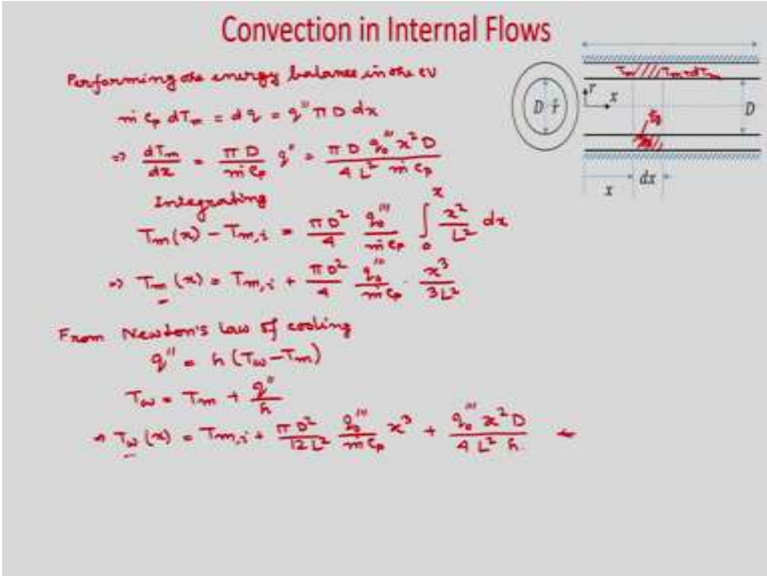
So, this is the volume and D is the diameter. So, pi D square by 4 into dx. So, this is the volume of this heating element cylindrical rod. So, this volume is $\frac{\pi D^2}{4}dx$. So, into q''' will give you the total heat living the surface. And now the circumferential area through which your heat is transferred to the fluid that area is πDdx. So, that is the heat transfer area and the heat flux, if it is q double prime then $q''' = q''\left(\pi Ddx\right)$.

So, what we will do? Now, you put this expression here because, $q_0'''\int_0^L \frac{x^2}{L^2}\frac{\pi D^2}{4}dx = \int_0^L q''\left(\pi D\right)dx$. So, this is the expression so, from here you can see. .

What is q''? $q'' = q_0'''\left(\frac{x}{L}\right)^2\frac{D}{4}$. So, this is your local heat flux local heat flux. So, now, if you calculate the; total heat transfer rate. So, the total heat transfer rate. What is that? So, $q = \int_0^L q''\pi Ddx$.

So, now if you put this one only the left hand side then, you will get $q_0'''\int_0^L \frac{x^2}{L^2}\frac{\pi D^2}{4}dx$. So, if you carry out the integration you will get, $q_0'''\frac{\pi}{12}D^2L$.

(Refer Slide Time: 34:30)



So, now you perform the energy balance in the control volume. So, performing the energy balance in the control volume. So, this is the control volume. So, whatever $\dot{E}_g$ is generated. So, dq is taken by the fluid. So, you can see you will get $\dot{m} C_p dT_m$ and dT_m is the change in the mean temperature.

So, $\dot{m} C_p dT_m = dq = q'' \pi D dx$. So, you can write $\frac{dT_m}{dx} = \frac{\pi D}{\dot{m} C_p} q''$ and you can write $q'' = q_0'''\left(\frac{x}{L}\right)^2 \frac{D}{4}$. So, if you write that expression. So, you will get $\frac{\pi D q_0''' x^2 D}{4L^2 \dot{m} C_p}$.

Now, if you integrate it, the mean temperature $T_m(x) - T_{mi} = \frac{\pi D^2}{4} \frac{q_0'''}{\dot{m} C_p} \int_0^x \frac{x^2}{L^2} dx$. So, your mean temperature variation you can see $T_m(x) = T_{mi} + \frac{\pi D^2}{4} \frac{q_0'''}{\dot{m} C_p} \frac{x^3}{3L^2}$. So, you have found the mean temperature variation.

Now, you have to find the wall temperature variation along the axial direction. So, that if you do this. So, from Newton's law of cooling you can write $q_w'' = h\left(T_w - T_m\right)$ and T_m already we have found from this expression.

So, $T_w = T_m + \frac{q_w^{"}}{h}$. So, you can see you can write $T_w(x) = T_{mi} + \frac{\pi D^2}{12L^2}\frac{q_0^{"'}}{\dot{m}C_p}x^3 + \frac{q_0^{"'}x^2 D}{4L^2 h}$.

So, $q^{"}$ if you write. So, you can write as this expression. So, it will be $q^{"} = q_0^{"'}\left(\frac{x}{L}\right)^2 \frac{D}{4}$.

So, you can see this is the expression of wall temperature variation along x.

So, today we solved total 5 example problems in most of the problem, you have seen that we have considered fully developed flow both hydro dynamically and thermally. But, before taking the Nusselt number for a fully developed condition first you cross check. What is the Reynolds number and the whether it is fully developed flow?

If Reynolds number < 2300, then the laminar the flow is laminar. And if you see the entrance length and this entrance length is less than the required length of the pipe, where you want to find the heat transfer coefficient or the heat flux. Then you can use the Nusselt number expression for thermally and hydro dynamically fully developed laminar flow condition.

So, you know that for uniform wall heat flux boundary condition, for flow through a pipe the Nusselt number is 4.364 and that we have used to calculate the heat transfer coefficient. In other case from the energy balance, you can calculate the heat transfer coefficient, but that heat transfer coefficient; obviously, it is a local heat transfer coefficient.

And after doing the integration you can calculate the average heat transfer coefficient. And these we have already studied in the initial lectures of these convection in internal flows. Also we have taken up one problem where inside the rod heat generation is taking place.

So, and heat generation per unit volume; we have considered and that we have equated with the heat flux at the wall, carried by the fluid. And at steady state you know whatever is heat is generated inside the heat generating element, that heat will be carried out by the fluid.

So, with that energy balance we have used and we have calculated the total heat transfer rate. And also we have done the energy balance in the fluid volume, where the mean temperature at inlet we have consider and mean temperature the outlet we have consider.

So, $\dot{m} C_p dT_m$ that is the fluid is carrying the energy equal to whatever the heat flux into the area is has come from the heat generating body. So, that we have equated and from that expression, we have calculated the variation of mean temperature along the axial direction and equating or using the Newton's law of cooling. We have calculated the variation of wall temperature in the axial direction.

So, if you have seen that in last three modules, we have solved several problems. Initially we considered thermally and hydro dynamically fully developed flow. Because, the calculation is easy the analysis is easy. And, also we started with the slug flow where you have a constant uniform velocity u m. And, then we considered fully developed condition, where the velocity profile is parabolic and also we considered thermally fully developed flow, where the non dimensional temperature of π does not change in the axial direction.

Later we considered fully hydro dynamically fully developed flow, but thermally developing flow. So, for this particular case also we considered two different types of boundary conditions constant wall heat flux and constant wall temperature boundary conditions.

In each cases we found the expression of local Nusselt number and the average Nusselt number. However, we have not studied where it is a thermally and hydro dynamically developing flow. So, it is thus analysis is more complicated, if you are interested you can refer some books convection heat transfer book and you can derive the Nusselt number for this particular case as a homework.

Thank you.

Fundamentals of Convective Heat Transfer

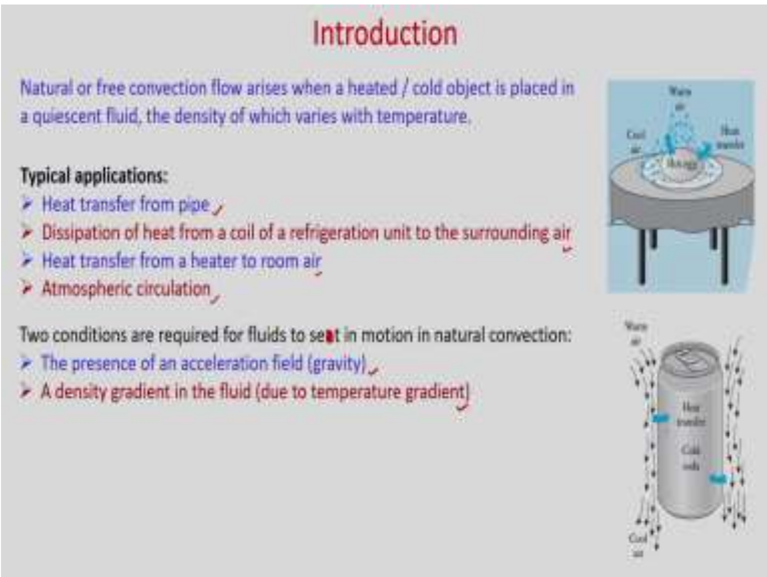
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 08
Natural Convection
Lecture - 26
Introduction and scale analysis

Hello everyone. So, till now we have considered forced convection. In forced convection the fluid is forced to flow over a surface or inside a tube by external means like using pump or blower. Today, we will start natural convection which is also known as free convection.

In natural convection fluid motion starts in natural way due to the temperature difference and hence, there will be density difference and in the presence of acceleration like gravitational acceleration.

(Refer Slide Time: 01:16)



So, you can see natural or free convection flow arises when a heated or cold object is placed in a quiescent fluid. The density of which varies with temperature. So, these are some typical applications; heat transfer from pipe, dissipation of heat from a coil of a refrigeration unit to the surrounding air, heat transfer from a heater to room air, and atmospheric circulation.

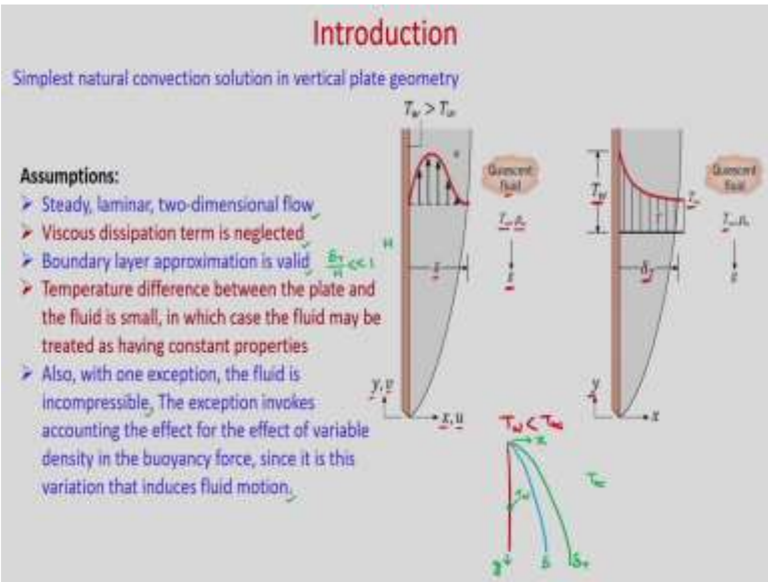
So, as I told before the two conditions are required, for fluids to set in motion in natural convection, the presence of an acceleration field like gravity and a density gradient in the fluid which may occur due to temperature gradient. So, you can see here in the right hand side figure. So, if you keep one hot egg in a plate then; obviously, surrounding fluids temperature is lower than the hot egg.

So, the fluid which is coming into contact with hot egg will have higher temperature and it will have lower density and it will go up. So, warm air will go up and cool air will come to the hot egg. Again it will get heated and it will go up. So, in this way you can see that fluid motion starts.

Another example of this cold soda; so, it is actually warmed up in presence of the ambient air. So, you can see when you keep it in the ambient air, warm air will come into contact, there will be heat transfer its density will increase and it will go down.

So; obviously, warm air will come into contact in the upper half and there will be heat transfer due to that a density will increase and cool air will come down. So, these are some examples of natural convection. We can have simplest natural convection solution for flow over vertical plate.

(Refer Slide Time: 03:32)



So, if you consider this vertical plate which is maintained at temperature T_w and the quiescent fluid is having temperature T_∞ and its density is $\rho\ \infty$ there will be acceleration field and that is your gravity.

So, we are taking x as normal to this vertical plate and velocity in that direction we will consider as u and along the plate we are taking y coordinate and velocity in that direction will be v. As $T_w > T_\infty$; obviously, your fluid will go up and thermal boundary layer and hydrodynamic boundary layer will form over this plate. So, here you can see in the first figure.

So, there will be a hydrodynamic boundary layer formation, whose hydrodynamic boundary layer thickness is δ and you can see the velocity distribution; obviously, for no slip condition, on the vertical wall velocity will be 0 and outside ambient is quiescent fluid so; obviously, at the edge of the boundary layer velocity will be almost 0. So, maximum velocity will occur inside this hydrodynamic boundary layer.

If you consider the thermal boundary layer; so, similarly it will also start from the bottom and δ T is the thermal boundary layer thickness. It will grow as you go up; that means, with increase in y δ T will also increase and $T_\infty < T_w$.

So, this will be the temperature distribution. So, at the wall you have T_w and at the edge of the thermal boundary layer you have temperature T_∞. If you consider $T_w < T_\infty$ so in that case your boundary layer will start forming from the top edge. So, that will be your δ and also you have thermal boundary layer δ_T. So, this is your δ_T and this is your δ and this is your direction y and this is your x this is your T_w and quiescent medium is having temperature T_∞.

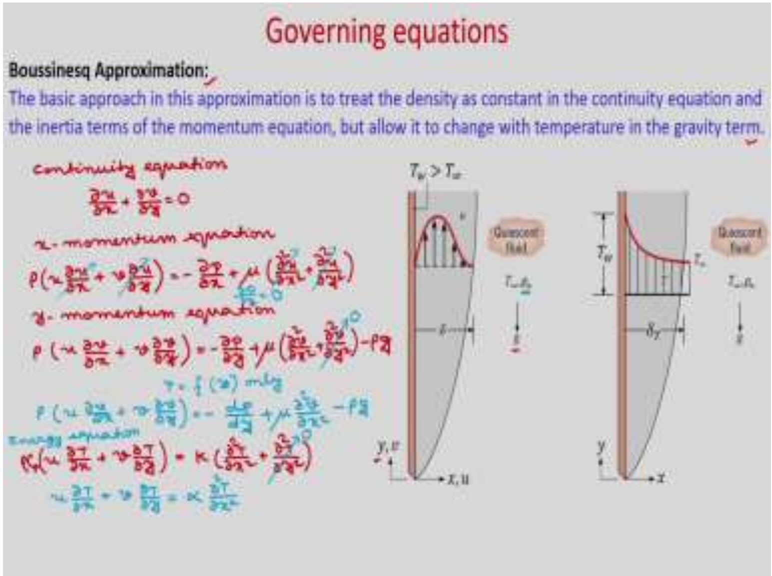
So, first we will write down the governing equations for flow over vertical plate. Let us first make the assumptions. So, you can see the flow is steady laminar and two dimensional viscous. Dissipation term is neglected and; obviously, we will have boundary layer approximation is valid.

And if we consider the height of the vertical plate as H then we can consider that $\frac{\delta_T}{H}$ will be << 1. Temperature difference between the plate and the fluid is small, in which case the fluid maybe treated as having constant properties. So, we are considering the temperature difference as low and the thermal physical properties like viscosity, thermal conductivity, specific heat, we will assume that these to be constant.

Also, with one exception the fluid is incompressible the exception invokes accounting the effect for the effect of variable density in the buoyancy force since it is this variation that induces fluid motion. So, you can see another important assumptions we are taking that we are considering ρ to be constant.

So, that it will become incompressible, but as temperature difference will be there, there will be density difference and that density difference effect will take into account in buoyancy term only and rest other terms like continuity equation and the in inertia term of the momentum equation, we will take ρ as constant.

(Refer Slide Time: 08:19)



So, in natural convection whatever we discussed just now that is known as Boussinesq approximation. The basic approach in this approximation is to treat the density as constant in the continuity equation and the inertia terms of the momentum equation, but allow it to change with temperature in the gravity term. So, this is important assumptions we are taking which is known as Boussinesq approximation.

So, in this approximation what we are telling; that we will take density as constant in the continuity equation as well as in inertia terms of momentum equation, but we will take the effect of its change in the buoyancy term. So, first let us write the governing equations. So, what are the governing equations? First is continuity equation.

So, we consider two dimensional flow. So, it will be $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$ and x momentum equation. So, $\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$ and y momentum equation $\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\rho g$.

Now, first write down the boundary layer equations. So, as we discussed in the forced convection in boundary layer approximation that δ_T << H k. So, in this particular case you can see that δ is the hydrodynamic boundary layer thickness and δ_T is your thermal boundary layer thickness and that will be much-much smaller than the height of the plate.

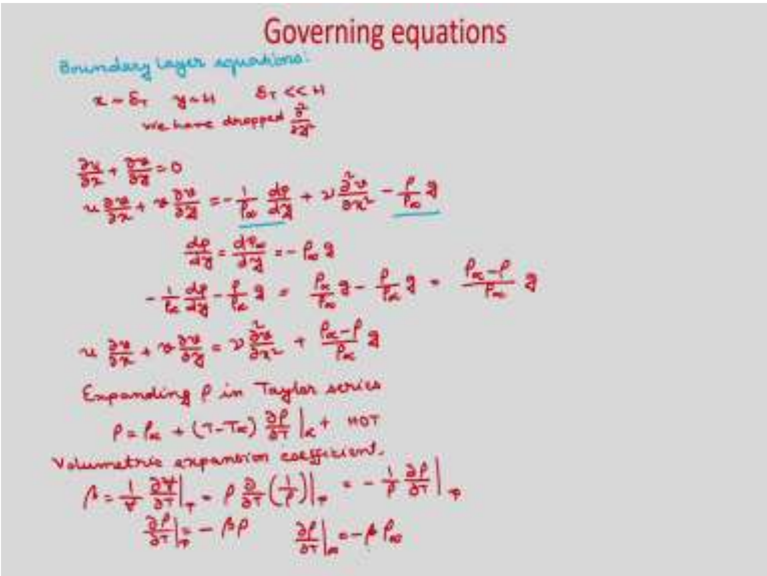
So, in this case now you can have the assumptions. So, if you have the similar way as we have done in the forced convection you can neglect $\frac{\partial^2 v}{\partial y^2}$ and here in x momentum equation all these terms will become negligible only from here so, all this terms will become negligible and you will have $\frac{\partial p}{\partial x}$ =0. And as $\frac{\partial p}{\partial x}$ is 0, p is function of y only, then y momentum equation you can write as $\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{dp}{dy}+\mu\frac{\partial^2 v}{\partial x^2}-\rho g$. So, this is your boundary layer equation for flow over vertical plate and this is your y momentum equation. In x momentum equation you will get $\frac{\partial p}{\partial x}$ =0. Now, let us write the energy equation.

So, in general your energy equation is $\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$. So, if you write the boundary layer equation then it will be 0. So, this is your energy equation.

So, you can write $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}$. Now, you see the y momentum equation. In the y momentum equation in the left hand side we have density ρ. So, invoking the Boussinesq approximation will take ρ as constant and it will be same as ρ_∞ .

So, in the first impression whatever you have ρ_∞ which is actually quiescent medium. So, that will be $\rho = \rho_\infty$. So, invoking the Boussinesq approximation will take the ρ in the left hand side of the y momentum equation as ρ_∞ which is your quiescent medium density ρ_∞. So, you can see this ρ_∞.

(Refer Slide Time: 14:07)



So, you can write down all these boundary layer equations now. So, you can write these boundary layer equations. So, whatever we invoked we have taken x ~ δ_T, y ~H and we have assumed δ_T << H and we have dropped $\frac{\partial^2}{\partial y^2}$ these terms.

So, your continuity equation is, your continuity equation will remain same $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$ and invoking the Boussinesq approximation and rearranging you can write the y momentum equation as $u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho_\infty}\frac{dp}{dy}+\nu\frac{\partial^2 v}{\partial x^2}-\frac{\rho}{\rho_\infty}g$.

Now, you have buoyancy term. So, what is your buoyancy term. So, buoyancy term will become $-\frac{\rho}{\rho_\infty} g$. So, now here you can see how you will calculate the $\frac{dp}{dy}$. So, $\frac{dp}{dy}$ you can write as $\frac{dp}{dy} = \frac{dp_\infty}{dy}$, because $\frac{\partial u}{\partial x} =$ 0. So, whatever pressure is there outside the boundary layer so that will be impressed inside the boundary layer. So, now, in the quiescent medium so you can write the hydrostatic pressure right. So, what will be the $\frac{dp_\infty}{dy}$?

So, this you can write as $-\rho_\infty g$. So, this is your from hydrostatic pressure distribution you can write it. So; obviously, you can see now these two terms together you can write this term and this term you can write as $-\frac{1}{\rho_\infty}\frac{dp}{dy} - \frac{\rho}{\rho_\infty} g = \frac{\rho_\infty}{\rho_\infty} g - \frac{\rho}{\rho_\infty} g$.

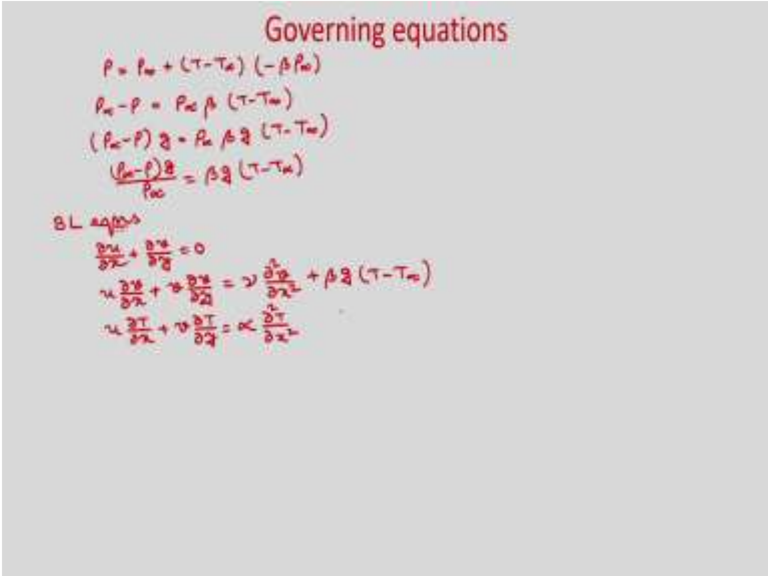
So, it will become $\frac{\rho_\infty - \rho}{\rho_\infty} g$. So, your momentum equation buoyancy term you can write as, $\frac{\rho_\infty - \rho}{\rho_\infty} g$ and now you can expand the ρ in Taylor series, expanding ρ in Taylor series. So, what you can write? $\rho = \rho_\infty + \left(T - T_\infty\right)\frac{\partial \rho}{\partial T}\Big|_\infty + HOT$.

So, neglect this high order term and you also you can have volumetric expansion coefficient as volumetric expansion coefficient, you can write $\beta = \frac{1}{\forall}\frac{\partial \forall}{\partial T}\Big|_p$.

So, here this is the $\forall$ volume at constant pressure. So, in terms of density if you write. So, it will be $\rho \frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)\Big|_p$. So, it will be $-\frac{1}{\rho}\frac{\partial \rho}{\partial T}\Big|_p$. So, what you can write $\frac{\partial \rho}{\partial T}\Big|_p = -\beta\rho$.

So, similarly you can write $\frac{\partial \rho}{\partial T}\Big|_\infty = -\beta\rho_\infty$. So, if you invoke these in this equation so what you will get? You will get $\rho = \rho_\infty + \left(T - T_\infty\right)\left(-\beta\rho_\infty\right)$.

(Refer Slide Time: 19:19)



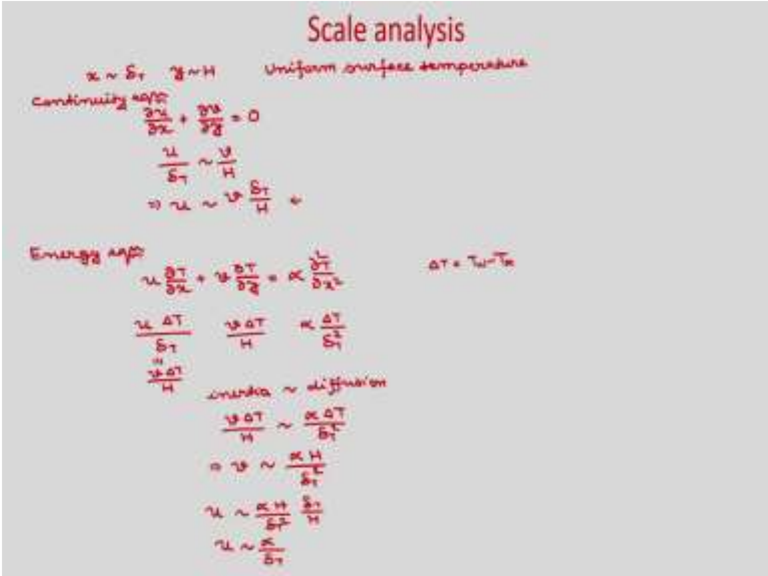
So, you can rearrange and you can write $\rho_\infty - \rho = \rho_\infty \beta (T - T_\infty)$ and $(\rho_\infty - \rho) g = \rho_\infty \beta g (T - T_\infty)$. So, now, you can see this term $\frac{(\rho_\infty - \rho) g}{\rho_\infty} = \beta g (T - T_\infty)$. So, this is your buoyancy term.

So, now your boundary layer equations will become continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, your x momentum equation will become $u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial x^2} \beta g (T - T_\infty)$. Now, the buoyancy term $\beta g (T - T_\infty)$ and your energy equation $u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2}$.

So, these are the boundary layer equations for flow over vertical plate, for the case of natural convection and we have assumed density to be constant in continuity equation and in the inertia terms of y momentum equation and we have taken the change of density only in the buoyancy term. So, these are the equations.

Now, we will do the scale analysis and we will try to find what is the order of heat transfer coefficient and the Nusselt number and which are the forces is dominating for high panel number fluids and low panel number fluids for constant surface temperature. So, first we will consider uniform surface temperature case, then we will find the order of magnitude of this heat transfer coefficient and the Nusselt number.

(Refer Slide Time: 22:15)



So, inside the thermal boundary layer your x ~ δ_T, y ~ H, where H is the height of the plate and we are assuming that uniform surface temperature and the continuity equation if you see that will be $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. So, if you see the order of magnitude of u. So, $\frac{u}{\delta_T} \sim \frac{v}{H}$.

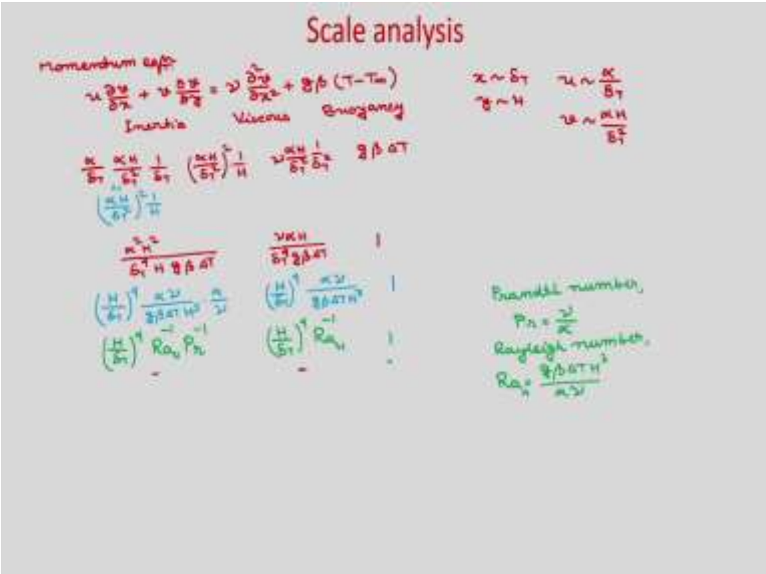
So, from here you can see $u \sim v\frac{\delta_T}{H}$, from energy equation. Now; so, this is your continuity equation now from energy equation $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2}$. So, the order of $\Delta T = T_w - T_\infty$ and this Δ T ~δ T. So, you can see here you can write $\frac{u\Delta T}{\delta_T}$.

So, this is the first inertia term the second inertia term. So, you can write $\frac{v\Delta T}{H}$ and the diffusion term will be $\alpha\frac{\Delta T}{\delta_T^2}$. So, now, you can see if you put $u \sim v\frac{\delta_T}{H}$. So, if you put it here you can see it will be equivalent to $\frac{v\Delta T}{H}$. So, you can see these inertia terms are comparable, because these are same $\frac{v\Delta T}{H}$ and this is your diffusion term.

So, now inertia should be comparable with the diffusion term. So, if you do that. So, you will get $\frac{v\Delta T}{H} \sim \alpha \frac{\Delta T}{\delta_T^2}$. So, from here you can see the, what is the scale of v. So, $v \sim \frac{\alpha H}{\delta_T^2}$. So, we have found the scale for v and similarly from this equation you can find the scale for u.

So, $u \sim \frac{\alpha H}{\delta_T^2}\frac{\delta_T}{H}$. So, $u \sim \frac{\alpha}{\delta_T}$. So, now, from using continuity equation and the energy equation, we have found the scale for velocity u and v. Now, let us consider momentum equation.

(Refer Slide Time: 25:59)



So, momentum equation we have $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\beta\left(T - T_\infty\right)$. So, these are inertia terms, this is your viscous term, and this is your buoyancy term and what is the order we have? x ~ δ_T, y ~H, $u \sim \frac{\alpha}{\delta_T}$, and $v \sim \frac{\alpha H}{\delta_T^2}$. So, using this scale now let us see each term.

So, the first inertia term if you see so, if we put $\frac{\alpha}{\delta_T}\frac{\alpha H}{\delta_T^2}\frac{1}{\delta_T}$. So, this is your first inertia term. Second inertia term now v so, it will be $\left(\frac{\alpha H}{\delta_T^2}\right)^2\frac{1}{H}$. So, first look at this two terms. So, this is equivalent to this. So, you can see this two inertia terms are of same order, because both are having the same order $\left(\frac{\alpha H}{\delta_T^2}\right)^2 \frac{1}{H}$. now, let us see the viscous term. So, in the viscous term now we have ν. So, $v \sim \frac{\alpha H}{\delta_T^2}$ and x ~ δ_T and this buoyancy term, so $g\beta\Delta T$.

So, now what we will do we will just divide this terms with $g\beta\Delta T$. So, that it will become 1 buoyancy. So, we are dividing by $g\beta\Delta T$. So, you will get so, this term if you see. So, it will be $\frac{\nu\alpha H}{\delta_T^3 g\beta\Delta T}$. So, this is your viscous term and now you divide this term. So, these two terms are same. So, I will write only one term.

So, it will be $\frac{\alpha^2 H^2}{\delta_T^4 H g\beta\Delta T}$. Now, we will rearrange it. So, if you rearrange it so, you can see the viscous term . We will write it as so this will be 1 buoyancy term, it will be $\frac{\alpha\nu}{g\beta\Delta T H^3}$. So, now, H cube we have just divided. So, we will multiply in the numerator H.

So, it will be $\left(\frac{H}{\delta_T}\right)^4 \frac{\alpha\nu}{g\beta\Delta T H^3}\frac{\alpha}{\nu}$.

So, you can see, you can write it as now Prandtl number, you know right what is the Prandtl number? So, $\Pr = \frac{\nu}{\alpha}$ and we will define another non dimensional number that is known as Rayleigh number. So, Rayleigh number we are defining as $\mathrm{Ra}_H = \frac{g\beta\Delta T H^3}{\alpha\nu}$.

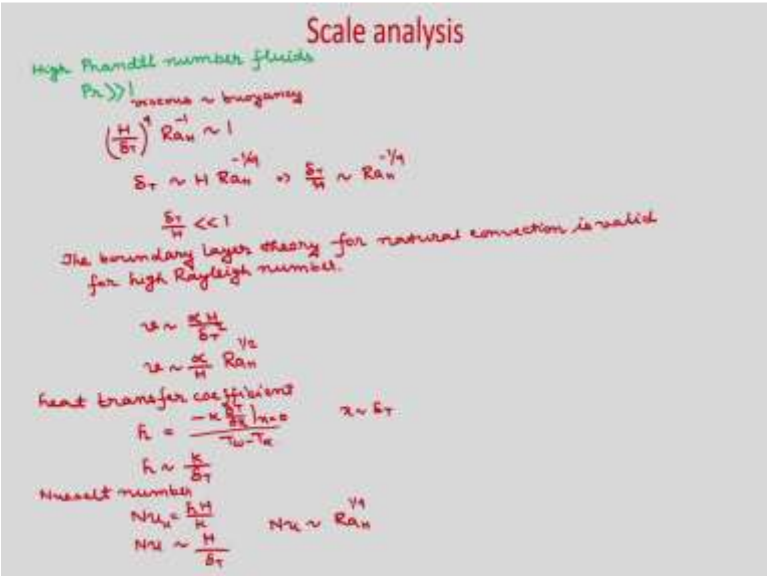
So, what you can do now, this you can write as $\left(\frac{H}{\delta_T}\right)^4 \mathrm{Ra}_H^{-1}\Pr^{-1}$. This is your $\left(\frac{H}{\delta_T}\right)^4 \mathrm{Ra}_H^{-1}$ and Rayleigh number we have defined based on the height of the plate. So, this is you can write H, here also you can write H and this is the buoyancy term this is 1.

Now, we will consider two different cases one is high Prandtl number fluids and low Prandtl number fluids. So, for high Prandtl number fluids, Prandtl number will be >>1 and you can see in this particular case. So, it is $\frac{1}{\Pr}$. So, if it is $\frac{1}{\Pr}$ and Prandtl number is very high. So, this term will be dominant term .

So, this term will be dominant term for the high Prandtl number fluids and when you have low Prandtl number fluids so; obviously, this will be your dominant term, but in each cases buoyancy term should be present.

So, either your inertia term will be comparable with buoyancy term or your viscous term will be comparable buoyancy term, because to have the natural convection you should have the buoyancy term present.

(Refer Slide Time: 33:48)



So, in two different cases now we will consider high Prandtl number fluids and low Prandtl number fluids. So, first we will consider high Prandtl number fluids. So, Prandtl number >> 1. So, you can see your $\left(\frac{H}{\delta_T}\right)^4 \mathrm{Ra}_H^{-1} \sim 1$.

So; that means, viscous term will be comparable with buoyancy term, because Prandtl number is very high. So, the inertia term you can see you have $\frac{1}{\Pr}$. So, and Prandtl number is very high. So, that will be negligible compared to the viscous term and for high Prandtl number fluids you can compare the viscous force with the buoyancy force. So, that we are doing.

So, with this now you can find what is δ_T. So, $\delta_T \sim H\,\mathrm{Ra}_H^{-1/4}$ and we know that $\frac{\delta_T}{H}$ we are assumed that it is >> 1. From here you can see $\frac{\delta_T}{H} \sim \mathrm{Ra}_H^{-1/4}$.

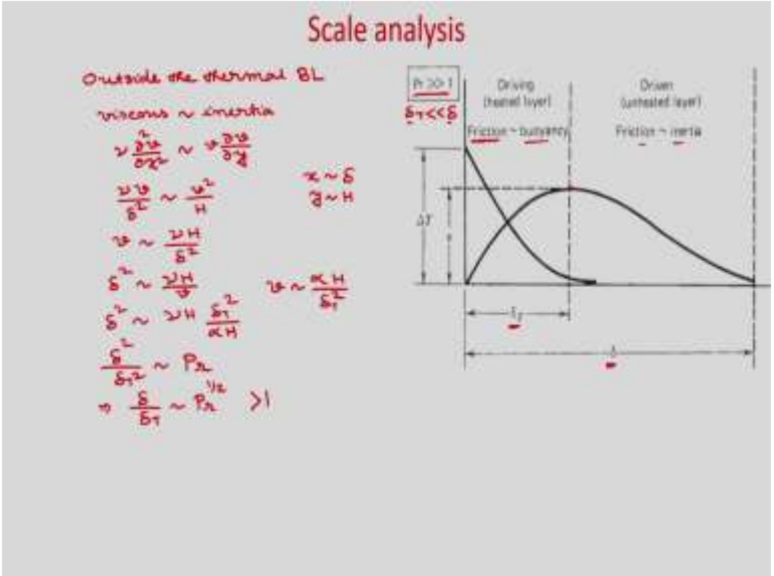
If $\frac{\delta_T}{H}$<<1 then this Rayleigh number should be very high, because it is of the order of $\mathrm{Ra}_H^{-1/4}$. So, it is minus is there. So, you can write the boundary layer theory for natural convection is valid for high Rayleigh number fluids. So, now, you can find what is the scale of velocity v. So, we know $v \sim \frac{\alpha H}{\delta_T^2}$. So, $\frac{\delta_T}{H} \sim \mathrm{Ra}_H^{-1/4}$.

So, from here you can write $v \sim \frac{\alpha}{H}\mathrm{Ra}_H^{1/2}$. So, this is the scale for velocity v for high Prandtl number fluids. Now, let us find what is the heat transfer coefficient. So, for heat transfer coefficient so, you know $h = \frac{-K \frac{\partial T}{\partial x}\big|_{x=0}}{T_w - T_\infty}$.

So, as you know $x \sim \delta_T$. So, you can write $h \sim \frac{K}{\delta_T}$. So, now, Nusselt number. So, it will be $Nu_H = \frac{hH}{K}$. So, based on height H so Nusselt number will be order of; so you can see, $Nu_H \sim \frac{H}{\delta_T}$. So, $Nu_H \sim \mathrm{Ra}_H^{1/4}$.

So, you can see from the scale analysis we have found the scale for δ_T, thermal boundary layer thickness, your velocity v, and heat transfer coefficient and Nusselt number. You can see $Nu_H \sim \mathrm{Ra}_H^{1/4}$. So, later when we will do the analytical solution for the boundary layer equations for natural convection you will find that $Nu_H \sim \mathrm{Ra}_H^{1/4}$.

(Refer Slide Time: 38:35)



Now, let us see that for high Prandtl number fluids so; obviously, for high Prandtl number fluids $\delta_T << \delta$ right, where δ_T is your thermal boundary layer thickness and δ is your hydrodynamic boundary layer thickness. So, in this case you can see that this is your thermal boundary layer thickness.

So, temperature will vary. It will be high at the wall and gradually it will decrease and in the quiescent medium; obviously, there will be temperature is T_∞ and δ_T will almost will become 0 . So, this is the variation of temperature inside the thermal boundary layer, but as $\delta >> \delta_T$.

So, the effect of velocity will be still there outside this thermal boundary layer and you can see in the outside there will be effect of this velocity and you will get maximum velocity; obviously, it will be inside the thermal boundary layer. So, maximum velocity you will get inside the thermal boundary layer. So, inside the thermally boundary layer if you consider so; obviously, your viscous force will be order of buoyancy force; that means, friction force.

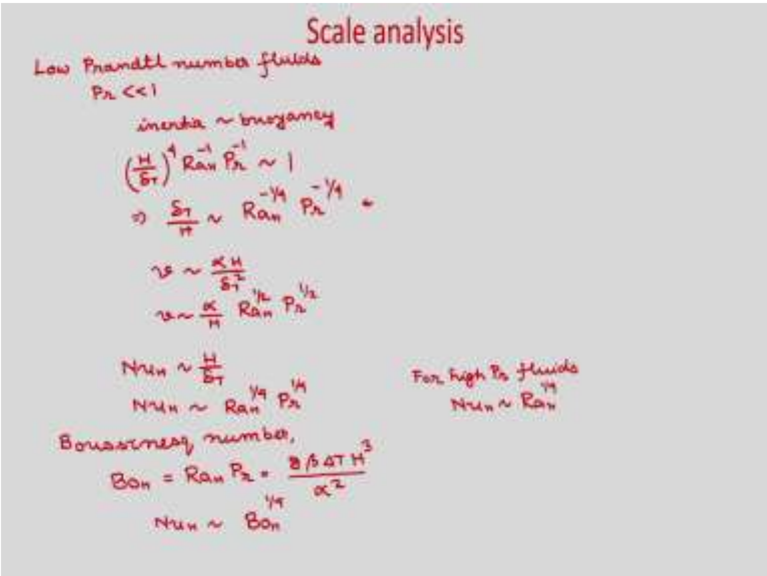
So, because you it is near to the solid wall. So, the viscous effect will be there and viscous effect will be comparable to the buoyancy force, but in the unheated layer. So, you can see; obviously, your buoyancy will be absent, because there is no temperature difference outside this thermal boundary layer. So, your buoyancy force will be absent and your viscous force will be comparable with the inertia force.

Now, you can see that outside the thermal boundary layer. So, you will have viscous force will be comparable with inertia force. So, you can see viscous force, viscous force will be comparable with the so this is your viscous and this is your inertia.

So, from here you can see. So, viscous term will be your $\nu \dfrac{\partial^2 v}{\partial x^2}$ will be comparable with the inertia term. So, that will be any term you can say. So, it will be let us say $v \dfrac{\partial v}{\partial y}$. So, in this case now it will be $\dfrac{\nu v}{\delta^2} \sim \dfrac{v^2}{H}$. So, from here you can see the velocity $v \sim \dfrac{\nu H}{\delta^2}$ and $\delta^2 \sim \dfrac{\nu H}{v}$ and v we have already found, it will be $v \sim \dfrac{\nu H}{\delta^2}$. So, if you put the value of v. So, it will be $\delta^2 \sim \nu H \dfrac{\delta^2}{\alpha H}$. So, you can see $\dfrac{\delta^2}{\delta_T^2}$.

So, this H H will get cancelled and $\Pr = \dfrac{\nu}{\alpha}$. So, you can see $\dfrac{\delta}{\delta_T} \sim \Pr^{1/2}$ which will be >> 1 right, because Prandtl number > 1 and from here you can see that $\delta > \delta_T$.

(Refer Slide Time: 43:17)



Now, let us consider low Prandtl number fluids; that means, Prandtl numbers << 1, low Prandtl number fluids. So, in this particular case now as Prandtl number is low so inertia force will be dominant . So, you can neglect the viscous force. So, inertia force will be comparable with the buoyancy force right. So, in this case inertia force will be comparable with buoyancy force so; obviously, it will be $\left(\frac{H}{\delta_T}\right)^4 \mathrm{Ra}_H^{-1}\,\mathrm{Pr}^{-1} \sim 1$.

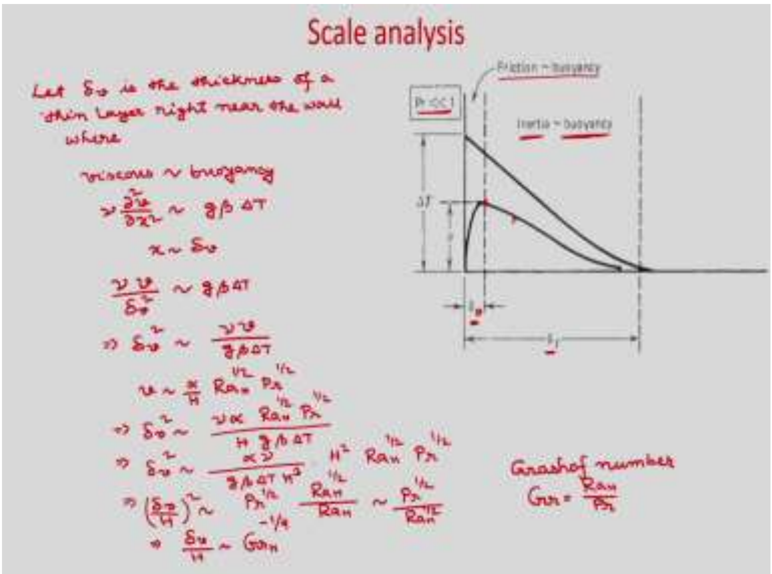
So, from here you can see $\frac{\delta_T}{H} \sim \mathrm{Ra}_H^{-1/4}\,\mathrm{Pr}^{-1/4}$. So, this is your $\frac{\delta_T}{H}$ and what will be the scale for velocity v; $v \sim \frac{\alpha H}{\delta_T^2}$. So, $v \sim \frac{\alpha}{H}\mathrm{Ra}_H^{1/2}\,\mathrm{Pr}^{1/2}$.

So, now you can see your Nusselt number. So, $Nu_H \sim \frac{H}{\delta_T}$, we have already shown. So, Nusselt number H from here you can see it will be $Nu_H \sim \mathrm{Ra}_H^{1/4}\,\mathrm{Pr}^{1/4}$.

So, you can see, we can conclude Rayleigh number into Prandtl number plays the same rule for low Prandtl number fluids as Rayleigh number plays for high Prandtl number fluids, because for high Prandtl number fluids, we have shown that $Nu_H \sim \mathrm{Ra}_H^{1/4}$ and this is for low Prandtl number fluids and from here we will define another non dimensional number that is known as Boussinesq number.

So, this is $Bo_H = Ra_H\,\mathrm{Pr}$. So, you can write it as $\frac{g\beta\Delta T H^3}{\alpha^2}$. So, $Nu_H \sim Bo_H^{1/4}$.

(Refer Slide Time: 46:22)



So, now, in this case you see for low Prandtl number fluids so; obviously, your this is your thermal boundary layer thickness δ_T. So, at the wall you will have high temperature difference and at the edge of the boundary layer; obviously, the $\delta_T \rightarrow 0$.

So, the δ_T variation and the velocity so; obviously, as inside the thermal boundary layer, buoyancy effect is there and there will be fluid motion and as fluid motion will be there. So, you can see your velocity effect. So, this is the velocity.

So, this is the velocities v. So, velocity effect will continue till the thermal boundary layer thickness. So, near to the thermal boundary layer thickness so; obviously, it effects will extend to the edge of thermal boundary layer.

So, the effect of velocity or the fluid motion effect will extend to the edge of the thermal boundary layer so; obviously, it is due to the buoyancy effect inside the thermal boundary layer. So, you cannot say that your hydrodynamic boundary layer thickness will be much-much smaller than δ_T for this low Prandtl number fluids, but you can see you have the maximum velocity near to the wall.

So, this is this will be your maximum velocity and if this thickness you can see, this is δ_v. So, viscous effect will be acting only near to the solid wall and your viscous force will be balancing to the buoyancy force inside this layer δ_v, but outside you can see viscous force will be less and here inertia will be comparable with the buoyancy force.

So, whatever we are introducing 1 thickness δ_v you have maximum velocity. So, inside this your viscous force will balance with the buoyancy force, but outside it your viscous force will be less and you can compare inertia force with the buoyancy force.

So, now, you can see that let δ_v is the thickness of a thin layer right near to the wall, where viscous force will be comparable with the buoyancy force, because already we have shown that for low Prandtl number fluids inertia force will be comparable with the buoyancy force and we have derived the thermal boundary layer thickness and the Nusselt number.

But near to this wall now, viscous force will be balancing with the buoyancy force and from here you can see $\nu \frac{\partial^2 v}{\partial x^2} \sim g\beta\Delta T$. So, we know that $x \sim \delta_v$.

So, you can write $\frac{\nu v}{\delta_v^2} \sim g\beta\Delta T$. So, you can write $\delta_v^2 \sim \frac{\nu v}{g\beta\Delta T}$. So, what is the order of v?

So, now, $v \sim \frac{\alpha}{H} Ra_H^{1/2} \Pr^{1/2}$. So, if you put this v here so you will get $\delta_v^2 \sim \frac{\nu\alpha Ra_H^{1/2} \Pr^{1/2}}{Hg\beta\Delta T}$.

So, now, we will rearrange it. So, if you rearrange. So, you will get $\delta_v^2 \sim \frac{\alpha\nu}{g\beta\Delta T H^3} H^2 Ra_H^{1/2} \Pr^{1/2}$.

So, here you can see this is you can write as Rayleigh number. So, $\left(\frac{\delta_v}{H}\right)^2 \sim \Pr^{1/2} \frac{Ra_H^{1/2}}{Ra_H} \sim \frac{\Pr^{1/2}}{Ra_H^{1/2}}$. So, now, we will define another non dimensional number that is known as Grashof number.

$Gr = \frac{Ra_H}{\Pr}$. So, now, we are defining another non dimensional number that is your Grashof number. So, $Gr = \frac{Ra_H}{\Pr}$. So, from here so, you can see, you can write $\frac{\delta_v}{H} \sim Gr_H^{-1/4}$.

(Refer Slide Time: 53:08)

And from here now we have $\frac{\delta_v}{H} \sim Gr_H^{-1/4}$ and also we have $\frac{\delta_T}{H} \sim Ra_H^{-1/4} \Pr^{-1/4}$.

So, from here you can see $\frac{\delta_v}{\delta_T} \sim \mathrm{Pr}^{1/2}$ and; obviously, it will be less than 1. So, $\delta_v << \delta_T$.

So, you can see that it has to be noted that δ_v is not same as the hydrodynamic boundary layer thickness δ . So, δ_v is the thickness where your viscous force is comparable with the buoyancy force very near to the wall and for low Prandtl number fluids this $\delta_v << \delta_T$. So, in today's class we have introduced the natural convection.

So, natural convection will occur in presence of some acceleration like gravity and density change and density change may occur due to the temperature difference. So, in the beginning we have shown some applications of this natural convection, then we considered the simplest case flow over vertical wall and we considered uniform wall temperature case starting from the Navier-Stokes equation and using the boundary layer approximation we have written down the boundary layer equations and from there we invoked the Boussinesq approximation.

So, what is Boussinesq approximation? In Boussinesq approximation we assume that density to be constant in continuity equation as well as in the inertia terms of the momentum equations, but the change of density effect we take into account in the buoyancy term.

From there we have used the scale analysis and using scale analysis for two different cases low Prandtl number fluids and high Prandtl number fluids, we have shown the scale for thermal boundary layer thickness, velocity v, heat transfer coefficient and the Nusselt number. And for low Prandtl number fluids we have also defined 1 thickness δ_v which is very near to the wall where viscous force is comparable with the buoyancy force.

Thank you.

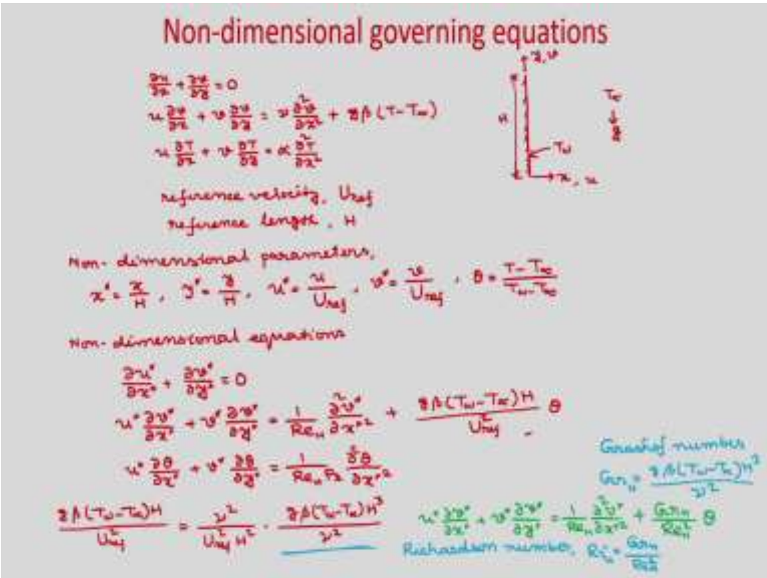
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 08
Natural Convection - I
Lecture - 27
Natural convection over a vertical plate: Similarity Solution

Hello everyone. So, in last class, we derived the laminar boundary layer equations for a Natural convection over a vertical plate. We derived continuity equation, y momentum equation and energy equation.

So, we have three partial differential equations. In today's class, we will use similarity method and we will convert these three partial differential equations to ordinary differential equations. Let us write down the governing equations, whatever we derived in last class in non-dimensional form.

(Refer Slide Time: 01:19)



So, if you see that we have derived in last class; continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$; then, we derived the momentum equation as $u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 v}{\partial x^2}+g\beta\left(T-T_\infty\right)$ and also, we have Energy equation $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}$.

So, if you know that we have consider vertical plate and this is your x and this is your y. In x direction, we have velocity u and y direction, we have velocity v. Ambient fluid is quiescent and it is having temperature T_∞.

We have gravitational acceleration in negative y direction that is g and this wall is maintained at uniform surface temperature that is your T_w. So, now, we will choose to reference scale, one is for length and one is for velocity. Let us choose that your reference velocity is U_{ref}.

We do not have any free stream velocity in this case. So, we are choosing some reference velocity that is your U_{ref} and reference length, let us choose H; where, H is the height of the fluid or you can choose any other reference length. Now, we will use this non-dimensional parameters, x* is the non-dimensional x coordinate, $x^*=\frac{x}{H}$, $y^*=\frac{y}{H}$, then $u^*=\frac{u}{U_{ref}}$, $v^*=\frac{v}{U_{ref}}$; then temperature, we will take $\theta=\frac{T-T_\infty}{T_w-T_\infty}$.

So, with this now, if you put it here and if you rearrange, you will get the non-dimensional equations. Non-dimensional equations as $\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0$, then we have $u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=\frac{1}{\mathrm{Re}_H}\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{g\beta\left(T_w-T_\infty\right)H}{U_{ref}^2}\theta$. So, using these non-dimensional parameters, you put all those things in the momentum equation. Then, if you rearrange, you will get the equation in this form; where in the viscous term, you will get 1 by Reynolds number and the buoyancy term you will get like this.

So, we will simplify it later. First let us write the non-dimensional energy equation, $u*\frac{\partial \theta}{\partial x*}+v*\frac{\partial \theta}{\partial y*}=\frac{1}{\mathrm{Re}_H \Pr}\frac{\partial^2 \theta}{\partial x*^2}$. So, if you see these term in the buoyancy term, so if you rearrange it, you will get $\frac{g\beta\left(T_w-T_\infty\right)H}{U_{ref}^2}=\frac{\nu^2}{U_{ref}^2 H^2}\frac{g\beta\left(T_w-T_\infty\right)H^3}{\nu^2}$.

So, these term is now one another non-dimensional number that is known as Grashof number ; Grashof number.

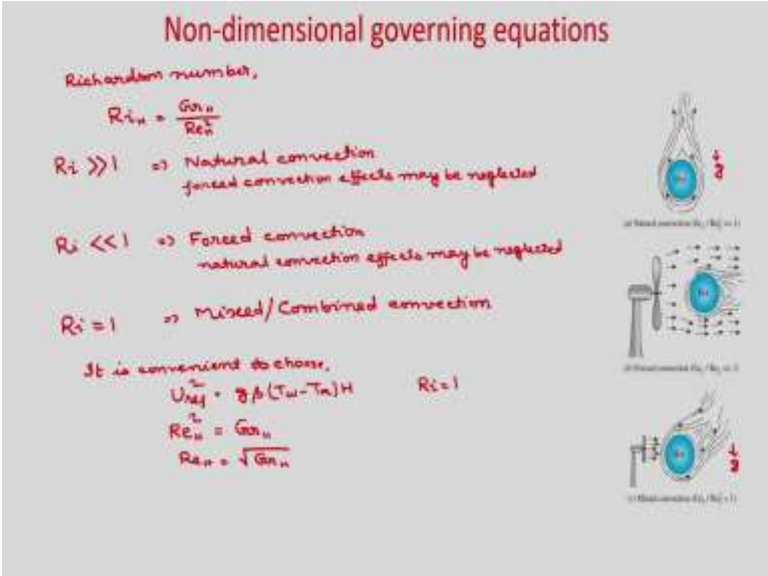
So, how Grashof number is defined Grashof number is the ratio of buoyancy force to the viscous force because in the numerator, you can see this is coming from the buoyancy force and in the denominator ν is there, so it is coming from the viscous force. So, Grashof number is the ratio of buoyancy force to the viscous force. So, and it is written in this from $Gr_H=\frac{g\beta\left(T_w-T_\infty\right)H^3}{\nu^2}$.

So, Grashof number based on H . So, now, you can write this term. So, momentum equation you can write as $u*\frac{\partial v*}{\partial x*}+v*\frac{\partial v*}{\partial y*}=\frac{1}{\mathrm{Re}_H}\frac{\partial^2 v*}{\partial x*^2}+\frac{Gr_H}{\mathrm{Re}_H^2}\theta$.

So, this is your momentum equation. So, here we will define another non-dimensional number which is your Grashof number by Reynolds number square and it is known as Richardson number.

So, this is known as Richardson number and denoted as, $Ri_H=\frac{Gr_H}{\mathrm{Re}_H^2}$. So, this Richardson number has physical significance that it compares which force is dominant; buoyancy force or your inertia force.

(Refer Slide Time: 09:45)



So, Richardson number ok. So, this is $Ri_H = \dfrac{Gr_H}{\mathrm{Re}_H^2}$. So, you can see Richardson number, so it is kind of buoyancy force to the inertia force . So, if it is so, now you can see three different cases. If Richardson number >> 1, so what will happen?

So, if Richardson number >> 1 , so now, you can see that Richardson number >> 1; that means, $Gr_H \gg \mathrm{Re}_H^2$. So, what does it mean? It means that buoyancy force will dominant.

So, in that case obviously, buoyancy force is dominating, then you can consider that it is natural convection. So that means, forced convection effects may be neglected. If Ri<<1, so in that case you can see inertia force will dominant .

So that means, it will be forced convection. So, in this case, your natural convection effects may be neglected and if Ri = 1, then what will happen? Your buoyancy force is comparable to inertia force. So, you cannot neglect either buoyancy force or inertia force.

So, you will have both natural convection and forced convection and that is known as mixed convection or combined convection. So, for Ri = 1, so you will get mixed convection which is known also as combined convection. So, in this case, you need to consider both forced and natural. So, you can see this example.

So, if natural convection is happening, say you have a sphere hot sphere, then $\frac{Gr_H}{\text{Re}_H^2} = Rz_H$ >> 1. L is the any characteristic length. So, in this case you will have only natural convection. So, due to buoyancy, you can see the you will have a flume and it will go up.

Now, if Ri << 1, then natural convection can be neglected and you have a forced convection. So, you can see this velocity is generated using some fan and over this hot sphere, this flow will takes place and heat transfer will takes place and that is your forced convection.

But if you have Ri = 1, then you will have both natural convection and buoyancy effect will be there. So, in this case, g is there and also in this case g is there. So obviously, you can see you will have buoyancy effect. So, this flume will go up, but you have forced convection also in the x direction. So, you will see the flume is go up in a inclined manner. So, it is known as mixed convection.

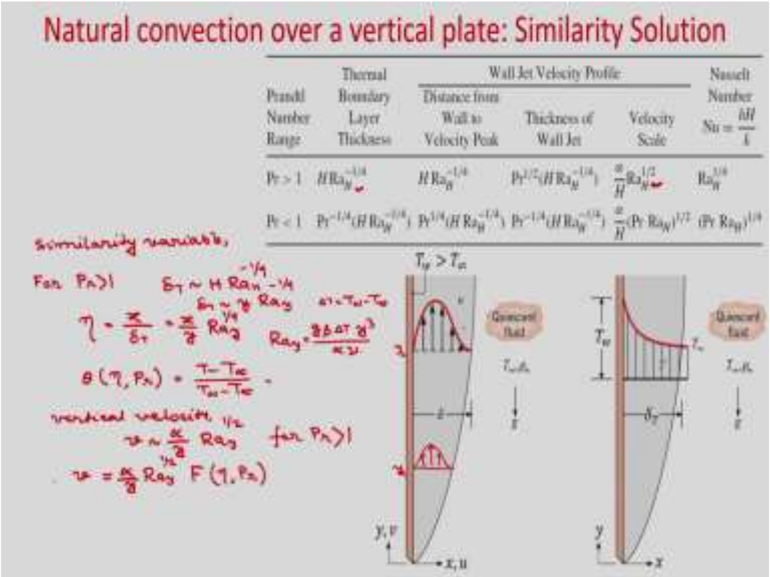
Now from here, you can see that if you choose the velocity scale in such a way that the coefficient in the buoyancy term in the θ will become 1. So, in that case, you can see because reference velocity is arbitrary right. So, it can be chosen to simplify the form of the equation.

So, it is convenient to choose; convenient to choose $U_{ref}^2 = g\beta\left(T_w - T_\infty\right)H$ so that your the term with the θ, so you can see this term. So, if $U_{ref}^2 = g\beta\left(T_w - T_\infty\right)H$, then this will become 1. If it becomes 1; then obviously, it is kind of Richardson number will become 1.

And for this case you can see your $\text{Re}_H^2 = Gr_H$ right. That is we have already defined because this is your ratio right. $Rz_H = \frac{Gr_H}{\text{Re}_H^2}$ and $\text{Re}_H = \sqrt{Gr_H}$. So, you can see that Grashof number place the role same role in the natural convection, what Reynolds number place the role in forced convection right.

Now, let us use similarity transformation method and convert this three partial differential equation to ordinary differential equation so that we can have the numerical solution. Now, you can see in last class, we have found scales for different quantities like thermal boundary layer thickness, then velocity v and also, Nusselt number and those are tabulated in this table.

(Refer Slide Time: 16:03)



So, you can see for Prandtl number >> 1, the boundary layer thickness $\delta_T \sim HRa_H^{-1/4}$. And also, velocity scale $\frac{\alpha}{H}Ra_H^{1/2}$ and $Nu_H \sim Ra_H^{1/4}$ and you can see that the velocity looks like this.

So, because at the wall, you will have 0 velocity and at the edge of the boundary layer, we will have 0 velocity. So, your maximum velocity occur in between this boundary layer and this is the known as wall jet velocity profile.

So, you can see at different location, you will get this type of profile and if you can say if you see at this location; let us say it is y_1 and this is your y_2. So, you will get some profile like this and you can see that if you scale down this velocity profile with a proper scale, then you can bring down to this profile.

So that means, if somehow we can convert this partial differential equation to ordinary differential equation using some scale factor, then your similarity transformation exists and we can write this partial differential equation to ordinary differential equation.

So, now how to choose these similarity variable? So, you here in that table, you can see that we have this $\delta_T \sim HRa_H^{-1/4}$. So, this scale we can take for the similarity variable eta and we have also velocity scale $\frac{\alpha}{H} Ra_H^{1/2}$.

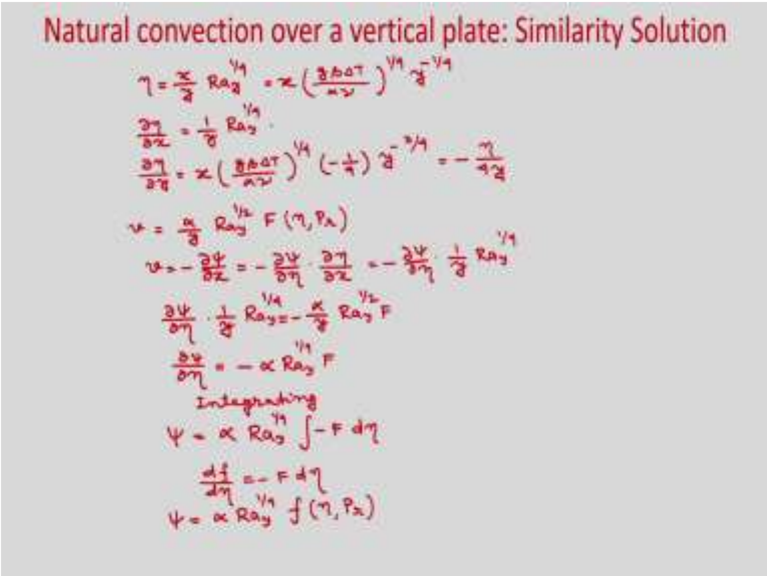
So, what we can do? You can so, this similarity variable η; similarity variable $\eta = \frac{x}{\delta_T}$. So, we can write for Prandtl number > 1 , so thermal boundary layer thickness you can see $\delta_T \sim HRa_H^{-1/4}$.

So, we can write $\eta = \frac{x}{\delta_T}$. So, you can write; so, this $\delta_T \sim yRa_y^{-1/4}$. So, in this case we can write $\eta = \frac{x}{\delta_T} = \frac{x}{y} Ra_y^{1/4}$. So, it is a in the numerator we have written. So, we are equally depend + 1/4.

So, here $Ra_y = \frac{g\beta\Delta Ty^3}{\alpha\nu}$. So, now, you can see $\theta(\eta, \Pr) = \frac{T - T_\infty}{T_w - T_\infty}$. So, in this case, you can see this is the temperature profile and this also you can use some scale factor so that it will become same or all the temperature profile will collapse, if you plot it with η.

So, theta you can write this way and we know the vertical velocity scale as $v \sim \frac{\alpha}{y} Ra_y^{1/2}$ for Pr > 1. This we are considering. So, this we can write velocity scale $v = \frac{\alpha}{y} Ra_y^{1/2} F(\eta, \Pr)$. So, this is your dimensionless similarity profile of the wall jet. So, this scale we can use and we can now find what is the value of Ψ from this expression .

(Refer Slide Time: 20:53)



So, now we will use $\eta = \frac{x}{y} Ra_y^{1/4}$ and if you put the value of Ra_y, then you can write in terms of y as , $x\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4} y^{-1/4}$. So, now, you can find $\frac{\partial \eta}{\partial x} = \frac{1}{y} Ra_y^{1/4}$ and $\frac{\partial \eta}{\partial y} = x\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4}\left(-1/4\right) y^{-5/4}$. So, now, you rearrange it and then, you can simplify it and you will get $\frac{\partial \eta}{\partial y} = -\frac{\eta}{4y}$ after simplification.

Now, we have v. So, $v = \frac{\alpha}{y} Ra_y^{1/2} F(\eta, \Pr)$. Now, we will find the stream function from this velocity v and once stream function is known, then you will be able to calculate the velocity u and velocity gradient like $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial x^2}$.

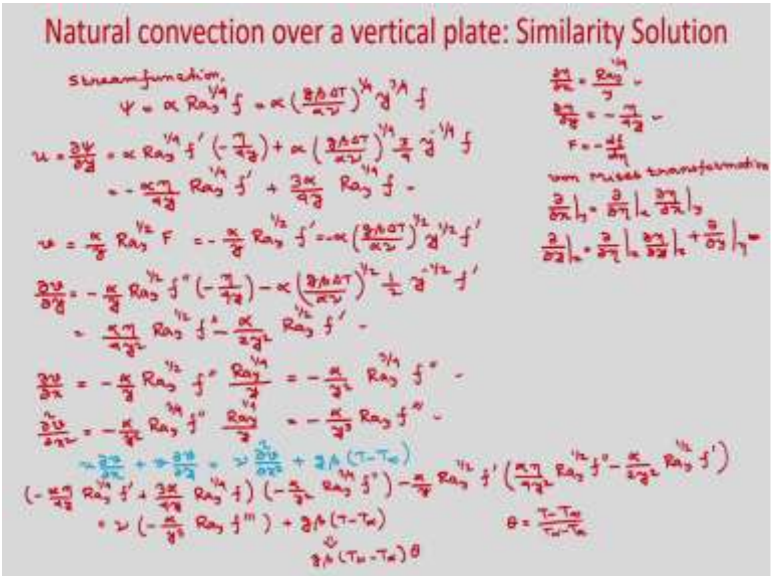
So, now, let us find what is stream function Ψ. So, you can define $v = -\frac{\partial \psi}{\partial x}$. So, if you put it here, you will get; so, this you can write $-\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x}$. So, $\frac{\partial \eta}{\partial x}$ already we have found here. So, you can write $-\frac{\partial \psi}{\partial \eta}\frac{1}{y} Ra_y^{1/4}$.

So, now, if you put these. So, you can find $\frac{\partial \psi}{\partial \eta}\frac{1}{y}Ra_y^{1/4} = -\frac{\alpha}{y}Ra_y^{1/2}F$. So, now if it is so, now we can write $\frac{\partial \psi}{\partial \eta} = -\alpha Ra_y^{1/4}F$. Now you can integrate it. So, integrating you can write $\psi = \alpha Ra_y^{1/4}\int -Fd\eta$ and the integration constant is put inside this integral.

So, now, we will define a new variable small f. So, that your $\frac{df}{d\eta} = -Fd\eta$. So, if it is so, now you can write $\psi = \alpha Ra_y^{1/4} f\left(\eta, \mathrm{Pr}\right)$.

So, you can see from here F will be just $-Fd\eta$ integration constant is involved here. So, you can write $f\left(\eta, \mathrm{Pr}\right)$. So, now, we have found the stream function Ψ. So, you will able to calculate the velocity u right.

(Refer Slide Time: 25:19)



So, here just let us write $\frac{\partial \eta}{\partial x} = \frac{Ra_y^{1/4}}{y}$ and $\frac{\partial \eta}{\partial y} = -\frac{\eta}{4y}$ and we have used $F = -\frac{df}{d\eta}$.

And von Mises transformation, again we will use it. So, let us write these von Mises transformation. So, in this case it is a vertical plate, so we will use $\frac{\partial}{\partial x}|_y = \frac{\partial}{\partial \eta}|_x \frac{\partial \eta}{\partial x}|_y$, we will write as $\frac{\partial}{\partial y}|_x = \frac{\partial}{\partial \eta}|_x \frac{\partial \eta}{\partial y}|_x + \frac{\partial}{\partial y}|_\eta$.

So, we will use these a transformation while finding the derivative. So, stream function you have found $\psi = \alpha Ra_y^{1/4} f$ and in terms of y if you write, then you will get $\alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4} y^{3/4} f$. So, that we have written outside.

So, now velocity $u = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial y}$ if you write, then this transformation you have to use. So, if you use these transformation, you can see so $\frac{\partial \psi}{\partial y} = \alpha Ra_y^{1/4} f' + \left(-\frac{\eta}{4y}\right) + \alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/4} \frac{3}{4} y^{-1/4} f$.

So, if you rearrange it. So, you will get $-\frac{\alpha\eta}{4y} Ra_y^{1/4} f' + \frac{3\alpha}{4y} Ra_y^{1/4} f$. So, we have found the velocity u.

Now, let us write the velocity v in terms of f. So, in terms of capital F already we have written. So, you know that we have written $v = \frac{\alpha}{y} Ra_y^{1/2} F$ and this you can see $F = \frac{d\alpha}{d\eta}$.

So, you can write $-\frac{\alpha}{y} Ra_y^{1/2} f'$ and in terms of y if you take $-\alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/2} y^{1/2} f'$.

So, we are writing in this form because we need to find the derivative with respect to y that is why we have taken y outside. So, that it will be easiest to integrate. Now, you can see in the momentum equation, we have the velocity gradient $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial x^2}$. So, that let us find from this expression.

So, if you use $\frac{\partial v}{\partial y}$. So, again this von Mises transformation this one we will use. So, you can see, it will be $\frac{\partial v}{\partial y} = -\frac{\alpha}{y} Ra_y^{1/2} f''\left(-\frac{\eta}{4y}\right) - \alpha\left(\frac{g\beta\Delta T}{\alpha\nu}\right)^{1/2} \frac{1}{2} y^{-1/2} f'$. Now, let us find the derivative of v with respect to x.

So that means, $\frac{\partial v}{\partial x} = -\frac{\alpha}{y} Ra_y^{1/2} f'' \frac{Ra_y^{1/4}}{y}$.

So, if you simplify it, you will get $-\frac{\alpha}{y^2} Ra_y^{3/4} f''$ and again, $\frac{\partial^2 v}{\partial x^2} = -\frac{\alpha}{y^2} Ra_y^{3/4} f''' \frac{Ra_y^{1/4}}{y}$.

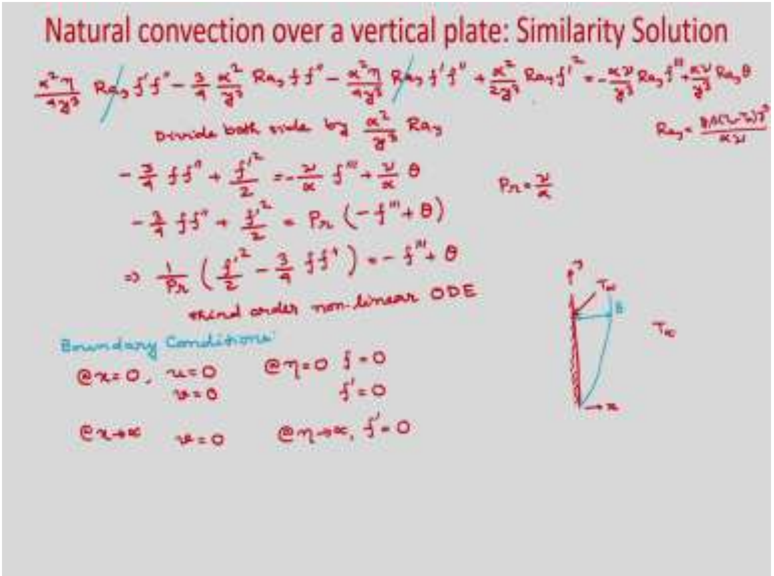
So, if you simplify it, you will get $-\frac{\alpha}{y^3} Ra_y f'''$. So, now let us put all these in the momentum equation. What is your momentum equation? $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\beta\left(T - T_\infty\right)$.

So, now, if you put the value here. So, first u; So, this if u if you put, so you will get $\left(-\frac{\alpha\eta}{4y} Ra_y^{1/4} f' + \frac{3\alpha}{4y} Ra_y^{1/4} f\right)\left(-\frac{\alpha}{y^2} Ra_y^{3/4} f'\right) - \frac{\alpha}{y} Ra_y^{1/2} f'\left(\frac{\alpha\eta}{4y^2} Ra_y^{1/2} f'' - \frac{\alpha}{2y^2} Ra_y^{1/2} f'\right)$

So, left hand side, the inertia terms we have written; now in the right hand side, you write the viscous terms $\nu\left(-\frac{\alpha}{y^3} Ra_y f'''\right) + g\beta\left(T - T_\infty\right)$.

So, now, we have $\theta = \frac{T - T_\infty}{T_w - T_\infty}$. So, this term $g\beta\left(T_w - T_\infty\right)\theta$. So, you multiply it first, then simplify it.

(Refer Slide Time: 34:39)



So, this if you do, then you will get $\frac{\alpha^2\eta}{4y^3}Ra_y f'f'' - \frac{3}{4}\frac{\alpha^2}{y^3}Ra_y ff'' - \frac{\alpha^2\eta}{4y^3}Ra_y f'f'' + \frac{\alpha^2}{2y^3}Ra_y f'^2$.

Then, we have the viscous term, $-\frac{\alpha\nu}{y^3}Ra_y f''' + \frac{\alpha\nu}{y^3}Ra_y\theta$. You see $Ra_y = \frac{g\beta\left(T_w - T_\infty\right)y^3}{\alpha\nu}$. So, now, you can see this two terms will get cancelled; this term and this term. Now, you can write this equation; divide both side by $\frac{\alpha^2}{y^3}Ra_y$.

So, if you divide by these, then first term what you can write? So, you can see, it will be $-\frac{3}{4}ff'' + \frac{f'^2}{2} = -\frac{\nu}{\alpha}f''' + \frac{\nu}{\alpha}\theta$ and you know the definition of Prandtl number, $\Pr = \frac{\nu}{\alpha}$. So, if you put it here, so you can see it will be $-\frac{3}{4}ff'' + \frac{f'^2}{2} = \Pr\left(-f''' + \theta\right)$.

So, if you write in this way, so it will be $\frac{1}{\Pr}\left(\frac{f'^2}{2} - \frac{3}{4}ff''\right) = -f''' + \theta$. So, you can see that choosing the similarity variable, we could convert the three partial differential equations; continuity equation; momentum equation and energy equation to ordinary differential equation.

So, if you see that using the similarity approach, we could convert the momentum equation which is your partial differential equation to ordinary differential equation. You can see this is the third order differential equation and it is non-linear because you can see this is f''', so this is third order and you have ff'' that means, it is non-linear ordinary differential equation.

But here you can see in this ordinary differential equation, in the momentum equation also you have term θ which is your temperature . So that means, this equation also will depend on the temperature profile. So, what are the boundary conditions now?

So, for a vertical plate if you see, so this is your x, this is your y right and I am going to write T_∞ and this is your T_w which is uniform and there will be hydrodynamic boundary layer like this, which is your δ; at any location, you can find what is the δ. So, now, you can see what are the boundary conditions.

So, we can see at x = 0, you have no slip boundary condition; that means, u = 0 and v= 0. So that means, at η=0, if you put u = 0, so you can see the u_s scale. So, you can see from here. So, here you can see that if u = 0, then $f'=0$.

So, let us say that at η=0, so first find v = 0. So, if v = 0, then you can see from this expression. If v =0 from this expression, $f'=0$; from this expression, $f'=0$. So, you can see it will be v = 0, it will be $f'=0$.

So, if $f'=0$, then for u = 0. So, u is the this is the expression. So, f' is already 0. So, f will be 0. So, this is the stream function actually. So, you are putting stream function value as 0 because it is a vertical plate, so u_0 will give you the stream function and vertical plate, you can see it is a streamline and along a streamline your stream function will be constant; so, f = 0.

So, and x →∞. So, near to this edge of the boundary layer. Obviously, you will get the velocity 0. So that means, again you have v = 0. So, you have v = 0. So, at η→∞ you will get f' as 0.

Now, you can solve this ordinary differential equation using these boundary conditions, using some numerical technique and you can find the velocity distribution . But you cannot solve it alone because along with these you need to solve the energy equation, because theta is involved.

So, you have to solve in coupled way. So, you have to apply boundary condition for velocity in this equation and when you solve the energy equation, you need to invoke the temperature boundary condition and together, numerical you need to solve these two equations and you will find the velocity distribution as well as the temperature distribution.

So, if you see that when we wrote the governing equations for flow over vertical plate when with uniform temperature boundary condition, what are the assumptions we took? Obviously, it is two-dimensional steady and laminar flow and we have invoke the bossiness approximation and you can see the properties like thermal conductivity, specific heat and the viscosity are constant and the ambient temperature T_∞ is also constant and T_w which is your wall temperature that is also constant.

So, in the next class, we will solve the energy equation and we will convert this partial differential equation to ordinary differential equation, using the same similarity variable. So, in today's class, first we wrote the non-dimensional form of the boundary layer equations for natural convection over a vertical plate.

So, we have introduced with two non dimensional numbers; one is Grashof number which is the ratio of buoyancy force to the viscous force and another non-dimensional number that is Richardson number which is defined as Grashof number divided by Reynolds number square. This actually gives the ratio of buoyancy force to the inertia force. So, depending on the value of Richardson number, you can see which is the dominant force.

So, for Richardson number >> 1; obviously, you can neglect the forced convection and it will be purely natural convection. If Richardson number << 1, then you can see that inertia force will be dominant and it you can have the forced convection and effect of natural convection you can neglect.

But when Ri = 1, then both are significant natural convection as well as forced convection. So, this is known as mixed or combined convection. Then, we have used the similarity transformation approach and using the similarity variable eta and we have defined a velocity v from the scale of velocity v, we converted this momentum equation which is your partial differential equation to ordinary differential equation.

So, first we found the stream function from the velocity v and once you know the stream function; from there, we found the velocity u and the velocity gradient $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial x^2}$.

So, all those terms if you put in the momentum equation, then you could get third order non-linear ordinary differential equations and the boundary conditions already we have discussed that at the wall you have velocity v = 0 which will give f' is 0 and at the wall again u is 0 and it will give stream function as 0. So, f = 0 and as $\eta \to \infty$; that means, v will be 0 and f' will be 0.

So, in the next class, we will derive the ordinary differential equation for the energy equation.

Thank you.

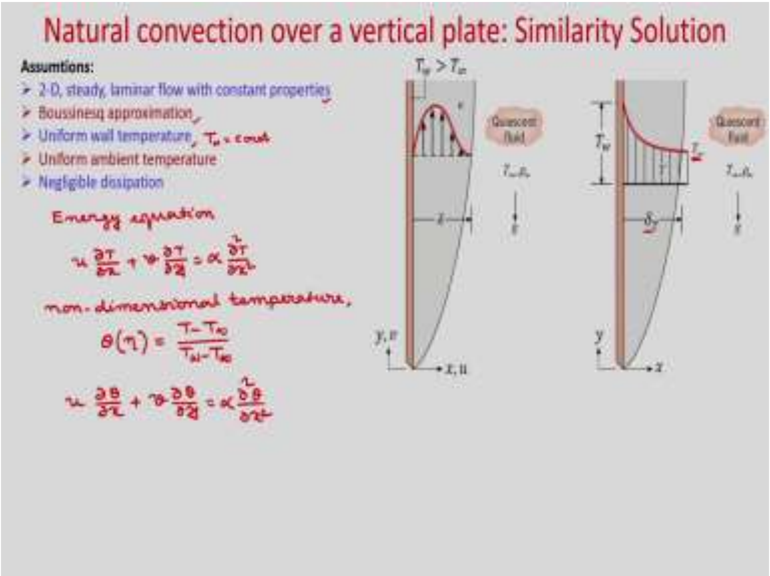
Fundamentals of Convective Heat Transfer

Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module-08
Natural Convection -I
Lecture – 28
Natural convection over a vertical plate: Similarity solution of energy equation

Hello everyone. So now, we are studying laminar Natural convection over a vertical plate. In last class, we started with the momentum equation and using similarity transformation method, we converted the PD to OD and we discussed about the boundary conditions. Now, let us consider energy equation and we will use similarity transformation method to convert this PD to OD.

(Refer Slide Time: 01:02)

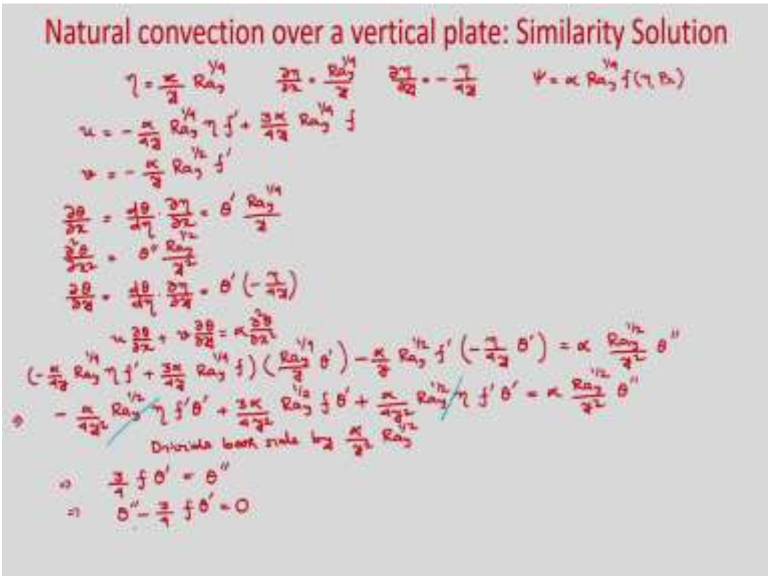


So, these are the assumptions we have already discussed in last class; 2 dimensional steady; laminar flow with constant properties, Boussinesq approximation is valid. We are considering uniform wall temperature; that means T_w is constant and uniform ambient temperature, $T\infty$ is also constant and we are neglecting the viscous dissipation. So, we can write the energy equation as $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2}$.

So, we are neglecting the viscous dissipation, and this is boundary layer energy equation. And you can see here δ_T is your thermal boundary layer thickness, and how the temperature is varying as $T_w > T_\infty$. So, temperature is maximum at the wall and gradually it is baring to the free stream where you have quiescent fluid temperature T_∞.

Now, let us consider non dimensional temperature $\theta(\eta) = \dfrac{T - T_\infty}{T_w - T_\infty}$. So, now, if you write this equation in terms of non dimensional temperature then you can write as; $u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial y} = \alpha \dfrac{\partial^2 \theta}{\partial x^2}$.

(Refer Slide Time: 03:23)



Now we will use the similarity variable eta and will convert this PD to OD.

So, in last class we have already shown that $\eta = \dfrac{x}{y} Ra_y^{1/4}$, and we know $\dfrac{\partial \eta}{\partial x} = \dfrac{Ra_y^{1/4}}{y}$ and $\dfrac{\partial \eta}{\partial y} = -\dfrac{\eta}{4y}$. This already we have derived in last class. And Ψ, the stream function; also we have defined as $\psi = \alpha Ra_y^{1/4} f\left(\eta, \mathrm{Pr}\right)$.

So now, you can write the velocity u and v as $u=-\frac{\alpha}{4y}Ra_y^{1/4}\eta f'+\frac{3\alpha}{4y}Ra_y^{1/4}f$. So, this we have derived in last class and velocity $v=-\frac{\alpha}{y}Ra_y^{1/2}f'$.

Now, let us find the derivative of θ with respect to x and y. So, you can see $\frac{\partial\theta}{\partial x}=\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial x}=\theta'\frac{Ra_y^{1/4}}{y}$.

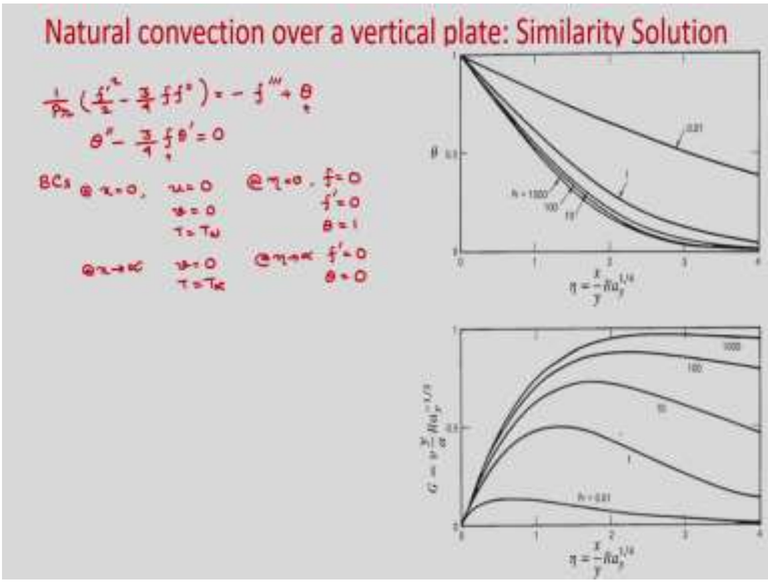
And $\frac{\partial^2\theta}{\partial x^2}=\theta''\frac{Ra_y^{1/2}}{y^2}$. And $\frac{\partial\theta}{\partial y}=\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y}=\theta'\left(-\frac{\eta}{4y}\right)$.

So now, we have energy equation $u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\alpha\frac{\partial^2\theta}{\partial x^2}$, if you write $\left(-\frac{\alpha}{4y}Ra_y^{1/4}\eta f'+\frac{3\alpha}{4y}Ra_y^{1/4}f\right)\left(\frac{Ra_y^{1/4}}{y}\theta'\right)-\frac{\alpha}{y}Ra_y^{1/2}f'\left(-\frac{\eta}{4y}\theta'\right)=\alpha\frac{Ra_y^{1/2}}{y^2}\theta''$. So, if you multiply, then you will get $-\frac{\alpha}{4y^2}Ra_y^{1/2}\eta f'\theta'+\frac{3\alpha}{4y^2}Ra_y^{1/2}f\theta'+\frac{\alpha}{4y^2}Ra_y^{1/2}\eta f'\theta'=\alpha\frac{Ra_y^{1/2}}{y^2}\theta''$.

So you can see here, here you can write, so these two terms you can see, this will cancel out. This term and this term will cancel out; Now divide both sides by, $\frac{\alpha}{y^2}Ra_y^{1/2}$; now you will get, $\frac{3}{4}f\theta'=\theta''$. So, you can see this is your equation $\theta''-\frac{3}{4}f\theta'=0$.

So you can see, this is ordinary differential equation. This is second order linear ordinary differential equation and you need the velocity distribution from the solution of momentum equation. Because here (Refer Time: 9:49) is there so; obviously, that will be your you can get the solution from the solution of momentum equation. But you can see, these both equations are coupled, because in the momentum equation temperature term is there θ, and in energy equation you have the term f which you will get from the velocity distribution. So, in the coupled way with the proper boundary condition you need to solve.

(Refer Slide Time: 10:15)



So you can see the whatever we have derived, so we have derived the momentum equation as $\frac{1}{\Pr}\left(\frac{f'^2}{2}-\frac{3}{4}ff''\right)=-f'''+\theta$. So, you will get the velocity profile from this equation and the energy equation is $\theta''-\frac{3}{4}f\theta'=0$.

So you can see, in this equation θ is there and in this equation f is there. So, these are coupled, so together you need to solve and get the velocity distribution from the first equation and temperature distribution from the second equation. And what are the boundary conditions?

So, already we have discussed for velocity at x = 0, you have u = 0; that means, at η=0 you will get f = 0 and v = 0. You will get at η= 0 f'=0. And what is temperature? Temperature T is T_w; so you will get tθas 1. And at x →∞; that means, at the edge of the boundary layer, you will get v = 0.

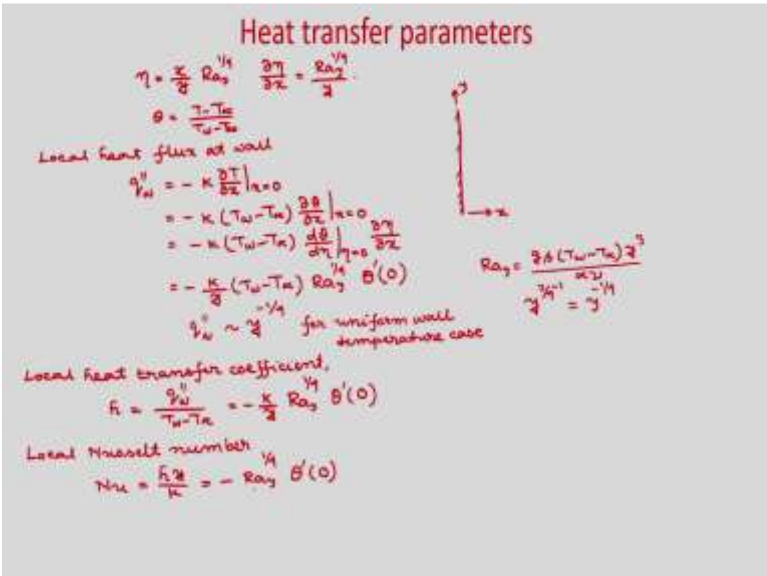
So, you will get at η→∞, f'=0 and T will be T_∞, so at η→∞, you will get θas 0. So, with suitable numerical integration technique you can solve these two; ordinary differential equation with given boundary condition. And you can find the velocity distribution as well as the temperature distribution. Once you get that, then you will be able to calculate the heat flux, heat transfer coefficient and Nusselt number.

So, you can see the solution of temperature ah, so you can see theta versus eta it is plotted for different Prandtl number. So, you can see as Pr→ ∞. All the temperature profile collapse into a single curve.

You can see here, and this is the solution of G, which actually gives the velocity versus eta. So, you can see that velocity v is actually we have written in terms of G, so this will give the velocity distribution; and here also you can see as η→ ∞, so or you can see that as Prandtl number increases because it is Prandtl number 0.01, 1, 10, 100 and 1000.

So, as Prandtl number increases, the velocity profile extends farther and farther into the isothermal fluid . So, once you know the temperature profile and the velocity profile, you will be able to calculate the heat flux and heat transfer coefficient and hence Nusselt number. So, let us find these parameters.

(Refer Slide Time: 13:39)



So, we know that $\eta = \frac{x}{y} Ra_y^{1/4}$ and $\frac{\partial \eta}{\partial x} = \frac{Ra_y^{1/4}}{y}$.

And $\theta = \frac{T - T_\infty}{T_w - T_\infty}$. So, now, you can see, we can write the local heat flux. So q_w'', local heat flux at wall. So, $q_w'' = -K \frac{\partial T}{\partial x}\big|_{x=0}$. Because if you have a vertical plate and this is your x and this is your y so; obviously, normal gradient is $\frac{\partial T}{\partial x}|_{x=0}$ you need to find the heat flux.

So, now you can write $-K\left(T_w - T_\infty\right)\frac{\partial \theta}{\partial x}|_{x=0}$. So, now we can write $-K\left(T_w - T_\infty\right)\frac{d\theta}{d\eta}|_{\eta=0}\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial x}$ this you can write. So you can write $-\frac{K}{y}\left(T_w - T_\infty\right)Ra_y^{1/4}\theta'(0)$.

So, can you tell how this heat flux varies with y. So you can see from this equation. So, in the Rayleigh number expression, what is the Rayleigh number expression? $Ra_y = \frac{g\beta\left(T_w - T_\infty\right)y^3}{\alpha\nu}$. So, in the Rayleigh number you see y^3, so $Ra_y^{1/4}$ means $y^{3/4}$.

And, here in the denominator y is there. So, it will be -1. So, you can see it will be $y^{-1/4}$. So, $q_w^{"} \sim y^{-1/4}$ for uniform wall temperature case.

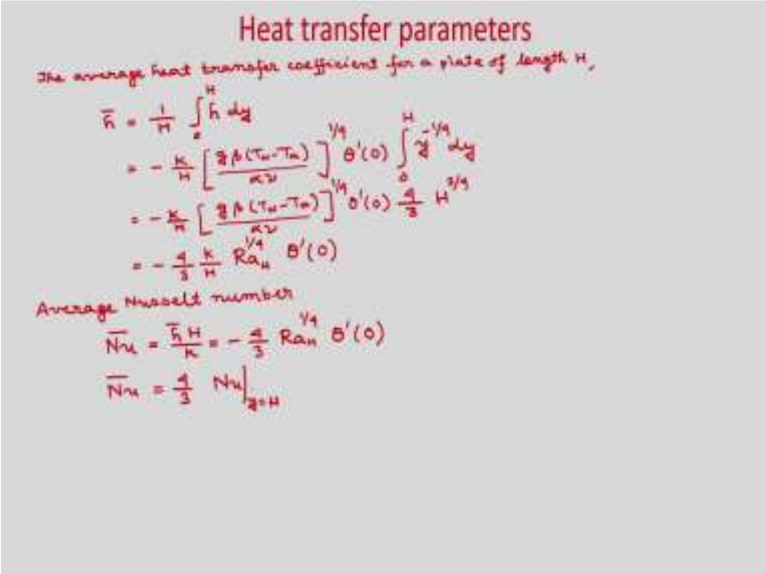
Now, once you know the heat flux at the wall, you will able to calculate the local heat transfer coefficient because local heat transfer coefficient, you can calculate from the Newton's law of cooling and h will be your q double prime w divided by the temperature difference.

So, local heat transfer coefficient . So, from Newton's law of cooling you can write $h = \frac{q_w^{"}}{T_w - T_\infty}$. So, from this expression directly you can write $-\frac{K}{y}Ra_y^{1/4}\theta'(0)$.

So once you know the temperature gradient at the wall, you will be able to calculate the heat transfer coefficient. Now, you can calculate the local Nusselt number. So, $Nu = \frac{hy}{K}$. So, from this expression you can see it will be $-Ra_y^{1/4}\theta'(0)$.

So, from the solution of the ordinary differential equation, once you get the temperature gradient at the wall which is function of Prandtl number as well, then you will be able to write the Nusselt number.

(Refer Slide Time: 18:26)



Now, similarly you can calculate the average heat transfer coefficient and average Nusselt number. So, the average heat transfer coefficient for a plate of length H, here vertical plate so the height of the plate is H.

You can write $\bar{h} = \frac{1}{H}\int_0^H h dy$. Where h is the; local heat transfer coefficient; so that we have already found. So, you can write $-\frac{K}{H}\left[\frac{g\beta\left(T_w - T_\infty\right)}{\alpha\nu}\right]^{1/4}\theta'(0)\int_0^H y^{-1/4}dy$.

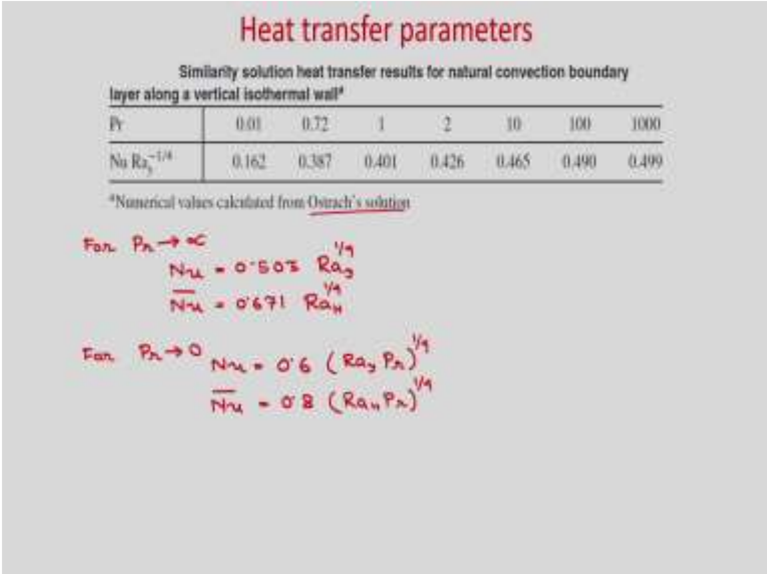
So, if you integrate it then you will get $-\frac{K}{H}\left[\frac{g\beta\left(T_w - T_\infty\right)}{\alpha\nu}\right]^{1/4}\theta'(0)\frac{4}{3}H^{3/4}$.

So, this H^3 you insert in this bracket, so that you can get back the Rayleigh number. So, if you write that, then you can write $-\frac{4}{3}\frac{K}{H}Ra_H^{1/4}\theta'(0)$.

So, hence you can write average Nusselt number. So, average Nusselt number you can write as $\overline{Nu} = \frac{\bar{h}H}{K}$.

So, you can write $-\frac{4}{3}Ra_H^{1/4}\theta'(0)$. So, in this case you can see, the $\overline{Nu} = \frac{4}{3}Nu\big|_{y=H}$.

(Refer Slide Time: 21:20)



So, you can see the similarity solution heat transfer results for natural convection, boundary layer along a vertical isothermal wall. So, $NuRa_y^{-1/4}$, where Nusselt number is the local Nusselt number. So, for different Prandtl number, these are the values.

So, this is obtained from the Ostrach's solution. So, once you know these values then you will be able to calculate the Nusselt number.

So, at the extreme case for Pr → ∞, you can write Nusselt number, $Nu = 0.503Ra_y^{1/4}$ and $\overline{Nu} = 0.671Ra_H^{1/4}$.

And for Pr → 0 you can write $Nu = 0.6\left(Ra_y \Pr\right)^{1/4}$, and $\overline{Nu} = 0.8\left(Ra_H \Pr\right)^{1/4}$.

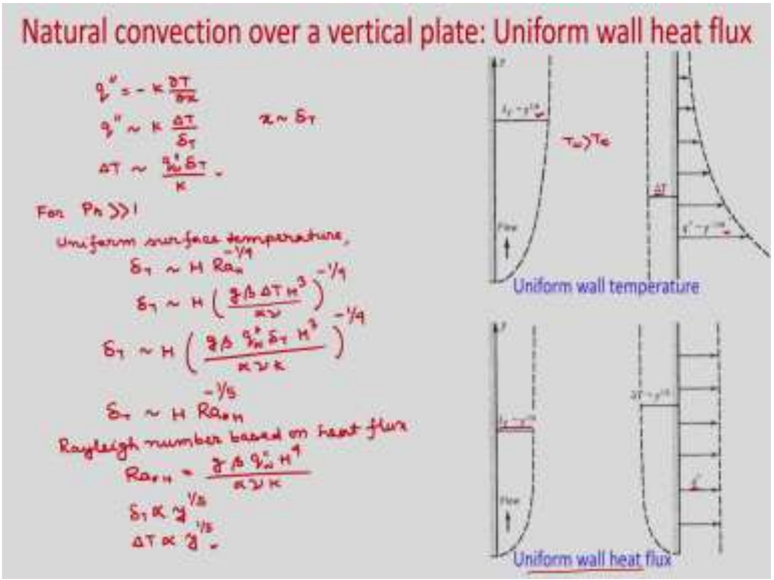
So you can see here, that in this case you have for Pr → ∞; that means, the case where Prandtl number is very high so; obviously, from the scale analysis we have shown that $Nu \sim Ra_y^{1/4}$.

And, when Pr < 1 then we have shown from the scale analysis that Rayleigh that $\left(Ra_y \Pr\right)^{1/4}$ and from this expression you can see that.

So, till now we considered uniform wall temperature case so; that means, T_w is constant. We can also consider where in some applications you can have the uniform wall heat flux condition. Like, when the solar radiation falls on a body then you will get uniform wall temperature case and in industrial application also we can have use of this uniform wall heat flux case.

So, the simplified case we will take. So we will consider again, vertical flat plate keeping the other assumptions constant, only we will assume that heat flux at the wall will be kept constant.

(Refer Slide Time: 24:13)



So, you can see that from the uniform wall temperature case, already we have shown $\delta_T \sim y^{1/4}$. So, it is a vertical wall, $T_w > T_\infty$ so; obviously, you will get the thermal boundary layer like this and your $\delta_T \sim y^{1/4}$. And, also we have shown that as your $\Delta T = T_w - T_\infty$, we are keeping constant $q'' \sim y^{1/4}$. So this also we have shown today.

Now, for uniform wall heat flux case what is the heat flux? Generally, we tell $q'' = -K\dfrac{\partial T}{\partial x}$. So, what will be the order? $q'' \sim K\dfrac{\Delta T}{\delta_T}$, because x $\sim\delta_T$.

So, from here we can get the $\Delta T \sim \dfrac{q_w''\delta_T}{K}$. Obviously, your q''at the wall we are considering. So, this is your at the wall.

So, from here, you can see we can find for Pr> 1, we have already derived for uniform surface temperature case. What is the δ_T? $\delta_T \sim HRa_H^{-1/4}$.

So, we can write $\delta_T \sim H\left[\frac{g\beta\Delta TH^3}{\alpha\nu}\right]^{-1/4}$. So, this we have derived for the uniform wall temperature case. Now, in this expression you see the temperature difference Δ T. We will put $\Delta T \sim \frac{q_w^{"}\delta_T}{K}$.

So, you will get $\delta_T \sim H\left[\frac{g\beta q_w^{"}\delta_T H^3}{\alpha\nu K}\right]^{-1/4}$. So, from here you can see if you bring this δ_T in this side, so now, you can write after rearrangement, $\delta_T \sim HRa_{*H}^{-1/5}$.

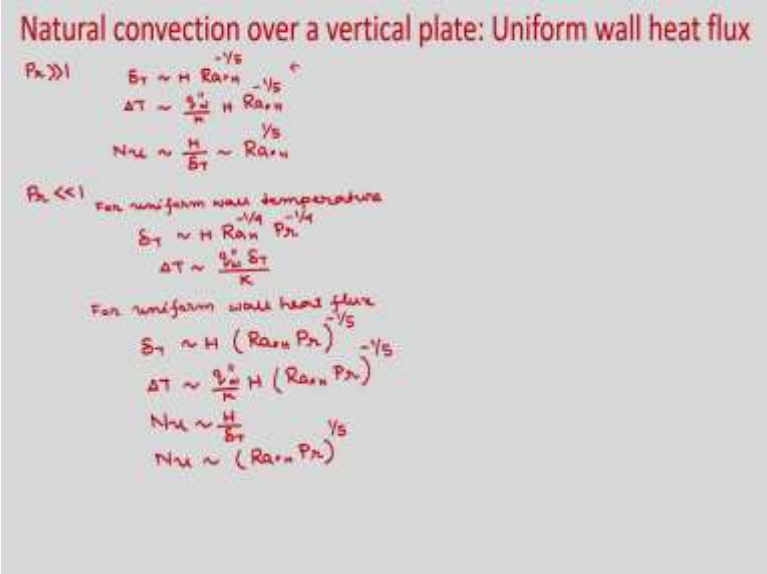
So, you do the rearrangement and here, this Rayleigh number based on $q_w^{"}$ is defined as, $Ra_{*H} = \frac{g\beta q_w^{"}H^4}{\alpha\nu K}$.

So it is non dimensional number right. So, from this expression you can see your δ_T. So, δ_T what is the order? So, it will be $\delta_T \propto y^{1/5}$.

And what is about your δ_T. So, $\Delta T \sim \frac{q_w^{"}\delta_T}{K}$. So, $\Delta T \propto y^{1/5}$. So, you can see in this figure, for uniform wall heat flux case, when flow is taking place you can see your thermal boundary layer thickness will vary as $y^{1/5}$ whereas, for uniform wall temperature it is $y^{1/4}$.

And as it is uniform wall heat flux condition. Your $\Delta T \propto y^{1/5}$. So, that we have. So now, and for uniform wall temperature ΔT is constant and $q" \sim y^{1/4}$. So, now, we can see; what is the order of Nusselt number in case of uniform wall heat flux case. So, δ_T we have already found.

(Refer Slide Time: 29:39)



So, $\delta_T \sim HRa_{*H}^{-1/5}$ and this is the case for Pr>> 1 and $\Delta T \sim \frac{q_w''}{K} HRa_{*H}^{-1/5}$.

And we know, we have shown earlier that $Nu \sim \frac{H}{\delta_T}$. So from this expression, the first expression you can write $Nu \sim Ra_{*H}^{1/5}$. So here, you remember that this Rayleigh number is modified Rayleigh number based on wall heat flux.

Similarly, now for Pr<<1 . We can write, for uniform wall temperature case. Already we have found $\delta_T \sim HRa_H^{-1/4}\,\mathrm{Pr}^{-1/4}$. And $\Delta T \sim \frac{q_w''\delta_T}{K}$.

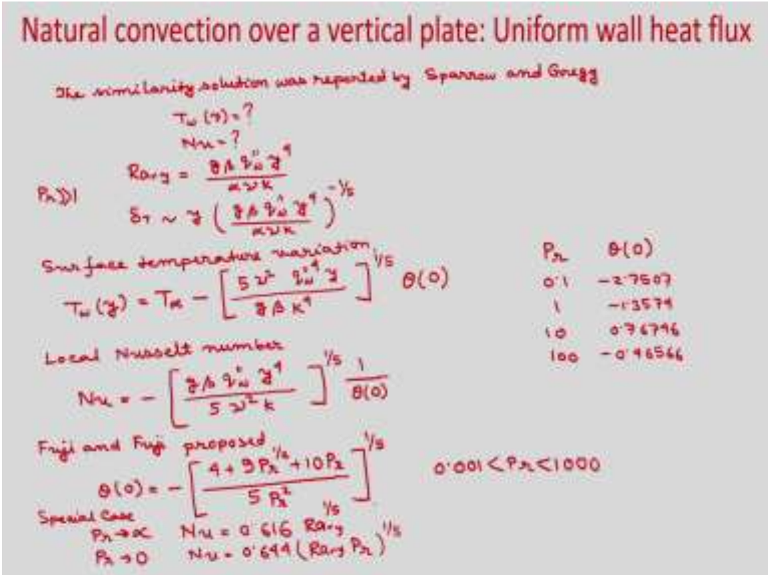
So, if you put these ΔT in the expression of Rayleigh number where Δ T is there, then for uniform wall heat flux case, you can write $\delta_T \sim H\left(Ra_{*H}\,\mathrm{Pr}\right)^{-1/5}$.

And Nusselt number and $\Delta T \sim \frac{q_w''}{K} H\left(Ra_{*H}\,\mathrm{Pr}\right)^{-1/5}$ and $Nu \sim \frac{H}{\delta_T}$. So, $Nu \sim \left(Ra_{*H}\,\mathrm{Pr}\right)^{1/5}$.

So, we have shown the $Nu \sim \left(Ra_{*H}\,\mathrm{Pr}\right)^{1/5}$ for Pr << 1 and for high Prandtl number fluids it depends on Rayleigh number. So, $Ra_H^{1/5}$.

So, using similarity method you can solve this problem, but we will not go into details, just we will show some results for uniform wall heat flux case using similarity variable.

(Refer Slide Time: 33:04)



So, the similarity solution was reported by Sparrow and Gregg; So, what we need to find? We need to find T_W as a function of y and Nusselt number right. So, from here, you can see that in this case $Ra_{*y} = \dfrac{g\beta q_w'' y^4}{\alpha \nu K}$.

And δ_T for high Prandtl number case, we have defined as $\delta_T \sim y\left[\dfrac{g\beta q_w'' y^4}{\alpha \nu K}\right]^{-1/5}$.

So, from the solution the surface temperature variation is given as follows. $T_w(y) = T_\infty - \left[\dfrac{5\nu^2 q_w''^4 y}{g\beta K^4}\right]^{1/5}\theta(0)$. $\theta(0)$ is the temperature, at η= 0. So these depend on Prandtl number. And you can see, Prandtl number for different Prandtl number what is the $\theta(0)$. So, at 0.1 you have - 2.7507, 1 it is - 1.3574, 10 it is 0.76746 and for 100 it is - 0.46566. And local Nusselt number is given by, $Nu = -\left[\dfrac{g\beta q_w'' y^4}{5\nu^2 K}\right]^{1/5}\dfrac{1}{\theta(0)}$.

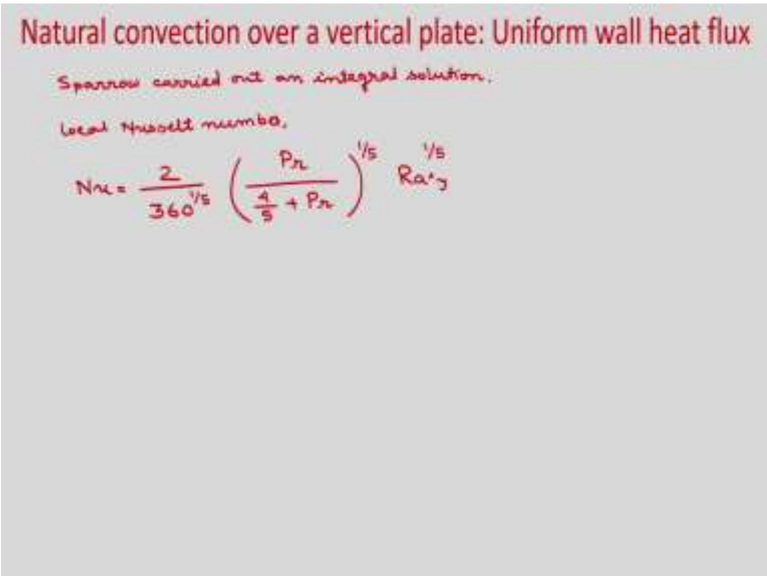
So, in literature you will find many correlations. So, Fuji and Fuji proposed this correlations to find the temperature at θ = 0. So, Fuji and Fuji proposed, $\theta(0) = -\left[\frac{4+9\,\mathrm{Pr}^{1/2}+10\,\mathrm{Pr}}{5\,\mathrm{Pr}^2}\right]^{1/5}$.

So, this is valid in the range of $0.001 < \mathrm{Pr} < 1000$. And as a special case, for Prandtl number special case $\mathrm{Pr}\to\infty$. $Nu = 0.616 Ra_{*y}^{1/5}$ and you can see from the scale analysis also, we have found that $Nu \sim Ra^{1/5}$ and $\mathrm{Pr}\to 0$.

$Nu = 0.644\left(Ra_{*y}\,\mathrm{Pr}\right)^{1/5}$, this also we have shown for $\mathrm{Pr} < 1$ the Nusselt number varies as $\left(Ra_{*y}\,\mathrm{Pr}\right)^{1/5}$.

So, these results we have shown from the similarity solution, but using integral solutions also many researchers have proposed correlations we will just write down what Sparrow proposed.

(Refer Slide Time: 38:33)



So, Sparrow carried out an integral solution and propose the local Nusselt number as, $Nu = \frac{2}{360^{1/5}}\left(\frac{\mathrm{Pr}}{\frac{4}{5}+\mathrm{Pr}}\right)^{1/5} Ra_{*y}^{1/5}$.

So, in today's class we started with the energy equation and using the similarity variable, we converted this PD to OD and we have shown the boundary conditions for both momentum and energy equations and we have shown the solution of θ and g which is your representation of velocity v versus η. And we have also shown the local heat transfer coefficient and local Nusselt number and average heat transfer coefficient and average Nusselt number for the case of uniform wall temperature case.

Later, we considered uniform wall heat flux case, and from the scale analysis we have found the order of Nusselt number and δ_T. Then we have shown some similarity solution what are available in the literature, and also we have shown the local Nusselt number expression from the integral solution of Sparrow.

Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 09
Natural Convection – II
Lecture – 29
Natural convection over a vertical plate: Integral solution

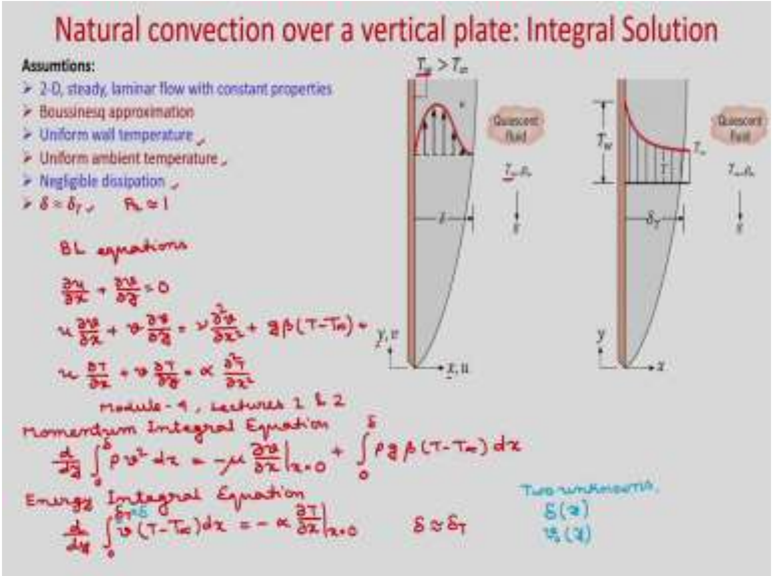
Hello everyone. So, in today's class we will solve natural convection over a vertical flat plate using integral method. We have already learned this integral method in module 4, lecture 1. We have derived the momentum integral equation, in module 4, lecture 2 we have derived the energy integral equation.

So, please refer these derivations. In today's class we will use the momentum integral equation and energy integral equation with some modification, because we have a buoyancy term in the boundary layer equations for natural convection flow and we will solve for the unknown variables δ.

So, if you remember in post convection we have two unknowns; one is hydro (Refer Time: 01:31) boundary layer thickness δ and thermal boundary layer thickness δ_T and we had two integral equations and we solve for δ and δ_T. In natural convection will assume that $\delta = \delta_T$. For these we solve this equations and we will find the unknown variable δ and another unknown variable velocity profile.

Although, we are assuming that $\delta = \delta_T$, but later will show that the solutions; whatever we will be deriving using this integral analysis it will be valid for wide range of Prandtl number.

(Refer Slide Time: 02:17)



So, let us consider natural convection over a vertical flat plate. The plate is maintained at uniform wall temperature T_w and the quiescent medium temperature is T_∞ and that is also maintained at constant temperature. Here, we have these assumptions 2 dimensional steady laminar flow with constant properties, we have Boussinesq approximation valid and T_w is constant, T_∞ is constant and we are neglecting the viscous dissipation.

So, in this integral solution we will assume $\delta \approx \delta_T$; that means, $Pr \approx 1$, but we will write the solution in terms of Prandtl number and we will show that the solution is reasonable reasonably valid for wide range of Prandtl number.

So, first let us write the boundary layer equations for a natural convection over a flat plate. So, we have already derived these equations, boundary layer equations. So, continuity equation, the momentum equation is $u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 v}{\partial x^2}+g\beta\left(T-T_\infty\right)$. So, in this case x is perpendicular to the wall and y is along the vertical wall and energy equation, $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial x^2}$.

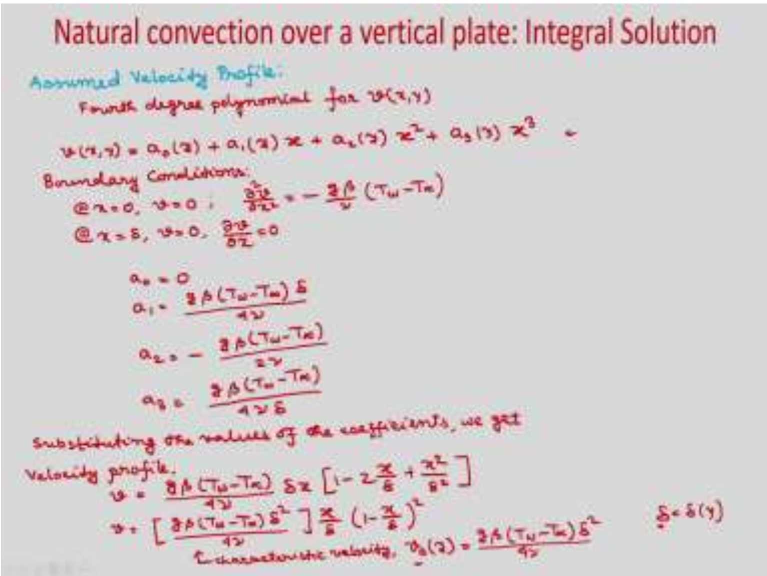
So, you refer module 4, lecture 1 and 2 for derivation of these integral equation. So, if you derive you will get momentum integral equation as; so, using the continuity equation you can derive the momentum integral equation from this momentum equation as

$\frac{d}{dy}\int_0^{\delta}\rho v^2 dx = -\mu \frac{\partial v}{\partial x}|_{x=0} + \int_0^{\delta}\rho g\beta\left(T - T_{\infty}\right)dx$ and energy integral equation you can write as $\frac{d}{dy}\int_0^{\delta_T} v\left(T - T_{\infty}\right)dx = -\alpha\frac{\partial T}{\partial x}|_{x=0}$. So, you can see $-K\frac{\partial T}{\partial x}|_{x=0}$ will give you wall heat flux and here alpha is thermal diffusivity.

So, now, we will assume $\delta \approx \delta_T$. So, here the integral, we will integrate up to δ. We will solve for two unknowns; one is δ which is function of y and another is unknown velocity profile. So, some will derive later, some velocity v_0, so which will be function of y. So, these are the two unknowns we will use this two integral equations and will solve for $\delta(y)$ and $v_0(y)$.

So, what is the next step while using the integral method? So, you have to assume the velocity profile and temperature profile, then once you get the velocity and temperature profile you have to invoke those in the integral equations.

(Refer Slide Time: 07:27)



So, first let us see what is the assumed velocity profile. So, we will use here fourth degree polynomial for velocity. So, will use,

$v\left(\text{x}, y\right) = a_0\left(y\right) + a_1\left(y\right)x + a_2\left(y\right)x^2 + a_3\left(y\right)x^3$.

So, we have to find these coefficients a_0, a_1, a_2, and a_3. So, there are four coefficients. So, how many boundary conditions we need? So, we need four boundary conditions, two boundary condition you know easily; one is the wall you have velocity 0 at x →∞ you have velocity 0 and also at x →∞ you have velocity gradient is 0, because it is a quiescent medium. Another boundary condition you have to derive; so, from the momentum equation. So, that is the derived boundary condition, these already we have discussed in module 4.

So, boundary conditions at x = 0, you have v = 0, at x = δ, you have v = 0, also the velocity gradient is 0, because you have a quiescent medium. So, $\frac{\partial v}{\partial x}$=0 and another boundary condition that is derived from the momentum equation. So, at x = 0 you can write $\frac{\partial^2 v}{\partial x^2} = -\frac{g\beta}{\nu}\left(T_w - T_\infty\right)$.

So, in a momentum equation the inertia term will become 0, because u ,v at 0. So, this is your viscous term and at wall you have T = T_w; so, you can write $\frac{\partial^2 v}{\partial x^2} = -\frac{g\beta}{\nu}\left(T_w - T_\infty\right)$.

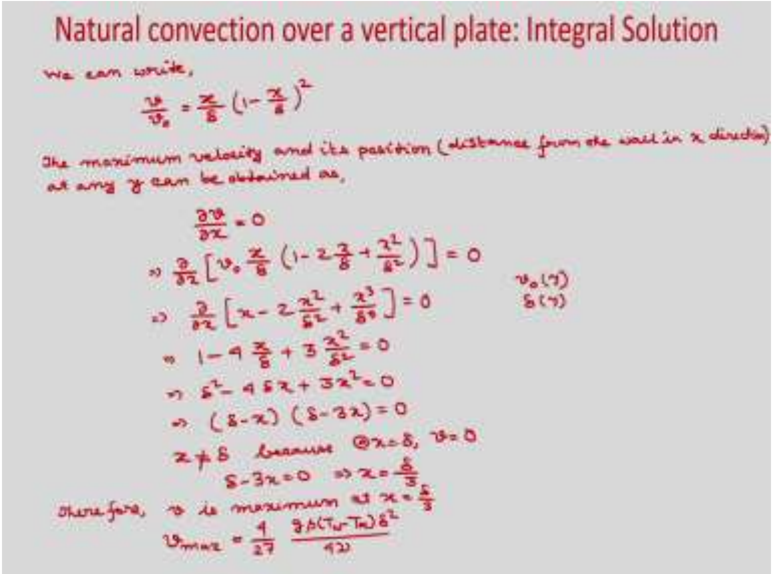
So, if you use invoke this boundary condition and find the coefficients you will get a_0 = 0 $a_1 = \frac{g\beta\left(T_w - T_\infty\right)\delta}{4\nu}$, $a_2 = -\frac{g\beta\left(T_w - T_\infty\right)}{2\nu}$ and $a_3 = -\frac{g\beta\left(T_w - T_\infty\right)}{4\nu\delta}$.

So, now this coefficient if you substitute in the assumed velocity profile, then you will get the velocity profile. So, if you see substituting the values of the coefficient we get; so, if you substitute here, so we will get the velocity profile as $v = \frac{g\beta\left(T_w - T_\infty\right)}{4\nu}\delta x\left[1 - 2\frac{x}{\delta} + \frac{x^2}{\delta^2}\right]$.

If you rearrange it, you will get $v = \left[\frac{g\beta\left(T_w - T_\infty\right)\delta^2}{4\nu}\right]\frac{x}{\delta}\left(1 - \frac{x}{\delta}\right)^2$. So, you can see the first term in the right hand side in the inside the bracket. So, these term can form the characteristic velocity.

Now, we will say that it is the characteristic velocity and v_0 which is function of y and this is to be found from the solution. So, this is another unknown $v_0(y) = \dfrac{g\beta\left(T_w - T_\infty\right)\delta^2}{4\nu}$. So, you can see δ is function of y. So, v_0 is also function of y. So, this is the second unknown, first unknown is the δ we have to find and another unknown v_0 we have to find .

(Refer Slide Time: 13:27)



So, now you can write the velocity profile as $\dfrac{v}{v_0} = \dfrac{x}{\delta}\left(1-\dfrac{x}{\delta}\right)^2$. Now, you know the velocity profile; assume velocity profile and that from that we can find the maximum velocity location. So, at which location you will get the maximum velocity. So, if you take the derivative of velocity v, then make it 0, then we will be able to find at which location you will get the maximum velocity, at which x location right.

So, the maximum velocity and its position, distance from the wall in x direction at any y can be obtained as. So, you can write $\dfrac{\partial v}{\partial x}$ =0 so; that means, $\dfrac{\partial}{\partial x}\left[v_0 \dfrac{x}{\delta}\left(1-2\dfrac{x}{\delta}+\dfrac{x^2}{\delta^2}\right)\right] = 0$. So, you see v_0 is function of y and δ is also function of y. So, you can take it outside the derivative.

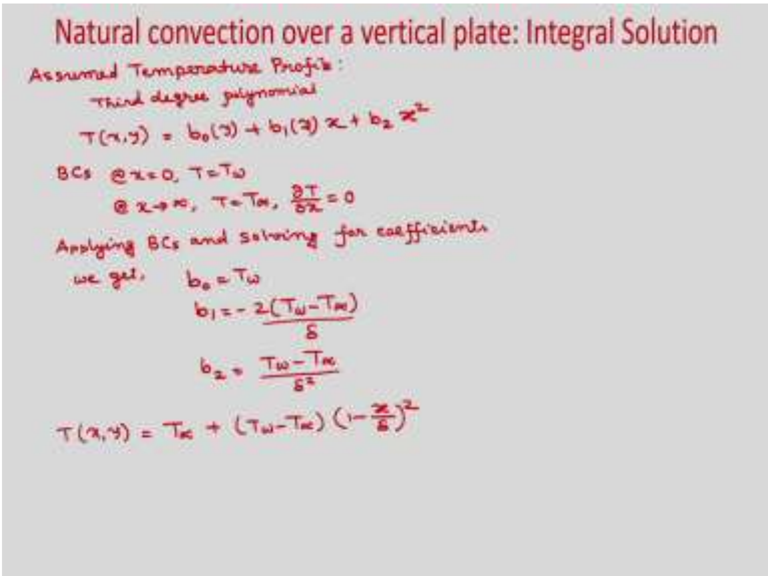
So, you can write $\frac{\partial}{\partial x}\left[x-2\frac{x^2}{\delta^2}+\frac{x^3}{\delta^3}\right]=0$, because v_0 is function of y and δ is function of y right and from here if you see; so, you will get, $1-4\frac{x}{\delta}+3\frac{x^2}{\delta^2}=0$.

If you rearrange it you will get $\delta^2-4\delta x+3x^2=0$ and you will get $(\delta-x)(\delta-3x)=0$. You can see x = δ you have velocity,v= 0.

So, there will not be maximum velocity. So, x ≠δ, because at x = δ you have v = 0 right. So, you have $\delta-3x$=0; that means, $x=\frac{\delta}{3}$. So, you can see at $x=\frac{\delta}{3}$ you will get the maximum velocity. So, therefore, v is maximum at $x=\frac{\delta}{3}$ and its value if you find, it will be v_{max} , after simplification I am writing $v_{max}=\frac{4}{27}\frac{g\beta(T_w-T_\infty)\delta^2}{4\nu}$.

So, we have found the velocity distribution now, let us find the temperature profile, assume temperature profile. So, for that also we will use a third degree polynomial.

(Refer Slide Time: 17:43)



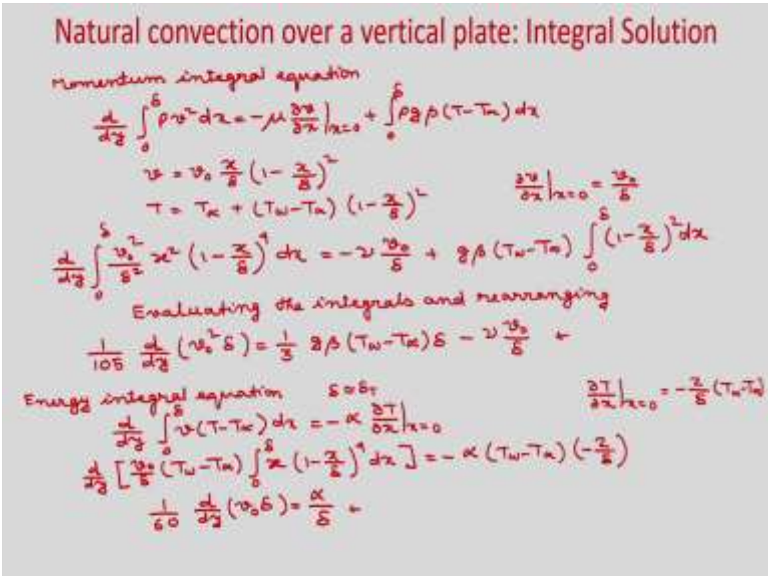
So, assumed temperature profile we will use third degree polynomial. So, if it is so then you can write $T(x,y)=b_0(y)+b_1(y)x+b_2x^2$.

So, now you need three boundary conditions to find the three unknowns b_0 , b_1 and b_2. So, you know at wall you have temperature T_w, at $x \rightarrow \infty$ you have temperature T_∞ which is the quiescent free temperature, as well as $x \rightarrow \infty$ your temperature gradient is 0. So, you can write boundary conditions at $x = 0$ you have $T = T_w$, at $x \rightarrow \infty$ you have $T=T_\infty$ and also the temperature gradient is 0.

So, applying boundary conditions and solving for the coefficients we get, applying boundary conditions and solving for coefficients we get $b_0 = T_w$, $b_1 = -\frac{2\left(T_w - T_\infty\right)}{\delta}$ and $b_2 = \frac{\left(T_w - T_\infty\right)}{\delta^2}$. So, if you put these values in the assumed temperature profile, then you will get $T\left(x, y\right) = T_\infty + \left(T_w - T_\infty\right)\left(1 - \frac{x}{\delta}\right)^2$.

So, now, we have assumed the velocity profile as well as the temperature profile. Now, you want to put this profiles into the integral equations. So, because you have the momentum equation where you have temperature profile as well as you have the velocity profile. So, in the momentum integral equation you need the temperature profile as well as the velocity profile.

(Refer Slide Time: 20:43)



So, if you put there; so, you have momentum integral equation so that is your, $\frac{d}{dy}\int_0^\delta \rho v^2 dx = -\mu \frac{\partial v}{\partial x}|_{x=0} + \int_0^\delta \rho g \beta \left(T - T_\infty\right) dx$. Now, put the velocity profile and temperature profile. So, you have $v = v_0 \frac{x}{\delta}\left(1-\frac{x}{\delta}\right)^2$ and temperature profile $T = T_\infty + \left(T_w - T_\infty\right)\left(1-\frac{x}{\delta}\right)^2$.

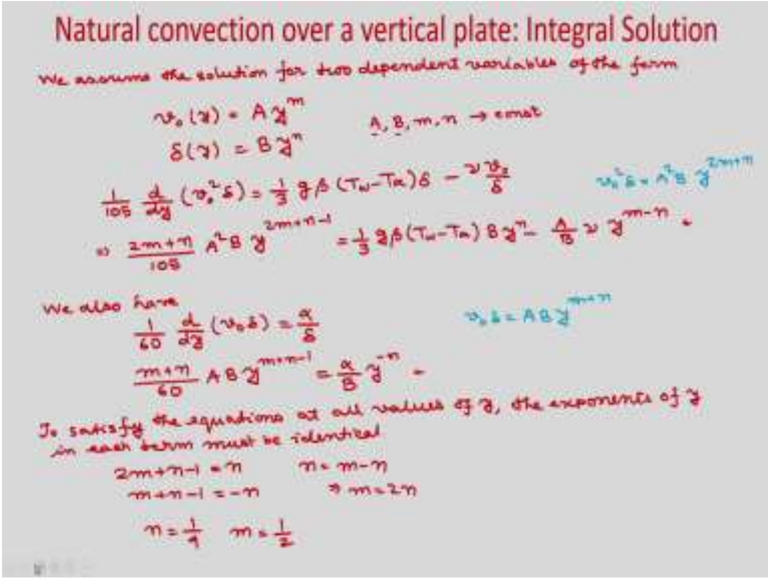
So, here you can see you have T -T_∞. These equation if you divide both sides by ρ then if you rearrange you will get $\frac{d}{dy}\int_0^\delta \frac{v_0^2}{\delta^2}x^2\left(1-\frac{x}{\delta}\right)^4 dx = -\nu\frac{v_0}{\delta} + g\beta\left(T_w - T_\infty\right)\int_0^\delta\left(1-\frac{x}{\delta}\right)^2 dx$.

So, now, evaluating the integrals and rearranging, you will get, $\frac{1}{105}\frac{d}{dy}\left(v_0^2\delta\right) = \frac{1}{3}g\beta\left(T_w - T_\infty\right)\delta - \nu\frac{v_0}{\delta}$.

Now, similarly you put the value of temperature profile in the energy integral equation. So, energy integral equation if you remember. So, it has $\frac{d}{dy}\int_0^\delta v\left(T - T_\infty\right)dx = -\alpha\frac{\partial T}{\partial x}\Big|_{x=0}$.

So, put the velocity profile and the temperature profile here. So, you will get $\frac{d}{dy}\left[\frac{v_0}{\delta}\left(T_w - T_\infty\right)\int_0^\delta x\left(1-\frac{x}{\delta}\right)^4 dx\right] = -\alpha\left(T_w - T_\infty\right)\left(-\frac{2}{\delta}\right)$. So, from here you can see, $\frac{\partial T}{\partial x}\Big|_{x=0} = \left(-\frac{2}{\delta}\right)\left(T_w - T_\infty\right)$. So, just rearranging, you will get $\frac{1}{60}\frac{d}{dy}\left(v_0\delta\right) = \frac{\alpha}{\delta}$. So, you can see we have got first order ordinary differential equation. So, this is one equation and this is another equation. These are first order ordinary differential equation and we need to find v_0 and δ from here.

(Refer Slide Time: 26:21)



So, we assume the solution for two dependent variables of the form v_0 which is function of y. So, we will use $v_0(y) = Ay^m$, because it is function of y and $\delta(y) = By^n$ where A, B, m, n, are constants .

So, now if you put these values in the ordinary differential equation; so for momentum equation we got this ordinary differential equation and find the value of A, B and m and n. So, you will get $\frac{1}{105}\frac{d}{dy}\left(v_0^2\delta\right) = \frac{1}{3} g\beta\left(T_w - T_\infty\right)\delta - \nu\frac{v_0}{\delta}$.

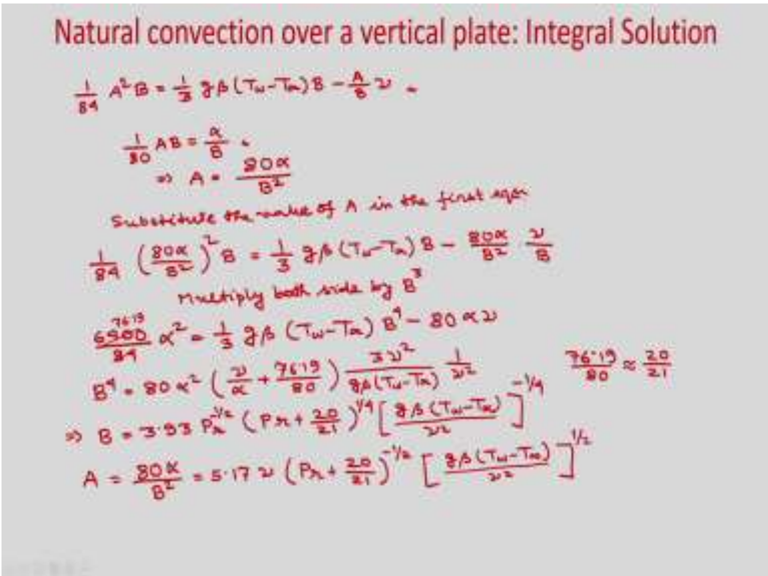
So, put the v_0 and δ expression here. So, you will get $v_0^2\delta = A^2By^{2m+n}$. So, if you take the derivative with respect to y, then you will get, $\frac{2m+n}{105}A^2By^{2m+n-1} = \frac{1}{3} g\beta\left(T_w - T_\infty\right)By^n - \frac{A}{B}\nu y^{m-n}$ and another ordinary differential equation we have, so that is your $\frac{1}{60}\frac{d}{dy}\left(v_0\delta\right) = \frac{\alpha}{\delta}$. So, if you see $v_0\delta = ABy^{m+n}$. So, you can take the derivative with respect to y.

So, you will get $\frac{m+n}{60}ABy^{m+n-1} = \frac{\alpha}{B}y^{-n}$. So, you can see, we have these equation and we have these equation and to satisfy the equations at all values of y, the exponents of y in each term must be identical.

So, to satisfy the equations at all values of y the exponents of y in each term must be identical. So, if it is so then you can write $2m+n-1=n$ and also you can write n =m - n and you can write $m+n-1=-n$.

So, from here you can find so, you can see from here you can find m = 2 n and if you put it here. So, you will get n = 1 /4 and m =1/2. So, now, you can see we have found the exponent m and n, m =1/2 and n = 1 /4. So, now, we have to find other two constants that is A and B.

(Refer Slide Time: 31:11)



.

So, now these exponents you put in those equations. After simplification you will get $\frac{1}{84}A^2B=\frac{1}{3}g\beta\left(T_w-T_\infty\right)B-\frac{A}{B}\nu$ and if you simplify it, so you will get; so, this is one equation you will get an another equation you will get $\frac{1}{80}AB=\frac{\alpha}{B}$. So, what we are doing? So, we are substituting the value of m and n in this equation and in this equation.

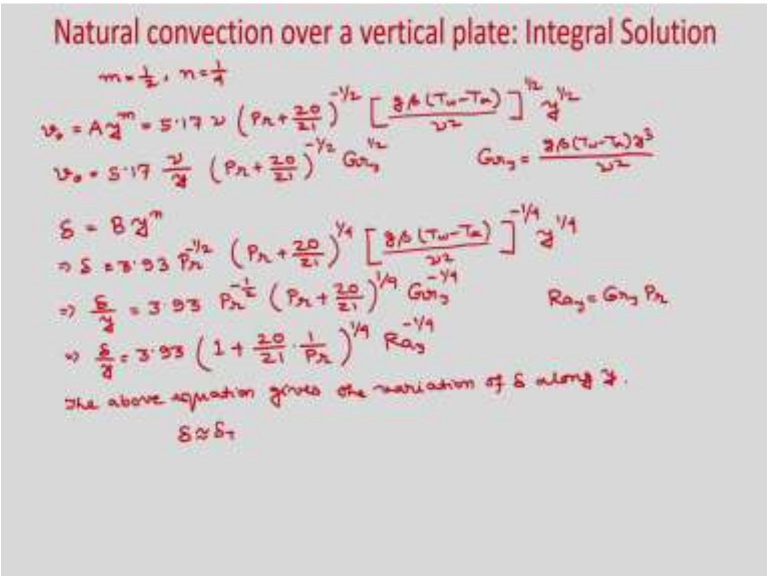
So, from the first equation you are getting this and from the second equation you are getting this. So, from here you can write $A=\frac{80\alpha}{B^2}$ and this you substituted in the first equation. So, what you will get? So, substitute the value of A in the first equation, top equation. So, what you will get? $\frac{1}{84}\left(\frac{80\alpha}{B^2}\right)^2B=\frac{1}{3}g\beta\left(T_w-T_\infty\right)B-\frac{80\alpha}{B^2}\frac{\nu}{B}$.

So, what you do? You multiply both side by B^3. So, if you rearrange it, so you will get $\frac{6400}{84}\alpha^2 = \frac{1}{3} g\beta(T_w - T_\infty)B^4 - 80\alpha\nu$. So, from here you will find that B^4; so, if you see it will come almost 76.19.

So, we are writing 76.19. So, $B^4 = 80\alpha^2\left(\frac{\nu}{\alpha}+\frac{76.19}{80}\right)\frac{3\nu^2}{g\beta(T_w - T_\infty)}\frac{1}{\nu^2}$. After rearrangement you will get these and this $\frac{76.19}{80} \approx \frac{20}{21}$.

So, we can write the value of $B = 3.93\,\text{Pr}^{-1/2}\left(\text{Pr}+\frac{20}{21}\right)^{1/4}\left[\frac{g\beta(T_w - T_\infty)}{\nu^2}\right]^{-1/4}$. So, now, you will be able to find the value of $A = \frac{80\alpha}{B^2}$.So, B value substitute and rearrange you will get $A = 5.17\nu\left(\text{Pr}+\frac{20}{21}\right)^{-1/2}\left[\frac{g\beta(T_w - T_\infty)}{\nu^2}\right]^{1/2}$. So, now you know the value of A and B, m and n; so, you will be able to write the velocity profile and the boundary layer thickness δ.

(Refer Slide Time: 35:55)



We have m =1/2, n =1/4 the expression of A, B. So, you will be able to find the velocity, $v_0 = Ay^m$. So, if you put all these values and you will get, $5.17\nu\left(\mathrm{Pr}+\frac{20}{21}\right)^{-1/2}\left[\frac{g\beta\left(T_w-T_\infty\right)}{\nu^2}\right]^{1/2} y^{1/2}$.

So, now what we will do? You can see if we put inside here y^3 then it will represent the Grashof number $Gr_y = \frac{g\beta\left(T_w-T_\infty\right)y^3}{\nu^2}$, so that will give you the non dimensional number Grashof number. So, we will put inside y^3, so it will be and we will subtract from here. So, after rearrangement you will get $v_0 = 5.17\frac{\nu}{y}\left(\mathrm{Pr}+\frac{20}{21}\right)^{-1/2} Gr_y^{1/2}$.

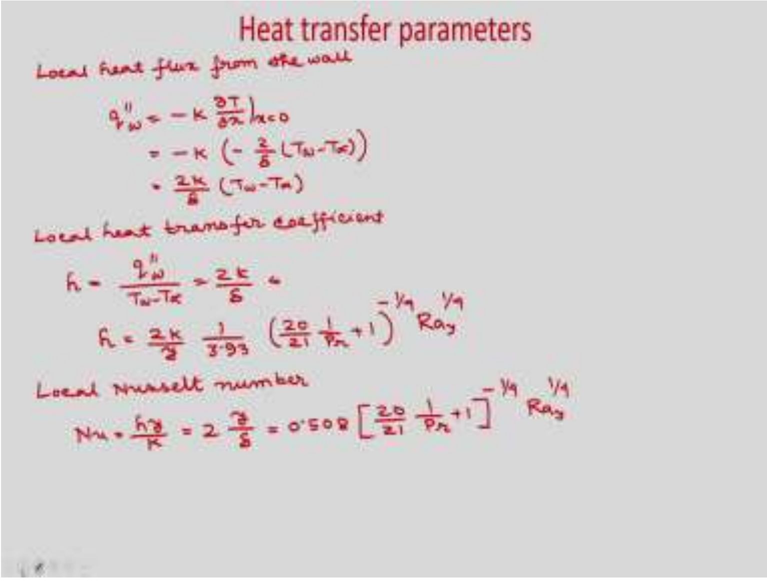
Where $Gr_y = \frac{g\beta\left(T_w-T_\infty\right)y^3}{\nu^2}$ and we have $\delta = By^n$.

So, $\delta = 3.93\,\mathrm{Pr}^{-1/2}\left(\mathrm{Pr}+\frac{20}{21}\right)^{1/4}\left[\frac{g\beta\left(T_w-T_\infty\right)}{\nu^2}\right]^{-1/4} y^{1/4}$.

And similar way here we will also put inside y^3 and we will rearrange and we will get $\frac{\delta}{y} = 3.93\,\mathrm{Pr}^{-1/2}\left(\mathrm{Pr}+\frac{20}{21}\right)^{1/4} Gr_y^{-1/4}$. So, Prandtl number and into Grashof number what it will give? It will give Rayleigh number . So, $Ra_y = Gr_y\,\mathrm{Pr}$. So, if you write it you will get, $\frac{\delta}{y} = 3.93\left(1+\frac{20}{21}\frac{1}{\mathrm{Pr}}\right)^{1/4} Ra_y^{-1/4}$.

So, you can see this equation will give you the boundary layer thickness along Y. So, the above equation gives the variation of δ along Y and if you remember we have assumed $\delta \approx \delta_T$, but we have written the expression in terms of Prandtl number .

(Refer Slide Time: 40:27)



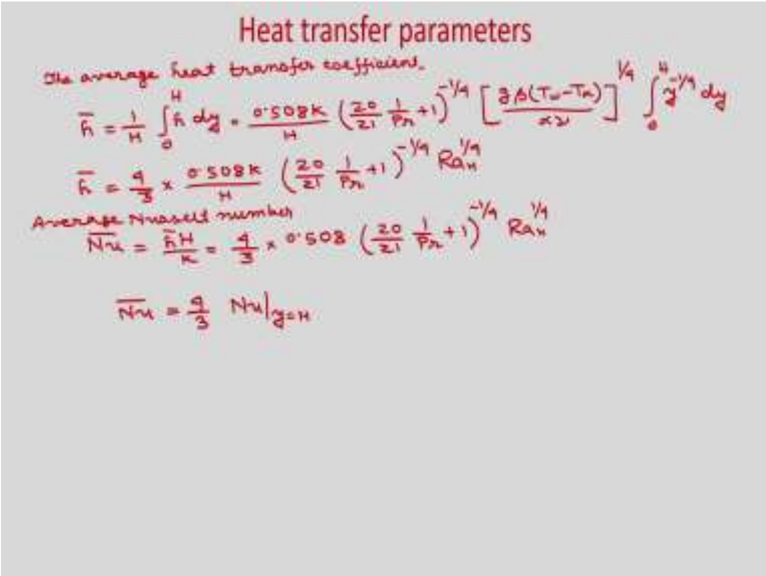
So, now we will find the heat transfer parameter. So, we will find the local heat flux then heat transfer coefficient and the Nusselt number. So, let us write the local heat flux from the wall.

So, we have $q_w^" = -K\frac{\partial T}{\partial x}|_{x=0}$ and we have shown that you will get, $q_w^" = -K\frac{\partial T}{\partial x}|_{x=0} = -K\left(-\frac{2}{\delta}\left(T_w - T_\infty\right)\right)$. So, this we will get $\frac{2K}{\delta}\left(T_w - T_\infty\right)$. So, the local heat transfer coefficient you can write $h = \frac{q_w^"}{T_w - T_\infty}$. So, this is from Newton's law of cooling. So, you will get h=$\frac{2K}{\delta}$.

So, δ expression you know. So, if you put the value then you will get $h = \frac{2K}{y}\frac{1}{3.93}\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_y^{1/4}$. So, this we are writing from this expression. So, you can see we have δ(y). So, this expression, from this expression we are writing the value of heat transfer coefficient.

Now, you can write the local Nusselt number . So, you will get $Nu = \frac{hy}{K}$ so, it will be $2\frac{y}{\delta}$. So, if you put this value of $\frac{\delta}{y}$, then you will get $Nu = 0.508\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_y^{1/4}$.

(Refer Slide Time: 43:23)



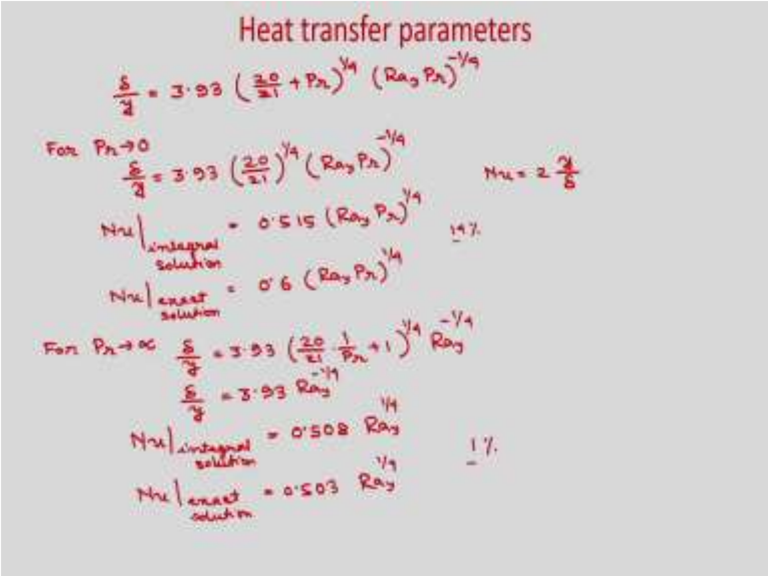
So, now the average heat transfer coefficient; the average heat transfer coefficient you can write as $\bar{h} = \frac{1}{H}\int_0^H h dy$ and put the values then you will get, $\frac{0.508K}{H}\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4}\left[\frac{g\beta\left(T_w - T_\infty\right)}{\alpha\nu}\right]^{1/4}\int y^{-1/4}dy$.

So, if you integrate it and rearrange you will get $\bar{h} = \frac{4}{3}\times\frac{0.508K}{H}\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_H^{1/4}$. So, now, Nusselt number, average Nusselt number you can write $\overline{Nu} = \frac{\bar{h}H}{K}$ so, you will get, $\frac{4}{3}\times 0.508\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{-1/4} Ra_H^{1/4}$. From here you can see the $\overline{Nu} = \frac{4}{3}Nu\big|_{y=H}$.

So, from here you can see that your $\overline{Nu} = \frac{4}{3}Nu\big|_{y=H}$. So, now we will discuss about two limiting cases; one is the $\Pr \to 0$ and another is $\Pr \to \infty$. You know when $\Pr << 1$ your

Nusselt number will be function of Rayleigh number and Prandtl number and when Pr→∞ that means when you have a high Prandtl number fluids then your Nusselt number will be function of only Rayleigh number. So, already we have found the value of $\frac{\delta}{y}$ first let us write that K expression.

(Refer Slide Time: 46:33)



So, $\frac{\delta}{y} = 3.93\left(\frac{20}{21} + \Pr\right)^{1/4} \left(Ra_y \Pr\right)^{-1/4}$ So, now, consider the limiting case for Pr→ 0. So, if Pr→ 0, you can see this term you can put 0.

So, you can write $\frac{\delta}{y} = 3.93\left(\frac{20}{21}\right)^{1/4} \left(Ra_y \Pr\right)^{-1/4}$. So, from here now you can write the expression of Nusselt number .You know $Nu = 2\frac{y}{\delta}$. So, these expression if you put local Nusselt number now, you can write from the integral solution. After rearranging, you will get $Nu\big|_{\substack{integral\\solution}} = 0.515\left(Ra_y \Pr\right)^{1/4}$.

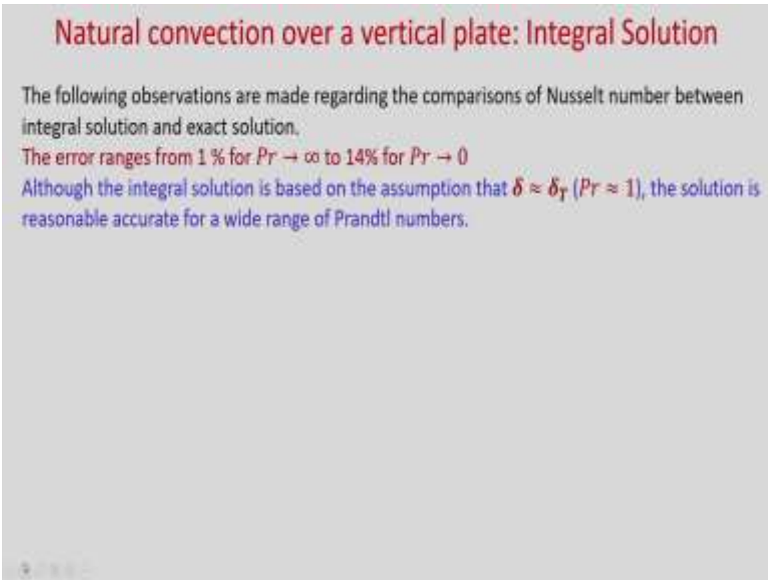
And if you remember these Nusselt number from the exact solution from the similarity solution we have written $Nu\big|_{\substack{exact\\solution}} = 0.6\left(Ra_y \Pr\right)^{1/4}$ and for Pr→ ∞ another limiting case we can write the expression of $\frac{\delta}{y} = 3.93\left(\frac{20}{21}\frac{1}{\Pr}+1\right)^{1/4} Ra_y^{-1/4}$. So, as Pr→ ∞, you can see these term will become 0, because 1 /Pr is there.

So, we can write $\frac{\delta}{y} = 3.93 Ra_y^{-1/4}$; obviously, now $Nu = 2\frac{y}{\delta}$. So, from integral solution, if you write it will get $Nu\big|_{\substack{integral\\solution}} = 0.508 Ra_y^{1/4}$ and you can see for high Prandtl number fluids, it depends on only the Rayleigh number and if you remember we have written the exact solution from the similarity solution as $Nu\big|_{\substack{exact\\solution}} = 0.503 Ra_y^{1/4}$.

So, you can see here and this is almost 1 % variation, for Pr→ ∞there is a 1 % variation and maximum variation occurs as Pr→ 0; so, these around 14 % you will get. So, in these two limiting cases you can see these variation of this Nusselt number between the integral solution and the exact solution varies between 1 % to 14 %.

So, you can see although, we have assumed that $\delta \sim \delta_T$, but still this gives a reasonable good solution for the different Prandtl numbers.

(Refer Slide Time: 50:11)



So, the observation whatever we have made you can see the error ranges from 1 % for Pr→ ∞ to 14 % for Pr→ 0 and although the integral solution is based on the assumptions that $\delta \approx \delta_T$ the solution is reasonable, accurate for a wide range of Prandtl numbers.

So, in today's class we have solved the boundary layer equations using integral method. So, first we integrated the momentum equation and the energy equation and we have written the momentum integral equation and energy integral equation, then we have assumed the velocity profile and temperature profile and invoking those expression in the integral equation.

We have written two ordinary differential equations then we have written the expression for δ and v_0 and we have put in the ordinary differential equation, and we have found the value of $\frac{\delta}{y}$ and from there we have found the heat transfer parameters like heat transfer coefficient Nusselt number.

And we have also shown the two limiting cases where $Pr \rightarrow 0$ and $Pr \rightarrow \infty$ and we have shown that in these limiting cases your error of integral solution compared to the exact solution varies between 1 % to 14 %.

Thank you.

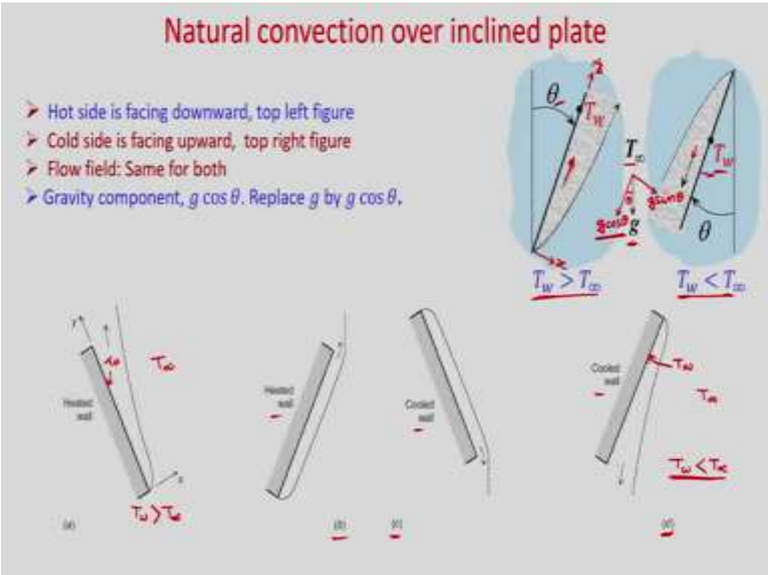
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 09
Natural Convection - II
Lecture – 30
Natural convection over inclined plate and mixed convection

Hello everyone, so we have already derived the Nusselt number expression for Natural convection over a vertical flat plate using integral solution as well as similarity solution, but you will find the natural convection in other configurations as well.

Say if you have an inclined plates or horizontal plate, and in horizontal plate if top side is 1 or bottom side is 1, then you will get different types of flow and Nusselt number expression will be different. Even if you consider horizontal circular cylinder, then also you will get different Nusselt number expression. So, today we will discuss about these configurations, and later we will just discuss about mixed convection flow.

(Refer Slide Time: 01:25)



So, you can see in this case, say if it is an inclined plate where θ is measured from the vertical line and T_w is the wall temperature and ambient temperature is T_∞, and gravity is acting in negative y-direction. And if $T_w > T_\infty$, that means, your wall is maintained at higher temperature than the ambient temperature, then obviously, you will get this upward flow. And if your wall temperature $T_w < T_\infty$, then obviously you can see you will get downward flow.

So, you can see it depends hot side is facing downward. So, you can see here in this case top side is facing downward and $T_w > T_\infty$, and you are getting flow like this. And in this case, you can see your cold side is facing upward. So, this is the cold plate because $T_w < T_\infty$.

So, cold side is facing upward. So, here $T_w < T_\infty$, and flow is happening in downward direction. And here flow is happening in upward direction due to natural convection.

But if you see these two cases are same, because you will get the similar boundary layer and similar expression for the Nusselt number. So, you can see now if gravity is acting in negative y-direction, then in the flow direction, so you need to take gravity accordingly. So, when you are defining the buoyancy term in place of g, you just put g cosθ.

So, you will get two components. So, you can see this is your θ, and this will get g cosθ, and in this direction g sinθ. So, you can see in this particular case if y is along the plate, then in y momentum equation you will get g cosθ. So, you can see in place of g, you just write g cosθ in the buoyancy term.

Now, you can see here four different cases. In this case, the first case you can see heated wall tilted upward, so heated wall tilted upward. And the case d, you can see in this case cooled wall tilted downward.

So, when the natural convection is happening for this particular case, so here in this case $T_w > T_\infty$, and this is your T_w wall temperature and ambient temperature is T_∞. And similarly this is your T_w, and ambient temperature is T_∞, and $T_w < T_\infty$.

So, you can see already we have discussed here both the cases are same. You can see the effect of the angle θ is to thicken the tail end of the buoyancy layer and to give the wall jet a tendency to separate from the wall. So, you can see as it is going up, so you are at the tail end boundary layer is thicken. And in this case where $T_w < T_\infty$ also it is similar observation you can see.

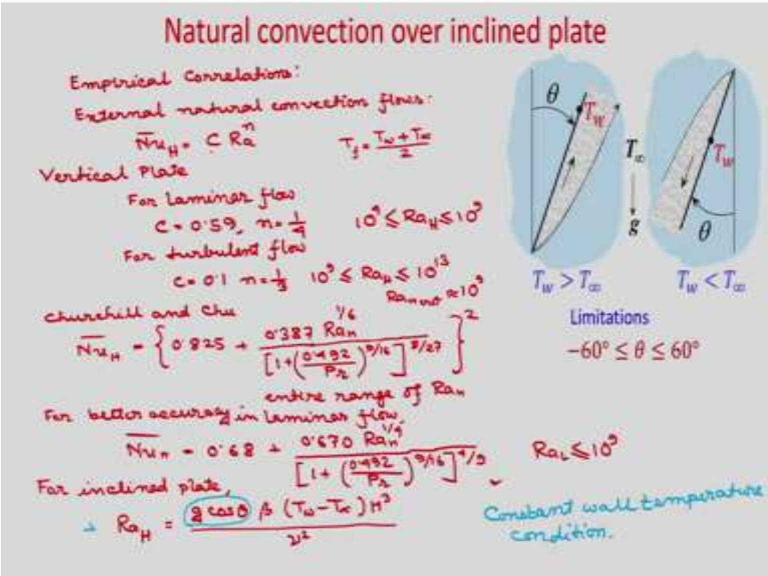
The next one you can see here heated wall tilted in downward direction and cooled wall is tilted in upward direction . So, it is the opposite effect you will get. In this case, you can see the wall jet is pinched as it flows over the trailing edge, because in this case as it is going up you can see it is the jets is pinched as it flows over the trailing edge .

So, this is the different cases you can see when natural convection is taking place over a flat plate, but depending on the wall condition, you will get different flow physics . So, in this case b and c are similar.

So, although using the similarity solution and integral solution, we could find the Nusselt number expression for natural convection over a vertical flat plate, but you can also write some empirical relations. So, in the literature, different researchers performed experiments; and based on that they are proposed some relations which are known as empirical relations.

For these particular cases whatever we discussed, the similar expression you can use whatever we had already discussed from the similarity solution and integral solution just replacing g with the g cosθ. So, in the buoyancy term or the non-dimensional number whatever you are considering Grashof number or Rayleigh number. So, there you replace g with g cosθ then you will get the Nusselt number as same for this cases.

(Refer Slide Time: 07:10)



So, first let us write empirical correlations for external natural convection flows. So, Nusselt number, average Nusselt number based on some characteristic length, you can write as $\overline{Nu}_H = CRa^n$.

So, Ra^n, so where the properties are evaluated at film temperature you know that is $T_f = \frac{T_w + T_\infty}{2}$. So, it is the average temperature. So, that is known as film temperature. So, at that temperature, you need to find the properties and for different types of flows you will get different constant C and this power n.

So, for vertical plate, for laminar flow, you will get C = 0.59 and n = 1 / 4 in the range of $10^4 \le Ra_H \le 10^9$, or in this case let us write H because we have used H.

So, generally in natural convection the flow becomes turbulent if $Ra_H \ge 10^9$. So, for turbulent flows you can write C =0.1, and n =1 / 3 in the range of $10^9 \le Ra_H \le 10^{13}$.

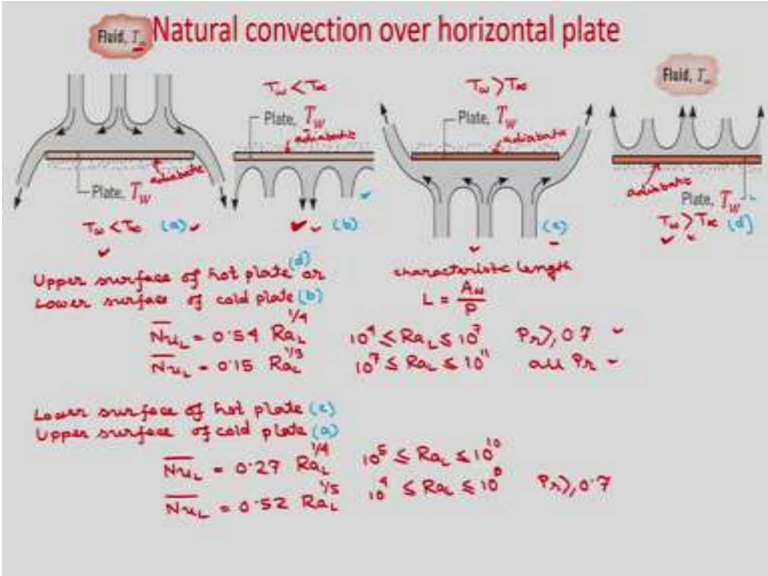
So, generally critical Reynolds number for turbulent flows, it is around 10^9. I hope that you will get the turbulent flows. And Churchill and Chu, they proposed for any range of Rayleigh number average, so it is, $\overline{Nu}_H = \left\{ 0.825 + \frac{0.387 Ra_H^{1/6}}{\left[1 + \left(\frac{0.492}{\Pr}\right)^{9/16}\right]^{8/27}} \right\}^2$.

So, these actually works good in reasonably good in the all range of Rayleigh number. So, it is entire range of Rayleigh number. But for better accuracy, in laminar flow, you can use this relation. $\overline{Nu}_H = 0.68 + \frac{0.670 Ra_H^{1/4}}{\left[1 + \left(\frac{0.492}{\Pr}\right)^{9/16}\right]^{4/9}}$. So, in the laminar flow range, you will get good accuracy if you use this correlation .

So, you can see these are for vertical plate. And as we are discussing inclined plate, so for inclined plate just you replace g = g cosθ. So, where you have Rayleigh number you just for inclined plate Rayleigh number will be $Ra_H = \frac{g \cos\theta \beta \left(T_w - T_\infty\right) H^3}{\nu^2}$.

So, you can see for inclined plate we have put g cosθ in place of g. And you remember in this case it is for all these relations we have used for constant wall temperature case. So, all these relations, you can use for inclined plate just using these Rayleigh number expression.

(Refer Slide Time: 13:22)



Now, if you consider horizontal plate, and in the horizontal plate if top side is warm or bottom side is warm, so depending on the thermal conditions, you will get different flow physics and you will get different Nusselt number expression. So, you can see here in this case say this is the horizontal plate and bottom side is adiabatic.

So, it is adiabatic. And topside is your T_w, and $T_w < T_\infty$. So, how, what is the flow physics you will get? See you can see as $T_w < T_\infty$ the fluid whatever is coming, so it will get cooled and these high density fluid will just go down. So, this way you will get the flow.

And now if topside is adiabatic, so there is no heat transfer across the top wall. And now bottom side temperature is T_w, and $T_w < T_\infty$, so that means it is cooled wall. So, in this particular case, obviously, the fluid coming into contact with the wall, so the density will become higher and it will go down. So, this way you will get that the high density fluid will be drained down.

Next case you can see it is also adiabatic top wall, but now $T_w > T_\infty$. If $T_w > T_\infty$ the fluid which coming into contact with the wall obviously the density will be lighter and it will go up.

So, this way you can see it is going up. And for this particular case, now this side is adiabatic and top side is now warm warmer than the ambient, so $T_w > T_\infty$. So, obviously, lighter fluid will go up, so this way. So, you will get this type of flow physics.

So, now, we will write the correlations for these cases. You can see this case and this case are almost similar. So, these are similar flow physics you are getting. And this case and this case you are getting similar flow physics. So, we can write the correlations as same for these two cases, and also for these two cases. So, you can see upper surface of hot plate or lower surface of cold plate.

So, if I say that this is the case a, this is the case b, this is the case c and this is the case d. Then upper surface of hot plates, that means, this is the case, so this is your case d . And lower surface of cold plate so; that means, this is lower surface of cold plate, so this is the case b .

So, for these two cases, you can write the correlation Nusselt number based on some characteristic length L. I am telling characteristic length L because $L = \frac{A_w}{P}$. So, this is the characteristic length, and it will be calculated as the heat transfer area – A_w divided by the perimeter , $L = \frac{A_w}{P}$.

So, for these you will write $\overline{Nu}_L = 0.54 Ra_L^{1/4}$ in the range of $10^4 \le Ra_L \le 10^7$, and Pr ≥ 0.7.

And for higher Rayleigh number, you can write Nusselt number. So, these are average $\overline{Nu}_L = 0.15 Ra_L^{1/3}, 10^7 \le Ra_L \le 10^{11}$ for all Prandtl number. So, you can see the first expression for laminar flow and it is for turbulent flow.

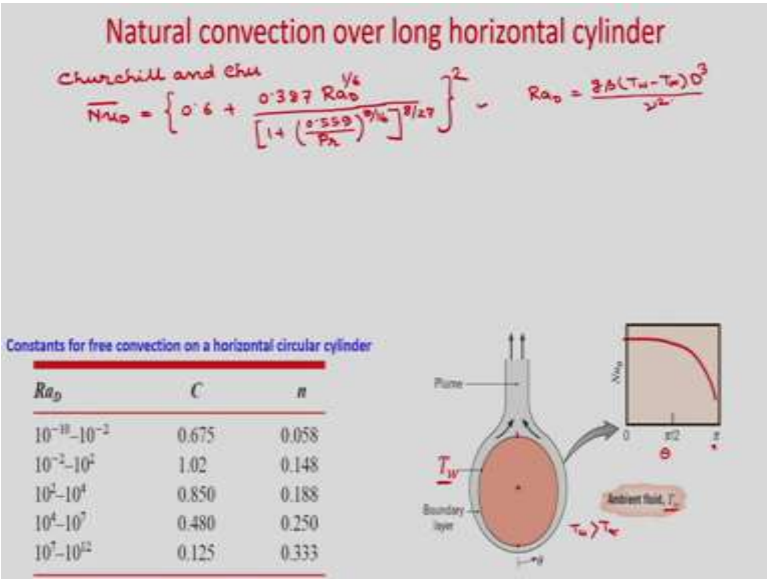
And now other two cases, you see this is the case a where upper surface of cold plate, and this is the bottom surface of the hot plate. So, the other cases you can write lower surface of hot plate. So, this is the case c. And you have upper surface of cold plate. So, this is the case.

So, you can write the case this is as c, and this is the case a. So, for these cases, this is the empirical relation you can use. $\overline{Nu}_L = 0.27 Ra_L^{1/4}$ in the range of $10^5 \le Ra_L \le 10^{10}$.

And also another expression is there $\overline{Nu_L} = 0.52 Ra_L^{1/5}$. This is also correlation. This is valid in the range of $10^4 \le Ra_L \le 10^9$, and $\Pr \ge 0.7$.

So, these are the correlations you can use, but here characteristic length you have to calculate as heat transfer area divided by the perimeter; it is not the just length of the plate. So, this way you can calculate. So, obviously, for different surfaces if it is a circular disc if it is a circular disc also you can use this relations using the characteristic length, $L = \dfrac{A_w}{P}$.

(Refer Slide Time: 20:56)



Now, let us consider flow over a horizontal circular cylinder. So, you can see this is the horizontal circular cylinder, where wall temperature is T_w, and ambient temperature is T_∞. In this case, $T_w > T_\infty$. So, obviously, you can see boundary layer will be developed over this cylinder surface. And due to buoyancy effect, it will go up. And this lighter fluid will go up.

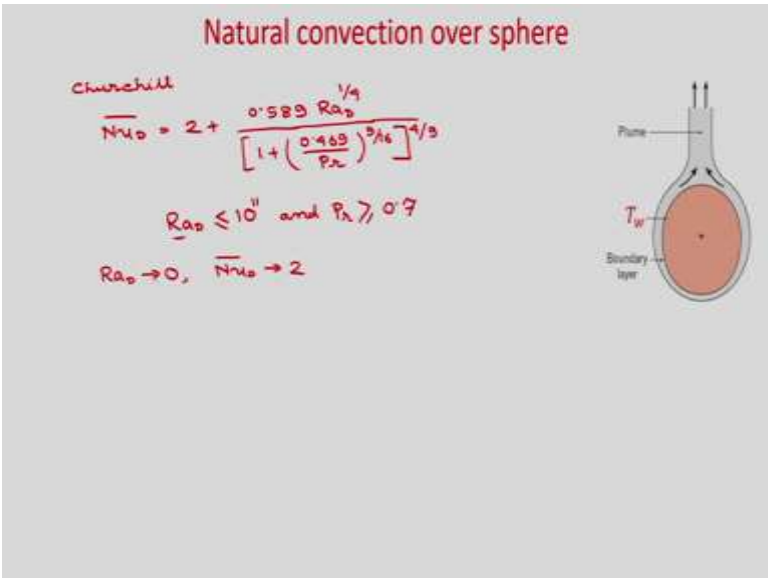
So, if you plot the Nusselt number along θ from here, so it is symmetric about this vertical line. So, along θ, if you see the Nusselt number, obviously, at θ = 0. That means, at this point you will get maximum Nusselt number.

Why, because you have low boundary layer or lowest boundary layer thickness at this point, and temperature gradient will be highest, and obviously, you will get maximum Nusselt number at θ =0. This is θ. So, θ = 0 we will get maximum Nusselt number.

As you go along this surface , so your Nusselt number will decrease. So, you can see, so obviously, minimum Nusselt number will get at this point at θ =π . So, the correlations proposed by Churchill and Chu, for this long horizontal cylinder case as Nusselt number based on the diameter D.

It is $\overline{Nu}_D = \left\{ 0.6 + \frac{0.387 Ra_H^{1/6}}{\left[1 + \left(\frac{0.559}{\text{Pr}} \right)^{9/16} \right]^{8/27}} \right\}^2$. So, this is the empirical relation for natural convection over long horizontal cylinder, and Rayleigh number is based on diameter. So, it is $Ra_D = \frac{g\beta\left(T_w - T_\infty\right)D^3}{\nu^2}$.

(Refer Slide Time: 23:53)

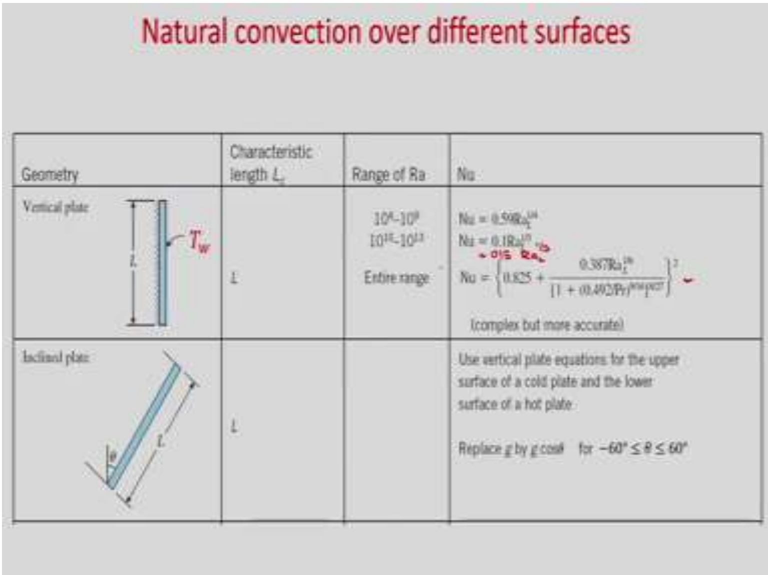


Now, if you consider a sphere and natural convection over a hot sphere, then we can write the correlation proposed by Churchill. So, this is also correlation. So, Churchill recommended this Nusselt number correlation as $\overline{Nu}_D = 2 + \dfrac{0.589 Ra_D^{1/4}}{\left[1+\left(\dfrac{0.469}{\Pr}\right)^{9/16}\right]^{4/9}}$.

And it is valid in the range of $Ra_D \leq 10^{11}$, and $\Pr \geq 0.7$. So, you can see this is the sphere and diameter of the sphere is D, and based on that Rayleigh number is defined.

Now, in this expression if you put $Ra_D \rightarrow 0$, then what will happen? So, if you put, $Ra_D \rightarrow 0$, then $\overline{Nu}_D \rightarrow 2$. So, what does it mean? It means that heat transfer by conduction between a spherical surface and stationary infinite medium. So, only conduction will be taking place as $Ra_D \rightarrow 0$.

(Refer Slide Time: 25:42)


Natural convection over different surfaces

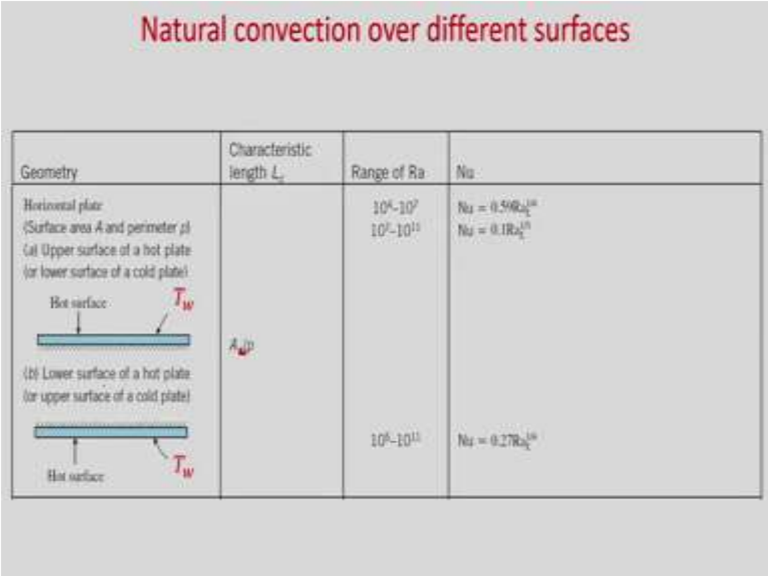
Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical plate	L	10^4–10^9 10^{10}–10^{13} Entire range	$Nu = 0.59 Ra_L^{1/4}$ $Nu = 0.1 Ra_L^{1/3}$ $\approx 0.15\, Ra_L^{1/3}$ $Nu = \left\{0.825 + \dfrac{0.387 Ra_L^{1/6}}{[1+(0.492/Pr)^{9/16}]^{8/27}}\right\}^2$ (complex but more accurate)
Inclined plate	L		Use vertical plate equations for the upper surface of a cold plate and the lower surface of a hot plate Replace g by $g\cos\theta$ for $-60° \leq \theta \leq 60°$

Now, let us summarize whatever we have discuss about the empirical relations, based on the experiments natural convection over different surfaces. So, vertical plate, you can see if you see characteristic length L, so in the range of 10^4 to 10^9, $Nu = 0.59 Ra_L^{1/4}$.

So, L is the characteristic length. And if it is high Rayleigh number in the range of turbulent flows, then it will be 0.1 or 0.15 also you can write 0.15, $Nu = 0.15 Ra_L^{1/3}$.

And for entire range of Rayleigh number, so this is the Churchill and Chu proposed. So, this is the expression you can use. If it is inclined plate and θ is measured from the vertical line, and just replace g by g cos θ, in the relations whatever we discussed for this vertical plate. And you can see we have a restriction that this θ should be between -60^0 to 60^0. So, in that range, you will get a good match.

(Refer Slide Time: 27:06)

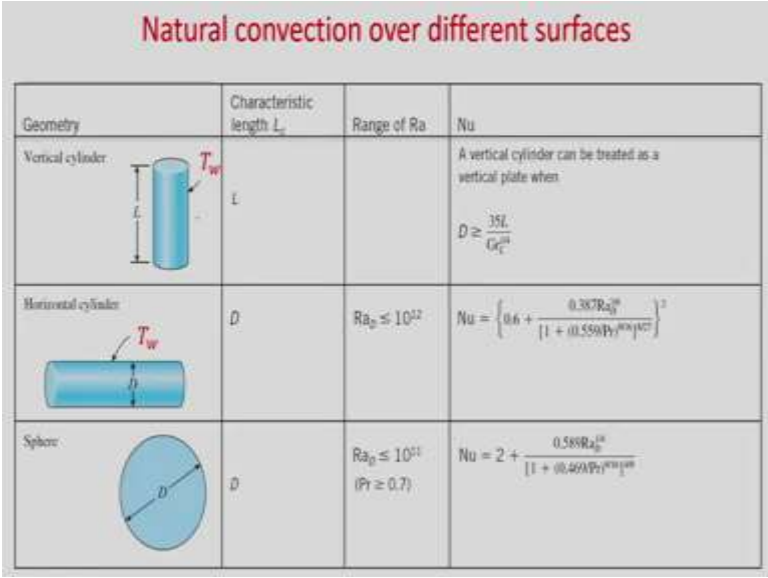


Natural convection over different surfaces

Geometry	Characteristic length L_c	Range of Ra	Nu
Horizontal plate (Surface area A and perimeter p) (a) Upper surface of a hot plate (or lower surface of a cold plate)	A_s/p	10^4–10^7 10^7–10^{11}	$Nu = 0.59Ra_L^{1/4}$ $Nu = 0.1Ra_L^{1/3}$
(b) Lower surface of a hot plate (or upper surface of a cold plate)		10^5–10^{11}	$Nu = 0.27Ra_L^{1/4}$

So, for horizontal plate you can see that you can calculate the characteristic length as $L = \frac{A_w}{P}$. And when you have a hot surface on the top upper surface of a hot plate or lower surface of a cold plate, so you can use these Nusselt number expression for these range of Rayleigh number. And similarly lower surface of a hot plate or upper surface of a cold plate, you can use this relation.

(Refer Slide Time: 27:37)



Natural convection over different surfaces

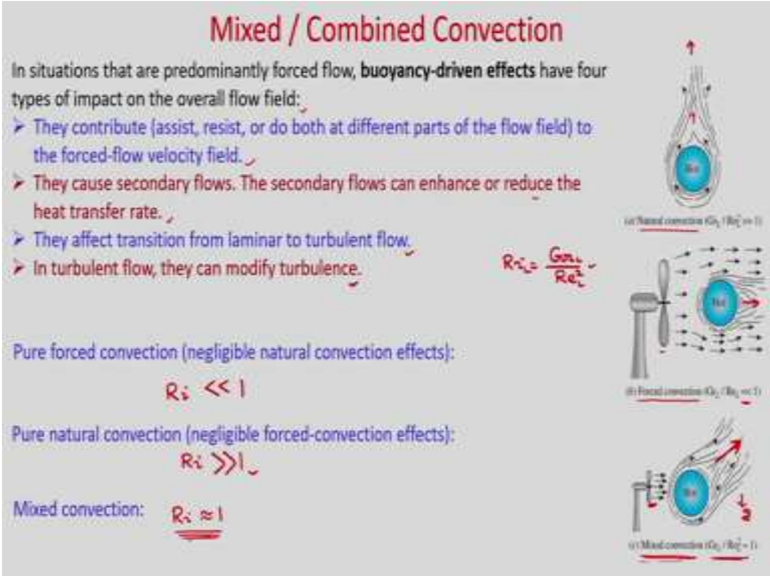
Geometry	Characteristic length L_c	Range of Ra	Nu
Vertical cylinder	L		A vertical cylinder can be treated as a vertical plate when $D \geq \frac{35L}{Gr_L^{1/4}}$
Horizontal cylinder	D	$Ra_D \leq 10^{12}$	$Nu = \left\{0.6 + \frac{0.387Ra_D^{1/6}}{[1 + (0.559/Pr)^{9/16}]^{8/27}}\right\}^2$
Sphere	D	$Ra_D \leq 10^{11}$ $(Pr \geq 0.7)$	$Nu = 2 + \frac{0.589Ra_D^{1/4}}{[1 + (0.469/Pr)^{9/16}]^{4/9}}$

If it is a vertical cylinder, then you can write a vertical cylinder can be treated as a vertical plate when, $D \geq \frac{35L}{Gr_L^{1/4}}$. So, the same relations for vertical plate whatever we have discussed you can use if this relation is satisfied.

For a horizontal cylinder, $Ra_D \leq 10^{12}$, you can use this Churchill and Chu correlation. And for a sphere also we have discussed for $Ra_D \leq 10^{11}$ and $Pr \geq 0.7$, you can use this Nusselt number relation.

So, now, we will discuss about the mixed convection which is also known as combined forced and natural convections. So, in this particular case, we have already discussed that both the forces forced and buoyancy are important and both are significant. So, we have shown earlier that Richardson number actually determines whether flow is natural convection, or forced convection, or mixed convection.

(Refer Slide Time: 28:56)



So, if you see that if Richardson number, we defined obviously as Grashof number by Reynolds number square right. So, if you some characteristic length if we define, then the $Ri_L = \frac{Gr_L}{\mathrm{Re}_L^2}$. So, this is the Richardson number. So, based on the Richardson number , we will determine whether flow is natural convection, or forced convection, or mixed convection.

So, there are many applications of mixed convection although if fluid velocity is less, then this forced convection and natural convection both will be significant. And in the application of electronics cooling or few heat exchangers, you will see that application of mixed convection.

In situations that are predominantly forced flow, buoyancy driven effects have four types of impact on the overall flow field. So, we can see they contribute either assist, resist, or do both at different parts of the flow field to the forced flow velocity field. They cause secondary flows.

The secondary flows can enhance or reduce the heat transfer rate . So, in presence of buoyancy effect whether it is assisting or resisting, based on that you will get the heat transfer enhancement or heat transfer reduction.

They affect transition from laminar to turbulent flows. And in turbulent flow, they can modify turbulence. So, we know that for pure forced convection, where you can neglect the natural convection effect. So, for this particular case, you can see that your Ri << 1, then you will get pure forced convection.

So, we can see example. So, you have a hot sphere and flow is taking place in the horizontal direction forced flow. So, it is a forced convection. So, where you have Ri<<1. For pure natural convection where you can neglect the forced convection effect, so their Ri>> 1.

So, this is the case you can see in the absence of forced convection, you will get only the natural convection. And due to buoyancy effect the flow will go up and flume will go up in this direction.

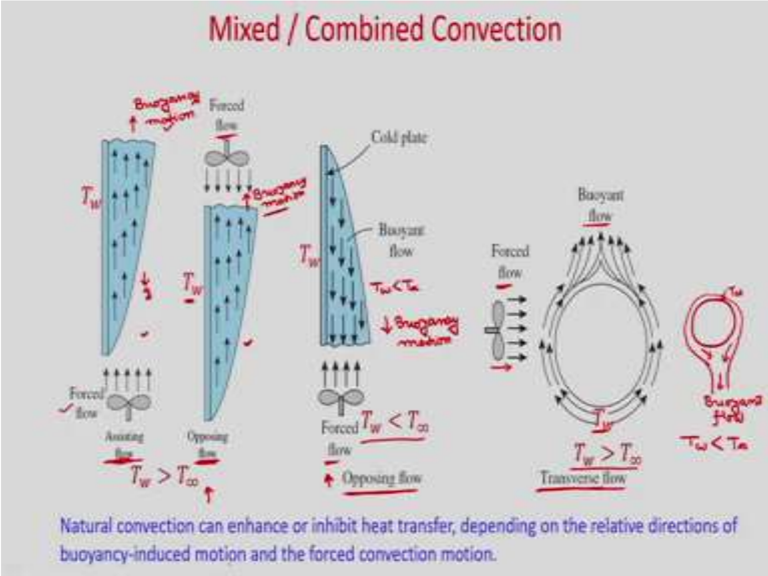
So, here actually with negligible forced convection effects, you can write or you can consider that it is a natural convection and the condition is Ri>> 1. But if Ri=1 or Ri ~ 1, then obviously, you will get the effect of both natural and forced convection, so that is known as mixed convection or combined convection.

So, you can see in this case there is a forced flow. And also you have buoyancy effect, and due to that the there will be effect of natural convection as well as forced convection here.

And it is known as mixed convection where Ri ~ 1. And you can see in this case forced convection flow will become only horizontal direction. In natural convection as you have absence of forced convection, flow will occur at the vertical direction due to the buoyancy effect. But when you have a mixed convection, so the both effect if you consider then you can see the flume will go up with an inclination. So, in this direction, it is going.

So, in this for a forced convection, you can see flow is occurring in this direction; in natural convection it is going in this direction. But for a mixed convection as it is having the combined effect of forced convection and natural convection, flow is occurring with an inclination. So, now, let us discuss whether this effect of buoyancy will assist or resist the flow.

(Refer Slide Time: 33:36)



So, you can see here this particular case, you have a hot plate. And from the bottom, you are giving a forced flow. You have a fan; and from the fan this flow is going up. And buoyancy g is acting in this direction, and obviously, you can see buoyancy effect will be in this direction. Buoyancy emotion will be in this direction.

So, obviously, you can see that the forced flow and the buoyancy motion are in same direction, so that means, it is kind of assisting flow. So, this buoyancy effect is also assisting the forced flow. So, this is known as assisting flow.

And consider this case. So, you have a hot plate, and obviously, the buoyancy motion in this direction. So, in this particular case, flow is coming in opposite direction of the buoyancy motion. So, it is trying to resist the flow. So, this forced flow is trying to resist the flow, and that is why it is known as opposing flow.

So, you can see in the assisting flow, buoyancy motion and the forced flow are in same direction. But in opposing flow your forced flow and buoyancy motion are in opposite direction. In forced flow and assisting flow, the buoyancy motion and the forced flow are in same direction. So, this is known as assisting flow, and this is known as opposing flow where buoyancy motion and forced flow are in opposite direction.

If you consider now cold plate, so this is your cold plate where $T_w < T_\infty$, so obviously, your buoyancy motion will be in this directions buoyancy motion. Why? Because the fluid which will come into contact with the cold plate, its density will be higher and it will go down, and the flow will occur in the downward direction.

But you have a forced flow. So, now, you can see the buoyancy motion and forced flow are in opposite direction. So, obviously, it is also known as opposing flow. So, these two cases you see. In this case, $T_w > T_\infty$. So, flow is natural flow is occurring in upward direction, but forced flow is trying to resist it . Here forced flow and buoyancy motion are in opposite direction.

If you consider the if you consider these case also in this case $T_w < T_\infty$. And you are due to natural convection this buoyant flow is occurring in the downward direction. And in this case also forced flow and buoyancy motion are in opposite direction, and this is also opposing flow.

So, depending on the thermal condition and the direction of forced flow and the buoyant flow, you will define whether it is assisting flow or opposing flow. Assisting flow, sometime it is known as also aiding flow. And now if you see that forced flow is occurring in perpendicular direction of the buoyancy flow, then it is known as transverse flow.

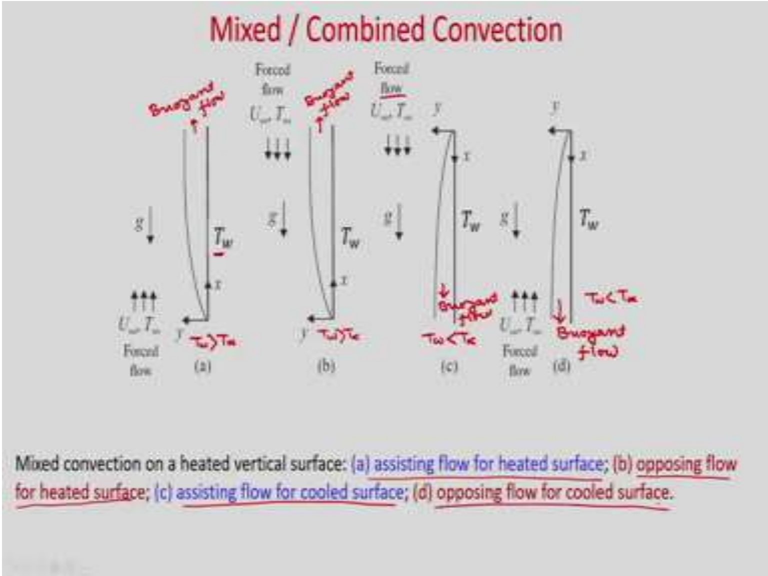
So, this particular case, you see this is let us say you have a sphere at temperature T w. And $T_w > T_\infty$, obviously, your buoyant flow will occur in upward vertical direction.

So, in this case, it is due to natural convection this will happen, but you have forced flow in perpendicular direction of the buoyant flow. So, this is kind of cross flow. So, this is known as transverse flow. This is known as transverse flow because buoyant flow and the forced flow are occurring in perpendicular direction.

And if $T_w < T_\infty$, this is also transverse flow because you can see if it is your sphere and T_w is the wall temperature, and if $T_w < T_\infty$, then obviously you can see that your flow will occur buoyant flow will occur in the downward direction.

So, in this direction, the flow will occur. And you have this is your buoyant flow. And perpendicular to this, you can see you have a forced flow. So, this is also transverse flow. So, you can see natural convection can enhance or inhibit heat transfer depending on the relative directions of buoyancy induced motion and the forced convection motion.

(Refer Slide Time: 38:57)



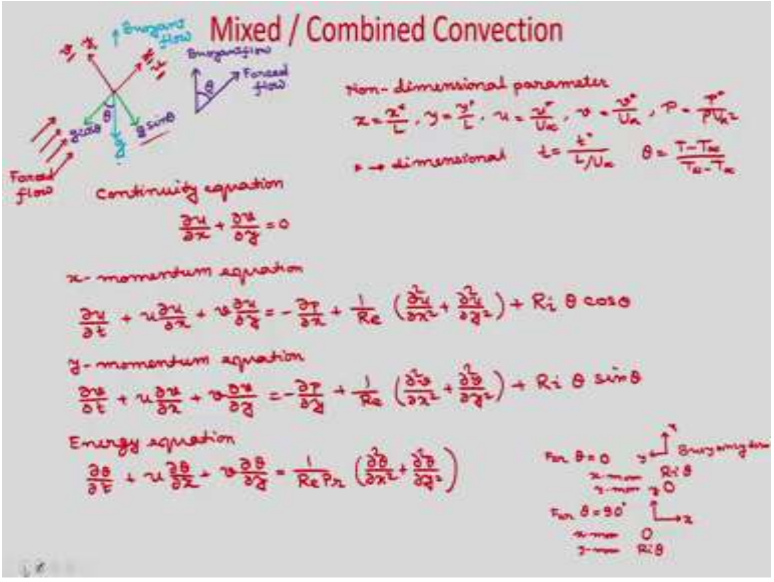
So, if you consider only flow over vertical plate, you can define the assisting or opposing flow for different thermal conditions. So, you can see in this case if it is heated surface, so $T_w > T_\infty$ in obviously, if it is forced flow direction in upward direction, then it is a assisting flow for heated surface.

In this case, here also $T_w > T_\infty$, and forced flow is just opposite to the buoyant motion, because buoyant motion will in these direction. These are buoyant motion or buoyant flow ok. So, you can see this is resisting the flow. So, this is known as opposing flow of heated surface.

Now, if you consider cooled surface where $T_w < T_\infty$ and forced flow is happening from top to bottom, then in these case buoyant flow in downward direction, and you will get assisting flow.

Assisting flow for cooled surface. Now, case d, if you see, here also it is a cooled surface and forced flow is occurring from bottom to upward direction, and buoyant flow is happening in these direction. So, these are in opposite direction. So, this is also opposing flow for cooled surface.

(Refer Slide Time: 40:45)



Now, let us write down the governing equations for this mixed convection. So, in this particular case, you can see your pressure gradient will also be there as well as there will be a buoyancy term. And this buoyancy term you can derive you can see from any book, and I am just writing the continuity equation momentum equation and the energy equation.

So, let us say that you have a forced flow in these direction. This is your forced flow. And if you define the axis x in the direction of forced flow, and perpendicular to this it is y. So, we are defining the coordinate system, in this way in the direction of forced flow x-direction and perpendicular to the direction of forced flow is y-direction.

So, now you can see obviously, your gravity will act in downward direction. So, this is your gravity. It is acting in downward direction. So, buoyant flow will be always in the upward direction. So, this is your buoyant flow. So, we will write the governing equations in x-direction and y-direction considering the buoyancy term.

So, now let us take two components of this gravity in x-direction and y-direction, and accordingly you can add it in the momentum equations. So, you can see in the if you take two components, this is one component in the negative x-direction you will get, and another component you will get in the negative y. So, and if you define the angle, so this will be your θ, so defining the θ,. So, you can see θ were defining, this is your buoyant flow; and this is your forced flow and this is the angle θ .

So, you can see this is your forced flow and this is your buoyant flow. So, this angle is we are defining as θ. So, you can have two components. So, one is g cosθ in the negative x-direction; and in negative y-direction, you have g sin θ. So, you see the definition of θ. And accordingly we are taking the component of g in negative x-direction and negative y-direction. So, and x-direction is due x-direction, we are taking in the direction of forced flow.

So, if you see that we have already written the non-dimensional equations using some non-dimensional parameters, and the governing equations in non-dimensional form we are going to write. So, you will see your continuity equation, you will get $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. So, we are taking the velocity u in the x-direction, and velocity v in y-direction. So, this you have to consider.

So, x momentum equation if you see, it will be $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+Ri\theta\cos\theta$, which is your non-dimensional temperature. And this velocity is u, v also we have written in non-dimensional form, and you have cos θ , because in x-direction you can see g cosθ right -g cos θ, so that you have written Ri θ cos θ.

And y momentum equation, so you will can see it will be, $\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Ri\theta\sin\theta$. So, you see gravity; in negative y-direction, it is g sin θ.

So, if you consider the buoyancy term. So, sin θ will be there and energy equation you will get, $\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\mathrm{RePr}}\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right)$.

So, whatever non-dimensional number we are getting, it depends on that how you have chosen the non-dimensional parameters. For getting this non-dimensional equations, we have used the following non-dimensional parameters.

So, non-dimensional parameters we have used, non-dimensional parameter we have used, so $x = \frac{x*}{L}$ now in this particular case I am just writing that star quantities dimensional. Because we have written the governing equations in non-dimensional form where u b we have given non-dimensional velocity so star quantities are dimensional.

So, and L is the characteristic length; $y = \frac{y*}{L}$. You do not get confused because earlier we defined $x = \frac{x*}{L}$ where x was the dimensional quantity here we are writing x_* is the dimensional quantity.

Just we have written the governing equations in non star form $u* = \frac{u}{U_\infty}$; U_∞ is the characteristic velocity ; $v* = \frac{v}{U_\infty}$, then $P* = \frac{P}{\rho U_\infty^2}$, no these we have to write star do not get confused, and $t = \frac{t*}{L/U_\infty}$. So, and $\theta = \frac{T - T_\infty}{T_w - T_\infty}$.

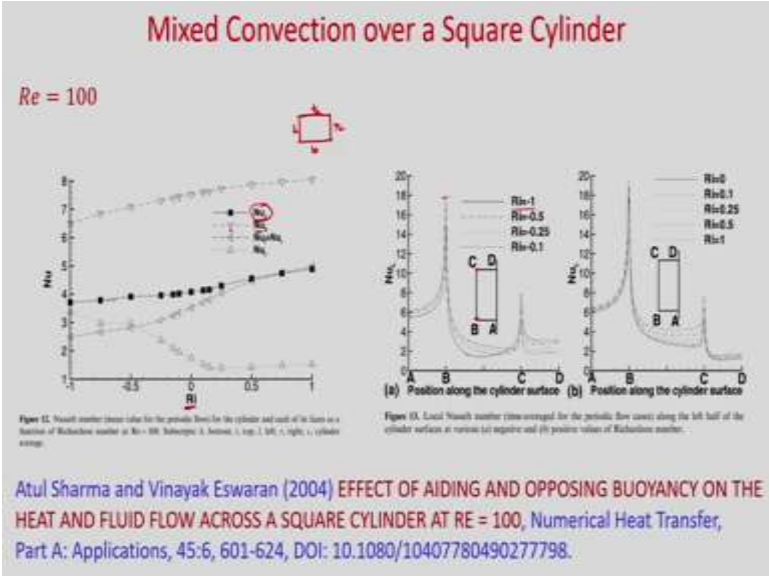
So, using these non-dimensional parameters, you can write the non-dimensional form of the governing equations as written here. So, if θ becomes 0, then what will happen? So, you will get sin θ as 0 and cos θ as 1.

And if θ is 0, then obviously you can see for θ = 0, you will get x and y like this; and you can see in the buoyant force and buoyant flow and forced flow will be in the same direction. And your x momentum equation will have Ri θ; and y momentum it will be 0 because sin θ is 0 and for $\theta = 90^0$.

So, what will happen? You will get x in this direction and y in this direction. So, in this case, x momentum equation, if it is 90^0 ,so x-momentum equation you will get buoyancy term, this is your buoyancy term buoyancy term. So, buoyancy term, we will get as 0, because $\cos 90^0 = 0$.

And in y momentum equation, you can see sin θ will be 1, so you will get desertion number into θ. So, you can use these equations for solving the mixed convection problem.

(Refer Slide Time: 51:12)

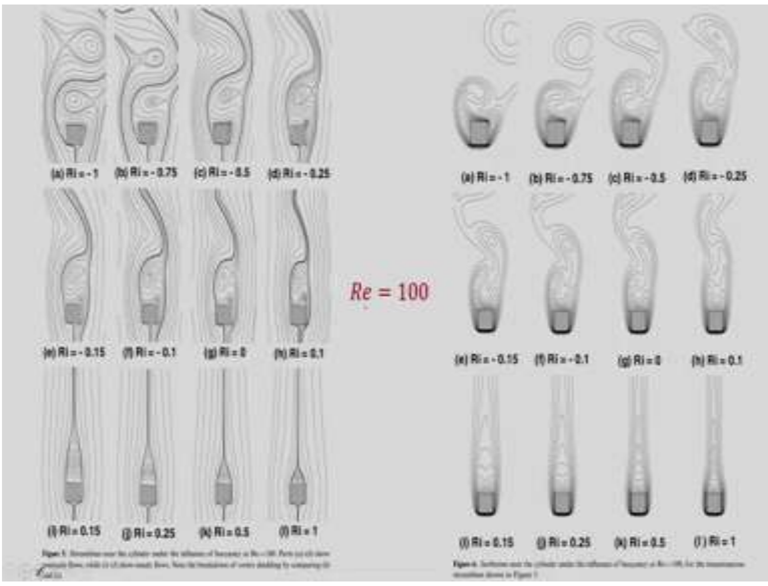


So, we will not go into details about finding the Nusselt number using some analytical approach, but we will show some results of mixed convection flow which are actually shown using numerical simulations.

So, first problem, you can see mixed convection over a square cylinder. So, you have a square cylinder maintained at temperature T_w and this square cylinder having the non-dimensional dimension as 1. And you have the vertical flow v_∞ and a main temperature is T_∞.

And these are the boundary conditions are used. And for Reynolds number 100 and for different Richardson number, this flow is calculated. So, it is taking this is taken from this paper Atul Sharma and Vinayak Eswaran Effect Of Aiding And Opposing Buoyancy On The Heat And Fluid Flow Across A Square Cylinder at Reynolds number 100 published in Numerical Heat Transfer, Part A, Applications.

(Refer Slide Time: 52:20)



So, you can see that for Reynolds number 100, and Prandtl number is 0.7. You can see the flow physics. These are the streamlines near the cylinder under the influence of buoyancy. So, you can see these cases are from Richardson number 0.15 to 1, you can see you have a steady flow.

There is no setting of vertices behind these cylinder, so these are steady flow. But if you see in these ranges of Rayleigh number where it is 0.1, 0, and in the negative direction -0.1 to -1, you can see you have vortex shedding behind the cylinder and this is unsteady flow.

And if you see the isotherms in right side figures , so in the isotherms also, you can see that in this range Richardson number 0.15 to 1, you have steady flow. And you can see the clustering of the isotherms near to this section, obviously, you will get maximum Nusselt number here . So, and here also you can see the isotherms. So, these are also instantaneous isotherms for at shown at different Richardson number.

Now, if you see the Nusselt number, so this is average Nusselt number is shown with Richardson number. Nu_C means it is a combined Nusselt number means if you consider a square cylinder, so you have a bottom, you have a top, this is your left, and this is your right.

So, for these you can see different surfaces, these are the average Nusselt number. So, you can see Nusselt number bottom, this is the symbol. So, this is the Nusselt number variation with Richardson number. So, obviously, with in this case you can see increase with Richardson number your for the bottom wall, this Nusselt number is increasing.

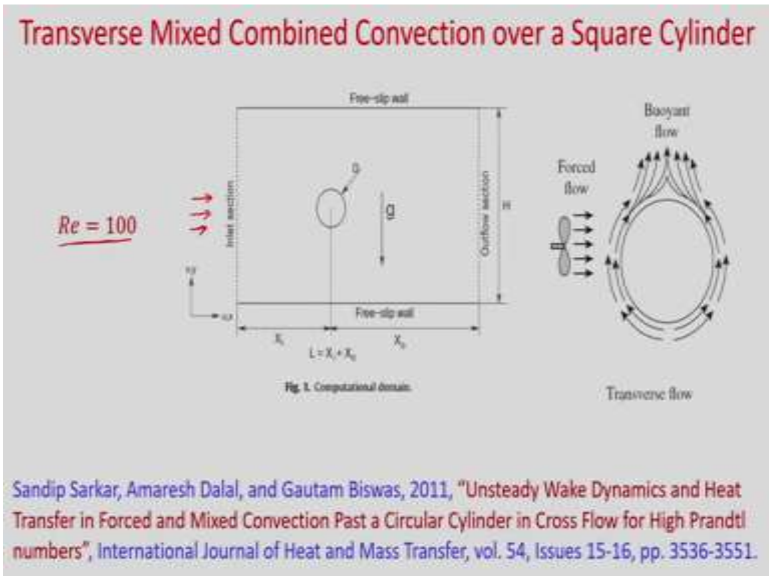
And this is the average Nusselt number for the whole wall considering these bottom top left and right. So, this is the Nu_C. So, this is the combined for all the combining all the surfaces. So, this is the average Nusselt number. And with increase of Richardson number, it is increasing.

And this is the local Nusselt number along these ABCD, half of the domain it is shown because it is the time average Nusselt number is shown over the surfaces ABCD. And this is the local Nusselt number and we can see for Ri = 1, this is the plot solid line.

So, how the variation is happening? So, at A point, so you have a Nusselt number. And it is at this corner point, as you have clustering of the temperature, you will get a high Nusselt number, then again it will decrease along the C. And at C there will be another peak, because you have a corner and again it will decrease.

And it is from for different Richardson number 0 to 1, it is also shown. So, you have seen for either assisting or resisting buoyancy flow the flow over a square cylinder case.

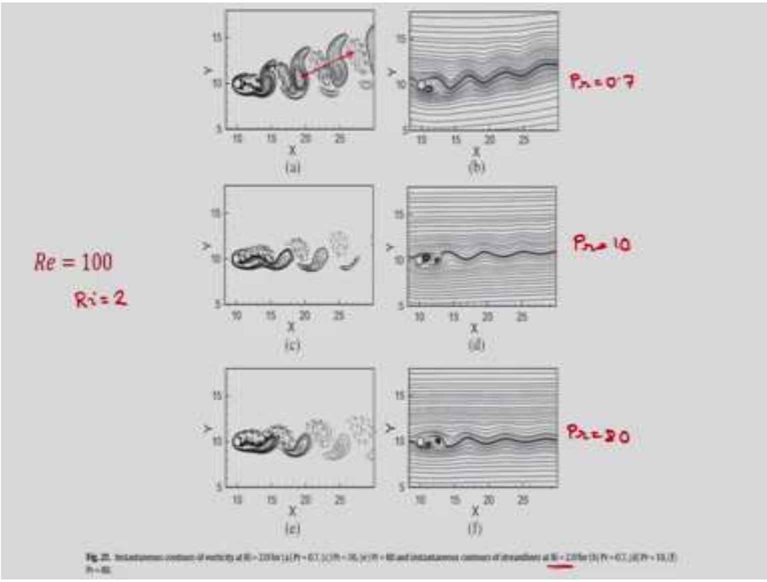
(Refer Slide Time: 56:16)



Now, if you have a transverse flow , so we are considering a transverse flow over a circular cylinder. So, this is taken from Sandip Sarkar, Amaresh Dalal and Gautam Biswas, Unsteady Wake Dynamics and Heat Transfer in a Forced and Mixed Convection Past a Circular Cylinder in Cross Flow for High Prandtl number.

So, this is a cross flow is happening. So, flow is happening in this direction. And you can see depending on the thermal boundary conditions, your buoyancy flow will act in perpendicular to this forced flow direction. And for Reynolds number 100, it is calculated. So, this is kind of transverse flow, or cross flow.

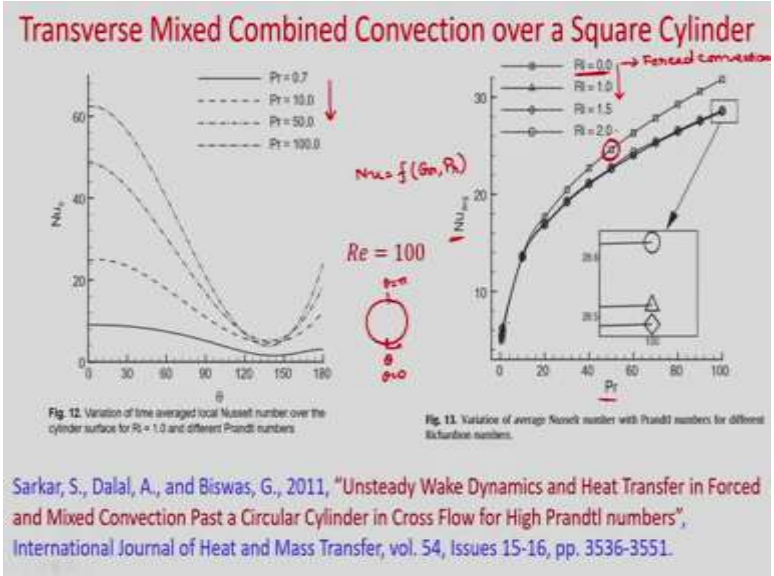
(Refer Slide Time: 56:44)



Now, you can see instantaneous contours of verticity, this left side figures; and in right side figure instantaneous contours of streamlines at Ri = 2. So, this is Ri=2, Reynolds number 100, and for different Prandtl number . So, Prandtl number is 0.7, this is your Pr=10, and this is for Pr = 80. So, you can see the effect of buoyancy .

So, when this vertices are shaded behind the cylinder , it is not going exactly in the horizontal direction, it is going in some inclined direction. You can see in this direction it is going. And these are the streamlines from streamlines also you can see similar thing. So, this is the transverse flow and obviously, with increase of Prandtl number, you can see the effect. So, this inclination is decreasing. And for higher Prandtl number, you can see it is going almost in the horizontal direction.

(Refer Slide Time: 57:51)



Now, if you see the Nusselt number, so this is the variation of time average local Nusselt number over the cylinder surface. So, you have a cylinder surface. And this from here this is your θ, and θ is measured from here to this. So, θ = 0 here, and θ =π here .

So, obviously, you can see for this particular case there is no corner. So, you will get maximum Nusselt number at this point. This is so you can see at θ = 0 you are getting maximum Nusselt number at any Prandtl number and it is decreasing. And obviously, again near to this π, it is increasing due to the effect of this buoyancy.

Now, you can see as you are increasing the Prandtl number, as you are increasing the Prandtl number, you can see your Nusselt number also increasing. So, you can see that Nusselt number is function of Grashof number and Prandtl number. And obviously, Prandtl number to the power something. So, as Prandtl number increases, Nusselt number also increases. So, you can see the average Nusselt number with Prandtl number for different Richardson number.

So, you can see; obviously, with increase of Prandtl number, your average Nusselt number is increasing. And at a particular Prandtl number, you can see at Richardson number 0. What is what does it mean? Richardson number 0, Richardson number 0 means your buoyancy effect is 0, that means, it is a forced convection. So, Richardson number 0 means it is a forced convection; natural convection is absent.

Here at a particular panel number if you see, you are getting maximum Nusselt number at a particular panel number you can see that you are getting a maximum Nusselt number at Richardson number 0. As you are increasing the Richardson number in this case of transverse flow, your Nusselt number is decreasing.

So, you can see that you also have a reduction in the heat transfer with increase of Richardson number, with increase of Richardson number, although your buoyancy effect is increasing with increase of Richardson number, but your Nusselt number is decreasing. So, in some flow, you will get reduction in some flow also we will get increase in the heat transfer.

So, in today's class, we first wrote some empirical relations for vertical flat plate cases and also inclined plates. If you replace the g with the g cos θ for the expression of vertical flat plate cases Nusselt number, you will can use those in the range of $60^0 \geq \theta \geq -60^0$.

Also we have discuss about correlations for flow over horizontal cylinder, flow over sphere. Then we introduced with the mixed convection. In mixed convection, we discussed about the assisting flow, opposing flow and the transverse flow.

Then we have written the governing equations in non-dimensional form for the case of mixed convection. And two different cases mixed convection flow over a square cylinder case we have discussed about the flow physics as well as the Nusselt number.

And later case also we have shown one example of transverse flow where you have mixed convection, but forced flow and the buoyant flow are perpendicular in direction. And for transverse flow over a circular cylinder, we have discussed about the flow physics as well as the Nusselt number.

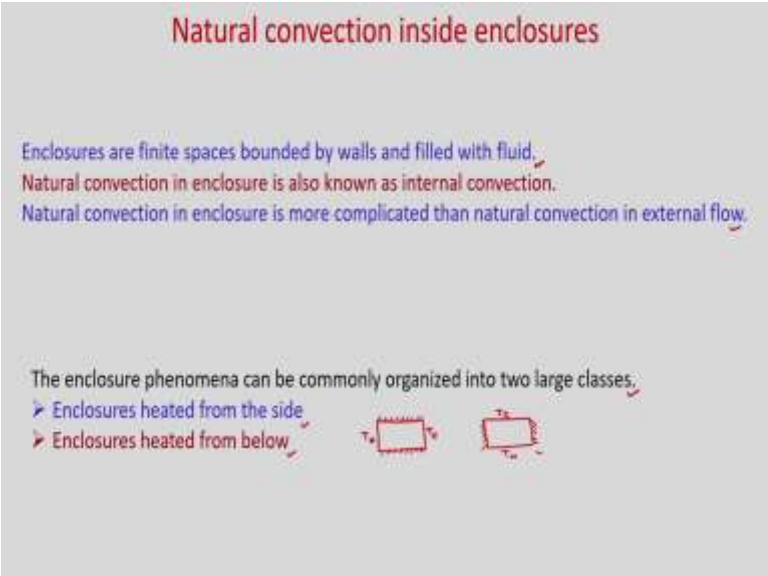
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 09
Natural Convection - II
Lecture – 31
Natural convection inside enclosures

Hello everyone, so in today's class, we will study Natural convection in enclosures. Natural convection in enclosure is classical problem and it is also known as internal natural convection.

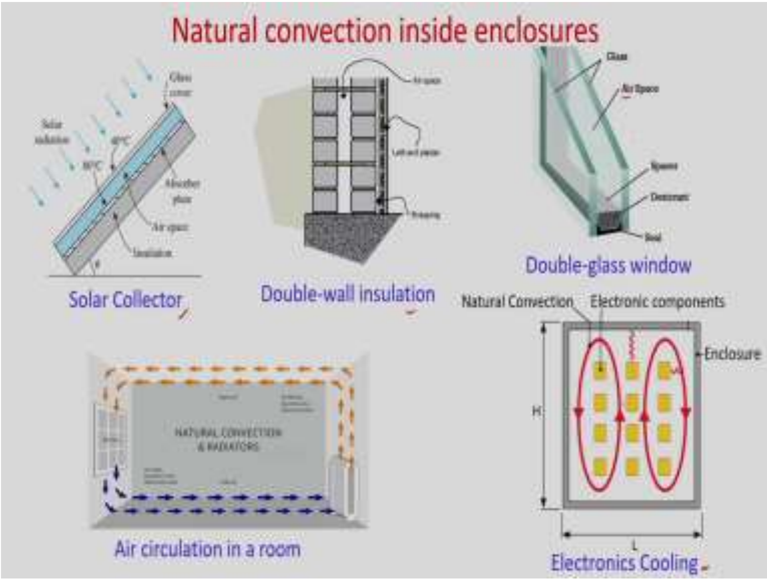
(Refer Slide Time: 00:48)



So, you can see what is enclosures? Enclosures are finite spaces bounded by walls and filled with fluid. And natural convection in enclosure is more complicated than natural convection in external flows. Because you know that natural convection in external flows this boundary layer will grow, and it will continue to grow, and it will go to turbulent flow regime. But, when we consider internal natural convection, then its growth will be limited due to the height or length of the enclosure.

The enclosure phenomena can be commonly organized into two large classes; enclosures heated from the side and enclosures heated from the below. So, you can see if you have enclosure and top and bottom walls are adiabatic and sidewalls are heated say let us say it is maintained at hot temperature and it is your cold temperature, then these are you can see heated from the side. So, this enclosure is heated from the side. Another problem you can have, where enclosures heated from below. So, if you have the bottom wall is maintained at higher temperature than the top wall and side walls may be insulated or maybe it may be insulated, then this is another class of problem.

(Refer Slide Time: 02:40)

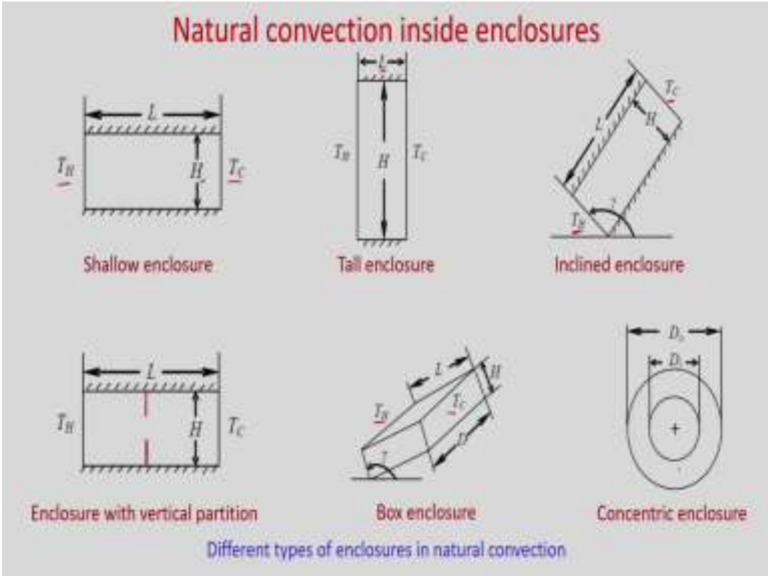


There are many applications of natural convection in enclosures; in solar collectors, then double wall insulator, also we can have application in some type of heat exchangers. So, you can see here. So, application of natural convection inside enclosure this is your solar collector, you have double wall insulation so, these are the bricks and inside you have air space, you know air is a common insulator. So, air is placed inside and double glass window so, you can see here.

So, these are double glass window inside air is placed, heat loss from the inside to outside can be prevented using these insulated air keeping in between the glasses. Also, we have air circulation in a room. So, you can see you have a radiator and obviously, from radiator hot air will go up, it will travel horizontally, then it will come to window, where it will be cooled and it will again travel like this.

So, these you can see that air circulation in a room it is also application of this natural convection of enclosure. Also, in electronics cooling, you will get application. So, these chips will be mounted and if it is kept horizontally, then there will be natural circulation inside the enclosure.

(Refer Slide Time: 04:04)

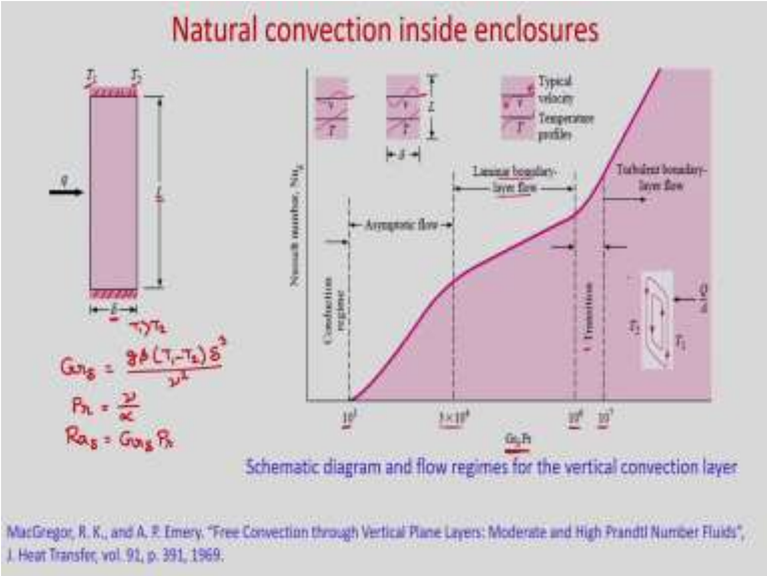


So, when we were talking about enclosures and with two different thermal boundary conditions so, you can see here we have shown different types of enclosures and different orientation. So, you can see this is known as shallow enclosure, where length is very large compared to the distance between these two plates, then this enclosure is known as shallow enclosure, and it is sidewalls are heated; sidewalls are heated and top and bottom walls are adiabatic.

Here, top and bottom walls are adiabatic, but height is much much greater than length. So, you can see it is known as a tall enclosure and it is the case where sidewalls are heated. Now, this shallow enclosure if you tilt it, then you will get inclined enclosure and obviously, you can see these are differentially heated. Enclosure with vertical partition. So, you can have these partitions you can see, this is the partition.

So, this is known as enclosure with vertical partition. In three-dimension if you consider this box, then this is known as box enclosure and you can see the side walls are heated T_H and T_C and other four walls are adiabatic. You can have concentric enclosure if two cylinders are there concentric cylinders or spheres so, inside you can have this natural convection so, which is your internal natural convection.

(Refer Slide Time: 05:48)



So, in this figure, you can see the different regimes of natural convection in enclosures. So, here you can see, here this is the enclosure, where δ is the length and L is the height and differentially heated. So, top and bottom walls are adiabatic, there is no heat transfer across these walls and left wall is maintained at constant temperature T_1 and right wall is maintained at temperature T_2 and heat transfer will take place and $Ra_\delta = Gr_\delta \Pr$.

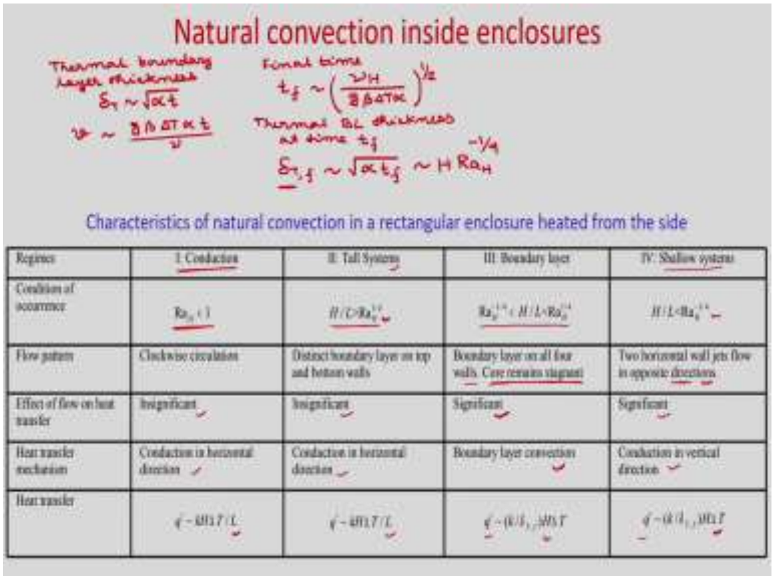
So, $Gr_\delta = \dfrac{g\beta\left(T_1 - T_2\right)\delta^3}{\nu^2}$, if $T_1 > T_2$, then $\dfrac{g\beta \Delta T \delta^3}{\nu^2}$. So, this is your Grashof number and $\Pr = \dfrac{\nu}{\alpha}$ and we know that $Ra_\delta = Gr_\delta \Pr$. So, in this particular case x axis is your Rayleigh number and y axis is a Nusselt number. So, you can see how the Nusselt number varies with Rayleigh number in an enclosure.

If Ra < 10^3, then it is almost conduction dominated. So, temperature will vary linearly and there will be less convection cell. So, velocity will be very very less, and heat transfer is dominated by conduction mode of heat transfer only. So, we can see velocity is very small and temperature is a linearly varying from T_1 to T_2.

But, if you go above this Rayleigh number, then Rayleigh number 10^3 to 3 x10^4 regime, then it is kind of asymptotic flow. Here also temperature you can see in the central region, it is almost linear, but near to the wall there is variation and velocity is comparatively a low. Now, in the range of Rayleigh number 3 x10^4 and 10^6, you will get laminar boundary layer flow, and you can see gradually there will be increase in the Nusselt number, and you can see there will be a temperature profiles like this.

So, here you can see in central region, you have almost constant temperature, but maximum temperature variation you will get near to the wall, and similarly in the velocity. So, near to the central region, you will get very low velocity, but near to the wall, you will get very high velocity jet. Here, you can see this is the velocity jet. So, this is a laminar boundary layer flow. But, if you go in between $10^6$and 10^7, it will be transition and then after Rayleigh number 10^7, you will get turbulent boundary layer flow.

(Refer Slide Time: 09:23)



So, we will not go into details in scale analysis derivation, but we will show; what is the scale of thermal boundary layer and the velocity as well as the heat flux. For details you can refer the "Heat Transfer" book by Adrian Bejan. Here, we will just show the scale of different parameters. So, you can see your thermal boundary layer thickness will be, $\delta_T \sim \sqrt{\alpha t}$, where t is the time and alpha is the thermal diffusivity. So, the velocity, $v \sim \dfrac{g\beta\Delta T\alpha t}{\nu}$.

Now, if you can see that in some cases, this natural convection will be steady state, but when it is starting the inside the enclosure, your fluid velocity will be 0, and temperature will be maintained at a film temperature or some temperature. Now, if you just start the heating of these two walls, then there will be velocity generation due to the temperature difference, and there will be density difference and there will be recirculation and from there, you will get the velocity scale.

Now, in this case, you can see that after some time this will become steady state. There will be a recirculation and temperature control will not vary no more. So, in that case to reach from time 0 to sometime t_f final time to reach steady state that scale we can write as; $t_f \sim \left(\frac{\nu H}{g\beta\Delta T\alpha} \right)^{1/2}$ and at this time, thermal boundary layer; thermal boundary layer thickness at time t_f it will be $\delta_{T,f} \sim \sqrt{\alpha t_f}$ and it will be $\delta_{T,f} \sim \sqrt{\alpha t_f} \sim HRa_H^{-1/4}$.

So, you can see at steady state, this thermal boundary layer thickness no longer increase with time. So, that is your $HRa_H^{-1/4}$. So, this analysis is given in detail in the "Heat Transfer" book by Bejan. So, you can see here in this table. So, there are four different regimes you will get. So, in I regime we will get conduction; where obviously, Ra < 1 and there will be clockwise circulation, but velocity will be very low and effect of flow on heat transfer will be insignificant and heat transfer mechanism is conduction in horizontal direction.

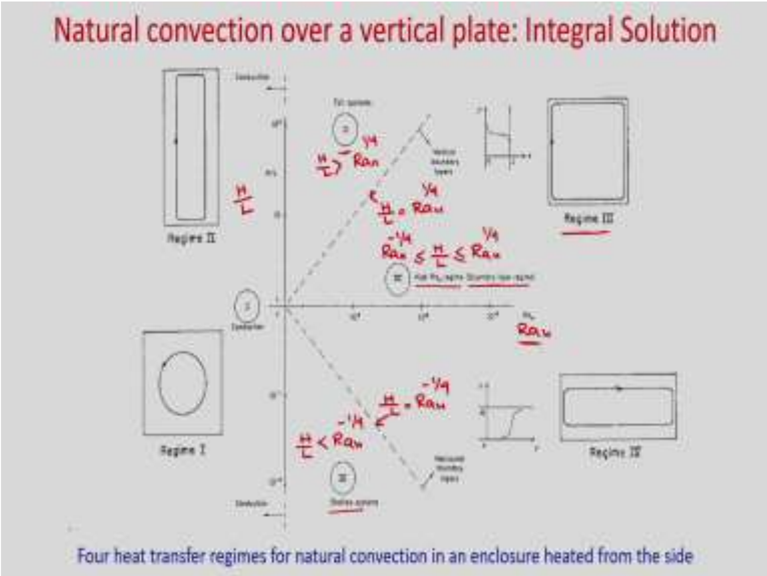
At, if you consider the heat transfer, then from the scale analysis you can show that $q' \sim kH\Delta T/L$. Now, another regime we will get tall system, where $H/L > Ra_H^{1/4}$ and flow pattern will be distinct boundary layer on top and bottom walls.

So, you will find distinct boundary layers on top and bottom walls, but effect of flow on heat transfer will be insignificant and conduction in horizontal direction and $q' \sim kH\Delta T/L$. So, this is the heat transfer rate per unit width.

Regime III you will get a boundary layer where $Ra_H^{-1/4} < H/L < Ra_H^{1/4}$. So, in this regime, you will get boundary layer on all four walls and core remains stagnant. Core remains stagnant and all four walls you will get. So, effect of flow on heat transfer it will be significant and you will get boundary layer convection and the heat transfer rate will be, $q' \sim \left(k/\delta_{T,f} \right) H\Delta T$.

Now, the IVth regime you will get the shallow system, where $H/L < Ra_H^{-1/4}$. So, two horizontal wall jets flow in opposite direction. So, you will get that type of flow and effect of flow on heat transfer is significant and conduction in vertical direction you will get and the heat transfer rate you will get $q' \sim \left(k/\delta_{T,f} \right) H\Delta T$.

(Refer Slide Time: 15:31)



So, schematically it is shown in the book of Adrian Bejan. You can see these are the four-regime shown. So, in this figure, you can see in the x axis, you have Rayleigh number and in y axis you have H/L. So, this is your y axis this is H/L and this is the x axis Rayleigh number and different regimes are shown here and you can see two dotted lines which you have vertical boundary layers and this is your $H/L = Ra_H^{1/4}$ and this is your horizontal boundary layer you will get and this is the line where $H/L = Ra_H^{-1/4}$.

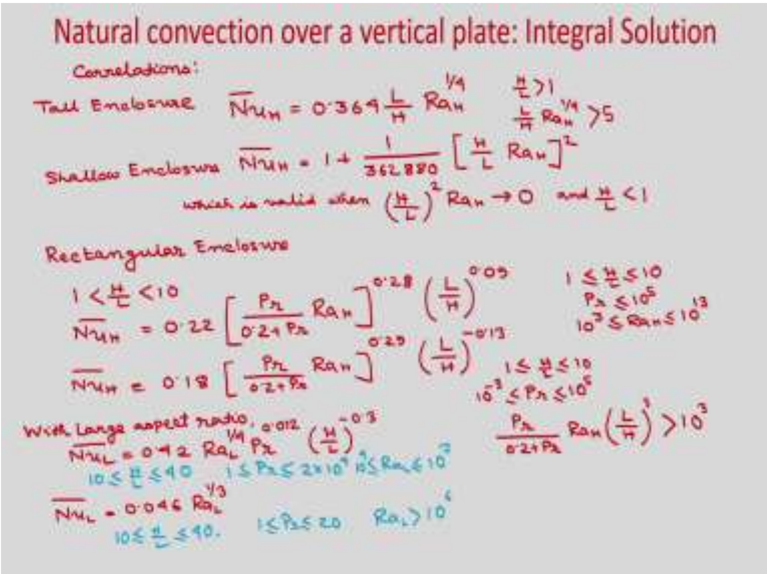
So, you can see you will get at very low Rayleigh number, $Ra_H < 1$ you will get the conduction regime and here, you can see a velocity will be very low and conduction dominated heat transfer will get and temperature will vary from left to right wall linearly. And, in IInd regimes, you will get in this regime. So, if you see in the table where $H/L > Ra_H^{1/4}$. So, in this regime, you will get that tall system so, this is the tall system.

So, in this particular case, you will see that $H/L > Ra_H^{1/4}$ and we discussed here that distinct boundary layers on top and bottom wall you will get. Then, in the IVth regime which is your shallow system. So, you can see this is your shallow system and here H/L you will get; $H/L < Ra_H^{-1/4}$ so, this is your shallow system. In shallow system, you can see two horizontal wall jets flow in opposite direction you will get and conduction in vertical direction you will get.

And in this regime, this is the high Rayleigh number regime, which is known as boundary layer regime. So, in a horizontal walls and vertical walls in both directions you will get boundary layer. So, here boundary layer heat transfer will take place and it is in the regime of $Ra_H^{-1/4} \leq H/L \leq Ra_H^{1/4}$.

So, you can see this regime is your boundary layer regime and $Ra_H^{-1/4} \leq H/L \leq Ra_H^{1/4}$. So, you can see there are distinct four regimes based on the length and height ratio and in this regime, in the IIIrd regime which is your boundary layer regime, here you will get a significant heat transfer, because you will have the boundary layer heat transfer. So, you will get a boundary layer convection.

(Refer Slide Time: 19:31)



So, based on some empirical relations for different type of configuration, let us write the expression for a Nusselt number. So, these are the correlations. So, for tall enclosure, the

Nusselt number you will get $\overline{Nu}_H = 0.364\frac{L}{H}Ra_H^{1/4}$, and it is valid for $H/L > 1$ and $\frac{L}{H}Ra_H^{1/4} > 5$.

Then, for shallow enclosure, the Nusselt number expression you will get as $\overline{Nu}_H = 1 + \frac{1}{362880}\left[\frac{H}{L}Ra_H\right]^2$ and this is valid; which is valid when $\left(\frac{H}{L}\right)^2 Ra_H \to 0$, and $H/L < 1$.

Now, let us consider rectangular enclosure, where we can have $1<\frac{H}{L}<10$, you can see Nusselt number you can write as, $\overline{Nu}_H = 0.22\left[\frac{\Pr}{0.2+\Pr}Ra_H\right]^{0.28}\left(\frac{L}{H}\right)^{0.09}$. And, it is valid for $1\leq \frac{H}{L} \leq 10$, $\Pr \leq 10^5$ and $10^3 \leq Ra_H \leq 10^{13}$.

Another Nusselt number correlation you will get, $\overline{Nu}_H = 0.18\left[\frac{\Pr}{0.2+\Pr}Ra_H\right]^{0.29}\left(\frac{L}{H}\right)^{-0.13}$, and this is valid in the range of $1\leq \frac{H}{L} \leq 10$, $10^{-3} \leq \Pr \leq 10^5$ and $\frac{\Pr}{0.2+\Pr}Ra_H\left(\frac{L}{H}\right)^3 > 10^3$. So, it will give regime with good results.

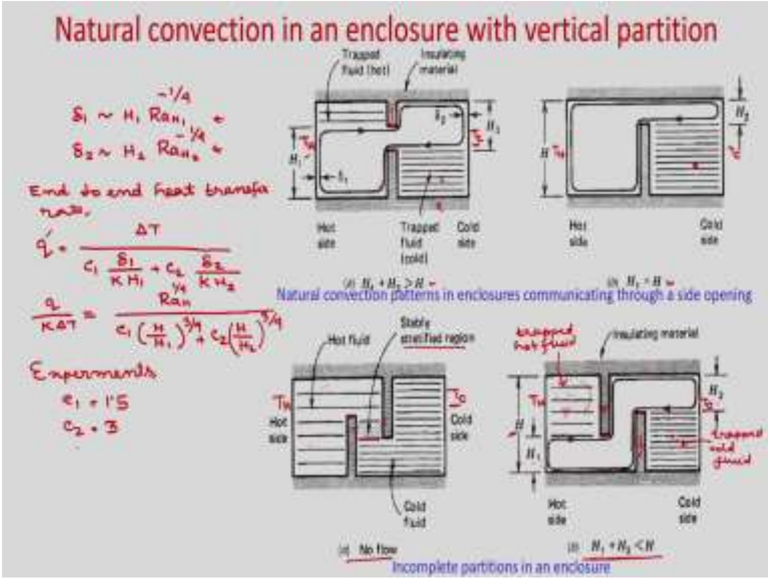
And, with large aspect ratio, Nusselt number these are all average Nusselt number and this is you can see it is based on the length L. Earlier, all these expression we have written based on the height, but here we are writing based on the length.

So, $\overline{Nu}_L = 0.42Ra_L^{1/4}\Pr^{0.012}\left(\frac{H}{L}\right)^{-0.3}$ and this is valid for $10 \leq \frac{H}{L} \leq 40$, $1 \leq \Pr \leq 2\times 10^4$ and $10^4 \leq Ra_L \leq 10^7$.

And, another expression you will get for Nusselt number for large aspect ratio so, here $\overline{Nu}_L = 0.046Ra_L^{1/3}$ and it is valid in the range of $10 \leq \frac{H}{L} \leq 40$, $1 \leq \Pr \leq 20$ and $Ra_L > 10^6$.

So, in this cases, you have seen that you have either shallow enclosure or tall enclosure or rectangular enclosure.

(Refer Slide Time: 25:35)



Now, you can have partition in this enclosure. So, you can see here how the natural convection takes place, in an enclosure with vertical partition. So, here you can see this is differentially heated, top and bottom walls are insulated and you can see here left side is hot wall T_H; left wall is hot wall T_H and right side is cold wall. So, what will happen? If it is hot wall, then obviously, the fluid will go up and it will try to move in this direction and when it till come into contact with the wall whose temperature is T_C, so this fluid will become heavier, density will be larger and it will come down.

But you can see as there are partitions, so in here, there will be trapped fluid which will be cold because it is a cold side so, these are a heavier fluid and it will sit just on the bottom, because you have this is adiabatic wall and it is cold wall. So, here cold fluid will be trapped and this heavier fluid will go this way and it will come here and again it will come into contact with the hot wall and again it will go up.

But here, you can see this is also trapped hot fluid, because its density is lighter because of this partition, it will be trapped here and there will be a hot fluid. So, you can see for this partition where this is your height here it is H_1 and this side from the top is H_2 and for this case, you will get a trapped fluid in here which is hot and here trapped fluid you will get which is a cold, and there will be circulation of the fluid in this region and δ_1 is the boundary layer thickness on the hot wall and here boundary layer thickness δ_2 in cold wall. Here, $H_1 + H_2 > H$, H is the distance between these two walls.

Now, another configuration you can see where H_1 is H; that means, there is no partition from the top. So, here you will get and this is your H_2 and in this particular case; obviously, there will be no trap fluid, because it will go straight away like this and it will come into contact with the cold fluid and it will be heavier and cold fluid will be trapped here, here cold fluid will be trapped and it will go come this way, it will go down as it is heavier it will come down and again when it will come into contact with the hot wall, then it will go up.

So, you can see how the natural convection is taking place inside the enclosure. So, in these two cases, you can see natural convection patterns in enclosure communicating through a side opening.

Now, you see in these two cases. In these case, now partitions are not in line. So, they are staggered. So, here this is the partition, here it is this is the partition, and this is your hot side, and this is your cold side. So, what will happen? You can see the cold side, there will be heavier fluid due to the cold wall so, here cold fluid will be trapped in this region, cold fluid will be trapped.

In the hot side, you can see the although initially the fluid will go up, but it cannot go this way, because this is hot fluid so, it will be trapped from the top side and gradually it will be stratified , and you can see hot fluid will trapped in the left side and here, you will get stably stratified region. So, here there is no flow is occurring due to the orientation of these partitions.

Now, if you change the orientation of this partition. So, now, you have brought these this side and now this you have brought in the right side. So, you can see this is the partition, and this is the partition, from the top this is the partition attached with the bottom, this temperature is T_H and right wall temperature is T_c. So, in this particular case,$H_1 + H_2 < H$. So, this is the H_1 the distance from the bottom to this and this is from top to this it is H_2, then $H_1 + H_2 < H$.

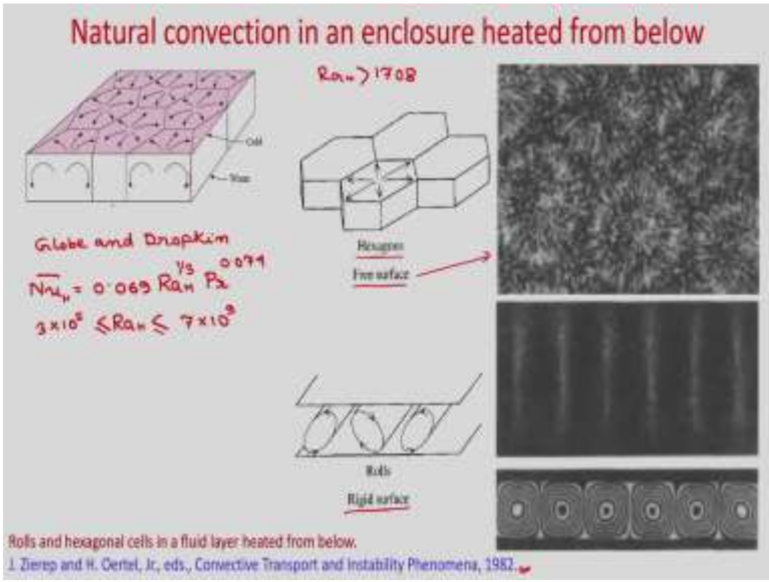
In this particular case, top and bottom walls are insulated, then the fluid which is in touch with the hot side; obviously, this hot fluid will be trapped here. So, it will be a stratified fluid and here also, the cold fluid will be trapped. So, this is cold fluid, trapped cold fluid and this is your trapped hot fluid.

And, natural convection now will take place so, it will go up and it will travel horizontally again as it is density is low, it will go up, then again it will travel horizontally and as it will come into contact with the cold wall so, it density will be larger. So, heavier fluid will come down and as trapped cold fluid is here so, it will flow horizontally, again it will come down, as it is heavier it will flow like this, again it will go up. So, this way your flow will take place.

So, you see the orientation of partition how it works. In this case, there is no flow , but if you change the orientation, then you will get the natural convection flow. So, in this particular case if you see so, if $\delta_1 \sim H_1 Ra_{H_1}^{-1/4}$ and $\delta_2 \sim H_2 Ra_{H_2}^{-1/4}$ and end to end heat transfer rate will get here; end to end heat transfer rate you will get as, $q' = \dfrac{\Delta T}{C_1 \dfrac{\delta_1}{KH_1} + C_2 \dfrac{\delta_2}{KH_2}}$.

So, here, if you rearrange, you will get $\dfrac{q'}{K\Delta T} = \dfrac{Ra_H^{1/4}}{C_1\left(\dfrac{H}{H_1}\right)^{3/4} + C_2\left(\dfrac{H}{H_2}\right)^{3/4}}$ and from experiments, these constants are found as C_1 =1.5 and C_2 =3.

(Refer Slide Time: 34:29)



Now, let us discuss the enclosure when the bottom wall is heated. So, you can see when the bottom wall is heated, then the fluid obviously, will go up and there will be formation of cells which are known as Benard convection cell. It is very well-known problem and very complicated.

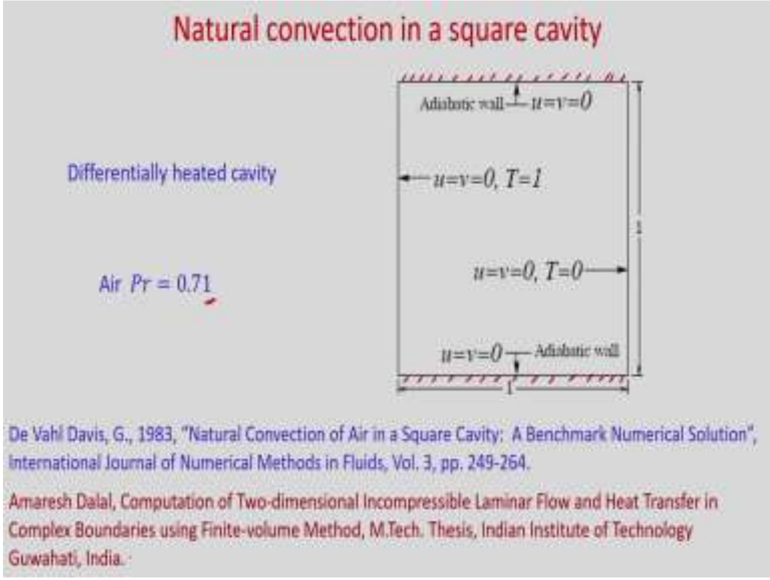
And, you can see in this case so, these are the in three-dimension if you see so, there will be formation of these as cells. So, we will go up, it will go again it will go down so, these were different cells you will get. You can see there will be roles also for the rigid surface and if it is a free surface, then you will get hexagon type cells. So, you can see the experimental visualization.

So, this is kind of free for free surface; So, you can see in this particular case, there will be kind of hexagon kind cells, here you can see and for rigid surface top and bottom walls are rigid, then you can see these are the cells you will get. So, these are experimental visualization and these are the Rayleigh-Benard cells and from the top if you see so, these are the rolls you can see; these are the roles. So, it is continuous in the other direction. So, this is from the top view and this is from the side view and you can see the rolls.

So, obviously, you know that this you will get when the Ra_H >1708 and in this case, Globe and Dropkin proposed this empirical relation of Nusselt number.

So, $\overline{Nu}_H = 0.069 Ra_H^{1/3} \Pr^{0.074}$ and it is valid in the range of $3\times10^5 \le Ra_H \le 7\times10^9$. So, this $\frac{H}{L}$ should be sufficiently large, so that the effect of the side walls can be negligible, because there will be side walls so, that effect you can neglect. So, these are some correlations, based on the experiments, we have written the Nusselt number expression.

(Refer Slide Time: 36:48)

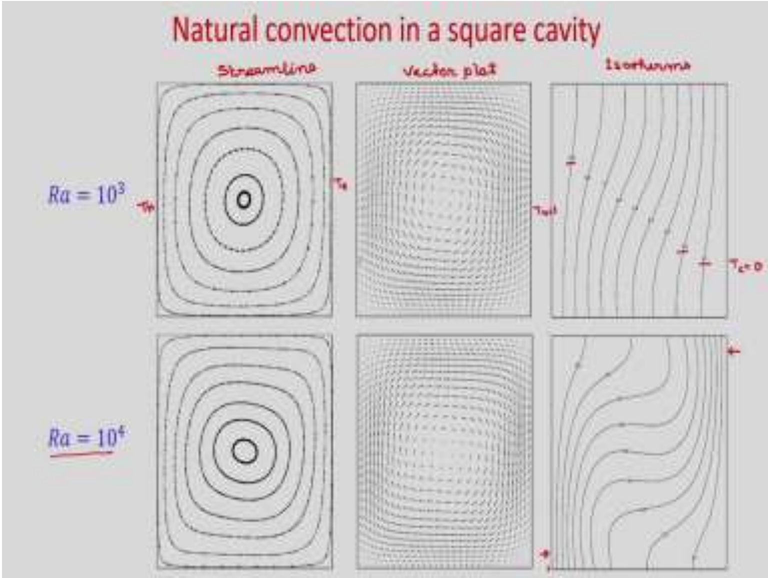


Now, we will show some visualization from the numerical simulations for different cases. One well-known problem is the differential heated cavity. So, natural convection taking place in a square cavity and these are sidewalls are heated. So, this is known as differential heated cavity. So, De Vahl Davis is having a well-known paper on this problem where he considered air where Pr = 0.71. So, this is the well-known paper.

De Vahl Davis natural convection of air in a square cavity; a benchmark numerical solution. So, what is this problem? Top and bottom walls are adiabatic; top and bottom walls are adiabatic and in non-dimensionally these are solved. So, you can see this is a square cavity so, length and height is 1, velocity is obviously, you have 0 in all walls, because it is cavity and these are differentially heated so, $T_H = 1$ and $T_C = 0$. So, 0 and 1. So, numerically, it is solved and now I will show some results from my M.Tech thesis.

So, you can see that for low Rayleigh number range in the range of Ra $=10^3$ or less than that, you will get conduction dominated flow. So, fluid velocity will be very low and heat transfer will take place mainly by conduction, and if you plot the temperature variation, you will get almost linear. But if it is higher Rayleigh number, then obviously, convection cell will be formed and heat transfer you will get by boundary layer convection.

(Refer Slide Time: 39:04)

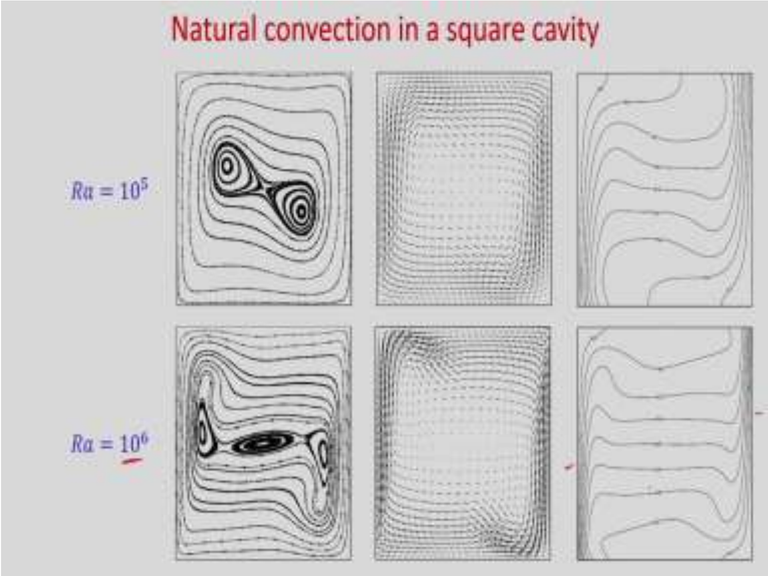


So, you can see Ra = 10^3, fluid velocity is are just having this like this and it is very low velocity and this is the streamline plot at Ra = 10^3 and this is your vector plot. So, velocity vector you can see how it is rolling. So, as you have left wall is T_H and right wall is T_C, you can see these fluid is going up and it is moving horizontally, then again it is coming down, because these fluids are getting heavier, again it is flowing horizontally and going up. So, this way you can see the convection is taking place.

And, the temperature profile you can see how it is varying from $T_C = 0$ to $T_H = 1$. So, you can see it is 0.1, 0.2, 0.3 and point this is your 0.9. So, you can see these are some kinky is there, but almost it is linear. But as you increase Rayleigh number, then fluid velocity increases. So, and you can see the isotherms, these are isotherms that means, temperature control. Isotherms means temperature control.

So, you can see how the temperature control looks like. So, there will be clustering of the isotherms near right top wall and left bottom wall and if you see the Nusselt number variation, if you see the local Nusselt number variation along this wall on the hot wall from bottom to top will be similar to the local Nusselt numbers distribution in the right wall from top to bottom, because you can see it is symmetric profile. So, you will get similar local Nusselt number variation.

(Refer Slide Time: 41:09)

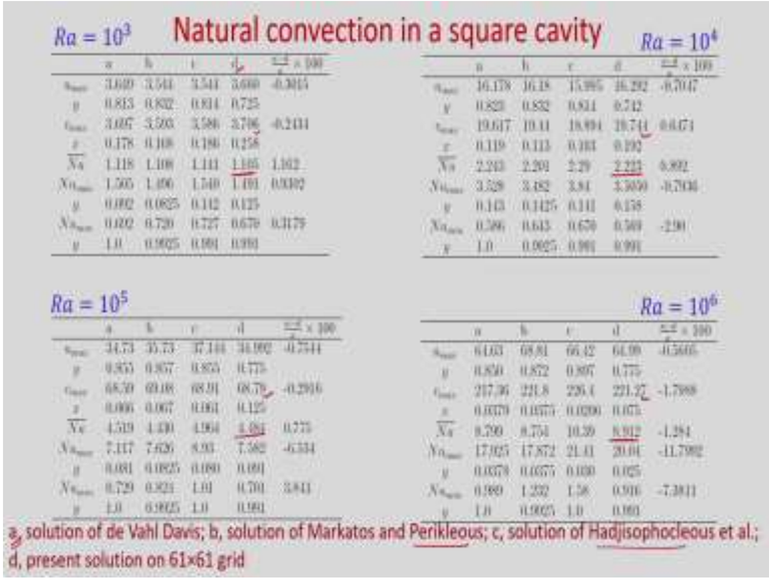


Now, if you further increase the Rayleigh number, then Ra = 10^5 you can see there is some cells formed near to the central region and it is very low velocity you can see and this is magnitude of the velocity vector so, it is very low velocity, but some circulations are there and near to the wall, you can see now velocity jet is there, you can see higher velocity is coming near to the wall.

And, you will can see the isotherms so, the isotherms are more clustered near to the top right wall and left bottom wall. And, if you further increase a Ra = 10^6, you can see from two cells now you got three cells in central region and these are two in near to the side walls. So, you can see it is almost near to the central region, it is stagnant fluid and more velocities yet you are getting.

You see velocity, vector, magnitude is increasing and more velocity you are getting and how the flow physics is happening you can see and here also, you can see clustering of the isotherms near to the cold wall and hot wall. And, in the central region, you can see it almost remain same, the temperature almost remains same in the central region you can see here, it is almost horizontal. So, near to the central region, you can see the temperature this is the constant temperature region it is 0.5, 0.6, 0.7 so, you can see vertically it is almost linear in the central region.

(Refer Slide Time: 42:48)

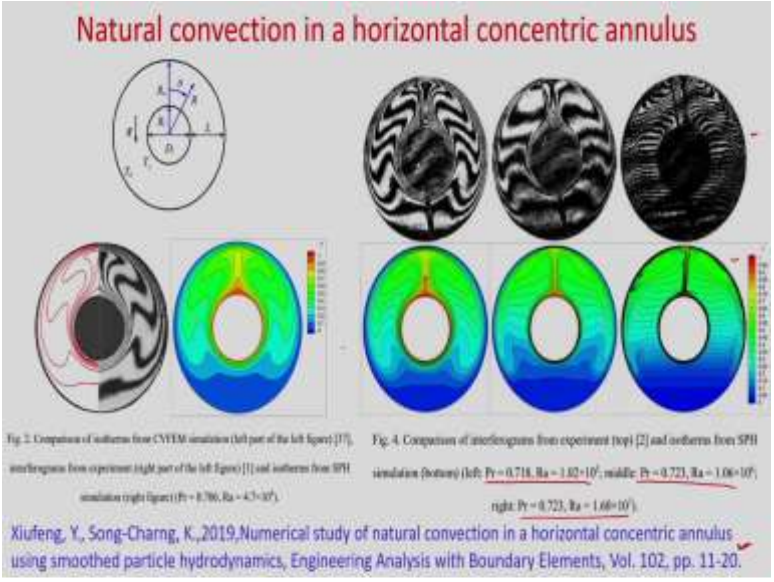


Now, if you see the average Nusselt number it will be same in cold wall and hot wall and maximum velocity and in these tables, the results are compared with the solution of De Vahl Davis and Markatos and Perikleous and Hadjisophocleous. So, a is the solution of the De Vahl Davis.

So, you can see Ra = 10^3, u_{max} and location of u_{max}, v_{max}, location of v_{max}, aboriginal number, maximum Nusselt number, location of maximum Nusselt number, minimum Nusselt number and max a location of minimum Nusselt number these are plotted and d, d is the present solution; that means, from my M.Tech thesis, whatever solution I got this is your d and a, b, c are from literature and you can see the last column is the percentage difference.

So, in this case, you can see obviously, Nusselt number, average Nusselt number is increasing as Rayleigh number is increasing. So, you can see average Nusselt number is increasing as Rayleigh number is increasing and also the velocity. So, if you see v max it is also increasing. So, you can see as Rayleigh number increases, your vertical jet velocity increases and average Nusselt number also increases.

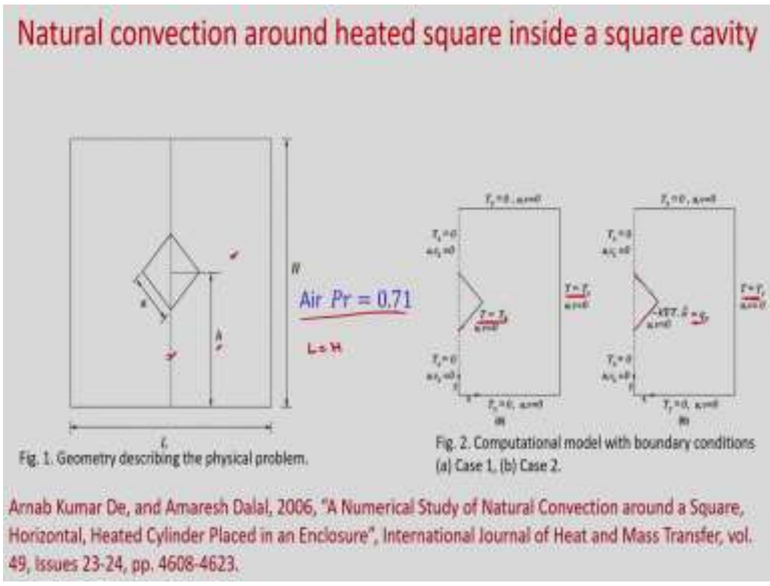
(Refer Slide Time: 44:26)



Now, you see natural convection in a horizontal concentric annulus. So, these are concentric annulus, and in this case, you can see these are the experimental result right-hand side and left side is the numerical result of this paper. So, these are isotherms you can see so, almost it is looking same and you can see these are the isotherms at different Prandtl and Rayleigh number. So, these are experimental results, and this is your numerical results.

So, what will happen? As it is a hot inner cylinder so, the fluid will go up and there will be a band flow in this direction and it will go up and again as you have outer cylinder is cold so, the fluid will come down; so, we will come down. So, there will be a convection like this, and it is symmetric about the vertical centerline up to certain Rayleigh number.

(Refer Slide Time: 45:52)

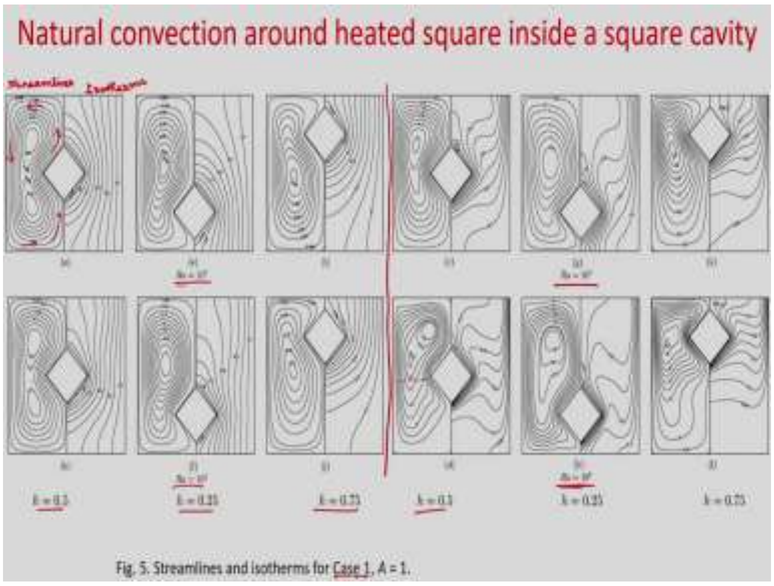


Now, let us consider a hot square cylinder inside and square enclosure. So, here also, air is considered and this results we are showing from this paper Arnab Kumar De and Amaresh Dalal, "A Numerical Study of Natural Convection around a Square, Horizontal, Heated Cylinder Placed in an Enclosure". So, you can see this is one square enclosure so, in this way, it is kept inside the enclosure and inside you have air, ok. So, Pr = 0.71. So, it is a square enclosure.

So, L = H and this H is varied from the bottom. So, this is solved as about the vertical centerline, it is symmetric so, half of the domain is solved with two different boundary conditions, one is this wall is maintained at a constant wall temperature T_H and this wall is maintained at a cold temperature T_C.

And another boundary condition is this hot cylinder is kept with uniform wall heat flux ; uniform wall heat flux and this is cold wall at with constant temperature T_C. So, these are the two cases. So, this is your case 1 with a where the heated square cylinder is maintained at a constant wall temperature and this heated square cylinder is maintained at a constant wall heat flux.

(Refer Slide Time: 47:21)

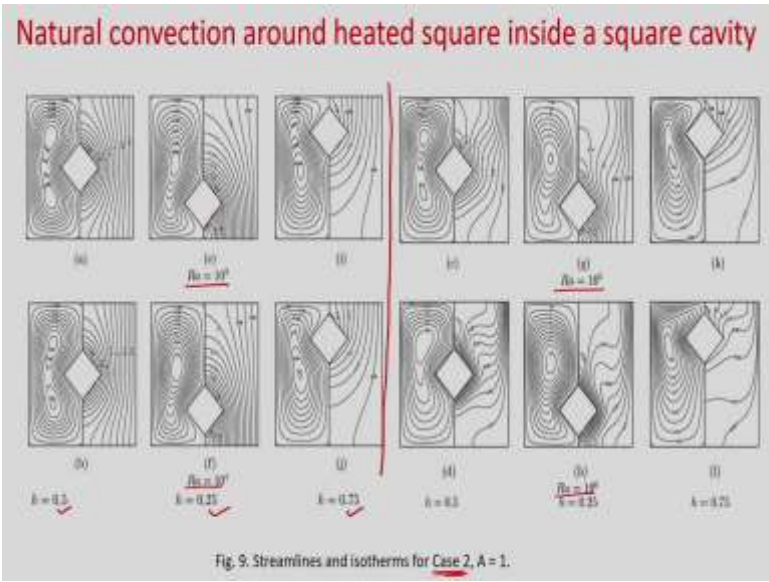


So, now, you can see left-hand side these are streamlines and right side it is isotherms, for a different Rayleigh number, so and with a different height. So, you can see height how it is varied h= 0.5, 0.25,0.75 for these if you see the left half so, for Ra = 10^3 and 10^4 ,it is shown and it is Ra = 10^5 and Ra = 10^6 with different height.

You can see how the convection S L is formed, because left hand side is you can see the streamlines; so, obviously, this is hot wall so, it will go up, then it will flow horizontally, it will come down, again it will flow horizontally and it will go up. So, this way your flow is taking place and these are the isotherms. As Ra = 10^3 so, you can see it is almost conduction dominated, but higher Rayleigh number it is convection dominated.

And now, how the cells are formed you can see and when you have Ra = 10^6 for h = 0.5, you can see you got only the two cells in the upper half ; in the upper half, but here it is you can see one is on the top half, one is in the lower half. So, obviously, with increase of Rayleigh number, you can see that your heat transfer also will increase and these are shown for the case 1.

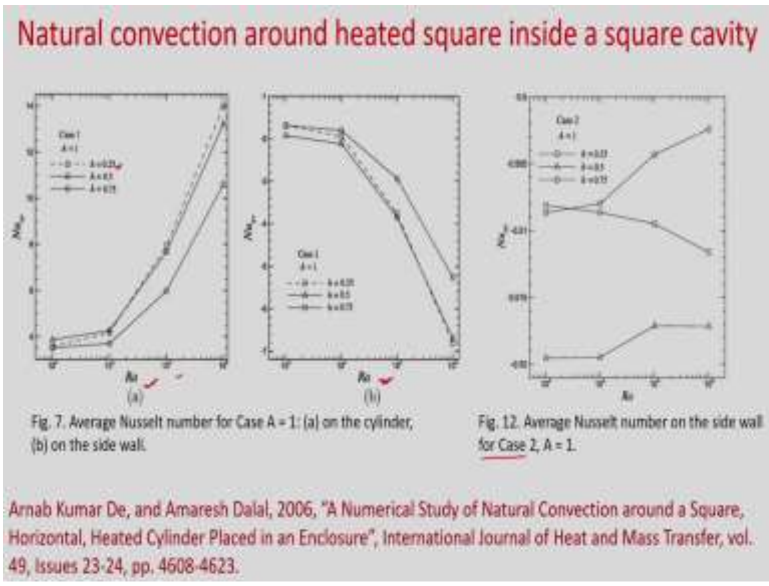
(Refer Slide Time: 49:06)



And this is for case 2, where heated square cylinder is maintained at constant heat flux. So, you can see this is for Ra = 10^3, Ra = 10^4, Ra = 10^5 and Ra = 10^6 for different h.

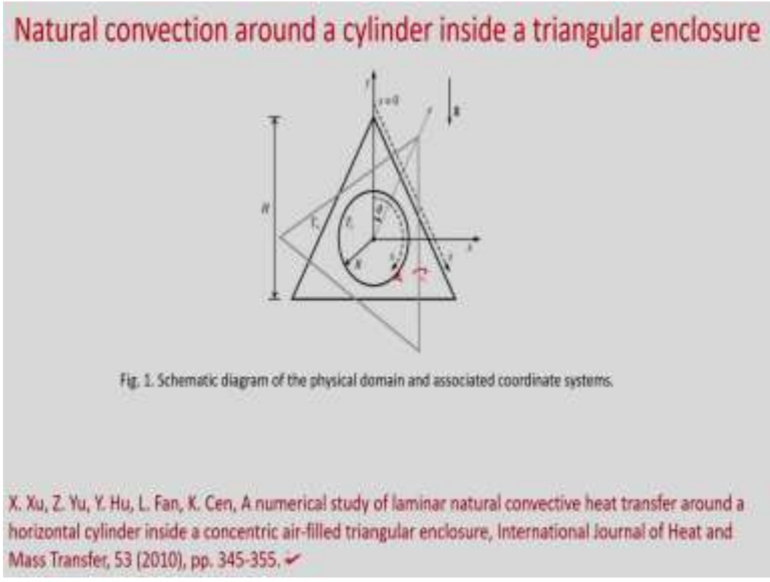
So, here, you can see the temperature ; temperature will vary along the surface, you can see temperature is varying along the surface, because it is maintained at a constant wall temperature as it is maintained as a constant wall heat flux, then obviously, temperature will vary along the surface.

(Refer Slide Time: 49:46)



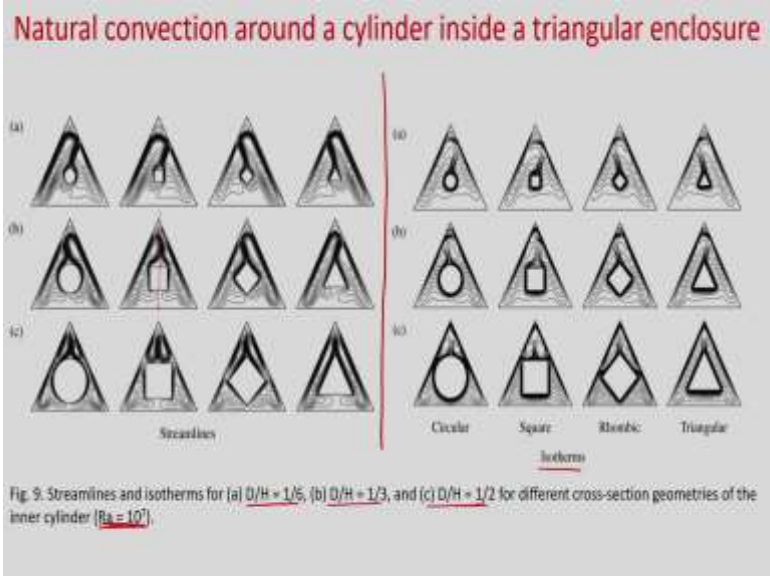
Now, you can see the average Nusselt number for case A this is and on the cylinder and on the side walls, on the cold wall. So, obviously, with increase of Rayleigh number, you can see your heat transfer is increasing and maximum heat transfer you are getting for h=0.25. And in the cold wall, this is the Nusselt number, average Nusselt number variation with Rayleigh number and this is average Nusselt number on the side wall for case 2 how it is varying.

(Refer Slide Time: 50:30)



Another configuration you see where you have a triangular enclosure, and one cylinder is kept inside this triangular enclosure with different shape. So, in this case, you can see this is the triangular enclosure and here, you have inside you have some cylinder of different shapes and inside you have fluid. So, these results are shown from this paper.

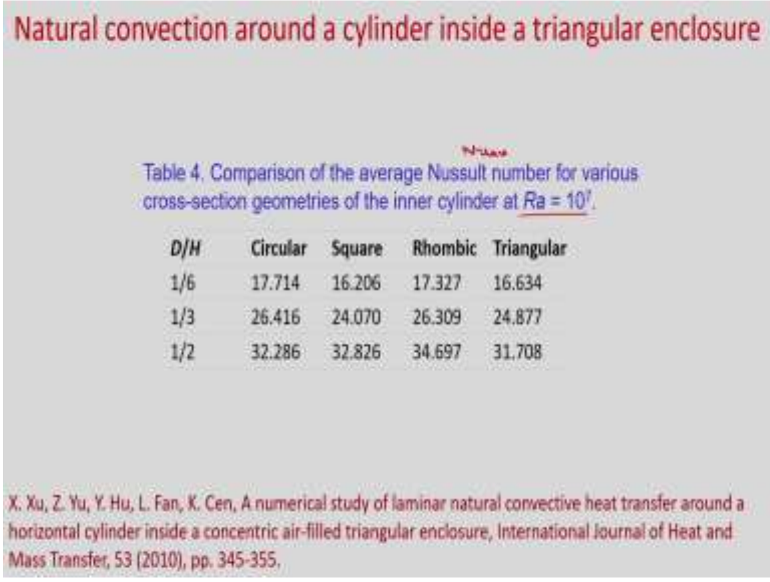
(Refer Slide Time: 50:49)



So, you can see these are left-hand sides, you have streamlines plot and right-side you have isotherms, ok. For different configurations you can see it is circular cylinder, it is square cylinder, it is rhombus, this is your triangular cylinder and D by H ratio are shown, ok. So, (a) is $\frac{D}{H}=\frac{1}{6}$, (b) $\frac{D}{H}=\frac{1}{3}$ and (c) $\frac{D}{H}=\frac{1}{2}$.

So, now you can see that, this is for this case a, b, it is not symmetric about the vertical centerline ; it is not symmetric about the vertical centerline, in these cases also isotherms you can see. So, how the flow physics is happening you can see so, it is going up, it is coming down, but however, it is not symmetric and Ra $=10^7$. So, that is why you can see you have lost the symmetry.

(Refer Slide Time: 51:52)



Natural convection around a cylinder inside a triangular enclosure

Table 4. Comparison of the average Nussult number for various cross-section geometries of the inner cylinder at $Ra = 10^7$.

D/H	Circular	Square	Rhombic	Triangular
1/6	17.714	16.206	17.327	16.634
1/3	26.416	24.070	26.309	24.877
1/2	32.286	32.826	34.697	31.708

X. Xu, Z. Yu, Y. Hu, L. Fan, K. Cen, A numerical study of laminar natural convective heat transfer around a horizontal cylinder inside a concentric air-filled triangular enclosure, International Journal of Heat and Mass Transfer, 53 (2010), pp. 345-355.

So, these are comparison of average Nusselt number for various cross section geometries of the inner cylinder at Ra $=10^7$. So, $\frac{D}{H}=\frac{1}{6}$, $\frac{D}{H}=\frac{1}{3}$ and $\frac{D}{H}=\frac{1}{2}$ and these are circular square rhombic and triangular. So, these are the average Nusselt number.

So, in today's class, we have studied four different regimes for the case of enclosures with side heated walls. So, those four regimes are tall regime, shallow regime, then you have conduction regime as well as you have high Rayleigh number regime. Then, we have written down the empirical relations for these four cases.

Then, we have also discussed about the case, where for the enclosure heated from the below and that is known as Rayleigh Benard convection. After that, we have shown some numerical results of differentially heated cavity and also we have discussed about the variation of Nusselt number. Then, we discussed about some heated cylinder kept inside an enclosure, and for different configurations, we have shown the streamlines and isotherms and also we discussed about the Nusselt number variations.

Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

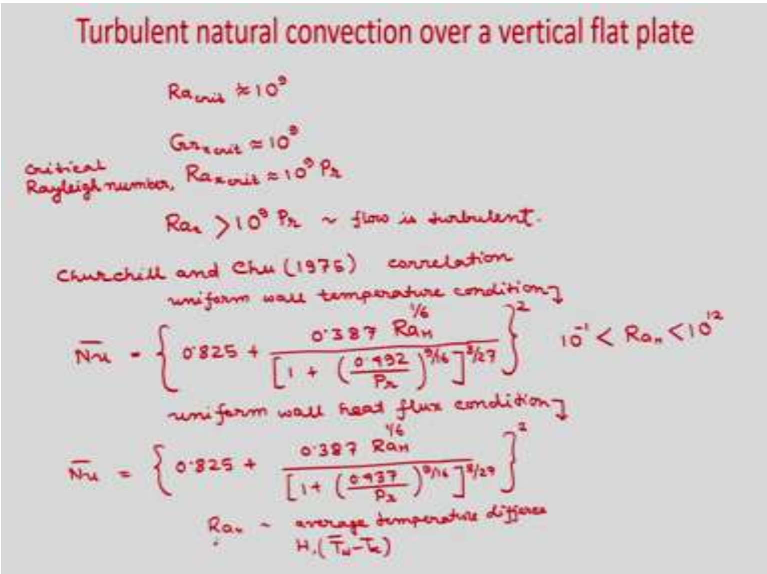
Module – 09
Natural Convection - II
Lecture – 32
Solution of example problems

Hello everyone. So, today we will solve few problems on Natural Convection Flows. Although, we will discuss about turbulent flows in post convection in detail in week 11, but today we will just show few correlations for turbulent natural convection flows.

So, generally if Ra > 10^9, then natural convection flows is to be said as turbulent flows. Whatever heat transfer relations we have derived in last 3 lectures, so those are valid for laminar flows, where Ra > 10^9.

Later, Bejan proposed that Rayleigh number is not the criteria for determining whether it is turbulent or laminar flows, it is the Grashof number. So, if Gr > 10^9, then it is turbulent natural convection flows, and few correlations will also write based on the experiments.

(Refer Slide Time: 01:47)



Turbulent natural convection over a vertical flat plate for these critical Rayleigh number, earlier it was proposed as based on your characteristic length was of the order of 10^9. But later Bejan proposed now it is the Grashof number, which actually determines this is the critical Grashof number which determines whether it is laminar or turbulent flow. So, $\mathrm{Gr}_{xcrit} = 10^9$.

So, as you know that Rayleigh number is the product of Grashof number and Prandtl number, so $Ra_{xcrit} = 10^9 \,\mathrm{Pr}$.

So, depending on Prandtl number, so Rayleigh number will vary. So, this is the critical Rayleigh number. So, it will determine whether the flow is laminar or turbulent. If $Ra_x > 10^9 Pr$ then the flow is turbulent. So, obviously, it is natural convection flow.

So, Churchill and Chu in 1975, they proposed one correlation for turbulent flows and let us write that correlation first. So, Churchill and Chu relations in 1975, so they proposed for uniform wall temperature condition.

$$\overline{Nu} = \left\{ 0.825 + \frac{0.387 Ra_H^{1/6}}{\left[1 + \left(\frac{0.492}{\mathrm{Pr}}\right)^{9/16}\right]^{8/27}} \right\}^2$$

. So, this is for turbulent flows and this is derived based on the experimental value. So, this is known as correlation. So, Churchill and Chu correlation.

And for uniform wall heat flux boundary condition, so this 0.492 you can replace with 0.437. So, this is the same relation except the value of these for 0.492. So, it will be,

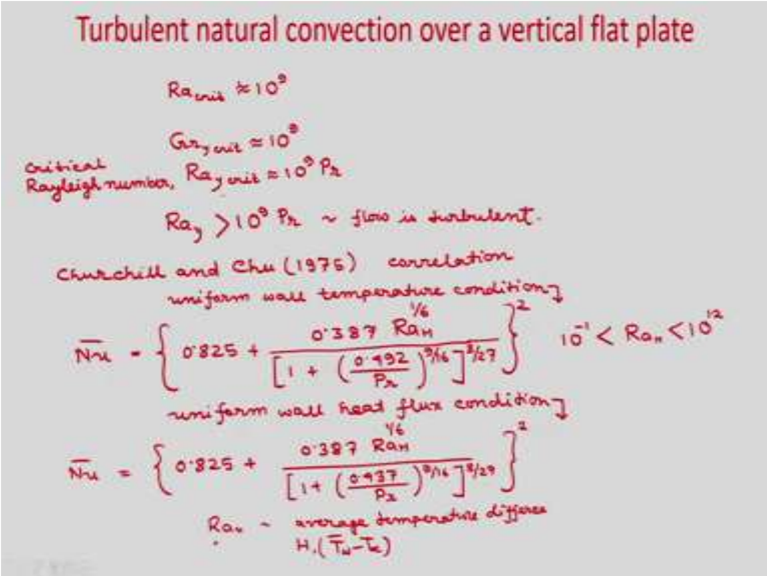
$$\overline{Nu} = \left\{ 0.825 + \frac{0.387 Ra_H^{1/6}}{\left[1 + \left(\frac{0.437}{\mathrm{Pr}}\right)^{9/16}\right]^{8/27}} \right\}^2$$

. And this relation is valid for $10^{-1} < Ra_H < 10^{12}$.

So, you can see this is also valid for $Ra_H < 10^9$. So, it will give reasonably good results. And you can see in case of uniform wall heat flux condition, this Rayleigh number will be based on your height of the plate and the temperature difference that will be average temperature difference because for uniform wall heat flux condition you know that temperature will vary along the plate.

So, obviously, your it will be based on H and your T_W -T_∞. So, this wall temperature you have to consider as average wall temperature. So, then it will be average temperature difference, based on that this Rayleigh number is defined. Even for laminar flow Gr < 10^9 , Churchill and Chu proposed another relation.

(Refer Slide Time: 07:24)



So, for laminar flow, where Gr < 10^9 another corelation was proposed by Churchill and Chu in 1975, and it is valid for both uniform surface temperature and uniform wall heat flux, and uniform wall heat flux conditions. So, that is given as,

$$\overline{Nu} = 0.68 + \frac{0.67 Ra_H^{1/4}}{\left[1 + \left(\frac{0.492}{\Pr}\right)^{9/16}\right]^{4/9}}.$$

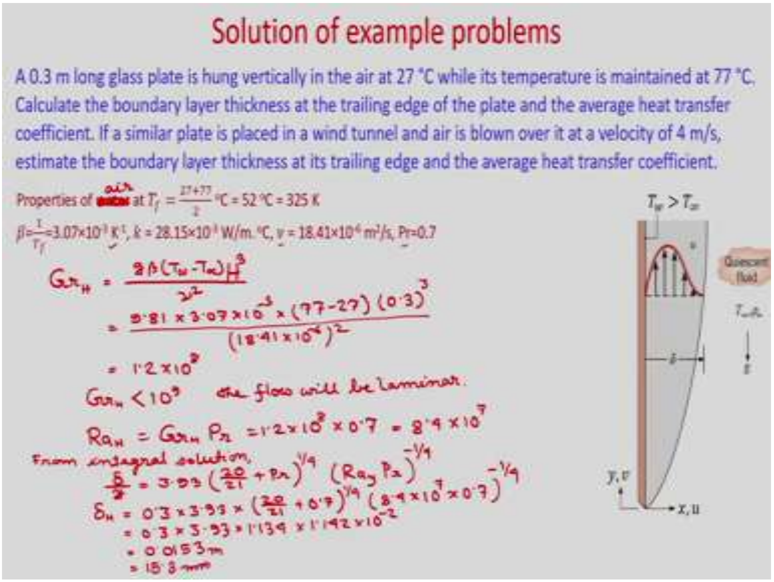
And Eckert and Jackson also part from the integral analysis for turbulent natural convection flows, and they proposed this Nusselt number relations. So, let us write that. So, using integral relation or using integral analysis for turbulent natural convection; Eckert and Jackson proposed.

So, Eckert and Jackson derived the $Nu = 0.0295\left[\frac{\Pr^7}{\left(1 + 0.494 \Pr^{2/3}\right)^6}\right]^{1/5} Gr_y^{2/5}$.

And $\overline{Nu}_H = 0.834 Nu\big|_{y=H}$. Because generally you have written in terms of y. So, this also you can write y, y because height of the plate, along that direction you have taken y, so this will be y.

So, while solving the problem first you have to determine the Rayleigh number and depending on the Rayleigh number you have to see whether it is laminar flow or turbulent natural convection flows. Then, you have to use suitable formula to calculate the average heat transfer coefficient or the heat transfer coefficient.

(Refer Slide Time: 11:22)



So, first let us solve this problem. If 0.3 m long glass plate is hung vertically in the air at 27^0C, while its temperature is maintained at 77^0C. Calculate the boundary layer thickness at the trailing edge of the plate and the average heat transfer coefficient.

If a similar plate is placed in a wind tunnel and air is blown over it at a velocity of 4 m/s estimate the boundary layer thickness at its trailing edge and the average heat transfer coefficient.

So, you can see first using natural convection you have to find the boundary layer thickness, and the heat transfer coefficient, and later case if it is a post convection flow, then you have to calculate the boundary layer thickness and the heat transfer coefficient.

So, in this case, first you have to know the properties. So, generally properties are calculated at the film temperature that means, average temperature and thermal expansion coefficient $\beta = \frac{1}{T_f}$, where T_f is the film temperature. And this film temperature you have to write in K.

So, this is air at T_f. So, $T_f = \frac{27+77}{2}$ = 52^0C, so 325 K. So, $\beta = \frac{1}{T_f}$ as this and the thermal conductivity, the kinematic viscosity, and Prandtl number are given.

So, first you find what is the Grashof number, so Grashof number you can calculate as, $Gr_H = \frac{g\beta\left(T_w - T_\infty\right)H^3}{\nu^2}$. So, you can see you can calculate $\frac{9.81 \times 3.07 \times 10^{-3} \times \left(77-27\right)\left(0.3\right)^3}{\left(18.41 \times 10^{-6}\right)^2}$. So, you will get it as 1.2×10^8 k. So, you can see obviously, Gr_H < 10^9, so the flow will be laminar.

And, once you know Grashof number you can calculate the Rayleigh number. So, it will be $Ra_H = Gr_H \Pr$. So, it is $1.2 \times 10^8 \times 0.7 = 8.4 \times 10^7$.

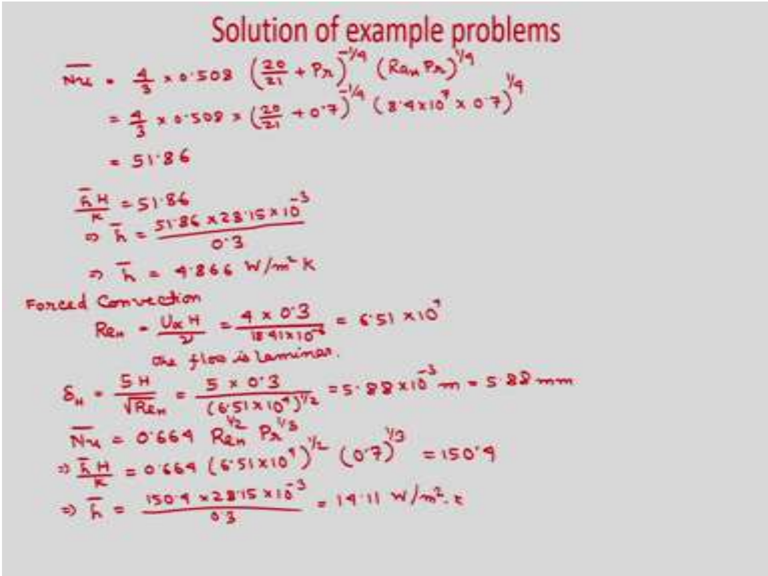
So, using integral solution we have derived the boundary layer thickness $\frac{\delta}{x}$, and also, we have found what is the Nusselt number and from there you can calculate the heat transfer coefficient. So, we will use those relations here. So, from integral solution we have found, $\frac{\delta}{y} = 3.93\left(\frac{20}{21} + \Pr\right)^{1/4}\left(Ra_y \Pr\right)^{-1/4}$.

So, in this case, at the trailing edge you have to calculate the boundary layer thickness. So, we will write y = H. So, you can write,

$\delta_H = 0.3 \times 3.93 \times \left(\frac{20}{21} + 0.7\right)^{1/4}\left(8.4 \times 10^7 \times 0.7\right)^{-1/4}$.So, this you can write, $0.3 \times 3.93 \times 1.134 \times 1.142 \times 10^{-2}$. So, finally, you will get 0.0153 m. So, if you write in terms of mm then, so 15.3 mm.

Now, you can write the average heat transfer coefficient. So, first let us write the expression for average Nusselt number and from there we will calculate the heat transfer coefficient.

(Refer Slide Time: 16:58)



So, you know the relation from the integral solution as, $\overline{Nu} = \frac{4}{3} \times 0.508 \left(\frac{20}{21} + \Pr \right)^{-1/4} (Ra_H \Pr)^{1/4}$.

So, if you put these values, so $\frac{4}{3} \times 0.508 \times \left(\frac{20}{21} + 0.7 \right)^{-1/4} \left(8.4 \times 10^7 \times 0.7 \right)^{1/4}$. So, if we evaluate these values you will get finally, as 51.86. So, now, you can write, $\frac{\bar{h}H}{K} = 51.86$.

So, average heat transfer coefficient you can write as $\bar{h} = \frac{51.86 \times 28.15 \times 10^{-3}}{0.3}$, so you will get, $\bar{h} = 4.866 W/m^2K$.

Now, you have to consider the post convection and first you calculate the Reynolds number, then check whether it is laminar or turbulent flows. You know that if $Re_H < 10^5$ then flow will be laminar for flow over flat plate case. And then, you use the suitable relation.

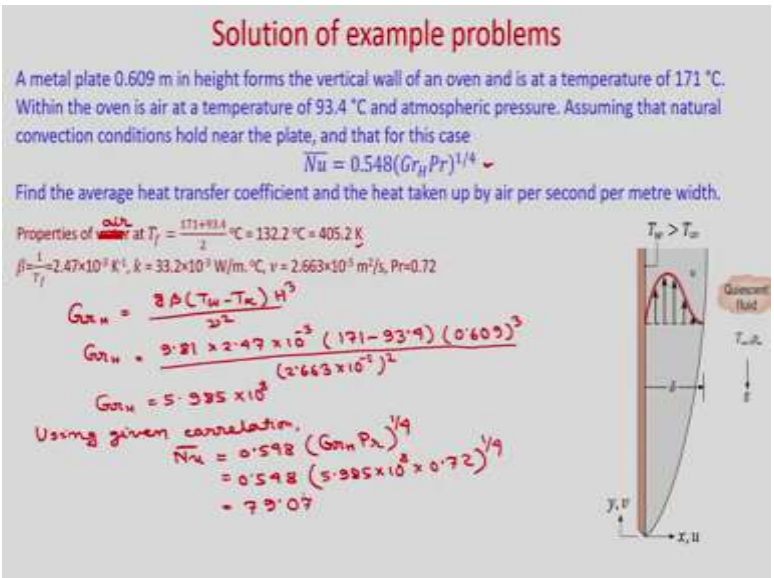
So, for forced convection $\mathrm{Re}_H = \frac{U_\infty H}{\nu}$. So, it will be $\frac{4 \times 0.3}{18.41 \times 10^{-6}}$. So, you will get 6.51×10^4, so the flow is laminar. And if you remember from the (Refer Time: 19:30) solution we calculate $\delta_H = \frac{5H}{\sqrt{\mathrm{Re}_H}}$.

So, you will be able to calculate $\frac{5 \times 0.3}{\left(6.51 \times 10^4\right)^{1/2}}$. So, this you will get as 5.88×10^{-3} m, so it will be 5.88 mm. So, you can see in case of post convection flow your boundary layer thickness is much much smaller than the natural convection boundary layer thickness.

So, now calculate the Nusselt number. So, $0.664 \mathrm{Re}_H^{1/2} \mathrm{Pr}^{1/3}$. So, you can see you can write $0.664 \times \left(6.51 \times 10^4\right)^{1/2} (0.7)^{1/3}$. So, if you calculate these you will get 150.4 and $\bar{h} = \frac{150.4 \times 28.15 \times 10^{-3}}{0.3}$. So, you will get as 14.11 W/m^{2}K

So, you can see here that in force convection flow obviously, your heat transfer is higher than the natural convection flow.

(Refer Slide Time: 21:30)

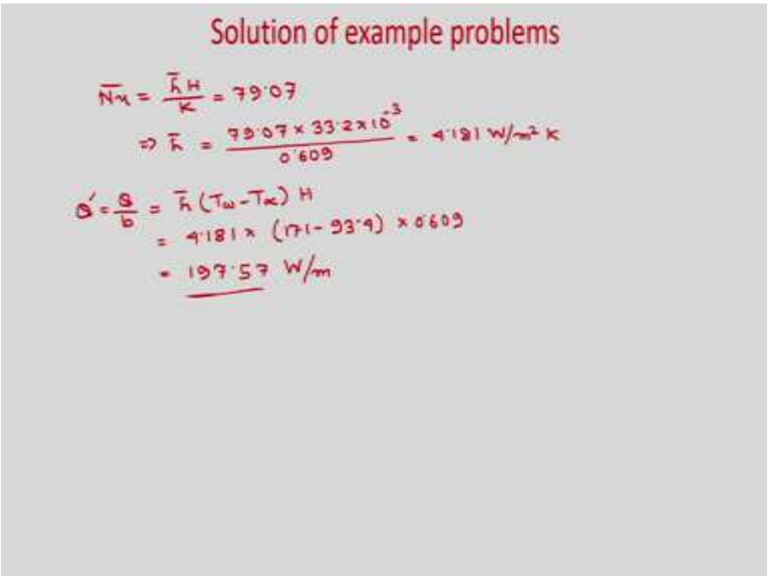


Now, let us consider the next problem. A metal plate 0.609 m in height forms the vertical wall open oven and is at a temperature of 171 ^{0}C. Within the oven is air at a temperature of 93.4 ^{0}C and atmospheric pressure. Assuming that natural convection conditions hold near the plate and that is for this case the average Nusselt number is given as this relation. Find the average heat transfer coefficient and the heat taken up by air per second per meter width.

So, the relation for the Nusselt number is given, so you have to use it. And properties for air, so it is not water, is evaluated at a film temperature 4.502 K and β thermal conductivity kinetic viscosity and Prandtl number are given.

So, first you calculate the Grashof number. Check whether flow is laminar or turbulent. So, you can see it is $Gr_H = \frac{g\beta(T_w - T_\infty)H^3}{\nu^2}$. So, Grashof number you can write as, $\frac{9.81 \times 2.47 \times 10^{-3} \times (171 - 93.4)(0.609)^3}{(2.663 \times 10^{-5})^2}$. So, if you evaluate it you will get Grashof number as 5.985×10^8. So, you can consider the flow is laminar however, the Nusselt number relation is given. So, you have to use it. So, using given correlation you can write $\overline{Nu} = 0.598(Gr_H \Pr)^{1/4}$. So, from here you can see you can calculate $\overline{Nu} = 0.548(5.985 \times 10^8 \times 0.72)^{1/4}$. So, Nusselt number you will get as 79.07.

(Refer Slide Time: 24:31)

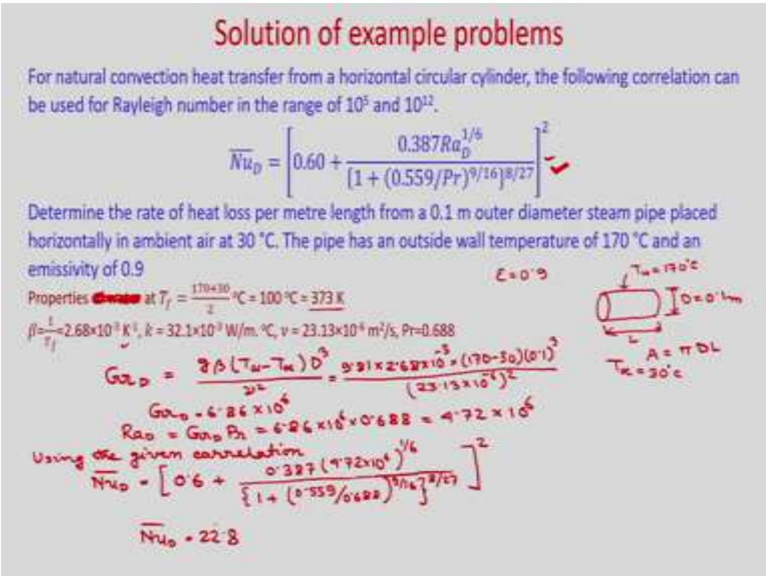


So, now Nusselt number you can write as $\frac{\bar{h}H}{K}$, so it will be 79.07. So, $\bar{h} = \frac{79.07 \times 33.2 \times 10^{-3}}{0.609}$. So, from here if you evaluate these you will get 4.181 W/m^{2}K.

So, average heat transfer coefficient we have calculated, now we are asked to calculate the heat transfer rate, per unit width of the plate. So, that you can use the Newton's law of cooling. So, we will write the heat transfer rate per unit width as $Q' = \frac{Q}{b}$, so per meter actually.

So, it will be $\bar{h}\left(T_w - T_\infty\right)H$. So, it is $4.181 \times \left(171 - 93.4\right) \times 0.609$. So, if you evaluate it you will get 197.57 W/m. So, this is the heat transfer rate.

(Refer Slide Time: 26:13)



So, now let us discuss about the next problem. For natural convection heat transfer from a horizontal circular cylinder, the following correlation can be used for Rayleigh number in the range of 10^5 and 10^{12}. So, Nusselt number correlation is given for the natural convection heat transfer from a horizontal circular cylinder. So, horizontal circular cylinder you have. So, from here this is the correlation given.

Determine the rate of heat loss per meter length from a 0.1 m outer diameter. So, outer diameter is given as 0.1 m and if this is the length then A= π DL. And per unit length it will be π D.

So, determine the rate of heat loss per meter length from a 0.1 m outer diameter steam pipe placed horizontally in ambient air at 30 ^{0}C. So, T_∞ is 30 ^{0}C. The pipe has an outside wall temperature of 170 ^{0}C. So, T_w is given as 170 ^{0}C. And emissivity also given, so you have to find what is the radiative heat transfer rate.

So, again the properties are calculated at the film temperature. So, you can see the film temperature is 373 K and $\beta = \frac{1}{373}$. So, you can calculate the β and thermal conductivity, kinetic viscosity and Prandtl number are given. So, you can now calculate the Grashof number first.

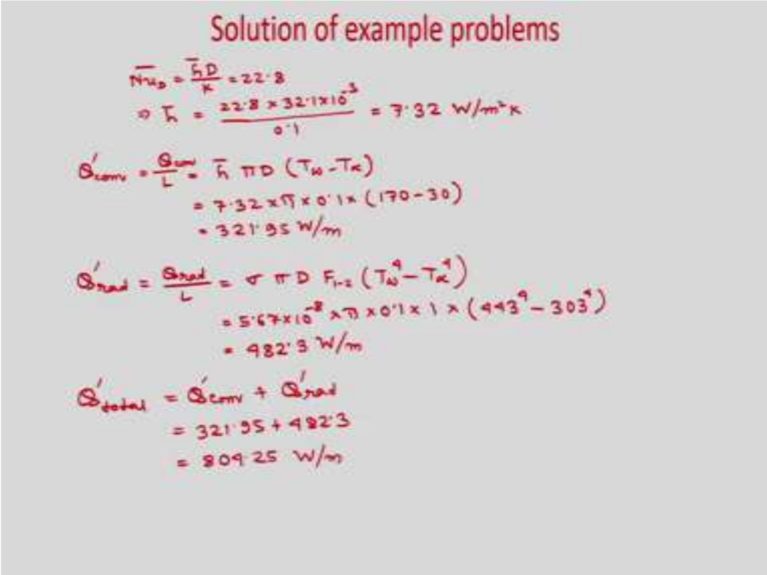
So, Grashof number based on the diameter, $Gr_D = \frac{g\beta\left(T_w - T_\infty\right)D^3}{\nu^2}$. So, you can see $\frac{9.81\times 2.68\times 10^{-3}\times\left(170-30\right)\left(0.1\right)^3}{\left(23.13\times 10^{-6}\right)^2}$.

So, Grashof number you will get as 6.86×10^6 and $Ra_D = Gr_D \Pr$. So, it will be $6.86\times 10^6 \times 0.688$ and you will get as 4.72×10^6.

Now, using the given correlation you can calculate the Nusselt number and from there you can calculate the average heat transfer coefficient. Once you know the average heat transfer coefficient, then you will be able to calculate the heat transfer rate.

So, from this relation, you calculate the Nusselt number using the given correlation, $\overline{Nu}_D = \left[0.6 + \frac{0.387\left(4.72\times 10^6\right)^{1/6}}{\left\{1+\left(\frac{0.559}{0.688}\right)^{9/16}\right\}^{8/27}}\right]^2$. So, if you evaluate it you will get as 22.8.

(Refer Slide Time: 30:56)



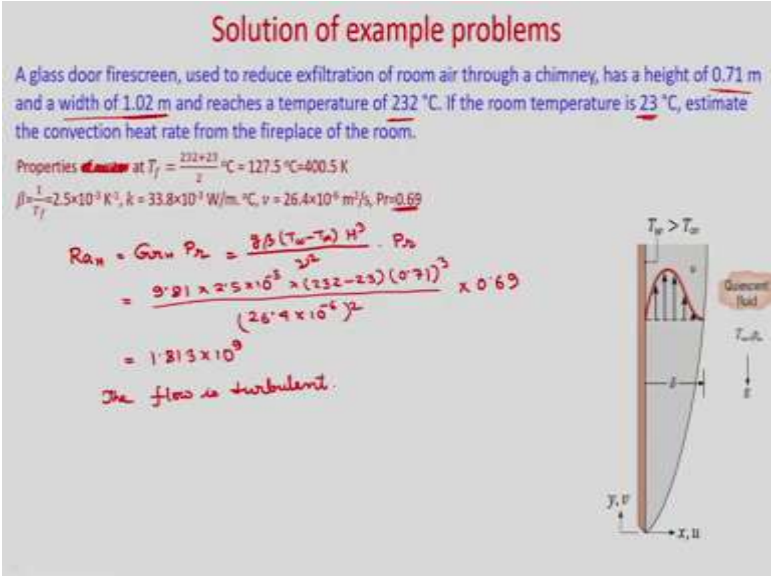
So, now you calculate the heat transfer coefficient. So, $\overline{Nu}_D = \frac{\bar{h}D}{K}$. This your average heat transfer coefficient 22.8. So, $\bar{h} = \frac{22.8 \times 32.1 \times 10^{-3}}{0.1}$. So, you can calculate these, $\bar{h} = 7.32 W/m^2K$.

Now, you can calculate the heat transfer rate per unit length, because A= π DL. So, you will write the per unit length, so it will be π D. So, $Q^{'}_{conv} = \frac{Q_{conv}}{L}$. So, it will be, $\bar{h}\pi D\left(T_w - T_\infty\right)$. So, this is a $7.32 \times \pi \times 0.1 \times (170 - 30)$, so you will get 321.95 W/m.

So, we have to calculate the heat transfer rate from the natural convection. Now, you have to calculate the heat transfer rate due to radiation. So, you can write $Q^{'}_{rad} = \frac{Q_{rad}}{L}$. So, this is $\sigma \pi D F_{1-2}\left(T_w^4 - T_\infty^4\right)$. So, these temperature we have to write in K. So, $5.67 \times 10^{-8} \times \pi \times 0.1 \times 1 \times \left(443^4 - 303^4\right)$. So, you will get 482.3 W/m.

So, now, you have calculated heat transfer rate due to convection and radiation, so $Q^{'}_{total} = Q^{'}_{conv} + Q^{'}_{rad}$. So, it will be 321.95 + 482.3, so you can write as 804.25 W/m.

(Refer Slide Time: 34:00)



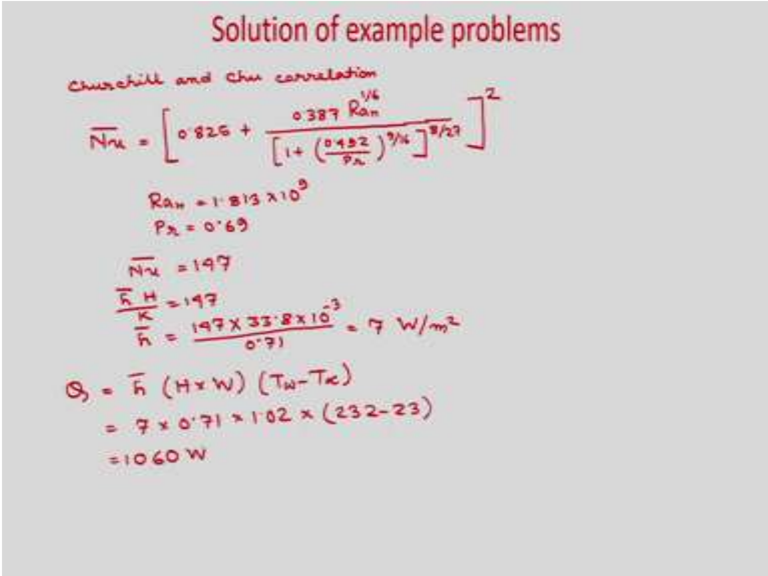
So, next let us consider one problem of turbulent flows. A glass door fire screen used to reduce exfiltration of room air through a chimney has a height of 0.71 m and width of 1.02 m and reaches a temperature of 232 ^{0}C. If the room temperature is 23 ^{0}C estimate the convection heat rate from the fires place of the room.

So, you can see your wall temperature will be 232 ^{0}C and ambient temperature is 3 ^{0}C. And height of the plate will be 0.71 m in this case width of the plate is given, 1.02 m. So, you can calculate the area as product of this height into width.

So, properties at film temperature you can calculate for the air, so it is 400.5 K, this is the film temperature. So, other properties are given here. So, the $Ra_H = Gr_H \Pr$.

So, $\frac{g\beta(T_w - T_\infty)H^3}{\nu^2}\Pr$. So, you can calculate Rayleigh number as $\frac{9.81\times 2.5\times 10^{-3}\times(232-23)\times(0.71)^3}{(26.4\times 10^{-6})^2}\times 0.69$. So, it is 1.813×10^9 and the flow is turbulent. So, you can use Churchill and Chu correlation for these turbulent flows.

(Refer Slide Time: 36:39)



So, Churchill and Chu correlation is given as $\overline{Nu_D} = \left[0.825 + \frac{0.387\left(Ra_H\right)^{1/6}}{\left\{1+\left(\frac{0.429}{\Pr}\right)^{9/16}\right\}^{8/27}}\right]^2$.

So, you put the values of Rayleigh number and Prandtl number here. So, you will get, $Ra_H = 1.813 \times 10^9$ and Pr = 0.69. So, if you put this values and evaluate it you will get $\overline{Nu} = 147$.

So, from here now you calculate the average heat transfer coefficient and from there you will be able to calculate the heat transfer rate. So, we can write $\frac{\bar{h}H}{K} = 147$. So, $\bar{h} = \frac{147 \times 33.8 \times 10^{-3}}{0.71}$. So, you will get as 7 W/m^2.

So, your heat transfer rate you can calculate as $\bar{h}A$. So, $Q = \bar{h}\left(H \times W\right)\left(T_w - T_\infty\right)$. So, $7 \times 0.71 \times 1.02 \times \left(232 - 23\right)$. So, if you evaluate it you will get 1060 W.

So, in today's class first we discuss about the turbulent natural convection flows. And if the Gr > 10^9, then you can say that the flow is turbulent. So, critical $Gr_{crit} = 10^9$ to determine whether the flow is laminar or turbulent.

Then, we considered 4 problems. So, while solving the problem first you need to calculate the Grashof number and check whether the flow is laminar or turbulent, and suitably you use the relation for the Nusselt number. And one problem also we have solved using Churchill and Chu correlation for the turbulent flows. You can solve more problems from any textbook.

Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

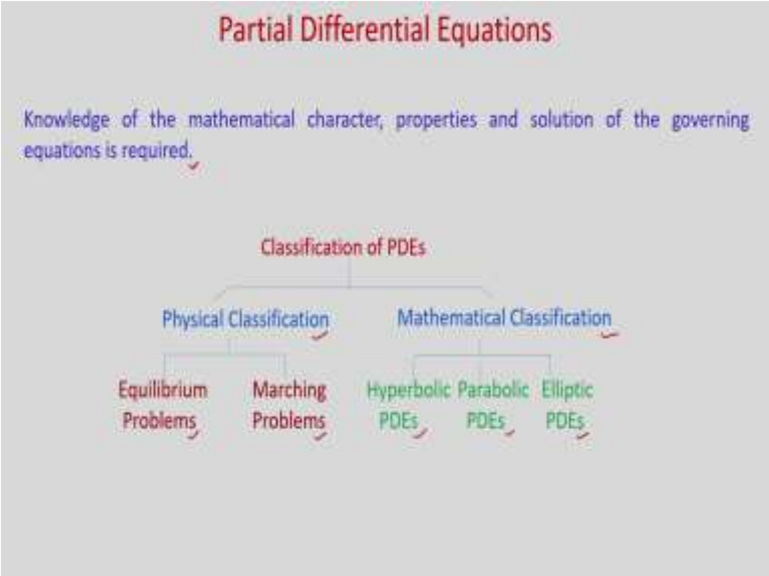
Module – 10
Numerical Solution of Navier–Stokes and Energy Equations
Lecture – 33
Basics of finite difference method

Hello, everyone. So, we have already solved analytical solution for external flows and for internal flows, for fully developed condition we could get some analytical solution, but if you have developing internal flows then obviously, it is very difficult to study because we cannot have the analytical solution, but we can solve the governing equations numerically.

So, using computational fluid dynamics technique we can solve the Navier–Stokes equations as well as the energy equation with viscous dissipation or with heat generation term in the energy equation. So, in this module we will study the solutions of Navier–Stokes equations and energy equations using numerical technique. There are three different types of discretization method – finite difference method, finite volume method and finite element method. In this course we will use finite difference method to discretize the Navier–Stokes equations and energy equation.

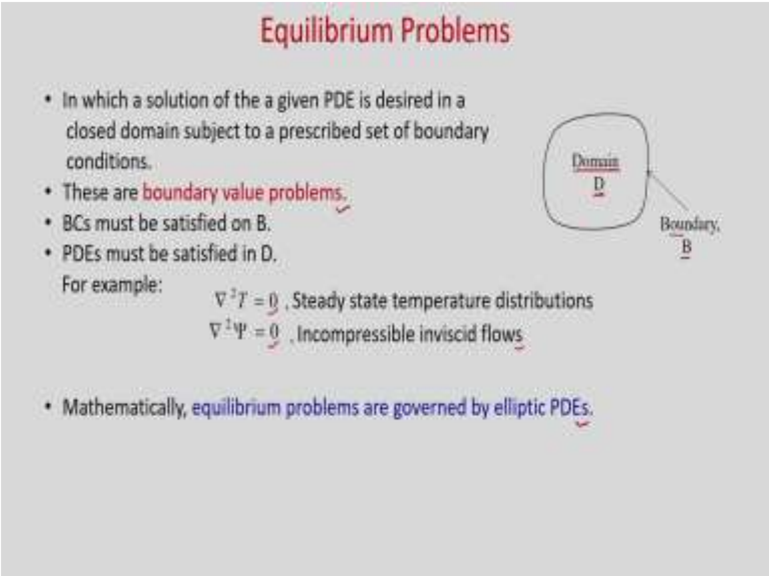
So, in today's class we will study the basics of finite difference method. So, before starting the basics of finite difference method let us know the classification of partial differential equations because ultimately we are going to discretize the partial differential equations and before using some numerical skills it is very much needed to know the classification of these PDEs.

(Refer Slide Time: 02:23)



So, knowledge of mathematical character properties and solution of the governing equations is required. So, you can see we can classify the PDEs as physical classification and mathematical classification. In physical classification we can have equilibrium problems and marching problems; and, in mathematical classification we have hyperbolic partial differential equations, parabolic partial differential equations and elliptic partial differential equations.

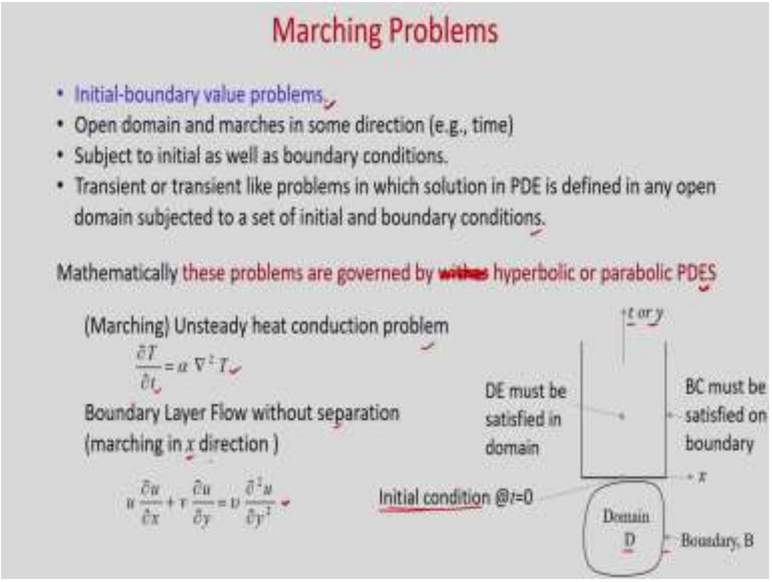
(Refer Slide Time: 02:52)



So, first let us discuss about the equilibrium problems. Let us say that we have a domain D and this is the boundary B. So, obviously, whatever partial differential equation you need to solve you will solve inside this domain and apply the boundary condition at the boundary. So, these problems are known as boundary value problem like you have steady state temperature distribution.

So, the equation is heat conduction equation $\nabla^2 T = 0$ or incompressible inviscid flows $\nabla^2 \psi = 0$. So, these you can see that these equations you can solve in the domain and with proper boundary condition you can apply in the boundary. So, mathematically equilibrium problems are governed by elliptic partial differential equations.

(Refer Slide Time: 03:43)



Marching problems: so, marching problems are also known as initial boundary value problems. In marching problem so, you have domain where you need to solve the partial differential equations and at the boundary you will apply the boundary conditions, but it will march in some direction maybe in time or maybe in space.

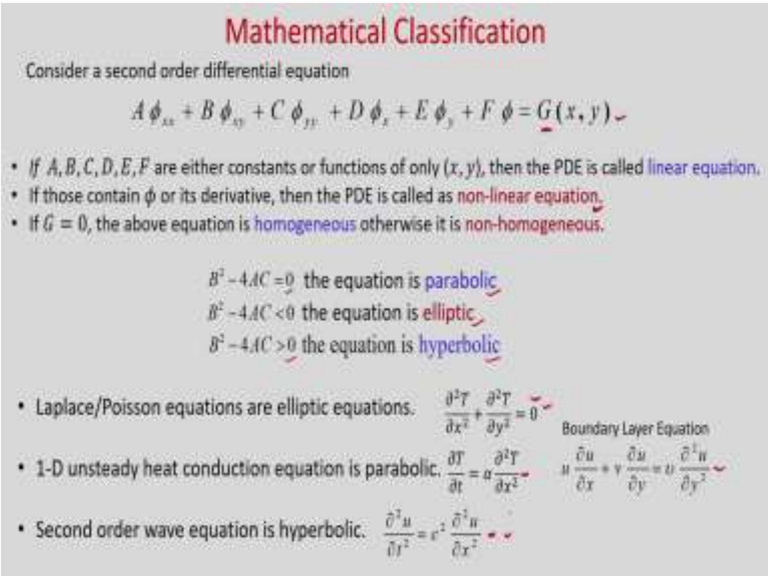
So, if it is a unsteady problem obviously, it will march in time and at time T = 0 you need to specify the value of that variable inside the domain. So, you can see here. So, this is the domain. So, governing equation at each time level will solve in this domain and also you will apply the boundary condition at boundary B and you can see at t = 0, you will apply the initial condition and you will march in time or in some direction.

So, differential equation must be satisfied in the domain and boundary condition must be satisfied on boundary. So, these are tangent or tangent like problems in which the solutions in PDE is defined in any open domain subjected to a set of initial and boundary conditions. So, mathematically these problems are governed by hyperbolic or parabolic partial differential equations.

So, you can have the example like unsteady heat conduction equation. So, $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$. So, you can see that in this case you will march in time, but if you have a boundary layer flow without separation then you can have the boundary layer equation. So, this equation is also parabolic and it you will march in the direction x because at x = 0 you will specify the free stream velocity infinity and in the x direction you will march.

Now, let us discuss about the mathematical classification. First let us consider a second order partial differential equation.

(Refer Slide Time: 06:03)



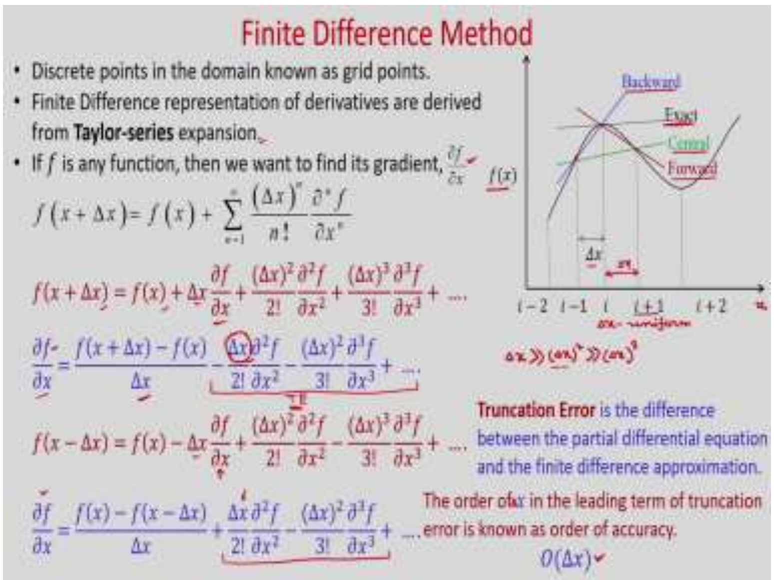
So, consider this equation $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi = G(x, y)$. So, you can see these coefficients A, B, C, D, E, F if these are constants or function of only x, y, then this partial differential equation is called as linear equation. If these coefficients contain Φ or its derivative, then the partial differential equation is called as non-linear equation and if G = 0, then this equation is homogeneous otherwise it is non-homogeneous.

So, for this equation if you find the discriminant so, if $B^2 - 4AC = 0$, then mathematically this equation is known as parabolic equation; $B^2 - 4AC < 0$, the equation is known as elliptic equation and $B^2 - 4AC > 0$, then the equation is hyperbolic. So, you can have some example.

So, you can have this Laplace equation, heat conduction equation. So, if you find $B^2 - 4AC < 0$, then it is elliptic. 1-D unsteady heat conduction equation, this is if you find you will get $B^2 - 4AC = 0$. So, it is parabolic; this is equation also is parabolic. And, second order wave equation is hyperbolic. So, this equation if you find $B^2 - 4AC > 0$ and this equation is hyperbolic in nature.

So, the detail of mathematical and physical classification you can study from any CFD book. Now, let us introduce the finite difference method. So, in finite difference method we considered the partial differential equation and its derivative we expressed in terms of its values using Taylor series expansion. First let us consider any function f which is a function of x.

(Refer Slide Time: 08:29)



So, you can see this is the function f(x) and x we are plotting in x-direction. If you take this variation of f with x and at this point let us say this is the discrete point i and this is i+1 with a distance Δx and in left side if you take another discrete point i - 1. So, this distance is also Δx. Let us say Δx is uniform. So, everywhere Δx is same.

So, now if you want to find the derivative at this point , the derivative of f at this point what you will do? Just you will draw one tangent. So, that is the exact determination of this derivative. So, this tangent will give you the $\frac{\partial f}{\partial x}$. Now, if you use two points to find it is derivative, let us say we are using one forward point forward point means i + 1, at point i we want to find the derivative. So, this point and this point we are using as you are using forward point so, it is a forward difference.

So, if you join this line then you will get a forward difference approximation; if you use one backward point so, we want to find the $\frac{\partial f}{\partial x}$ at this point and we are using another point at i -1 which is your backward point. So, if you join these two points, then you will get backward approximation and if you use one forward and one backward point and if you join then, you will get the central approximation to find the derivative.

So, finite difference representation of derivatives are derived from Taylor series expansion and from this graph you can see that central approximation is more closer to the exact because you see in backward and forward it is more deviated, but central is almost parallel to the exact. So, that means, it gives determination of this $\frac{\partial f}{\partial x}$ close to the exact solution.

Now, let us consider the Taylor series and expand it. So, if, $f\left(x+\Delta x\right)=f\left(x\right)+\sum_{n=1}^{\infty}\frac{\left(\Delta x\right)^{n}}{n!}\frac{\partial^{n} f}{\partial x^{n}}$. So, if you take one forward point and if you go in forward direction Δx, then you can write,

$$f\left(x+\Delta x\right)=f\left(x\right)+\Delta x\frac{\partial f}{\partial x}+\frac{\left(\Delta x\right)^{2}}{2!}\frac{\partial^{2} f}{\partial x^{2}}+\frac{\left(\Delta x\right)^{3}}{3!}\frac{\partial^{3} f}{\partial x^{3}}+.......$$

So, now if you want to find the first gradient this derivative, $\frac{\partial f}{\partial x}=\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}-\frac{\Delta x}{2!}\frac{\partial^{2} f}{\partial x^{2}}-\frac{\left(\Delta x\right)^{2}}{3!}\frac{\partial^{3} f}{\partial x^{3}}+.......$

So, you can see that these derivative we have expressed in terms of the discrete points using a forward point at i + 1 that is $f\left(x+\Delta x\right)$. So, the approximation using finite difference method of this derivative is $\frac{f\left(x+\Delta x\right)-f\left(x\right)}{\Delta x}$ and these higher order terms you can neglect. So, these terms are known as truncation error.

So, if you see if Δx is very small then Δ x > $\left(\Delta x\right)^2$ and $\left(\Delta x\right)^3 << \left(\Delta x\right)^2$. So, anyway you can neglect the higher order terms and this is whatever you are actually neglecting that is your truncation error. So, what is the definition of truncation error? You can see from this expression the truncation error is the difference between the partial differential equation and the finite difference approximation.

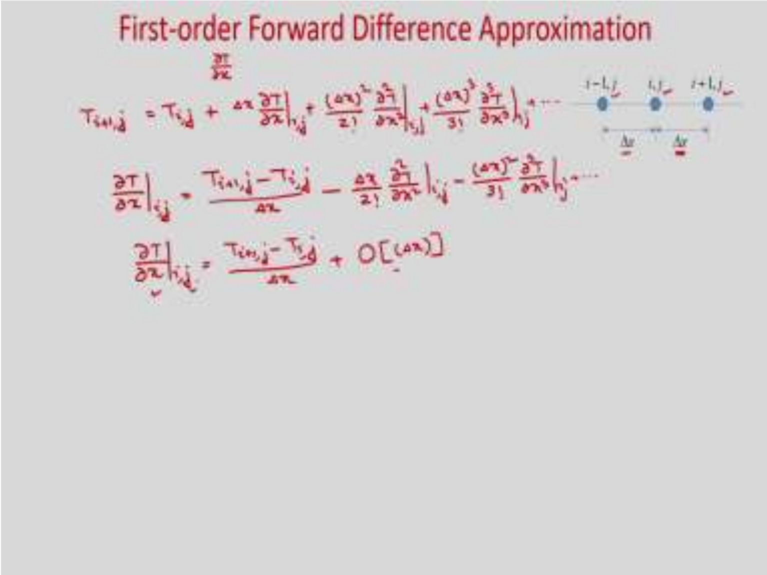
If you see the order of x in the leading term of truncation error and that is known as order of accuracy. So, if you can see in this truncation error the leading term is this one and here the order of x is Δx, so its order of accuracy is Δx and it is first order accurate approximation.

Similarly, now if you use one backward point and if you use $f\left(x-\Delta x\right)$, then you can write from the Taylor series expansion as, $f\left(x-\Delta x\right)=f\left(x\right)-\Delta x\frac{\partial f}{\partial x}+\frac{\left(\Delta x\right)^2}{2!}\frac{\partial^2 f}{\partial x^2}-\frac{\left(\Delta x\right)^3}{3!}\frac{\partial^3 f}{\partial x^3}+.......$

So, now if we want to approximate this gradient $\frac{\partial f}{\partial x}$ using one backward point then you can write $\frac{\partial f}{\partial x}=\frac{f\left(x-\Delta x\right)-f\left(x\right)}{\Delta x}+\frac{\Delta x}{2!}\frac{\partial^2 f}{\partial x^2}-\frac{\left(\Delta x\right)^2}{3!}\frac{\partial^3 f}{\partial x^3}+......$ So, you can see the finite difference approximation of this $\frac{\partial f}{\partial x}$ is $\frac{f\left(x-\Delta x\right)-f\left(x\right)}{\Delta x}$.

So, you have represented these $\frac{\partial f}{\partial x}$ using two discrete points f(x) and $f\left(x-\Delta x\right)$ and this is your truncation error and in the leading order term the order of Δ x. So, this is order of Δx. So, order of Δx is 1. So, its order of accuracy is order of Δx.

(Refer Slide Time: 15:27)



Now, let us consider the derivative of t with respect to x and the discrete points are i, j; i+1, j and i -1, j. So, let us consider these points i, j; i + 1, j and i - 1, j and you have a uniform grid. So, the step size is Δx and we went to find the derivative $\frac{\partial T}{\partial x}$, where T is the temperature.

So, you can use the Taylor series expansion. So, you can write $T_{i+1,j} = T_{i,j} + \Delta x \frac{\partial T}{\partial x}\Big|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + ...$

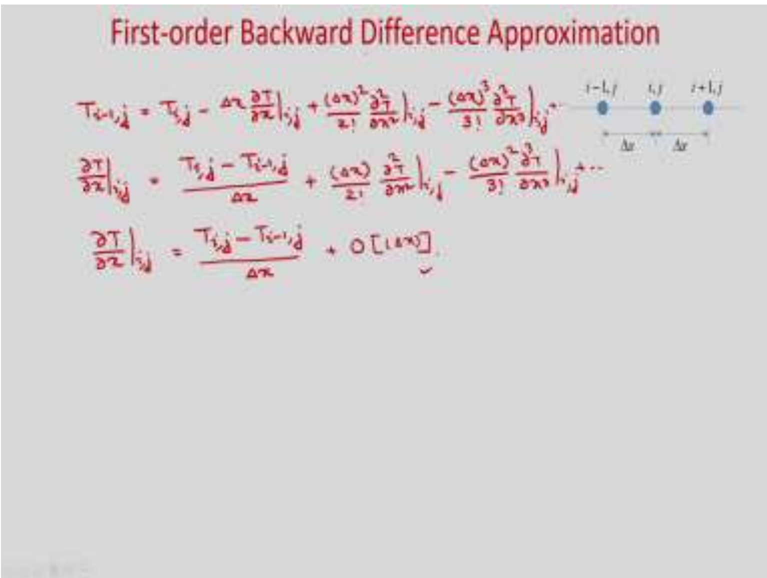
So, if you want to find the derivative $\frac{\partial T}{\partial x}\Big|_{i,j}$ so, you can rearrange and write,

$$\frac{\partial T}{\partial x}\Big|_{i,j} = \frac{T_{i+1,j} - T_{i,j}}{\Delta x} - \frac{\Delta x}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} + \frac{(\Delta x)^2}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + ...$$

So, you can see your $\frac{\partial T}{\partial x}$ so, you can represent using forward point $T_{i+1,j}$ and $T_{i,j}$. So, what will be the order of accuracy? So, if you neglect this truncation error then order of accuracy is Δx. So, it is a first order accurate discretization and it is known as first order forward difference approximation because we are using one forward point and order of accuracy is first order.

And, you can see the distance between $T_{i+1,j}$ and $T_{i,j}$ is Δ x; so, $\frac{T_{i+1,j} - T_{i,j}}{\Delta x}$. So, it is the finite difference approximation of the first gradient $\frac{\partial T}{\partial x}\Big|_{i,j}$.

(Refer Slide Time: 18:06)

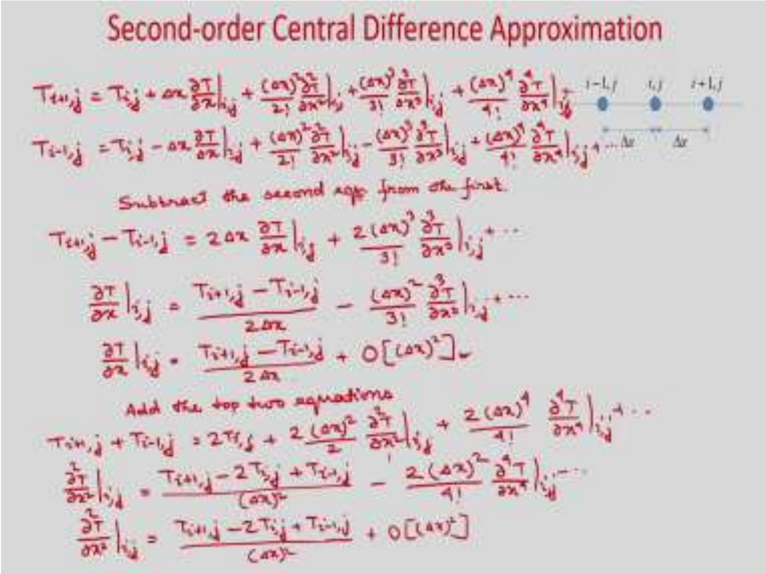


Now, similarly if you consider one backward point i - 1, j then you can find the derivative of T with respect to x as. So, you use Taylor series expansion, $T_{i-1,j} = T_{i,j} - \Delta x \frac{\partial T}{\partial x}\Big|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + \ldots\ldots$.

So, now if you want to represent this first gradient then you can rearrange it and you can write, $\frac{\partial T}{\partial x}\Big|_{i,j} = \frac{T_{i,j} - T_{i-1,j}}{\Delta x} + \frac{(\Delta x)}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} - \frac{(\Delta x)^2}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + \ldots\ldots$.

So, you can write $\frac{\partial T}{\partial x}\Big|_{i,j} = \frac{T_{i,j} - T_{i-1,j}}{\Delta x}$. So, you can see this is the finite difference approximation of this first gradient using one backward point and order of accuracy is Δx so, this is known as first order backward difference approximation. Now, use one backward and one forward points and find the approximation of this first derivative $\frac{\partial T}{\partial x}$ and the second derivative $\frac{\partial^2 T}{\partial x^2}$.

(Refer Slide Time: 20:20)



So, using Taylor series expansion you can write $T_{i+1,j} = T_{i,j} + \Delta x \frac{\partial T}{\partial x}\Big|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 T}{\partial x^4}\Big|_{i,j} + ...$ and other term and $T_{i-1,j} = T_{i,j} - \Delta x \frac{\partial T}{\partial x}\Big|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 T}{\partial x^4}\Big|_{i,j} + ...$. So, we are writing the Taylor series expansion. So, you can have the higher order terms. So, what you do? So, you first subtract the second one from the first equation. So, we want to find the $\frac{\partial T}{\partial x}$. So, if you subtract the second equation from the first.

So, if you subtract what you will get? You can see you will get, $T_{i+1,j} - T_{i-1,j} = 2\Delta x \frac{\partial T}{\partial x}\Big|_{i,j} + 2\frac{(\Delta x)^3}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + ...$

So, now if you write the approximation of first derivative $\frac{\partial T}{\partial x}$. So, you can see it will be $\frac{\partial T}{\partial x}\Big|_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x} - \frac{(\Delta x)^2}{3!}\frac{\partial^3 T}{\partial x^3}\Big|_{i,j} + ...$. So, you can write $\frac{\partial T}{\partial x}\Big|_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x} + O\left[(\Delta x)^2\right]$. So, it is a second order accurate.

So, it is second order central difference approximation because we are using one forward and one backward point and we are finding the derivative at i, j and you can see these two points are separated by a distance 2 Δx. So, $\frac{\partial T}{\partial x}\Big|_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta x} + O\left[(\Delta x)^2\right]$.

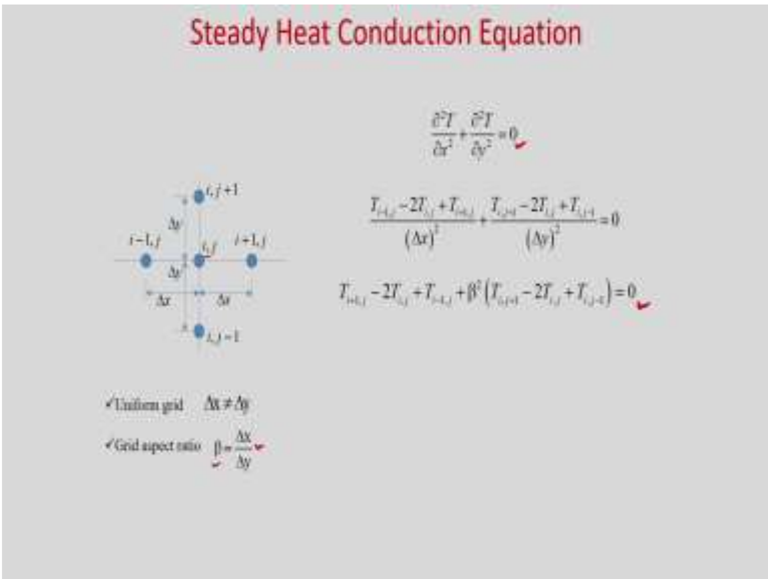
Now, you add these two equations so, and find the second derivative of T. So, if you add. So, if you add it you will get $T_{i+1,j} + T_{i-1,j} = 2T_{i,j} + 2\frac{(\Delta x)^2}{2!}\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} + 2\frac{(\Delta x)^4}{4!}\frac{\partial^4 T}{\partial x^4}\Big|_{i,j} + ...$

So, now you can see you can represent this second derivative using three points, $\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} - 2\frac{(\Delta x)^4}{4!}\frac{\partial^4 T}{\partial x^4}\Big|_{i,j} + ...$

So, you see the finite difference approximation of the second derivative about point i, j as $\frac{\partial^2 T}{\partial x^2}\Big|_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + O\left[(\Delta x)^2\right]$. It is second order. So, generally whatever equations we will consider we will have first derivative and second derivative. You can see that in Navier–Stokes equations, as well as in the energy equations you will get the first derivative of temperature and the second derivative of this temperature.

So, we have derived $\frac{\partial^2 T}{\partial x^2}$ so similar way you can write $\frac{\partial^2 T}{\partial y^2}\Big|_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} + O\left[(\Delta y)^2\right]$ because in the y-direction you are taking the derivative.

(Refer Slide Time: 26:48)



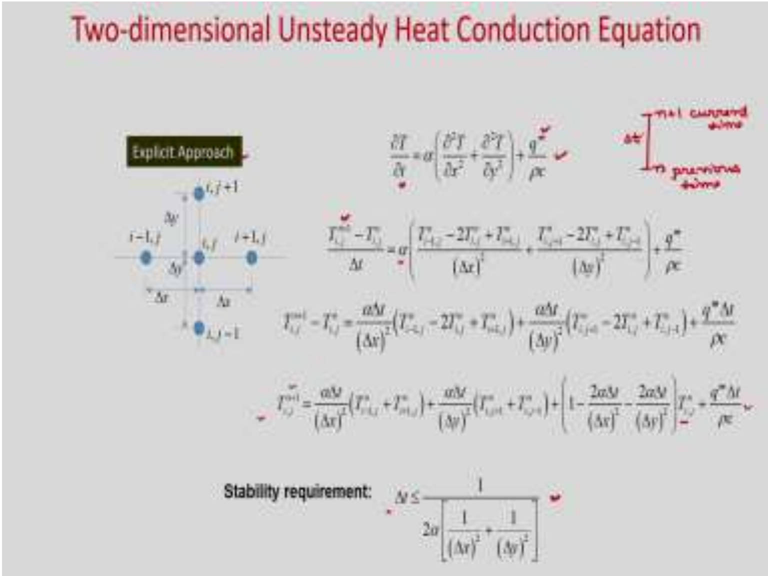
Now, we have learned how to use the Taylor series expansion to find the first derivative and second derivative. Now, let us consider steady state heat conduction equation and discretize using finite difference method. So, you can see we have this equation 2-dimensional steady state heat conduction equation these are the discrete points. So, this is known as grid.

So, we will discretize this equation about this point i, j. So, we have i +1, j; i -1, j these are the index and i, j + 1 and i, j -1 we have a uniform grid Δx in the x-direction and in y-direction Δy. So, Δx may not be same as Δy and grid aspect ratio we are defining $\beta=\frac{\Delta x}{\Delta y}$.

So, if you use the second order central difference approximation then you can write $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$ and $\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\left(\Delta x\right)^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\left(\Delta y\right)^2}=0$. If you rearrange it you can write, $T_{i+1,j}-2T_{i,j}+T_{i-1,j}+\beta^2\left(T_{i,j+1}-2T_{i,j}+T_{i,j-1}\right)=0$.

So, these equation is your discretized equation. So, this is your algebraic equation. So, that we have discretized about this point i, j. So, this is the equation we have written for this i, j. So, you can write similar equation for each point. So, you will get a system of algebraic equations. So, once you get the system of algebraic equation at each point, then you will you need to solve using some iterative method.

(Refer Slide Time: 28:48)



Now, let us consider two-dimensional unsteady heat conduction equation. So, this is the unsteady heat conduction equation with your heat generation term and we will use explicit approach; that means, there will be one unknown. So, we will use forward time discretization.

So, now, we need to march in the direction of time. So, we will march in time from n to n + 1. So, n is your previous time and this is your current time or present time. So, we are just marching in time from n to n +1.

So, these derivative now we are using forward time. So, $\frac{T_{i,j}^{n+1}-T_{i,j}^{n}}{\Delta t}=\alpha\left(\frac{T_{i-1,j}^{n}-2T_{i,j}^{n}+T_{i+1,j}^{n}}{\left(\Delta x\right)^{2}}+\frac{T_{i,j+1}^{n}-2T_{i,j}^{n}+T_{i,j-1}^{n}}{\left(\Delta y\right)^{2}}\right)+\frac{q'''}{\rho c}$.

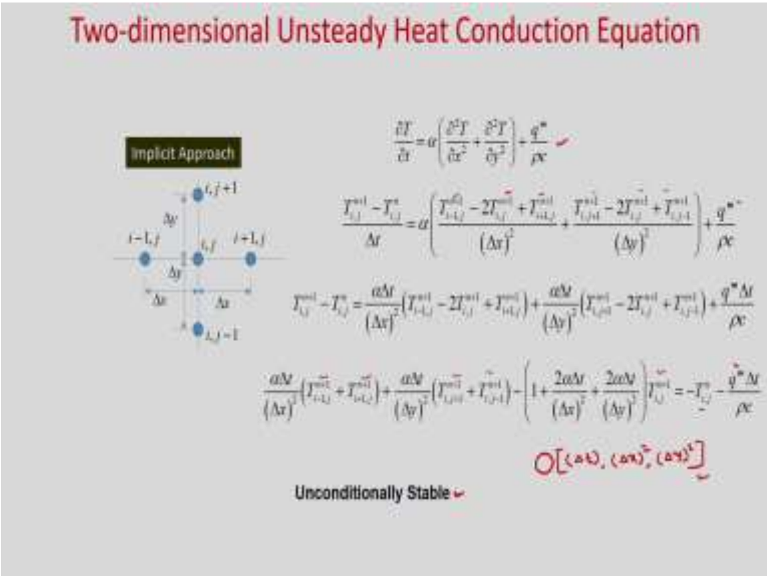
So, now you can write you can see here you have one unknown term n + 1 and all are at time level n. So, you are you can express,

$$T_{i,j}^{n+1}=\frac{\alpha\Delta t}{\left(\Delta x\right)^{2}}\left(T_{i-1,j}^{n}+T_{i+1,j}^{n}\right)+\frac{\alpha\Delta t}{\left(\Delta y\right)^{2}}\left(T_{i,j+1}^{n}+T_{i,j-1}^{n}\right)+\left(1-\frac{2\alpha\Delta t}{\left(\Delta x\right)^{2}}-\frac{2\alpha\Delta t}{\left(\Delta y\right)^{2}}\right)T_{i,j}^{n}+\frac{q'''\Delta t}{\rho c}.$$

So, this is your equation and it is you see you have only one unknown and all are known from the previous time level and this is known as explicit approach, but limitation of using the time is you will get from the stability criteria and here ΔT you have to choose from this stability requirement. So, $\Delta t \le \dfrac{1}{2\alpha\left[\dfrac{1}{(\Delta x)^2}+\dfrac{1}{(\Delta y)^2}\right]}$.

So, obviously, you can see if you refine the grid then Δx Δy will be smaller and accordingly, you have to decrease the time step ΔT and when you are going from n to n+1 so, this is the time step ΔT.

(Refer Slide Time: 31:14)



Now, we will use backward time and in that you will get more than one unknown and you will get a system of linear algebraic equations and this method is known as implicit method. So, the same equation we are considering two-dimensional unsteady heat conduction equation.

Now, if you discretize using first order this is your $\dfrac{T_{i,j}^{n+1}-T_{i,j}^{n}}{\Delta t}=\alpha\left(\dfrac{T_{i-1,j}^{n+1}-2T_{i,j}^{n+1}+T_{i+1,j}^{n+1}}{(\Delta x)^2}+\dfrac{T_{i,j+1}^{n+1}-2T_{i,j}^{n+1}+T_{i,j-1}^{n+1}}{(\Delta y)^2}\right)+\dfrac{q'''}{\rho c}$

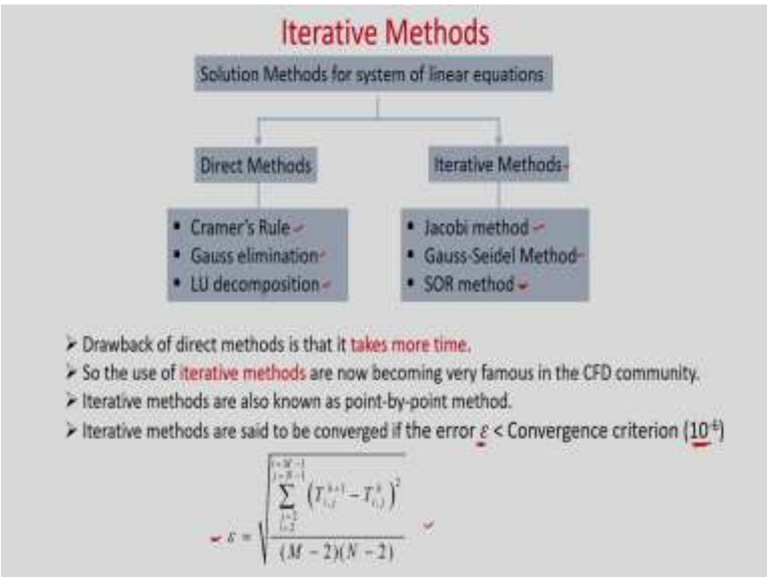
So, if you rearrange it so, you will get in the left hand side we have written all unknown terms you see n + 1 and in right hand side $T_{i,j}^{n}$ is known and the source term is also known from the known heat generation per unit volume term.

So, this you can see this you if you write the system of equations you will get a printer diagonal matrix. So, that you need to solve using some method and the advantage of this implicit method is that it is unconditionally stable. If you do the 1-dimension stability analysis you will get it is unconditionally stable.

So, what is the order of accuracy of this discretization? So, the order of accuracy for this explicit method and implicit method is order of ΔT because it is first order accurate in time and second order accurate in space, $(\Delta x)^2$ and $(\Delta y)^2$. Because you can see this is order of $(\Delta x)^2$, this is the order of $(\Delta y)^2$ and this is the order of ΔT. So, this is the order of accuracy of this method.

Using the finite difference discretization method we have converted the partial differential equation to system of algebraic equations. So, that, you need to solve at each discrete point.

(Refer Slide Time: 33:24)



So, there are many iterative methods are there. So, one is direct methods are very time consuming. So, direct methods like Cramer's rule, Gauss elimination or LU decomposition. So, generally these are not used in CFD. In iterative methods you have Jacobi method, Gauss-Seidel method and successive over election method. So, these are mostly point by point method.

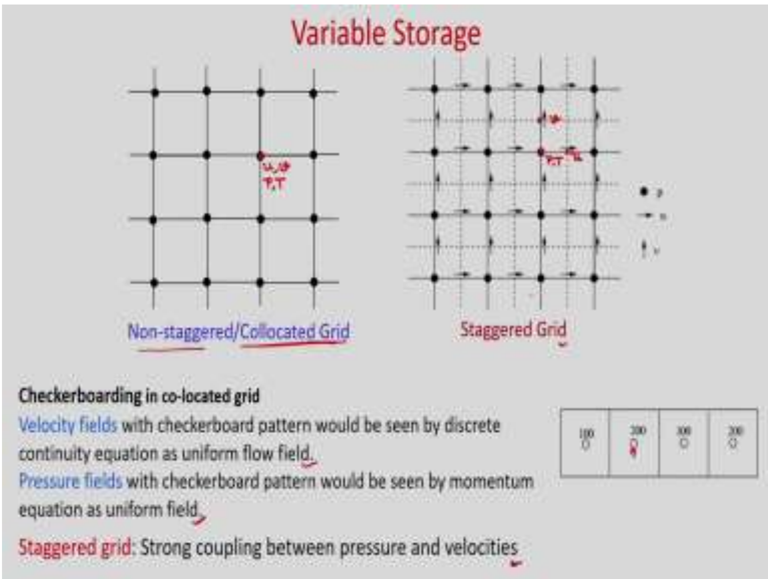
And, also nowadays you can use some advance iterative method like conjugate gradient or by conjugate gradient method, and using these iterative methods you need to solve those algebraic equations. So, you can see iterative methods are said to be converged if the error epsilon is less than the convergence criteria.

So, you have to set the convergence criteria let us say 10^{-6} or 10^{-5} and this ε you need to calculate from the value of the discrete points. So, at each point if you have, $\varepsilon = \sqrt{\dfrac{\sum_{\substack{i=2 \\ j=2}}^{\substack{i=M-1 \\ j=N-1}} \left(T_{i,j}^{k+1} - T_{i,j}^{k}\right)^2}{(M-2)(N-2)}}$. Then, you find the ε if it decreases with time and when it will go below the given convergence criteria let us say 10^{-6}, then your iteration will stop.

Now, we will discuss about the variable storage, say when you are solving Navier–Stokes equations and the energy equation how many variables you have? You have velocity u, v, w in 3-dimensions and temperature and you have pressure p because pressure is appearing in momentum equations.

So, obviously, you have total five unknowns; three velocities, pressure and temperature. So, where will you store or where will you solve these variables? So, depending on the storage variable storage we can have staggered grid and collocated grid.

(Refer Slide Time: 35:53)



So, first let us discuss about the collocated grid or non-staggered grid. So, you can see you can divide the domain into discrete points. So, that is known as grid and these are the grid points and in each grid point if you solve all the variables in two-dimension you solve u, v, pressure, temperature or any other species like mass fraction.

So, if you solve at same grid point then this is known as collocated grid. So, all the variables you are solving at the same point. But, it is having some disadvantage. So, that is known as pressure velocity decoupling. So, when you are solving the Navier–Stokes equation and energy equation, you will get pressure velocity decoupling; that means, pressure and velocity will not talk each other.

So, if you see here the let us say this is the pressure distribution. So, you have discrete points and at discrete points you have the pressure distribution 100, 200, 100 and 200. So, for some physical situation it may arise that your pressure is varying like this. Now, at this point if you want to find the pressure gradient using collocated grid for the Navier–Stoke equation, then you will find $\frac{\partial p}{\partial x}$.

So, $\frac{\partial p}{\partial x}$ is the if you central difference you can see $\frac{100-100}{2\Delta x}$. So, it will become 0. So, you can see pressure gradient is 0, but pressure gradient is the driving force for the velocity, but here pressure gradient is becoming 0. So, that is known as pressure velocity decoupling.

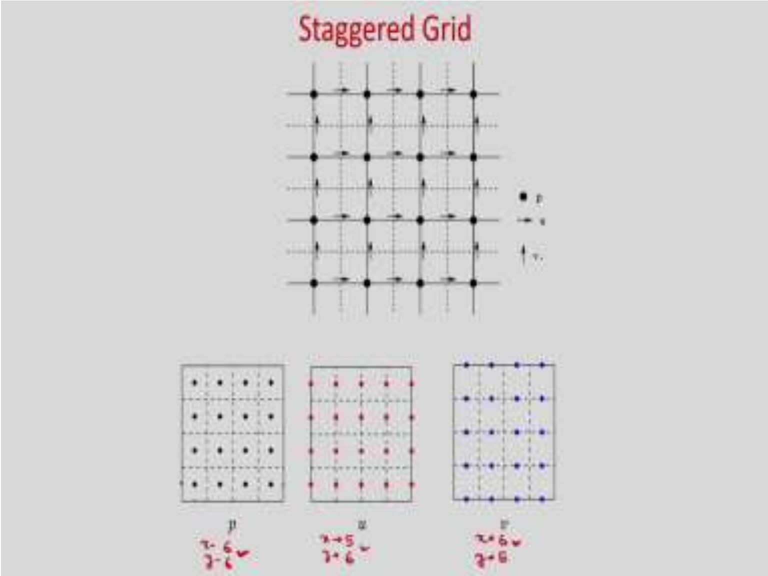
So, you can see for a checkerboard kind of pressure distribution or velocity distribution in collocated grid velocity fields with checkerboard pattern would be seen by discrete continuity equation as uniform flow field and pressure fields with checkerboard pattern would be seen by momentum equation as uniform field because it seems like a uniform field so that you do not have the pressure gradient. But, you have pressure gradient, but it is varying.

So, similarly continuity equation you see if we apply $\frac{\partial u}{\partial x}$ here it will become 0. So, it will field that it is a you have a uniform flow field, but you have a velocity variation. So, that is the disadvantage of collocated grid, but you have some way to overcome this problem and you can use momentum interpolation like Eddie and Joe proposed the moment of interpolation. So, that, you can use to overcome these velocity pressure decoupling problem.

Another grid is staggered grid. So, in the staggered grid the velocities are solved in staggered grid point ok. So, you can see here. So, this is the grid. So, if you see this is the grid point. In staggered grid, pressure and temperature or any other scalar like mass fraction these are solved at this main grid point, but velocities are solved in a staggered point. So, here velocity u is solved and here velocity v is solved.

So, this is the velocity u and this is the velocity v and at these point pressure is solved and any other scalar like temperature are solved. So, you can see velocities are solved in staggered way that is why it is not it is known as staggered grid and these pressure velocity decoupling is avoided using these staggered grid. And, that is why in staggered grid you will get strong coupling between pressure and velocities.

(Refer Slide Time: 40:05)



So, you can see in the staggered grid you have pressure at these black dot velocity u in this red dot and v velocity you will have in these blue dots. But, here you can see you have some problem in book keeping. So, for pressure you can see you have interior points 1, 2, 3, 4 and if you consider the boundary points then here you have one boundary point, here another boundary points. So, you have five.

But, when you are considering u so, in x-direction how many points are there including the boundary points? 1, 2, 3, 4, 5 and in y direction you if you consider the boundary points so, 1, 2, 3, 4, 5, 6. If you consider v in x-direction if you consider the boundary point 1, 2, 3, 4, 5, 6, but in y-direction 1, 2, 3, 4, 5 and for pressure in x-direction 1, 2, 3, 4, 5, 6 and in y-direction 1, 2, 3, 4, 5, 6including the boundary points.

So, you can see you have total number of points for solving the velocities u v are different in x and y direction and it is also different than the p. So, this book keeping is the difficult in staggered grid, but you need to be careful while solving these velocities and pressure and accordingly, you need to show for the interior grid points.

So, that you need to be very careful while solving the equations using staggered grid. But, the main advantage of staggered grid that you can avoid this pressure velocity decoupling and obviously, you will get a strong coupling between pressure and velocity.

So, in today's class we have introduced the partial differential equation and we have discussed about different classification of partial differential equation; physical classification and mathematical classification. In mathematical classification, we have elliptic equations, parabolic equations and hyperbolic equations and in physical classification you have equilibrium problems and marching problems.

So, in equilibrium problems are mostly governed by elliptic equations and marching problems are generally governed by parabolic and the hyperbolic equations. Then we have introduced the finite difference approximation using the Taylor series expansion we have used a forward point and backward point and we have written the expression for the first derivative and also using central difference we have written the approximation of the first derivative and second derivative.

So, you can see the forward difference approximation and backward difference approximation the order of accuracy is 1. So, that is the order of Δx, but when you use central difference method then order of accuracy is 2 because order of $(\Delta x)^2$. Then, we discretize the partial differential equation of steady state heat conduction equation and unsteady heat conduction equation.

So, after discretizing you have seen that you will get system of algebraic equations and you need to use suitable iterative method to solve the system of algebraic equations, because at each grid point if you write that equation then you will get a system of algebraic equations.

And, for explicit method you will have only one unknown, but Δ T has the restriction due to the stability criteria, but in most of the time in unsteadies two-dimensional heat conduction equation, we have seen that this implicit method is unconditionally stable.

Then we have discussed about the variable storage, depending on that we have classified the grid as staggered grid and collocated grid. In collocated grid all the variables are stored at the same point, but it is having the problem of velocity and pressure decoupling. To avoid that problem you can use staggered grid where pressure and any scalar like temperature and species is stored at the main grid point and velocities u, v are stored in a staggered manner. So, in the staggered grid you will get a strong coupling between pressure and velocity.

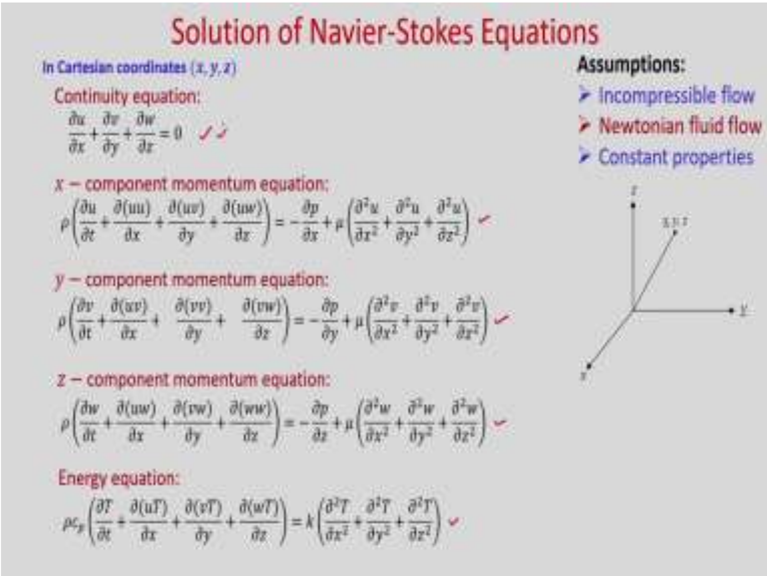
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 10
Numerical Solution of Navier-Stokes and Energy Equations
Lecture – 34
Solution of Navier-Stokes equations

Hello everyone, today we will consider unsteady three dimensional Navier-Stoke equations for laminar incompressible flows. And we will discretize this equation using finite difference method. In last class you have learnt how to discretize the first derivative and second derivative of any variable, we will use a famous technique Marker and Cell proposed by Harlow and Welch to discretize this Navier-Stoke equations using finite difference method.

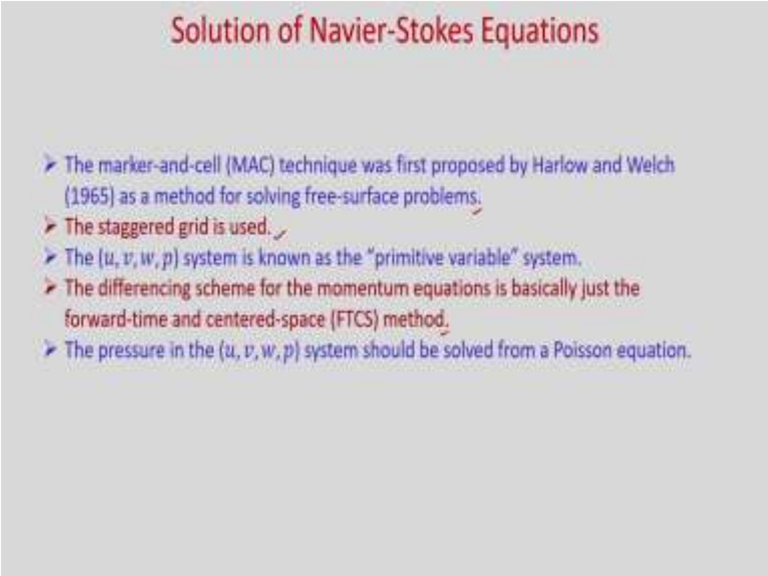
(Refer Slide Time: 01:05)



So, let us consider these equations in Cartesian coordinate. So, for incompressible Newtonian fluid flow with constant properties, this is the continuity equation. This is the x component of momentum equation, and this is y component of momentum equation, and this is the z component of momentum equation, and this is the energy equation. In today's class we will just discretize the Navier-Stoke equations and you can see how many variables are there? u, v, w, p,. And if you solve for energy then will be temperature.

So; obviously, you can see we have u, v, w for the Navier-Stoke equations and continuity equation. So, 4 equations and 4 unknowns u, v, w, and p. So, to find the pressure, there are no obvious equation we need to derive equation per to determine the pressure from the continuity equation.

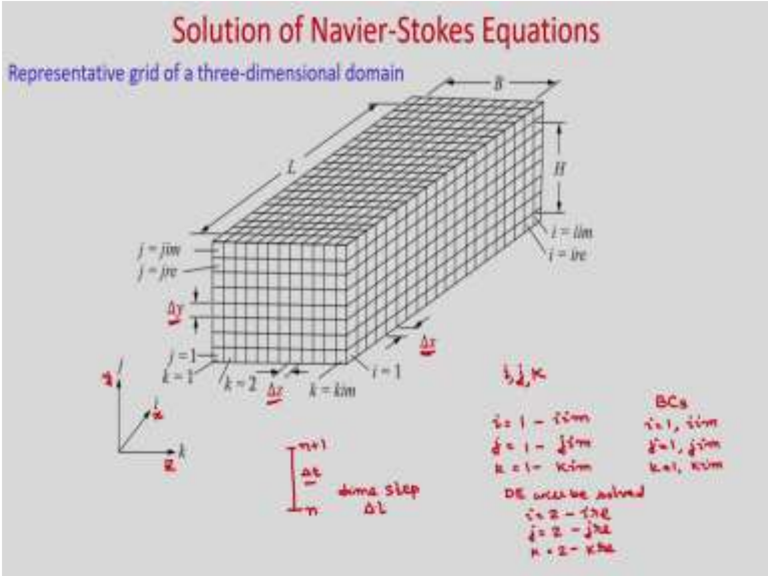
(Refer Slide Time: 02:07)



We will use the marker and cell technique first proposed by Harlow and Welch, for solving free-surface problems. And as we discussed in last class we will use staggered grid; you know that in staggered grid pressured and temperature will solve at cell center and velocities will solve in a staggered way, what is the advantage of using staggered grid? Because we will get a strong coupling between pressure and velocity.

The differencing scheme for the momentum equations is basically just the forward time and centered space FTCS method. And if you solve the system using u, v, w, p and temperature, then these system known as primitive variable approach and here we will solve equation for pressure separately.

(Refer Slide Time: 03:06)



So, you know that first we need to discretize the domain into grids so, that we can solve the discretized equation in a discrete point. So, if you consider this three-dimensional domain. And you can see this is divided into grid x direction is in these direction, this is the y direction, and this is the z direction and the grid we are giving the index with i, j, and k. So, you can see this is your j = 1 and it is varying in this direction and it is j = j i m and previous cell is j r e.

Similarly, k = 1, this is k = 2 the last cell is k i m and here we will apply the boundary condition at the last cell and previous cell is your k r e. Similarly, in x direction this is the i = 1, i = 2 and this is the i = i r e and this is the i i m.

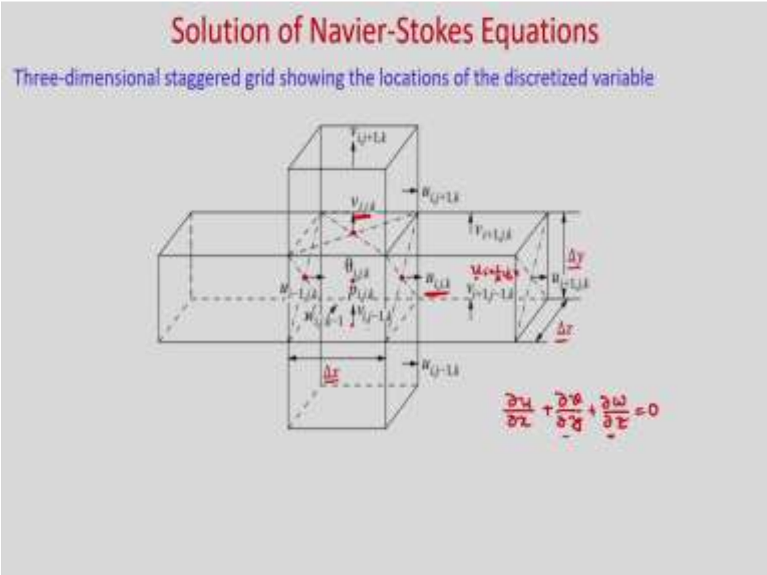
So, you can see your grid is varying i_1 to i i m, j_1 to j i m and k_1 to k i m. And boundary conditions will be applied at i = 1 and i i m and j = 1 and j i m, k = 1 and k i m. And the discretized equation we will solve in the interior domain; discretized equation will be solved and i = 2 to i r e, j = 2 to j r e, and k = 2 to k r e.

And you can see that in x direction we have uniform spacing of Δx, and in y direction we have uniform spacing with Δy, and in z direction we have uniform spacing Δz. So, grid size you can see this is Δx, Δ y and Δz. So, this is the discretization of this domain.

Now, we are considering unsteady equation. So, we need to march in time so, we will march from n to n + 1, where n is the previous time, and n + 1 is the current time or present time.

So, the variables at previous time n are already known from the previous solution and n+1 we need to determine. So, in MAC algorithm we use explicit method; that means, there will be only one unknown and all other terms will be known from the previous time n. When we will go in time marching so, we will go from n to n +1 and your time step will be Δt. So, here grid size is Δx, Δy, Δ z and your time step is Δt.

(Refer Slide Time: 06:49)



So, now let us see the cell, where we need to discretize the continuity equation as well as the energy equation. And the staggered grid where we need to solve the velocities. So, we can see this is the main cell, where at center you will solve for P and this cell is i j k index is i j k, and we need to solve for temperature, so, $t_{i,j,k}$ or $\theta_{i,j,k}$, where θ is non-dimensional temperature.

And velocities now you see we will solve in staggered way so, this is the point where we need to solve for the velocity $u_{i,j,k}$, and this is the staggered grid, where we need to solve for velocity $v_{i,j,k}$, and you can see in other direction we need to solve for $w_{i,j,k}$.

So, in that direction in inside you will solve for $w_{i,j,k}$ and this is the grid for $w_{i,j,k-1}$. And now if it is $u_{i,j,k}$ so; obviously, this will be your $u_{i-1,j,k}$ and if it is $v_{i,j,k}$ so, here you will get $v_{i,j-1,k}$. And in the other face the rear side you will solve for $w_{i,j,k}$ and this is the $w_{i,j,k-1}$.

So, we can see when we are solving for $u_{i,j,k}$. So, its neighbors are $u_{i-1,j,k}$ and $u_{i+1,j,k}$ and for $v_{i,j,k}$ this is the $v_{i,j+1,k}$ and this is $v_{i,j-1,k}$. And you can see now in this main cell we will satisfy the continuity equation. And when we satisfy the continuity equation so, velocities just difference will be just $\frac{\partial u}{\partial x}$ in x direction.

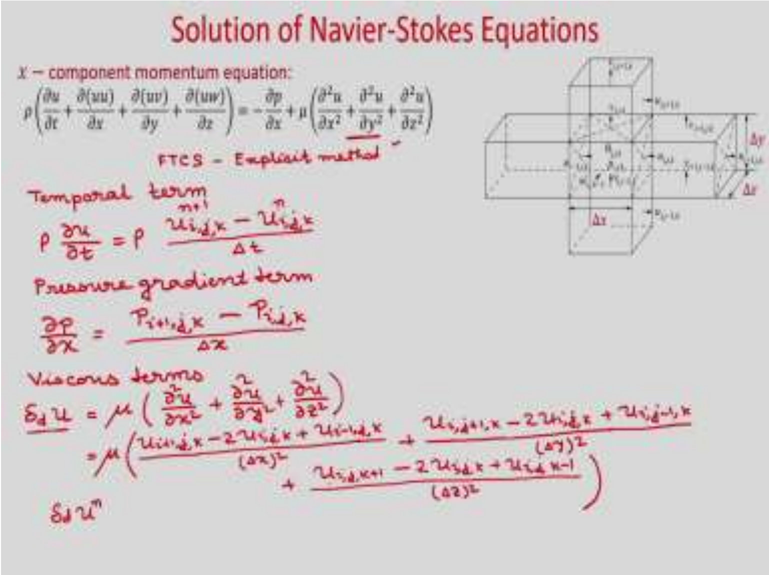
So, you can see your continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$. So, this equation we need to satisfy in this cell, where now $\frac{\partial u}{\partial x}$ you can see that it will be $\frac{u_{i,j,k}-u_{i-1,j,k}}{\Delta x}$, because that is the first derivative discretization.

Similar way you see this is $\frac{v_{i,j,k} - v_{i,j-1,k}}{\Delta y}$ so, you will get $\frac{\partial v}{\partial y}$. Similarly, you will can find $\frac{\partial w}{\partial z}$ and; obviously, you can see this is the grid size Δx this is your Δy and this is the Δz.

Here you notice that when you have $u_{i,j,k}$ and this is your $u_{i+1,j,k}$ and at the center. If you need to find the $u_{i,j,k}$ so, that will be $u_{i+\frac{1}{2},j,k}$ and this is unknown. So, this unknown will be just solved using the average velocity, using the neighbor points $u_{i,j,k}$ and $u_{i+1,j,k}$ and similarly for v and w.

So, we have now discussed about the staggered grid and how the variables are stored in the grid. Now, let us consider the x momentum equation and discretize term by term using finite difference method.

(Refer Slide Time: 10:41)



So, this is the x component momentum equation. So, first let us discretize temporal term so, we will use forward time central space and you know that it is explicit method. So, first let us discretize temporal term. So, what is temporal term? This is $\rho\frac{\partial u}{\partial t}$. So, now we will use forward time so, this is first order accurate scheme if we use then you will can write $u_{i,j,k}^{n+1}-u_{i,j,k}^{n}$. So, we are discretizing with respect to time. So, that is why we are writing $\rho\frac{\partial u}{\partial t}=\rho\frac{u_{i,j,k}^{n+1}-u_{i,j,k}^{n}}{\Delta t}$.

Next, let us discretize the pressure term so, pressure term if you discretize or pressure gradient term. So, that is $\frac{\partial p}{\partial x}=\frac{P_{i+1,j,k}-P_{i,j,k}}{\Delta x}$. Now, let us consider the viscous term in the right hand side. So, if you consider that term viscous terms so, this is the viscous term.

So, we can write this will denote as $\delta_d u$ so, when you will write the discretize equation we will use these notation. So, this is $\delta_d u=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$. So, you can see this is the second derivative of u, with respect to x, y and z.

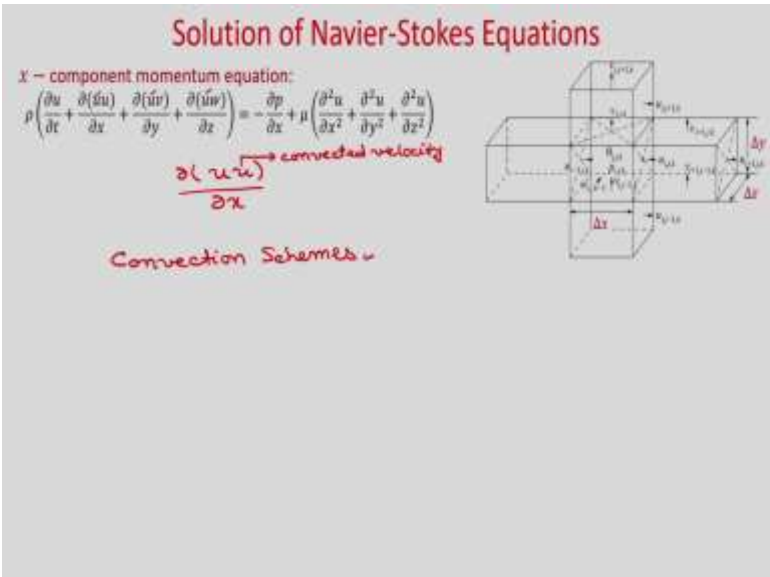
So, you know the central difference scheme. So, this is the second order accurate so, we can use the second order central difference approximation for this second derivative. So, you can write,

$$\delta_d u = \mu\left(\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2} + \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta z)^2}\right).$$

So, this is the discretized from of this equation, and when we will use the explicit scheme, we will write $\delta_d u^n$; that means, all these terms velocities will be from the nth time level. So, those will be known right.

Now, let us consider the convection terms. So, in the conversation terms you can see you have $\frac{\partial(uu)}{\partial x}$, similarly you have $\frac{\partial(uv)}{\partial y}$ and $\frac{\partial(uw)}{\partial z}$. So, here 2 velocities are coming. So, one velocity is convected velocity so, by the velocities u,v,w here u is convected right, because this is the x momentum equation so, your we are solving for the velocity u, but velocity u is convected by velocity u in x direction by velocity v in y direction and by velocity w in the z direction.

(Refer Slide Time: 16:15)

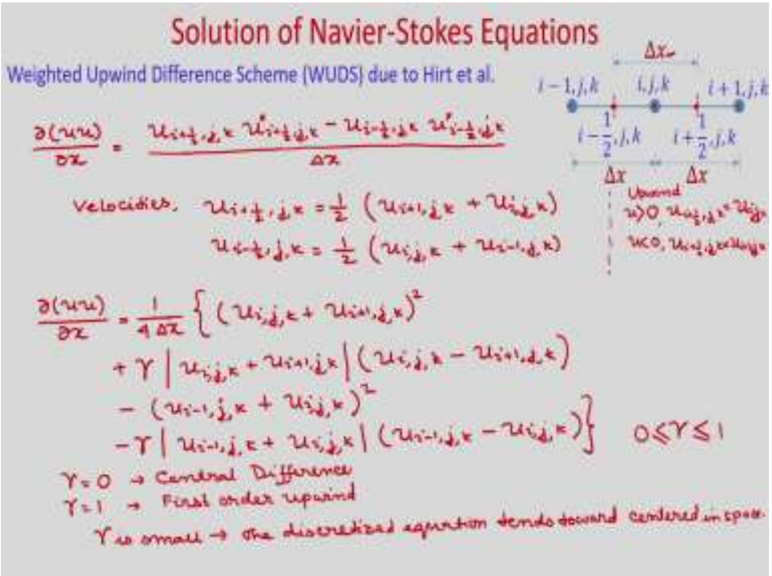


So, you can see this term, if you see $\frac{\partial(uu)}{\partial x}$ so, one is your convected velocity. So, here all these u are convected velocities and another u is just velocities, which actually convecting this variable convected velocity. So, if you use central difference then how you can use for this cell.

So, this is your $u_{i,j,k}$. So, this we will learn how to discretize this equation so, here we will use some convection schemes, because you have 2 velocities. So, one velocity we will calculate just at this phase centered, where it will be just average of the neighboring cells, but other velocity we will calculate using some convective schemes.

So, there are different convective schemes are available like first order accurate; first order upwind, second order upwind, third order upwind like quick even you have central difference, which is your second order accurate and you can have the combination of these within with some weighted function. So, here we will use convective scheme proposed by Hirt et. al.

(Refer Slide Time: 18:03)



So, this is the scheme we will use weighted upwind difference scheme, which is known as WUDS due to Hirt et. al. So, you can see we have $\frac{\partial(uu)}{\partial x}$. So, what we will see so, this is to be discretized for the velocity here i,j,k. So, we will use $\frac{\partial(uu)}{\partial x}=\frac{u_{i+\frac{1}{2},j,k}u^{*}_{i+\frac{1}{2},j,k}-u_{i-\frac{1}{2},j,k}u^{*}_{i-\frac{1}{2},j,k}}{\Delta x}$.

So, here you see here at this point we are discretizing this u u term and we are discretizing so, you can see this is the midpoint between i, j, k and i +1, j k. So, this point we are considering and this point, which is your $i-\frac{1}{2}, j, k$. So, you can see so, the distance between these two points is Δx , but here we are velocity the velocities we are writing $u_{i+1,j,k}$ and another is stared quantity u*.

So, for the u * quantity we will use some convection scheme and for $u_{i+1,j,k}$ we will just take the average of velocities at i, j, k and i + 1 j, k. So, you can see in this case. So, these velocities $u_{i+\frac{1}{2},j,k}$ just we will use the average value, at this point taking the value from i, j, k and i +1 j, k.

So, you can write $u_{i+\frac{1}{2},j,k} = \frac{1}{2}\left(u_{i+1,j,k} + u_{i,j,k}\right)$. Similarly, $u_{i-\frac{1}{2},j,k} = \frac{1}{2}\left(u_{i,j,k} + u_{i-1,j,k}\right)$. So, you can see it is just average value at this point we have considered.

Now, this convected variable u*, we will use some convection scheme and we are going to use this weighted upwind difference scheme. So, you can write this equation like $\frac{\partial(uu)}{\partial x} = \frac{1}{4\Delta x}\left\{\begin{array}{l}\left(u_{i,j,k}+u_{i+1,j,k}\right)^2 + \gamma\left|u_{i,j,k}+u_{i+1,j,k}\right|\left(u_{i,j,k}-u_{i+1,j,k}\right) - \left(u_{i-1,j,k}+u_{i,j,k}\right)^2 \\ -\gamma\left|u_{i-1,j,k}+u_{i,j,k}\right|\left(u_{i-1,j,k}-u_{i,j,k}\right)\end{array}\right\}$.

So, you see this factor gamma it varies between 0 and 1. So, if you put γ=0 what you are going to get. So, if you put γ=0 so, this term and this term we will get 0. So, you are going to get actually central difference, because if you use central difference you will get this value for this convected variable.

So, if you use that one then it will be 2 and another 2 is there from this value. So, it will be $\frac{1}{4\Delta x}$. So, this square minus this square so, these values. So, for γ=0 . so, 0≤γ≤1.

So, if γ=0 you will get central difference method, this is your convection scheme. And if you put γ= 1, so, γ=1 you will get upwind scheme, first order upwind you will get what is upwind scheme let us discuss.

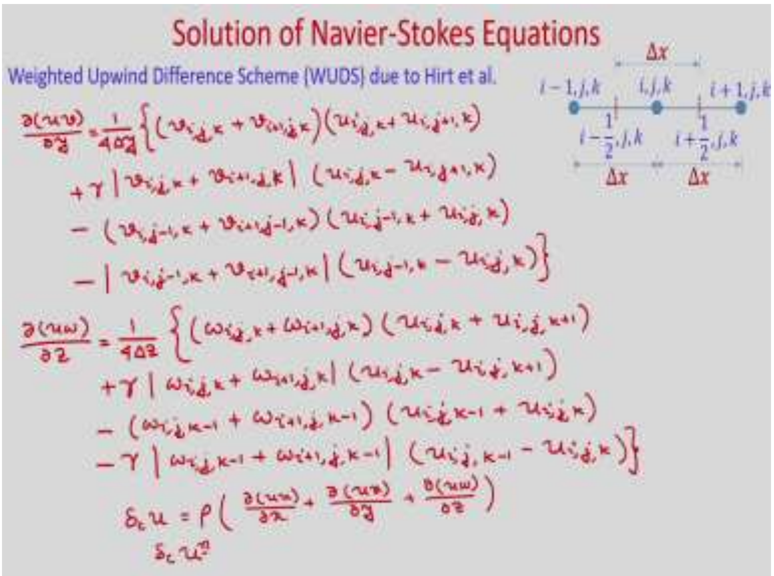
So, if you see this value at $i+\frac{1}{2}, j,k$ you want to find, let us say u . So, now if at this point if velocity is greater than 0 u > 0, then you consider the value of $u_{i+\frac{1}{2},j,k}$ as $u_{i,j,k}$ and if u < 0 then you consider $u_{i+\frac{1}{2},j,k}$; that means, if it is less than 0.

So, velocity is coming from right to left. So, you take the value of u at this point as same as this point $u_{i+1,j,k}$; so, it will be $u_{i+1,j,k}$. So, this is known as upwind. So, we are taking the upwind point from the direction in which the velocities are coming.

So, you can see u > 0 so, $u_{i+\frac{1}{2},j,k}$ you are taking $u_{i,j,k}$ value and if it is u < 0 so, you are considering this value. So, similarly here if u > 0 these value you consider as these value $u_{i-1,j,k}$. And if it u < 0, then you consider u at this point as $u_{i,j,k}$.

So, this is known as upwind and if you put γ=1 you can see that you will get first order upwind and if γ is small then; obviously, you can see it will be towards central difference. So, the discretized equation tends to centered in space. Now, similarly you discretize the other convection terms, $\frac{\partial(uv)}{\partial y}$ and $\frac{\partial(uw)}{\partial z}$.

(Refer Slide Time: 26:24)



So, similarly you can write $\frac{\partial(uv)}{\partial y}=\frac{1}{4\Delta y}\begin{Bmatrix}\left(v_{i,j,k}+v_{i+1,j,k}\right)\left(u_{i,j,k}+u_{i,j+1,k}\right)+\\ \gamma\left|v_{i,j,k}+v_{i+1,j,k}\right|\left(u_{i,j,k}-u_{i,j+1,k}\right)\\ -\left(v_{i,j-1,k}+v_{i+1,j-1,k}\right)\left(u_{i,j-1,k}+u_{i,j,k}\right)\\ -\gamma\left|v_{i,j-1,k}+v_{i+1,j-1,k}\right|\left(u_{i,j-1,k}-u_{i,j,k}\right)\end{Bmatrix}$. And this is the modulus you know that if these value is ≥ 0 then it will return this value, otherwise if it is < 0, then it will return the negative value.

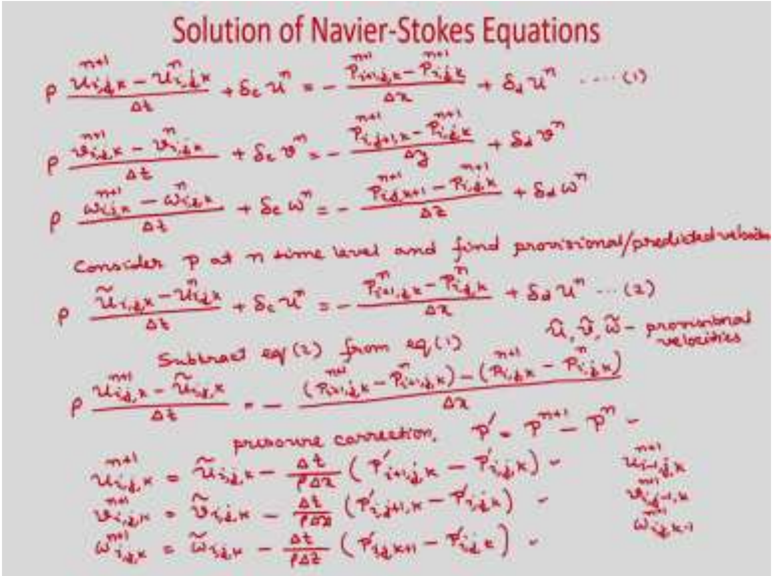
Similarly, you can write $\frac{\partial(uw)}{\partial z}=\frac{1}{4\Delta z}\begin{Bmatrix}\left(w_{i,j,k}+w_{i+1,j,k}\right)\left(u_{i,j,k}+u_{i,j,k+1}\right)+\\ \gamma\left|w_{i,j,k}+w_{i+1,j,k}\right|\left(u_{i,j,k}-u_{i,j,k+1}\right)\\ -\left(w_{i,j,k-1}+w_{i+1,j,k-1}\right)\left(u_{i,j,k-1}+u_{i,j,k}\right)\\ -\gamma\left|w_{i,j,k-1}+w_{i+1,j,k-1}\right|\left(u_{i,j,k-1}-u_{i,j,k}\right)\end{Bmatrix}$.

So, you can see that using weighted upwind difference scheme, we have discretized the convection terms, we can write considering all the convection terms as, $\delta_c u=\rho\left(\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\right)$.

And all these discretized equation we have already written so, that you can write and if you write $\delta_c u^n$; that means, we will consider u from the previous time level n, and in this discretized equation you can see whatever velocities are coming you consider from the known values at previous time level.

So, now we have already discretized the temporal term, convection terms, pressure gradient term, and the viscous terms. So, if you write down for the x component of momentum equations, this discretized equation then you can write like.

(Refer Slide Time: 31:18)



So, we are using forward time so, it is explicit method. So, this is the temporal term, now we have convection term, at time level n.

So, we have already discretized this term and we have $\rho \dfrac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n}}{\Delta t} + \delta_c u^n = -\dfrac{P_{i+1,j,k}^{n+1} - P_{i,j,k}^{n+1}}{\Delta x} + \delta_d u^n$.

Similarly, you can write the discretized equation for y component and z component of momentum equation. So, we can write, $\rho \dfrac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta t} + \delta_c v^n = -\dfrac{P_{i,j+1,k}^{n+1} - P_{i,j,k}^{n+1}}{\Delta y} + \delta_d v^n$.

And for z component of momentum equation you can write, $\rho \dfrac{w_{i,j,k}^{n+1} - w_{i,j,k}^{n}}{\Delta t} + \delta_c w^n = -\dfrac{P_{i,j,k+1}^{n+1} - P_{i,j,k}^{n+1}}{\Delta z} + \delta_d w^n$. We need to solve these equations the pressure at current time level n +1 unknown, right.

So, you cannot solve this equation unless you know the value of pressure. So, what we will do we will assume the pressure, some value. So, with this gas pressure field; obviously, you will get the some provisional velocity. So, let us say that we are taking the pressure value from the previous time level n and whatever velocity we will get that is not the correct velocity field, but we will get some predicted or provisional velocity from the previous time level pressure values.

So, now consider p at n level n time level and find provisional or predicted velocity. So, for u momentum equation, if you write then you can write as ρ and whatever predicted velocity is there so, we will denote as, $\rho \frac{\tilde{u}_{i,j,k} - u^n_{i,j,k}}{\Delta t} + \delta_c u^n = -\frac{P^n_{i+1,j,k} - P^n_{i,j,k}}{\Delta x} + \delta_d u^n$.

So, let us say this is equation 1, and this is equation 2. Now, you subtract this equation 2 from equation 1. And you see that your $\tilde{u}, \tilde{v}, \tilde{w}$ if you write for other momentum equations so, these are provisional velocities .

So, if we subtract you see this, $\rho \frac{u^{n+1}_{i,j,k} - \tilde{u}_{i,j,k}}{\Delta t} = -\frac{\left(P^{n+1}_{i+1,j,k} - P^n_{i+1,j,k}\right) - \left(P^{n+1}_{i,j,k} - P^n_{i,j,k}\right)}{\Delta x}$.

So, finally, if you subtract equation 2 from the equation 1, you will get this expression. Now, let us tell that your corrected pressure so, the difference between the pressure at time level n plus 1 and time level n will denote as a pressure correction and this pressure correction you can denote as $P' = P^{n+1} - P^n$.

So, at i, j, k if you write then it will be just, $u^{n+1}_{i,j,k} = \tilde{u}_{i,j,k} - \frac{\Delta t}{\rho \Delta x}\left(P'_{i+1,j,k} - P'_{i,j,k}\right)$.

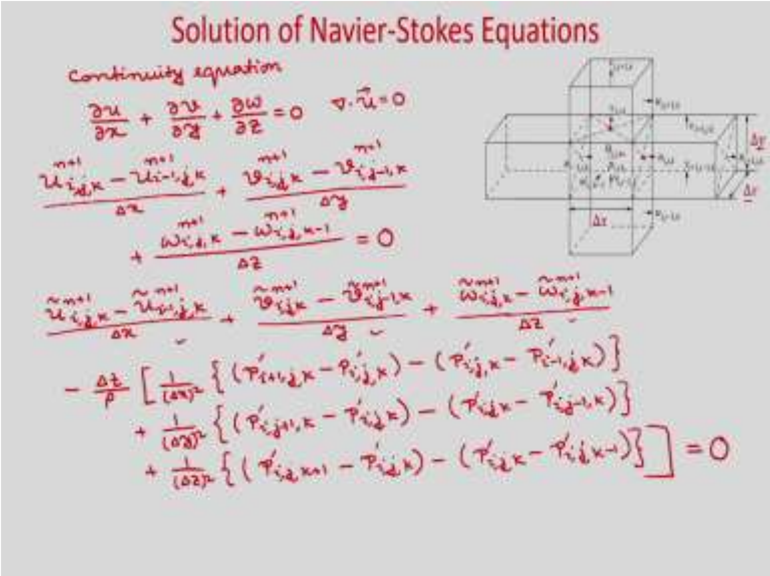
Now, similarly you can write this equation for velocity v and w, $v^{n+1}_{i,j,k} = \tilde{v}_{i,j,k} - \frac{\Delta t}{\rho \Delta y}\left(P'_{i,j+1,k} - P'_{i,j,k}\right)$. And similarly, $w^{n+1}_{i,j,k} = \tilde{w}_{i,j,k} - \frac{\Delta t}{\rho \Delta z}\left(P'_{i,j,k+1} - P'_{i,j,k}\right)$.

So, you can see that we have found the velocities at time level n +1 at point i, j, k in terms of the provisional velocity and the pressure correction term. So, you can see these $u_{i,j,k}$ you can write also in other grid point $u_{i-1,j,k}$ and $v_{i,j-1,k}$ and $w_{i,j-1,k}$.

Let us write the continuity equation and satisfy it in the main cell, and from there we will substitute these velocities and we will find the equation for pressure correction because that is unknown, right.

So, once we know the pressure correction value, then we can correct the pressure $P^{n+1} = P^n + P'$. So, we can correct the value of pressure.

(Refer Slide Time: 41:07)



So, now you can see the continuity equation, we will satisfy in this main cell, here . We will satisfy the continuity equation in this main cell. So, when you will satisfy when you write, $\frac{\partial u}{\partial x} = \frac{u_{i,j,k}^{n+1} - u_{i-1,j,k}^{n+1}}{\Delta x}$.

Similarly, $\frac{\partial v}{\partial y} = \frac{v_{i,j,k}^{n+1} - v_{i,j-1,k}^{n+1}}{\Delta y}$. And in z direction now $\frac{\partial w}{\partial z} = \frac{w_{i,j,k}^{n+1} - w_{i,j,k-1}^{n+1}}{\Delta z}$.

So, you can see here u velocity we have found at this point; at this point v and in the rear surface here w and similarly you have $u_{i-1,j,k}$ and $u_{i+1,j,k}$ here $v_{i,j+1,k}$ and $v_{i,j-1,k}$ and similarly $w_{i,j,k-1}$. So, now, satisfy this continuity equation at this main cell. So, your continuity equation is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} = 0$.

So, now, you write as, $\frac{u_{i,j,k}^{n+1} - u_{i-1,j,k}^{n+1}}{\Delta x} + \frac{v_{i,j,k}^{n+1} - v_{i,j-1,k}^{n+1}}{\Delta y} + \frac{w_{i,j,k}^{n+1} - w_{i,j,k-1}^{n+1}}{\Delta z} = 0$.

So, the value of u, v, w at n + 1, we have found from this relation you can see. So, the value of u, v, w at n + 1 you can write in terms of provisional velocity and the pressure correction. And similarly you can write for $u_{i-1,j,k}^{n+1}$, just you put here u_{i-1} and here, similarly you just change the pressure.

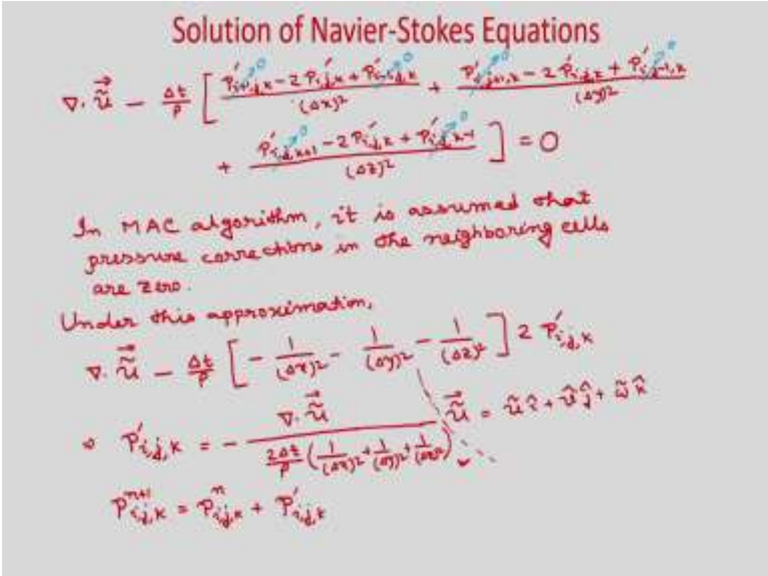
Similarly, $v_{i,j-1,k}^{n+1}$ you can find. And $w_{i,j,k-1}^{n+1}$. Similarly, you can write similar expression and if you; and if you substitute it here, $u_{i,j,k}$ and $u_{i-1,j,k}^{n+1}$, you are going to get.

So, in terms of provisional velocity now we are writing $\frac{\overset{n+1}{u}_{i,j,k}-\overset{n+1}{u}_{i-1,j,k}}{\Delta x}+\frac{\overset{\sim n+1}{v}_{i,j,k}-\overset{\sim n+1}{v}_{i,j-1,k}}{\Delta y}+\frac{\overset{n+1}{w}_{i,j,k}-\overset{n+1}{w}_{i,j,k-1}}{\Delta z}$ and you will have the pressure correction terms ok.

So, that will be $-\frac{\Delta t}{\rho}\left[\begin{array}{l}\frac{1}{(\Delta x)^2}\left\{\left(P'_{i+1,j,k}-P'_{i,j,k}\right)-\left(P'_{i,j,k}-P'_{i-1,j,k}\right)\right\}\\+\frac{1}{(\Delta y)^2}\left\{\left(P'_{i,j+1,k}-P'_{i,j,k}\right)-\left(P'_{i,j,k}-P'_{i,j-1,k}\right)\right\}\\+\frac{1}{(\Delta z)^2}\left\{\left(P'_{i,j,k+1}-P'_{i,j,k}\right)-\left(P'_{i,j,k}-P'_{i,j,k-1}\right)\right\}\end{array}\right]$.

So, you can see the first three terms, this term this term and this term this is actually the continuity equation for the velocity for the provisional velocity, but; obviously, when you are starting the solution provisional velocity will not satisfy the continuity equation. But when the solution will converge then u^{n+1} will be your $\overset{n+1}{u}$, v^{n+1} will be $\overset{\sim n+1}{v}$ and w^{n+1} will be $\overset{n+1}{w}$, because the provisional velocities will be same as the velocities at time level n +1 so, it will satisfy the continuity equation. So, you can write these three terms as a divergence form so, if you rewrite it.

(Refer Slide Time: 47:51)



And you can write after rearranging

$$\nabla.\vec{u}-\frac{\Delta t}{\rho}\left[\frac{P^{'}_{i+1,j,k}-2P^{'}_{i,j,k}+P^{'}_{i-1,j,k}}{(\Delta x)^2}+\frac{P^{'}_{i,j+1,k}-2P^{'}_{i,j,k}+P^{'}_{i,j-1,k}}{\frac{1}{(\Delta y)^2}}+\frac{P^{'}_{i,j,k+1}-2P^{'}_{i,j,k}+P^{'}_{i,j,k-1}}{(\Delta z)^2}\right]=0.$$

So, now in right hand side it will be 0 so, you can see here we will assume that pressure correction in the neighboring cells are 0. So, in MAC algorithm, it is assumed that the pressure correction in the neighboring cells are 0. So, under this approximation, so, you can see so, all these terms you can make it 0. So, this term is 0, this term because neighboring cell pressure correction we are just taking 0 for simple calculation and that is your MAC algorithm.

So, if you put 0 then you can write the equation for pressure. So, $\nabla.\vec{u}-\frac{\Delta t}{\rho}\left[-\frac{1}{(\Delta x)^2}-\frac{1}{(\Delta y)^2}-\frac{1}{(\Delta z)^2}\right]2P^{'}_{i,j,k}$.

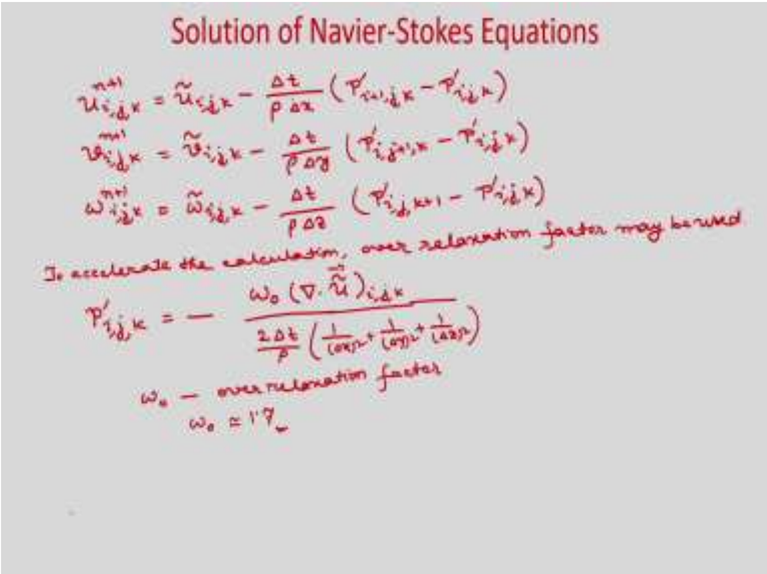
So, you can see now this is the equation for pressure. So, you will get the pressure value from this equation so, you can write $P'_{i,j,k}$. So, you can find it

so, $$P'_{i,j,k} = -\frac{\nabla.\vec{u}}{\frac{2\Delta t}{\rho}\left(\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta z)^2}\right)}.$$

So, you can see that we have already discretize the equations for the momentum and substituting those in the continuity equation in the main cell, we have found the equation for pressure using MAC algorithm; so, in the MAC algorithm we are neglecting or assuming the neighboring cell value of the pressure correction as 0 and following that you can write the equation for pressure as this.

So, once you find p prime then; obviously, you can correct it. So, you can correct $P^{n+1}_{i,j,k} = P^{n}_{i,j,k} + P'_{i,j,k}$. So, this if you solve then; obviously, you will be able to find.

(Refer Slide Time: 53:49)



The values for the velocity $u^{n+1}_{i,j,k}$, once you know the pressure correction value then you can find from the provisional velocity $u^{n+1}_{i,j,k} = u_{i,j,k} - \frac{\Delta t}{\rho \Delta x}\left(P'_{i+1,j,k} - P'_{i,j,k}\right)$ then $v^{n+1}_{i,j,k} = \tilde{v}_{i,j,k} - \frac{\Delta t}{\rho \Delta y}\left(P'_{i,j+1,k} - P'_{i,j,k}\right)$ and $w^{n+1}_{i,j,k} = w_{i,j,k} - \frac{\Delta t}{\rho \Delta z}\left(P'_{i,j,k+1} - P'_{i,j,k}\right)$.

So, once you know the pressure correction values and from the known provisional velocities, you will be able to calculate the velocities at n + 1 level in the pressure equation to accelerate the calculation $P'_{i,j,k}$ some over relaxation factor is used.

So, you can write $P'_{i,j,k} = -\dfrac{\omega_0 \left(\nabla.\vec{u}\right)_{i,j,k}}{\dfrac{2\Delta t}{\rho}\left(\dfrac{1}{(\Delta x)^2}+\dfrac{1}{(\Delta y)^2}+\dfrac{1}{(\Delta z)^2}\right)}$. So, here this ω_0 is known as over relaxation factor.

So, to accelerate the calculation, over relaxation factor may be used and ω_0 value generally of the; obviously, it will be > 1. So, you can write of the order of 1.5 or 1.6 in that range.

So, in today's class we have used finite difference method to discretize the Navier-Stoke equations using a MAC algorithm, we have actually used staggered grid to discretize these unsteady Navier-Stoke equations. We have used finite difference method and written the difference equation for first derivative and second derivative using Taylor series expansion.

And we have used forward time and central space which is your explicit method after discretizing the each terms temporal term, pressure gradient term, viscous term and the convection term, we have written the governing equations, but pressure at n +1 level is unknown so, we have assume the pressure from the previous time level and we have solved for the provisional velocities first.

Then once you know the provisional velocities then you substitute it in the continuity equation in the main cell. Once you substitute it you can get the pressure Poisson equation, but if you neglect the pressure correction of neighboring cells. Then you will get the equation for pressure.

Once you can find the equation for pressure correction, then you can find the pressure at n + 1 time level after correcting the values from the previous time level p value and you can find the velocities at n + 1 time level from the provisional velocities, and the pressure correction terms. Sometime, this pressure correction equation you can use the over relaxation factor to accelerate the solution and these over relaxation factor generally or commonly, it is used as 1.5 to 1.8.

Thank you.

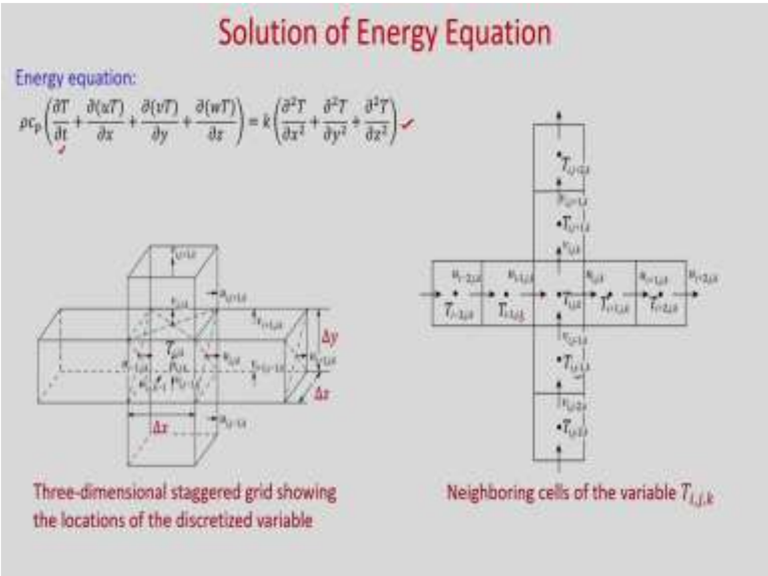
Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module - 10
Numerical Solution of Navier-Stokes and Energy Equations
Lecture – 35
Solution of energy equation

Hello everyone. So, in last class we discretized the Navier-Stokes equations using finite difference method, also we have written the equation for pressure correction from the continuity equation. Today we will discretize the energy equation using finite difference method and then we will discuss about the boundary conditions, then we will discuss about the solution algorithm. As you know that we are using staggered grid and MAC algorithm to discretize the equations.

So, in staggered grid we solve pressure and temperature in the main cell and the velocities in staggered way. So, you can see when we will discretize the energy equation in the main cell the velocities will be available from their main control volume respectively.

(Refer Slide Time: 01:41)

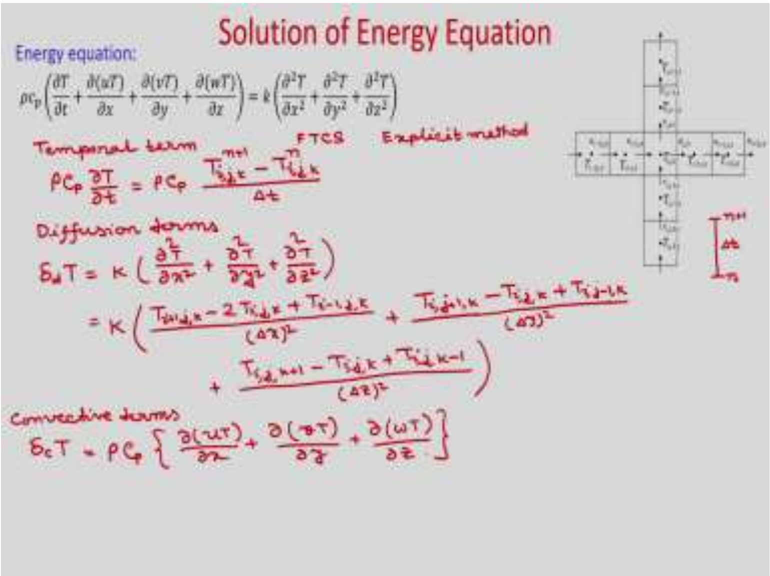


So, you can see this is the energy equation, this is the temporal term this is the convection terms and this is the diffusion terms. So, you can see temperature and pressure is solved in the main control volume.

So, when you will discretize this equation you can see that the velocities are available at this point because u we have already solved at this point and velocity v we have solved at this point and similarly for w velocity. If you see at a particular k then in two- dimension if you look then it will be easier to visualize. So, temperature we need to solve at $T_{i,j,k}$ at this main control volume and when you will use central difference you can use $\frac{T_{i+\frac{1}{2},j,k} - T_{i-1,j,k}}{\Delta x}$.

So, but the velocities at this point you can see these are available right. So, you do not need to interpolate the velocities at this point. So, this is your $u_{i,j,k}$ this is your $u_{i+1,j,k}$ and similarly for the temperature these are the neighbours $T_{i+1,j,k}$,$T_{i,j+1,k}$ $T_{i-1,j,k}$ and $T_{i,j-1,k}$ and also you will find in k direction similarly.

(Refer Slide Time: 03:13)



So, now, let us discuss the terms one by one first, let us consider the temporal term and we will use forward time central space and explicit method. So, temporal term we can discretize like this $\rho C_p \frac{\partial T}{\partial t}$. So, we are using forward time central space. So, this is explicit method and we will solve for only $T_{i,j,k}^{n+1}$ so; obviously, you know that when we are marching in time we are going from previous time n to the current time or present time at n + 1 and the time step size is Δt.

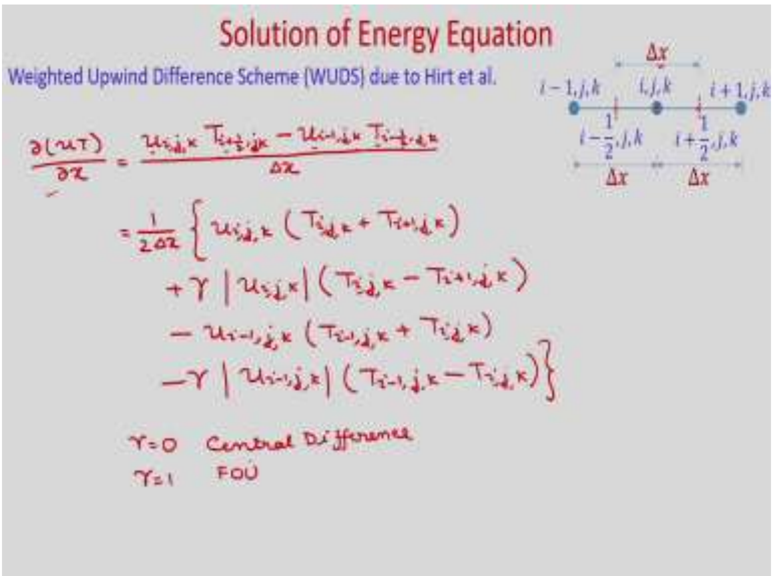
So, here you can write $\rho C_p \frac{\partial T}{\partial t} = \rho C_p \frac{T_{i,j,k}^{n+1} - T_{i,j,k}^{n}}{\Delta t}$. Now, let us consider the diffusion term, in diffusion term we have the second derivative of temperature. So, we will use central difference. So, diffusion terms. So, we can have $\delta_d T = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$. So, using central difference method if you discretize what you will get?

So, it will be,

$$\delta_d T = K\left(\frac{T_{i+1,j,k} - 2T_{i,j,k} + T_{i-1,j,k}}{(\Delta x)^2} + \frac{T_{i,j+1,k} - T_{i,j,k} + T_{i,j-1,k}}{(\Delta y)^2} + \frac{T_{i,j,k+1} - T_{i,j,k} + T_{i,j,k-1}}{(\Delta z)^2}\right).$$

So, when will you write $\delta_d T$ at time level n then all these temperature will be at time level n which is your previous time. Now when you write the convective term so, you can write $\delta_c T$ convective terms. So, $\delta_c T = \rho C_p \left\{\frac{\partial(uT)}{\partial x} + \frac{\partial(vT)}{\partial y} + \frac{\partial(wT)}{\partial z}\right\}$.

(Refer Slide Time: 06:52)



So, now let us take the term one by one and discretize using weighted upwind difference scheme. So, already we have discussed in last class in detail about this scheme, we will use to discretize the convective term in the energy equation. Now we have $\frac{\partial(uT)}{\partial x}$. So, simply you can see you can use central difference and discretize using this.

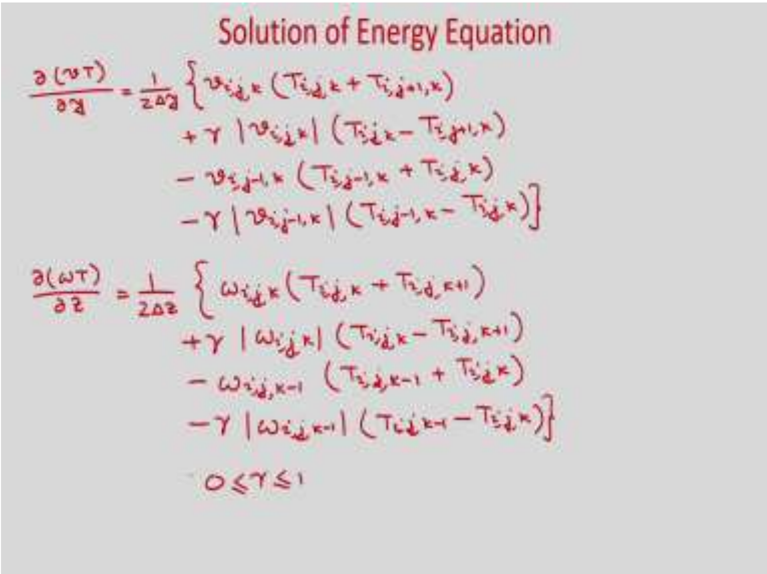
So, here you we are discretizing this term. So, we can write u. So, you can see at this point we have the value of u. So, $\frac{\partial(uT)}{\partial x} = \frac{u_{i,j,k} T_{i+\frac{1}{2},j,k} - u_{i-1,j,k} T_{i-\frac{1}{2},j,k}}{\Delta x}$ Now using this weighted upwind difference scheme you can write this.. So, if you use central difference then you will get like this, but as you are using weighted upwind difference scheme with a factor γ which varies between 0 and 1 and this is your for upwind.

So, if u > 0 it will return that positive value, if u < 0 then negative of that value then,

$$\frac{1}{2\Delta x}\left\{\begin{matrix} u_{i,j,k}\left(T_{i,j,k}+T_{i+1,j,k}\right)+\gamma\left|u_{i,j,k}\right|\left(T_{i,j,k}-T_{i+1,j,k}\right) \\ -u_{i-1,j,k}\left(T_{i-1,j,k}+T_{i,j,k}\right)-\gamma\left|u_{i-1,j,k}\right|\left(T_{i-1,j,k}-T_{i,j,k}\right)\end{matrix}\right\}$$

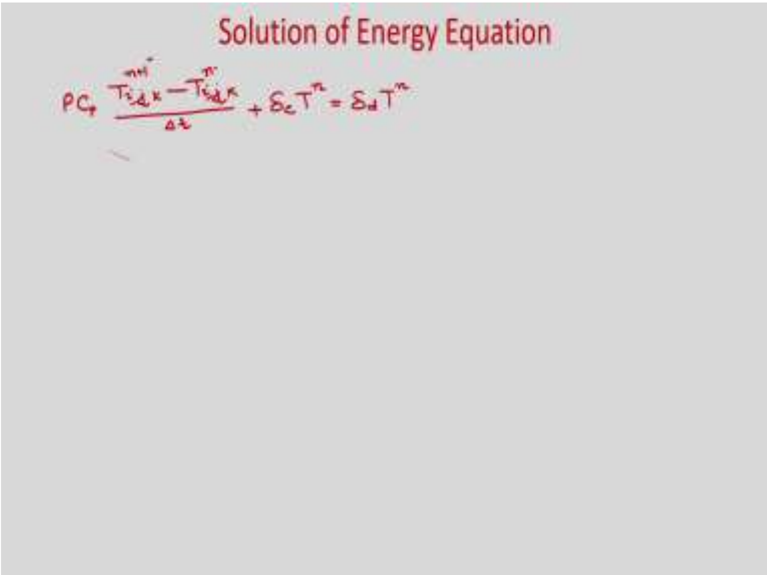
So, this is the discretization of this convective term using weighted upwind difference scheme and you know that if γ=0 then it is central difference and if γ=1 then upwind scheme.

(Refer Slide Time: 10:44)



Similarly, now let us discretize the term $\frac{\partial(vT)}{\partial y}$ and $\frac{\partial(wT)}{\partial z}$. So, you can write $\frac{\partial(vT)}{\partial y}=\frac{1}{2\Delta y}\left\{\begin{array}{l} v_{i,j,k}\left(T_{i,j,k}+T_{i,j+1,k}\right) \\ +\gamma\left|v_{i,j,k}\right|\left(T_{i,j,k}-T_{i,j+1,k}\right) \\ -v_{i,j-1,k}\left(T_{i,j-1,k}+T_{i,j,k}\right) \\ -\gamma\left|v_{i,j-1,k}\right|\left(T_{i,j-1,k}-T_{i,j,k}\right) \end{array}\right\}$. Similarly you can do the discretization of $\frac{\partial(wT)}{\partial z}=\frac{1}{2\Delta z}\left\{\begin{array}{l} w_{i,j,k}\left(T_{i,j,k}+T_{i,j,k+1}\right) \\ +\gamma\left|w_{i,j,k}\right|\left(T_{i,j,k}-T_{i,j,k+1}\right) \\ -w_{i,j,k-1}\left(T_{i,j,k-1}+T_{i,j,k}\right) \\ -\gamma\left|w_{i,j,k-1}\right|\left(T_{i,j,k-1}-T_{i,j,k}\right) \end{array}\right\}$, where γ varies between 0 and 1.

(Refer Slide Time: 13:17)



So, now, we have discretized the each terms. So, now, we can write the discretized form of the full energy equation temporal term we have discretized like this n + 1 and n then the convection term $\delta_c T$.

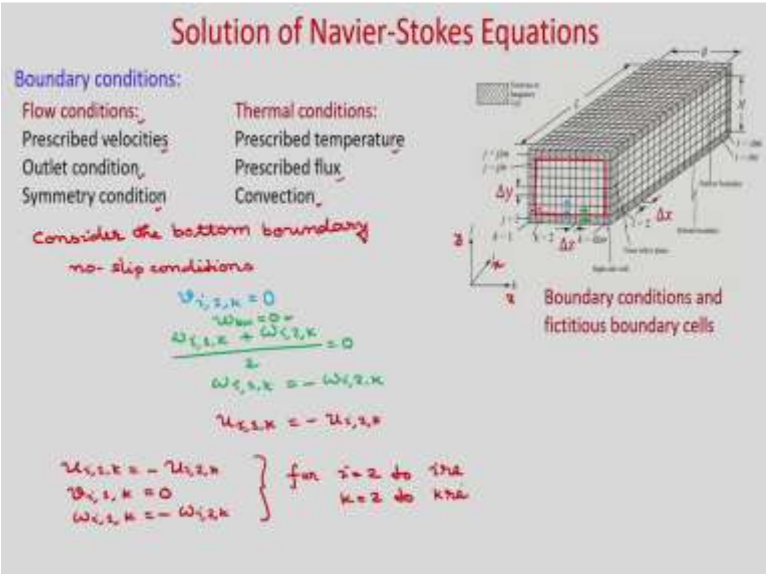
So, as you are using explicit method. So, it will be $\rho C_p \frac{T_{i,j,k}^{n+1}-T_{i,j,k}^{n}}{\Delta t}+\delta_c T^n=\delta_d T^n$. So, now, once you discretize all these terms you can put it here and you can find $T_{i,j,k}^{n+1}$

because this is the unknown all are known from the previous time level n. So, this you can find easily from this equation.

Now, if you have natural convection or heat generation in the energy equation so that will come as a source term in the governing equation. Say, if you have a buoyancy term so that is a source term you can put it in the x momentum equation, y momentum equation depending on the orientation of your problem. So, that source term you can put and you can solve the governing equation.

Similarly, in energy equation you if you have heat generation term per unit volume say q''' then you just add here q'''. So, that is that will come as a source term. So, I am not writing those equations, but easily you can put those source terms in these discretized equations. Now let us discuss about the thermal and flow boundary conditions.

(Refer Slide Time: 15:08)



So, you can see that in flow boundary conditions say at inlet we have prescribed velocities or on the solid wall we can have no slip boundary conditions means velocities will be 0. Outlet condition we can have where we can assume that fully developed condition and the gradient we can make as 0, symmetry condition also we can have.

And in thermal conditions we can have prescribed temperature or prescribed flux or we can have convection because from Newton's law of cooling at the wall your whatever heat is conducted that will be convected so that will be convection.

Now, whatever we have discussed for this geometry that i = 1, j = 1 and k = 1 all these are fictitious boundary cells. So, we are applying the boundary conditions through these fictitious boundary cells. So, whatever has cells are shown those are boundary cells and similarly i = i,i,m, j = j,i,m and k = k,i,m these are also boundary cells. So, we will use these fictitious boundary cells and we will apply the boundary conditions at the wall .

So, actually we have the boundaries here right. So, we have boundaries here. So, we have to apply the boundary conditions at wall and we will use these fictitious cells as well as your interior cells. So, we are solving the discretized equation inside this domain ok, but we need to apply the boundary conditions at the boundary fictitious cells, but at the boundary we will apply the boundary conditions and we will find the suitable values at the fictitious cells so that it will satisfy the boundary condition at the wall.

So, here you can see that i is in this direction; that means, that is your x direction, this is your y direction and this is your z direction for this geometry and if you think that it is a channel then at inlet where you have i = 1 there we can have a prescribed velocity.

And if you have a wall sidewalls then you can apply no slip boundary conditions, if you have a symmetry boundary conditions then you can apply symmetry boundary condition and at the outlet where i = i,i,m you can apply outflow boundary conditions. So, we will consider few boundary conditions flow boundary conditions and we will discuss or discretize that equations and we will find the value at the fictitious cells.

Now consider the bottom boundary consider the bottom boundary. So, if it is a bottom boundary you can see. So, j = 1 and i will vary and k will also vary. So, if you apply no slip condition, say no slip condition now what is no slip condition because the velocities will be 0 at the wall or the velocity of the fluid particle sitting on this wall will be the same will have the same velocity as the wall. So, that is your no slip conditions. So, if you see that u will be 0 at this particular point.

So, when we apply no slip condition; that means, velocities at the wall will be 0. So, now, you have u velocity, v velocity and w velocities will be 0 at the wall, but we are solving the velocities in staggered way. So, we have to find where we are solving the velocities u, v and w, from this figure you can see that your v velocity v velocity in j direction.

So, v velocity we are solving in staggered way right this is your v velocity. So, now, we can see v velocity is falling at the wall itself. And this is your v velocity in the interior points because we are solving in staggered way. So, this is your y direction. So, v velocity so, you can write $v_{i,j} = 1, k = 0$ because v is falling at the boundary itself.

Now if you see w velocity. So, you are solving in z direction. So, this is your z direction. So, for this you can see that this is your interior point w velocity, this is your u velocity at the fictitious cell. So, it is not falling at the boundary. So, what would happen?

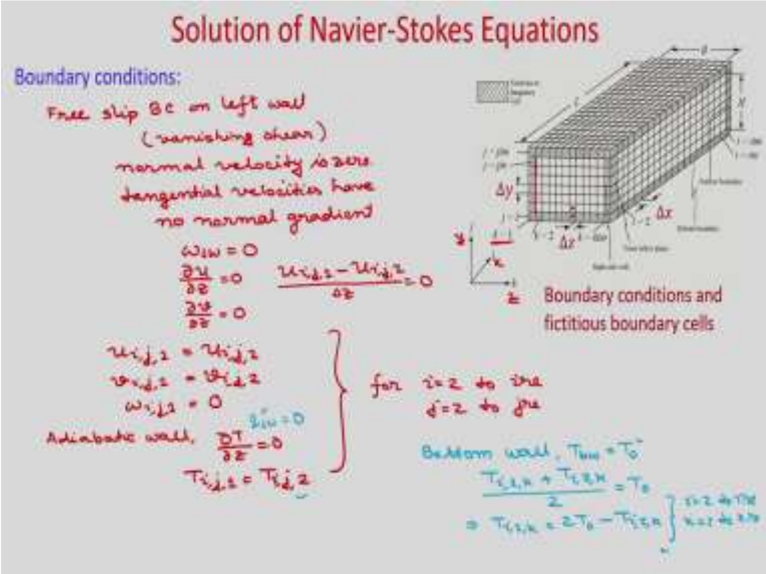
So, you have to take the average of these velocities such that the velocity at the wall will become 0 so; that means, you can write $\frac{w_{i,1,k} + w_{i,2,k}}{2} = 0$; that means, you are satisfying w at bottom wall is 0. So, at this wall you are putting the velocity 0, but you are solving w in staggered way. So, you are solving in these two points.

So, that is your average velocity you are taking 0. So, that $w_{i,1,k} = -w_{i,2,k}$. So, if you use this then essentially you will get w = 0 on the wall. Similarly, you will get for u velocity $u_{i,1,k} = -u_{i,2,k}$ because u velocity you are solving in staggered way. So, it will not fall on the wall ok. So, you have this boundary condition.

So, now from the boundary condition you can see for no slip condition on the bottom wall if you have applied then you will get $u_{i,1,k} = -u_{i,2,k}$, $v_{i,1,k} = 0$, $w_{i,1,k} = -w_{i,2,k}$. So, these you are varying the i and k. So, for i = 2 to ire and k = 2 to kre.

Now, another boundary condition let us take let us consider the left boundary as free slip boundary condition; that means, shear stress will be 0 on this wall.

(Refer Slide Time: 22:49)



So, for free slip boundary condition on left wall . So, this is your boundary this is your boundary on the left wall now you are having on the left wall you see k = 1 and j is varying and i is varying . So, at this wall if you use the free slip boundary condition; that means, vanishing shear. There will be no shear on this wall. So, normal velocity will be 0 is 0.

So, what is normal velocity at this wall, you see in k direction. So, this is the normal velocity; that means, w will be 0 on this wall and tangential velocities have no normal gradient tangential velocities have no normal gradient.

So, you can see for velocity u and v this normal gradient will be 0; that means, you have w_{lw} = 0 and you have $\frac{\partial u}{\partial z} = 0$ because that is your normal gradient in z direction, this is your x, this is your y and this is your z direction. So, and you have also $\frac{\partial v}{\partial z} = 0$.

So, now, if you satisfy these conditions on this wall then you can find the velocities at the fictitious cells in terms of the interior cells. So, if you do that you will get $u_{i,j,1} = u_{i,j,2}$.

So, you can see $\frac{\partial u}{\partial z}$ what you can do. So, if you discretize it will be $\frac{u_{i,j,1} - u_{i,j,2}}{\Delta z} = 0$.

So, distance is Δz = 0. So, if it is so, now, $u_{i,j,1} = u_{i,j,2}$. Similarly, $v_{i,j,1} = v_{i,j,2}$ and $w_{i,j,1} = 0$.

And if it is adiabatic wall so, normal gradient of temperature also will be 0. So, $\frac{\partial T}{\partial z} = 0$.

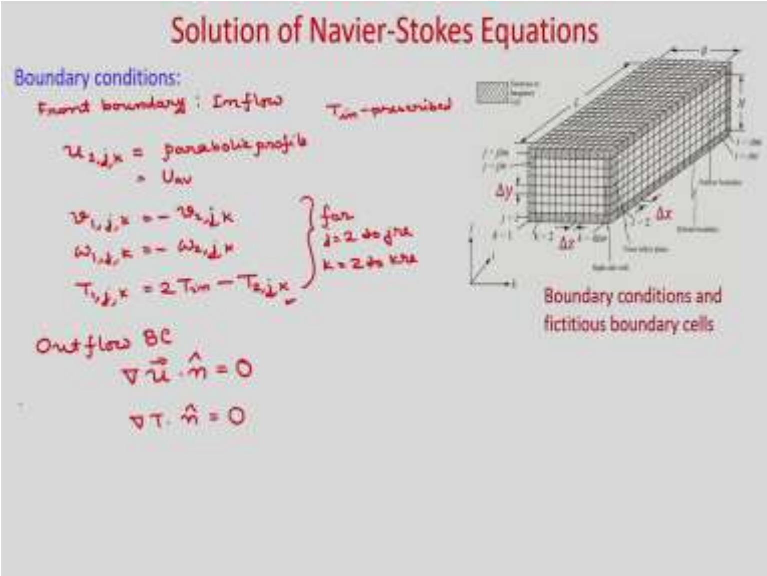
So, if it is so, then you can write $T_{i,j,1} = T_{i,j,2}$ because $\frac{\partial T}{\partial z}$ normal gradient is 0 adiabatic wall no heat flux across the wall. So, for this you can see you are varying for i = 2 to ire and j = 2 to jre and for the bottom wall if you consider that it is a constant wall temperature.

So, you can see here, you have temperature and this is your temperature. So, for a special you if you consider that bottom wall you have isothermal wall; that means, $T_{bw} = T_0$. So, now, you can see this is your interior point and this is your fictitious point, but you need to satisfy $T_{bw} = T_0$ at this point.

So, if you satisfy that then you can write, $\frac{T_{i,1,k} + T_{i,2,k}}{2} = T_0$. So, from here you can see $T_{i,1,k} = 2T_0 - T_{i,2,k}$ and this you need to vary i = 2 to ire and k = 2 to kre. So, this is for constant wall temperature boundary condition if you know the temperature on the wall T_0 and it is for adiabatic wall where q'' at left wall is 0 or left boundary.

So, it is symmetric condition also you can see that $\frac{\partial T}{\partial z} = 0$. So, this also will be valid. Now similarly for other boundary conditions also you can write say if you have an inlet. So, velocity inlet so, you can have constant velocity or you can have a parabolic profile. So, that you can write for this particular domain if at i = 1 you have a inflow condition then we can write as front boundary.

(Refer Slide Time: 28:41)



So, this is the front boundary so, you have inflow. So, if it is so, far front boundary in flow you can see it is i = 1 and j and k will vary. So, $u_{1,j,k}$ you can see $u_{1,j,k}$ will fall at the boundary itself. So, if it is so, then you can specify the parabolic profile for $u_{1,j,k}$ and if it is some constant value that also you can specify U_{av} let us say and for v velocity and w velocity it will be in staggered way.

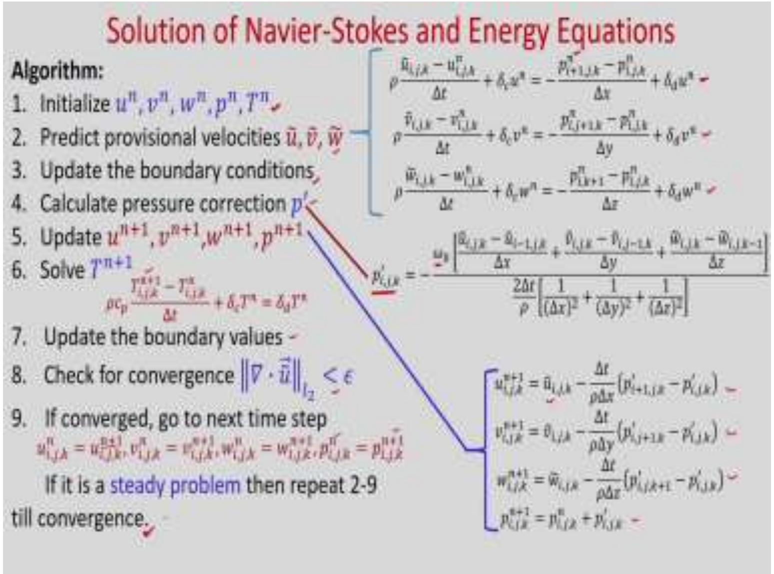
So, similarly way you can write $v_{1,j,k} = -v_{2,j,k}$ and $w_{1,j,k} = -w_{2,j,k}$ and for temperature. So, temperature if you do so, for temperature you will get $T_{1,j,k}$ let us say at the inflow your in $T_{infinity}$ is prescribed $T_{infinity}$ or let us say let us say at the inlet you have T_{in} is prescribed.

So, if it is so, you can write $T_{1,j,k} = 2T_{in} - T_{2,j,k}$ and here j will vary j = 2 to jre and k =2 to kre ok. So, prescribed temperature you have T_{in} at the inlet and you can find the T_{in} it is at the boundary. So, at the fictitious cell you will have the this temperature $T_{1,j,k} = 2T_{in} - T_{2,j,k}$ k.

Let us say that at i = iim which is your outlet. So, there you have outflow boundary condition. So, if you have outflow boundary condition. So, outflow boundary condition generally we say that normal gradients are 0. So, in general you can write n = 0 and accordingly you can write and discretize the gradient and write in terms of $T_{iim, j,k}$. So, you can just do as homework and for temperature also the normal gradient will be 0 ok. So, $\nabla T.\hat{n} = 0$ at the outflow boundary condition.

So, now we have the discretized equations for Navier-Stokes equations, we have pressure correction equations we have pressure correction equation and also we have discretized energy equation. So, now, let us discuss about the solution algorithm. So, we have already discretized these Navier-Stokes equations.

(Refer Slide Time: 32:13)



So, first you initialize the velocities at T = 0 and or previous time level you can find the values from the solution and first we initialize the values at T =0 then you predict the provisional velocities, you have seen that in the discretized equation the pressure at n +1 level is not available. So, we have used the previous pressure value previous time level pressure value and we have found the provisional velocities from these equations.

So, these are the equations taking the pressure value from the time level n ok. So, u tilde, v tilde and w tilde you find, then update the boundary conditions then you calculate the pressure equation. So, from this expression already we have derived from the continuity equation. So, you can see this is the equation. So, $P'_{i,j,k}$ you can find, omega naught is your overall relaxation factor and; obviously, this is the $\nabla, \vec{u}$.

So, this equation you can use to find P', once you know the $P'$and the provisional velocities you can update the velocities and pressure from these equations. So, you can see u tilde and the pressure corrections are known. So, you can use these equations to update the velocities and pressure you can update as $P^{n+1} = P^n + P'$, once you calculate this then you solve for the temperature because you can see in the temperature velocity is adequate.

So, these are the velocities at that time level. So, you can use those velocities and solve this equation to find T^{n+1}. Then again you update the boundary values then check for convergence, for the velocities you can check as well as for energy equation also similarly you can check, then if converged then go to the next time step. So, when you will go to the next time step you put these n plus 1 values at u n so that you can use these values.

So, all u v w your and for pressure you are taking from the p n value you are putting P^{n+1}. So, that it will be your initial in the initial values in the next time step. So, this way you will continue for the unsteady problem, but if it is a steady problem then repeat 2 to 9 till convergence.

For steady problem you can use a unsteady solver and you can merge in a pseudo tangent way and till convergence you can repeat 2 to 8 and you can find the converged solution for velocities and temperature, but if it is an unsteady problem then; obviously, you see that it will you have to go to the next time step and it depends on you that how much time you want to march.

If you have mixed convection problem or natural convection problem then; obviously, you can see temperature and velocities are coupled because when you will solve the momentum equations you have the temperature there and anyway in that energy equation you have the velocities so; obviously, you have to solve in a coupled way.

So, that is why you are solving the temperature after just after the velocity solution in each time step, but if it is only forced convection where both coupling is not there; that means, in when you solve the momentum equations you do not need the temperature then you first converged for a steady state problem, first you converge the velocities then separately you just solve energy equation. So, that is known as segregated method.

For forced convection problem you can use segregated method because after solution of the velocities separately you can solve the energy equation, but if it is buoyancy driven flow means natural or mixed convection then; obviously, you need to solve the energy equation along with the velocities because these are both way coupled.

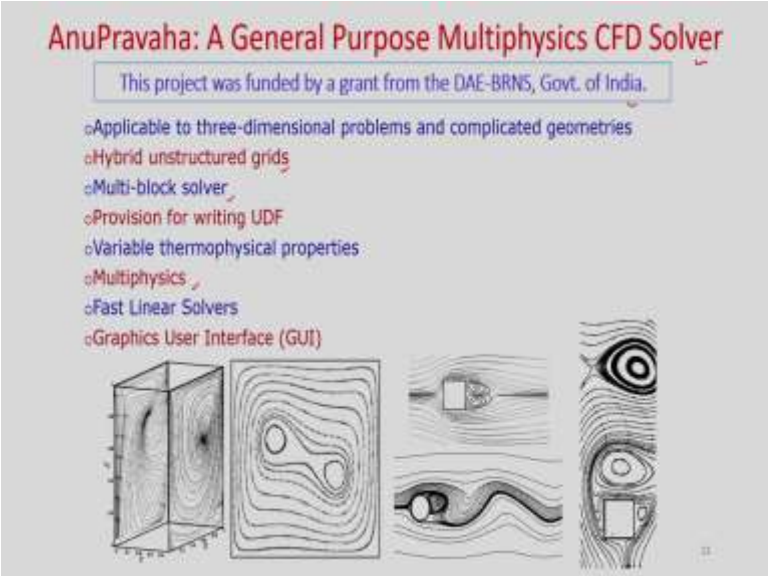
So, we have discretized the Navier-Stokes equations and energy equations using finite difference method and we have used MAC algorithm to solve these equations you can solve these equations numerically when you cannot have the analytical or exact solutions of the governing equations.

So, you see that if you consider a channel flow, in channel flow when it is developing both thermal and hydrodynamic boundary layers are developing then; obviously, it is very difficult to study analytically, but you can use numerical techniques and find the velocity distribution as well as you can find the heat transfer parameters like heat transfer coefficient and Nusselt number easily.

You can find local Nusselt number as well as average Nusselt number in the developing region and if you have a complicated geometry or where you cannot have a simplified form of the governing equations then you cannot have the analytical solutions. So, it is more convenient to use numerical techniques for these kind of problems.

Now, I will show some solutions of heat transfer problems which are solved using CFD using numerical techniques; however, the results whatever I will be showing we have used in house solver AnuPravaha which is actually finite volume based solver.

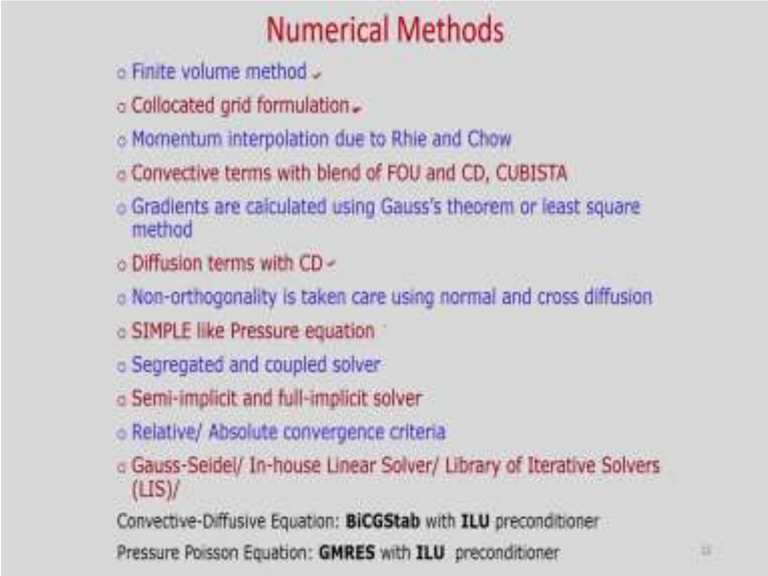
(Refer Slide Time: 38:38)



So, you can see AnuPravaha we have this in house solver which is a general purpose multi physics CFD solver and this project was funded by a grant from the DAE- BRNS, Government of India and we have these features it is applicable to three- dimensional problems and complicated geometry we have used unstructured grids, we have multi block solvers and it is multi physics ok. We can solve different kinds of problem.

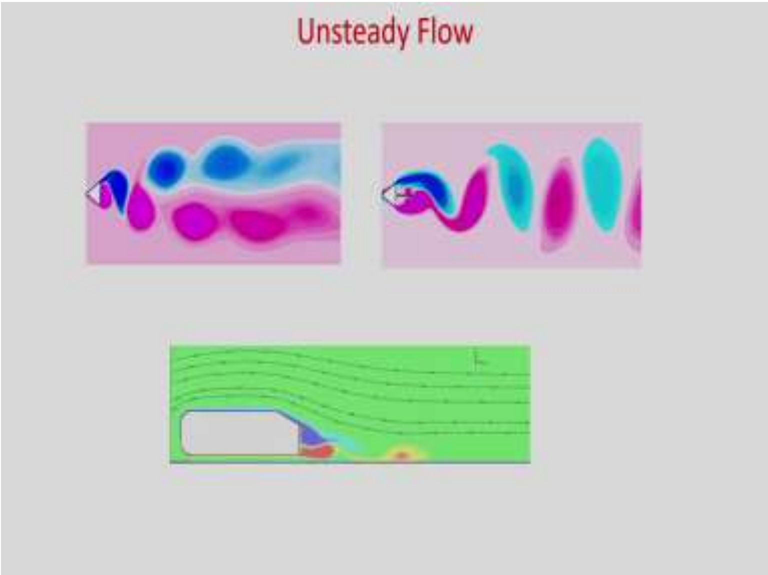
So, you can see these are the some results of natural convection and lid driven cavity or unsteady flow, you can see flow over square cylinder, circular cylinder, mixed convection over a square cylinder.

(Refer Slide Time: 39:20)



So, we have used actually finite volume method here and collocated grid formulation we have used and as collocated grid you know that there is a problem of velocity and pressure decoupling and we have used momentum interpolation, and diffusion terms with central difference we have discretized.

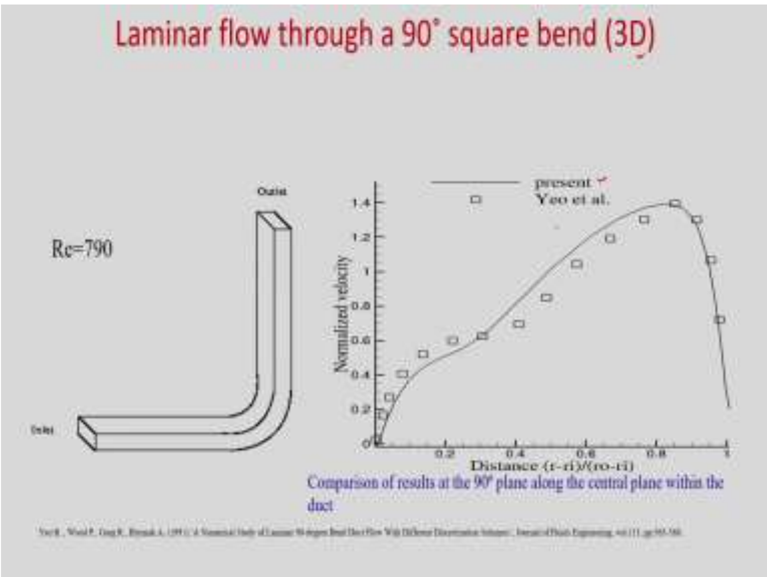
(Refer Slide Time: 39:41)



And we have used simple algorithm simple like algorithm. So, you can see these are some unsteady flows these simulations are carried out using AnuPravaha solver and you can see the animation. So, flow over a triangular cylinder, this is flow over a triangular cylinder with a splitter plate. So, here you can see one splitter plate is there and how the von Karman vortex streets are formed behind the cylinder and it is seated behind the cylinder.

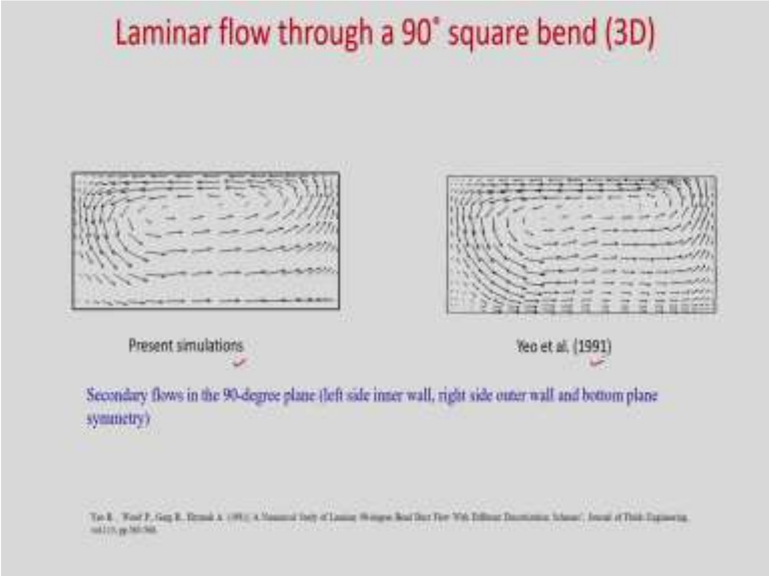
So, you can see these are vorticity contours and how it is transporting behind the cylinder and this is your kind of armoured body problem. So, behind this body it is moving in the negative x direction so; obviously, you can see the flow physics behind this body and interacting with the wall.

(Refer Slide Time: 40:27)



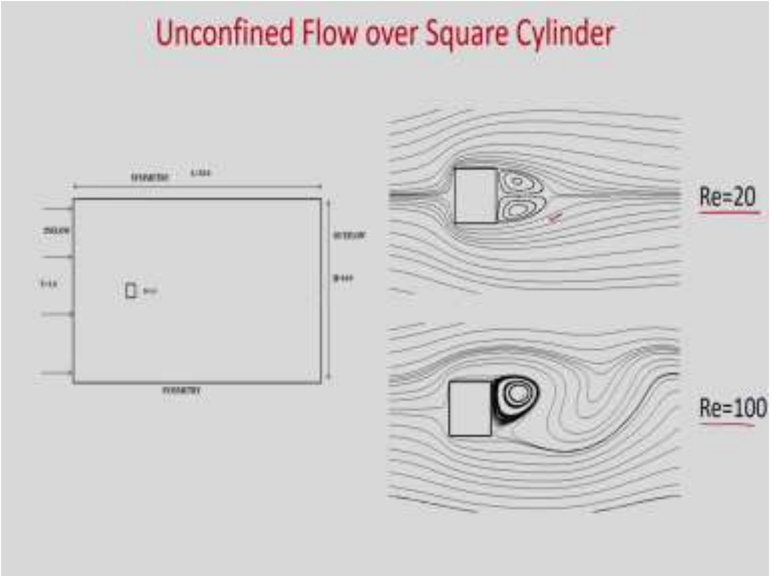
So, this is your wall. So, you can see this is laminar flow through a 90^0 square bend. So, this is your inlet, this is your outlet, you have Reynolds number 790. So, normalized velocities in distance $\frac{(\mathrm{r}-\mathrm{r}_0)}{(r_0-r_i)}$ this is your solid line is our result and these square is your Yeo et. al.

(Refer Slide Time: 40:55)



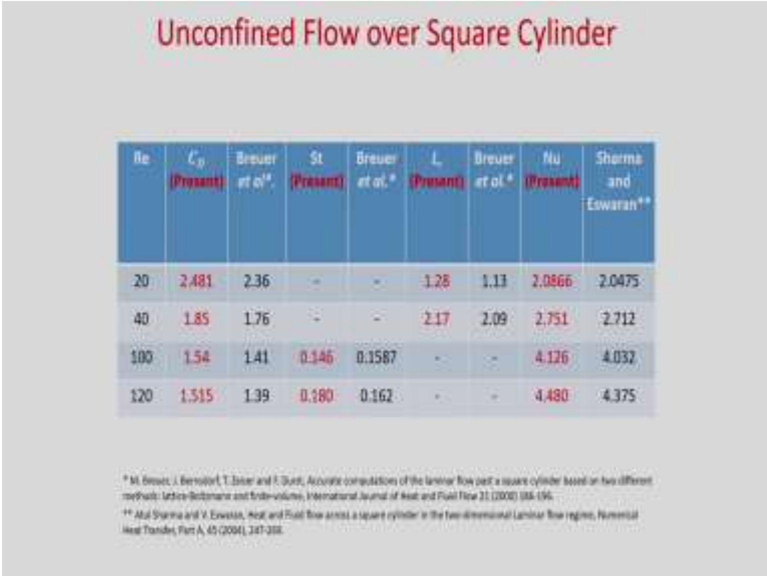
And if you see the flow physics at 90^0 plane left side inner wall right side outer wall and bottom plane is symmetry.

(Refer Slide Time: 41:09)



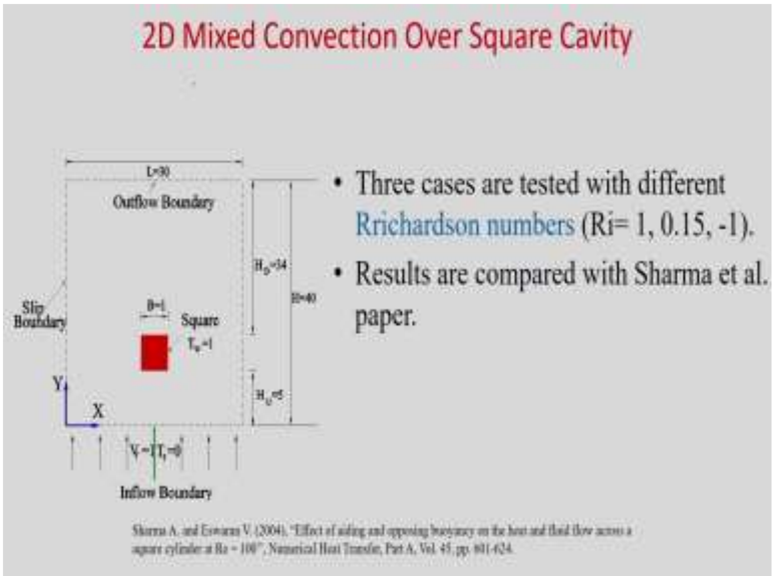
So, you can see how the flow physics is looking. So, this is your present simulation and this is your Yeo et al simulation. Now you consider the convective flow over a square cylinder and you know that after a certain Reynolds number it will become unsteady flow. So, you can see this is Reynolds number 20, this is your symmetry vertices are formed this is in steady region and for Reynolds number 100 this is an unsteady region.

(Refer Slide Time: 41:32)



And these are some drag coefficients 12 number for the unsteady flows and the Nusselt number on the cylinder you can see. So, we have calculated for different Reynolds number 20 and 40 are steady flows and Reynolds number 100, 120 unsteady flows. So, and these are also compared with some literature. Now, you can see 2-D mixed convection over a squared cylinder.

(Refer Slide Time: 41:52)



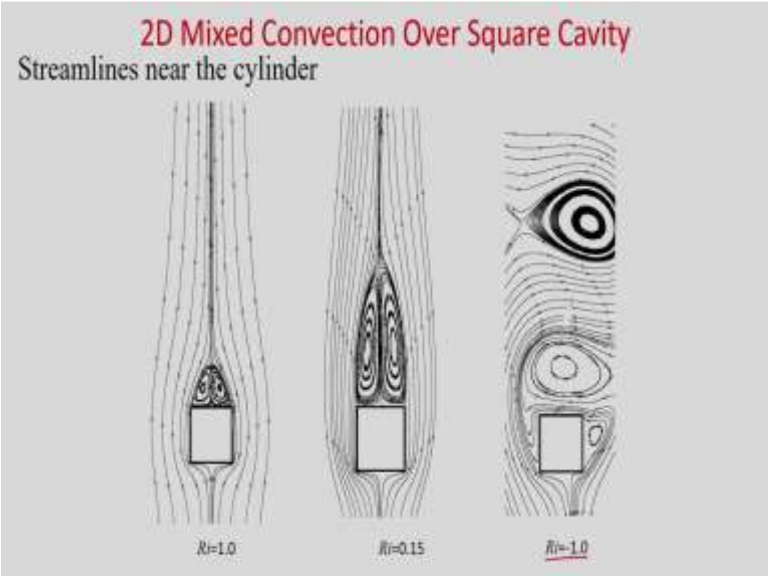
So, we have a mixed convection you have already studied it, for Richardson number 1, 0.15 and -1 these are simulated for Reynolds number 100.

(Refer Slide Time: 42:06)

So, you can see the isotherms near the cylinder. So, Ri = 1, Ri= 0.15, Ri=- 1.

So, Ri=- 1you can see it becomes unsteady.

(Refer Slide Time: 42:19)

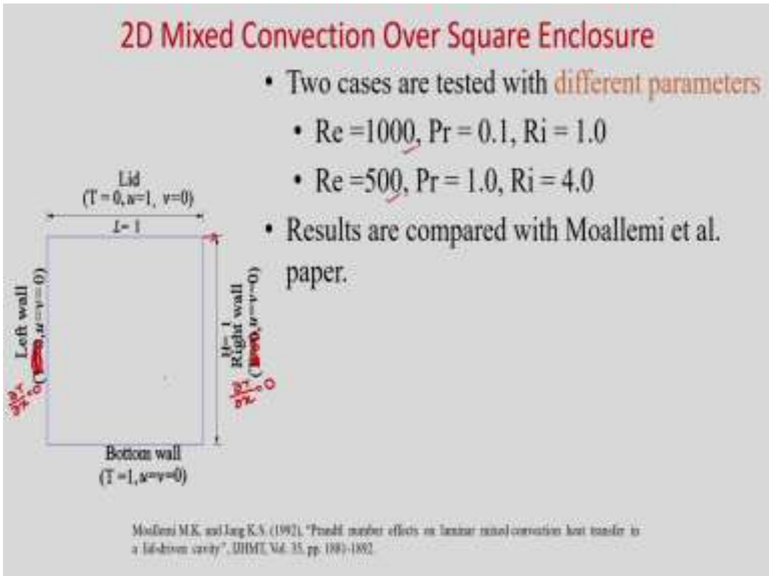


These are streamlines and from streamline plot you can see for Ri= - 1these vortices are seated behind the square cylinder.

(Refer Slide Time: 42:29)

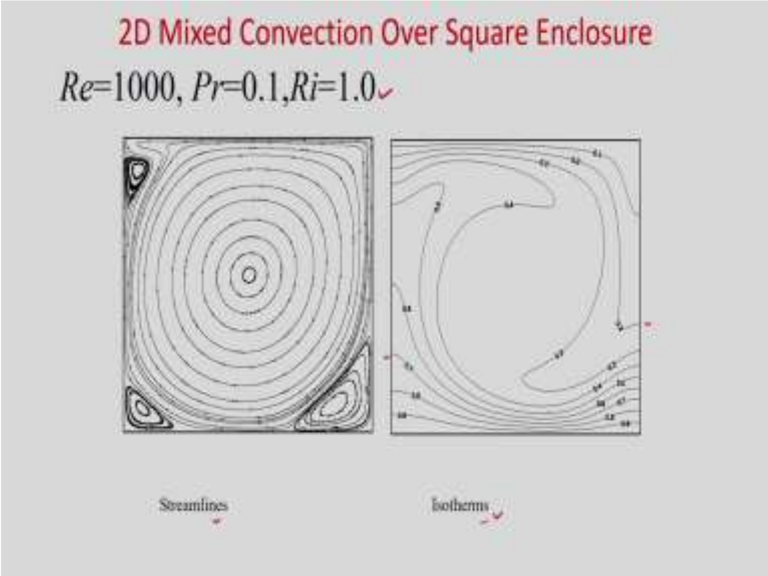
These are some comparison of drag coefficient with the literature and the Nusselt number at the square cylinder.

(Refer Slide Time: 42:38)



Then, we have 2-D mixed convection over a square enclosure. So, you can see this is your cavity the upper lid is moving in the x direction and temperature is T_0 and bottom wall is T_1 and these are also maintained at T_0 sidewalls. So, Reynolds number 1000 and 500 for different Prandtl number, Richardson number these are simulated.

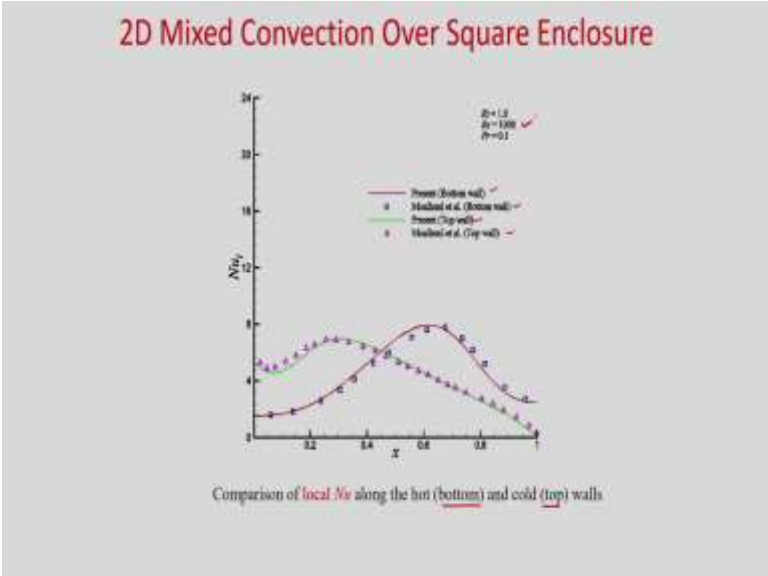
(Refer Slide Time: 43:08)



And you can see for these non-dimensional numbers, this is the streamlines and this is the isotherms. So, here the at the sidewalls you have the temperature not T = 0 this is your adiabatic wall. So, $\frac{\partial T}{\partial x} = 0$ and this right wall also you have adiabatic wall. So, $\frac{\partial T}{\partial x} = 0$.

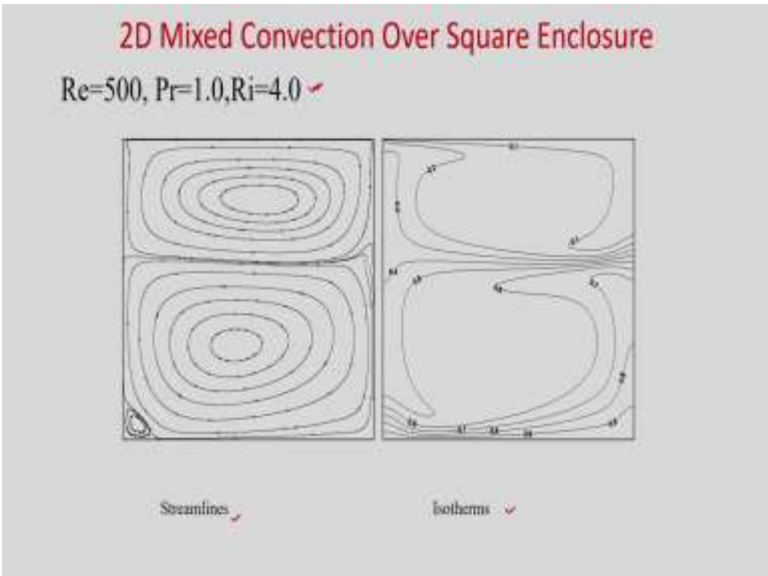
Now, for these boundary conditions you can see the stream lines for this non-dimensional numbers. So, in the stream lines you can see there are vortices near to the corners and these are the isotherms and you can see sidewalls are adiabatic. So, temperature contour is cutting this walls normally then you will get because $\frac{\partial T}{\partial x} = 0$. So, you can see from the simulations and top wall is actually 0 and bottom wall is 1. So, we can see how the isotherms are varying.

(Refer Slide Time: 44:06)



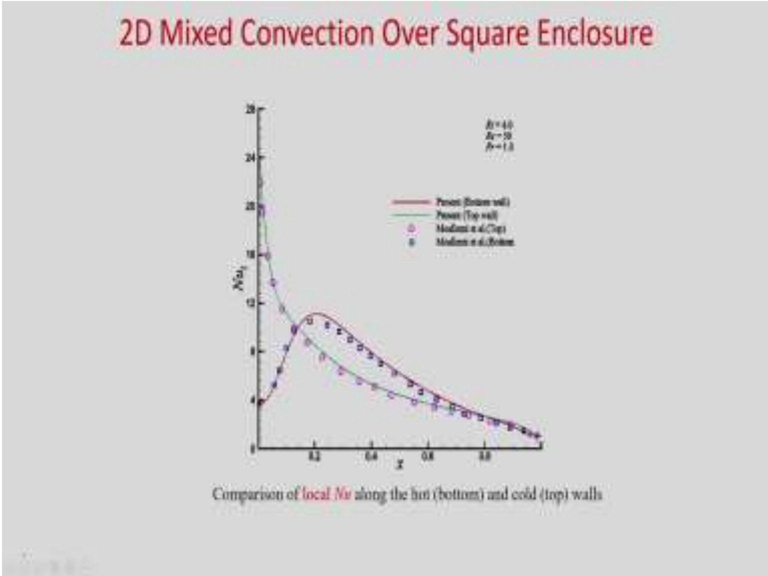
And this is the local Nusselt number, on the bottom and top wall. So, you can see red colour is your bottom wall this is the present simulation from the literature and top wall is this green colour and this is your from literature, you can see there is a good match for these non- dimensional numbers.

(Refer Slide Time: 44:26)



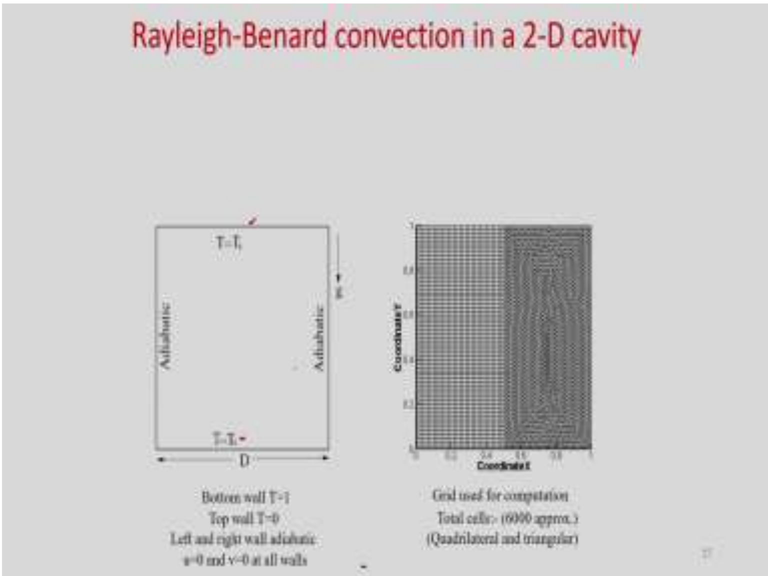
So, this is for another set of parameters this is a streamlines and this is the isotherms.

(Refer Slide Time: 44:34)



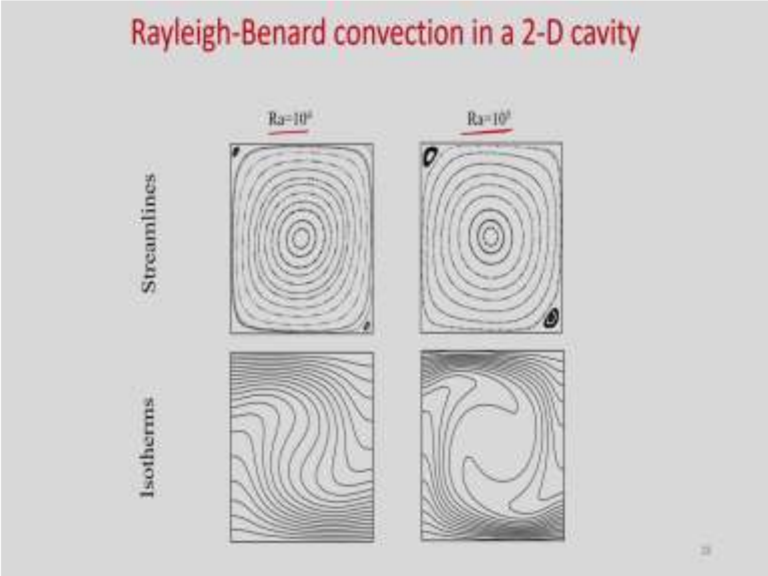
And these are the local Nusselt number variation. Then, we have a Rayleigh-Benard condition in 2-dimensional cavity.

(Refer Slide Time: 44:37)



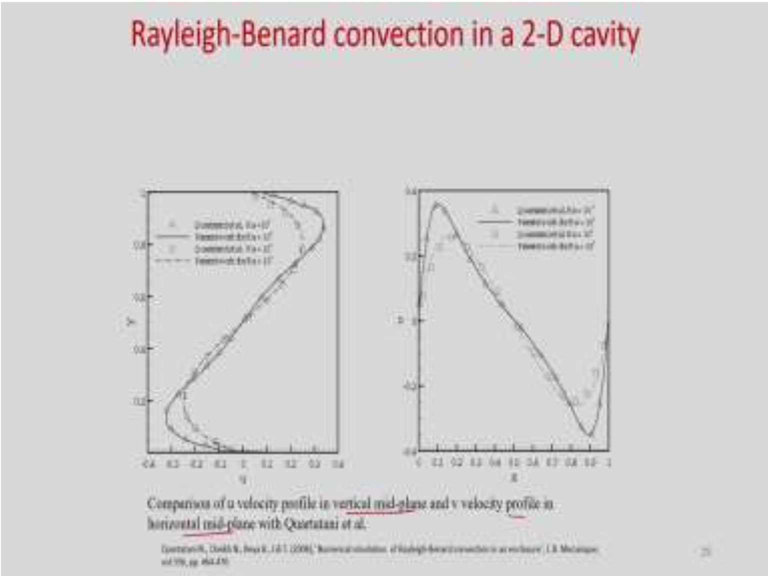
So, Rayleigh-Benard convection you know that bottom wall is maintained at T_h , this is your higher temperature than the top wall temperature and sidewalls are adiabatic.

(Refer Slide Time: 44:54)



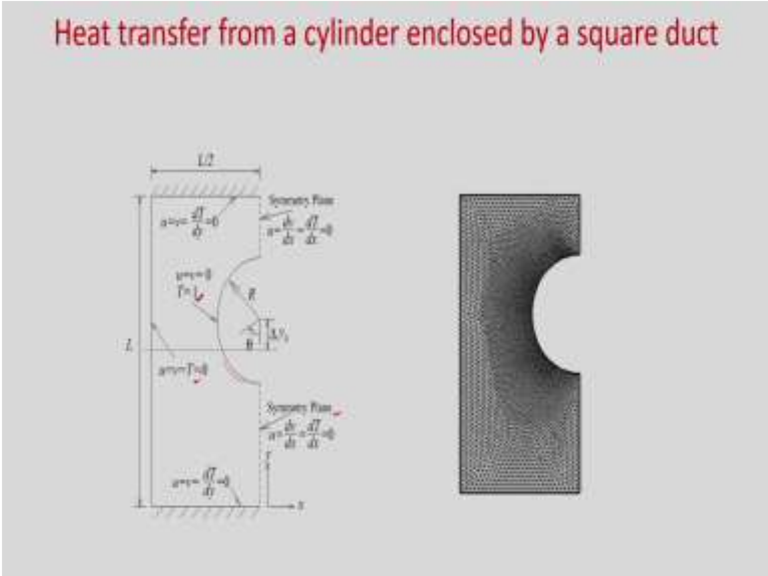
And these are the streamlines and isotherms for different Rayleigh number 10^4 and 10^5. So, right walls are adiabatic. So, the isotherms are cutting the wall normally.

(Refer Slide Time: 45:07)



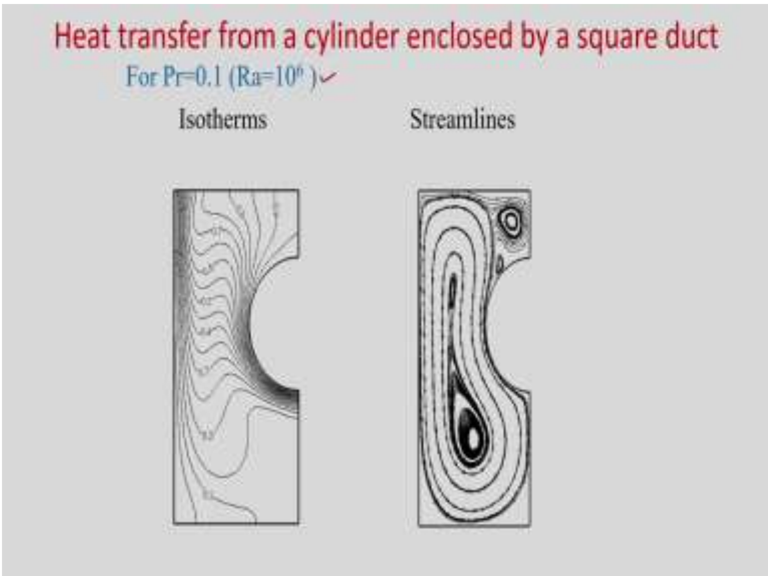
So, these are some velocities u and v. So, u velocity profile in vertical mid-plane and y velocity profile in horizontal mid-plane compared with the literature.

(Refer Slide Time: 45:19)



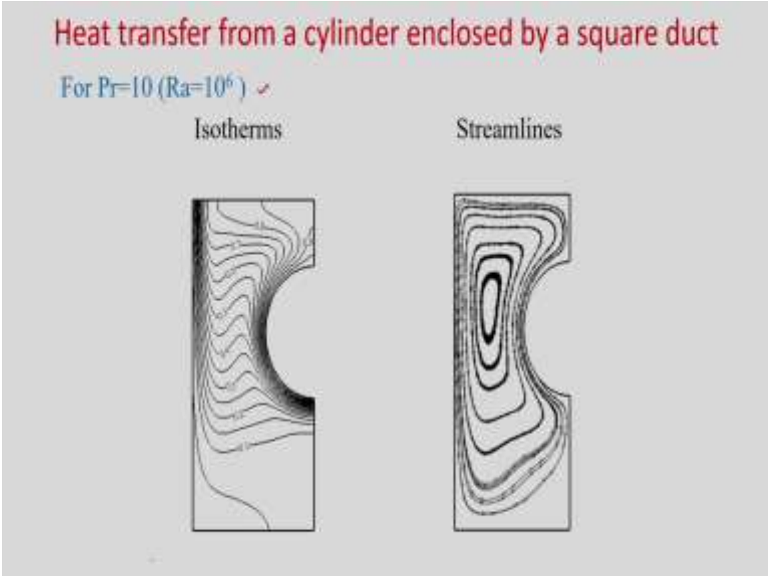
This is a heat transfer from a cylinder enclosed by a square. So, half of the geometry is shown here because due to symmetry we have solved only half of the domain and this is your symmetry plane and top and bottom walls are adiabatic, this cylinder circular cylinder is maintained at hot temperature T_1 and sidewall is maintained at temperature T_0.

(Refer Slide Time: 45:43)



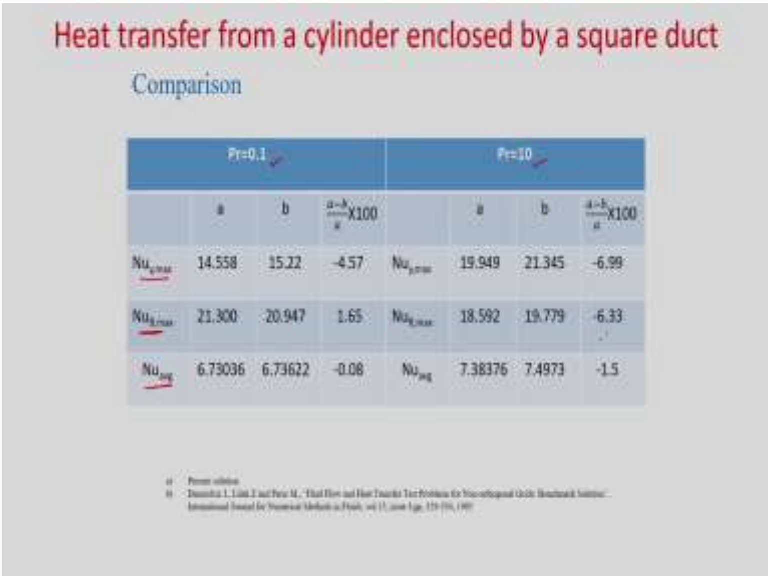
So, you can see for this setup non dimensional parameters, this is the isotherms and these are the streamlines.

(Refer Slide Time: 45:50)



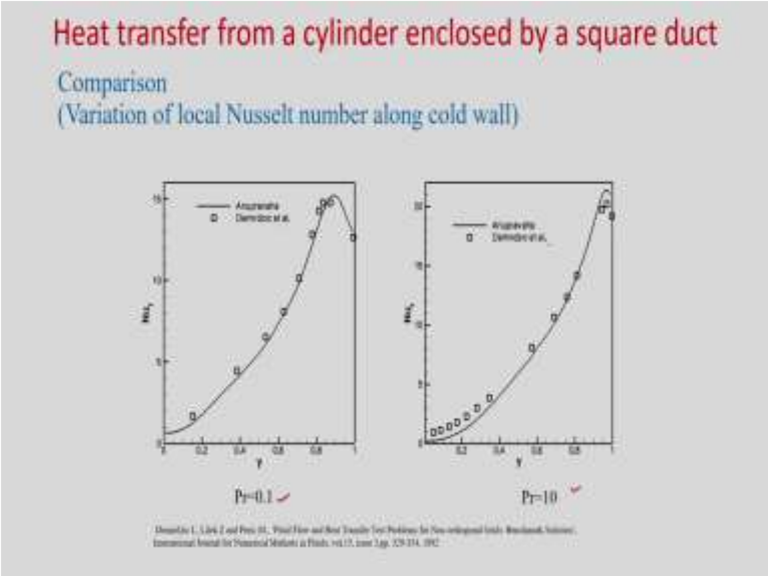
And for Pr = 10 and Ra= 10^6 these are the isotherms and streamlines.

(Refer Slide Time: 45:58)



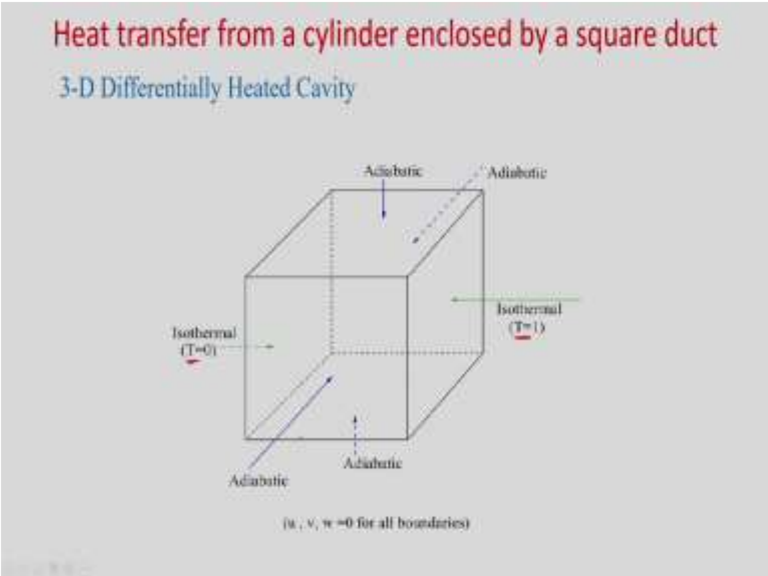
And this is some comparison, local average Nusselt number and at sidewall what is the maximum Nusselt number and on the cylinder what is the maximum Nusselt number. So, a is our result and b is from this literature and the percentage difference in the result you can see for two different Prandtl numbers.

(Refer Slide Time: 46:21)

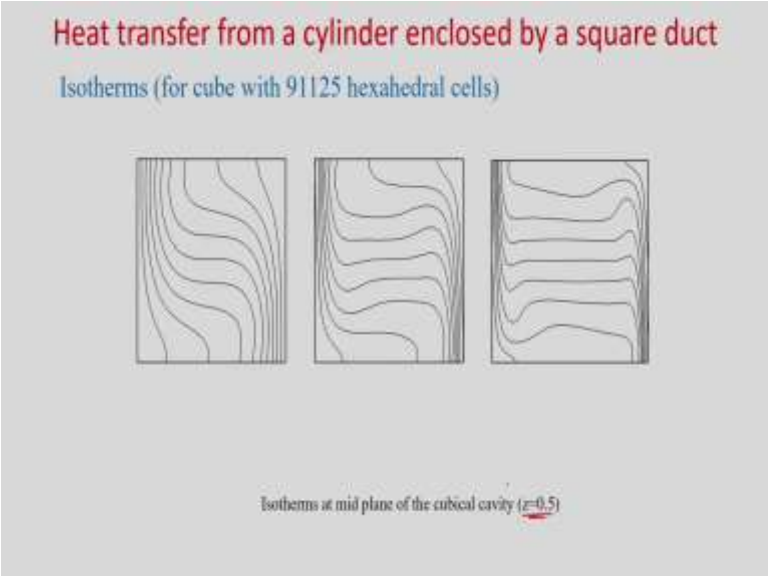


So, this is some local Nusselt number variation along the cold wall for two different Prandtl numbers and it matches well with the Demirci et al. These differentially heated cavity so, now, 3-D problem. So, these two walls are differentially heated you can see this is your T = 1 and T = 0 and other 4 walls are adiabatic.

(Refer Slide Time: 46:29)

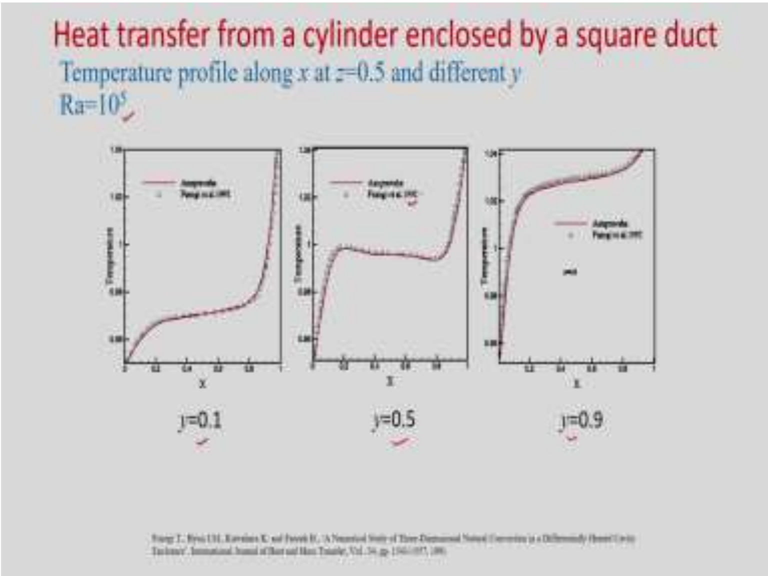


(Refer Slide Time: 46:42)



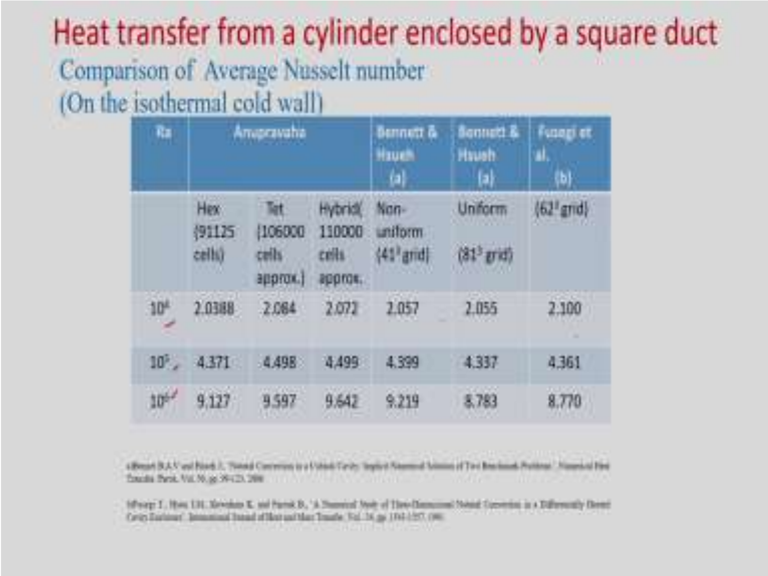
So, for this you can see the isotherms at mid-plane of the cubical cavity at z = 0.5.

(Refer Slide Time: 46:48)



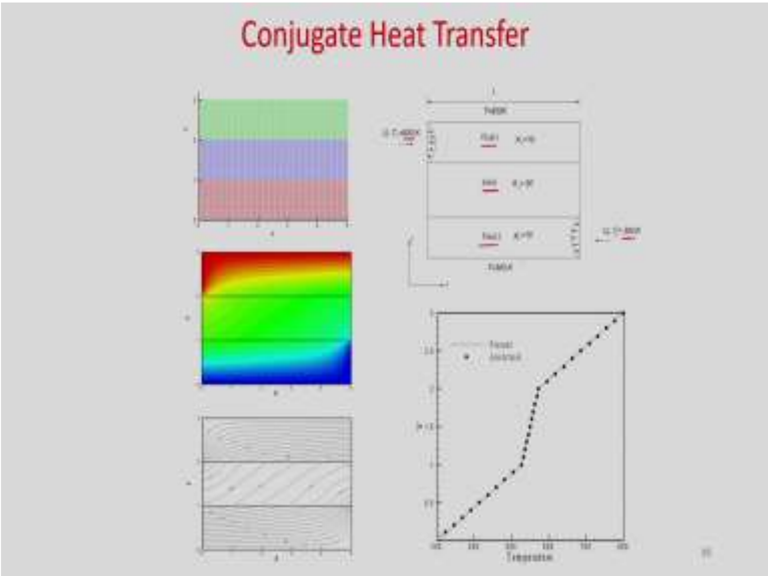
And temperature profile along x at z = 0.5 at different y Ra= 10^5. So, you can see y =0.1, 0.5, 0.9 how the temperature is varying along x with the and this results are compared with Fusegi et al.

(Refer Slide Time: 47:10)



These are some comparison of average Nusselt number at different Rayleigh number. So, this is our solution using different types of grid and these are from literature and you can see there is a good match.

(Refer Slide Time: 47:26)



Also we can solve conjugate heat transfer, what is conjugate heat transfer? In conjugate heat transfer we solve the heat conduction equation in the solid domain and in the fluid domain we solve the fluid flow equations and the energy equation. So, like in heat exchanger. So, parallel or cross fluid heat exchanger you can see that it will be separated by solid walls. And in the solid walls we calculate the heat transfer.

So, you can see this is your solid in the middle and this is your fluid domain fluid is flowing from left to right and this is the fluid two domain where fluid flow is taking place from right to left and temperature are maintained at T = 300 K on the bottom 800 K at the top and this is the inlet temperature here and this is the inlet temperature here.

So, you can see how the isotherms are varying, in the solid also you can see how the isotherms are varying. So, here you can see. So, this is your solid and these are the fluid domains and you can see the temperature along y so how it is varying.

In today's class first we solved the energy equation using finite difference method, then we have discussed about the thermal and flow boundary conditions. We have used fictitious cell method and found the velocities and the temperature at the fictitious cell. Then we discussed the solution algorithm where you are solving Navier-Stokes equations as well as the energy equation. Then we have shown some heat transfer results use for the problems solving from the in house solver AnuPravaha.

So, we have shown for forced convection and mixed convection, results in terms of isotherms, streamlines as well as local Nusselt number and average Nusselt number. So, you can use these numerical simulations for the problems where you cannot have the analytical solutions and if you have a complicated geometry also you can use numerical technique because you cannot have the analytical solution available.

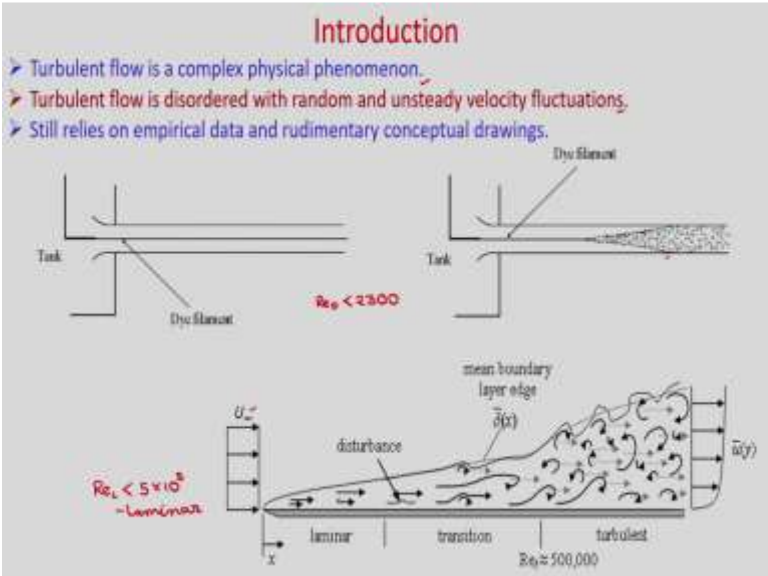
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 11
Turbulent Flow and Heat Transfer
Lecture – 36
Derivation of Reynolds Averaged Navier – Stokes Equations

Hello everyone. So, today we will study convection in turbulent flow most flows in nature and in industrial applications are turbulent. You will find applications in mixing of the flows, then in combustion processes as well as in heat exchangers.

(Refer Slide Time: 01:09)



Generally turbulent flow occurs relatively at high Reynolds number and in turbulent flows velocity temperature fluctuate with time. Turbulent flow is a complex physical phenomena, turbulent flow is disordered with random and unsteady velocity fluctuations. In laminar flows we already had the exact solution of many flows with certain assumptions, but in turbulent flows it is very difficult to have the exact solutions.

So, mostly whatever correlations will write that depends on the experimental values and that is why we will write the empirical correlations. You know about the famous experiment carried by Reynolds. So, you can see here Reynolds did this experiment say the ink is injected here in a tank and this is the pipe, when it passes through this pipe you can see this dye filament is almost straight line.

You will get this type of flow when you have low velocity, but if you increase the velocity here then you can see these dye filament will diffuse in other directions. So, it is due to the fluctuation of the velocities at high Reynolds number and it becomes turbulent flows.

So, you will get this kind of structure you know that in pipe flow when Reynolds number based on the diameter is < 2300 then it will be laminar, then it will transform to transition and turbulent flow.

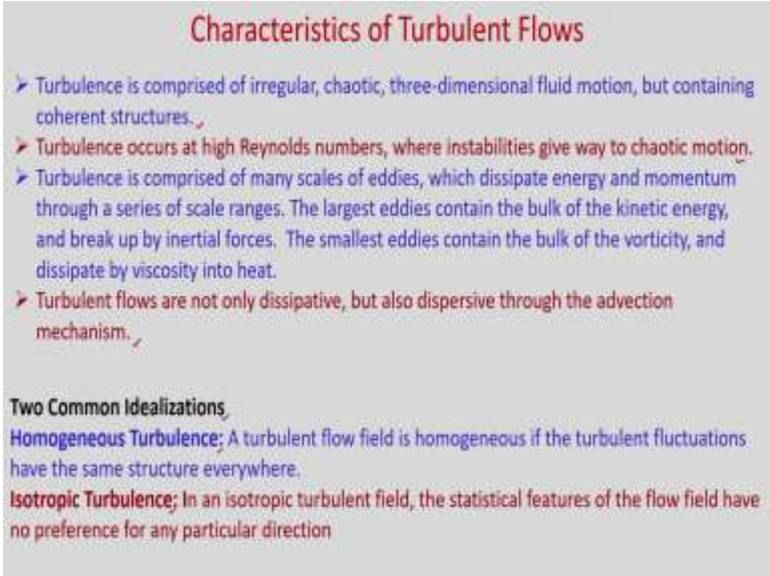
If you consider flow over flat plate which already we have done the exact solutions in earlier classes in external flows, you have seen that near to the leading edge of the flat plate we get laminar flows and we have done the study of this laminar flow.

But if you increase the length then you will find that it will become a transitionism then turbulent. So, you can see here. So, you have a free stream velocity U_∞; when it will come and flow over this flat plate so; obviously, you know due to the viscous effect there will be formation of boundary layer and near to the leading edge you will get laminar flows.

You will see that very streamline flow but after that there will be some disturbances near to the wall and it will propagate away from the wall slowly and after if you go ahead and you will get a fully turbulent flows and in this case you see if Reynolds number based on the length of the flat plate if it is > 5×10^5, then you will get laminar flows.

So, turbulent flow is very complex and chaotic. So, it is very difficult to define turbulent flows. So, most of the researchers they are given the characteristic of the turbulent flows.

(Refer Slide Time: 04:41)



So, you see turbulence is comprised of irregular chaotic three dimensional fluid motion, but containing coherent structures. So, you can see that turbulent flow inherently three dimensional and unsteady.

But with certain assumptions again we may consider as two dimensional flow as well as steady flow. When we consider the velocity components or temperature as comprised of mean value, time average mean value and the fluctuating components.

Problems occurs at high Reynolds number where instabilities give way to chaotic motion we have already seen in external and internal flows; obviously, it is a high Reynolds number flow then the turbulence occur.

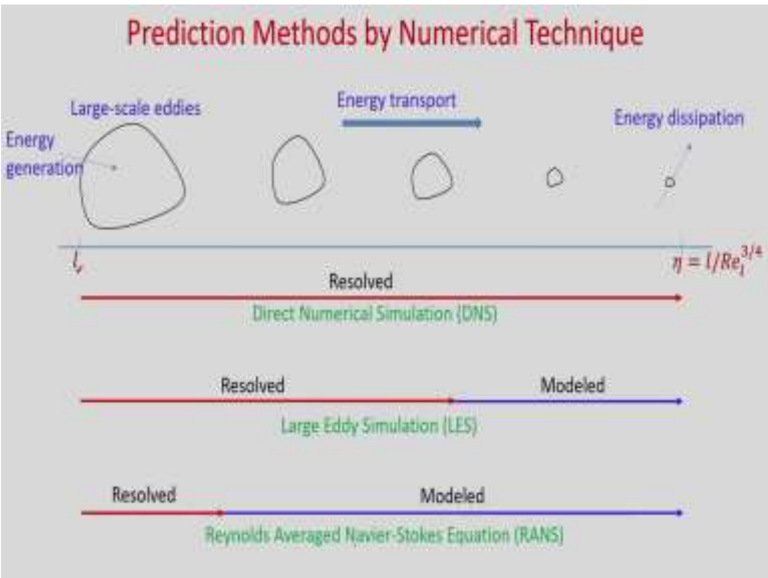
Turbulence is comprised of many scales of eddies which dissipate energy and momentum through a series of scale ranges. The largest eddies contain the bulk of the kinetic energy and break up by inertial forces, the smallest eddies contain the bulk of the vorticity and dissipate by viscosity into heat.

So, you can see that in general you will have a larger eddy which will contain bulk of the kinetic energy and it will be transported by the velocities and during this transport it will divide into smaller eddies and it will keep on decreasing the size of the eddies and those eddies will contain bulk of the vorticity and then it will actually dissipate into heat by the viscous effect. Turbulent flows are not only dissipative, but also dispersive through the advection mechanism.

So, when we study the turbulence we make two common idealizations one is homogeneous turbulence and another is isotropic turbulence. What is homogeneous turbulence?

If the turbulence has the same structure quantitatively in all parts of the flow field then the turbulence is said to be homogeneous turbulence and in isotropic turbulence, the statistical features have no directional preference, then it is called isotropic turbulence. So, when we do the numerical simulation of this turbulent flows, we need to have some prediction methods.

(Refer Slide Time: 07:32)



So, here you can see as we discussed you have a large scale eddies let us say its scale is of the order of l. So, it contains bulk kinetic energy due to the inertial effect it will be transported and due to the during the transport it will divide into smaller eddies.

Then again it will become smaller and after that it cannot be smaller than this eddy and that time this eddy will contain bulk of the vorticity and these vorticity or this eddy will dissipate heat due to the viscous effect and the scale of this eddy is known as Kolmogorov scale and η is known as Kolmogorov scale and $\eta = l / \mathrm{Re}_l^{3/4}$.

So, you see Re is based on the large scale eddy length and if Reynolds number is very high then η which is your Kolmogorov scale will be very low and to actually compute or to capture these smaller eddies it is very difficult because you need to have very very small grid size to capture the small eddy.

So, you can see if you solve the Navier stokes equations using direct numerical simulation without modelling any of the eddies, then you need to have very fine mesh so, that you can capture the smaller eddies in the Kolmogorov scale length scale.

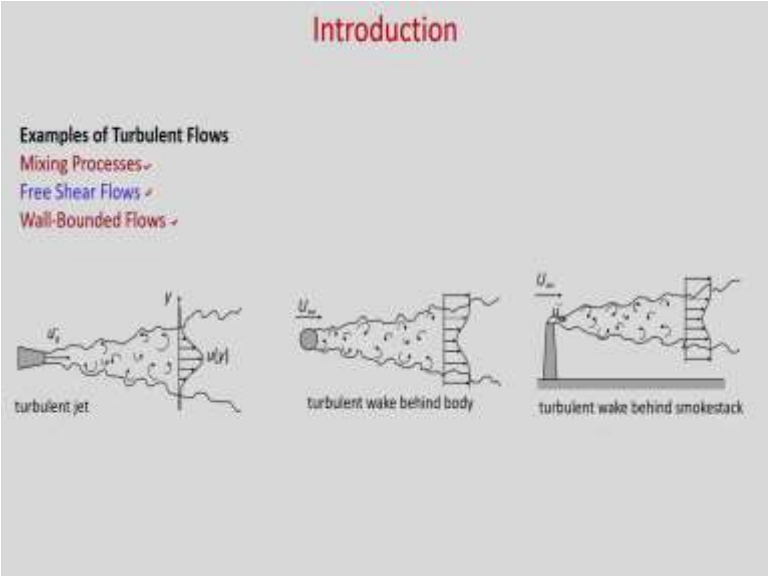
Now, direct numerical simulation for three dimensional flows and unsteady flows obviously, it is very difficult computational time will be huge and generally for industrial fluid flow direct numerical simulation is very difficult.

We can have some smaller scales we can model and rest of the eddies we can compute using large eddy simulation. So, we can see in the large eddy simulation up to certain eddy size we resolve and rest we model and another simplified way to solve this turbulent flow is just using the Reynolds average Navier stokes equation.

So, whatever velocity and temperature you have you can decompose into two components one is time average min component and another is fluctuating component and we can solve these governing equations in a average sense. So, here you will model only the large scale eddies, but other you need to model it.

So, in this course we will derive the Reynolds average Navier stokes equations and later we learn how to solve for flow over flat plate and the pipe flow.

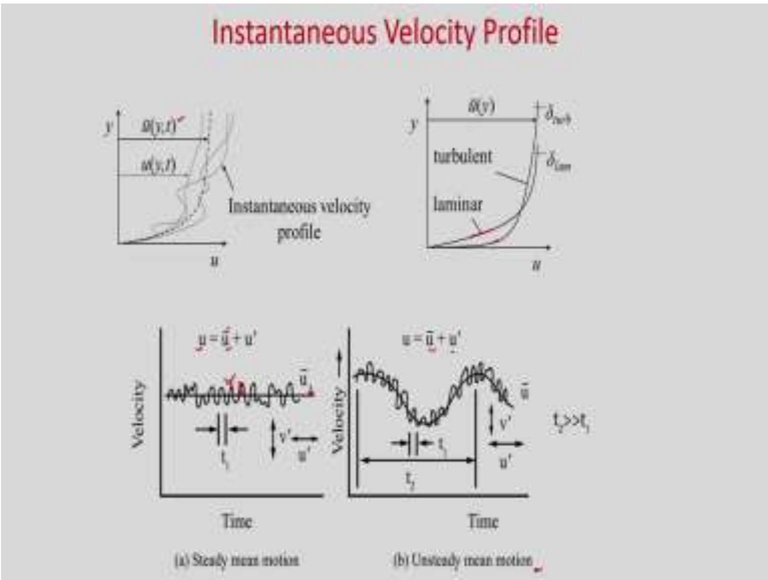
(Refer Slide Time: 11:03)



You can see examples of turbulent flows in mixing processes, free shear flows, wall bounded flows. So, these are some examples you can see in turbulent jet turbulent wake behind a body.

So, if this circular cylinder is heated then obviously, there will be convection in turbulent flow and here obviously, you can see turbulent wake behind smokestack. So, in the smokestack you will have high temperature and ambient will be at lower temperature and there will be convection and that is your turbulent convection you will get.

(Refer Slide Time: 11:43)



Here you can see the instantaneous velocity profile. So, velocity will vary with time . So, these are the velocity profile over a flat plate and if you time average the quantity then you will get this dotted line.

So, you can see this is the time average velocity profile and obviously, you will get a smooth curvature of this velocity and here we have compared the velocity profile for laminar and turbulent flow. So, this is your laminar flow velocity profile and this is your turbulent flow velocity profile.

So, you can see in turbulent flow near to the wall you have more gradient. So, obviously, you can see from this velocity profile now we have discussed that these velocity and temperature fluctuate with time and we can decompose into two components one is time average mean component and another is your fluctuating component deviating from the mean value.

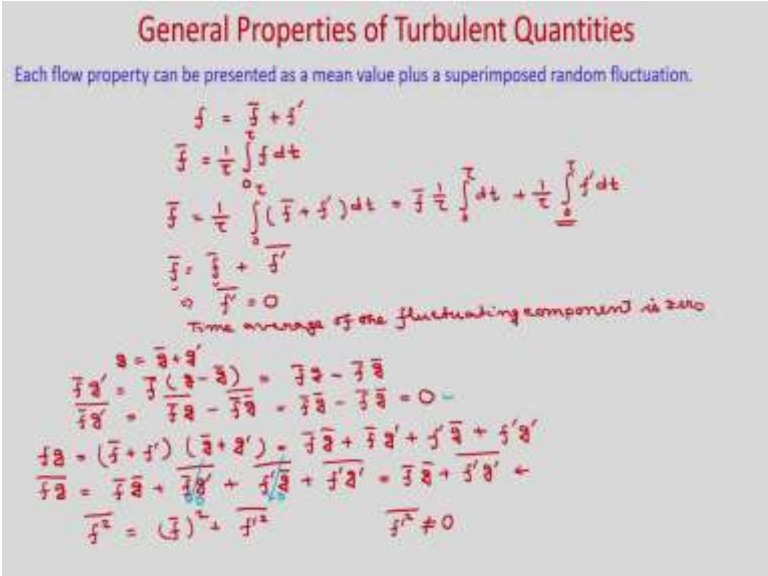
So, you can see this is some velocity varying with time and we can decompose this velocity into two components one is time average mean component. So, we will denote with u bar and plus your fluctuating component that is your u'. So, you can see if we take the mean. So, this is your u bar. So, this straight line is u bar and whatever fluctuating component you have whatever it is deviating from the mean value.

So, this is known as steady mean motion because your mean velocity is not varying with time, but you can have unsteady mean motion as well. So, here you can see these are fluctuating components and if the time period is t_1 of the fluctuation. So, here if you time average in this period and you will get that your mean value will also vary with time.

So, in this case you can see that t_2 is the time period of mean velocity variation and t_1 is the time period for the fluctuating velocity variation and you can see here velocity is also varying with mean velocity is varying with time. So, this is your unsteady mean motion. So, Reynolds first proposed that we can have this decomposition of this velocity into two; one is mean velocity and the fluctuating velocity.

So, this is known as Reynolds decomposition. So, first we will see that you if you take any scalar f and if we decompose into 2, then what are the identities we can have?

(Refer Slide Time: 15:07)



So, if we consider a scalar property f, then we can have $f=\overline{f}+f\,'$ where, $\overline{f}$ which is your time averaged value and $f\,'$ which is your fluctuating component.

So, you can see this is superposition of this time average and fluctuating components. Now if you calculate the mean value; that means, it is if you have a time period τ, then it is defined as, $\overline{f}=\frac{1}{\tau}\int_0^{\tau} fdt$. So, this way actually you calculate the time average component. So, you can see in this case that if you put this $f=\overline{f}+f\,'$ then what you will get?

So, now $\overline{f}=\frac{1}{\tau}\int_0^{\tau}\left(\overline{f}+f\,'\right)dt$ then what we can have? So, you can see. So, $\overline{f}$ which is your average quantity. So, you can bring it outside the integral. So, $\overline{f}\frac{1}{\tau}\int_0^{\tau} dt$ .So, we assume that in that time period $\overline{f}$ is constant.

So, you can take it out of the integral and we can have $\frac{1}{\tau}\int_0^{\tau} f\,'dt$. So, you can see you can write $\overline{f}$. So, this $\overline{f}\frac{1}{\tau}\int_0^{\tau} dt$ will give only $\overline{f}$ and what does it mean? This actually give the definition of this average quantity; that means, it is $\overline{f'}$. So, you can see this quantity and this quantity are same.

So, you can see $\overline{f'}=0$; that means, time average of the fluctuating component is 0. So, time average of the fluctuating components is 0. Now, let us consider another scalar property $g=\overline{g}+g'$. So, if you calculate $\overline{f}g'$ what does it mean?

So, $\overline{f}g'=\overline{f}\left(g-\overline{g}\right)=\overline{f}g-\overline{f}\,\overline{g}$. So, if you take the time average of this quantity left hand side then what you will get? So, you will get $\overline{\overline{f}g'}=\overline{\overline{f}g}-\overline{\overline{f}\,\overline{g}}$. So, you can show that this quantity will be $\overline{f}\,\overline{g}-\overline{f}\,\overline{g}$.

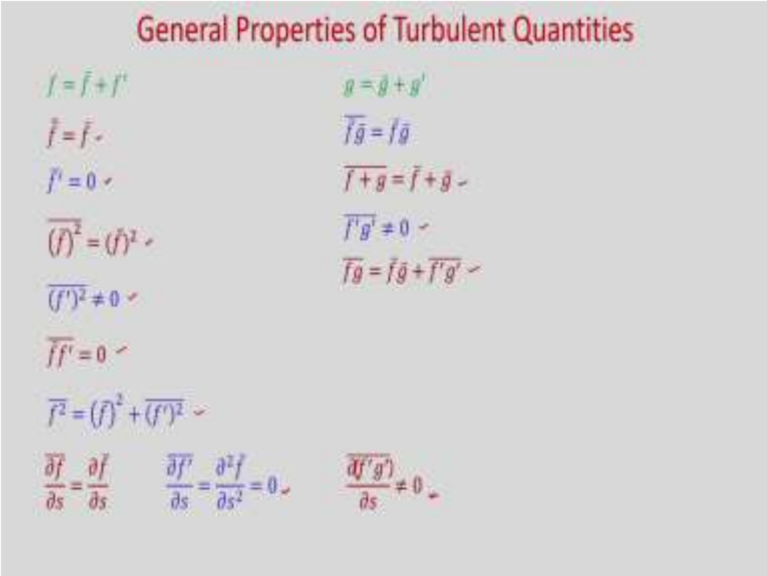
So, this is equal to 0. So, you can see $\overline{\overline{f}g'}=0$ and now if you calculate f g; that means, $fg=\left(\overline{f}+f'\right)\left(\overline{g}+g'\right)$. So, what you will get? $\overline{f}\,\overline{g}+\overline{f}g'+f'\overline{g}+f'g'$. Now, you take the time average of this quantity.

So, if you take the time average of this quantity. So, this will give $\overline{fg}=\overline{f}\,\overline{g}+\overline{\overline{f}g'}+\overline{f'\overline{g}}+\overline{f'g'}$. So, you can see this quantity already we have shown that it is 0.

So, this is 0 and this is 0. So, you will get, $\overline{fg}=\overline{f}\,\overline{g}+\overline{f'g'}$ and if you write f^2. So, from here you can write $\overline{f^2}=\left(\overline{f}\right)^2+\overline{f'^2}$ and $\overline{f'^2}\neq 0$, the time average of the fluctuating component is 0 that we have already shown.

That means, $\overline{f'}=0$, but $\overline{f'^2}\neq 0$ why? Because this is a fluctuating component. So, it is the deviation from the mean value. So, it is having the positive value as well as negative value so, but f'^2 where when you are making. So, it is always positive. So, $\overline{f'^2}\neq 0$.

(Refer Slide Time: 20:34)

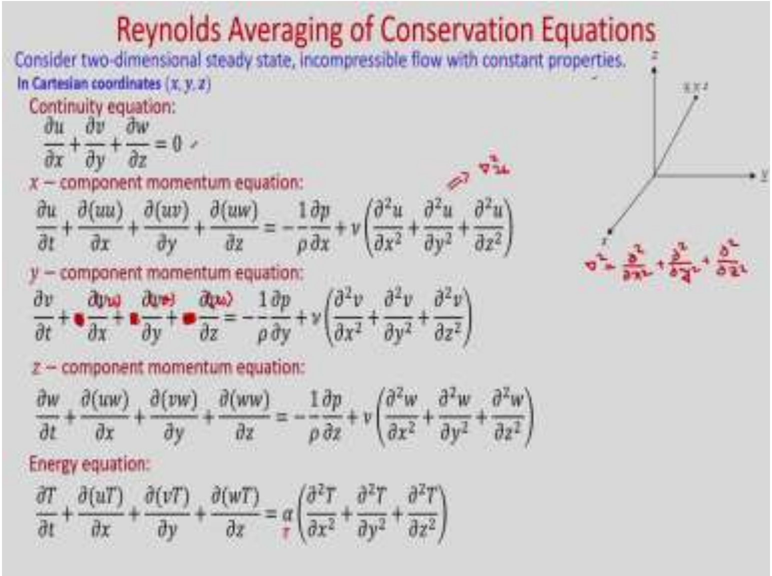


So, now you can see the general properties of turbulent quantities. So, if we have two scalars f and g and we have $f = \overline{f} + f', g = \overline{g} + g'$. So, you can see $\overline{\overline{f}} = \overline{f}$, $\overline{f'} = 0$ that we have already shown then $\overline{\left(\overline{f}\right)^2} = \left(\overline{f}\right)^2$.

But $\overline{f'^2} \neq 0$ and $\overline{\overline{f} f'} = 0$ that we have already shown $\overline{f^2} = \left(\overline{f}\right)^2 + \overline{\left(f'\right)^2}$. So, that also we have shown now if you take $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}$, $\overline{f+g} = \overline{f} + \overline{g}$, $\overline{f'g'} \neq 0$.

And $\overline{fg} = \overline{f}\,\overline{g} + \overline{f'g'}$ and if you take the derivative s is any special direction, then $\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}$, $\overline{\frac{\partial f'}{\partial s}} = \frac{\partial^2 \overline{f}}{\partial s^2} = 0$ and $\overline{\frac{\partial f'g'}{\partial s}} \neq 0$.

(Refer Slide Time: 22:05)



So, now these properties we can use for the when we will decompose the velocities and temperature into mean quantity and the fluctuating component. So, first we will start with the Navier stokes equations, we will write the Navier stokes equation in weak conservative form. So, you can see if you consider two dimensional steady state incompressible flow with constant properties.

So, in Cartesian coordinate you can write continuity equation as $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$, x component momentum equation $\frac{\partial u}{\partial t}+\frac{\partial (uu)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}$. So, this is your convection terms and you can see we have written in weak conservative form.

So, if you invoke continuity equation you can put from non-conservative form to this weak conservative form and in right hand side you will get $-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$. So, this is the diffusion term. So, this last term actually you can write also $\nabla^2 u$.

So, in the bracket whatever quantities there. So, we can write $\nabla^2 u$ because you know, $\nabla^2 = \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$.

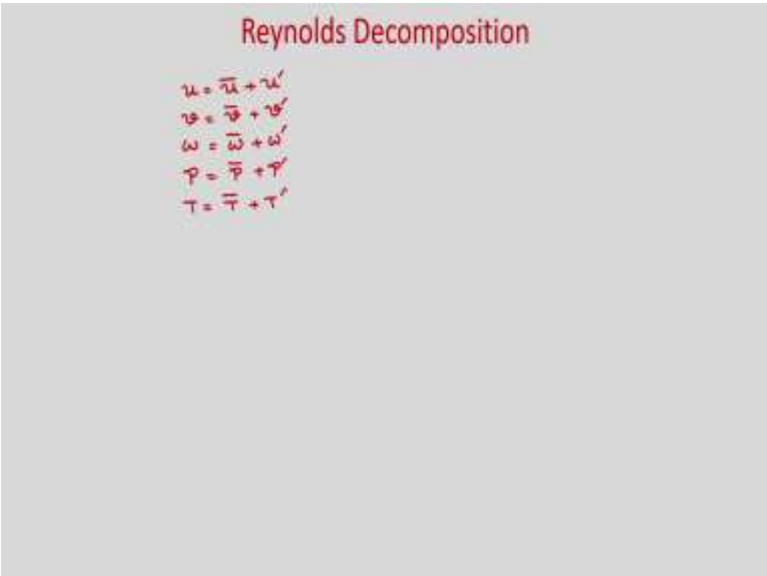
Similarly y momentum equation you can write in conservative form. So, this you can write in conservative form u v w.

So, this is $\frac{\partial v}{\partial t}+u\frac{\partial (vu)}{\partial x}+v\frac{\partial (vv)}{\partial y}+w\frac{\partial (vw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$ and this is the energy equation in weak conservative form.

So, right hand side is your diffusion term we have neglected the viscous dissipation and α is the thermal diffusivity and obviously, ν is your kinematic viscosity and ρ is the density.

Now, we will use the Reynolds decomposition. So, we will write the velocities u v w and temperature t as superposition of mean quantity as well as the fluctuating quantity.

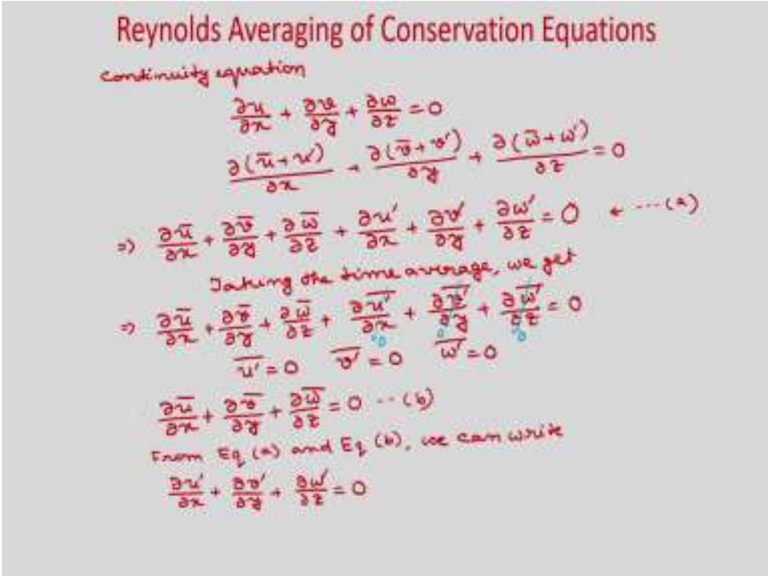
(Refer Slide Time: 24:36)



So, we can use $u=\bar{u}+u'$, $v=\bar{v}+v'$, $w=\bar{w}+w'$ you have pressure. So, $P=\bar{P}+P'$ and temperature $T=\bar{T}+T'$.

So, first let us consider only x momentum equation and we will do the derivation and similar way you can do for the y and z momentum equations.

(Refer Slide Time: 25:13)



So, we have the continuity equation. So, continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$.

So, if you substitute you get, $\frac{\partial\left(\overline{u}+u'\right)}{\partial x}+\frac{\partial\left(\overline{v}+v'\right)}{\partial y}+\frac{\partial\left(\overline{w}+w'\right)}{\partial z}=0$. So, you can write it as $\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}+\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$ now you take the time average.

So, taking the time average we get. So, you can see if you take the time average $\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}+\frac{\partial \overline{u'}}{\partial x}+\frac{\partial \overline{v'}}{\partial y}+\frac{\partial \overline{w'}}{\partial z}=0$ and if you take the time average, then you can write, $\frac{\partial \overline{u'}}{\partial x}+\frac{\partial \overline{v'}}{\partial y}+\frac{\partial \overline{w'}}{\partial z}=0$.

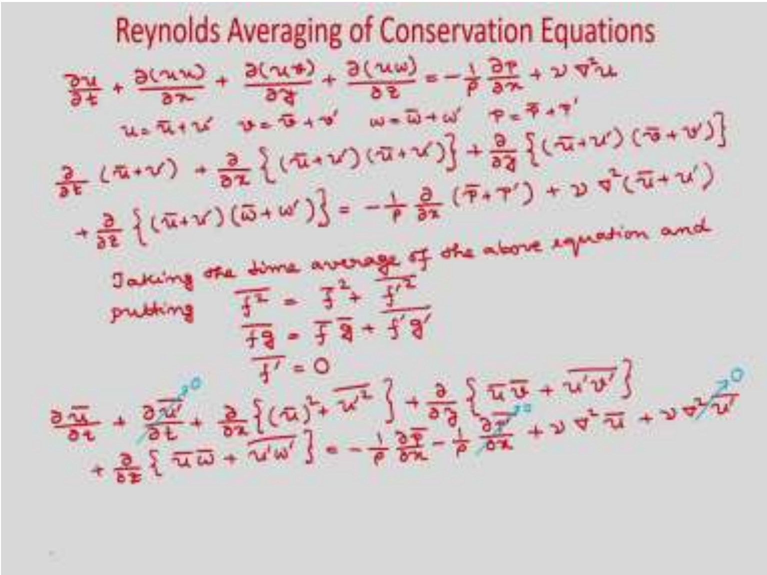
We have already shown that the time average of the fluctuating components are 0. So; that means, $\overline{u'}=0$, $\overline{v'}=0$, $\overline{w'}=0$ so; that means, these quantities will be 0. So, this will be 0, this will be 0 and this will be also 0. So, you can have $\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}=0$.

So, you can see the time average velocities satisfy the continuity equation. So, this is the continuity equation you need to solve when you are using Reynolds average Navier stokes equations.

So, here you can see if you invoke these in this equation what you will get. So, let us say this is your (a) and this is your (b). So, from equation (a) and equation (b) we can write. So, if you put this equal to 0.

So, first three terms will become 0. So, you will get $\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$ what does it mean? It means that the fluctuating component satisfy the continuity equation now let us consider x momentum equation.

(Refer Slide Time: 28:47)



So, let us write the x component momentum equation, $\frac{\partial u}{\partial t}+\frac{\partial (uu)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$.

So, now you invoke $u=\bar{u}+u'$, $v=\bar{v}+v'$, $w=\bar{w}+w'$, $P=\bar{P}+P'$. So, if you put it here. So, you will get,

$$\frac{\partial}{\partial t}\left(\bar{u}+u'\right)+\frac{\partial}{\partial x}\left\{\left(\bar{u}+u'\right)\left(\bar{u}+u'\right)\right\}+\frac{\partial}{\partial y}\left\{\left(\bar{u}+u'\right)\left(\bar{v}+v'\right)\right\}+\frac{\partial}{\partial z}\left\{\left(\bar{u}+u'\right)\left(\bar{w}+w'\right)\right\}=-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\bar{P}+P'\right)+\nu\nabla^2\left(\bar{u}+u'\right)$$

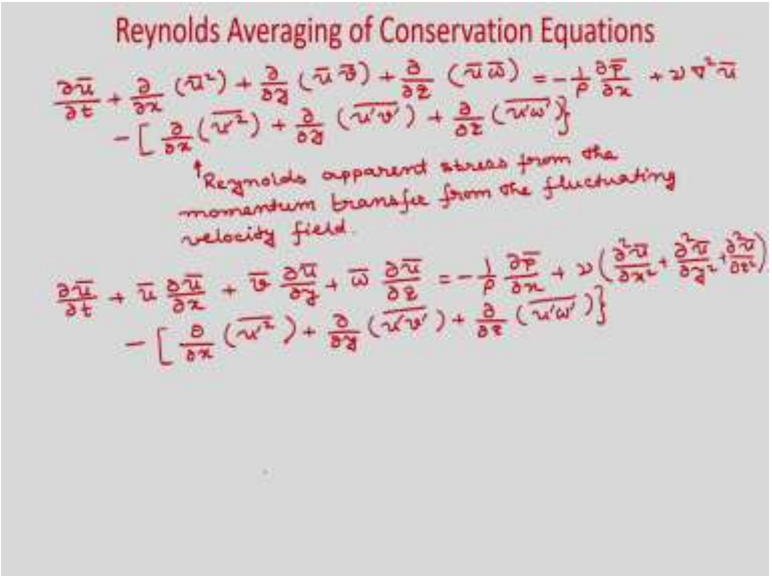
So, now, you take the time average of this equation and invoke some properties which we have already derived. So, taking the time average of the above equation and putting, $\overline{f^2}=\bar{f}^2+\overline{f'^2}$.

And also you write $\overline{fg}=\bar{f}\,\bar{g}+\overline{f'g'}$ and $\overline{f'}=0$. So, we will use these properties and we will simplify the above equation. So, if you put it. So, you will get this as,

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial \overline{u'}}{\partial t}+\frac{\partial}{\partial x}\left\{\left(\bar{u}\right)^2+\overline{u'^2}\right\}+\frac{\partial}{\partial y}\left\{\bar{u}\bar{v}+\overline{u'v'}\right\}+\frac{\partial}{\partial z}\left\{\bar{u}\bar{w}+\overline{u'w'}\right\}=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}-\frac{1}{\rho}\frac{\partial \overline{P'}}{\partial x}+\nu\nabla^2\bar{u}+\nu\nabla^2\overline{u'}$$

So, here you can see this will be 0 because time average of the fluctuating components will be 0, this will be 0 and this will be 0.

(Refer Slide Time: 33:11)



So, now you can write this equation as,

$$\frac{\partial \bar{u}}{\partial t}+\frac{\partial}{\partial x}\left(\bar{u}\right)^2+\frac{\partial}{\partial y}\left(\bar{u}\bar{v}\right)+\frac{\partial}{\partial z}\left(\bar{u}\bar{w}\right)=-\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}+\nu\nabla^2\bar{u}-\left[\frac{\partial}{\partial x}\left(\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right].$$

So, you can see these are the new unknowns. So, these three terms are not zero because we have already shown that your $\overline{f'g'}\neq 0$.

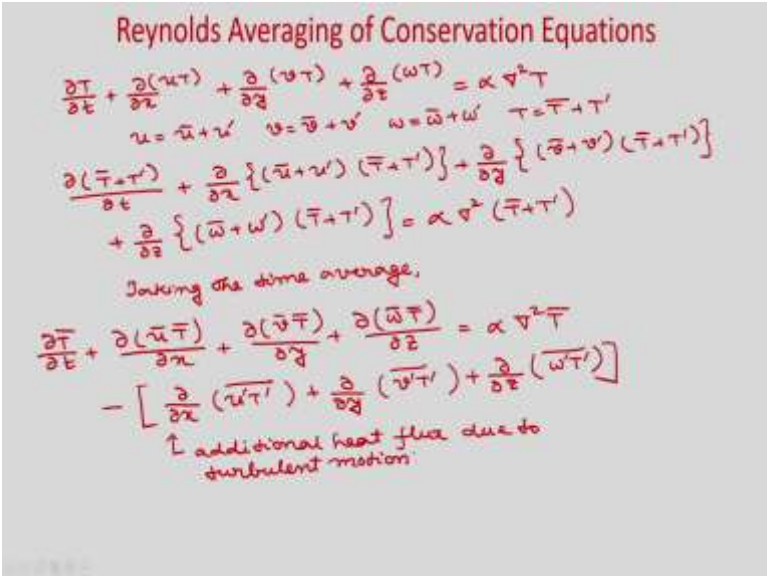
So, as these are not zero. So, these three unknowns are appearing during the Reynolds decomposition due to the fluctuating components of the velocities. So, you can see these are known as Reynolds apparent stress later we will discuss in detail Reynolds apparent stress from the momentum transfer from the fluctuating components.

So, now these we can write in nonconservative form. So, you can write,

$$\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x}+\nu\left(\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\partial^2 \overline{u}}{\partial z^2}\right)-\left[\frac{\partial}{\partial x}\left(\overline{u'^2}\right)+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\overline{u'w'}\right)\right]$$

Now, let us consider the energy equation and we will use the Reynolds decomposition in similar way whatever we have done for the x momentum equation.

(Refer Slide Time: 36:46)



So, your energy equation is $\frac{\partial T}{\partial t}+\frac{\partial}{\partial x}(uT)+\frac{\partial}{\partial y}(vT)+\frac{\partial}{\partial z}(wT)=\alpha\nabla^2 T$. So, now, you use

$u=\overline{u}+u'$, $v=\overline{v}+v'$, $w=\overline{w}+w'$ and $T=\overline{T}+T'$.

So, if you invoke and do the similar way if you take the time average of these quantities. So, you can write,

$$\frac{\partial\left(\overline{T}+T'\right)}{\partial t}+\frac{\partial}{\partial x}\left\{\left(\overline{u}+u'\right)\left(\overline{T}+T'\right)\right\}+\frac{\partial}{\partial y}\left\{\left(\overline{v}+v'\right)\left(\overline{T}+T'\right)\right\}+\frac{\partial}{\partial z}\left\{\left(\overline{w}+w'\right)\left(\overline{T}+T'\right)\right\}=\alpha\nabla^2\left(\overline{T}+T'\right)$$

So, taking the time average and using the properties you can write, $\frac{\partial \overline{T}}{\partial t}+\frac{\partial\left(\overline{u}\overline{T}\right)}{\partial x}+\frac{\partial\left(\overline{u}\overline{T}\right)}{\partial y}+\frac{\partial\left(\overline{u}\overline{T}\right)}{\partial z}=-\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x}+\nu\nabla^2\overline{T}-\left[\frac{\partial}{\partial x}\left(\overline{u'T'}\right)+\frac{\partial}{\partial y}\left(\overline{v'T'}\right)+\frac{\partial}{\partial z}\left(\overline{w'T'}\right)\right]$.

So, these are again unknown terms. So, you have additional heat flux due to turbulent motion.

(Refer Slide Time: 39:42)

Reynolds Averaged Navier-Stokes Equations

Continuity equation:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$

x – component momentum equation:

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)-\rho\frac{\partial(\overline{u'u'})}{\partial x}-\rho\frac{\partial(\overline{u'v'})}{\partial y}-\rho\frac{\partial(\overline{u'w'})}{\partial z}$$

y – component momentum equation:

$$\rho\left(\frac{\partial \bar{v}}{\partial t}+\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial y}+\mu\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}+\frac{\partial^2 \bar{v}}{\partial z^2}\right)-\rho\frac{\partial(\overline{u'v'})}{\partial x}-\rho\frac{\partial(\overline{v'v'})}{\partial y}-\rho\frac{\partial(\overline{v'w'})}{\partial z}$$

z – component momentum equation:

$$\rho\left(\frac{\partial \bar{w}}{\partial t}+\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial z}+\mu\left(\frac{\partial^2 \bar{w}}{\partial x^2}+\frac{\partial^2 \bar{w}}{\partial y^2}+\frac{\partial^2 \bar{w}}{\partial z^2}\right)-\rho\frac{\partial(\overline{u'w'})}{\partial x}-\rho\frac{\partial(\overline{v'w'})}{\partial y}-\rho\frac{\partial(\overline{w'w'})}{\partial z}$$

In tensor form,

$$\rho\left(\frac{\partial \bar{u}_i}{\partial t}+\frac{\partial \overline{\bar{u}_i\bar{u}_j}}{\partial x_j}\right)=-\frac{\partial \bar{p}}{\partial x_i}+\mu\frac{\partial^2 \bar{u}_i}{\partial x_j^2}-\rho\frac{\partial}{\partial x_j}\left(\overline{u_i'u_j'}\right)$$

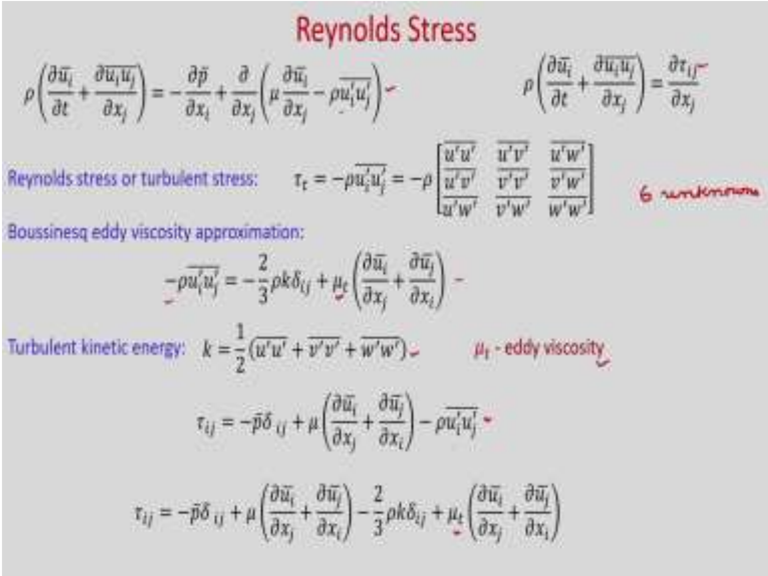
Energy equation:

$$\rho c_p\left(\frac{\partial \bar{T}}{\partial t}+\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}++\bar{w}\frac{\partial \bar{T}}{\partial z}\right)=k\left(\frac{\partial^2 \bar{T}}{\partial x^2}+\frac{\partial^2 \bar{T}}{\partial y^2}++\frac{\partial^2 \bar{T}}{\partial z^2}\right)-\rho c_p\frac{\partial(\overline{u'T'})}{\partial x}-\rho c_p\frac{\partial(\overline{v'T'})}{\partial y}-\rho c_p\frac{\partial(\overline{w'T'})}{\partial z}$$

So, now if you carry out the time averaging of y momentum and z momentum equation similar way then we can write the Reynolds average Navier stokes equation as. So, this is your continuity equation this is the x component momentum equation and these are the three additional terms in y component momentum equation these are the three additional terms, in z component momentum equation these are the three additional terms.

So, now, these three equations you can write in tensor form as $\rho\left(\frac{\partial \overline{u_i}}{\partial t}+\frac{\partial \overline{u_i u_j}}{\partial x_j}\right)=-\frac{\partial \overline{p}}{\partial x_i}+\mu\frac{\partial^2 \overline{u_i}}{\partial x_j^2}-\rho\frac{\partial}{\partial x_j}\left(\overline{u_i^{'}u_j^{'}}\right)$. So, you can see this is the additional term and energy equation also you can see these are the additional terms.

(Refer Slide Time: 40:31)



So, if you see you can rewrite the momentum equation in tensor form as these where $\frac{\partial}{\partial x_j}$ we have taken. So, $\mu\frac{\partial \overline{u_i}}{\partial x_j}-\rho\left(\overline{u_i' u_j'}\right)$.

So, these right hand side quantity if we put as the total stress $\frac{\partial \tau_{ij}}{\partial x_j}$, then this stress term this is known as Reynolds stress or turbulent stress. So, we can write $\tau_t=-\rho\left(\overline{u_i' u_j'}\right)=-\rho\begin{pmatrix}\overline{u'u'} & \overline{u'v'} & \overline{u'w'}\\ \overline{u'v'} & \overline{v'u'} & \overline{v'w'}\\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'}\end{pmatrix}$. So, this is the Reynolds stress tensor and here you can see how many unknowns are there because it is a symmetric tensor.

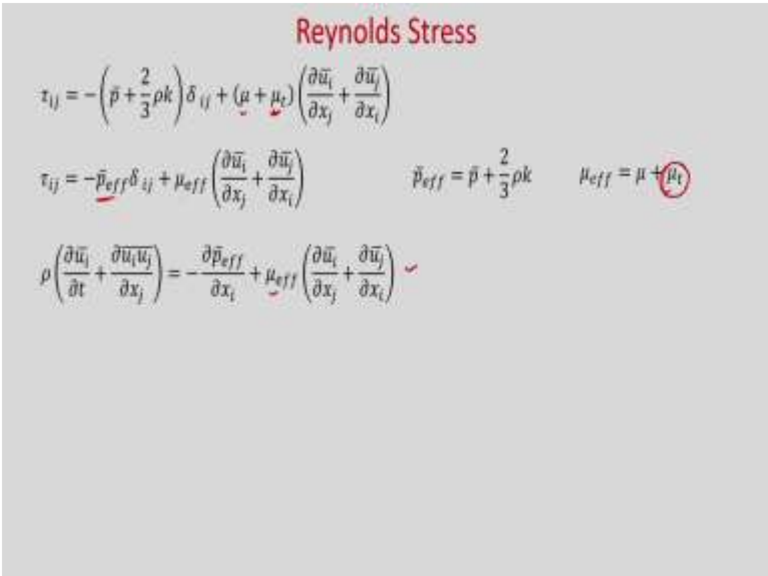
So, there are six unknowns there are six unknowns. So, 1, 2, 3, 4, 5 then 6. So, these six unknowns now we have to model to find these unknowns. So, we can use Boussinesq eddy viscosity approximation where this Reynolds stress is written in this way $-\rho\overline{u_i' u_j'}=-\frac{2}{3}\rho k\delta_{ij}+\mu_t\left(\frac{\partial \overline{u_i}}{\partial x_j}+\frac{\partial \overline{u_j}}{\partial x_i}\right)$ where k is the turbulent kinetic energy and δ_{ij} is the Kronecker delta.

Kronecker delta you know that if i = j then its value is 1 otherwise 0 +μ_t which is your turbulent viscosity or eddy viscosity it is known as eddy viscosity it is known as eddy viscosity and in terms of your the gradient of the mean velocity. So, you can write these Reynolds stress in terms of the gradient of the mean velocity with a unknown parameter eddy viscosity.

So, you can see now this μ_t is unknown. So, that you need to model. So, here turbulent kinetic energy is given by this expression and if you see the total stress, then total stress will be your laminar stress plus the turbulent stress we already know from the constitutive relation and this is the additional stress. So, that is your turbulent stress.

So, this expression if you put it here and rearrange, then you will get in terms of the time average velocity gradient and mu t is unknown.

(Refer Slide Time: 43:28)

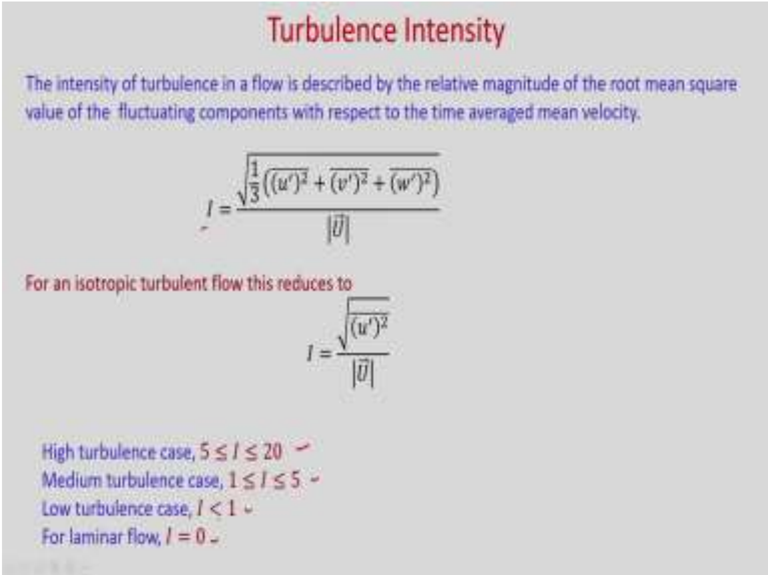


So, if you rearrange it. So, you can see $\tau_{ij} = -\left(\overline{p} + \frac{2}{3}\rho k\right)\delta_{ij} + (\mu + \mu_t)\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$.

So, you can see here only unknown is μ_t. So, we have expressed the Reynolds stress in terms of the gradient of a time average velocity. So, this $\overline{p}_{eff} = \overline{p} + \frac{2}{3}\rho k$ you can write as and μ_{eff} which is your dynamic viscosity molecular viscosity and this is your turbulent viscosity.

So, these two together you can write as μ_{eff}. So, if you invoke in the tensor equation. So, you will get in this form where $\mu_{eff} = \mu + \mu_t$ and this μ_t is unknown. So, this μ_t is to be modelled in this regard we will discuss about the turbulence intensity.

(Refer Slide Time: 44:45)



So, it is a some measure about the turbulence and you can see it is defined as the intensity of turbulence in a flow is described by the relative magnitude of the root mean square value of the fluctuating components with respect to the time average mean velocity. So, if I is the turbulent intensity, then it is root mean square value of this fluctuating component.

So, $I = \frac{\sqrt{\frac{1}{3}\left(\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}\right)}}{|\vec{U}|}$. So, if you have a flow over flat plate then it will become u_∞ and if you consider isotropic turbulent flow. So, it is not having any directional preference. So, $\overline{u'} = \overline{v'} = \overline{w'}$.

So, you will get this intensity $I = \frac{\sqrt{\overline{(u')^2}}}{|\vec{U}|}$. So, you can see if this intensity is between 5 and 20, then you can say that it is a high turbulent case if it is in between 1 and 5, then it is medium turbulence case and if it is > 1 then it is low turbulence case and; obviously, you can see for laminar flow these fluctuating components will be 0 then your turbulent intensity will be 0.

So, when you numerically solve this Reynolds average Navier stokes equations then you need to give the turbulence intensity at the inlet so, that you can impose some turbulence at the entry.

So, in today's class we started with the characteristic of the turbulent flows then we have discussed about two main characteristic of the turbulent flows one is a homogeneous turbulence, another is isotropic turbulence, then we discuss about the Reynolds decomposition and we started with the continuity equation and using the Reynolds decomposition we have shown that mean velocity satisfy the continuity equation as well as the fluctuating components also satisfy the continuity equation.

When we used the Reynolds decomposition for x momentum equation and taking the time average of this equation there are three additional unknowns appear.

So, these three unknowns are coming from the fluctuating components of the velocity and if you have these three momentum equations u, v and w momentum equations then obviously, you will get nine additional terms out of which six are unknowns and you can write in as a Reynolds stress or the apparent stress.

We have also cut out the time averaging of the energy equation and using Reynolds decomposition we have written the time average energy equation there also we have seen there are additional three unknowns.

Later we use the constitutive relation and the Boussinesq approximation for the Reynolds stress and we have written the Reynolds average Navier stokes equation in tensor form in terms of the eddy viscosity. So, you can see that here it resembles with the laminar flow Navier stokes equations except one additional term in the viscous term.

So, that is your Reynold stress and from the Reynold stress using this Boussinesq approximation and the constitutive relation we have shown that you will get the effective viscosity as the as a summation of molecular viscosity and the turbulent or eddy viscosity and here you can see only one unknown term is there that is your eddy viscosity μ_t. So, in later classes we will try to find these unknown eddy viscosity.

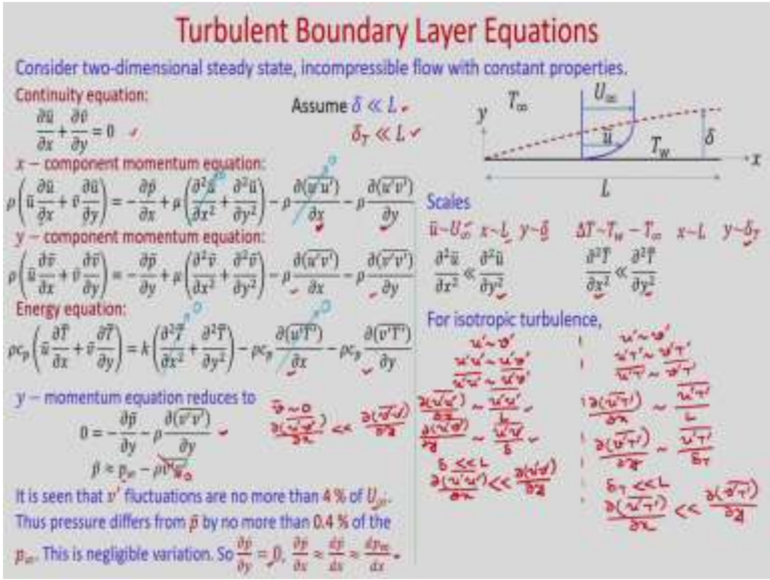
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 11
Turbulent Flow and Heat Transfer
Lecture – 36
Convection in Turbulent External Flow

Hello everyone, in last class we derived the Reynolds average Navier-Stokes equations as well as the time averaged energy equation. Today, we will consider external flows and we will derive the turbulent boundary layer equations. Then we will discuss about different turbulent layers inside the boundary layer, and we will find the universal velocity profile and universal temperature profile.

(Refer Slide Time: 01:03)



So, you can see that if we consider two-dimensional steady state incompressible flow with constant properties. Then you have this continuity equation. This is the x component momentum equation. You can see we have additional terms. And in y momentum equation, here also we have two additional terms. And in energy equation, we have also two additional terms these are coming due to the fluctuations in velocities and temperature.

So, now in boundary layer approximation, if we assume that δ << L and δ_T which is your thermal boundary layer is much much smaller than the length of the plate, then using scale analysis we have already carried out for external flows we can do the similar analysis and we can derive the boundary layer equations. Here the other terms will have the similar derivation as we did earlier, but the fluctuating terms we will see here specially.

So, you can see if you use the scales of velocity $\bar{u}$ ~U_∞, x ~ L, and y ~δ_T, then obviously, this you can show right that $\frac{\partial^2 \bar{u}}{\partial x^2} \ll \frac{\partial^2 \bar{u}}{\partial y^2}$. So, you can see that these already we have derived earlier, and these inner shear terms will be comparable, so we cannot neglect this.

But if you do the similar analysis for the energy equation and the $\Delta T \sim (T_w - T_\infty)$ and y as you scale of thermal boundary layer thickness δ_T, then obviously, you can show that $\frac{\partial^2 \bar{T}}{\partial x^2} \ll \frac{\partial^2 \bar{u}}{\partial y^2} T$.

Now, what about these terms. can we neglect any term from these two terms? So, let us see that. So, first let us assume that it is a isotropic turbulence. So, for these you know that there is no preferred direction of the fluctuation. So, we can write $u' \sim v'$.

So, now if we multiply u' both side, so you can write $u'u' \sim u'v'$. And if you take the average of this quantity, then you can write $\overline{u'u'} \sim \overline{u'v'}$. So, this will be same order, but now let us see what will be this gradient, the order of these gradients then you can see that this first fluctuating term if you write.

So, $\frac{\partial\left(\overline{u'u'}\right)}{\partial x} \sim \frac{u'u'}{L}$. And this term if you see, then it will be $\frac{\partial\left(\overline{u'v'}\right)}{\partial y} \sim \frac{u'u'}{\delta}$. So, now you can see we have already assumed that δ << L. So, obviously, if you compare these two terms, so you can say that you have $\frac{\partial\left(\overline{u'u'}\right)}{\partial x} \ll \frac{\partial\left(\overline{u'v'}\right)}{\partial y}$.

So, you can see that in the x momentum equation, the last term you can consider because it is having a higher order than the this term. So, you keep this term and this term you can neglect. Now, you can also show in the y momentum equation that $\bar{v} \sim 0$.

And from there you can see for this particular case, your y momentum equation keeping the $\bar{v} \sim 0$, you can also show that $\frac{\partial\left(\overline{u'v'}\right)}{\partial x} \ll \frac{\partial\left(\overline{v'v'}\right)}{\partial y}$. So, from there, you can reduce this y momentum equation as this ok.

In laminar flow, it was $\frac{\partial p}{\partial y} = 0$. But as it is turbulent flows, you will have this term. So, you can see $\bar{p}$ will be if you integrate this equation, so at outside the boundary layer you have the free stream pressure p_∞. So, $\bar{p} \approx p_\infty - \rho\left(\overline{v'v'}\right)$. But it is seen that v' fluctuations are no more than 4 % of the free stream velocity U_∞.

Thus pressure differs from $\bar{p}$ by no more than 0.4 % of the p_∞. So, you can actually neglect this term, and you can write from here that $\frac{\partial \bar{p}}{\partial y} = 0$. So, you can see as laminar flow we have derived that pressure does not vary perpendicular to the wall.

So, $\frac{\partial \bar{p}}{\partial y} = 0$. And say hence you can write that $\frac{\partial \bar{p}}{\partial x} \approx \frac{d\bar{p}}{dx}$. And you can equate it also with the free stream pressure p_∞. So, we have written $\frac{d\bar{p}}{dx} \approx \frac{dp_\infty}{dx}$.

Similarly, now if you do the scale analysis of these two terms. So, you can see from for isotropic turbulence $u' \sim v'$. Then you can write $u'T' \sim v'T'$. So, if you take its time average, so you can write $\overline{u'T'} \sim \overline{v'T'}$.

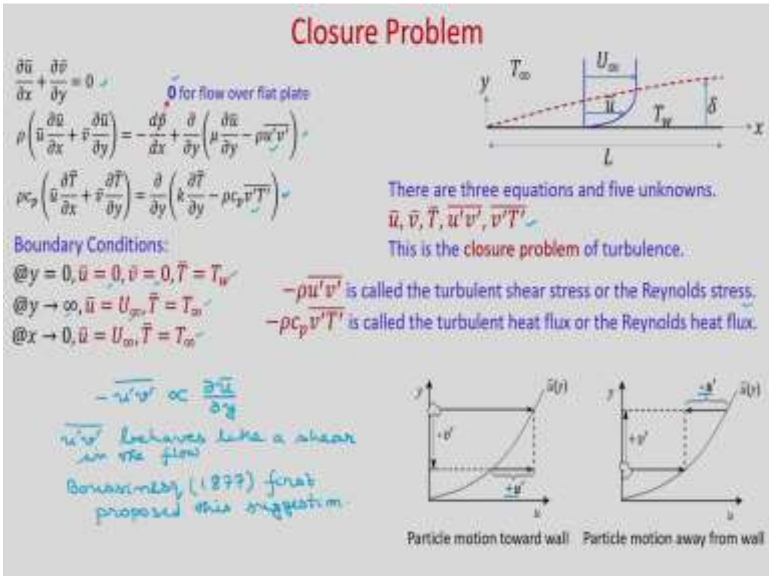
So, now, let us see the derivative , what is the scale. So, $\frac{\partial\left(\overline{u'T'}\right)}{\partial x} \sim \frac{\overline{u'T'}}{L}$. And here you can write, $\frac{\partial\left(\overline{u'T'}\right)}{\partial y} \sim \frac{\overline{u'T'}}{\delta_T}$.

So, now, we have already assumed that δ_T << L. Hence, you can write $\frac{\partial\left(\overline{u'T'}\right)}{\partial x} \ll \frac{\partial\left(\overline{v'T'}\right)}{\partial y}$.

So, now you can see whatever equations we have written for two-dimensional steady turbulent flow equations, we can neglect few terms for the boundary layer equations. So, you can see, here we can drop this term. We can drop this term from scale analysis we have shown. And these you can write as $\frac{d\bar{p}}{dx}$ and anyway all these term will become 0.

So, $\frac{\partial \bar{p}}{\partial y} = 0$. And in energy equation, similarly you can neglect this term and also this term ok, because its magnitude is very small compared to the other terms. So, if you can neglect, then you can write the continuity equation as this. This is the x momentum equation and this is the energy equation after dropping low magnitude order terms.

(Refer Slide Time: 09:32)



But if you consider flow over flat plate, so if you consider a flow over flat plate of length L, your $\bar{u}$ will be the mean velocity which varies from 0 to U_∞, U_∞ is the free stream velocity, and free stream temperature is T_∞, and wall temperature is T_W . So, if you consider flow over flat plate, obviously, $\frac{d\bar{p}}{dx}$ = 0.

Now, let us discuss about the boundary conditions. So, what are the boundary conditions? So, obviously, at x = 0, you have U = U_∞, and T = T_∞ . And y = 0 this is the wall, so again $\bar{u}$ = 0 and $\bar{v}$ = 0, and $\bar{T} = T_w$.

And y → ∞ ; you have free stream velocity and free stream temperature. So, you can see at y =0, you have $\bar{u}$ = 0, $\bar{v}$ = 0, $\bar{T} = T_w$; y →∞ $\bar{u}$ = U_∞ , $\bar{T} = T_\infty$, and x →0 ; $\bar{u}$ = U_∞ , $\bar{T} = T_\infty$.

Now, if you see these equations, there are three equations. And how many unknowns are there? You can see u', v', then you have T'. You can see there are three equations, and we have how many variables unknown variables $\bar{u}$, $\bar{v}$, $\bar{T}$. And you have two more terms $\overline{u'v'}$, and $\overline{v'T'}$. So, you can see these are the five unknowns and we have three equations.

So, this is known as closure problem of turbulence. So, we need to model these two terms these $\overline{u'v'}$ and v prime $\overline{v'T'}$. So, you can see these term in the momentum equation is called the turbulent shear stress or the Reynolds stress. And this term in the energy equation is called the turbulent heat flux or the Reynolds heat flux.

Now, these we need to model with the known parameters. Now, you see in the turbulent flows in the inside the boundary layer, one particle is here. Now, due to fluctuation it is forced to move at this position. So, you will have the v' will be negative and it will come here. So, the particle if you see here it has higher velocity than here. So, your local velocity is low and this particle will obviously feel low velocity when it will come, but it is having higher velocity than the local velocity.

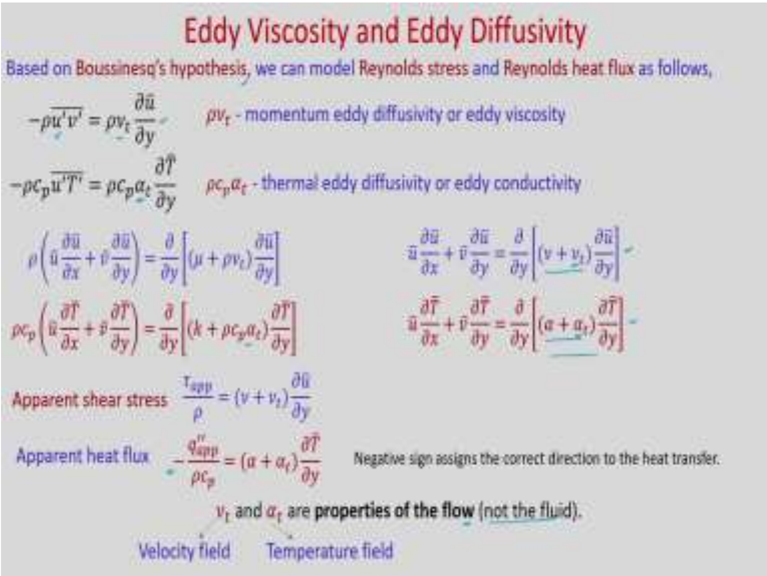
So, it will have some fluctuation of plus u', so that means, you can see when this particle is coming towards the wall, obviously, it is experiencing one velocity fluctuation as plus u'. So, obviously, the value of this u' will depend on the velocity gradient. So, we can model this $\overline{u'v'}$ with the velocity gradient of the time average velocity.

Similarly, if particle motion away from the wall, if it is going from here to here, so you can see obviously here when it will come, it will experience a higher velocity. So, it will have a minus u prime fluctuation, and it will also depend on the velocity gradient. So, you can see this $u'v'$, obviously, will give a negative value.

Because when v' is positive your u' is negative, and when v' is negative u' is positive. So, $u'v'$ is a itself a negative quantity because if one is positive other will be negative.

So, from these analysis, we can say that $-\overline{u'v'} \propto \frac{\partial \bar{u}}{\partial y}$, so that means, $\overline{u'v'}$ behaves like a shear in the flow. So, this suggestion was first made by Boussinesq, Boussinesq first proposed this suggestion. So, now you can see that we can actually write $\overline{u'v'}$ in terms of the velocity gradient.

(Refer Slide Time: 15:18)



So, based on Boussinesq hypothesis we can model Reynolds stress and Reynolds heat flux as follows. So, this is your Reynolds stress. So, $-\rho\overline{u'v'}$ that we are relating with the velocity gradient $\frac{\partial \bar{u}}{\partial y}$ and $\rho\nu_t$. So, $\rho\nu_t$ is known as momentum eddy diffusivity, and it is known as eddy viscosity also.

And $-\rho c_p \overline{u'T'}$, you can model it as $\rho c_p \alpha_t \frac{\partial \bar{T}}{\partial y}$. And this $\rho c_p \alpha_t$ is the thermal eddy diffusivity or eddy conductivity. So, this is your eddy viscosity commonly known, and this is commonly known as eddy diffusivity.

Now, you can see still here ν_t and α_t are unknown . So, now, our task is to model these eddy viscosity and eddy diffusivity. Now, if you put this in the momentum equation and energy equation, what you will get? So, in the right hand side, now this term you are replacing with this term.

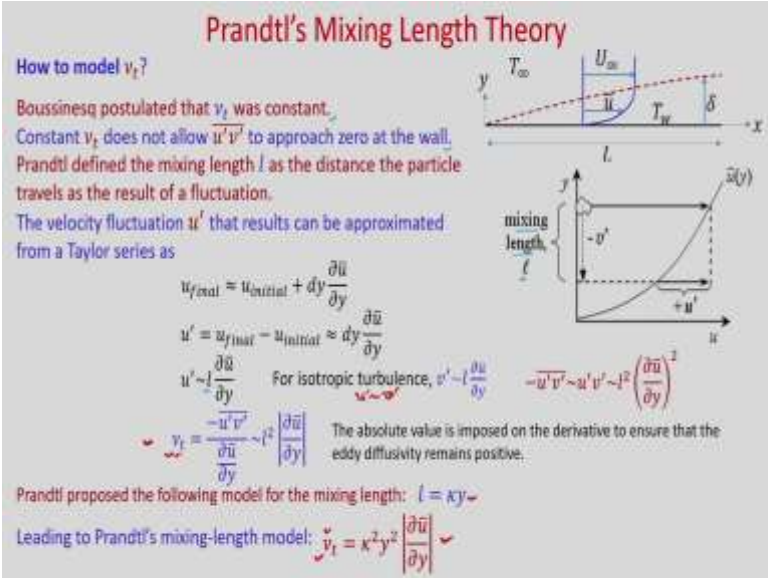
So, you can write $\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)=\frac{\partial}{\partial y}\left[\left(\mu+\rho\nu_t\right)\frac{\partial \bar{u}}{\partial y}\right]$. So, you can see we have written in terms of some shear stress. And similarly you can see in the energy equation if we take this term, then we can write $\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}\right)=\frac{\partial}{\partial y}\left[\left(k+\rho c_p\alpha_t\right)\frac{\partial \bar{T}}{\partial y}\right]$. So, this also represents some heat flux.

So, if you rearrange it. So, after rearranging you can write, $\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\frac{\partial}{\partial y}\left[\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial y}\right]$. And in energy equation, $\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}=\frac{\partial}{\partial y}\left[\left(\alpha+\alpha_t\right)\frac{\partial \bar{T}}{\partial y}\right]$ where terms ν_t and α_t are unknown.

So, these now together you can say that this is the apparent shear stress, $\frac{\tau_{app}}{\rho}=\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial y}$. And this you can say that it is apparent heat flux it is $-\frac{q_{app}}{\rho c_p}=\left(\alpha+\alpha_t\right)\frac{\partial \bar{T}}{\partial y}$. So, you can see here we have put negative sign, because it assigns the correct direction to the heat transfer.

So, now we can see now ν_t and α_t are properties of the flow. You remember not the fluid. So, this eddy viscosity and eddy diffusivity are properties of flow, because ν_t depends on the velocity field and α_t depends on the temperature field. Now, the question is that how to model this ν_t and α_t? So, first let us discuss how to model ν_t.

(Refer Slide Time: 18:40)



So, first Boussinesq postulated that ν_t was constant . If ν_t is constant, then you can see near to the wall it will have some constant value, but there your it should be 0 right the fluctuation velocity fluctuation should be 0.

So, this model will give problem when you go closer to the wall. So, constant ν_t does not allow $\overline{u'v'}$ to approach zero at the wall. So, Prandtl defined the mixing-length L as the distance the particle travels as the result of a fluctuation.

So, the velocity fluctuation u prime that results can be approximated from a Taylor series as. So, you can see now this particle is forced to move here. So, as we discussed earlier, so obviously, it will have negative v velocity and due to that it will have some fluctuation in u, and that will be u'. And whatever distance it travels, so that is known as mixing length L.

So, now if you see if you tell that velocity here is u_{final} and it is $u_{initial}$ using Taylor series, you can write $u_{final} \approx u_{initial} + dy\frac{\partial \bar{u}}{\partial y}$ and neglect the higher order terms. So, the difference between u_{final} and $u_{initial}$ will be your velocity fluctuation. So, $u' = u_{final} - u_{initial} \approx dy\frac{\partial \bar{u}}{\partial y}$.

So, this distance now whatever it travelled that we are telling mixing length, so $u' \sim l\frac{\partial \bar{u}}{\partial y}$.

So, now, for isotropic turbulence, you know that for isotropic turbulence you know that $u' \sim v'$. So, obviously, $v' \sim l\frac{\partial \bar{u}}{\partial y}$. So, you can see this minus $-\overline{u'v'} \sim u'v' \sim l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$.

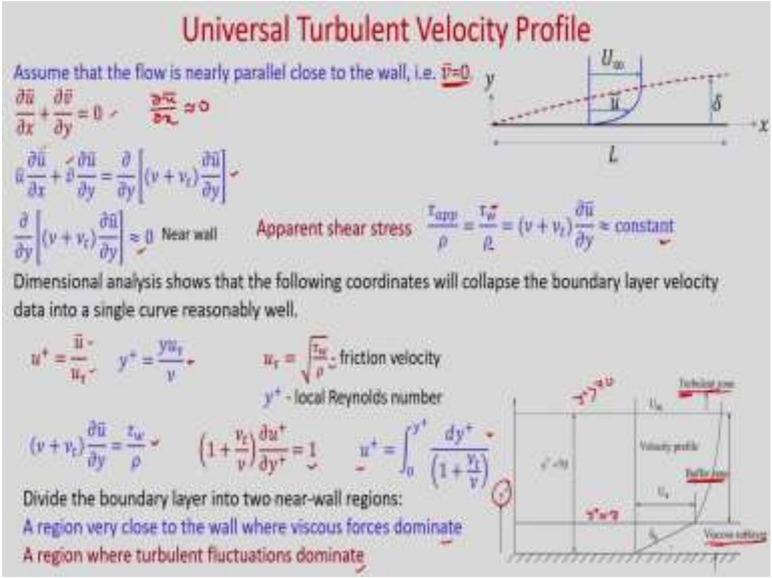
So, now if you write ν_t, so you can write $\nu_t = \frac{-\overline{u'v'}}{\frac{\partial \bar{u}}{\partial y}} \sim l^2\left|\frac{\partial \bar{u}}{\partial y}\right|$. So, these modulus we are giving or absolute value is imposed on the derivative to ensure that the eddy diffusivity remains positive because this is a positive quantity. So, it remains positive.

So, this is the way we can model using Prandtl's mixing-length theory. It is the simplest model we can have. And Prandtl propose the following model for the mixing-length $l = \kappa y$, and κ is constant. And it differs for different types of flows. And leading to Prandtl mixing length model now ν_t, you can write Prandtl proposed the following model for the mixing-length $l = \kappa y$.

So, this κ value depends on different types of flow. And leading to Prandtl's mixing-length model now eddy diffusivity, you can model as $\nu_t = \kappa^2 y^2\left|\frac{\partial \bar{u}}{\partial y}\right|$. So, you can see this is the simplest model for to determine the eddy diffusivity or eddy viscosity.

So, now, we have found the eddy viscosity ν_t using the Prandtl's mixing-length hypothesis. Inside the boundary layer close to the wall, you can see you can neglect the fluctuating velocities $u'v'$. And if you are away from the wall, then the effect of molecular viscosity can be neglected. So, in based on that, you can differentiate two different layers inside the boundary layer.

(Refer Slide Time: 22:56)



So, you can see if this is the flat plate and you can have one layer very close to the wall where you can neglect the velocity fluctuations, and you can have the viscous sub layer. And away from the wall where you can have fully turbulent layer, where you can neglect the effect of this molecular viscosity. And in between this zone is known as buffer layer or buffer zone.

So, you can see that we have these equations. This is the continuity equation and this is the momentum equation. Now, we are assuming that the flow is nearly parallel close to the wall. So, if close to the wall if we assume nearly parallel, that means, v bar will be 0 ok. And if $\bar{v}$ = 0, then from the continuity equation, you can say that $\frac{\partial \bar{u}}{\partial x}$ ~ 0.

So, now, you can see if this is 0, then in the momentum equation you can see this term is 0 and $\bar{v}$ = 0. So; obviously, inertia terms you can neglect. So, if you neglect the inertia term, so very near to the wall you can say that $\frac{\partial}{\partial y}\left[\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial y}\right]\approx 0$.

So, now we have already defined apparent shear stress as $\frac{\tau_{app}}{\rho}=\frac{\tau_w}{\rho}=\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial y}$. And obviously, τ for steady state flow, τ_w will be constant, density of the fluid is constant. So, this term will be constant.

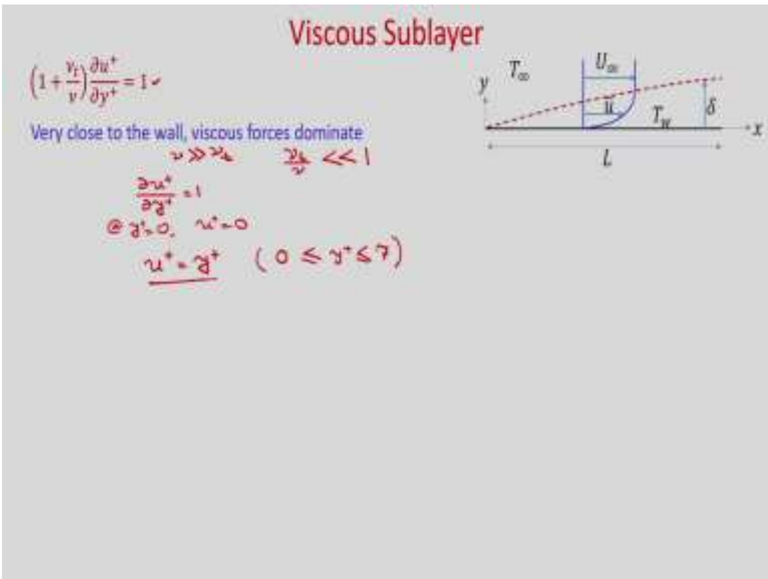
So, now, if we define this non-dimensional quantities $u^+ = \frac{\bar{u}}{u_\tau}$ where u_τ is known as friction velocity which is defined as $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ and τ_w is the shear stress at the wall. And $y^+ = \frac{y u_\tau}{\nu}$. So, this term you can see it is related to Reynolds number, because the $\mathrm{Re} = \frac{U_\infty L}{\nu}$. So, similarly it is known as local Reynolds number.

So, if you define it and this equation whatever we got because this if you integrate then you will get $(\nu + \nu_t)\frac{\partial \bar{u}}{\partial y} = \frac{\tau_w}{\rho}$. And these if you use these non dimensional quantities, you can write as $\left(1 + \frac{\nu_t}{\nu}\right)\frac{\partial u^+}{\partial y^+} = 1$. So, you can see that $u^+ = \int_0^{y^+} \frac{dy^+}{\left(1 + \frac{\nu_t}{\nu}\right)}$, then you will be able to find the velocity.

Now, we are actually dividing the boundary into two near wall regions, a region very close to the wall where viscous force dominant, and a region where turbulent fluctuation dominate. So, you can see away from the surface, it will be fully turbulent zone. And effect of molecular viscosity, you can neglect and turbulent fluctuation will dominate. And generally you can see this y^+ if you take in this way which is your non-dimensional coordinate.

So, here near to the wall , it is around $y^+ = 7$, you can say that it is viscous sub layer. Then away from 70 if $y^+ > 70$, then it is fully turbulent layer. And between 7 and 70, you will get buffer layer. Now, let us consider the near wall region which is your known as viscous sub layer. So, in the viscous sub layer, we can neglect the fluctuating components. So, generally your viscous effect will dominate the flow.

(Refer Slide Time: 27:07)

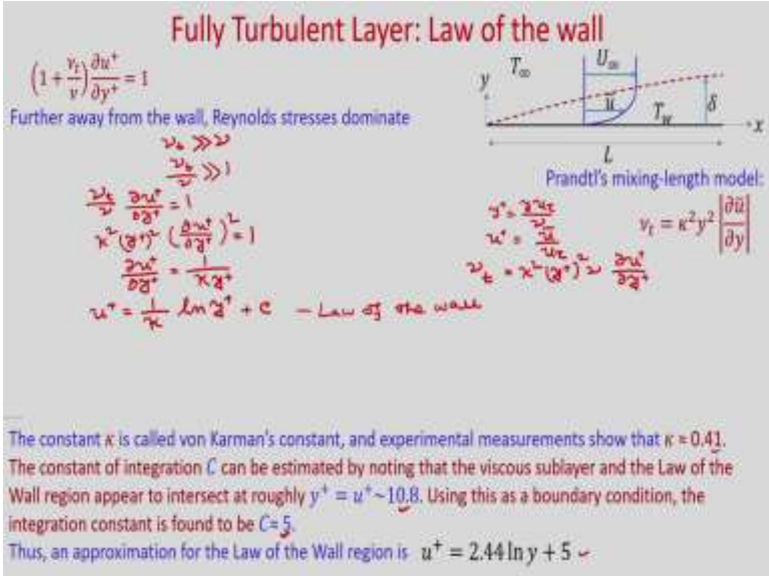


So, in this equation, now you can say that $\nu >> \nu_t$. If $\nu >> \nu_t$, then this equation you can write as $\frac{\partial u^+}{\partial y^+}=1$, because $\frac{\nu_t}{\nu}$ << 1. So, you can neglect this term. So, you will get, $\frac{\partial u^+}{\partial y^+}=1$.

So, now if you integrate it as and put the boundary condition as $y^+ = 0$, obviously, your $u^+ = 0$. So, if you integrate it, you will get $u^+ = y^+$. So, it is valid in the viscous sub layer in the range of $0 \le y^+ \le 7$.

And you can see it is a linear profile in terms of non-dimensional quantities. So, you can see here this varies linearly in the viscous sub layer, because if it is u^+ and this is your y^+, then it varies linearly inside the viscous sub layer. Now, if you go further away from the wall, then your fluctuations will dominate, fluctuating velocities will dominate.

(Refer Slide Time: 28:35)



So, in that case, you can say that $\nu_t >> \nu$. So, if it is so, then you can write that $\frac{\nu_t}{\nu}$ >> 1.

So, this equation we can write as $\frac{\nu_t}{\nu}\frac{\partial u^+}{\partial y^+}=1$. Now, ν_t we know from the Prandtl's mixing-layer hypothesis. What is that? $\nu_t = \kappa^2 y^2 \left|\frac{\partial u^+}{\partial y^+}\right|$.

So, now if you use the non dimensional quantities, $y^+ = \frac{y u_\tau}{\nu}$, and $u^+ = \frac{\bar{u}}{u_\tau}$. Then this, $\nu_t = \kappa^2 \left(y^+\right)^2 \nu \frac{\partial u^+}{\partial y^+}$. So, this ν_t value now you put .

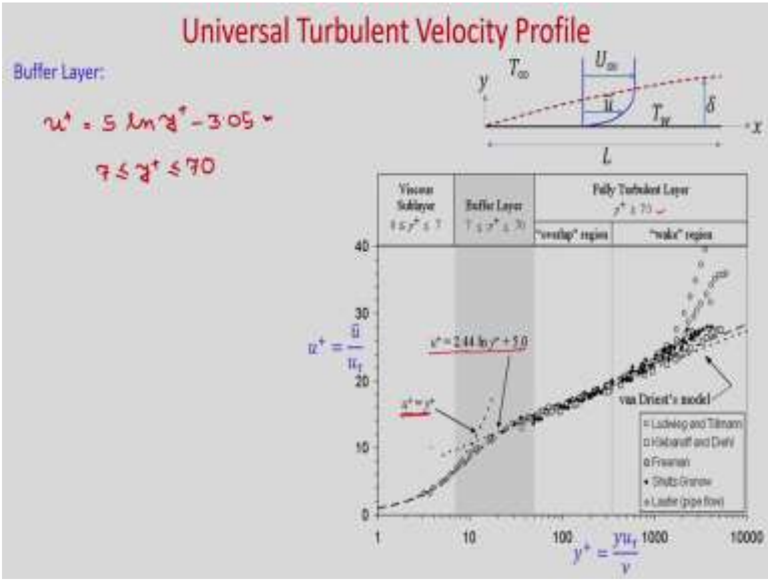
So, $\frac{\nu_t}{\nu}$ will be this quantity. So, you can write it as $\kappa^2 \left(y^+\right)^2 \left(\frac{\partial u^+}{\partial y^+}\right)^2 = 1$. And you can write, $\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+}$.

Now, if you integrate it, you will get $u^+ = \frac{1}{\kappa}\ln y^+ + C$. So, this is known as law of the wall. Now, how to find this κ and C? So, this you need to find empirically you need to find it from the experimental conditions. So, you can see that in this equation you need to know the value of κ as well as the constant C.

The constant κ is called von Karman's constant and experimental measurements, so that κ = 0.41. And the constant of integration C can be estimated by noting that the viscous sub layer and the Law of the Wall region appear to intersect at roughly $y^+ = u^+ \sim 10.8$.

So, if you put $y^+ = 10.8$, then you will be able to find the constant $C \approx 5$, so that is an approximation for the Law of the Wall region is putting the values of κ and see you can get $u^+ = 2.44 \ln y + 5$. So, now, you can see that these two layers viscous sub layer as well as fully turbulent layer will intersect through the buffer layer.

(Refer Slide Time: 31:56)

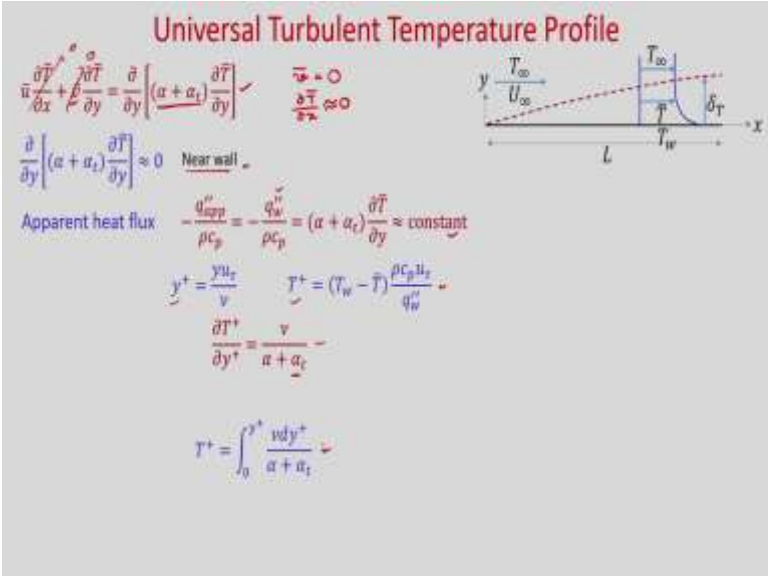


So, you can see in this curve u^+ verses y^+. So, here you can see that this is your viscous sub layer where $y^+ < 7$, and here $u^+ = y^+$. And you have fully turbulent layer, so that is already we have derived as $u^+ = 2.44 \ln y + 5$. So, you can see that these two model meets here in the buffer layer right.

So, in the buffer layer because this is valid in the range of $y^+ > 70$. So, in between 7 and 70, we have buffer layer. And in the buffer layer, you can use $u^+ = 5 \ln y^+ - 3.05$ in the range of $7 \leq y^+ \leq 70$.

So, this is actually connecting your viscous sub layer as well as fully turbulent layer model. Now, we have discussed about the universal velocity profile. Now, let us discuss about the universal temperature profile. So, now, we need to find the eddy diffusivity which is your α_t.

(Refer Slide Time: 33:23)



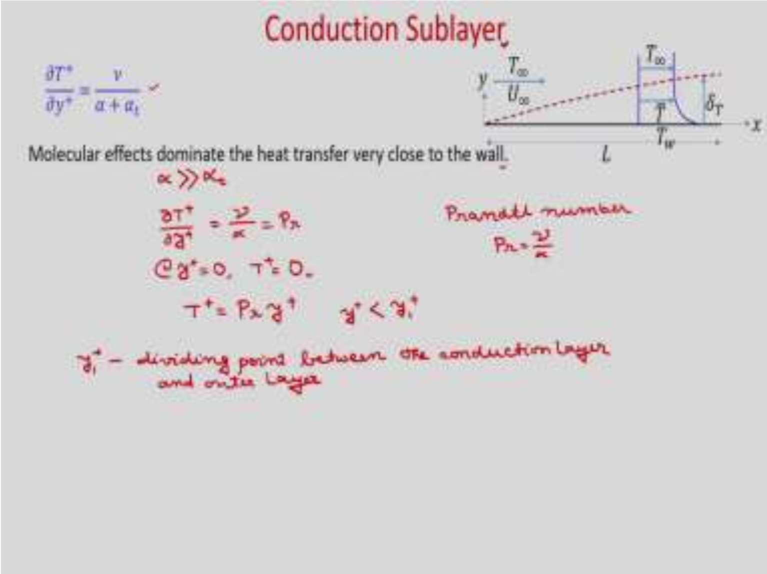
So, you can see this is the equation we have derived. So, this is your $\alpha + \alpha_t$. And similarly in near wall region, we can have this $\bar{v}$ = 0 , because nearly flow is parallel flow is nearly parallel. So, $\bar{v}$ = 0. And your axial heat conduction you can neglect. And you see the variation of $\bar{T}$ along x is very small, so $\frac{\partial \bar{T}}{\partial x}$ ~0.

So, if this is 0 and this is 0, then you can write $\frac{\partial}{\partial y}\left[\left(\alpha + \alpha_t\right)\frac{\partial \bar{T}}{\partial y}\right] \approx 0$ near to the wall. So, obviously, you can write in terms of apparent heat flux $-\frac{q''_{app}}{\rho c_p} = -\frac{q''_w}{\rho c_p}$ which is your wall heat flux, and obviously $\left(\alpha + \alpha_t\right)\frac{\partial \bar{T}}{\partial y} \approx$ constant.

Now, similarly you define y^+, and $T^+ = \left(T_w - \bar{T}\right)\frac{\rho c_p u_\tau}{q''_w}$. So, using these non-dimensional quantities if you put it here, you are going to get $\frac{\partial T^+}{\partial y^+} = \frac{\nu}{\alpha + \alpha_t}$. So, you can see here α_t is unknown . So, $T^+ = \int_0^{y^+} \frac{\nu dy^+}{\alpha + \alpha_t}$.

Now, similarly we can have the conduction layer which is very very near to the wall, where you can neglect the fluctuating components. And away from the wall, you can have fully turbulent region the fluctuating components will dominate.

(Refer Slide Time: 35:29)



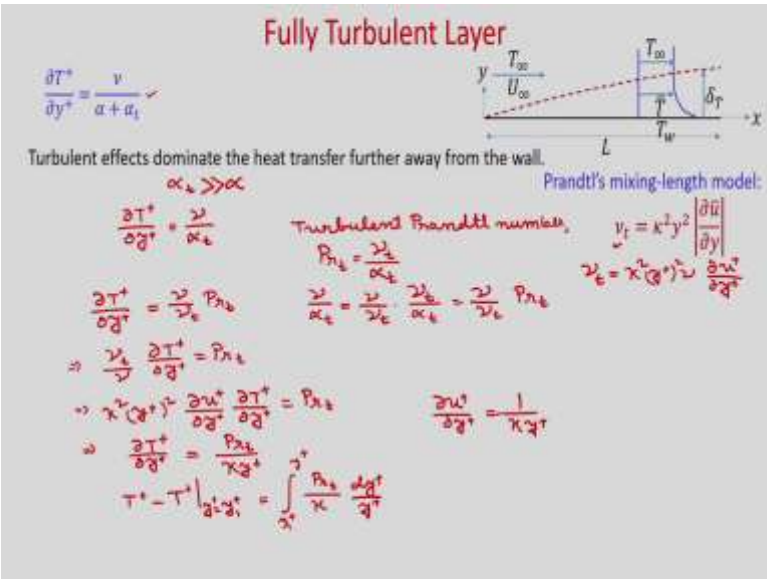
So, with this you can see the conduction sub layer which is very near to the wall. So, this is the equation. So, we are telling that molecular effects dominate the heat transfer very close to the wall. So, fluctuating components you can neglect, that means, here $\alpha \gg \alpha_t$.

So, from here you can see that if $\alpha \gg \alpha_t$, then you can write $\frac{\partial T^+}{\partial y^+} = \frac{\nu}{\alpha}$.

And what is $\frac{\nu}{\alpha}$ you know, Prandtl number right? So, Pr= $\frac{\nu}{\alpha}$. So, you can write this is equal to Prandtl number. Now, if you integrate it and put the boundary condition at y^+ =0 T^+ = 0 because T^+ is having one quantity $\overline{T} - T_w$. So, at y = 0, you have T = T_w. So, T^+ will become 0.

So, if you integrate it, you will get T^+ = Pr y^+. And let us say that it is valid in the range of $y^+ < y_1^+$, where y_1^+ is the dividing point between the conduction layer and outer layer. So, now let us consider the outer layer. So, in the outer layer, obviously, your it is a fully turbulent flow and fluctuating components dominate.

(Refer Slide Time: 37:25)



So, in this case you can see this is the equation. So, turbulent effects dominate the heat transfer further away from the wall. So, you can see that $\alpha_t \gg \alpha$. So, you can see you can write $\frac{\partial T^+}{\partial y^+} = \frac{\nu}{\alpha + \alpha_t}$. So, α_t is unknown. So, now, we will write α_t in terms of turbulent Prandtl number. And we will use the Prandtl mixing-length model and we will substitute this ν_t.

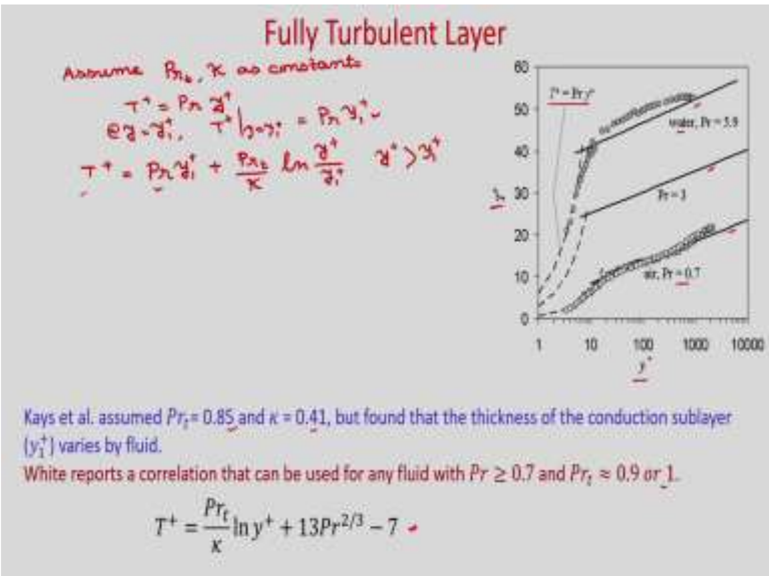
So, we can see here turbulent Prandtl number we are defining as general $\Pr = \frac{\nu}{\alpha}$. So, turbulent $\Pr_t = \frac{\nu_t}{\alpha_t}$. So, you can see here $\frac{\nu}{\alpha_t}$. You can write as $\frac{\nu}{\alpha_t} = \frac{\nu}{\nu_t}\frac{\nu_t}{\alpha_t}$. So, you can write $\frac{\nu}{\nu_t}\Pr_t$. So, and this ν_t in non-dimensional form if you write it will be, $\nu_t = \kappa^2 \left(y^+\right)^2 \nu \frac{\partial u^+}{\partial y^+}$.

So, now if you write this equation, $\frac{\partial T^+}{\partial y^+} = \frac{\nu}{\nu_t}\Pr_t$. If you take in the left hand side you get, $\frac{\nu_t}{\nu}\frac{\partial T^+}{\partial y^+} = \Pr_t$. So, now you see, $\kappa^2 \left(y^+\right)^2 \frac{\partial u^+}{\partial y^+}\frac{\partial T^+}{\partial y^+} = \Pr_t$.

Now, from universal velocity profile for the fully turbulent layer, $\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+}$. So, we will use $\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y^+}$. So, if you put it here, so you are going to get as, $\frac{\partial T^+}{\partial y^+} = \frac{\Pr_t}{\kappa y^+}$.

So, now we have $\frac{\partial T^+}{\partial y^+} = \frac{\Pr_t}{\kappa y^+}$. Now, if you integrate it, so you will get $T^+ - T^+\big|_{y^+=y_1^+} = \int_{y_1^+}^{y^+} \frac{\Pr_t}{\kappa} \frac{dy^+}{y^+}$. So, you see if we assume $\Pr_t$ and κ as constant, then you will be able to integrate it.

(Refer Slide Time: 41:00)



So, let us assume $\Pr_t$ and κ as constants. And you can see we have in the conduction layer $T^+ = \Pr y^+$. So, at $y = y_1^+$, you can write $T^+\big|_{y^+=y_1^+} = \Pr y_1^+$.

So, if you put all these values and if you integrate keeping $\Pr_t$ and κ constant, then you will get $T^+ = \Pr y_1^+ + \frac{\Pr_t}{\kappa} \ln \frac{y^+}{y_1^+}$ and it is valid for $y^+ > y_1^+$.

So, you can see that your temperature profile depends on this fluid that means for Prandtl number, and also it depends on $\Pr_t$ and κ. So, Kays et al. assumed this $\Pr_t$ = 0.85 and κ =

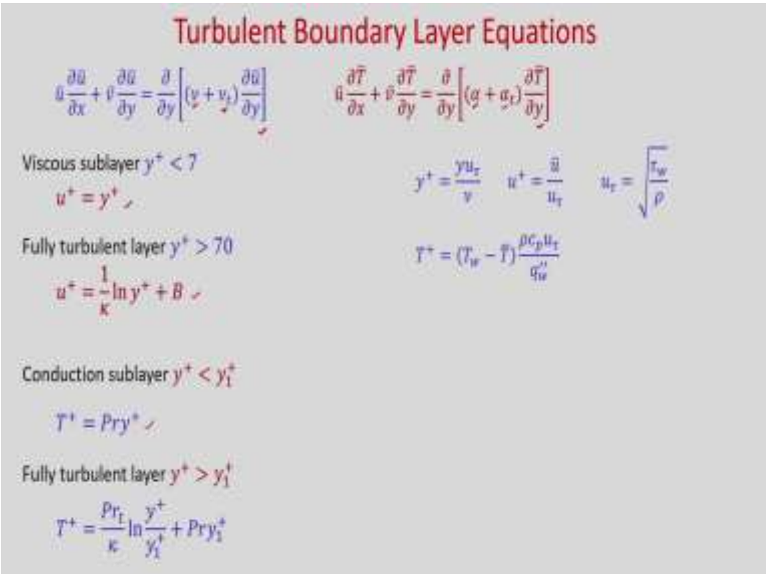
0.41, but found that the thickness of the conduction sub layer y_1^+ varies by fluid. So, if you have different fluid, this y_1^+ varies. So, depending on the value of y_1^+, you can use these conduction layer model as well as fully turbulent layer model.

Why it reports a correlation that can be used for any fluid with Pr ≥ 0.7, and $\Pr_t \approx 0.9$ or 1. So, if you put all these values, you will get $T^+ = \frac{\Pr_t}{\kappa}\ln y^+ + 13\Pr^{2/3} - 7$.

So, you can see here in this curve T^+ versus y^+. So, in viscous sub layer region or in conduction sub layer, you have $T^+ = \Pr y^+$; and in fully turbulent region, you can have this model.

So, for different Prandtl number, you can see for here 0.7. So, these are the solid line you can see for different Prandtl number Prandtl number 0.7, 0.3, and what are Prandtl number 5.9. So, this is the model for fully turbulent layer. And the results of kays et al. also here it is shown for air as well as water, and this is the Kays et al. model. So; obviously, you can see with increase of Prandtl number, your value of T^+ increases.

(Refer Slide Time: 44:03)



So, you can see in today's class we have derived these boundary layer equations for turbulent flows, where ν is your kinematic viscosity, ν_t is your eddy viscosity, and α is your thermal diffusivity, and α_t is your eddy diffusivity, and ν_t is your eddy viscosity.

So, you can see in viscous sub layer we have derived $u^+ = y^+$ and in fully turbulent layer; $u^+ = \frac{1}{\kappa}\ln y^+ + B$ where $y^+ = \frac{yu_\tau}{\nu}$, $u^+ = \frac{\bar{u}}{u_\tau}$ and $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$, and $T^+ = \left(T_w - \bar{T}\right)\frac{\rho c_p u_\tau}{q_w^{"}}$. And in conduction sub layer, we have derived this $T^+ = \Pr y^+$; and in fully turbulent layer, you have, $T^+ = \frac{\Pr_t}{\kappa}\ln\frac{y^+}{y_1^+} + \Pr y_1^+$.

In today's class, we considered a steady state two-dimensional in incompressible fluid flow equations and for turbulent flows, and we used the scale analysis and we have written the boundary layer equations for turbulent flow. When we write these boundary layer equations, we have seen that you have the fluctuating components $\overline{u'v'}$; and in energy equation we have $\overline{u'T'}$. So, these are the unknowns and that we need to model some way.

So, from the Prandtl mixing-length hypothesis, we have seen that your eddy viscosity ν_t you can write in terms of the mixing length. And this mixing length also you can write as κy, where κ is constant for any fluid flow and you need to determine experimentally.

Now, we have seen that to solve these equations, you need to know the eddy viscosity as well as the eddy diffusivity which are unknown. And these unknowns you need to find with some assumptions as well as from the experimental conditions. When we considered the velocity profile, we have taken two different layers; one region is very near to the wall where you can neglect the effect of the fluctuating components.

And that is known as a viscous sub layer. And one is away from the wall where your fluctuating components dominate and you can neglect the effect of viscosity or effect of wall in those region and in between you have you will have the buffer zone.

So, in these two different zones in viscous sub layer and fully turbulent layer, we have derived the non-dimensional velocity profile u^+. And similarly for the energy equation, we considered two layers one is very near to the wall that is your conduction layer and away from the wall that is your fully turbulent layer.

So, in these layers also we have derived the non-dimensional temperature profile T^+. And these velocity profile and temperature profiles are known as universal velocity profile and universal temperature profile.

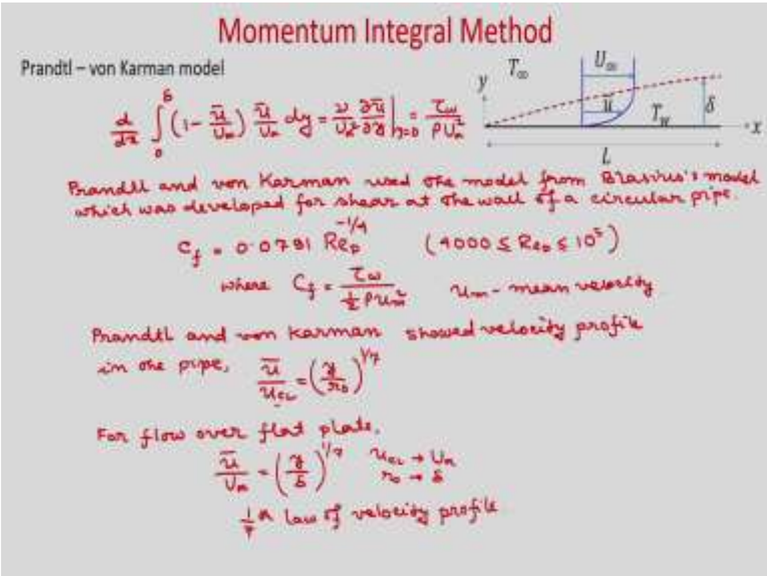
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 11
Turbulent Flow and Heat Transfer
Lecture – 38
Integral solution for turbulent boundary layer flow over a flat plate

Hello everyone. So, in today's class, first we will use momentum integral equation which we derived for laminar flows and we will find the friction coefficient for turbulent flows, for flow over flat plate and then, we will find the heat transfer coefficient and Nusselt number for flow over flat plate.

(Refer Slide Time: 00:54)



So, we can write the momentum integral equation which we derive for the laminar flows. So, this is $\frac{d}{dx}\int_0^{\delta}\left(1-\frac{\bar{u}}{U_\infty}\right)\frac{\bar{u}}{U_\infty}dy = \frac{\nu}{U_\infty^2}\frac{\partial \bar{u}}{\partial y}\bigg|_{y=0} = \frac{\tau_w}{\rho U_\infty^2}$. So, this equation also can be used for time average velocities, in turbulent flows.

So, you just replace $u = \bar{u}$. So, if you replace $u = \bar{u}$ then these momentum integral equation we can use for this turbulent flows.

Now, you know that to use this integral equation, we need to find or we need to assume some velocity profile. In turbulent flows, it is very difficult to assume the velocity profile, so Prandtl and Von-Karman; what they did? They used very crude and simple method, but it gives very accurate result for flow over flat plate or for external flows.

So, they used the solution of Blasius for circular pipe case and that they used the velocity profile for the flow over flat plate. So, you can see Prandtl and Von-Karman used the model from Blasius model which was developed for the shear at the wall of a circular pipe.

So, Blasius proposed for circular pipe based on the dimensional analysis and experimental data the $C_f = 0.0781 \mathrm{Re}_D^{-1/4}$. And it is valid in the range $4000 \le \mathrm{Re}_D \le 10^5$. So, this is for internal flows, pipe flow, where, $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho u_m^2}$, where u_m is your mean velocity.

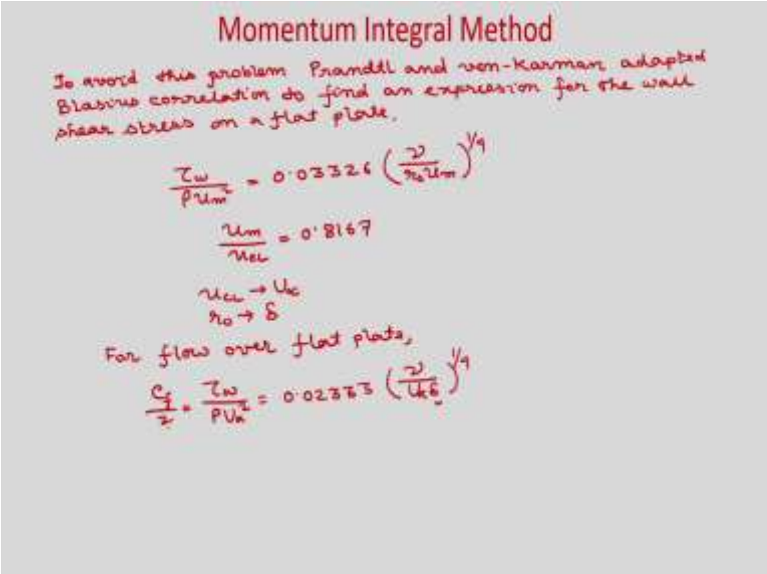
So, based on these empirical relation your Prandtl and Von-Karman developed the velocity profile inside a pipe as; so, Prandtl and Von-Karman showed the velocity profile in the pipe $\dfrac{\bar{u}}{u_{cL}} = \left(\dfrac{y}{r_0}\right)^{1/7}$. In this relation, y is measured from the pipe wall and u_{CL} is your centerline velocity, and r_0 is the radius of the pipe.

So, Prandtl and Von-Karman actually using this relation, they use the velocity profile for flow over flat plate just putting the r naught as δ, that is your boundary layer thickness. And u C L; u C L in this case ah there is no central line velocity, so u C L is substituted with the free stream velocity U_∞.

So, we will use for flow over flat plate. These velocity profile u by U ∞, so u_{cL} is actually replaced with U_∞ and $\dfrac{\bar{u}}{U_\infty} = \left(\dfrac{y}{\delta}\right)^{1/7}$, where r_0 is replaced with δ, boundary layer thickness to the power 1/ 7. So, you can see that it is a well-known one-seventh law of velocity profile; one-seventh law of velocity profile.

So, although they propose the velocity profile for the flow over flat plate like this, but it has some fundamental problem. So, if you calculate the shear stress at the wall, it will become almost ∞. So, to avoid this problem, they used the correlation for the C_f from the pipe flow relation.

(Refer Slide Time: 06:58)



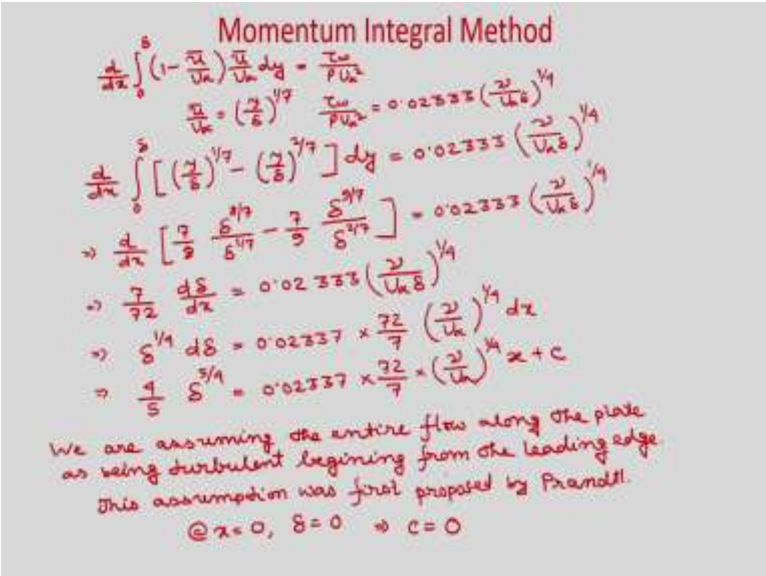
To avoid this problem Prandtl and Von-Karman adapted Blasius correlation to find an expression for the wall shear stress on a flat plate. So, they used $\frac{\tau_w}{\rho u_m^2} = 0.03326\left(\frac{\nu}{r_0 u_m}\right)^{1/4}$.

So, now you can substitute u_{CL} as U_∞ and r_0 as δ, but the mean velocity u_m. So, $\frac{u_m}{u_{cL}}$ it can be found for one-seventh law velocity profile inside a pipe as $\frac{u_m}{u_{cL}} = 0.8167$. So, these are valid for pipe flow. So, now you can use u_{CL} you can substitute with U_∞, and $r_0$you can substitute with δ.

So, now, if you substitute then you will get the shear stress relation for flow over flat plate. You can use now $\frac{C_f}{2} = \frac{\tau_w}{\rho U_\infty^2}$ after putting all these values you can rearrange and you will get as, $0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$. So, you see for flow over flat plate this $\frac{C_f}{2}$ is given in terms of the boundary layer thickness δ.

Now, you can use the momentum integral equation and we can substitute the velocity profile, one-seventh law velocity profit, and this shear stress relation, and we can find what is the boundary layer thickness.

(Refer Slide Time: 09:47)



So, we have the momentum integral equations as $\frac{d}{dx}\int_0^\delta \left(1-\frac{\bar{u}}{U_\infty}\right)\frac{\bar{u}}{U_\infty}dy = \frac{\tau_w}{\rho U_\infty^2}$. So, now we have, $\frac{\bar{u}}{U_\infty}$, Prandtl and Von-Karman proposed as $\left(\frac{y}{\delta}\right)^{1/7}$ and you have, $\frac{\tau_w}{\rho U_\infty^2} = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$.

Now, you substitute these two in the momentum integral equation and find the value of δ. And once you know the value of δ you will be able to find, the skin friction coefficient because in the skin friction coefficient you have the unknown parameter δ.

So, if you substitute it you will get $\frac{d}{dx}\int_0^\delta\left[\left(\frac{y}{\delta}\right)^{1/7}-\left(\frac{y}{\delta}\right)^{2/7}\right]dy=0.02333\left(\frac{\nu}{U_\infty\delta}\right)^{1/4}$. So, if you integrate it what you will get? $\frac{d}{dx}\left[\frac{7}{8}\frac{\delta^{8/7}}{\delta^{1/7}}-\frac{7}{9}\frac{\delta^{9/7}}{\delta^{2/7}}\right]=0.02333\left(\frac{\nu}{U_\infty\delta}\right)^{1/4}$. So, this if you do the algebra you will get $\frac{7}{72}\frac{d\delta}{dx}=0.02333\left(\frac{\nu}{U_\infty\delta}\right)^{1/4}$.

So, this δ you take in the left hand side, so you will get, $\delta^{1/4}d\delta=0.02337\times\frac{72}{7}\left(\frac{\nu}{U_\infty}\right)^{1/4}dx$.

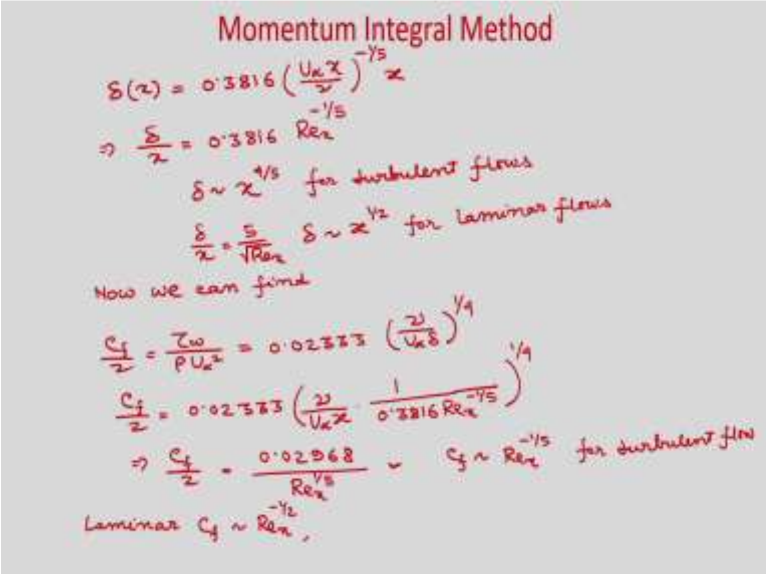
So, now if you integrate it you see ν, U_∞ are constant. So, you can integrate this. So, you will get $\frac{4}{5}\delta^{5/4}=0.02337\times\frac{72}{7}\left(\frac{\nu}{U_\infty}\right)^{1/4}x+C$. So, we will assume here that you have the turbulent flow from the leading edge of the flat plate.

So, first Prandtl proposed this assumption, so that we can use the condition at x = 0; that means, at the leading edge of the flat plate you have boundary layer thickness δ = 0. So, these are the assumptions you have to take.

So, if you take these assumptions, then you can use the boundary condition at x = 0, δ = 0. So, we are assuming that you have a turbulent flow over this flat plate starting from the leading edge.

So, we are assuming the entire flow along the plate as being turbulent beginning from the leading edge. So, this assumption was first proposed by Prandtl. So, if you assume this then you can put at x = 0, δ = 0.

(Refer Slide Time: 15:17)



So, that means, your constant C = 0. So, if we put C = 0 and if you rearrange it you will get $\delta(x) = 0.3816\left(\frac{U_\infty x}{\nu}\right)^{-1/5} x$. So, you can write $\frac{\delta}{x} = 0.3816\,\mathrm{Re}_x^{-1/5}$.

So, here you can see that your boundary layer thickness varies $\delta \sim x^{4/5}$. So, your, in this case you can see that your boundary layer thickness $\delta \sim x^{4/5}$ for turbulent flows.

For laminar flows, do you remember what was the δ? So, it was $\frac{5}{\sqrt{\mathrm{Re}_x}}$, so that means, $\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}}$. So, that means, $\delta \sim x^{1/2}$, for laminar flows.

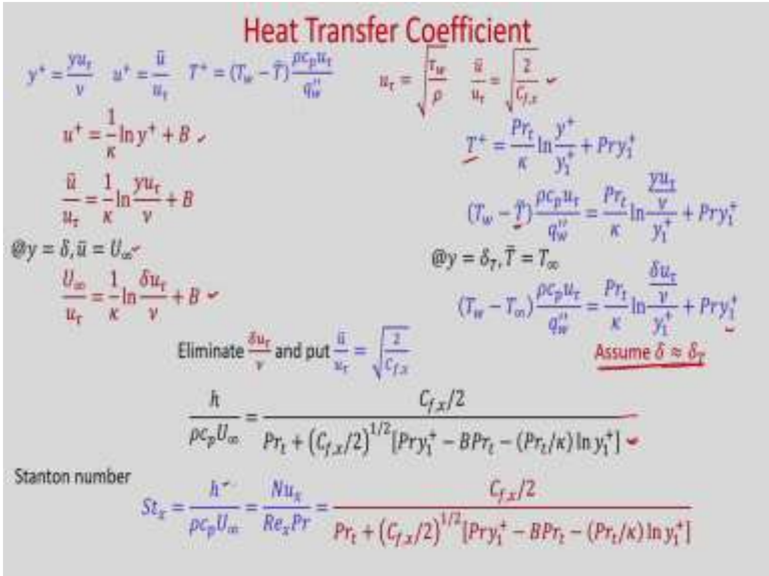
And now, if you find the C_f; $\frac{C_f}{2} = \frac{\tau_w}{\rho U_\infty^2} = 0.02333\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$. So, you substitute here this δ value. So, if you substitute it you will get $\frac{C_f}{2} = 0.02333\left(\frac{\nu}{U_\infty x}\frac{1}{0.3816\,\mathrm{Re}_x^{-1/5}}\right)^{1/4}$.

So, this you can write as, $\frac{C_f}{2} = \frac{0.02968}{\mathrm{Re}_x^{1/5}}$.

And for laminar flow you know $C_f \sim \mathrm{Re}_x^{-1/2}$, and in turbulent flows you can see $C_f \sim \mathrm{Re}_x^{-1/5}$, for turbulent flows.

So, now whatever expression we have derived for this $\frac{C_f}{2}$ that will use to find the heat transfer coefficient.

(Refer Slide Time: 19:06)



So, you can see we have already derived these for fully turbulent boundary layer $u^+ = \frac{1}{\kappa}\ln y^+ + B$. And, $T^+ = \frac{\mathrm{Pr}_t}{\kappa}\ln\frac{y^+}{y_1^+} + \mathrm{Pr}\, y_1^+$. Now, $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$. So, now, if you define, $\frac{\bar{u}}{u_\tau} = \sqrt{\frac{2}{C_{f,x}}}$.

So, here you can see $u^+ = \frac{\bar{u}}{u_\tau}$. So, $y^+ = \frac{yu_\tau}{\nu}$, plus the constant B. So, at y = δ, at the edge of the boundary layer you have free stream velocity U_∞.

So, $\frac{\bar{u}}{u_\tau}$ now this $\bar{u} = U_\infty$. So, $\frac{U_\infty}{u_\tau} = \frac{1}{\kappa}\ln\frac{\delta u_\tau}{\nu} + B$. So, this is we have derived.

Now, if you see here, here we will assume that $\delta \approx \delta_T$, so that our derivation will be simplified. So, with that we know that at the edge of the boundary layer we have $\overline{T} = T_\infty$.

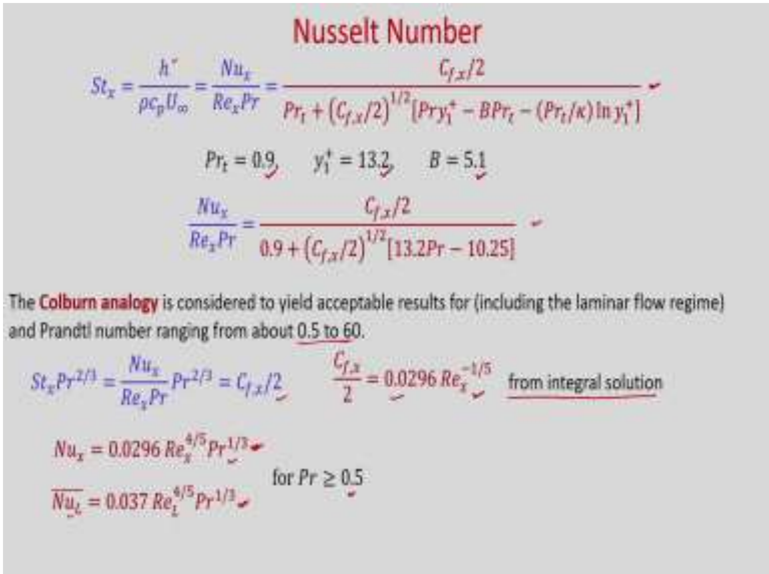
So, here if you put $\overline{T} = T_\infty$,so this you can write, $\left(T_w - T_\infty\right)\dfrac{\rho C_p u_\tau}{q_w''} = \dfrac{\Pr_t}{\kappa}\ln\dfrac{\delta u_\tau}{\nu} + \Pr y_1^+$ at y = δ.

So, now these two relations we have. Now, what you do you see in both the equations you have $\dfrac{\delta u_\tau}{\nu}$. Now, eliminate $\dfrac{\delta u_\tau}{\nu}$ and put $\dfrac{\overline{u}}{u_\tau} = \sqrt{\dfrac{2}{C_{f,x}}}$. So, if you do that then you will get,

$$\frac{h}{\rho c_p U_\infty} = \frac{C_{f,x}/2}{\Pr_t + \left(C_{f,x}/2\right)^{1/2}\left[\Pr y_1^+ - B\Pr_t - \left(\Pr_t/\kappa\right)\ln y_1^+\right]}.$$

So, now, we will define Stanton number. Already we have discussed about this. So, $St_x = \dfrac{Nu_x}{\mathrm{Re}_x \Pr}$. So, $St_x = \dfrac{h}{\rho c_p U_\infty}$. So, this if you put, then now we will be able to find what is the heat transfer coefficient and from there we will be able to find what is the Nusselt number.

(Refer Slide Time: 22:06)



So, now we have just expressed Stanton number with this. Now, we will assume turbulent Pr= $_t$ 0.9, y_1^+ = 13.2 and the B = 5.1. So, these are from empirical values, from the experiments these values are found.

Now, if you put all these here then you will get, $\frac{Nu_x}{\text{Re}_x \text{Pr}} = \frac{C_{f,x}/2}{0.9 + \left(C_{f,x}/2 \right)^{1/2} \left[13.2\,\text{Pr} - 10.25 \right]}$. So, you can see from these expression now you will be able to find what is the Nusselt number or also heat transfer coefficient.

Now, this we have derived using the two layers model because we have used for fully turbulent flows what is the u^+ and T^+ expression and from there we have derived the expression for Nusselt number here.

You can also use Colburn analogy. So, that already we have discussed. So, the Colburn analogy is considered to yield acceptable results for including the laminar flow regime and Prandtl number ranging from about 0.5 to 60.

So, this ah Colburn analogy if you use, so you know that, $St_x \text{Pr}^{2/3} = C_{f,x}/2$. And, $St_x = \frac{Nu_x}{\text{Re}_x \text{Pr}}$.

And from the integral solution, just in this class we have derived, $\frac{C_{f,x}}{2} = 0.0296\,\text{Re}_x^{-1/5}$.

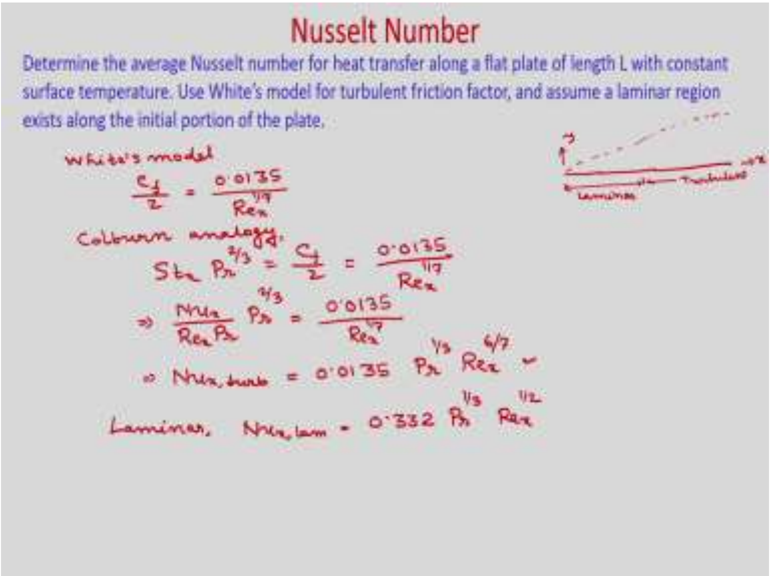
So, you can see in the Colburn analogy we will use this integra $\frac{C_{f,x}}{2}$ l solution of $\frac{C_{f,x}}{2}$, and if you put it here and if you find what is the Nusselt number you will get Nu_x in terms of Reynolds number and Prandtl number. So, you can see $Nu_x = 0.0296\,\text{Re}_x^{4/5}\,\text{Pr}^{1/3}$.

So, this is from Colburn analogy, just using the integral solution from, integral solution of $\frac{C_{f,x}}{2}$ you can find the Nusselt number. This is your local Nusselt number and it is valid for Pr $\geq$ 0.5.

And if you find the average Nusselt number just integrating from 0 to L, you will get average $\overline{Nu_L} = 0.037\,\mathrm{Re}_x^{4/5}\,\mathrm{Pr}^{1/3}$. So, this actually gives reasonable good results. This is simple simplified expression, but you can also use these expression as well.

So, now, if you have laminar region in the beginning then you have turbulent region. And you know that critical Re_{x_c} = 10^5, so in this region if you find the what is the average Nusselt number, considering both laminar and turbulent regime that now we will find.

(Refer Slide Time: 25:21)



So, determine the average Nusselt number for heat transfer along a flat plate of length L with constant surface temperature. Use White's model for turbulent friction factor, and assume a laminar region exist along the initial portion of the plate.

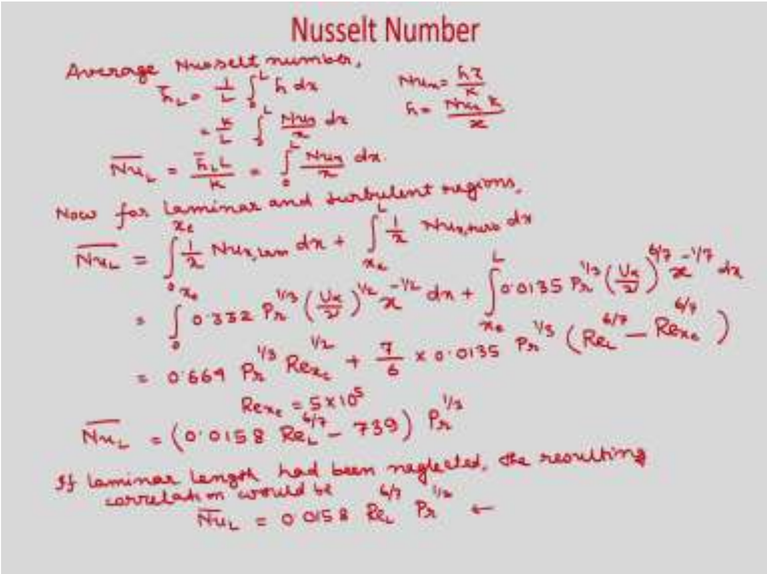
So, if you consider a flat plate. So, we are considering that initial region you have laminar flow, and then you have turbulent. So, if you see the, so this is your boundary layer thickness and this is your y, this is your x.

Now, we will use White's model. So, if you see what is White's model, White's model we have found, $\frac{C_f}{2} = \frac{0.0135}{\mathrm{Re}_x^{1/7}}$. And from Colburn analogy, Colburn analogy and you can find the Nusselt number in turbulent flow regime. So, $St_x\,\mathrm{Pr}^{2/3} = \frac{C_f}{2} = \frac{0.0135}{\mathrm{Re}_x^{1/7}}$.

So, $St_x = \frac{Nu_x}{\text{Re}_x \text{Pr}}$ and thus, $\frac{Nu_x}{\text{Re}_x \text{Pr}} \text{Pr}^{2/3} = \frac{0.0135}{\text{Re}_x^{1/7}}$. So, Nusselt number you can find. For turbulent flows as, $Nu_{x,turb} = 0.0135 \text{Pr}^{1/3} \text{Re}_x^{6/7}$.

So, using White's model, we found the Nusselt number in the turbulent regime as this. And you know for laminar region, we have already derived this Nusselt number, $Nu_{x,\text{lam}} = 0.332 \text{Pr}^{1/3} \text{Re}_x^{1/2}$. So, this we will use and we will find the average heat transfer coefficient for both laminar and turbulent regime.

(Refer Slide Time: 28:12)



So, how to calculate average Nusselt number? Average Nusselt number we calculate as, $\bar{h} = \frac{1}{L}\int_0^L h dx$. And now, $Nu_x = \frac{hx}{K}$.

So, you can find $h = \frac{Nu_x K}{x}$. You substitute it here. So, you will get $\bar{h} = \frac{K}{L}\int_0^L \frac{Nu_x}{x} dx$.

Now, if you find the Nusselt number, average Nusselt number, if you find the average, $\overline{Nu_L} = \frac{\overline{h_L} L}{K}$. So, you can simply see this will be, $\int_0^L \frac{Nu_x}{x} dx$.

Now, considering the laminar and turbulent region, for laminar and turbulent region, so, $\overline{Nu}_L = \int_0^{x_c} \frac{1}{x} Nu_{x,lam} dx + \int_{x_c}^{L} \frac{1}{x} Nu_{x,turb} dx$. So, we have already found the expression that you substitute it here.

So, you will get, $\overline{Nu}_L = \int_0^{x_c} 0.332 \Pr^{1/3} \left(\frac{U_\infty}{\nu} \right)^{1/2} x^{-1/2} dx + \int_{x_c}^{L} 0.0135 \Pr^{1/3} \left(\frac{U_\infty}{\nu} \right)^{6/7} x^{-1/7} dx$.

So, if you perform the integration you will get, $\overline{Nu}_L = 0.664 \Pr^{1/3} \mathrm{Re}_{x_c}^{1/2} + \frac{7}{6} \times 0.0135 \Pr^{1/3} \left(\mathrm{Re}_L^{6/7} - \mathrm{Re}_{x_c}^{6/7} \right)$.

If you consider, $\mathrm{Re}_{x_c} = 5 \times 10^5$ and substitute it here and you will get, $\overline{Nu}_L = \left(0.0158 \mathrm{Re}_L^{6/7} - 739 \right) \Pr^{1/3}$.

And if you neglect the laminar length, if laminar length had been neglected the resulting correlation would be, $\overline{Nu}_L = 0.0158 \mathrm{Re}_L^{6/7} \Pr^{1/3}$. So, it is for fully turbulent flow considering the turbulent flow from the beginning of the flat plate.

So, in today's class we started with the momentum integral equation which we derived for laminar flows and we substituted u as time average velocity $\bar{u}$. From there with the correlation of pipe flow, we could find the velocity distribution $\frac{\bar{u}}{U_\infty} = \left(\frac{y}{\delta} \right)^{1/7}$. So, this is known as one-seventh velocity profile.

And also, we saw that the problem is finding the tau w, because if you use these velocity profile as well y = 0 you will get infinite shear stress. So, to avoid that again from Blasius relation of C_f, we use the turbulent friction factor for the flow over flat plate.

Now, that we expressed in terms of the unknown parameter boundary layer thickness δ, now we substituted this velocity profile as well as the turbulent friction factor in the momentum integral equation and we found the value of turbulent boundary layer thickness δ. And once you know the δ, so you could find the value of $C_{f,x}$.

Then, using the relation for fully turbulent layer this u^+ and T^+ values, we found using the Colburn analogy the expression for Nusselt number. And also, for using this integral solution whatever we found the value of $C_{f,\,x}$ that we used and use the Colburn analogy and we found the simplified expression for the Nusselt number, local Nusselt number as well as the average Nusselt number.

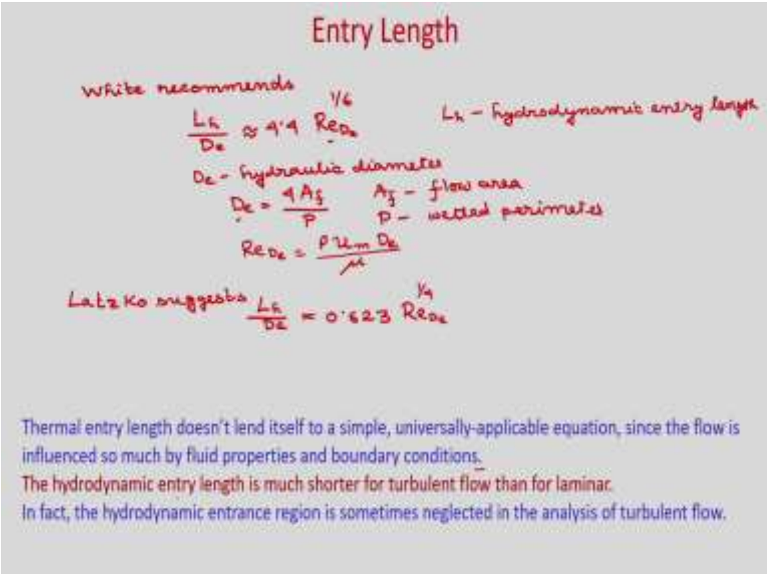
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 11
Turbulent Flow and Heat Transfer
Lecture – 39
Convection in turbulent pipe flow

Hello, everyone. So, today we will study Convection in turbulent pipe flow. In last classes we have already derived the universal velocity profile for flow over flat plate and also we have derived the heat transfer analogy relations or correlations. We will use those universal velocity profile for pipe flow with slight modifications. First let us discuss about the entry length for turbulent pipe flow.

(Refer Slide Time: 01:05)

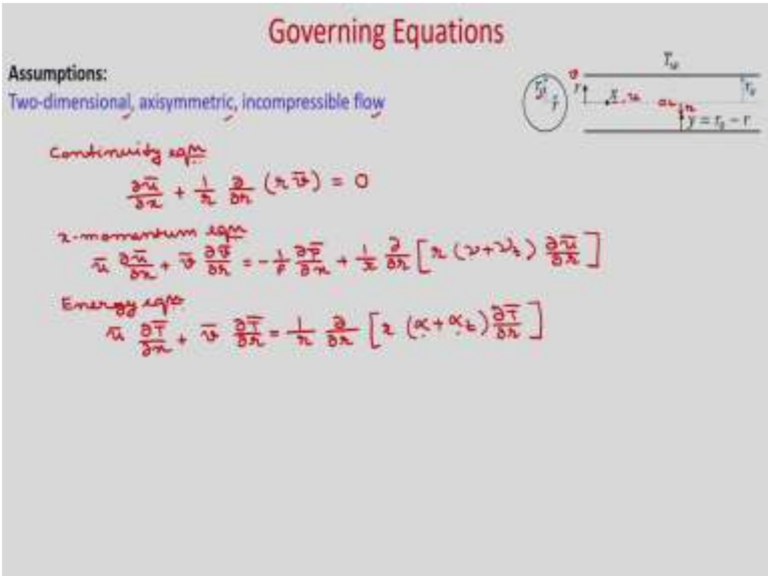


White recommends $\frac{Lh}{D_e} \approx 4.4\,\text{Re}_{D_e}^{1/6}$. So, you know that D_e is your hydraulic diameter and it is obviously, you know how it is defined. It is defined as $D_e = \frac{4A_f}{P}$, where A_f is your flow area and P is your wetted perimeter wetted perimeter and this Reynolds number is defined based on this hydraulic diameter. So, Re_{D_e} is defined as $\text{Re}_{D_e} = \frac{\rho u_m D_e}{\mu}$ and in this case we are considering internal flow.

Another scientist Latzko suggests $\frac{Lh}{D_e} = 0.623\,\mathrm{Re}_{D_e}^{1/4}$. So, L h obviously, it is hydrodynamic entry length hydrodynamic entry length. In general, in turbulent flows it is very small compared to the laminar flow and open this hydrodynamic entrance length is neglected. However, it is very difficult to calculate the thermal entrance length for turbulent flows.

Thermal entry length does not lend itself to a simple, universally-applicable equation, since the flow is influenced so much by fluid properties and boundary conditions. The hydrodynamic entry length is much shorter for turbulent flow than for laminar. In fact, the hydrodynamic entrance region is sometime neglected in the analysis of turbulent flow.

(Refer Slide Time: 03:39)



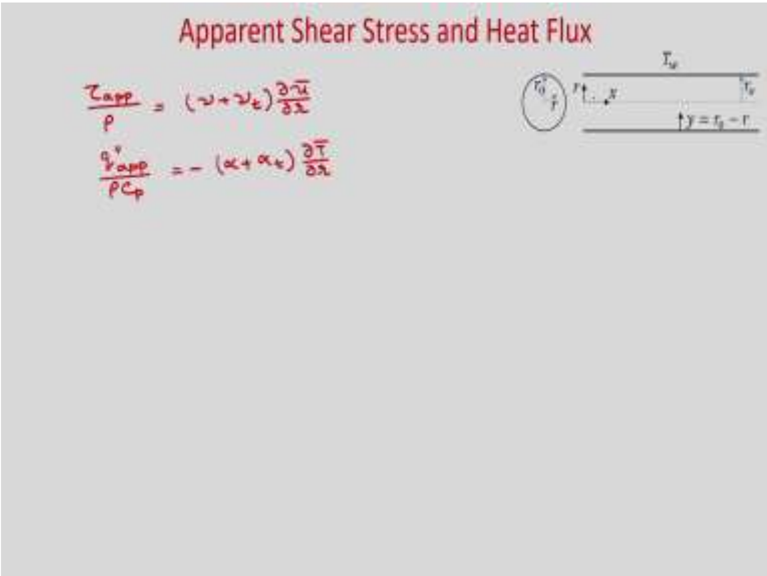
So, let us write the governing equation for this internal flow with these assumptions: two-dimensional, axisymmetric and incompressible flow. And, as we are considering boundary layer flows so, you can see this is the pipe of radius r_0 and x is measured in axial direction; r is measured from the centerline.

So, this is your C L centerline and y if we tell it is measured from the boundary then it is r_0 - r. So, obviously, y = r_0 - r. So, in this case you can see we will define the velocity u in axial direction and v velocity in r direction ; for convenience we are just defining these velocities u and v.

So, now, you can write the governing equations after Reynolds averaging. So, you will get continuity equation. So, we can write as $\frac{\partial \bar{u}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\bar{v}\right)=0$. And, Reynolds averaged x momentum equation you can write as $\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial r}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial r}\right]$. So, this is boundary layer flow so obviously, $\frac{\partial^2 u}{\partial x^2}$ we can neglect. And, Reynolds average energy equation will be $\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial r}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\alpha+\alpha_t\right)\frac{\partial \bar{T}}{\partial r}\right]$. So, these equations we have already derived for the flow over flat plate.

These we have written for the circular pipe case and you can see ν is your kinematic viscosity and ν_t is your eddy viscosity, and α is your thermal diffusivity and α_t is your eddy diffusivity and these are coming due to the turbulent fluctuations.

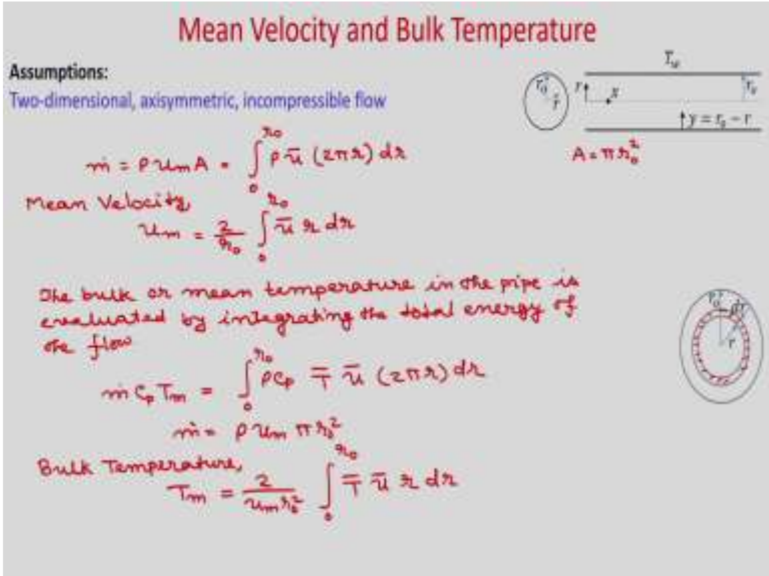
(Refer Slide Time: 06:43)



Now, whatever we have derived the apparent stress and apparent heat flux flow over flat plate those will be applicable for pipe flow. So, you can write $\frac{\tau_{app}}{\rho}=\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial r}$ and

$\frac{q''_{app}}{\rho C_p} = -(\alpha + \alpha_t)\frac{\partial \overline{T}}{\partial r}$. So, you can see for flow over flat plate we have derived it has $\frac{\partial \overline{u}}{\partial y}$. So, in this case we are writing $\frac{\partial \overline{u}}{\partial r}$.

(Refer Slide Time: 07:31)



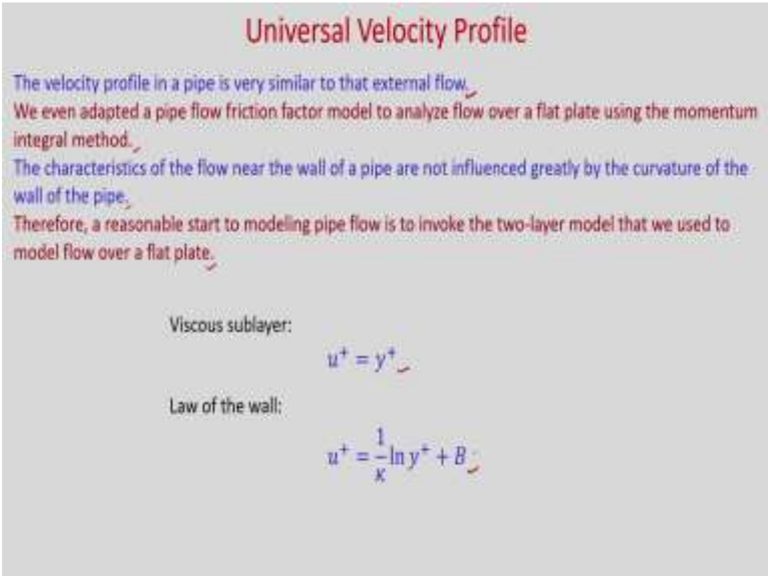
So, in pipe flow generally we deal with the mean velocity and bulk temperature. When we did find the Nusselt number we write it based on the mean bulk mean temperature, as well as when we define the Reynolds number we define based on the mean velocity. So, let us write the expression for mean velocity as well as the bulk temperature.

So, we can write mass flow rate $\dot{m} = \rho u_m A$. So, in this case you can see; obviously, $A = \pi r_0^2$. So, it is your flow area. So, this will be $\int_0^{r_0} \rho \overline{u} (2\pi r) dr$.

So, if you put here $A = \pi r_0^2$. So, you can write the mean velocity, $u_m = \frac{2}{r_0}\int_0^{r_0} \overline{u} r dr$. So, the bulk or mean temperature in the pipe is evaluated by integrating the total energy of the flow. So, you can write $\dot{m} C_p T_m = \int_0^{r_0} \rho C_p \overline{T}\,\overline{u} (2\pi r) dr$.

So, from here you know that $\dot{m}$. So, you can write, $\dot{m} = \rho u_m \pi r_0^2$. So, if you put it here and ρC_p are constant. So, you can cancel. So, you will get bulk temperature $T_m = \frac{2}{u_m r_0^2} \int_0^{r_0} \bar{T} \bar{u} r dr$. So, it is same expression as laminar only difference is that this velocity and temperature are evaluated as mean value.

(Refer Slide Time: 10:50)

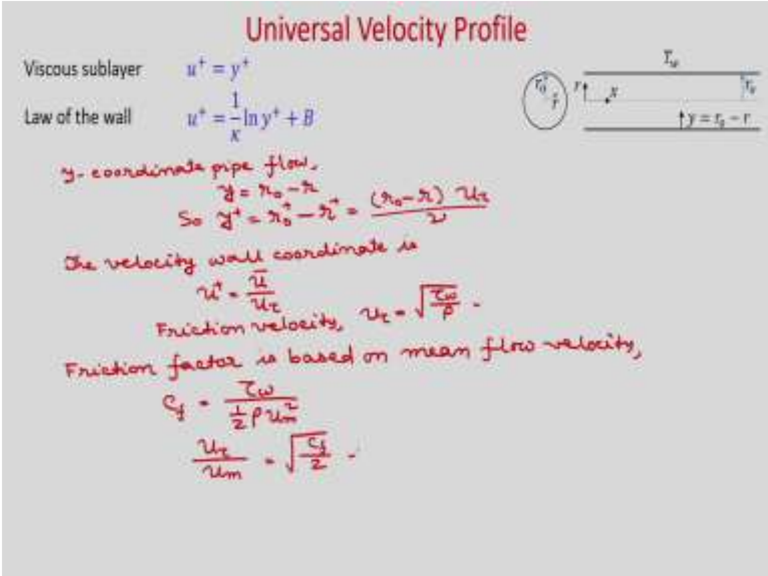


For flow over flat plate case already we have derived the universal velocity profile, we considered very small region near to the wall and we assumed that their shear stress remain constant. So, that is your viscous sub-layer and away from the wall you have law of the wall. So, the velocity profile in a pipe is very similar to the external flow.

We even adapted a pipe flow friction factor model to analyze flow over a flat plate using the momentum integral method. The characteristic of the flow near the wall of a pipe are not influenced greatly by the curvature of the wall of the pipe. Therefore, a reasonable start to modeling pipe flow is to invoke the two-layer model that we used to model flow over a flat plate.

So, you can see for viscous sub layer we have derived $u^+ = y^+$ and law of the wall $u^+ = \frac{1}{\kappa} \ln y^+ + B$. Here now, the definition of y^+ will be somewhat different in case of pipe flow.

(Refer Slide Time: 12:00)

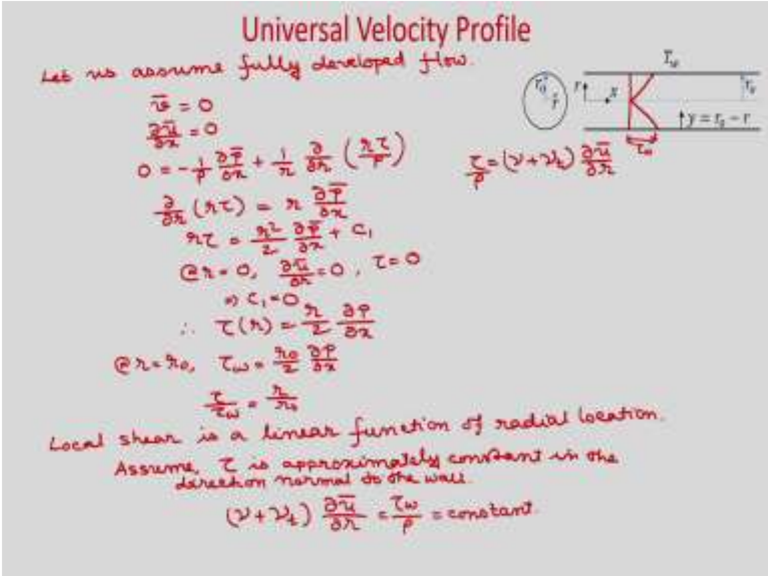


So, you can see in case of flat plate this y is measured from the wall. So, in this particular case now if you measure the distance from the wall, this is your y then you have to replace this $y = r_0 - r$. So, if you see y coordinate pipe flow so, $y = r_0 - r$.

So, $y^+ = r_0^+ - r^+ = \frac{(r_0 - r)u_\tau}{\nu}$. u_τ is your friction velocity and the velocity wall coordinate is, $u^+ = \frac{\bar{u}}{u_\tau}$. This expression is same. So, your fiction velocity is $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$.

And, friction factor based on the mean flow velocity you can write friction factor is based on mean flow velocity. So, $C_f = \frac{\tau_w}{\frac{1}{2}\rho u_m^2}$ and now you can write $\frac{u_\tau}{u_m} = \sqrt{\frac{C_f}{2}}$. So, if you put these expression in this friction velocity then you will get, $\frac{u_\tau}{u_m} = \sqrt{\frac{C_f}{2}}$. Now, let us see that in pipe flow how the shear stress varies inside the domain.

(Refer Slide Time: 14:17)



So, for that let us assume fully developed flow. Let us assume fully developed flow. So, if it is a fully developed flow obviously, the velocity $\bar{v}=0$ and from continuity equation you can write, $\frac{\partial \bar{u}}{\partial x}=0$. So, if you put these in the boundary layer equation whatever we have written so, you can write, $0=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r\tau}{\rho}\right)$.

So, $\frac{\tau}{\rho}=\left(\nu+\nu_t\right)\frac{\partial \bar{u}}{\partial r}$. So, τ is your shear stress. So, if you rearrange it you will get and ρ is constant. So, you can write $\frac{\partial}{\partial r}\left(r\tau\right)=r\frac{\partial \bar{p}}{\partial x}$. So, if you integrate it you will get, $r\tau=\frac{r^2}{2}\frac{\partial \bar{p}}{\partial x}+C_1$.

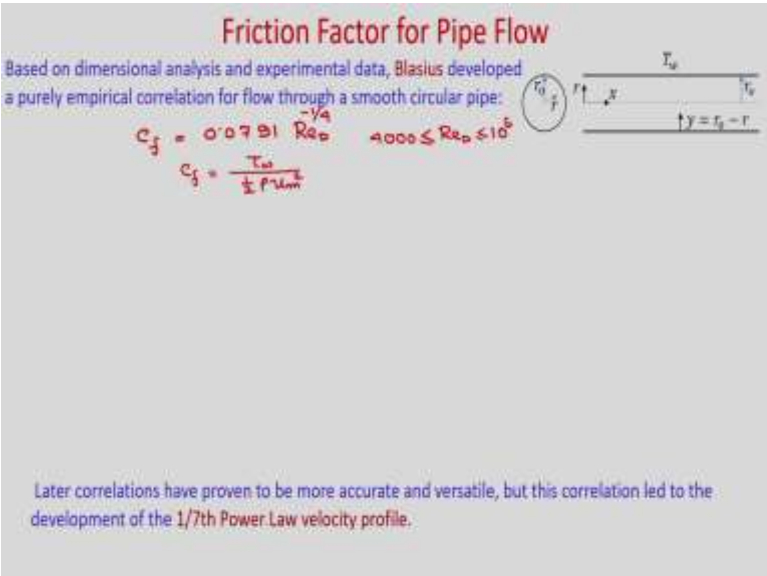
Now, you know at r = 0, $\frac{\partial \bar{u}}{\partial r}$ = 0, right? Because this is your at the center it is changing its gradient. So, obviously, $\frac{\partial \bar{u}}{\partial r}$ = 0 and hence shear stress will be 0. So, that means, C_1=0. So, if you see that $\tau\left(r\right)=\frac{r}{2}\frac{\partial p}{\partial x}$.

So, you can see that shear stress varies linearly inside the flow domain maximum will be at the wall and this will be your τ_w at $r = r_0$, τ will be τ_w and 0 will be at $r = 0$. So, at $r = r_0$. So, $\tau_w = \frac{r_0}{2}\frac{\partial p}{\partial x}$. So, the ratio $\frac{\tau}{\tau_w} = \frac{r}{r_0}$. So, local shear is a linear function of radial location.

So, here you can see that shear stress linearly varies with radius. So, it contradicts with the assumptions whatever we have taken for the flow over flat plate case. So, here also we will assume that wherein close to the wall shear stress remain constant and that is equal to τ_w. So, the assume that τ is approximately constant in the direction normal to the wall.

So, universal velocity profile that resulted from this assumption works well for flat plate flow as well as pipe flow. So, in this case we can write $(\nu + \nu_t)\frac{\partial \bar{u}}{\partial r} = \frac{\tau_w}{\rho}$ obviously, it is constant. So, this is the assumptions we are taking.

(Refer Slide Time: 18:52)

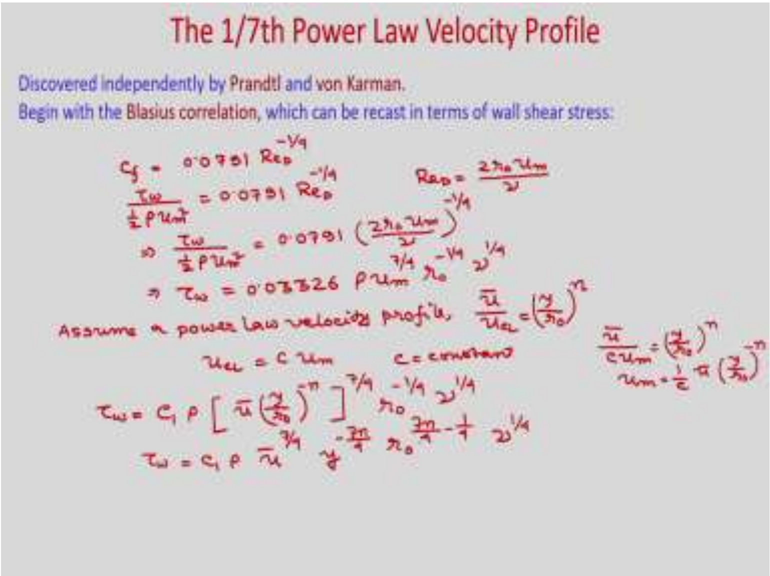


So, now, let us discuss about the friction factor for the pipe flow. So, we have already seen the Blasius correlation for friction factor right. So, based on dimensional analysis and experimental data, Blasius developed a purely empirical coordination for flow through a smooth circular pipe.

And you know that it is $C_f = 0.0791 \mathrm{Re}_D^{-1/4}$ and it is valid in the range $4000 \leq \mathrm{Re}_D \leq 10^5$ and C_f is defined based on the mean velocity. So, it will be $C_f = \dfrac{\tau_w}{\frac{1}{2}\rho u_m^2}$.

So, now if you use the 1/7th velocity profile then you can write the expression for the shear stress. So, later correlations have proven to be more accurate and versatile, but this correlation lead to a development of the 1/7th power law velocity profile.

(Refer Slide Time: 20:07)



So, let us assume that velocity profile mean velocity profile $\dfrac{\bar{u}}{u_{CL}} = \left(\dfrac{y}{r_0}\right)^n$ and let us find what is the value of this exponent n. So, we have already seen that $C_f = 0.0791 \mathrm{Re}_D^{-1/4}$ and if you put the expression of C_f, then you will write $\dfrac{\tau_w}{\frac{1}{2}\rho u_m^2} = 0.0791 \mathrm{Re}_D^{-1/4}$. So, this,

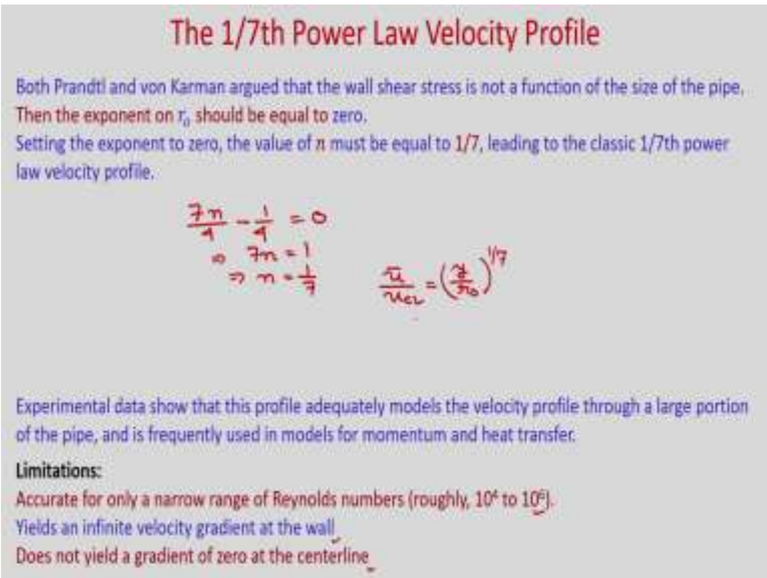
$\mathrm{Re}_D = \dfrac{2r_0 u_m}{\nu}$.

So, you can see that you can write as, $\frac{\tau_w}{\frac{1}{2}\rho u_m^2} = 0.0791\left(\frac{2r_0 u_m}{\nu}\right)^{-1/4}$. So, you can rearrange and you can write $\tau_w = 0.03326\rho u_m^{7/4} r_0^{-1/4}\nu^{1/4}$. Now, assume a power law velocity profile ok. So, we will assume that $\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_0}\right)^n$. So, let us find the value of this exponent n.

You put this velocity profile in the expression of shear stress and find the value of n. So, your centerline velocity will be $u_{CL} = C\, u_m$. So, now, you can see your C is your constant. So, what is u m? So, $u_m = \frac{u_{CL}}{C}$. So, you can see it will be. So, here if you put then, $\frac{\bar{u}}{Cu_m} = \left(\frac{y}{r_0}\right)^n$ and $u_m = \frac{1}{C}\bar{u}\left(\frac{y}{r_0}\right)^{-n}$. So, some constant. So, these constant will be involved here.

So, $\tau_w = C_1\rho\left[\bar{u}\left(\frac{y}{r_0}\right)^{-n}\right]^{7/4} r_0^{-1/4}\nu^{1/4}$. So, now, if you simplify, $\tau_w = C_1\rho\bar{u}^{-7/4} y^{-\frac{7n}{4}} r_0^{\frac{7n}{4}-\frac{1}{4}}\nu^{1/4}$.

(Refer Slide Time: 24:21)



Now, we need to find the value of exponent n both Prandtl and von Karman argued that the wall shear stress is not a function of the size of the pipe then the exponent on $r_0 = 0$.

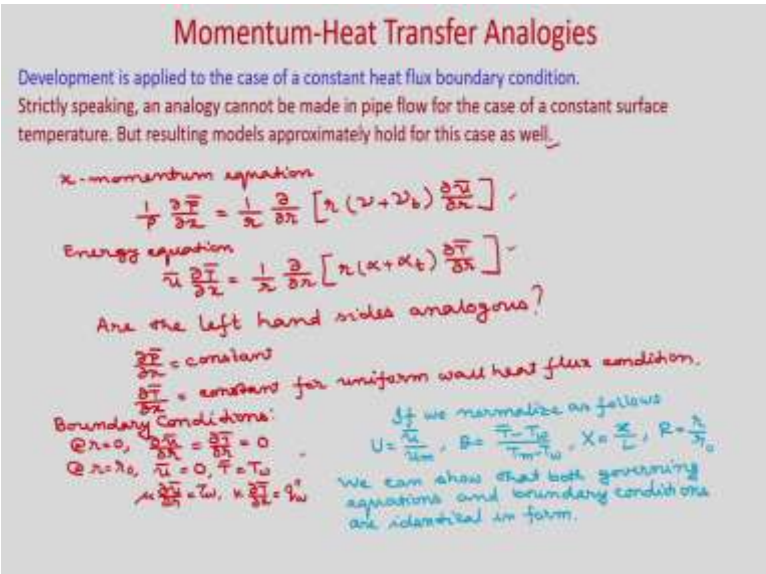
So, you can see in this relation whatever we have written it should not depend on the shear stress should not depend on the size of the pipe. So, here you can see only r_0 is there which is your radius of the pipe. So, we will put its exponent as 0. So, $\frac{7n}{4} - \frac{1}{4} = 0$.

So, setting the exponent to 0 the value of $n = \frac{1}{7}$ leading to the classic 1/ 7th power law velocity profile. $\frac{7n}{4} - \frac{1}{4} = 0$. So, you can see 7n = 1. So, $n = \frac{1}{7}$. So, you can see the velocity profile $\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_0}\right)^{1/7}$ and this is known as 1/7th power law velocity profile.

Experimental data show that this profile adequately models the velocity profile through a large portion of the pipe and is frequently used in models for momentum and heat transfer. But, it has some limitations. You can see that if you use this velocity profile the velocity gradient at r = 0 will not be 0. So, you cannot find the shear stress directly from these velocity profile.

So, the limitations are accurate for only a narrow range of Reynolds number roughly 10^4 to 10^6 yields an infinite velocity gradient at the wall and does not yield a gradient of 0 at the centerline . So, these are the limitations.

(Refer Slide Time: 26:52)



Now, let us discuss about the momentum and heat transfer analogies. So, we have already written the expression for apparent shear stress and apparent heat flux and, let us see that both are analogous to each other or not. Development is applied to the case of constant heat flux boundary conditions.

So, whatever we will be discussing, so, it is directly applicable for the thermal condition with uniform heat flux boundary condition. Strictly speaking, an analogy cannot be made in pipe flow for the case of constant surface temperature. But resulting models approximately hold for this case as well.

So, you can see that your x momentum equation whatever we have written we can write the inertia terms as 0. So, $\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nu+\nu_t\right)\frac{\partial \overline{u}}{\partial r}\right]$. And, if you write the energy equation, so, it is, $\overline{u}\frac{\partial \overline{T}}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\alpha+\alpha_t\right)\frac{\partial \overline{T}}{\partial r}\right]$.

So, you can see as we have assumed that it is fully developed flow so, obviously, $\overline{v}=0$. So, for that reason the second term in the energy equation is 0. So, now, the question is that are these left hand sides analogous the question is that are the left hand sides analogous? So, now, let us see if you consider pipe flow so, obviously, you see that pressure varies linearly in the axial direction. So, that means, your $\frac{\partial \overline{p}}{\partial x}$ = 0.

So, in the momentum equation left hand side is constant because $\frac{\partial \overline{p}}{\partial x}$ is constant. So, in x momentum equation left hand side is constant. Now, if you come to the energy equation we have derived while discussing about laminar internal flows that for a constant wall heat flux boundary condition $\frac{\partial \overline{T}}{\partial x}$ is constant.

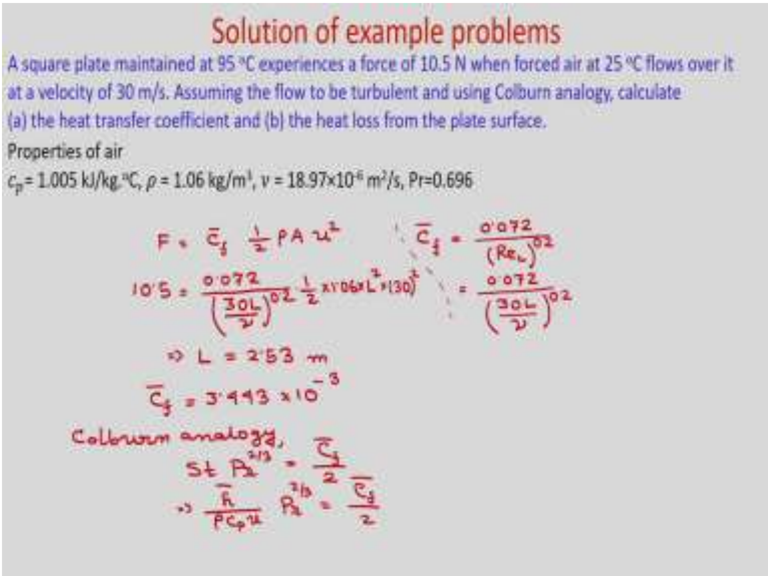
So, you can see $\frac{\partial \overline{p}}{\partial x}$ is constant and $\frac{\partial \overline{T}}{\partial x}$ is constant for uniform wall heat flux condition and now, let us check about the boundary conditions. So, boundary conditions if you check. So, boundary condition at r = 0, $\frac{\partial \overline{u}}{\partial r}$ = $\frac{\partial \overline{T}}{\partial r}$ = 0 at r = r_0 , $\overline{u}$ = 0 .

And, $\overline{T}$ = T_w as well as you have shear stress $\mu \frac{\partial \overline{u}}{\partial r} = \tau_w$ and we have, $k \frac{\partial \overline{T}}{\partial x} = q_w^{"}$. Then we can show that the both the governing equation and boundary conditions are identical in form.

So, if we normalize as follows, $U = \frac{\overline{u}}{u_m}; \theta = \frac{\overline{T} - T_w}{T_m - T_w}$; $X = \frac{x}{L}$ and $R = \frac{r}{r_0}$ we can show that both governing equations and boundary conditions are identical in form.

So, we can use the analogy whatever we are writing for momentum equation that also you can use for energy equation. Using these normalized variables, we can show that both governing equations and boundary conditions are identical in form. So, momentum heat transfer analogy is possible and we can apply analogy method for pipe flow.

(Refer Slide Time: 32:56)



Now, let us solve two problems. A square plate maintained at 95 ^{0}C experiences a force of 10.5 N when forced air at 25 ^{0}C flows over it at a velocity of 30 m/s. Assuming the flow to be turbulent and using Colburn analogy, calculate (a) the heat transfer coefficient and (b) the heat loss from the plate surface.

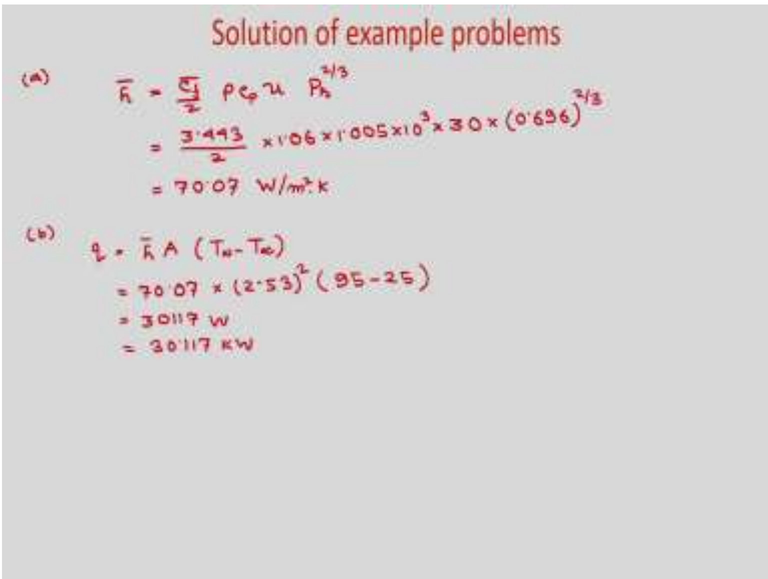
Properties of air are given – you can see c_p, ρ, ν and Pr. So, what we can do you can see the force is given. So, from here you will be able to calculate what is the friction coefficient. So, you can see that the force is given 10.5 N. So, you can write $F = \overline{C}_f \frac{1}{2}\rho A u^2$ and this C_f you know from the analogy that, $\overline{C}_f = \frac{0.072}{\left(\mathrm{Re}_L\right)^{0.2}}$.

So, this is average friction coefficient $\overline{C}_f = \frac{0.072}{\left(\frac{30L}{\nu}\right)^{0.2}}$. So, if you substitute it here from here you will be able to calculate the L. So, you can see $10.5 = \frac{0.072}{\left(\frac{30L}{\nu}\right)^{0.2}} \times \frac{1}{2} \times 1.06 \times L^2 \times (30)^2$. So, if you evaluate it you will get length as 2.53 m. So, once you know L then you will be able to calculate Reynolds number and $\overline{C}_f$. So, from here you can see your $\overline{C}_f$ you can calculate from here $\overline{C}_f$ average friction coefficient if you put the value of L=2.53m, you will get, $\overline{C}_f = 3.443 \times 10^{-3}$.

So, now you use the Colburn analogy. So, Colburn analogy if you use then you will be able to calculate the average heat transfer coefficient. So, this is your $St\,\mathrm{Pr}^{2/3} = \frac{\overline{C}_f}{2}$.

So, this Stanton number you can write as $\frac{\overline{h}}{\rho c_p u}\mathrm{Pr}^{2/3} = \frac{\overline{C}_f}{2}$. Now, you put the values ρ c_p u Pr and $\overline{C}_f$ are known so, you will be able to calculate the heat transfer coefficient.

(Refer Slide Time: 36:51)

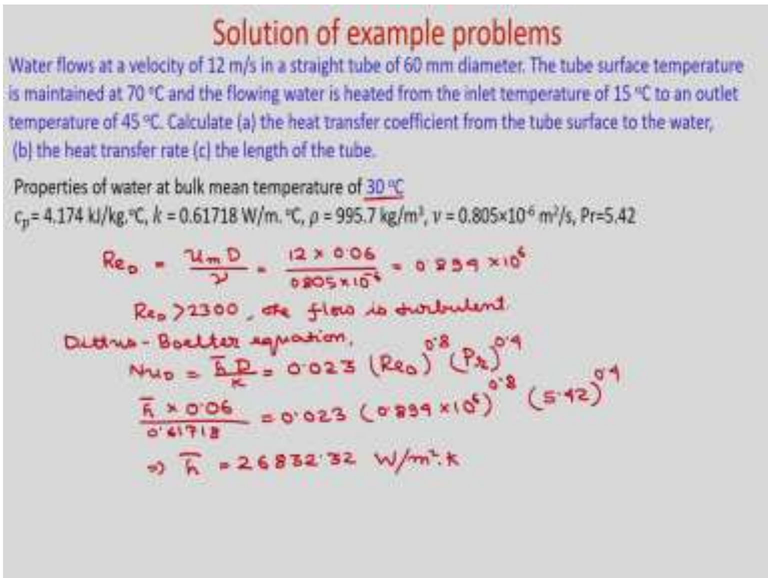


So, $\bar{h} = \frac{\overline{C}_f}{2} \rho c_p u \Pr^{2/3}$. So, you can calculate $\bar{h} = \frac{3.443}{2} \times 1.06 \times 1.005 \times 10^3 \times 30 \times (0.696)^{2/3}$.

So, if you calculate you will get $\bar{h} = 70.07$ W/m^{2}K. So, this first part we have already calculated. So, this is you're (a) heat transfer coefficient. Now, you have to calculate the heat loss from the plate surface. So, heat loss $q = \bar{h} A (T_w - T_\infty)$.

So, what is your temperature difference? $q = 70.07 \times (2.53)^2 (95 - 25)$. You will get as 30117 W or 30.117 kW.

(Refer Slide Time: 38:45)



Now, let us discuss about the next problem. Water flows at a velocity of 12 m/ in a straight tube of 60 mm diameter. The tube surface temperature is maintained at 70 ^{0}C and the flowing water is heated from the inlet temperature of 15 ^{0}C to an outlet temperature of 45 ^{0}C.

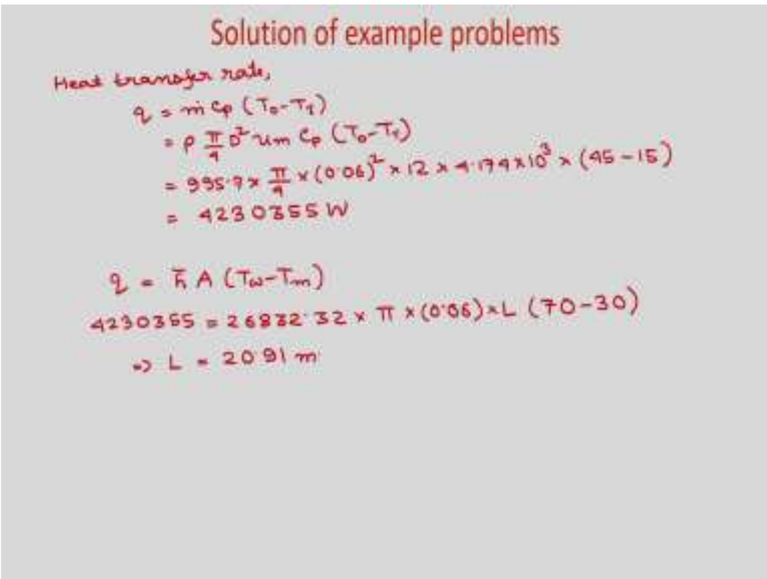
Calculate (a) the heat transfer coefficient from the tube surface to the water. Calculate the heat transfer coefficient from the tube surface to the water, the heat transfer rate and the length of the tube. Properties of water at bulk mean temperature of 30 ^{0}C are given. So, you can see bulk mean temperature is 30 ^{0}C. So, c_p, k, ρ, ν, Pr are given.

So, from here now first you calculate the Reynolds number. So, $\mathrm{Re}_D = \frac{u_m D}{\nu}$. So, based on mean velocity so, it will be 12 m/s, D= 60 mm. So, $\frac{12 \times 0.06}{0.805 \times 10^{-6}}$; so, it will be around 0.894×10^6. So, you can see your $\mathrm{Re}_D > 2300$.

So, obviously, the flow is turbulent. So, we discuss about Dittus-Boelter equation so, that we can use and find the heat transfer coefficient. So, Dittus-Boelter equation so, here you can see it is a heating case because T_w is higher. So, you can use $Nu_D = \frac{\bar{h} D}{K} = 0.023 (\mathrm{Re}_D)^{0.8} (\mathrm{Pr})^{0.4}$.

So, $\frac{\bar{h} \times 0.06}{0.61718} = 0.023\left(0.894 \times 10^{6}\right)^{0.8}(5.42)^{0.4}$. So, if you calculate $\bar{h} = 26832.32 W / m^{2} K$.

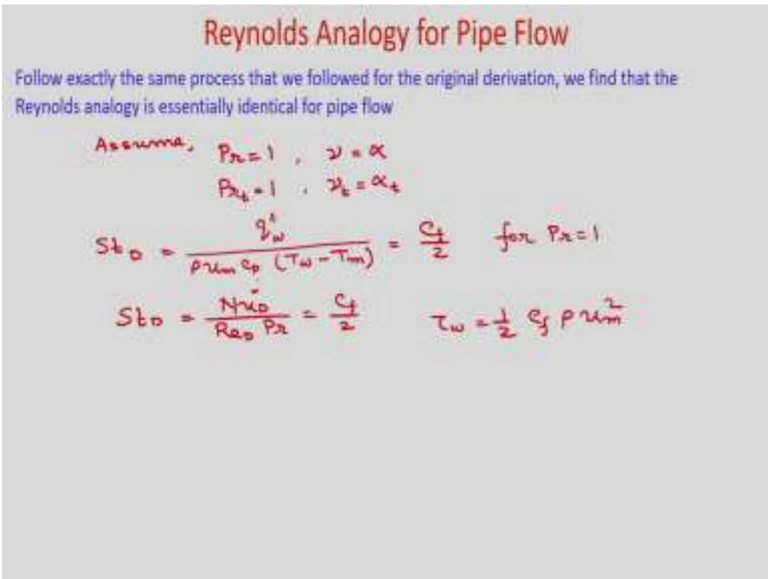
(Refer Slide Time: 41:56)



So, next you need to calculate the heat transfer rate. So, heat transfer rate you can calculate $q = \dot{m} c_p \left(T_o - T_i\right)$. So, $q = \rho \frac{\pi}{4} D^2 u_m c_p \left(T_o - T_i\right)$. So, you put all these values density as, $995.7 \times \frac{\pi}{4} \times (0.06)^2 \times 12 \times 4.174 \times 10^3 \times (45-15)$. So, if you calculate then you will get as 4230355 W.

Now, you need to calculate the length of the tube. So, we will use now the Newton's law of cooling. So, $q = \bar{h} A\left(T_w - T_m\right)$ because T_m is your bulk mean temperature it is given. So, $4230355 = 26832.32 \times \pi \times (0.06) \times L \times (70-30)$. So, if you calculate from here you will get length as 20.91 m.

(Refer Slide Time: 44:00)

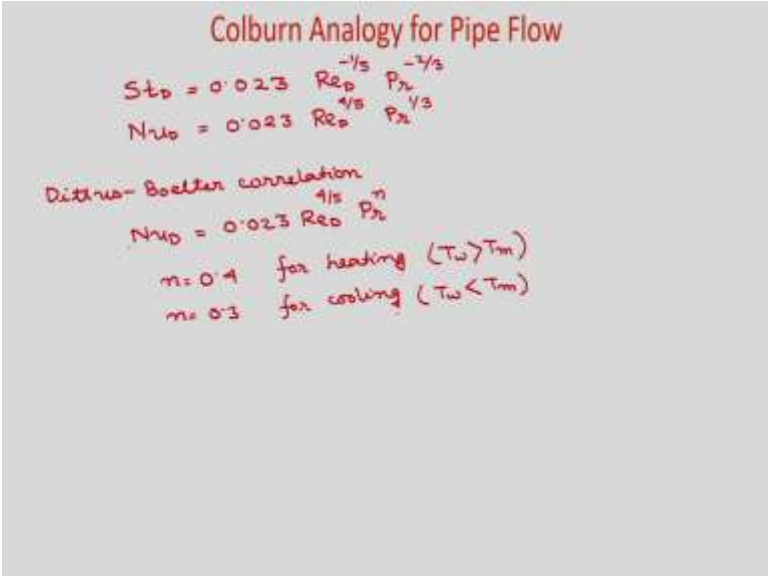


Now, first let us discuss about the Reynolds analogy because we have already derived for laminar flows and for a special case when Pr =1 and turbulent Pr_t =1; that means, your kinetic viscosity is equal to turbulent viscosity and also your thermal diffusivity is equal to your eddy diffusivity.

So, in that case you can use the Reynolds analogy. So, follow exactly the same process that we followed for the original derivation we find that the Reynolds analogy is essentially identical for pipe flow and you assume Pr =1; that means, your $\nu = \alpha$ and Pr_t =1; that means, your $\nu_t = \alpha_t$.

So, the, $St_D = \frac{q_w^{"}}{\rho u_m c_p \left(T_w - T_m\right)} = \frac{C_f}{2}$ for Pr =1. So, $St_D = \frac{Nu_D}{\text{Re}_D \text{Pr}}$. So, $\frac{C_f}{2}$ And, you know that tau w we have already found. So, $\tau_m = \frac{1}{2} C_f \rho u_m^2$. So, from here you will be able to find what is the Nusselt number in case of pipe flow. For Pr =1, you can use Colburn analogy that also we have discussed in detail when we considered laminar internal flow.

(Refer Slide Time: 46:24)



So, in this case you can write the, $St_D = 0.023\,\mathrm{Re}_D^{-1/5}\,\mathrm{Pr}^{-2/3}$ and $Nu_D = 0.023\,\mathrm{Re}_D^{4/5}\,\mathrm{Pr}^{1/3}$.

So, this is your Colburn analogy and you can use this relations when Pr ≠ 1. Another analogy you can write it is a popular correlation $Nu_D = 0.023\,\mathrm{Re}_D^{4/5}\,\mathrm{Pr}^{n}$ where n = 0.4 for heating.

So, when $T_w > T_m$ and n = 0.3 for cooling. So, this you can write as $T_w < T_m$. So, means depending on the whether wall temperature is greater than T_m that means, it is a heating case and if it is a cooling case $T_w < T_m$. So, you can use different value of n and it gives a reasonably good results using this correlation.

So, today we discussed about the convection in a turbulent pipe flow. We started with the universal velocity profile for the flow over flat plate case, and those are also applicable for the pipe flow. Then, we use the Blasius correlation for the friction factor and from there we have derive the exponent for the power law velocity profile. So, n = 1/ 7.

Then, we also we have seen the shear stress varies linearly inside the flow domain, but when we use the universal velocity profile we near to the wall we need to assume tau w as constant. After that we have discuss about the momentum and heat transfer analogy. So, we have seen that the equations governing equations and the boundary conditions in non-dimensional form both are identical.

So, we have used the Reynolds analogy for Pr =1 and we have found the Nusselt number expression; as well as for Pr $\neq$ 1, we use Colburn analogy and also we have written the expression for Nusselt number.

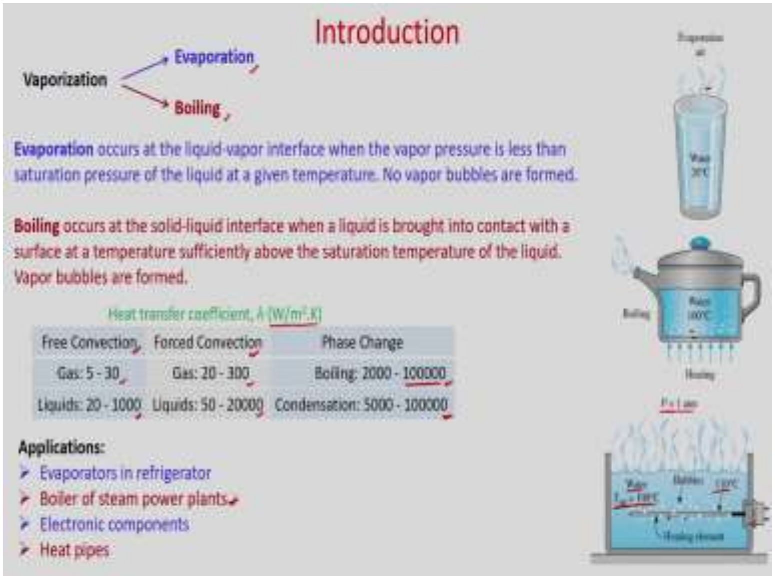
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 12
Boiling and Condensation
Lecture – 40
Boiling regimes and boiling curve

Hello everyone, till now we have studied the convective heat transfer in a single phase fluid flow. Today we will study Boiling and Condensation which is phase change process. Generally, we consider this phase change process because high heat transfer coefficient is involved in this heat transfer process.

(Refer Slide Time: 00:51)



So, first we will discuss about the vaporization. Vaporization is a phase change process where phase change occurs from liquid to vapor. So, vaporization can be classified as evaporation and boiling. Evaporation is a surface phenomena and boiling is a bulk phenomena.

So, you can see here, so vaporization can be classified as evaporation and boiling. So, evaporation is a surface phenomena, so, it occurs at the liquid-vapor interface. So, you can see you have water in a glass. So, outside you have vapor. So, obviously, this process is occurring at the liquid-vapor interface.

And the vapor pressure is less than the saturation pressure of the liquid at the given temperature. And in this particular case, obviously, no bubbles are formed. And when we consider boiling, so boiling occurs at the interface of solid and liquid. So, you consider here, you are heating bottom of this pan, so, obviously the boiling takes place at the interface of the solid and the liquid .

And the temperature of this wall should be higher than the saturation temperature of the liquid at that saturation pressure. So, if you consider water as a liquid, then water saturation temperature at atmospheric pressure you know that it is 100 ^{0}C. So, if surface temperature is higher than the saturation temperature, then your boiling will takes place. In this particular case, obviously, vapor bubbles will be formed and those will go up due to buoyancy.

Let us consider the heat transfer coefficient in different modes of convective heat transfer. You can see if free convection if you consider, so, the, for gas – the heat transfer coefficient is of the range of 5 to 30 W/m^{2}K, whereas, if you consider free convection in liquids, it is 20 to 1000 W/m^{2}K. In forced convection, obviously, fluid flow is taking place by external means.

So, heat transfer coefficient increases. In this particular case, if you consider gas, then the heat transfer coefficient will be of the range of 20 to 300 W/m^{2}K; whereas, if you consider liquids, then obviously it will be much higher than this. So, it will be in the range of 50 to 20,000 W/m^{2}K.

But when you consider phase change like boiling or condensation, your latent heat is involved. So, in this particular case, heat transfer coefficient will be very very high. So, you can see here if you consider boiling phase change process, then your heat transfer coefficient may vary 2000 to 100000 W/m^{2}K. So, you can see it is a very high process; it is a very high heat transfer coefficient.

And you cannot achieve it in forced convection in a single phase fluid. If you consider condensation, the heat transfer coefficient may vary between 5000 to 100000 W/m^{2}K. So, in many application, where you need to remove high heat transfer, then this phase change process is taken as mode of heat transfer. So, you can see there are many applications of this boiling and condensation.

Mainly if you consider boiling, you can see in household application like you boil the water. So, while preparing the tea, when you boil the water, you can observe that when you put the water it is generally the temperature will be lower than the saturation temperature of the water. So, it will just start heating the liquid. After that you will see that formation of small small bubbles from the surface heating surface.

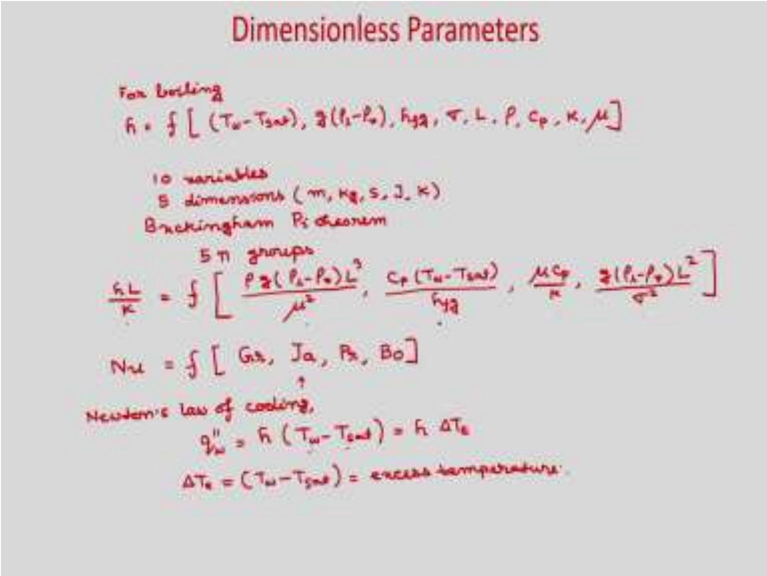
Then it will grow and it will go up due to buoyancy, and it will come to the free surface and it will bust. So, this is one example in our daily life. If you consider refrigerator, so in the evaporator, this refrigerant boiling in evaporator of refrigerator boiling of a refrigerant takes place and high heat transfer coefficient is involved in this particular case.

In electronic components if you consider, nowadays the electronic chip is becoming very smaller and smaller, so heat transfer area is becoming smaller, but you have high heat generation. So, to remove this high heat, you need to have this phase change process.

So, in many application, it is seen that on the chip micro channels are fabricated, and liquids are pass through this channels and heat is removed in phase change process. Another application of electronics cooling is heat pipe. So, heat pipe you can see in your laptop.

So, it is generally removes heat from the processor to the outside ambient by using heat pipes. So, here also phase change process occurs. Evaporator, this boiling process occurs and in condensation takes place at the condenser. And in boilers of steam power plants also you can see the boiling processes.

(Refer Slide Time: 07:15)



In case of phase change, this heat transfer coefficient is function of many parameters like temperature difference between the wall and the saturation temperature. Then you have the buoyancy force, you have the latent heat. And here most important another parameter we need to consider that is surface tension, because there will be liquid-vapor interface as well as it depends on the characteristic length of the geometry and also fluid properties.

So, if you see that your, for boiling, your heat transfer coefficient, $h = f\left[(T_w - T_{sat}), g(\rho_l - \rho_v), h_{fg}, \sigma, L, \rho, c_p, K, \mu\right]$. Then you have some buoyancy, because there will be density difference between the liquid and the vapor, then you have latent heat, surface tension, characteristic length, density of the liquid, specific heat, thermal conductivity and viscosity of the liquid. So, how many variables are there? You can see 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. So, there are 10 variables are there.

And how many fundamental dimensions you have in this particular case? It is 5, because you have meter, second, kg, and two more for energy – Joule, and temperature – Kelvin. So, there are 5 fundamental dimensions, and you have 10 variables. And if you use Buckingham-Pi theorem, then you can find the 5 π non-dimensional groups. So, you can see that there are 10 variables and you have 5 dimensions, which are those you have meter, kg, second, Joule and Kelvin.

So, if you use Buckingham-Pi theorem, then you will get 5 π groups these are non-dimensional groups. And you can show that this will be $\frac{hL}{K} = f\left[\frac{\rho g(\rho_l - \rho_v)L^3}{\mu^2}, \frac{C_p(T_w - T_{sat})}{h_{fg}}, \frac{\mu c_p}{K}, \frac{g(\rho_l - \rho_v)L^2}{\sigma^2}\right]$. So, these are all non-dimensional groups and you see you will get 5 π group, so 1, 2, 3, 4, and 5.

So, $\frac{hL}{K} = Nu$ and it is $Nu = f\left[Gr, Ja, \Pr, Bo\right]$, so this is you can define that it is the ratio of buoyancy force to the viscous force right. So, you can see that there is a density difference. Due to density difference is coming due to temperature difference, and that you can actually use Grashof number.

So, another definition in terms of the density that is similar to Grashof number is $\frac{C_p(T_w - T_{sat})}{h_{fg}}$. So, this is your in the numerator it is sensible energy absorbed by the vapor, and in the denominator it is latent energy absorbed by the vapor. So, this is your you can see that it is called Jakob number, and $\frac{\mu c_p}{K}$ you know it is Prandtl number, and this is the ratio of you can see buoyancy force to the surface tension force and this is known as Bond number .

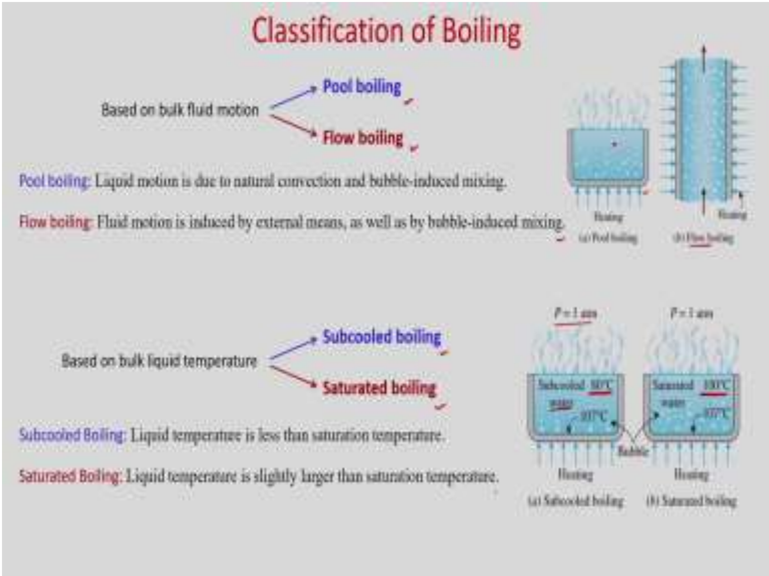
So, bond number is the ratio of buoyancy force to the surface tension force. So, you can see we have got 5 non-dimensional numbers. So, Nusselt number in case of boiling is function of Grashof number, Jakob number, Prandtl number and Bond number.

Generally, in Jakob number when you are considering this $\frac{C_p(T_w - T_{sat})}{h_{fg}}$ is very high. So, in this many cases, you can see that Jakob number can be neglected, it will be very small and it can be neglected. So, you can see that in boiling heat transfer, obviously, your you can write from the Newton's law of cooling what is the heat flux.

So, you can see from Newton's law of cooling you can write the heat flux at the wall $q_w^{"} = h(T_w - T_{sat})$. So, we define the heat flux based on the temperature difference of wall and the saturation temperature. And these temperature difference, generally known as excess temperature and denoted by ΔT_e. So, ΔT_e is the temperature difference between the wall and the saturation, and it is known as excess temperature.

Heat transfer in boiling is very complicated phenomena because there will be formation of bubbles, it will grow, it will collapse, and it will also rise due to buoyancy. So, first we will discuss about the boiling curve how the heat flux varies with the temperature difference that means the excess temperature. And later we will present the correlation to find the heat transfer, to find the heat transfer coefficient or the heat flux first.

(Refer Slide Time: 14:26)



Let us discuss about the classification of boiling. So, we can see based on bulk fluid motion, you can classified classify as boiling can be classified as pool boiling and flow boiling. In case of pool boiling, your liquid medium is quiescent. Once the heating starts, so there will be natural convection.

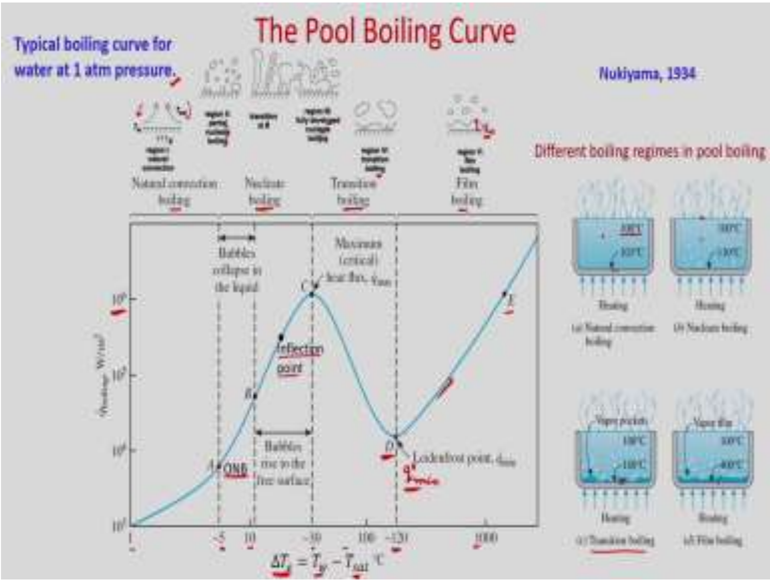
And if you increase more temperature, then there will be formation of bubbles and it will go up due to buoyancy. And due to that, naturally there will be some motion, but there will be no bulk motion in the in case of pool boiling. But when you consider flow boiling, so there will be an external bulk motion when it will pass through this heated surface.

So, there will be heat transfer between the hot surface to the liquid, and there will be formation of vapor, bubbles, and this is known as flow boiling. So, pool boiling – liquid motion is due to natural convection and bubble induced mixing; and in flow boiling – fluid motion is induced by external means as well as by bubble induced mixing.

Based on bulk liquid temperature, these boiling can be classified as sub-cooled boiling and saturated boiling. So, what is sub-cooled boiling? You can see here. So, if the liquid temperature initially it is less than the saturation temperature, then it is known as sub-cooled boiling. So, you can see water at 1 atmospheric pressure, your saturation temperature is 100 ^{0}C. So, water is at 80 ^{0}C, so it is known as sub-cooled boiling.

And when you consider saturated boiling, the liquid temperature initially will be slightly higher than the saturation temperature. In case of water you know saturation temperature is 100 ^{0}C, so the water will be just slightly higher than the saturation temperature. So, you can see sub-cooled boiling liquid temperature is less than the saturation temperature, but in saturated boiling liquid temperature is slightly larger than the saturation temperature.

(Refer Slide Time: 16:46)



So, now we will discuss about the boiling curve for the pool boiling. In this lecture, we will just consider pool boiling. And you can see that this boiling curve was first observed by scientist Nukiyama in 1934. And this typically he did the experiment for the water at 1 atmospheric pressure.

First, I will present the boiling curve what happens different regions of the boiling curve, and what happens when you increase the excess temperature. So, you see this the pool boiling curve the x-axis is your excess temperature, $\Delta T_e = (T_w - T_{sat})$; and in the y-axis, it is the heat flux W/m^{2}K.

When you have this saturation temperature or saturated boiling you consider that means the liquid is at the saturation temperature 100^0C because we are considering the water. So, in this case, when you increase this ΔT_e, that means, essentially you are increasing T_w, because T_{sat} is constant that is for water at 1 atmospheric pressure, it is 100^0C.

So, as you increase T_w, obviously, your excess temperature will increase. So, we can see in this region up to A, where ΔT_e varies from 1 to 5^0C you will get only natural convection boiling. So, what will happen? So, you can see your temperature of the surface is increasing because ΔT_e is increasing; and the liquid is at saturation temperature 100^0C. So, the fluid near to this surface, we will have higher temperature and density will decrease. And due to that, it will go up.

And you can see here. So, this low density fluid will go up and high density liquid will come down, and that way natural convection boiling will takes place. But there will be no formation of bubbles in this region, because you have very small excess temperature. When you go from this point A to C, these are known as nucleate boiling.

So, in the nucleate boiling, you can see it is the temperature range of 5 to 30 ^{0}C – this is the excess temperature 5 to 30^0C. In this particular case, you will find that there will be formation of the bubbles from the liquid and solid surfaces and those bubbles will go up due to buoyancy. When you see between these region A to B generally bubbles collapse in the liquid, because your these super heated water will be your condense and these bubbles will disappear.

After that, if you go in the range of 10 to 30^0C, then these bubbles will go to the free surface. So, in case of delta x, the excess temperature between 5 and 10, your bubbles generally will not go to the free surface because it will be absorbed in the liquid itself. But when you go excess temperature between 10 and 30, these bubbles formation frequency will increase and it will reach to the free surface.

In this region, there are two separate regions you can see A to B, where bubble collapse in the liquid itself in the range of 5 to 10 ^{0}C ; and B to C where bubbles rise to the free surface and nucleate boiling takes place. So, you can see this point A, where the bubbles formation takes place, this is known as onset of nucleate boiling.

This point is known as ONB or onset of nucleate boiling. When temperature increases, your heat flux increases. At some point here around 20 ^{0}C you can see the curvature changes its slope. So, here the curvature of q double prime was increasing. But after this point this curvature will decrease, and this point is known as inflection point.

So, what happens here you can see from when you travel from B, generally your heat flux increases, your heat transfer coefficient will also increase. But after this inflection point your $q_w^{"}$, your heat flux is increasing, but your heat transfer coefficient will decrease because the curvature changes when you go from B to C at this point.

So, at this point – inflection point, you can see after this your heat transfer coefficient will decrease although your heat flux increases then you reach to the point C. So, C is known as maximum heat flux or critical heat flux because beyond that you cannot increase the heat flux. So, it is very important to know what is the maximum heat flux for the design of industrial equipments.

So, in this particular case, you can see if you cannot increase more than these heat maximum heat flux, and it is of the range of 10 to the power 6 W/m^{2}K. So, researchers are finding how to increase this critical heat flux. After that, so in this range A to C you can see when you are in the range of A to B, generally partial nucleate boiling takes place. So, vapor bubbles will go up, it will be it will disappear inside the liquid.

At the point of B, so there will be transition, so these bubbles actually there will be more frequency of bubble formation. And this will coalescence vertically as well as in horizontal direction. So, you can see in vertical direction, this coalescence it will form the vapor column.

And if horizontal direction it coalescence, then it will form a bigger bubble. So, in the range of B to C, you can see this is a fully developed nucleate boiling. So, these bubbles will coalescence, and there will be formation of vapor bubble column, and heat transfer will increase.

After point C to D, you can see although your excess temperature is increasing, but heat flux continuous to decrease up to point D. So, you can see, this is known as transition boiling. So, this is the regime where partial nucleate boiling and partial film boiling takes place. So, this is actually you have an unstable region where it switch to nucleate and film boiling, this is known as transition boiling.

In this particular case, you can see your discrete vapor blanket will be formed over the surface. So, you can see here, if you see here in this figure transition boiling, so in the range of 30 to 120^0C, you can see your this vapor pockets will be formed over the surface, so that means, there will be no connection between liquid and the surface.

So, as the liquid is separated from the surface by this vapor, so your heat flux will continue to decrease, because there will be one vapor blanket and discrete points over the surface. So, it will act as the resistance to heat transfer. So, your heat transfer continues to decrease although your excess temperature increases. So, this is known as transition regime. And in this particular case, generally partial nucleate boiling and film boiling will takes place.

When you come to the bottom point D, so, you will get minimum heat flux. So, this will be q double prime minimum. So, this is this point is known as latent frost point, where you have a minimum heat flux. After that what happens when you further increase the excess temperature you can see that if you go beyond 120^0C, then your heat flux again starts increasing ok. So, you can see it will continue to increase.

So, in this particular regime, what happens? Although you have a vapor blanket over the surface, but it is a continuous vapor blanket on the surface, and your temperature of the surface is also increasing because you have more excess temperature. So, as your temperature of the wall is increasing, your, you will get radioactive heat transfer mode as a dominant heat transfer mode.

So, here radiation heat transfer will take place and heat flux will start increasing. So, you can see, in this regime, there will be a vapor blanket. But as temperature of the surface T w is much much higher, so you will get the radiation mode as a dominant heat transfer mode in this particular regime.

If you see here, so you can see your temperature will be more than 120 ^{0}C and there will be vapor film over the surface. So, there will be a radiation heat transfer will takes place and your heat flux will increase. But if you go up to point E, then at this point generally most of the material will have the melting point in this region.

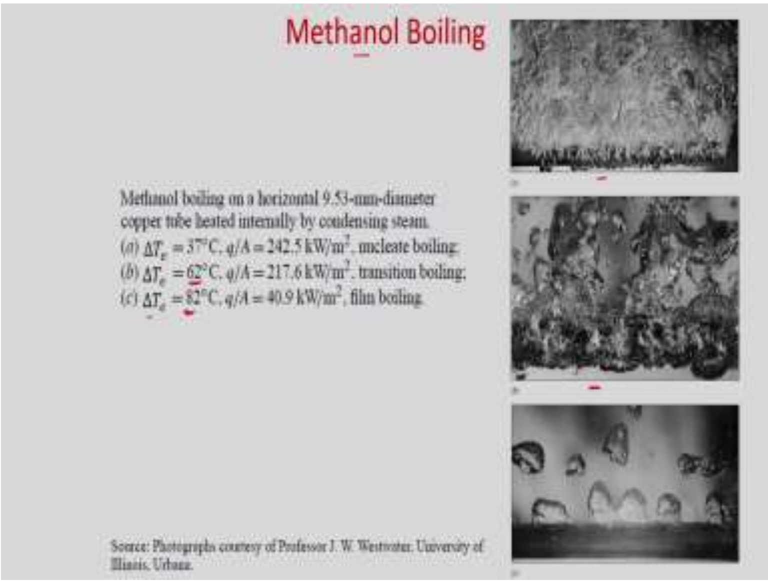
So, if melting point is in this region, then burn out will takes place, that means, your surface or the wall will melt. But if you use some material which is having higher melting point, then obviously, it will go up beyond the point E.

So, we can see when you considered the boiling for when you consider boiling curve for water at 1 atmospheric pressure, so we got 4 different regimes natural convection boiling which occurs between the excess temperature 1 and 5, then you get nucleate boiling in the range of 5 and 10, 5 and 30, then transition boiling between 30 and 120, and film boiling above 120.

And in the range of nucleate boiling, you can see a point is the onset of boiling, where vapor bubbles starts appearing. And at point B, you can see up to point B, there will be bubbles collapses in the liquid; but B to C, you will get bubbles rise to the free surface. And in between you will see that your this curve will change its curvature at this point that is why it is known as inflection point.

And beyond this point, your heat transfer coefficient will decrease although your heat flux increases. Once you reach at C, you will get the maximum heat flux. And the point D which is known as Leidenfrost point, so there is the minimum heat flux you will get. And after that radiation heat transfer mode will be dominant, and again heat transfer will increase.

(Refer Slide Time: 30:50)

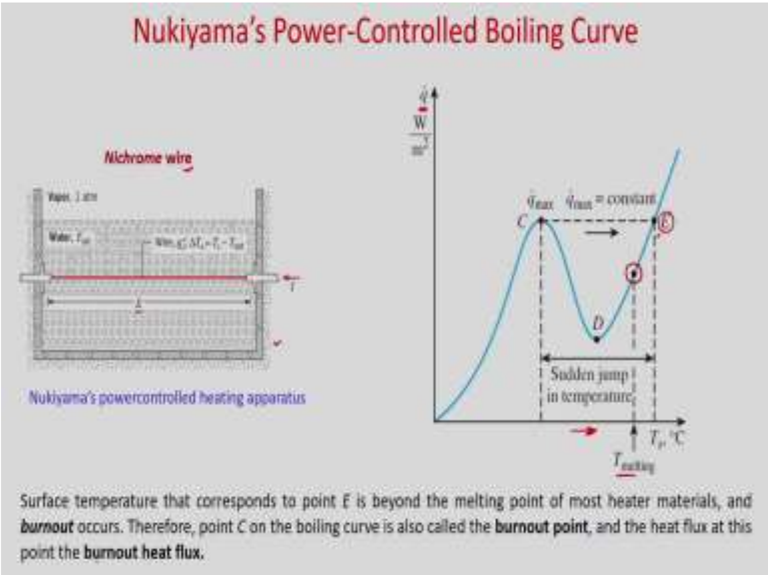


Here you can see the experimental figure of methanol boiling. So, you can see that methanol boiling on a horizontal 9.53 mm diameter copper tube heated internally by condensing steam. So, when you have excess temperature of 37^0C, you can see how the nucleate boiling takes place. So, formation of bubbles, it grows, it coalescence.

And when the temperature difference as 62^0C, then transition boiling takes place, so it will be unstable region where both film boiling and the nucleate boiling will takes place. And when ΔT_e is 82^0C, you can see it is a film boiling. So, here 82 ^{0}C because we are considering methanol. So, at the 82^0C, your film boiling will takes place you can see on the surface you have a vapor blanket.

So, we discuss the pool boiling curve in general. Now, let us see what Nukiyama did the experiment. So, Nukiyama in 1934 when he did the experiment he took first nichrome wire.

(Refer Slide Time: 32:16)



So, you can see it was his experimental setup. This is a power control heating. So, you have this voltage difference and the current is passing through this nichrome wire. And he was just increasing the heat flux on this nichrome wire by controlling the current.

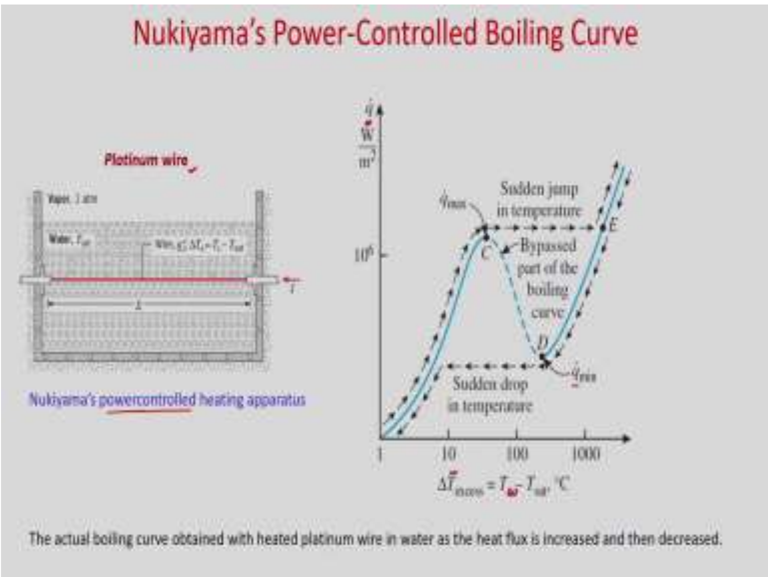
So, you can see this is your Nukiyama power control heating apparatus. So, when he did this experiment using nichrome wire, so we can see in the natural convection boiling nucleate boiling it goes. But when it reaches to the critical heat flux, then suddenly it jumps into the point E, where actually burn out takes place.

These region actually the temperature when you are controlling the heat flux, when you are controlling the heat flux, that means, you are increasing slowly slowly this heat flux, then what happens when you cross point C, there will be sudden jump in the temperature ok. So, suddenly your wall temperature will increase, and hence actually it will reach two point three where it will be the temperature which is higher than the melting temperature of the nichrome.

So, actually this nichrome wire, nichrome wire was burnt at this point. So, suddenly it will jump from here to here, because the melting temperature of the nichrome was lower than this point. So, your nichrome wire was burnt. So, he did not observe this curve from C to D, D to E because it suddenly goes from C to E.

So, surface temperature that corresponds to point E is beyond the melting point of most heater materials and burn out occurs. Therefore, point C on the boiling curve is also called the burnout point and the heat flux at this point the known as burn out heat flux. When this nichrome wire was burnt, then Nukiyama took platinum wire whose melting temperature is higher than the nichrome wire.

(Refer Slide Time: 34:46)



So, when he used platinum wire in the same experimental setup and he was controlling the power, that means, he was increasing the heat flux then he observed that you can see here. So, when he was increasing the temperature, it was following this natural convection regime, nucleate boiling regime.

And once it comes to the critical heat flux it suddenly jumps to point E ok, so that means, there will be sudden jump in the temperature , sudden jump in the water temperature, because you are increasing the heat flux. As you are increasing the heat flux you do not have any control to over the surface temperature.

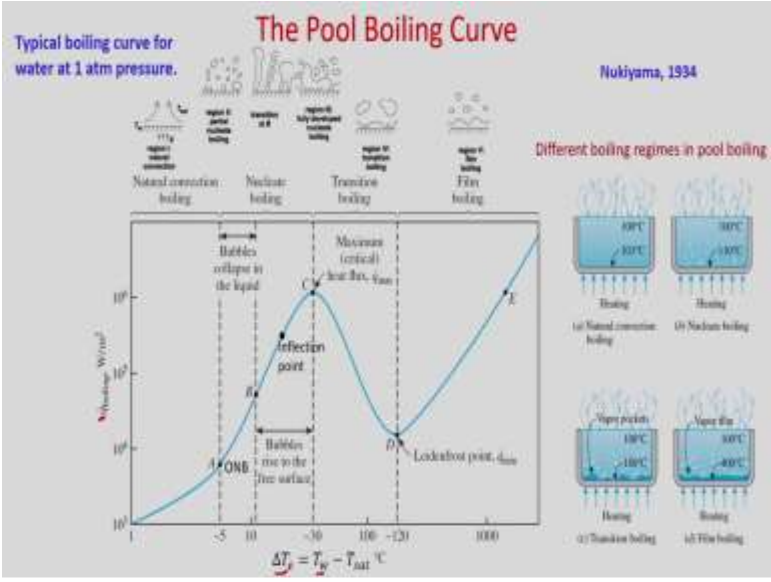
So, your surface temperature increases and this point from here, it suddenly jumps to this temperature E as your platinum wire temperature is higher than the 1000^0C. So, then you will see that your heat flux is increasing as your ΔT_e increases. And if you do the reverse, now you do the you decrease the heat flux. So, when you decrease the heat flux from this point.

So, as platinum wire is used, so burn out did not takes place. So, from that point if you decrease, the heat flux gradually decreases and it follows the path and it comes to the Leidenfrost point q_{min} at point D. But after that suddenly when you decrease the heat flux, suddenly it jumps from D to this point , then there will be sudden drop in the temperature.

So, you can see when this platinum wire was used and power controlled experiment was carried out. So, this C to D was missed because these phenomena tooks place so fast, that from C after point C suddenly there will be jump in the temperature and it will go to the point E. And once the heat flux is decreased gradually, it will follow the path up to D, but after D suddenly there will be a drop in the temperature. So, it will not follow D to C. So, this experiment was carried out by Nukiyama using power control method.

Now, somehow if you can control the temperature, how can you do that? So, you can use some phase change. Let us say condensation is taking place over the surface, so at constant temperature this phase change will occur. So, you can maintain the temperature. So, controlling the temperature, if you see the boiling curve, then you can see that it will follow whatever we have discussed.

(Refer Slide Time: 37:47)



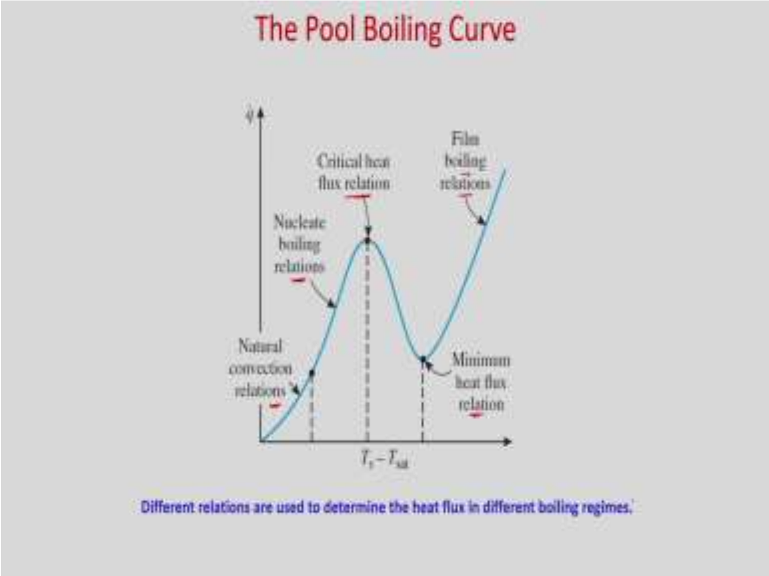
So, when you use temperature controlled curve, so that means, your controlling ΔT_e, that means, T_w you are increasing slowly and you are measuring the heat flux. In a, what Nukiyama did Nukiyama, initially he was actually maintaining the heat flux. So, he was increasing the current. So, heat flux was increasing and we discuss about the boiling curve.

Now, you are controlling the temperature that means you are slowly increasing the temperature. In that particular case, you can see it will follow the curve from A to B to C and C to D is also observed because it does not take place in short time. So, it is seen in this particular case and D to E.

And if you decrease the temperature also, it will follow E to D, then D to C and C to A. So, similar way, it will also follow when temperature controlled experiment if you perform. Although these boiling curve we have discussed for water, but the other liquids also follow the similar curve. But you can see that its shape may change because the temperature range will change – the excess temperature.

As you can see that it is a very complicated phenomena as you can see that boiling is a complicated phenomena, so it is difficult to derive the heat transfer coefficient or Nusselt number analytically. So, mostly these are experimental base that means we will present now the heat flux or the Nusselt number based on the correlation. So, from the experiment whatever researchers are carried out from there generally these relations are presented and those are known as correlations.

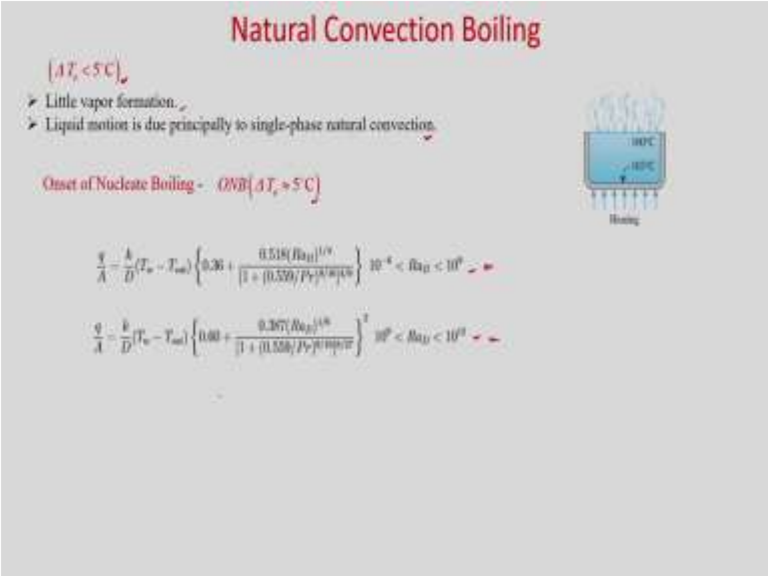
(Refer Slide Time: 39:50)



So, we can see we have natural convection relations where your natural convection takes place, then nucleate boiling relations, then critical heat flux relation where maximum heat transfer will take place, then minimum heat flux relation at the Leidenfrost point whatever heat flux you get, then film boiling relations. So, different relations are used to determine the heat flux in different boiling regimes using correlations.

Now, first regime is natural convection boiling. So, in a natural convection boiling, generally you have already studied in the natural convection. So, those relations you can use depending on the value of Rayleigh number.

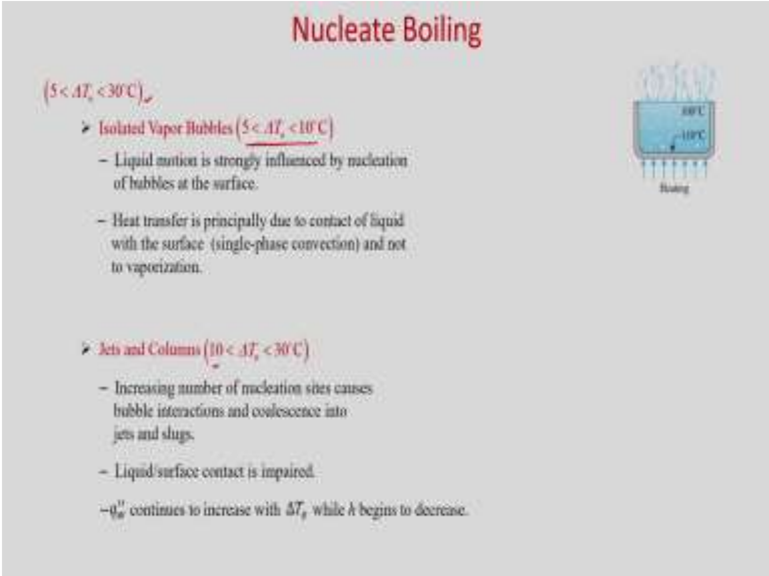
(Refer Slide Time: 40:36)



So, you can see this natural convection boiling takes place when your excess temperature is less than 5 ^{0}C. So, there will be little vapor formation liquid motion is due to principally due to single phase liquid motion is due principally to single phase natural convection. And onset of boiling takes place at 5 ^{0}C.

So, these are the heat flux relation you can use for different Rayleigh number regimes. So, it is 10^{-6} to 10^9, and this is 10^9 to 10^{12} that means it is in a turbulent regime. And this is you can say that Rayleigh number generally if it is less than 10^7 then it is the laminar regime. So, these relations can be used what you have studied in the natural convection.

(Refer Slide Time: 41:35)

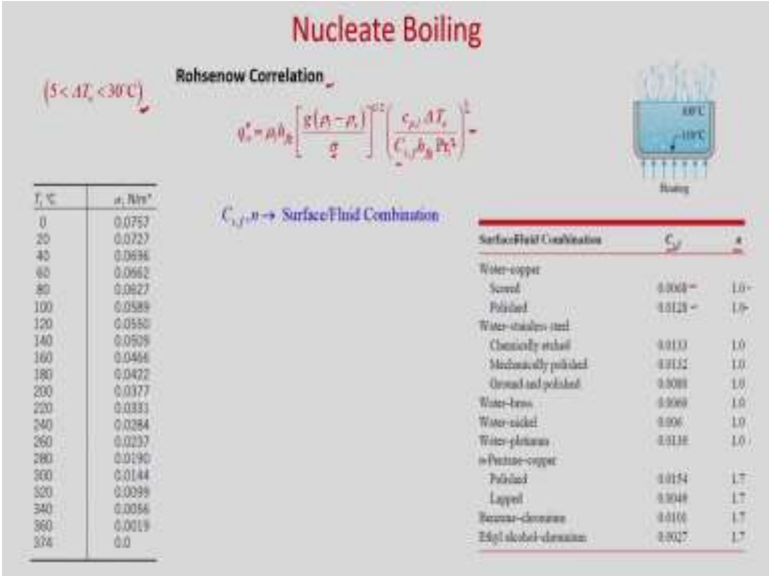


Next nucleate boiling. So, in the nucleate boiling takes place between excess temperature 5 degree centigrade and 30^0C, so you can see there will be isolated vapor bubbles. Liquid motion is strongly influenced by nucleation of bubbles at the surface, and heat transfer is principally due to contact of liquid with the surface. So, single phase convection and not to vaporization between 5 to 10^0C.

So, in the first regime of this nucleate boiling and when you go beyond 10 ^{0}C, so between 10 to 30^0C, there will be coalescence of the bubbles in the vertical direction and horizontal directions you will get jets and columns and increasing number of nucleation sites causes bubble interactions, and coalescence into jets and slugs.

Liquid and surface contact is impaired, and your heat flux continues to increase with ΔT_e with while h begins to decrease. So, you can see that their after the inflection point, your h begins to decrease.

(Refer Slide Time: 42:48)



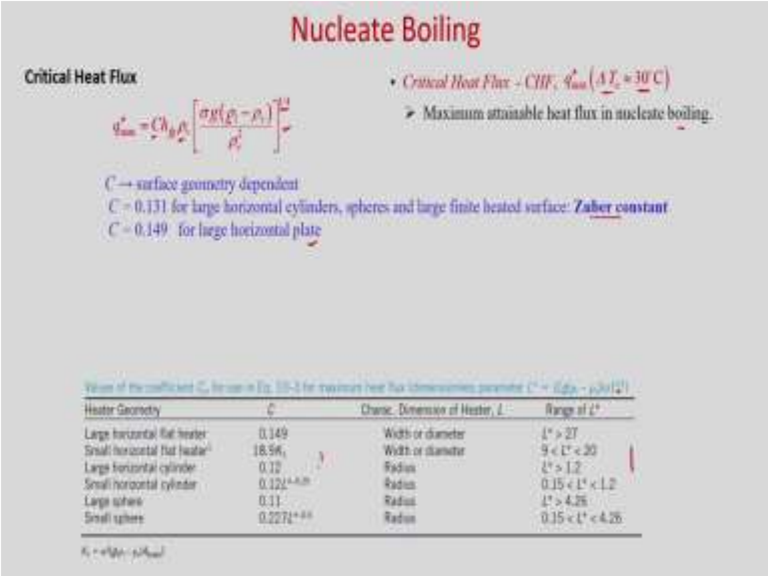
So, in the nucleate boiling generally these correlation is used that is known as Rohsenow correlation. So, you can use this correlation in the nucleate boiling regime. So, here you can see, $q''_w = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left(\frac{c_{p,l} \Delta T_e}{C_{s,f} h_{fg} \mathrm{Pr}_l^n} \right)^3$.

So, here you can see there are two constants, these are experimental constant. So, $C_{s,f}$ depends on the surface, and n depends on the fluid. So, you can see different surface fluid combination what are the values of this $C_{s,f}$ and n. So, if you consider water copper combination, so water is your fluid and copper is your surface.

So, there are scored and polished surfaces depending on that you will get different $C_{s,f}$ and n values is 1 for water. And for water mostly these value n value is 1. But if you take different liquid like pentane, benzene, ethyl alcohol, then it is 1.7. And $C_{s,f}$ also varies accordingly.

Here you can see your surface tension. So, surface tension is also depends on the temperature. So, at different temperature, you can see in this table the surface tension value in N/m.

(Refer Slide Time: 44:31)



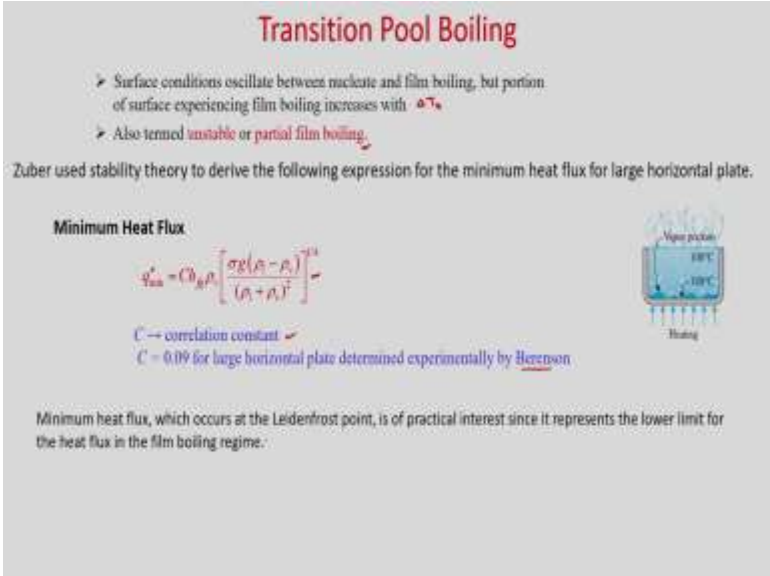
So, as a temperature increases, you can see surface tension decreases. Then at the in the nucleate boiling, you will get the critical heat flux. So, this critical heat flux you will get when your excess temperature is of the order of 30 ^{0}C. So, this is the maximum attainable heat flux in nucleate boiling.

So, this critical heat flux correlation is given by this relation $q''_{\max} = Ch_{fg}\rho_v\left[\frac{\sigma g\left(\rho_l-\rho_v\right)}{\rho_v^2}\right]^{1/4}$.

So, here ρ_v is your vapor density, and ρ_l is your liquid density. And this is the constant and it depends on surface geometry.

So, C is 0.131 for large horizontal cylinders, spheres and large finite heated surface. And this C is known as Zuber constant, and C is 0.149 for large horizontal plate. So, you can see here this C value for different surface geometry, these are the value of C in the range of L*, where $L^* = L\left[\frac{g\left(\rho_l-\rho_v\right)}{\sigma}\right]^{1/2}$. So, this is the L*. So, different L* regime, you will get different value of C.

(Refer Slide Time: 45:59)



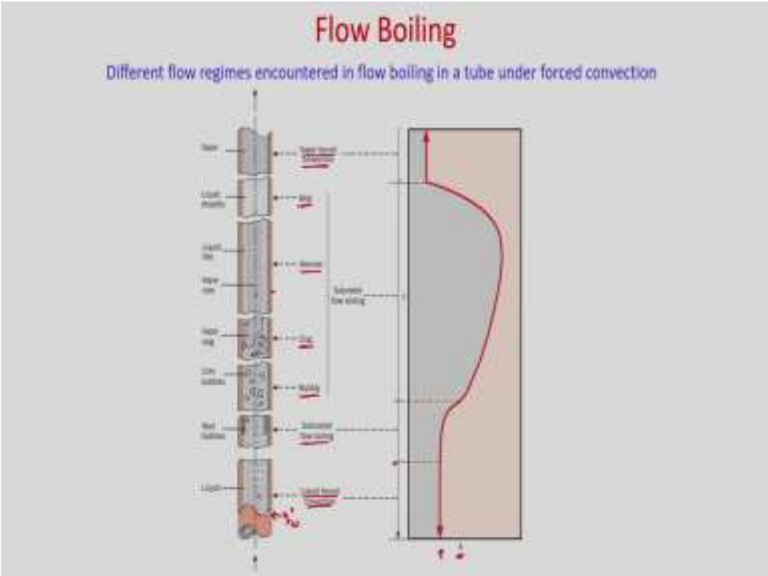
Now, if you go beyond that regime, then you will get transition pool boiling. So, in a transition boiling, what happens it shifts between nucleate boiling and film boiling. So, surface condition oscillates between nucleate and film boiling, but portion of surface experiencing film boiling increases with your ΔT_e, also termed unstable or partial film boiling.

So, Zuber used stability theory to derive the following expression for the minimum heat flux for large horizontal plate, and this minimum heat flux which actually happens in the Leidenfrost point, so that is $q''_{\min} = C h_{fg} \rho_v \left[\frac{\sigma g (\rho_l - \rho_v)}{(\rho_l + \rho_v)^2} \right]^{1/4}$. So, here C is your correlation constant it is generally 0.09.

For large horizontal plate determined experimentally by Berenson. So, maximum heat flux minimum heat flux which occurs at the Leidenfrost point is of practical interest since it represent the lower limit for the heat flux in the film boiling regime. So, in the pool boiling, we have seen when the vapor bubbles are formed those bubbles goes up due to buoyancy, and goes to the free surface.

But when we consider the flow boiling where fluid flow is there due to the external means like a pump then there will be no free surface to remove these bubbles. So, let us consider a vertical pipe where fluid flow is taking place from bottom to top, and the surface of the tube is maintained at a constant heat flux. So, you can see this is your vertical pipe.

(Refer Slide Time: 47:59)



So, water, subcooled water is entering here. Now, the surface of the tube is maintained at constant wall heat flux. So, when this liquid flows through this tube, there are different regimes are seen. You can see that when subcooled liquid enters here, so obviously, the there will be forced convection because the temperature of the liquid will get heated. Once you go up, then you will see that there will be subcooled flow boiling will takes place.

So, some vapor bubbles will be formed from the surfaces. And these vapor bubbles will come near to the central region, and it will form bubbles. And those region is known as bubbly region. If we go further, more heat transfer will take place. And these vapor bubbles will coleus each other and it will grow in size forming the slug region. So, vapor slug will be formed. If you go further, then this vapor slug will form a vapor core in the central region.

And liquid will be flowing between this vapor core, and the surface in the annular region of this vapor core and the tube surface. So, vapor liquid film will be just in contact with the heated surface, and vapor core will move in the central region. After that, if we go further these liquid film, there will be generation of small small vapor bubbles and it will form liquid droplets and homogeneous mixture will be formed, and that is known as mist.

If you go further, then due to heat transfer from the tube surface, these liquid droplets will convert into vapor, and pure vapor region you will get where only vapor forced convention will takes place. So, you can see when you are going from bottom to top, in the bottom you have liquid forced convection because it is a sub-cooled liquid is coming.

So, here single phase heat transfer takes place in this region. Then in between phase change occurs from liquid to vapor following the different regimes. Then at the end, you can see the liquid will be fully converted to vapor, and again you will get a vapor forced convection which is your single phase heat transfer will takes place.

If you see the heat transfer coefficient how it varies, so it is shown in this picture. So, this is your heat transfer coefficient along the length, and x is the liquid mass fraction. So, you can see when you go from this liquid forced convection – single phase heat transfer takes place, there will be not much increase in the heat transfer. When subcooled flow boiling, this will takes place there is will start increasing in bubbly and slug flow it will start increasing, you will get maximum heat transfer coefficient in the annular region.

So, you can see that, when it will come in the annular region you will get maximum heat transfer. Once you go to the mist, so slowly your heat transfer coefficient will decrease. And in the vapor core convection, you can see that there will be formation of hotspot because there is no liquid, so obviously, your low heat transfer will takes place. So, it will become minimum in this region.

So, you can see your liquid mass fraction will be 1 when it is entering; and when it is going to the vapor forced convection region, then obviously this will become 0 because it has become fully vapor. So, you can see this is a very complicated process goes through the different regimes, and different researchers proposed different correlation in different regimes. But those discussions is out of scope of this lecture.

So, you can see only in the sub-cooled region as well as in your vapor convection region, you can use a single phase heat transfer correlations, you can use heat transfer coefficient correlations. And in between you can use because it is a exhibits the combined effect of pool boiling and the forced convection. So, different correlations are proposed in this region considering the effect of pool boiling as well as the pure single phase convection.

So, today we discussed about the pool boiling curve. In pool boiling curve, first we considered the water at 1 atmosphere. With increase of excess temperature we have found four discrete regime; one is nucleate boiling, then first is natural convection boiling, then nucleate boiling regime, then transition boiling and film boiling. So, the point at which your first vapor bubble formation takes place, so that is known as onset of nucleate boiling. Then you will get the inflection point.

In the nucleate boiling regime, if you go beyond the inflection point, then your heat transfer coefficient decreases with increase of heat flux. Then you reach at the maximum heat flux which is known as critical heat flux. After that as you increase the ΔT_e , your heat flux starts decreasing because of formation of vapor blanket over the surface which acts as a resistance to the heat transfer.

Once you come to the Leidenfrost point which is your where you will get the minimum heat flux, after that regime your heat flux starts increasing with the increase of ΔT_e. Because in this particular regime, you will get which is your film boiling, you will get radiation as a major mode of heat transfer.

Due to that, there will be increase in the heat flux as you increase the excess temperature. As boiling phenomena is a very complicated phenomena. So, to calculate the heat flux or the heat transfer coefficient, generally correlations are used from different experiments. So, today we have presented the expression for heat flux in different regimes from where you will be able to calculate the heat transfer coefficient.

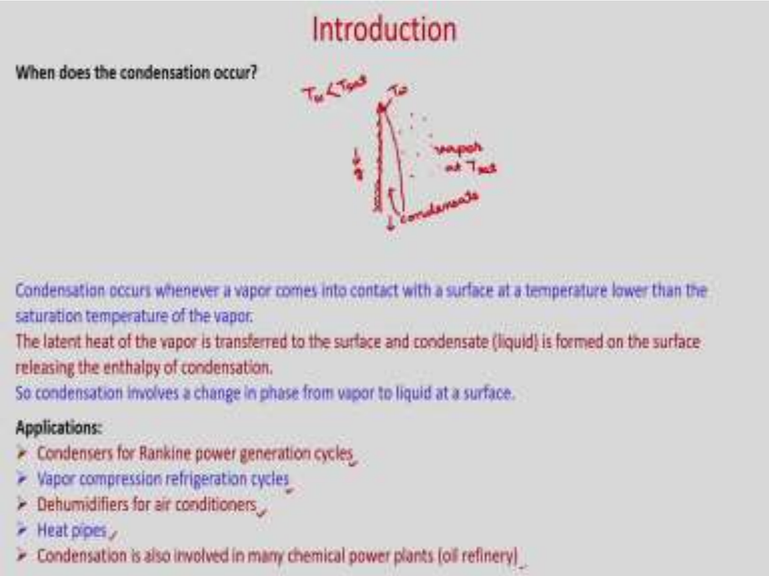
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 12
Boiling and Condensation
Lecture – 41
Laminar film condensation on a vertical plate

Hello, everyone. So, today we will study the condensation and particularly, in today's lecture we will study the Laminar film condensation on a vertical plate.

(Refer Slide Time: 00:44)



We can ask this question, when does the condensation occur? So, when the vapor comes into contact with a liquid or a surface which has less temperature than the saturated saturation temperature of that vapor, then condensation will occur. When the vapor comes into contact with a cold liquid then condensation may occur and it may form fog.

But, in our lecture we will consider only condensation on a surface. So, if you consider that one vapour regime is there and it is at saturation temperature T_{sat}. Now, if you bring one surface whose temperature is lower than the saturation temperature of the vapor, then condensation will take place. So, vapor will release the heat to the surface which is colder than the saturation temperature and condensation will take place and liquid will be formed on the surface of the plate.

So, if you consider one vapor this is your vapor at temperature T_{sat} . Now, if you bring one plate to this vapor whose temperature is T_w and $T_w < T_{sat}$. So, what will happen? This vapor will release the heat to this surface, and liquid will be formed on the surface which is known as condensate and this liquid will form a film on the surface depending on the nature of the surface.

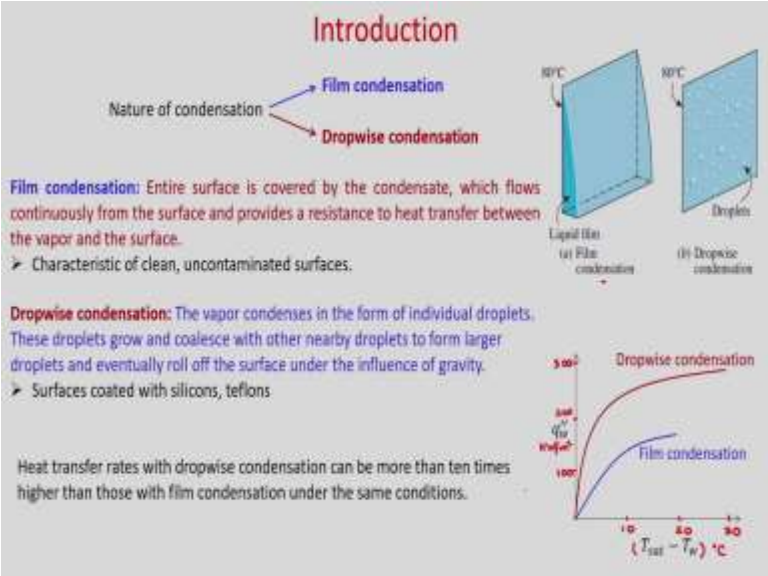
So, this is this liquid is known as condensate and due to gravity it will fall in the downward direction. So, you can see condensation occurs whenever a vapor comes into contact with a surface at a temperature lower than the saturation temperature of the vapor. The latent heat of the vapor is transferred to the surface and condensate is formed on the surface releasing the enthalpy of the condensation.

So, condensation involves a change in phase from vapor to liquid at a surface. So, one good example is that when you see a cold drinks bottle, you bring out from the refrigerator and keep in a ambient then you will see that condensation takes place on the outer surface of the cold drinks bottle. And, generally you have seen that droplets are formed and those droplets just come in downward direction. So, this is one example of condensation.

So, there are many applications of condensation in industry as well as in our daily life. So, you can see here we have noted few applications; condensers for Rankine power generation cycles, vapor compression refrigeration cycles, dehumidifier for air conditioners, heat pipes. So, in heat pipes there are evaporators and condensers. So, in condensers condensation takes place; condensation is also involved in many chemical power plants like oil refinery.

So, as in our lecture we are considering only the surface condensation. So, there are two ways condensation can takes place – one is film condensation and another is drop wise condensation and it depends on the nature of the surface.

(Refer Slide Time: 05:03)



So, you can see film condensation generally one film is formed on the surface and this film just come in downward direction due to gravity. So, this is known as film condensation. So, entire surface is covered by the condensate which flows continuously from the surface and provides a resistant to heat transfer between the vapor and the surface.

So, you can see when this condensation takes place on the surface this liquid film is formed. Once the liquid film is formed then this vapor is not in direct contact with the surface. So, this film acts as a thermal barrier. So, generally this vapor gives this heat to this liquid and this heat is transferred from this liquid to the surface.

So, generally, this film condensation is takes place for a clean and uncontaminated surface. Whereas, in dropwise condensation you can see here so, the small drops are formed after realising the latent heat of condensation; then these smaller droplets coalesces and form bigger droplets and these bigger droplets due to gravity just slides in downward direction.

So, you can see here you have more beer surface where more heat transfer can take place because the vapor is in direct contact with the surface and more heat transfer can takes place. But in case of film condensation, there is a resistance because there is a film thickness. So, you can see the vapor condenses in the form of individual droplets these droplets grow and coalesces with other nearby droplets to form larger droplets and eventually roll up the surface under the influence of gravity.

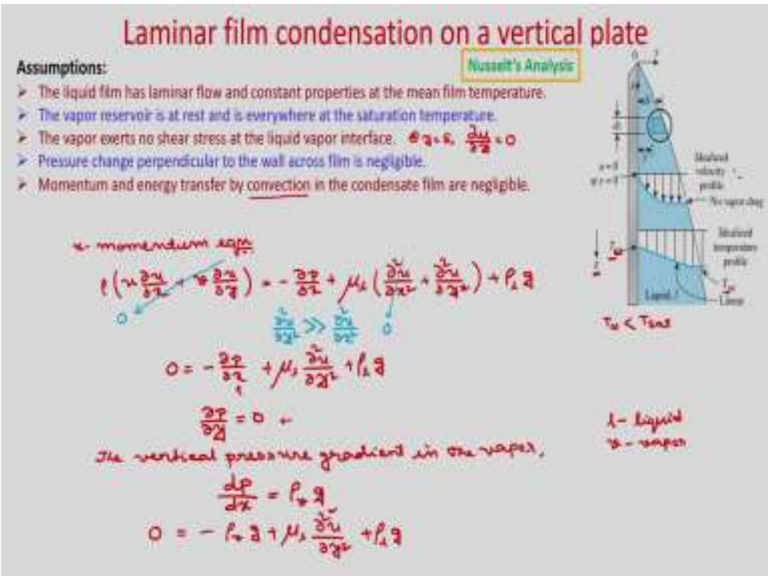
So, surfaces coated with silicones and Teflons. If you have those surfaces then this dropwise condensation can takes place. So, as I told that in case of dropwise condensation vapor is in direct contact with the surface and it can release the latent heat of condensation directly to the surface, but in case of film condensation there is a resistance in heat transfer. So, heat transfer is less compared to the dropwise condensation.

So, if you see the heat transferred on the wall due to condensation versus this the T_{sat} - T_w then you can see that if it is let us say 10 ^{0}C. So, this is your 20 ^{0}C, this is your 30 ^{0}C. So, this T_{sat} - T_w in ^{0}C and y-axis is your heat flux wall heat flux and let us say this is your 100 kW/m^2 this is your 200 and this is your 300 kW/m^2.

So, in case of film condensation so, if you increase this T_{sat} - T_w then gradually it increases. Whereas, you have dropwise condensation then it is much higher heat transfer takes place in case of dropwise condensation. So, you can see here how the heat flux on the wall varies with T_{sat} - T_w. So, heat transfer rates with dropwise condensation can be more than 10 times higher than those with film condensation under the same conditions.

So, now we will study the laminar film condensation on a vertical plate. So, we will consider a vertical plate and the vapor is in quiescent medium. So, when this vapor comes into contact with the surface which is at lower temperature than the T^{sat}, then condensation takes place. And, we are considering only the film condensation, we will use some assumptions.

(Refer Slide Time: 09:38)



So, these are the assumptions are made. The liquid film has laminar flow. It is steady laminar flow and constant properties at the mean film temperature. The vapor reservoir is at rest and is everywhere at the saturation temperature. The vapor exerts no shear stress at the liquid vapor interface; that means, if you have, this is the vertical plate.

So, y is measured perpendicular to this plate and x is measured along the vertical plate in the downward direction. Gravity acts in downward direction; the temperature of the plate is constant that is your T_w and the vapor is at rest and its temperature is T_{sat} everywhere. And, in this particular case $T_w < T_{sat}$. So, obviously, condensation will take place and a liquid film will be formed over this surface.

If you see the velocity distribution we will find in this class. So, here we are assuming that the vapor exerts no shear stress at the liquid vapor interface. So, if this is your liquid vapor interface. So, at this $y = \delta$, where δ is the film thickness. So, if δ is the film thickness then at $y=\delta$ you have no vapor drag; that means, at $y = \delta$ you have $\frac{\partial u}{\partial y} = 0$.

Then pressure change perpendicular to the wall across film is negligible. So, the film thickness is very small and you can neglect the pressure change perpendicular to this wall across film. One another important assumptions we will take in this particular case that momentum and energy transfer by convection in the condensate film are negligible.

So, we will see later that the film thickness is very very small and the velocity of the film in downward direction is very small. And due to that we can neglect the inertia terms in the momentum equation as well as in energy equation. And, obviously, if we assume it so, it will simplify the analysis and this analysis actually was first carried out by scientist Nusselt in 1916 and that is why this analysis is known as Nusselt analysis.

And, it is found that with these assumptions if you find the heat transfer coefficient and Nusselt number or the total heat transfer rate, generally it predicts well with the experiments. With this our objective is to find what is the heat transfer coefficient and the Nusselt number.

Both local heat transfer coefficient and average heat transfer coefficient we will calculate and from there we will calculate the local Nusselt number and average Nusselt number. We need to calculate again the condensation rate or the mass flow rate for this particular case. And, from there we need to calculate what is the film thickness δ because δ varies is in the x-direction. So, we need to calculate the thickness film thickness δ.

Let us write the x-momentum equation. So, $\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu_l\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho_l g$.

So, in this particular case as we have made these assumptions that your force inertia force is very very small. So, in the left hand side this you can neglect you can neglect and also you can show that the variation $\frac{\partial^2 u}{\partial y^2} >> \frac{\partial^2 u}{\partial x^2}$. Hence you can also neglect this term in the diffusion term.

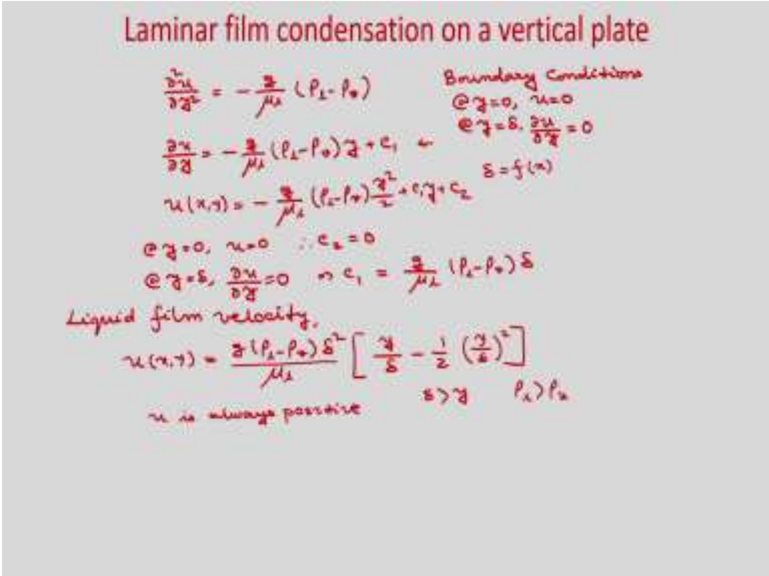
So, if you simplify it now you can write the governing equation as $=-\frac{\partial p}{\partial x}+\mu_l\frac{\partial^2 u}{\partial y^2}+\rho_l g$, and also you can see the liquid film on the surface is assumed to remain to remain thin. So, pressure change across the liquid film are neglected. So, from you can say the $\frac{\partial p}{\partial y}=0$.

So, you can see so, now, if you consider the vapor region. So, in the vapor region it is at rest. So, you can use the hydrostatic pressure. So, how do you calculate the hydrostatic pressure in the vapor region? So, in the vapor region if you calculate the hydrostatic pressure, so, the vertical pressure gradient in the vapor so, we will use this hydrostatic pressure equation.

So, this is nothing, but, $\frac{dp}{dx}=\rho_v g$. So, v is for vapor; so, density of the vapor. So, l is l suffix l is for liquid and suffix v is for vapor. So, as this vapor is at rest. So, you can see $\frac{dp}{dx}=\rho_v g$ because g is acting in the positive x-direction. So, you can write $\frac{dp}{dx}=\rho_v g$.

So, now this pressure gradient is imposed on the outer edge of the liquid film. So, if $\frac{\partial p}{\partial y}=0$, then $\frac{dp}{dx}$ this you can actually use in this equation. So, you can see that you can write $0=-\rho_l g+\mu_l\frac{\partial^2 u}{\partial y^2}+\rho_l g$. So, this we could do because $\frac{\partial p}{\partial y}=0$. So, you have imposed this pressure gradient inside the liquid film.

(Refer Slide Time: 17:30)



So, now from here you can see that you can write the simplified equation as $\frac{\partial^2 u}{\partial y^2} = -\frac{g}{\mu_l}(\rho_l - \rho_v)$. So, this is the equation now you can integrate it and find the velocity distribution, but here you can see what are the boundary conditions at the wall, where y = 0.

Obviously, velocity is 0 and at the edge of the film which is your y = δ you have shear stress is 0; that means, $\frac{\partial u}{\partial y}$ = 0 that assumptions already we have made that vapor is at rest and it exerts no shear stress on the film.

So, boundary conditions at y = 0, u = 0 and at y = δ you have $\frac{\partial u}{\partial y}$ = 0. So, if you integrate this equation so, you will get $\frac{\partial u}{\partial y} = -\frac{g}{\mu_l}(\rho_l - \rho_v)\,y + C_1$ and u which is function of x and y because as a function of δ where δ is function of x right.

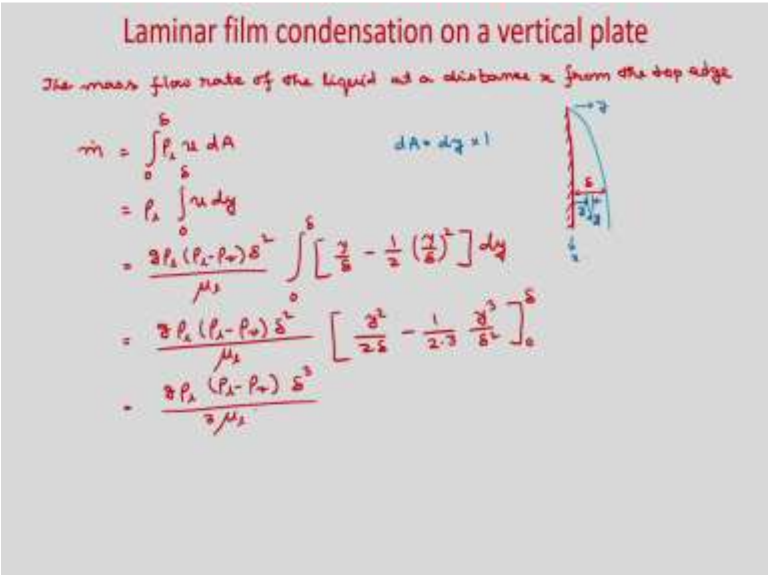
So, we will get $u(\mathrm{x}, y) = -\frac{g}{\mu_l}(\rho_l - \rho_v)\frac{y^2}{2} + C_1 y + C_2$. So, now you apply the boundary conditions. So, if you put at y = 0, u = 0 you will get C_2 as 0. So, at y = 0 u = 0; that means, C_2 = 0 and at y = δ, $\frac{\partial u}{\partial y}$ = 0.

So, if you see from here you can calculate $C_1 = \frac{g}{\mu_l}(\rho_l - \rho_v)\delta$ and δ is function of x. So, if you put all these C_1 and C_2 value there. So, you will get as $u(x, y) = \frac{g(\rho_l - \rho_v)\delta^2}{\mu_l}\left[\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$.

So, in this equation you can see in the right hand side $(\rho_l - \rho_v)$ is positive because $\rho_l >> \rho_v$ and $\frac{y}{\delta}$ is also positive. So, because $\delta > y$ and $\rho_l > \rho_v$.

So, you can see the right hand side term will be positive. So, u is always positive. So, u is always positive. Now, you want to calculate the mass flow rate. So, if you want to calculate the mass flow rate you can integrate this velocity. So, over the film thickness δ.

(Refer Slide Time: 21:26)



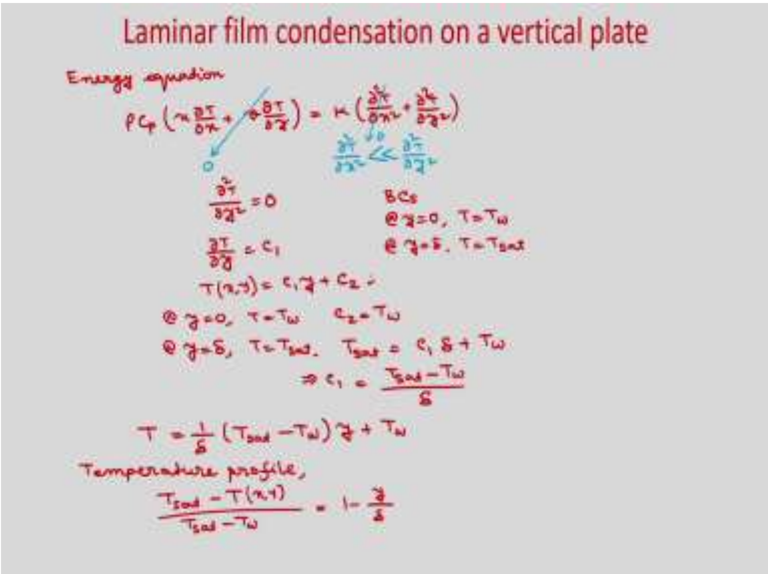
So, the mass flow rate of the liquid at a distance x from the top edge. So, you consider the vertical plate and this is your liquid film over this surface. Now, if you take a distance y from the surface and an elemental distance dy. So, in the perpendicular direction of the surface if you take unit width then your this elemental area will be dy × 1 and this is your y and this is your x.

So, what will be your area? dA = dy $\times 1$; 1 is your unit width . So, now if you want to calculate the mass flow rate, so, $\dot{m}$ you know that you integrate from 0 to δ. So, this is your film thickness δ; so, $\dot{m} = \int_0^\delta \rho_l dA$. So, you can see this will be ρ_l is constant, so $\dot{m} = \rho_l \int_0^\delta u dy$.

So, you can see if you put the expression of u and the constant if you take outside the integral, then you can write this, $\dot{m} = \frac{g\rho_l(\rho_l - \rho_v)\delta^2}{\mu_l}\int_0^\delta \left[\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]dy$. So, now you perform the integration and put the limits then you can find the mass flow rate as $\dot{m} = \frac{g\rho_l(\rho_l - \rho_v)\delta^2}{\mu_l}\left[\frac{y^2}{2\delta} - \frac{1}{2\cdot 3}\frac{y^3}{\delta^2}\right]_0^\delta$ and finally, you will get $\dot{m} = \frac{g\rho_l(\rho_l - \rho_v)\delta^3}{3\mu_l}$. So, you can see the in the expression of velocity as well as mass flow rate, δ is there and this δ is function of x and it is still unknown. So, we need to find this film thickness δ.

So, next we will find the temperature distribution. So, if you see in the energy equation we can neglect the inertia term. So, left hand side terms will be 0 and in this particular case also we can neglect the axial heat conduction because it is very very small compared to the $\frac{\partial^2 T}{\partial y^2}$ hence you will get the energy simplified energy equation as.

(Refer Slide Time: 25:23)



So, if you write energy equation $\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$. So, all these terms in the left hand side these are negligible because inertia terms and $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$. So, this term also you can neglect.

So, simplified energy equation you can write $\frac{\partial^2 T}{\partial y^2}$ =0 so; that means, only heat convection is taking place. So, what are the boundary conditions? At y = 0 you have wall temperature T_w and at y = δ you have T = T_{sat}.

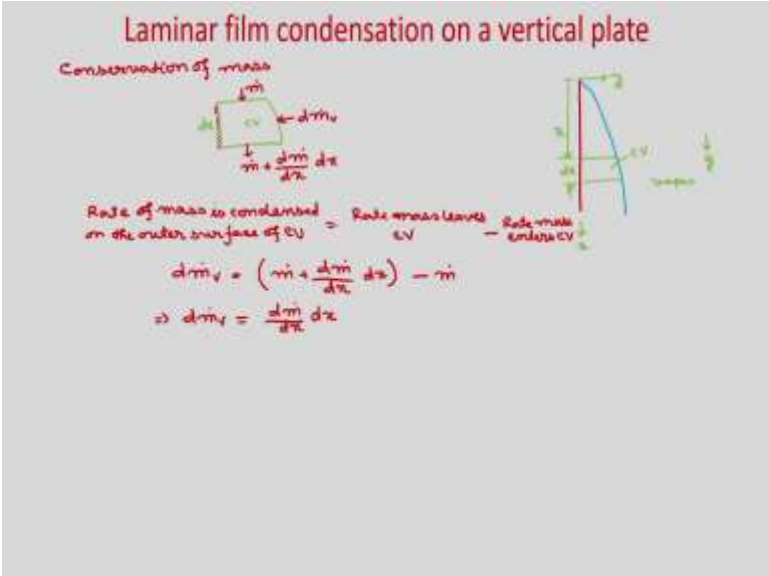
So, if you integrate this equation, what you will get? You will get $\frac{\partial T}{\partial y} = C_1$ or you can write C_3. So, after integrating this equation you will get $\frac{\partial T}{\partial y} = C_1$ and another time if you integrate then you will get, $T(x, y) = C_1 y + C_2$.

So, if you put the boundary condition at y = 0, T = T_w, then you can see that you will get, C_2=T_W and at y = δ, T =T_{sat} . So, you can see you will get $T_{sat} = C_1\delta + T_w$; that means, $C_1 = \frac{T_{sat} - T_w}{\delta}$.

So, if you put C_2 and C_1 value here you will get $T = \frac{1}{\delta}(T_{sat} - T_w)y + T_w$ and you can write temperature profile. So, you can see here only we considered the heat conduction inside the film. So, we will get, $\frac{T_{sat} - T_{(x,y)}}{T_{sat} - T_w} = 1 - \frac{y}{\delta}$.

So, if only heat conduction is taking place so, what do we expect the temperature profile? It will be linear and you can see here that it is a linear profile ok. So, it is a linear. So, it will vary linearly from T_w to T_{sat} because $T_{sat} > T_w$. Now, δ is unknown to calculate the δ, now we will use conservation of mass and conservation of energy inside an elemental volume in liquid film.

(Refer Slide Time: 28:44)



So, if you consider the vertical plate and this is your film thickness, this is your y, this is your x, g is acting in downward direction and if you consider these elemental volume where you can see if it is vapor. So, from the vapor it will release the latent heat through this interface and this will conduct to the surface and in the surface it will release the heat.

And, obviously, there will be mass flow rate is coming here and it will mass flow rate will go out and with that your energy also will come in and energy will go out. So, this is the elemental control volume we are considering and these control volume if you see so, these we are considering at a distance x of distance dx here it is dx and this is your control volume. So, we are using conservation of mass now.

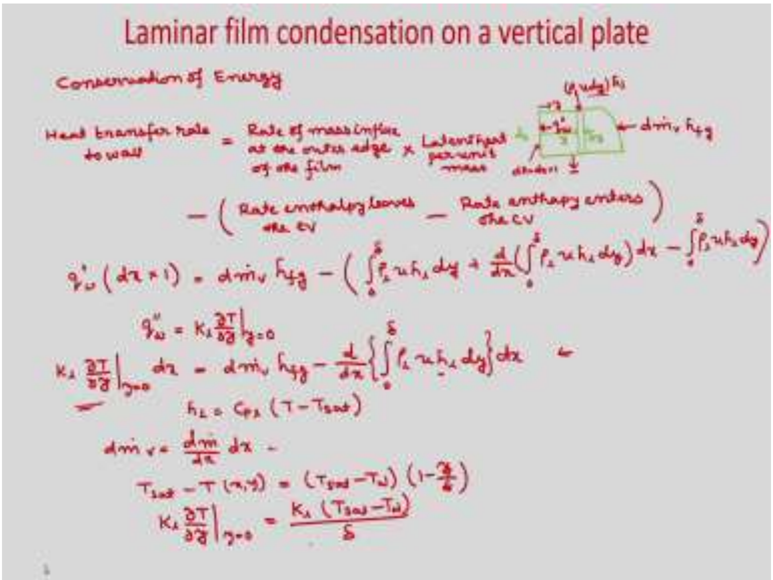
So, you can see from the vapor. So, $d\dot{m}_v$ will go because this will convert from vapor to the liquid releasing the latent heat. Your m dot is coming mass flow rate is coming and as $d\dot{m}_v$ is added. So, you will get m dot and if you use the Taylor series expansion then whatever $\dot{m}$ is coming so, that you can write at the exit of this control volume as $d\dot{m}_v = \left(\dot{m} + \frac{d\dot{m}}{dx}dx\right) - \dot{m}$.

So, this is we are using Taylor series expansion only and this is wall. So, there is no mass is going through the surface because this is wall. So, if you consider this and if you do the mass balance so, you will get rate of mass is condensed on the outer surface of CV is equal to rate of rate mass leaves the control volume minus rate mass enters in control volume.

So, what is that? In left hand side it is $d\dot{m}_v$, right hand side what is leaving? So, $\left(\dot{m}+\frac{d\dot{m}}{dx}dx\right)$ and whatever enters that is your $\dot{m}$. So, you can see your whatever vapor is converting to the liquid, $d\dot{m}_v = \frac{d\dot{m}}{dx}dx$.

So, now we will use the conservation of energy. In this particular case as liquid flame act as the thermal resistance, the vapor converts to the liquid releasing the latent heat to the film, and then this heat is conducted through the liquid to the surface. So, if you consider the latent heat whatever it is given to the liquid at the interface, after that there will be a temperature difference that we are assuming. So, that is known as sub cooling.

(Refer Slide Time: 33:16)



So, if you consider the effect of sub cooling and write the conservation of energy equation then we can write it as. So, whatever control volume we considered. So, this is the control volume we considered, this is your dx and at a distance y let us take a strip of distance dy.

So, now we will use the conservation of energy in this control volume. So, you can see the, whatever vapor is converted to the liquid at the liquid film interface so, that is your $d\dot{m}_v h_{fg}$.

And, there will be a sensible heat because whatever $d\dot{m}_v h_{fg}$ this is going and here on the surface your q''_w is released.

And, you have $\dot{m}$ is coming in and this $\dot{m}$ in this particular small elemental volume it is $\rho_l u h_l$ because we are considering the sensible heat. So, if you are considering a sensible heat then $\rho_l u h_l$ is coming here and obviously, again if you integrate this from 0 to δ because δ is your thickness. So, if you so 0 to δ if you integrate, then you will get the total energy coming in and that if you use the Taylor series expansion then you can write at the outlet.

So, when sub cooling of the liquid film is allowed then total enthalpy flowing into and out of the CV must be included in the energy balance. So, if you do not consider the sensible heat then you can see it is simply whatever heat is released; that means, $d\dot{m}_v h_{fg} = q''_w$. But, if you consider the effect of sub cooling and if you consider this sensible heat then this you need to consider whatever energy is coming in.

So, these and whatever is coming out so, that sensible heat you need to consider. So, conservation of energy now if you write so, it will be heat transfer rate to wall is equal to rate of mass in flux at the outer edge of the film in to latent heat per unit mass.

Now, minus you have to write rate enthalpy leaves the CV minus rate enthalpy enters the C V. So, now, left hand side so, heat transfer rate to the wall what is that? So, your heat flux is

$$q''_w\left(dx\times 1\right) = d\dot{m}_v h_{fg} - \left(\int_0^\delta \rho_l u h_l dy + \frac{d}{dx}\left(\int_0^\delta \rho_l u h_l dy\right)dx - \int_0^\delta \rho_l u h_l dy\right).$$

So, this we are considering if you are taking into account the sensible heat . So, this is known as effect of sub cooling. So, if you are considering that then only you need to add it. So, if you see that if you do not consider the sensible heat then it is just $q''_w\left(dx\times 1\right) = d\dot{m}_v h_{fg}$. This is the simple thing, but as we are considering the sensible heat you need to add this term and you have to integrate because u is function of y. So, x and y, so that you need to integrate.

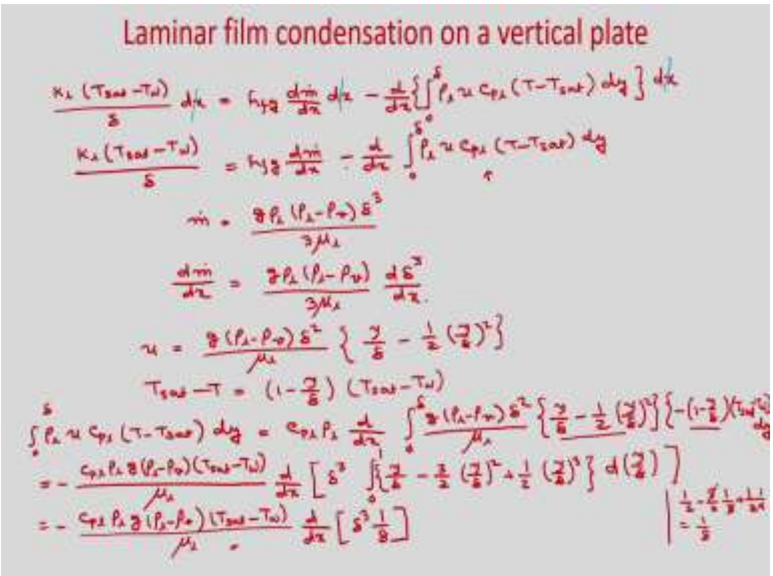
So, $q_w^{"} = K_l \left.\frac{\partial T}{\partial y}\right|_{y=0}$ you see we are writing plus because $q_w^{"}$ is in the negative x y direction because this is your y direction. So, obviously, this is negative into y direction. So, generally we write $q_w^{"} = -K \left.\frac{\partial T}{\partial y}\right|_{y=0}$, but here $q_w^{"}$ is negative to y. So, this is $+K_l \left.\frac{\partial T}{\partial y}\right|_{y=0}$.

So, this if you write here so, you will get, $K_l \left.\frac{\partial T}{\partial y}\right|_{y=0} = d\dot{m}_v h_{fg} - \frac{d}{dx}\int_0^{\delta} \rho_l u h_l dy dx$. Now, here the enthalpy you see the enthalpy of this sub cooled liquid is measured relative to that existing at the saturation temperature. I assuming that the liquid specific heat is constant and the enthalpy of the sub liquid film, $h_l = C_{pl}\left(T - T_{sat}\right)$.

So, this expression you can write. So, as we are calculating the enthalpy with respect to the enthalpy at the interface T at T_{sat}. So, that is why you are writing $h_l = C_{pl}\left(T - T_{sat}\right)$. So, obviously, your now, $d\dot{m}_v = \frac{d\dot{m}}{dx}dx$. So, this you can see your temperature distribution is $T_{sat} - T_{(x,y)} = \left(T_{sat} - T_w\right)\left(1 - \frac{y}{\delta}\right)$.

So, from here if you calculate $K_l \left.\frac{\partial T}{\partial y}\right|_{y=0} = \frac{K_l\left(T_{sat} - T_w\right)}{\delta}$. So, in this term you put this value, then d m dot v you put this value.

(Refer Slide Time: 41:48)



And, write the energy equation as $\frac{K_l\left(T_{sat}-T_w\right)}{\delta}dx = h_{fg}\frac{d\dot{m}}{dx}dx - \frac{d}{dx}\left\{\int_0^{\delta}\rho_l u C_{pl}\left(T-T_{sat}\right)dy\right\}dx$.

So, just putting all the terms here we wrote this expression.

Now, you can see in every term dx is there. So, these dx term you can cancel. So, you can write it as $\frac{K_l\left(T_{sat}-T_w\right)}{\delta} = h_{fg}\frac{d\dot{m}}{dx} - \frac{d}{dx}\int_0^{\delta}\rho_l u C_{pl}\left(T-T_{sat}\right)dy$.

So, now let us calculate $\frac{d\dot{m}}{dx}$ you know the mass flow rate $\dot{m}$. So, $\dot{m} = \frac{g\rho_l\left(\rho_l-\rho_v\right)\delta^3}{3\mu_l}$. So,

$\frac{d\dot{m}}{dx} = \frac{g\rho_l\left(\rho_l-\rho_v\right)}{3\mu_l}\frac{d\delta^3}{dx}$.

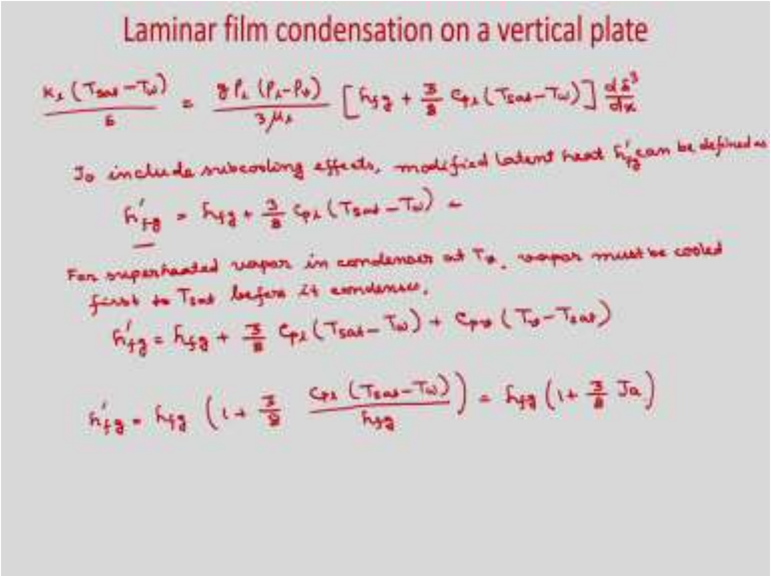
And, we know u, $u = \frac{g\left(\rho_l-\rho_v\right)\delta^2}{\mu_l}\left[\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^2\right]$ and here temperature is there. So, you know $T_{sat}-T = \left(1-\frac{y}{\delta}\right)\left(T_{sat}-T_w\right)$. So, if you put let us evaluate this term first. So, if you put u and T -T_{sat}, then you integrate it.

So, if you write this,

$\int_0^{\delta}\rho_l u C_{pl}\left(T-T_{sat}\right)dy = C_{pl}\rho_l\frac{d}{dx}\int_0^{\delta}\frac{g\left(\rho_l-\rho_v\right)\delta^2}{\mu_l}\left\{\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^2\right\}\left\{-\left(1-\frac{y}{\delta}\right)\left(T_{sat}-T_w\right)\right\}dy$ .So, now, you integrate this one. So, if you see this you can write as, $= -\frac{C_{pl}\rho_l g\left(\rho_l-\rho_v\right)\left(T_{sat}-T_w\right)}{\mu_l}\frac{d}{dx}\left[\delta^3\int_0^1\left\{\frac{y}{\delta}-\frac{3}{2}\left(\frac{y}{\delta}\right)^2+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}d\left(\frac{y}{\delta}\right)\right]$. So, you can see here δ^2 is there, we have divided by δ. So, it will be δ^3. So, now, easily you can integrate it.

You will get $= -\frac{C_{pl}\rho_l g\left(\rho_l-\rho_v\right)\left(T_{sat}-T_w\right)}{\mu_l}\frac{d}{dx}\left[\delta^3\frac{1}{8}\right]$. Because it will be, $\frac{1}{2}-\frac{3}{2}\frac{1}{3}+\frac{1}{2}\frac{1}{4}=\frac{1}{8}$.

(Refer Slide Time: 47:59)



And, $\frac{d\dot{m}}{dx}$ you put it here so, finally, you will get $\frac{K_l(T_{sat}-T_w)}{\delta} = \frac{g\rho_l(\rho_l-\rho_v)}{3\mu_l}\left[h_{fg}+\frac{3}{8}C_{pl}(T_{sat}-T_w)\right]\frac{d\delta^3}{dx}$. So, with that you can write as $h_{fg}+\frac{3}{8}C_{pl}(T_{sat}-T_w)$. So, you can see when we consider the sensible heat one additional term is coming. So, we can say that it is a modified latent heat of condensation h'_{fg} which is your h_{fg} plus the term which is coming considering the sub cooling effect.

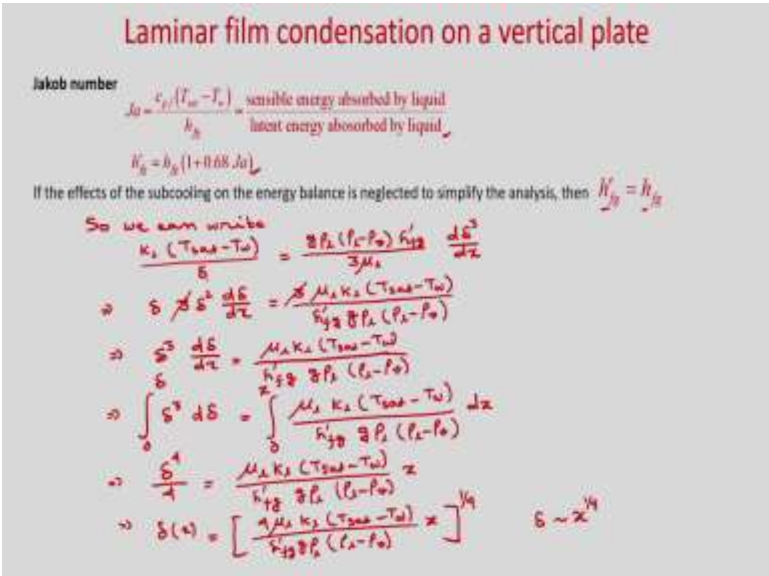
You can say that to include sub cooling effects modified latent heat h' can be defined as $h'_{fg} = h_{fg}+\frac{3}{8}C_{pl}(T_{sat}-T_w)$. So, if you do not consider the sensible heat part, then the last term will be 0. So, simply h_{fg} you can use and that will come from the energy balance where you can write that $d\dot{m}_v h_{fg} = q''_w(dA)$.

So, that is the energy balance, but when we considered the sub cooling effect. So, this additional term is coming. Now, if you consider that your vapor is at higher temperature than T_{sat}; that means superheated vapor then you have to bring this temperature to T_{sat} then only the condensation will take place.

So, that time if you consider the modified latent heat, then you need to consider that effect also to cool down the vapor from the superheated temperature to the saturation temperature. So, in that particular case you can write for superheated vapor in condenser at T_v, vapor must be cooled first to T_{sat} before it condenses. So, in that case, $h_{fg}^{'} = h_{fg} + \frac{3}{8} C_{pl} \left(T_{sat} - T_w \right) + C_{pv} \left(T_v - T_{sat} \right)$.

So, now here we will define one new non-dimensional number. So, when you considering the sub cooling effect $h_{fg}^{'} = h_{fg} \left(1 + \frac{3}{8} \frac{C_{pl} \left(T_{sat} - T_w \right)}{h_{fg}} \right) = h_{fg} \left(1 + \frac{3}{8} Ja \right)$, one new non-dimensional number which is known as Jakob number.

(Refer Slide Time: 52:26)



So, now, you can see that we can define $Ja = \frac{C_{pl} \left(T_{sat} - T_w \right)}{h_{fg}}$ where it is the ratio of sensible energy absorbed by liquid to the latent energy absorbed by the liquid. And, already so, if you define this Jakob number like this, then the modified latent heat $h_{fg}^{'} = h_{fg} \left(1 + 0.68 Ja \right)$.

And, if the effects of the sub-cooling on the energy balance is neglected to simplify the analysis then whatever now we will derive just in place of $h_{fg}^{'}$ you put h_{fg} then the sub-cooling effect will be neglected. So, now we have done the integration. So, let us put in the original equation and find what is the film thickness δ.

So, we can right, $\frac{K_l\left(T_{sat}-T_w\right)}{\delta}=\frac{g\rho_l\left(\rho_l-\rho_v\right)h_{fg}^{'}}{3\mu_l}\frac{d\delta^3}{dx}$.

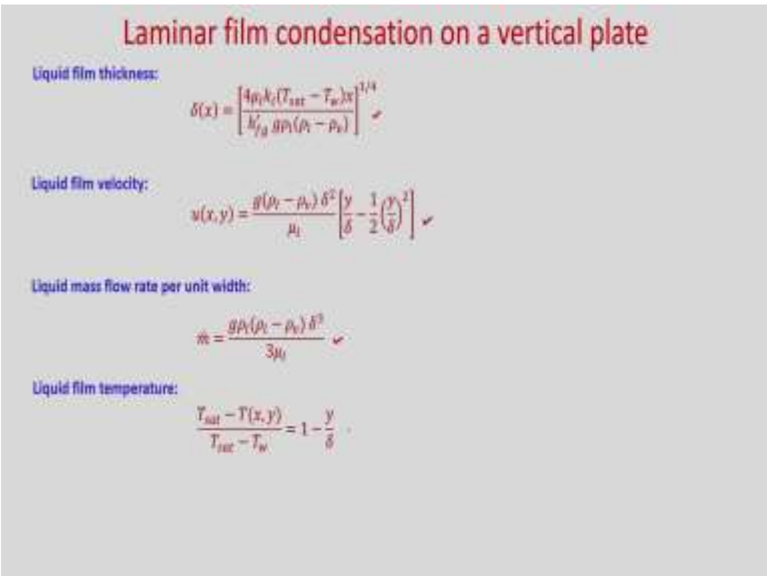
So, you can see if you take $\frac{d\delta^3}{dx}$, then you can write it as, $\delta 3\delta^2\frac{d\delta}{dx}=\frac{3\mu_l K_l\left(T_{sat}-T_w\right)}{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)}$.

So, you can see these 3, 3 will get cancel. So, $\delta^3\frac{d\delta}{dx}=\frac{\mu_l K_l\left(T_{sat}-T_w\right)}{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)}$. So, we can integrate this now. So, you can write $\int_0^\delta \delta^3 d\delta=\int_0^x \frac{\mu_l K_l\left(T_{sat}-T_w\right)}{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)}dx$.

So, you can see if you integrate from x = 0 to l; l is the length of the plate, then at x = 0 what is the thickness of the film? It is 0 and at x = l it is δ . So, let us integrate left hand side 0 to δ and this is 0 to x; x at any length x you can put. So, you can write $\frac{\delta^4}{4}=\frac{\mu_l K_l\left(T_{sat}-T_w\right)}{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)}x$. So, in general we have written the film thickness. So, you can see δ which is function of x you can write, $\delta\left(x\right)=\left[\frac{4\mu_l K_l\left(T_{sat}-T_w\right)}{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)}x\right]^{1/4}$.

So, you see this film thickness film thickness varies with x. So, what is the order? $x^{1/4}$. So, $\delta \sim x^{1/4}$ and when we will solve some example problems you will see that at the end of vertical plate this δ will be order of less than 1 mm. So, it is very very thin.

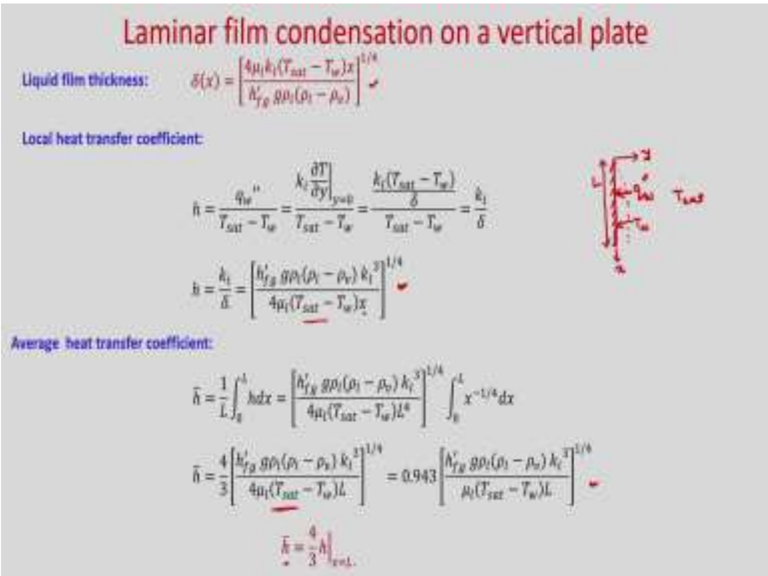
(Refer Slide Time: 58:03)



So, now we have found the film thickness δ. So, you will be able to calculate the velocity as well as temperature profile, then the mass flow rate because all these we have represented in terms of δ. So, you can see the δ(x) we have found now. So, the liquid film velocity is this one.

So, δ if you put you will be you will be getting the film velocity and now, m dot which is your mass flow rate per unit width. So, this is the expression where δ you can put this value and liquid film temperature is this one. So, now we are interested to find what is the local and average heat transfer coefficient and then local and average Nusselt number. So, you can see liquid film thickness we have found like this.

(Refer Slide Time: 58:55)



So, now, heat transfer coefficient which is your local heat transfer coefficient you can define as $h = \dfrac{q_w^{"}}{T_{sat} - T_w}$. So, you can see if this is your plate. So, $q_w^{"}$ is this one and this is your y direction. So, obviously, you can see the heat transfer coefficient you can write as the temperature of the plate is T_w, this is your film thickness and vapor temperature is T_{sat}.

So, obviously, the heat transfer coefficient you can write as $h = \dfrac{k_l \left.\dfrac{\partial T}{\partial y}\right|_{y=0}}{T_{sat} - T_w}$ and this is a positive sign; because y direction the $q_w^{"}$ whatever we have considered in negative y direction.

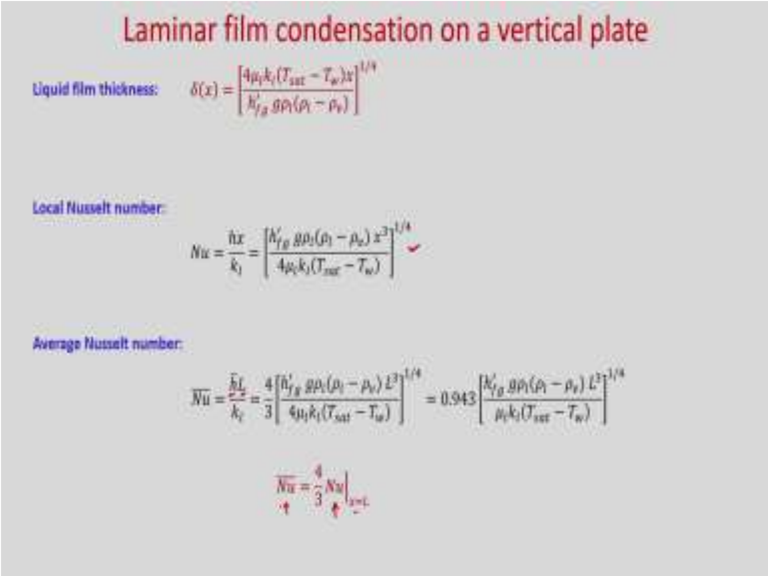
So, obviously, $\dfrac{\dfrac{k_l\left(T_{sat}-T_w\right)}{\delta}}{T_{sat}-T_w}$ .So, here you can see this temperature profile you know. So, you just find $\left.\dfrac{\partial T}{\partial y}\right|_{y=0}$. So, you can see heat transfer coefficient is $\dfrac{k_l}{\delta}$. So, δ if you put in this expression, then you will get the local heat transfer coefficient as this. So, you have to remember this expression.

And, average heat transfer coefficient now you have to just integrate it over the plate length L. So, if plate length L, x is measured from here in the downward direction. So, $\bar{h}=\dfrac{1}{L}\int_0^L hdx$.

So, $\bar{h}=\dfrac{1}{L}\int_0^L hdx=\left[\dfrac{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)k_l^3}{4\mu_l\left(T_{sat}-T_w\right)L^4}\right]^{1/4}\int_0^L x^{-1/4}dx$. So, if you put it then you will get the average heat transfer coefficient like this. So, $\bar{h}=\dfrac{4}{3}\left[\dfrac{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)k_l^3}{4\mu_l\left(T_{sat}-T_w\right)L}\right]^{1/4}$. So, if you evaluate it you will get around $0.943\left[\dfrac{h_{fg}^{'}g\rho_l\left(\rho_l-\rho_v\right)k_l^3}{4\mu_l\left(T_{sat}-T_w\right)L}\right]^{1/4}$. Now, you see this is your local heat transfer coefficient. If you put the local heat transfer coefficient x = L, so, you will get just x = L if you put you are going to get similar expression only outside there will be 4 by 3 because you see this is the expression is same as this inside the bracket except x equal to L. So, the average heat transfer coefficient we can write $\bar{h}=\dfrac{4}{3}h\Big|_{x=L}$.

Now, we can easily calculate the Nusselt number because as you have found the local and average heat transfer coefficient using that you will be able to calculate the Nusselt number.

(Refer Slide Time: 61:49)



So, you can see $Nu = \frac{hx}{k_l}$. So, h already we have found. So, this h if you put here and rearrange you will get this expression. So, now if you put this h bar whatever we have put already evaluated. So, average Nusselt number, $\overline{Nu} = \frac{\bar{h}x}{k_l}$.

So, these if you put here and rearrange you will get, $\overline{Nu} = 0.943\left[\frac{h_{fg}^{'} g\rho_l\left(\rho_l - \rho_v\right)L^3}{\mu_l k_l\left(T_{sat} - T_w\right)}\right]^{1/4}$ and remember here we have defined the Nusselt number based on the length of the plate. Similarly the average Nusselt number you can write $\overline{Nu} = \frac{4}{3}Nu\big|_{x=L}$. So, this is your local Nusselt number this is your local Nusselt number and this is your average Nusselt number.

So, today we considered laminar film condensation on a vertical plate and we neglected the inertia term in the left hand side. So, because the film thickness is very small and it travels very in very slow motion hence we neglected the convection term in the momentum equation and energy equation and from there we calculated the velocity distribution as well as temperature distribution.

Then we calculated the mass flow rate per unit width, then from there we have calculated equating the using the conservation of mass and conservation of energy we have found the film thickness. Once you know the film thickness you know the velocity distribution, temperature profile as well as mass flow rate.

Then we have calculated the local and average heat transfer coefficient as you know the film thickness δ. So, you will be able to calculate easily once you calculate the average heat transfer coefficient and local heat transfer coefficient you will be able to calculate the average Nusselt number and local Nusselt number and that we have found. Now, from there you can conclude that, $\overline{Nu} = \frac{4}{3} Nu\big|_{x=L}$.

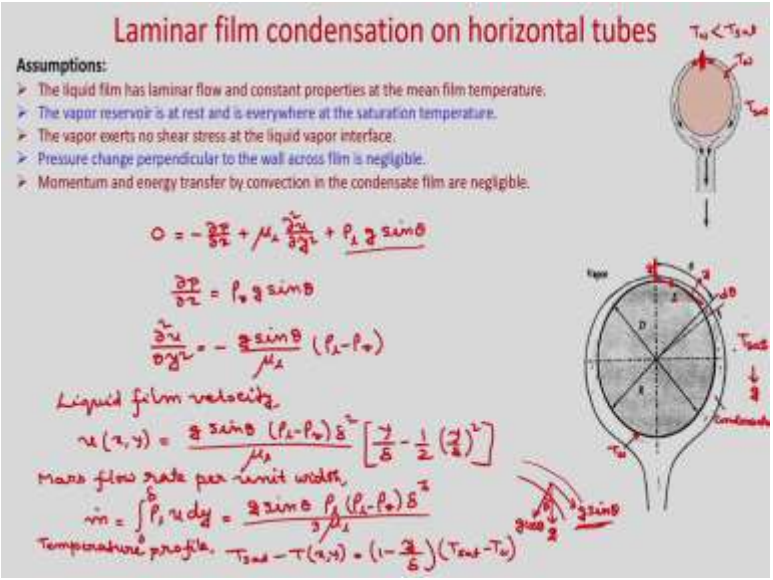
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 12
Boiling and Condensation
Lecture – 42
Laminar film condensation on horizontal tubes

Hello everyone. So, now, today we will consider the Laminar film condensation on horizontal tube. So, in many applications, industrial application, you will find that the condensers are horizontal and over that there will be film flow. So, condensation will take place over the circular pipe, horizontal pipe and those film will fall in downward direction due to gravity. So, in condenser used in power plant, so those are cooled using this technique.

(Refer Slide Time: 01:09)



So, first we will consider only single horizontal tube and similar to earlier analysis, whatever we have done for film condensation on vertical plate, we will use here and we will find the Nusselt number. So, you can see if you consider a single horizontal tube, so obviously if the surface temperature is T_w and vapor temperature is T_{sat} and $T_w < T_{sat}$; obviously, the condensation will take place on the surface of the tube and those film due to gravity, it will just travel in the downward direction.

So, the assumptions whatever we made in earlier study, we will use the same assumptions here and the at the vapor liquid interface, we will use no shear stress boundary condition. So, you can see in this particular case, θ will measure from here and x is measured along the surface in this direction. So, at any point, it is tangential direction and perpendicular to this surface, we are measuring y.

So, at any point if you draw the perpendicular directions, so that is your y and tangential direction is your x. So, at an angle θ, we are taking one elemental volume, so that is your dθ. So obviously, so this is your condensate. The surface temperature is T_w and vapor temperature is T_{sat} . So, in this particular case g is acting in this direction, vertically downward direction.

But when we consider the governing equation, you can see this will have two components; one in the tangential direction and another in perpendicular direction. So, when we calculate the pressure gradient from the hydrostatic relation for the vapor. So, these component; x direction component of the gravity, we have to take because we are considering the governing equation in x direction.

So, you can see that we can just left hand side in the momentum equation, it will be 0. You have $0 = -\frac{\partial p}{\partial x} + \mu_l \frac{\partial^2 u}{\partial y^2} + \rho_l g \sin\theta$. So, in this case if you see if you consider the film here. So, this is your x direction gravity and this is your y direction gravity and this is your g acting.

So obviously, you can see this will be g sin θ because this is your θ and this will be your g cos θ. So, in the x momentum equation, we have written $\rho_l \times g \sin\theta$; the x component of the gravity. So, here now you can write from the hydrostatic equation in the vapor $\frac{\partial p}{\partial x}$, you can consider.

So, $\frac{\partial p}{\partial x} = \rho_v g \sin\theta$ similar way, in the x direction we are considering. So, if you substitute it here, you will get $\frac{\partial^2 u}{\partial y^2} = -\frac{g \sin\theta}{\mu_l}\left(\rho_l - \rho_v\right)$. So, if you integrate this equation with the boundary condition that at y = 0, your velocity 0 and at y = δ, you have shear stress 0.

So, putting the boundary condition, I am going to write the final expression of the velocity profile. So, velocity profile we will get; it is the same expression as earlier case except g is replaced with g sin θ and next, we want to calculate the mass flow rate. So, we have to integrate from 0 to y. So, mass flow rate per unit width; so, $\dot{m} = \int_0^\delta \rho_l u dy$.

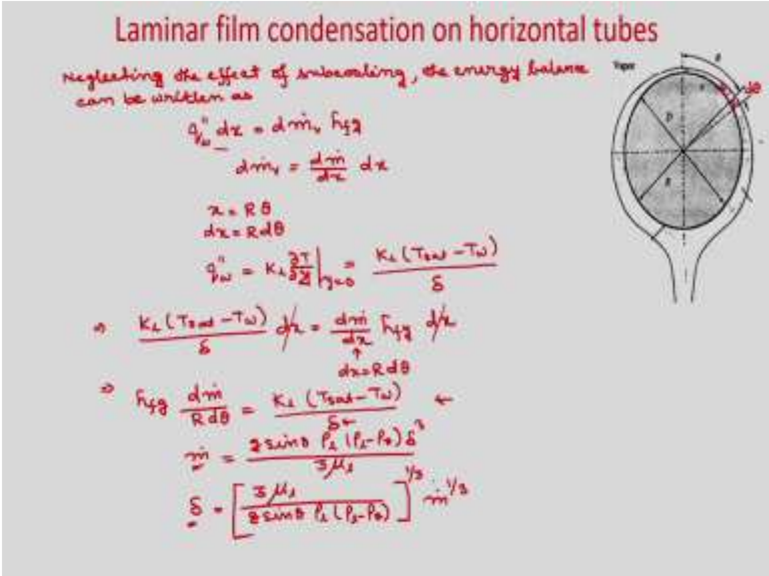
So, if you put u expression here finally, we will get $\frac{g \sin\theta \rho_l \left(\rho_l - \rho_v\right)\delta^3}{3\mu_l}$. And if you consider the energy equation, so you will get only $\frac{\partial^2 u}{\partial y^2}$ = 0and the temperature profile will remain same as the case of vertical plate. So, temperature profile will get, $T_{sat} - T\left(x, y\right) = \left(1 - \frac{y}{\delta}\right) T_{sat} - T_w$.

So, now, we have calculated the velocity mass flow rate and the temperature profile, now we need to calculate the film thickness or we are interested to calculate the heat transfer coefficient. So, in this particular case, you can see that although your mass flow rate is 0 at x = 0; that means, at this particular case, so, your mass flow rate is 0; condensation is taking place ok. But at this place your δ is not 0 . The film there is a finite film thickness at θ = 0.

So obviously, we do not know that but we know that their mass flow rate will be 0 because it is a symmetric line. Because you can see in this particular case at the vertical line, it is symmetry because whatever way it will flow here, this way also it will flow. So, we know that due to symmetry at this line, your mass flow rate will be 0.

So, in earlier case, we evaluated the film thickness; but here, it is difficult to find the film thickness because you do not know what is the thickness of δ at x = 0. So, hence, we will use a separate route, where we will find what is the mass flow rate.

(Refer Slide Time: 08:43)



In this particular case for simplicity, we will neglect the sub cooling effect. So, we will use just latent heat instead of modified latent heat. So, neglecting the effect of sub cooling, the energy balance can be written as you can see the in the left hand side, it will be $q_w^{"}dx$.

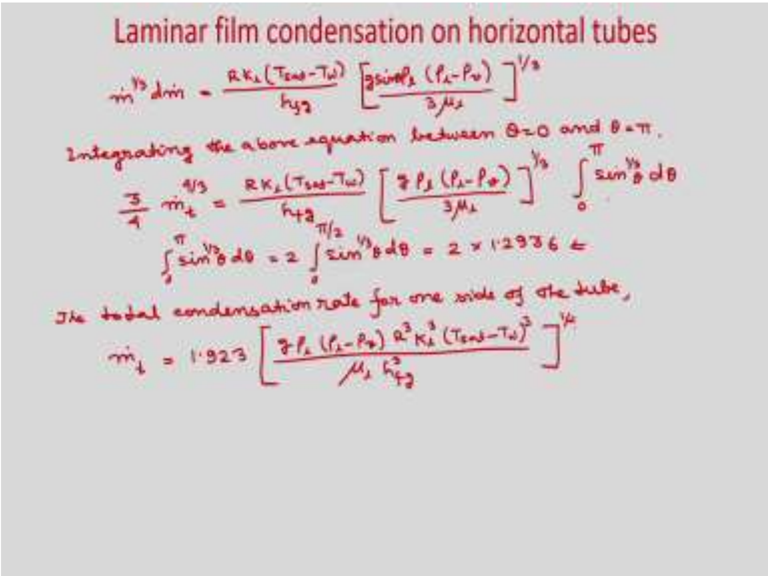
So, what is area? So, if you consider this elemental volume, so this is your dθ and in this direction, this is your x and this is your δ; this is your δ. So, $q_w^{"}dx = d\dot{m}_v h_{fg}$. So, the sensible heat part, we are not adding here; we are neglecting it and from the mass balance you can show that, $d\dot{m}_v = \frac{d\dot{m}}{dx}dx$.

Now, you see that in this case, so x =R θ. So, that means, dx = R dθ. So, you will get $q_w^{"} = K_l \left.\frac{\partial T}{\partial y}\right|_{y=0}$. If you evaluate it, you will get as, $\frac{K_l\left(T_{sat} - T_w\right)}{\delta}$.

So, you can see this we will put $\frac{K_l\left(T_{sat} - T_w\right)}{\delta}dx = \frac{d\dot{m}}{dx}h_{fg}dx$. So, these dx, this dx you cancel. So, you will get. So, this d x now you write, dx = R dθ and you take in the left hand side. So, you will get as $h_{fg}\frac{d\dot{m}}{Rd\theta} = \frac{K_l\left(T_{sat} - T_w\right)}{\delta}$. So, now we have already know that

$\dot{m} = \frac{g \sin\theta \rho_l \left(\rho_l - \rho_v\right) \delta^3}{3\mu_l}$. So, from here, you can find what is δ. So, this we have to replace in terms of $\dot{m}$ so that we can integrate it. So, $\delta = \left[\frac{3\mu_l}{g \sin\theta \rho_l \left(\rho_l - \rho_v\right)} \right]^{1/3} \dot{m}^{1/3}$. So, now, you put this expression of δ in this equation. So, what you we will get? Here $\dot{m}^{1/3}$ is there.

(Refer Slide Time: 12:45)



So, if you rearrange it, you are going to get $\dot{m}^{1/3} \, d\dot{m} = \frac{RK_l \left(T_{sat} - T_w\right)}{h_{fg}} \left[\frac{g \sin\theta \rho_l \left(\rho_l - \rho_v\right)}{3\mu_l} \right]^{1/3}$.

So, you can see that now it will be easier to integrate because you know that at x = 0, $\dot{m}$ is 0 because it is a symmetry line and at x = l that means, at the θ = π, it will be total mass flow rate.

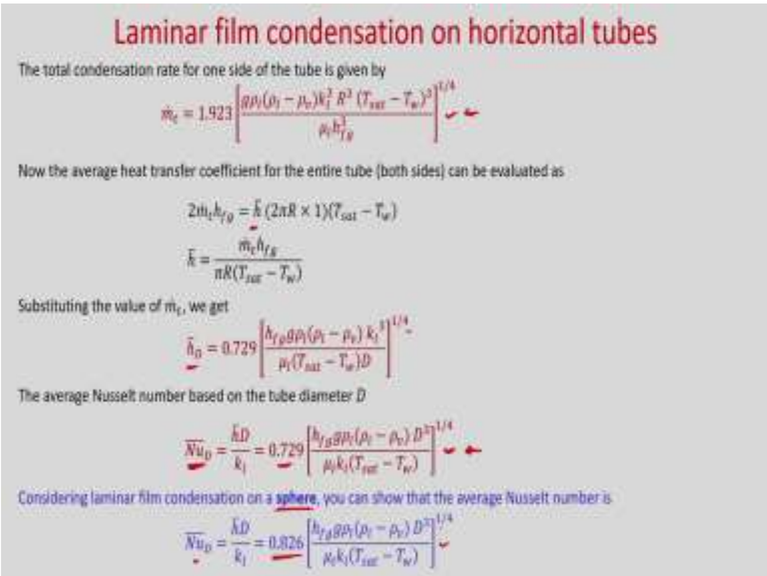
So, $\dot{m}$ total. Because all will be integrated from θ = π, so you will get the total mass flow rate at θ = π. It is for the one side. So, if you consider both sides, then it will be just 2 $\dot{m}$. So, here $\dot{m}$ is the mass flow rate per unit width in one-half; but you can see, it is due to symmetry in the other side also you will get $\dot{m}$. So, total mass flow rate at the bottom you will get 2$\dot{m}$ total. So, integrating the above equation between θ 0 and θ = π. So, what you will get?

So, now at $\theta = 0$ we are telling that at $\dot{m} = 0$ and $\theta = \pi$ it is $\dot{m}$ total. So, this $\dot{m}$ total will just represent as $\dot{m}_t$. So, you can if you integrate, you will get $\frac{3}{4}\dot{m}_t^{4/3} = \frac{RK_l\left(T_{sat}-T_w\right)}{h_{fg}}\left[\frac{g\rho_l\left(\rho_l-\rho_v\right)}{3\mu_l}\right]^{1/3}\int_0^{\pi}\sin^{1/3}\theta d\theta$. So, now you have to integrate this. So, this if you integrate, you will get $\int_0^{\pi}\sin^{1/3}\theta d\theta = 2\int_0^{\pi/2}\sin^{1/3}\theta d\theta$.

If you integrate it, you will get as 1.2936. So, now if you put these value here and rearrange it, so you will get the total condensation rate for one side of the tube, you will get as, $\dot{m}_t = 1.923\left[\frac{g\rho_l\left(\rho_l-\rho_v\right)R^3K_l^3\left(T_{sat}-T_w\right)^3}{\mu_l h_{fg}^3}\right]^{1/4}$.

So, now, we have calculated the total mass flow rate appearing at $\theta = \pi$ means at the bottom and in one side, we have written $\dot{m}$. So, if you consider both sides, then it will be 2 $\dot{m}$. So, now, you use the energy balance . So, if you use the conservation of energy, then you will be able to calculate the heat transfer coefficient.

(Refer Slide Time: 17:31)



So, you can see that this already we have derived. Now, the average heat transfer coefficient for the entire tube, both sides can be evaluated as so 2 $\dot{m}_t\, h_{fg}$. So, that is the total heat transfer rate is $2\dot{m}_t\, h_{fg} = \bar{h}\left(2\pi R\times 1\right)\left(T_{sat} - T_w\right)$.

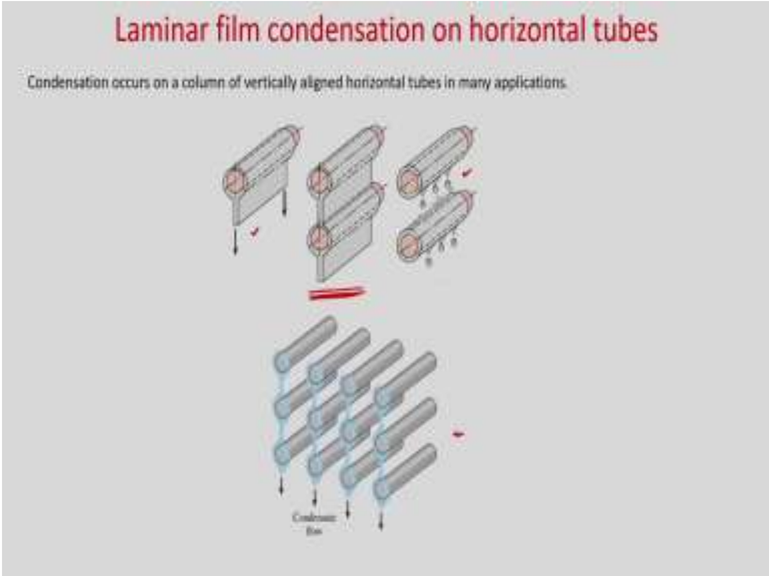
So, from here your average heat transfer coefficient, you can calculate as $\bar{h} = \dfrac{\dot{m}_t\, h_{fg}}{\pi R\left(T_{sat} - T_w\right)}$.

So, now, you see here $\dot{m}_t$ expression is here. So, you substituted it here and rearrange. So, you if you rearrange and based on diameter if you define the average heat transfer coefficient, then you will get, $\bar{h}_D = 0.729\left[\dfrac{h_{fg}\, g\rho_l\left(\rho_l - \rho_v\right)D^3}{\mu_l K_l\left(T_{sat} - T_w\right)}\right]^{1/4}$.

So, now we can calculate the average Nusselt number. So, $\overline{Nu}_D = \dfrac{\bar{h}D}{K_l}$. So, now, in this particular case, we have defined the average Nusselt number based on diameter. So, if you put it and rearrange it, so, you will get this expression. You if you consider, so this is the average Nusselt number for a single horizontal tube; based on the diameter of the tube.

So, if you compare this equation with the expression for the vertical tube, you can see D is replaced with l ; l^3 and this coefficient will be different, that is the just change. Now, if you consider laminar film condensation on a sphere and if you do the similar analysis, you can get the average Nusselt number, similar expression only this constant is 0.826; 0.826 and it is based on the diameter.

(Refer Slide Time: 19:52)

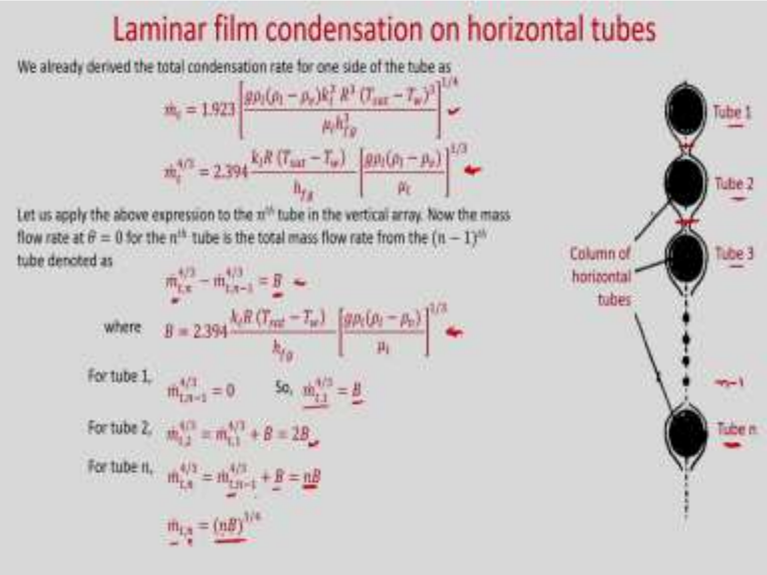


Now, this average Nusselt number, we have calculated for a single tube; but in industry you will get a different columns of tubes. So, in the power plant these are used. So, if you see for this single tube, we have already considered and this is the film falling vertically downward direction.

If you have 2 horizontal tubes, just one above the other; then, these film whatever is coming out from the first tube, it will go to the second tube and it will just fall. Sometimes, it will there will be no continuous film. So, you might get this type of phenomena. But now, if you consider that this columns of horizontal tubes and we will assume that you have a continuous film thickness like these.

So, for this condition, where you have a continuous film and it is falling on over the other; so, you can see here. So, it is falling to the next and next and so on. So, for these; so, for these we need to calculate what will be the heat transfer coefficient and Nusselt number.

(Refer Slide Time: 21:14)



So, you can see if you consider total n tubes in a single column; so, this is the first tube, then second tube, third tube, then this is the n^{th} tube. So, these are column of horizontal tubes. Now, you can see from here when the condensation is taking place; obviously, due to symmetry, mass flow rate will be 0. But when it will come and fall on the second, there is a some mass flow is coming from the tube 1.

So, similarly tube 3, it will get whatever condensation happened in tube 1 and tube 2. Similarly, when you consider tube n, so, some mass flow rate will come whatever is collected from the previous tubes. So, these analysis, now do in this way. We have already calculated this total condensation rate for a single tube and if you write in terms of $\dot{m}_t^{4/3} = 2.394\frac{K_l R\left(T_{sat}-T_w\right)}{h_{fg}}\left[\frac{g\rho_l\left(\rho_l-\rho_v\right)}{3\mu_l}\right]^{1/3}$ you will get this expression.

Now, let us apply the above expression to the n^{th} tube in the vertical array. Now, the mass flow rate at θ = 0 for the nth tube is the total mass flow rate from the $(n-1)^{th}$ tube denoted as $\dot{m}_t$; $\dot{m}_{t,n}^{4/3} - \dot{m}_{t,n-1}^{4/3} = B$; where B is represented with this expression. So, this is your B.

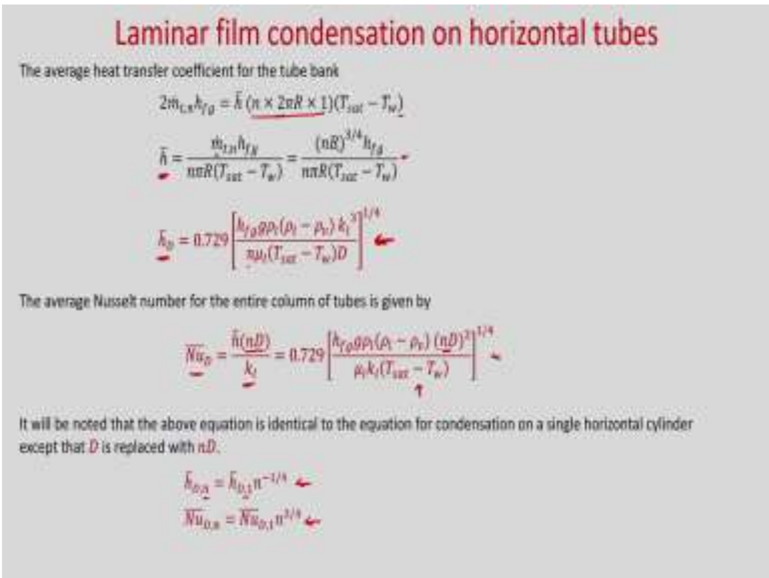
So, what we are telling that whatever total mass flow rate is coming from the previous tube, so that will be just at θ = 0, it will be it will fall to the n^{th} tube. So, if you consider this is the n^{th} tube, then in the nth tube whatever mass flow rate is there, so that will be just minus the previous n -1 tube whatever mass flow rate if you would deduct. So, you are going to get a single tube whatever condensation is taking place. So, that is the total mass flow rate and that is your B .

So, $B = 2.394\dfrac{K_l R\left(T_{sat} - T_w\right)}{h_{fg}}\left[\dfrac{g\rho_l\left(\rho_l - \rho_v\right)}{3\mu_l}\right]^{1/3}$. Now, you see when you are considering tube 1, this mass flow rate is 0. So, when you are coming here, so you can see that $\dot{m}_{t,1}$ from the tube 1 whatever you are getting, $\dot{m}_{t,1}^{4/3} = B$, from this expression because n -1 is 0. Now, if you consider tube 2; so, for tube 2, $\dot{m}_{t,2}^{4/3}$.

So, at this place whatever mass flow is coming, it will be just B plus whatever mass flow rate came here. So, $\dot{m}_{t,2}^{4/3} = \dot{m}_{t,1}^{4/3} + B = 2B$. And similar way if you go and if you go to tube n, then you can write $\dot{m}_{t,\mathrm{n}}^{4/3} = \dot{m}_{t,\mathrm{n}-1}^{4/3} + B = nB$ and this will be nB.

So, total will be nB. So, you can see that $\dot{m}_{t,n} = \left(nB\right)^{3/4}$. So, n is the n$^{\text{th}}$ tube. So, you can write $\dot{m}_{t,n} = \left(nB\right)^{3/4}$ and where, n is the number of tube and B is the this expression. Next, let us do the energy balance.

(Refer Slide Time: 25:01)



So, you can see the average heat transfer coefficient for the tube bank, we can calculate 2 into $\dot{m}_{t,n}$. So, at the nth tube whatever this mass flow rate is collected, so that is in both sides. So, $2\dot{m}_{t,n}h_{fg} = \bar{h}\left(n\times 2\pi R\times 1\right)\left(T_{sat}-T_w\right)$. So, this is the Newton's law of cooling. So, now, what is your total area? So, now, you have n number of tubes in a single column.

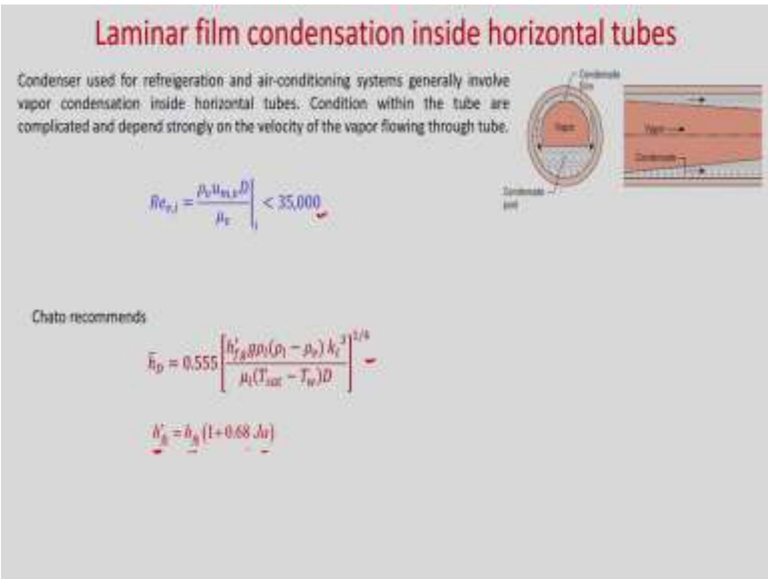
So, for a single tube, what is the area? This is $\left(2\pi R\times 1\right)$ per unit width. So, $\left(2\pi R\times 1\right)$ that is your for area for single tube. Now, in a single column, you have total number of tubes n; so, $\left(n\times 2\pi R\right)$. So, that we have written as the heat transfer area. So, from here your average heat transfer coefficient, just rearrange, you will get $\bar{h} = \dfrac{\dot{m}_{t,n}h_{fg}}{n\pi R\left(T_{sat}-T_w\right)}$ and $\dot{m}_{t,n} = \left(nB\right)^{3/4}$. So, this is the expression and B expression if you know, so that B expression you just put it and you get the average heat transfer coefficient based on the diameter as, $\bar{h}_D = 0.729\left[\dfrac{h_{fg}g\rho_l\left(\rho_l-\rho_v\right)K_l^3}{n\mu_l\left(T_{sat}-T_w\right)D}\right]^{1/4}$.

So, now let us calculate the average Nusselt number. So, here average Nusselt number based on the diameter we will consider; but how many numbers of tubes are there? So, there are n number of tubes. So, it will be based on nD. So, you can write $\overline{Nu}_D = \dfrac{\bar{h}\left(nD\right)}{K_l}$. So, now, this is your $\bar{h}D$. So, if you put this expression, then n if you take inside; so, it will be n^4 and in the denominator 1 n is there. So, it will be n^3. So, you can put it in D.

So, it will be $\overline{Nu}_D = 0.729\left[\dfrac{h_{fg}g\rho_l\left(\rho_l-\rho_v\right)\left(nD\right)^3}{\mu_l K_l\left(T_{sat}-T_w\right)}\right]^{1/4}$. So, you can see this is the expression for average Nusselt number for the entire column of tubes and the expression is similar to the single tube except D is replaced with n D and you can show that, $\bar{h}_{D,n} = \bar{h}_{D,1}n^{-1/4}$. Because this is your for the nth tube and if you put for a single tube whatever we have calculated, so that is the first tube. So, you can see this relation.

And similarly, for the Nusselt number also, you can write $\overline{Nu}_{D,n} = \overline{Nu}_{D,1}n^{3/4}$. So, till now you considered the condensation outside the tube. Now, if the steam is condensing inside the tube when it is flowing through the tube, then your condensation will take place inside the tube and the condensate will fall due to gravity in the downward direction.

(Refer Slide Time: 28:37)

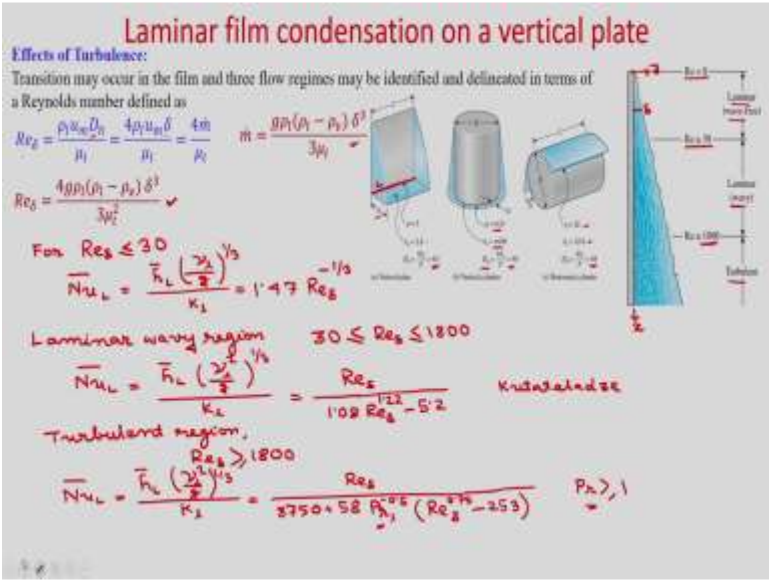


So, if you see condenser used for refrigerator and air conditioning systems generally involve vapor condensation inside horizontal tubes; condition within the tube are complicated and depend strongly on the velocity of the vapor flowing through the tube. So, you can see this is your tube inside this condensation will take place from the surface but due to gravity condensate will come and come down.

So, the film thickness obviously will increase in the downward direction and this is the condensate pool and the Reynolds number of the vapor at the inlet, $\mathrm{Re}_{v,i} = \left.\frac{\rho_v u_{m,v} D}{\mu_v}\right|_i$; i is for inlet, v for vapor. So, Reynolds number of the vapor at the inlet, you can calculate as density of the vapor into mean velocity of the vapor, the diameter of the tube divided by the viscosity of the vapor.

So, $\mathrm{Re}_{v,i} = \left.\frac{\rho_v u_{m,v} D}{\mu_v}\right|_i <$ 35,000, then scientist Chato recommends that, $\bar{h}_D = 0.555\left[\frac{h'_{fg} g \rho_l (\rho_l - \rho_v) K_l^3}{\mu_l (T_{sat} - T_w) D}\right]^{1/4}$ and here obviously, $h'_{fg} = h_{fg}(1 + 0.68 Ja)$. So, all these study, we assume that it is a laminar flow; but it may become turbulent flow, if your length of the vertical plate is more.

(Refer Slide Time: 30:19)



So, if you consider again the vertical plate and you see how the condensation is taking place at x = 0, you have film thickness 0 and gradually in the x direction, it will fall and perpendicular direction is y and δ is the film thickness. So, film thickness gradually, it will increase along the downward direction.

It is seen that the up to Reynolds number 30. So, how the Reynolds number is defined? Reynolds number is defined based on the film thickness $\mathrm{Re}_\delta = \frac{\rho_l u_{m,v} D_h}{\mu_v}$. So, what is hydraulic diameter here? So, if it is a vertical plate, you can see here. So, the perimeter, perimeter you can see that heat transfer a where it is taking place. So, that is L .

So, $L \times 1$ and what is the flow cross sectional area? So, if this is the L. So, you can see the L is the width of the vertical plate and the perimeter if you see, so it will be Lp; it will be l at any cross section and the flow area A_c will be $L \times \delta$ because at a particular x, if you see that thickness is δ; then, cross-sectional flow cross sectional area will be $L \times \delta$.

So, $D_h = \frac{4A_c}{p}$; p is the perimeter of the heat transfer. So, you can see at this particular position, if we consider so this is your L and this is your perimeter. In this particular case, this is your perimeter and at this place your flow cross sectional area is $L \times \delta$.

So, the heat transfer, where it is taking place, so that is your perimeter and these perimeter is in this particular case at any location, you can see that is L. So, $D_h = \frac{4A_c}{p}$. So, if you see, here it will be 4 δ. If you consider condensation on a vertical cylinder, so p = π D. At particular location if you see, it will be πD and flow cross sectional area is $\pi D \times \delta$.

So, D_h will be just 4 δ and if it is a horizontal tube, in this particular case, p = 2L. So, if it is your L, in both side if you consider, it will be 2L and flow cross sectional area will be $L \times \delta$ n; both side, it is $2 L \times \delta$. So, hydraulic diameter will be 4 δ.

So, Reynolds number, we are defining now this hydraulic diameter we are putting 4 δ and if you see that your in place of mass flow rate, if you write; so, it will be $\frac{4\rho_l u_m \delta}{\mu_l}$. So, that will be your $\frac{4\dot{m}}{\mu_l}$. So, this m dot we know, this expression, so if you put it here, you are going to get, $\mathrm{Re}_\delta = \frac{4g\rho_l(\rho_l - \rho_v)\delta^3}{3\mu_l^2}$. So, Reynolds number is defined in this way.

Now, if you see at x = 0, at this point what is the Re_δ is 0? So obviously, Reynolds number will be 0. So, Reynolds number 0. So, it is seen that up to Reynolds number 30, you will get a laminar and it is wave free. So, you can see this interface, it is almost straight line. So, there is no waviness on the interface.

If you see from $30 \le \mathrm{Re}_\delta \le 1800$, you will get a laminar flow; but on the interface, you will get some waviness. So, this is your laminar wavy region and if $\mathrm{Re}_\delta > 1800$, then it will be a turbulent flow. So, inside also you will see that gravity motion is taking place.

So, this is some. So, you will get turbulent zone. So, based on this Reynolds number, if you find the Nusselt number then just we will present what will be the Nusselt number in different regime. So, Nusselt number already we have calculated based on the length average. So, you can actually write in this way after rearranging $\overline{Nu}_L = \dfrac{\bar{h}_L\left(\dfrac{\nu_l}{g}\right)^{1/3}}{K_l}$.

So, this expression we have not derived, but in some book, you will get this expression and this you can write as four point will get $1.47\,\mathrm{Re}_\delta^{-1/3}$ and this is valid for laminar zone; wave free zone. So, for $\mathrm{Re}_\delta \leq 30$.

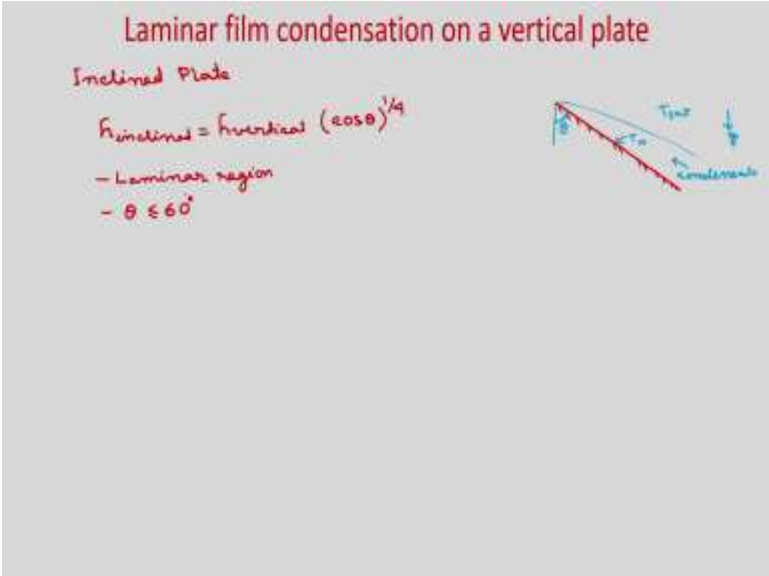
So, if your this temperature difference is unknown, $T_{sat} - T_w$ then, actually you can represent the Nusselt number in this way. So, you can see here temperature difference is not coming ok. So, that derivation you can do as a homework and you can show that wavy free laminar zone where $\mathrm{Re}_\delta < 30$, you can use this average Nusselt number expression.

Now, for laminar wavy zone; laminar wavy region in the range of $30 \leq \mathrm{Re}_\delta \leq 1800$. So, this you can use as, $\overline{Nu}_L = \dfrac{\bar{h}_L\left(\dfrac{\nu_l^2}{g}\right)^{1/3}}{K_l} = \dfrac{\mathrm{Re}_\delta}{1.08\,\mathrm{Re}_\delta^{1.22} - 5.2}$. So, this is the correlation actually.

So, it is proposed by the scientist Kutateladze and in turbulent zone, you can get for Re_δ >1800. So, you will get, $\overline{Nu}_L = \dfrac{\bar{h}_L\left(\dfrac{\nu_l^2}{g}\right)^{1/3}}{K_l} = \dfrac{\mathrm{Re}_\delta}{8750 + 58\,\mathrm{Pr}_l^{-0.5}\left(\mathrm{Re}_\delta^{0.75} - 253\right)}$.

So, in this expression for turbulent region, you can see in the expression Prandtl number is coming and this is valid for $\mathrm{Pr} \geq 1$.

(Refer Slide Time: 39:18)



Now, if you consider inclined plates, so you have a plate. So, your condensation will take place. So, you will get this the condensate. And if θ is measured from here, so, this is your T_w, this is your T_{sat} and this is your inclined plate, this is your condensate. So, you can see in this case only the gravity will be different. Because gravity is acting in the vertical downward direction, but plate is inclined with angle θ.

So, you can put $h_{inclined} = h_{vertical}\left(\cos\theta\right)^{1/4}$ and for it is valid for laminar zone, laminar region and $\theta \le 60^0$. So, this gives satisfactory results specially, for $\theta \le 60^0$.

So, today, we considered the laminar film condensation on a horizontal tube. So, we did the same analysis as we did for the vertical plate, except the gravity in the x direction we considered as g sinθ because we consider the x in the tangential direction on the surface of the tube and y in the perpendicular direction.

And from there, we calculate the velocity distribution, mass flow rate and temperature distribution. Then, we used the conservation of mass and conservation of energy. While using the conservation of energy, we evaluated the total mass flow rate at the bottom of the tube. Because in this particular case δ is the film thickness at x = 0 is unknown; but due to symmetry at x = 0, $\dot{m} = 0$; the mass flow rate $\dot{m}$ will be 0 at x = 0.

Hence, we evaluated we the $\dot{m}_t$ at the bottom of the tube and from there, we calculated the average heat transfer coefficient and average Nusselt number. Then, we considered column of vertical column of horizontal tubes. So, we considered n tubes and for that case also, we evaluated the average heat transfer coefficient and average Nusselt number.

Then, we considered the Reynolds number based on the film thickness δ and depending on different regime whether it is laminar region wave free or laminar region with waviness and turbulent region, we wrote the correlation for Nusselt number.

And finally, if we considered the inclined plate and in this particular case, you know that your gravity in the flow direction will be g sinθ and hence, h which is your locally transferred coefficient, you can write $h_{inclined} = h_{vertical} (\cos\theta)^{1/4}$ and generally, it gives satisfactory results for $\theta \leq 60^0$ and for laminar region.

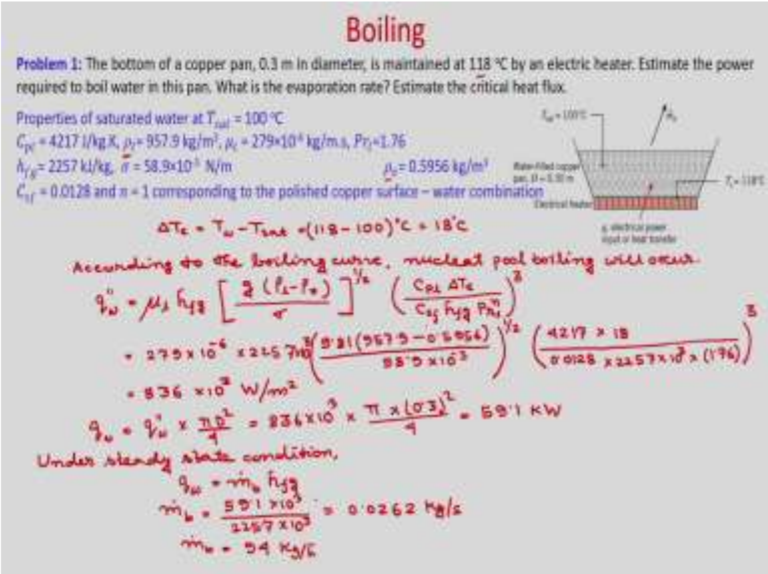
Thank you.

Fundamentals of Convective Heat Transfer
Prof. Amaresh Dalal
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Module – 12
Boiling and Condensation
Lecture – 43
Solution of example problems

Hello everyone. So, in last few classes, we have learnt Boiling and Condensation. Today we will solve few example problems on boiling and condensation.

(Refer Slide Time: 00:42)



So, let us take the first problem. The bottom of a copper pan, 0.3 meter in diameter, is maintained at 118 ^{0}C by an electric heater. Estimate the power required to boil water in this pan. What is the evaporation rate? Estimate the critical heat flux. So, you can see, this is the boiling phenomena and we need to find the evaporation rate and the critical heat flux, and also the power required to boil water in this pan.

So, you can see this is the copper pan, which is having this bottom diameter as 0.3 m and properties of saturated water at T_{sat} = 100 ^{0}C these are given; C_{pl}, ρ_l, ν_l, Pr_l for liquid. And from steam table, you will get h_{fg} as well as the vapour density ρ_v.

σ is the surface tension; you can see in this particular case, the $\rho_l \sim 1000$ and $\rho_v \sim 1$. So, you can see that $\rho_l >> \rho_v$. In this case, we also need the value of C_{sf} and the n corresponding to the polished copper surface and water combination and that is given.

Now, let us first see what is the excess temperature. ΔT is the excess temperature is just T_w - T_{sat}. So, in this particular case you can see, T_w is 118 ^{0}C and T_{sat} is 100 ^{0}C. So, 118 - 100 = 18^{0}C.

So, if you see the (Refer Time: 02:52) curve of boiling; then you can see for this excess temperature of 18 ^{0}C, it will fall in nucleate boiling. So, we need to use the expression of heat flux from this nucleate boiling region. So, according to the boiling curve, nucleate pool boiling will occur.

So, $q_w^{"} = \mu_l h_{fg} \left[\frac{g(\rho_l - \rho_v)}{\sigma} \right]^{1/2} \left(\frac{C_{pl} \Delta T_e}{C_{sf} h_{fg} \Pr_l^n} \right)^3$. So, for this particular case, you can see C_{sf} and the n value are given.

So, you, if you put all the values here so you see $279 \times 10^{-6} \times 2257 \times 10^3 \left(\frac{9.81(957.9 - 0.5956)}{58.9 \times 10^{-3}} \right)^{1/2} \left(\frac{4217 \times 18}{0.0128 \times 2257 \times 10^3 \times (1.76)} \right)^3$. So, you can write $\Pr_l$, because it is for liquid. So, to calculate this, you will get approximately $836 \times 10^3 W/m^2$. So, this is the heat flux on the wall.

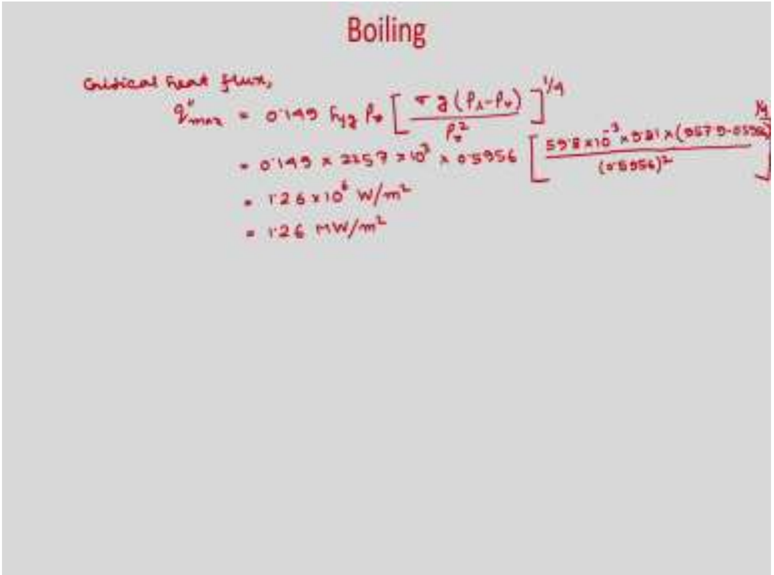
So, now we need to calculate the power required. So, power required will be just your heat flux into the heat transfer area. So, what is the heat transfer area in this particular case? It is the bottom of the pan. So, that is $\frac{\pi D^2}{4}$. So, $q_w = q_w^{"} \times \frac{\pi D^2}{4}$.

So, this is $836 \times 10^3 \times \frac{\pi \times (0.3)^2}{4}$. So, if you calculate it; it will come around 59.1 kW.

Next we need to calculate the evaporation rate and we will use the steady state condition. So, under steady state condition, the energy balance you can see that, q_w whatever is heat transfer rate will be just your mass flow rate, which is your evaporation rate into h_{fg}.

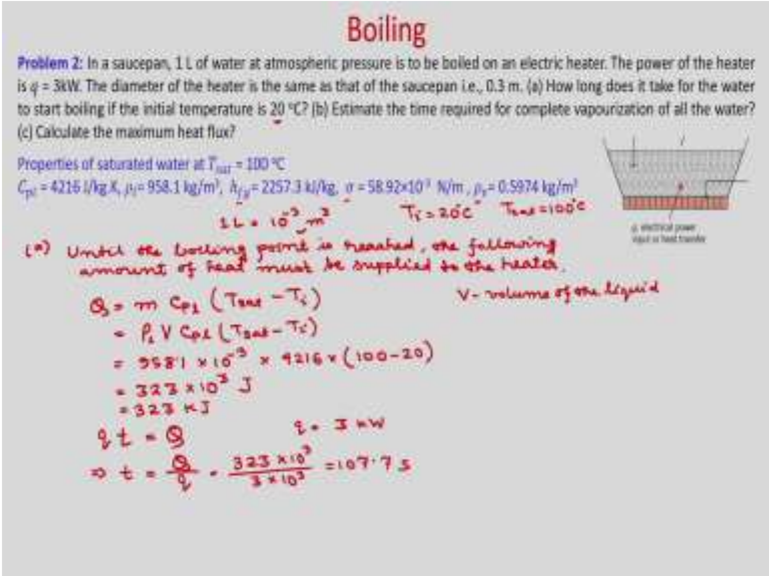
So, this is the latent heat. So, $q_w = \dot{m}_b h_{fg}$, that is your $\dot{m}_b = \frac{59.1 \times 10^3}{2257 \times 10^3}$, So, if you calculate this, you will get it as 0.0262 kg/s. If you calculate in kg /hr; then $\dot{m}_b$, you will get as 94 kg/hr. Now, we need to calculate the critical heat flux. So, you know the expression of it.

(Refer Slide Time: 08:41)



So, critical heat flux $q''_{max} = 0.149 h_{fg} \rho_v \left[\frac{\sigma g(\rho_l - \rho_v)}{\rho_v^2}\right]^{1/4}$. So, you put all the values. So, you will get, $0.149 \times 2257 \times 10^3 \times 0.5956 \left[\frac{59.8 \times 10^{-3} \times 9.81 \times (957.9 - 0.5956)}{(0.5956)^2}\right]^{1/4}$. So, if you calculate, you will get as $1.26 \times 10^6 W/m^2$. And this is 1.26 MW/m^2.

(Refer Slide Time: 10:28)



Now, let us take another problem on boiling. In a saucepan, 1 litre of water at atmospheric pressure is to be boiled on an electric heater. The power of the heater is q = 3 kW. The diameter of the heater is the same as that of the saucepan and that is 0.3 m. How long does it take for the water to start boiling if the initial temperature is 20 ^{0}C?

So, you can see in this particular case, the water is at 20 ^{0}C, so that is your initial temperature. So, to boil this water, you have to increase the temperature from 20 ^{0}C to 100 ^{0}C; then only the boiling phenomena will take place.

Next you have to estimate the time required for complete vaporization of all the water? Calculate the maximum heat flux? So, in this particular case, first you have to find what is the power required to increase the temperature of the water from 20 ^{0}C to 100 ^{0}C; 100 ^{0}C is the saturation temperature, then the boiling will start. Once boiling starts, then how much time it will take to evaporate all water, complete water? And we have to also find the critical heat flux.

The properties are given at T_{sat} = 100 ^{0}C; these are the properties and from the steam table, you can find h_{fg} and ρ_v and the surface tension σ is also given. And 1 litre of water; that means it is 10^{-3} m^3. So, this is the volume of the water.

So, first calculate what is the total heat supplied to increase the temperature from 20 ^{0}C to 100 ^{0}C. So, until the boiling point is reached, the following amount of heat must be supplied to the heater. So, that is your Q. So, it will be $Q = mC_{pl}\left(T_{sat} - T_i\right)$.

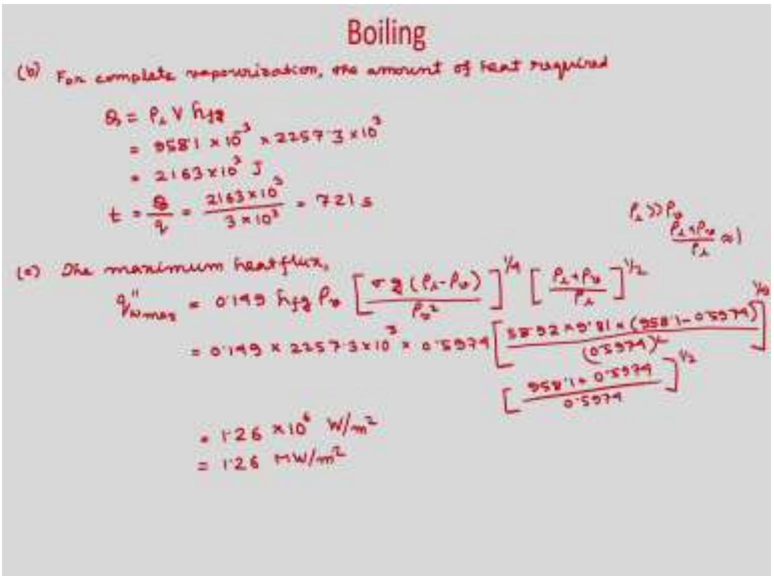
So, T_i let us say, where T_i in this case it is 20 ^{0}C. And T_{sat} obviously it is 100 ^{0}C. So, what is m? Total mass of the liquid. So it will be just ρ_l into volume of the liquid; so, $\rho_l \times$ V, where Vis the volume of the liquid.

So, in this particular case it is, $958.1 \times 10^{-3} \times 4216 \times (100 - 20)$. So, if you calculate it, you will get around 323×10^3 J.

So, you can write 323 kJ. So, now, you need to calculate the time; because how long does it take? So, you know that, heat transfer rate is given that q = 3 kW, from the bottom, so from the heater. So, you can write q t = Q. So, you can find, $t = \dfrac{Q}{q}$.

So, $\dfrac{323 \times 10^3}{3 \times 10^3}$. So, you will get around 107.7 s. So, now, the water has reached to temperature 100 ^{0}C. Now, we have to find the time for complete evaporation of the liquid. So, now as it has reached to the saturation temperature 100 ^{0}C, it will evaporate.

(Refer Slide Time: 15:36)



So, now, (b) for complete vaporization the amount of heat required. So, that is $Q = \rho_l V h_{fg}$. That is, $958.1 \times 10^{-3} \times 2257.3 \times 10^3$.

So, this if you calculate, you will get around 2163×10^3 J. So, time you can calculate. Similarly, $t = \frac{Q}{q}$. So, it will be $\frac{2163 \times 10^3}{3 \times 10^3}$. So, you will get around 721 s.

And the maximum heat flux will use the expression what we used for earlier problem. So, the maximum heat flux; $q''_{\max} = 0.149 h_{fg} \rho_v \left[\frac{\sigma g \left(\rho_l - \rho_v \right)}{\rho_v^2} \right]^{1/4} \left[\frac{\rho_l + \rho_v}{\rho_l} \right]^{1/2}$.

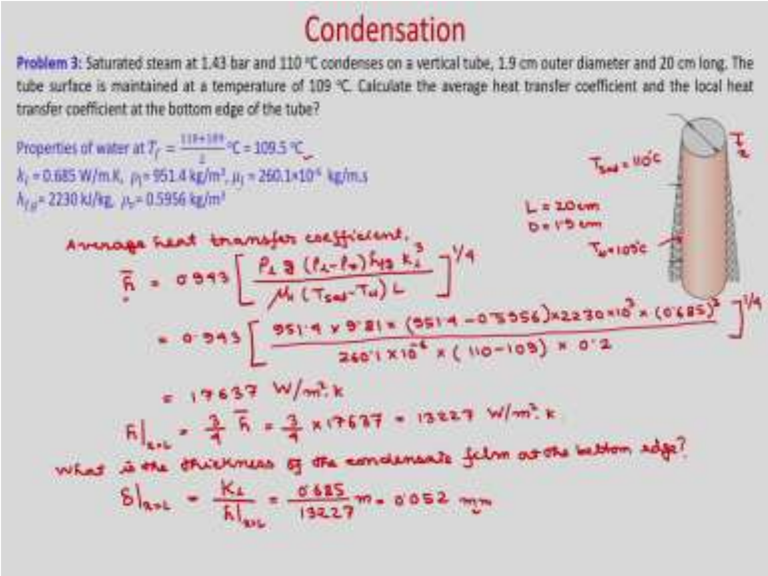
So, if you see in the earlier case, we have not taken $\left[\frac{\rho_l + \rho_v}{\rho_l} \right]^{1/2}$; because $\rho_l >> \rho_v$. So, obviously you can see that, $\frac{\rho_l + \rho_v}{\rho_l} \approx 1$. So, for that reason we did not consider.

But if you take also, there will be not much difference in results. So, we put all the values here. So, you will get,

$$0.149 \times 2257.3 \times 10^3 \times 0.5974 \left[\frac{58.92 \times 9.81 \times \left(958.1 - 0.5974 \right)}{\left(0.5974 \right)^2} \right]^{1/4} \left[\frac{958.1 + 0.5974}{0.5974} \right]^{1/2}.$$

You can calculate without these terms and you can see what is the difference you are getting in the result. So, if you calculate this, you will get around 1.26 $\times 10^6$ W/m^2 that is you can write 1.26 M W/m^2.

(Refer Slide Time: 19:36)



So, now let us take three problems on condensation. First we will discuss this problem; saturated steam at 1.43 bar and 110 ^{0}C condenses on a vertical tube, 1.9 cm outer diameter and 20 cm long. The tube surface is maintained at a temperature of 109 ^{0}C.

Calculate the average heat transfer coefficient and local heat transfer coefficient at the bottom edge of the tube? So, you can see the surface temperature. So, this is the surface temperature T_w is your 109 ^{0}C; whereas the saturation temperature of the steam is 110 ^{0}C.

So, this vapour will condense on the surface of the vertical tube, but you can see that it is a vertical tube; so you can use the same correlation of vertical plate. The expression whatever we derived, it will be same for this particular case; only the total volume of this condensate you can calculate in different way.

So, properties of water at film temperature, which is your mean film temperature as 109.5 ^{0}C; the thermal conductivity of the liquid, density of the liquid, dynamic viscosity of the liquid, are given. And from the steam table, you can find what is h_{fg} and the density of the vapour.

So, you know the average heat transfer coefficient. What is the expression? This is your $\bar{h} = 0.943\left[\frac{\rho_l g(\rho_l-\rho_v)h_{fg}K_l^3}{\mu_l(T_{sat}-T_w)L}\right]^{1/4}$. So, we put all the values. So, $0.943\left[\frac{951.4\times9.81\times(951.4-0.5956)\times2230\times10^3\times(0.685)^3}{260.1\times10^{-6}\times(110-109)\times0.2}\right]^{1/4}$. So, if you calculate, you will get this as 17637 W/m^{2}K.

So, you can see; when phase change occurs ok, the heat transfer coefficient is very high. So, you can see it is of the order of 18000 right, whereas when you see the heat transfer coefficient, average heat transfer coefficient for a fully developed case in side circular pipe, it is for constant heat flux 4.36.

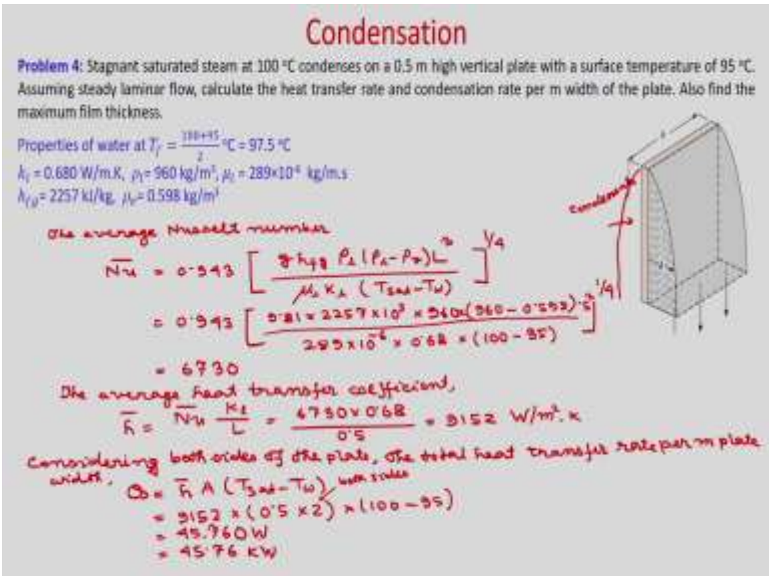
So, it is you can see for this case; when you consider the phase change, the heat transfer coefficient is much much higher than the single phase heat transfer. So, as you know the average heat transfer coefficient; now you will be able to calculate what is the heat transfer, local heat transfer coefficient at the bottom edge. So, at the bottom edge means, x we have measured from here.

So, obviously at x = l, you can find what is the local heat transfer coefficient and that you can calculate h, as $h|_{x=L} = \frac{3}{4}\bar{h}$. This relation already we have derived, right.

So, you can right $\frac{3}{4} \times 17637$ = 13227 W/m^{2}K. Now, if you are asked to find the thickness of this condensate at the bottom edge; then how will find it? So, if I ask that what is the thickness of the condensate film at the bottom edge.

So, $\delta|_{x=L} = \frac{K_l}{h|_{x=L}}$. So, that is $\frac{0.685}{13227}$. So, if you calculate, you will get around 0.052 mm.

(Refer Slide Time: 26:29)



So, it is very small thickness. So, next we will discuss about this problem. Stagnant saturated steam at 100 ^{0}C condenses on a 0.5 m high vertical plate with a surface temperature of 95 ^{0}C.

Assuming steady laminar flow, calculate the heat transfer rate and condensation rate per meter width of the plate? Also find the maximum film thickness?

So, properties of water at mean film temperature 97.5 ^{0}C; k_l, ρ_l, μ_l, h_{fg} and ρ_v. So, if you now calculate the average Nusselt number. So, $\overline{Nu} = 0.943\left[\frac{gh_{fg}\rho_l(\rho_l-\rho_v)L^3}{\mu_l K_l(T_{sat}-T_w)}\right]^{1/4}$.

So, if you put all these values. So,

$0.943\left[\frac{9.81\times2257\times10^3\times960\times(960-0.598)\times5^3}{289\times10^{-6}\times0.68\times(100-95)}\right]^{1/4}$. So, average Nusselt number if you calculate, you will get 6730. Now, you will be able to calculate the average heat transfer coefficient.

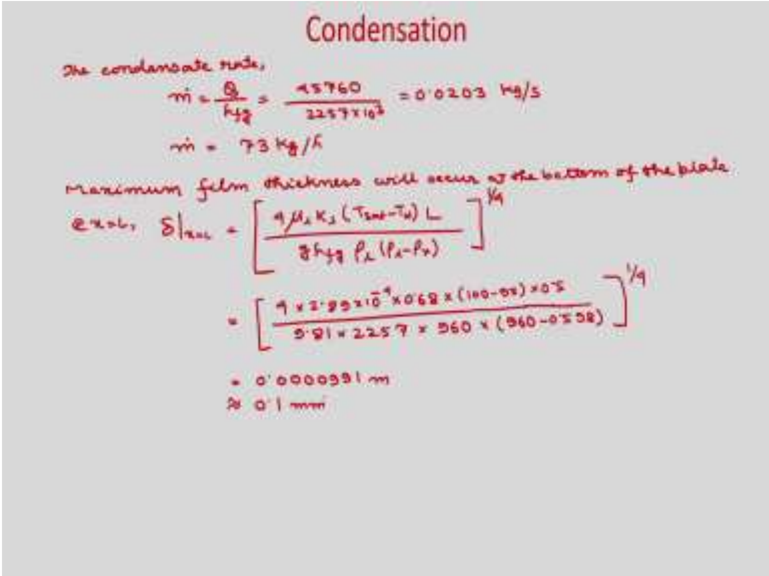
The average heat transfer coefficient, $\bar{h} = \overline{Nu}\frac{K_l}{L}$; because based on length we have calculated the Nusselt number. So, it will be $\frac{6730\times0.68}{0.5}$. So, you will get 9152 W/m^{2}K.

So, once you know the heat transfer coefficient, you will be able to calculate the heat transfer rate. So, here if you see, we have shown only in one side how the condensate is flowing downward direction; but in the other side also you can consider the similar condensate, other side of the plate.

So, in the both side of the plate if you calculate, the heat transfer rate, then you have to multiply by 2. So, considering both sides of the plate, the total heat transfer rate per meter plate width ok; we can write as, $Q=\bar{h}A(T_{sat}-T_w)$.

So, $9152\times(0.5\times2)\times(100-95)$. So, you will get it as 45760 W or around 45.76 kW. Now, you will be able to calculate the condensate rate; because you know at steady state, $Q=\dot{m}h_{fg}$.

(Refer Slide Time: 31:15)



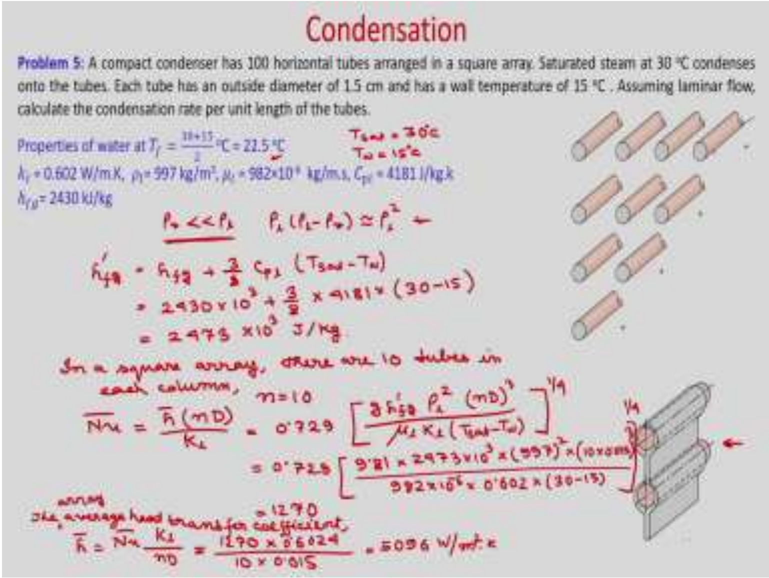
So, the condensate rate you can calculate as. So, $\dot{m} = \frac{Q}{h_{fg}} = \frac{45760}{2257 \times 10^3}$. So, you will get 0.0203 kg/s or you can write as 73 kg/hr. And now, maximum film thickness you can calculate maximum film thickness will occur at the bottom of the plate, ok.

So, at x = L , $\delta|_{x=L} = \left[\frac{4\mu_l K_l \left(T_{sat} - T_w\right) L}{g h_{fg} \rho_l \left(\rho_l - \rho_v\right)}\right]^{1/4}$. So, this is the expression. So, in earlier problem we have seen that, we calculated the local heat transfer coefficient at x = L, at the bottom of the plate.

So, from there easily we could calculate the film thickness; but in this particular case we did not calculate the heat transfer coefficient at x = L. So, for that reason just we are writing the full expression. So, if you put all the values; so you will get, $\left[\frac{4 \times 2.89 \times 10^{-4} \times 0.68 \times (100 - 95) \times 0.5}{9.81 \times 2257 \times 960 \times (960 - 0.598)}\right]^{1/4}$. So, if you calculate, you will get the film thickness at the bottom of the plate as 0.000991 m and you can write it is as ≈ 0.1 mm.

So, you can see that usually the condensate film thickness is very small; in both the cases, we have calculated the film thickness at the end of the plate and it is very very small.

(Refer Slide Time: 34:44)



So, now let us take this last problem. A compact condenser has 100 horizontal tubes arranged in a square array. So, you have 100 horizontal tubes and these are arranged in square array, ok. Saturated steam at 30 ^{0}C condenses on to the tubes. Each tube has an outside diameter of 1.5 cm and has a wall temperature of 15 ^{0}C. Assuming laminar flow, calculate the condensation rate per unit length of the tubes?

So, if there are 100 horizontal tubes and these are arranged in a square array; then you can see in 10 by 10 you can arrange. So, in the vertically, you will have 10 tubes and horizontally you can have 10 tubes. So, if you consider only one vertical array; then we will calculate the condensation rate, per unit length of the tubes.

And in this particular case, you can see that T_{sat} is your 30 ^{0}C and T_w temperature is 15 ^{0}C, and at mean film temperature 22.5 ^{0}C, the properties are given. So, you can calculate the here you can see that vapour density obviously, we you know most of the cases we have seen that $\rho_v << \rho_l$.

So, we will just use $\rho_l\left(\rho_l - \rho_v\right) \approx \rho_l^2$. So, in the expression we will use this one, as $\rho_v << \rho_l$; you can consider, but here we are going to neglect. So, in this particular case, the modified h_{fg} you have to calculate; because you have this wall temperature 15 ^{0}C and T_{sat} 30 ^{0}C.

So, here you need to consider the sensible heat. So, you can see $h_{fg}^{'} = h_{fg} + \frac{3}{8}C_{pl}\left(T_{sat} - T_w\right)$.

So, this if you see. So, $2430\times10^3 + \frac{3}{8}\times4181\times\left(30-15\right)$. So, if you calculate it, you will get 2473×10^3 J/kg.

So, you can see that in the square array. So, we will have 10 tubes in each column. So, in a square array, there are 10 tubes in each column. So, n will be 10. So, $\overline{Nu} = \frac{\bar{h}\left(nD\right)}{K_l}$.

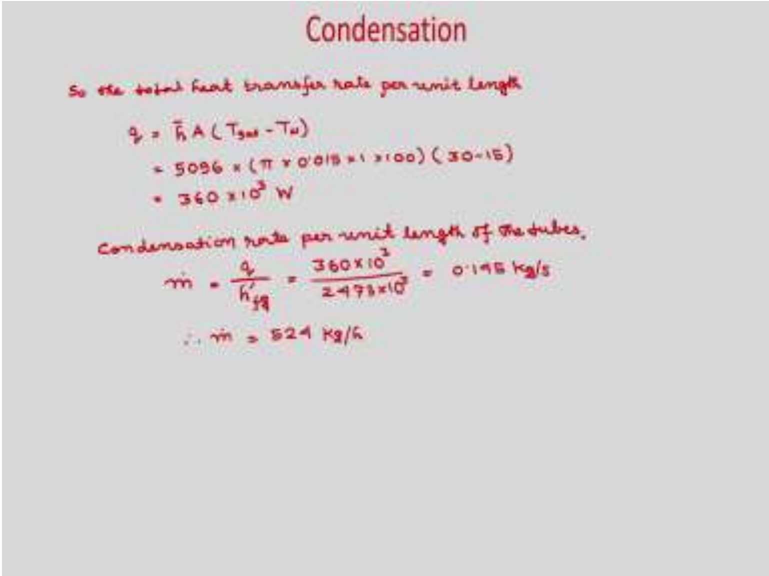
So, this you can write as $0.729\left[\frac{gh_{fg}^{'}\rho_l^2\left(nD\right)^3}{\mu_l K_l\left(T_{sat} - T_w\right)}\right]^{1/4}$. So, if you put all the values here. So, you will get $0.729\times\left[\frac{9.81\times2473\times10^3\times\left(997\right)^2\times\left(10\times0.015\right)^3}{982\times10^{-6}\times0.602\times\left(30-15\right)}\right]^{1/4}$. So, this is the average Nusselt number for this particular case considering a single column. So, you will get this around 1270. So, from here $\bar{h}$ you can calculate.

So, the average heat transfer coefficient. So, $\bar{h} = \overline{Nu}\frac{K_l}{nD}$. So, it will be $\frac{1270\times0.6024}{10\times0.015}$. So, this you will get as 5096 W/m^{2}K.

So, you can see that, we have calculated the average heat transfer coefficient. So, it is one for, one column. So, it is array average heat transfer coefficient. Now, we need to calculate the total heat transfer rate.

So, when we need to calculate total heat transfer rate is equal to your heat transfer coefficient into area into the temperature difference. So, in this particular case, what is the area? So, we have to consider total area per unit width. So, total area means, it is your $\pi D\times l$; 1 is the unit width and we have 100 cylinders, so we have to multiply with 100.

(Refer Slide Time: 41:27)



So, the total heat transfer rate per unit length. So, $q = \bar{h}A\left(T_{sat} - T_w\right)$. So, you can see this heat transfer coefficient already we have calculated. So, $5096 \times \left(\pi \times 0.015 \times 1 \times 100\right) \times \left(30 - 15\right)$. So, if you multiply, you will get approximately, 360 $\times 10^{3}$ W. So, now, if we need to calculate the mass flow rate; so we have to just use this q divided by your latent heat of condensation.

So, your condensation rate per unit length of the tubes, so that will be $\dot{m} = \frac{q}{h'_{fg}}$; so, this we have calculated as $\frac{360 \times 10^3}{2473 \times 10^3}$.So, if you calculate it, you will get 0.145 kg/s. And if you convert it into kg per hour, then you will get as 524 kg/hr. So, today we solved two problems from boiling and three problems from condensation. You have seen that to solve the boiling problems, you need to remember the expression of heat flux in a particular regime.

So, you need to calculate the excess temperature ΔT and you have to find that in which region it falls and that expression of heat flux you need to use. Also to calculate the critical heat flux, you need to remember the expression which we taught in the first lecture of this module.

When you solve the problems of condensation, then you need to remember the expression for heat transfer coefficient, both local and average and Nusselt number expression. Then from there, you can also calculate the mass flow rate or condensation rate and that expression you should remember, as well as from there you need to calculate the thickness of the film.

So, thickness of the film in terms of your heat transfer coefficient, you can remember the expression or as a whole whatever we have used in today's lecture; so those expression you need to remember to solve these problems. Few problems will be given in assignments. You solve those problems and practice more problems from any undergraduate heat transfer book.

Thank you.

THIS BOOK IS NOT FOR

SALE

NOR COMMERCIAL USE

(044) 2257 5905/08

nptel.ac.in

swayam.gov.in