

In[*]:= $v = v0[y] (x / \delta[y]) (1 - x / \delta[y])^2$

Out[*]=
$$\frac{x v0[y] \left(1 - \frac{x}{\delta[y]}\right)^2}{\delta[y]}$$

In[*]:= $T = T0 + (Tw - T0) (1 - x / \delta[y])^2$

Out[*]=
$$T0 + (-T0 + Tw) \left(1 - \frac{x}{\delta[y]}\right)^2$$

In[*]:= $lims = \{x, 0, \delta[y]\};$

$$momInt = \partial_y \int_0^{\delta[y]} v^2 dx == -v (\partial_x v /. x \rightarrow 0) + \int_0^{\delta[y]} g \beta (T - T0) dx$$

Out[*]=
$$\frac{2}{105} v0[y] \times \delta[y] v0'[y] + \frac{1}{105} v0[y]^2 \delta'[y] == -\frac{v v0[y]}{\delta[y]} - \frac{1}{3} g T0 \beta \delta[y] + \frac{1}{3} g Tw \beta \delta[y]$$

In[*]:= $egyInt = \partial_y \int_0^{\delta[y]} v (T - T0) dx == -\alpha (\partial_x T /. x \rightarrow 0) // Simplify[#, Tw - T0 > 0] \&$

Out[*]=
$$\frac{60 \alpha}{\delta[y]} == \delta[y] v0'[y] + v0[y] \delta'[y]$$

In[*]:= $rules = \{$
 $\quad v0[y] \rightarrow A y^m,$
 $\quad v0'[y] \rightarrow D[A y^m, y],$
 $\quad \delta[y] \rightarrow B y^n,$
 $\quad \delta'[y] \rightarrow D[B y^n, y]$
 $\}$

Out[*]=
$$\{v0[y] \rightarrow A y^m, v0'[y] \rightarrow A m y^{-1+m}, \delta[y] \rightarrow B y^n, \delta'[y] \rightarrow B n y^{-1+n}\}$$

In[*]:= $eq1 = momInt /. rules$

Out[*]=
$$\frac{2}{105} A^2 B m y^{-1+2 m+n} + \frac{1}{105} A^2 B n y^{-1+2 m+n} == -\frac{1}{3} B g T0 y^n \beta + \frac{1}{3} B g Tw y^n \beta - \frac{A y^{m-n} v}{B}$$

In[*]:= $eq2 = (egyInt /. rules)$

Out[*]=
$$\frac{60 y^{-n} \alpha}{B} == A B m y^{-1+m+n} + A B n y^{-1+m+n}$$

Comparing the powers of y since above equation has to be true for all values of y

In[*]:= $mnSol = Solve[\{2 m + n - 1 == n, n == m - n, m + n - 1 == -n\}, \{m, n\}]$

Out[*]=
$$\left\{\left\{m \rightarrow \frac{1}{2}, n \rightarrow \frac{1}{4}\right\}\right\}$$

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In[*]:= ABSol = Solve[{
  Simplify[eq1 /. mnSol, y > 0] // (#[[1]] &),
  Simplify[eq2 /. mnSol, {y > 0, Tw - T0 > 0}] // (#[[1]] &)},
  {A, B}]
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Out[*]=
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$$\left\{ \left\{ A \rightarrow \frac{4 \left(-\frac{\sqrt{35} g T_0 \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} + \frac{\sqrt{35} g Tw \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} \right)}{20 \alpha + 21 \nu}, \right. \right.$$

$$B \rightarrow -\frac{2 \left(\frac{5}{7} \right)^{1/4} (-20 \alpha^2 - 21 \alpha \nu)^{1/4}}{(g T_0 \beta - g Tw \beta)^{1/4}} \left. \right\},$$

$$\left\{ A \rightarrow \frac{4 \left(\frac{\sqrt{35} g T_0 \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} - \frac{\sqrt{35} g Tw \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} \right)}{20 \alpha + 21 \nu}, \right.$$

$$B \rightarrow -\frac{2 i \left(\frac{5}{7} \right)^{1/4} (-20 \alpha^2 - 21 \alpha \nu)^{1/4}}{(g T_0 \beta - g Tw \beta)^{1/4}} \left. \right\},$$

$$\left\{ A \rightarrow \frac{4 \left(\frac{\sqrt{35} g T_0 \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} - \frac{\sqrt{35} g Tw \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} \right)}{20 \alpha + 21 \nu}, \right.$$

$$B \rightarrow \frac{2 i \left(\frac{5}{7} \right)^{1/4} (-20 \alpha^2 - 21 \alpha \nu)^{1/4}}{(g T_0 \beta - g Tw \beta)^{1/4}} \left. \right\},$$

$$\left\{ A \rightarrow \frac{4 \left(-\frac{\sqrt{35} g T_0 \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} + \frac{\sqrt{35} g Tw \beta \sqrt{-\alpha (20 \alpha + 21 \nu)}}{\sqrt{g (T_0 - Tw) \beta}} \right)}{20 \alpha + 21 \nu}, B \rightarrow \frac{2 \left(\frac{5}{7} \right)^{1/4} (-20 \alpha^2 - 21 \alpha \nu)^{1/4}}{(g T_0 \beta - g Tw \beta)^{1/4}} \right\}$$

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In[*]:= ASol = Simplify[ABSol[[4]][[1]][[2]]]
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Out[*]=
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$$\frac{4 \sqrt{35} \alpha \sqrt{g (T_0 - Tw) \beta}}{\sqrt{-\alpha (20 \alpha + 21 \nu)}}$$

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In[*]:= BSol = Simplify[ABSol[[4]][[2]][[2]]]
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Out[*]=
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$$\frac{2 \left(\frac{5}{7} \right)^{1/4} (-\alpha (20 \alpha + 21 \nu))^{1/4}}{(g (T_0 - Tw) \beta)^{1/4}}$$

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In[*]:= h = -κ (D[T, x] /. x -> 0) / (Tw - T0) // (# /. δ[y] -> BSol y^(1/4) &)
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Out[*]=
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$$\frac{\left(\frac{7}{5} \right)^{1/4} (g (T_0 - Tw) \beta)^{1/4} \kappa}{y^{1/4} (-\alpha (20 \alpha + 21 \nu))^{1/4}}$$

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In[*]:= Nu = h y / κ
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Out[*]=
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$$\frac{\left(\frac{7}{5} \right)^{1/4} y^{3/4} (g (T_0 - Tw) \beta)^{1/4}}{(-\alpha (20 \alpha + 21 \nu))^{1/4}}$$

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In[*]:= NuMean = (4 / 3) Nu /. g → Ra √ α / (β (Tw - T0) y^3) // # /. √ → Pr α & //
Simplify[#, {α > 0, Tw - T0 > 0, y > 0, Pr > 0, Ra > 0}] & // N[#] &
```

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Out[*]=
1.45034  $\left( \frac{\text{Pr Ra}}{20. + 21. \text{Pr}} \right)^{1/4}$ 
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In[*]:= NuMean /. Pr → 0.71
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Out[*]=
0.547706 Ra1/4
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