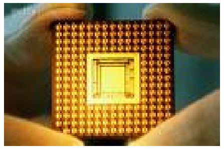
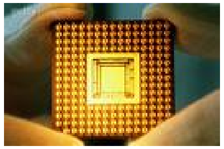


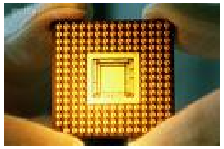
BSC Modul 4: Advanced Circuit Analysis

- Fourier Series
- Fourier Transform
- Laplace Transform
- Applications of Laplace Transform
- Z-Transform



BSC Modul 4: Advanced Circuit Analysis

- **Fourier Series**
- Fourier Transform
- Laplace Transform
- Applications of Laplace Transform
- Z-Transform



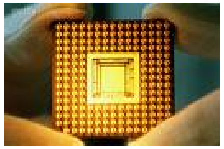
Trigonometric Fourier Series (1)

- The Fourier series of a periodic function *f(t)* is a representation that resolves *f(t)* into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

- Given a periodic function $f(t) = f(t+nT)$ where n is an integer and T is the period of the function.

$$\boxed{f(t) = \underbrace{a_0}_{dc} + \underbrace{\sum_{n=1}^{\infty}(a_0 \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{ac}}$$

where $\omega_0=2\pi/T$ is called the fundamental frequency in radians per second.



Trigonometric Fourier Series (2)

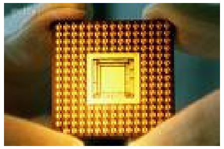
- and a_n and b_n are as follow

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_o t)dt \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_o t)dt$$

- in alternative form of *f(t)*

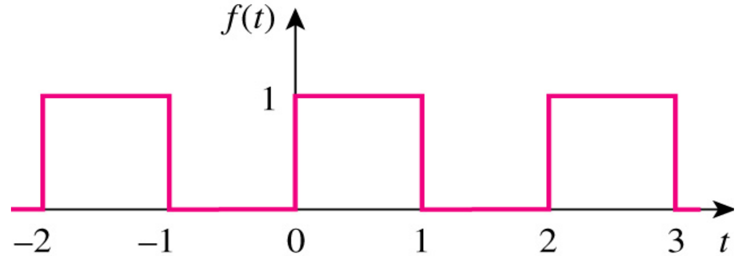
$$f(t) = \underbrace{a_0}_{dc} + \underbrace{\sum_{n=1}^{\infty}(c_n \cos(n\omega_0 t + \phi_n))}_{ac}$$

where $$c_n = \sqrt{a_n^2 + b_n^2}\,,\ \phi_n = -\tan^{-1}(\frac{b_n}{a_n})$$ (Inverse tangent or arctangent)



Fourier Series Example

Determine the Fourier series of the waveform shown right. Obtain the amplitude and phase spectra.



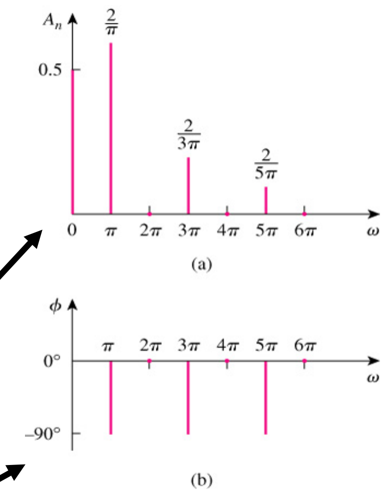
$$f(t)=\begin{cases}1, & 0<t<1\\ 0, & 1<t<2\end{cases} \quad \text{and } f(t)=f(t+2)$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt=0 \text{ and}$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt=\begin{cases}2/n\pi, & \mathrm{n=odd}\\ 0, & \mathrm{n=even}\end{cases}$$

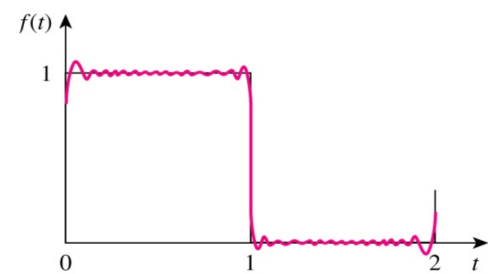
$$A_n=\begin{cases}2/n\pi, & \mathrm{n=odd}\\ 0, & \mathrm{n=even}\end{cases}$$

$$\phi_n=\begin{cases}-90°, & \mathrm{n=odd}\\ 0, & \mathrm{n=even}\end{cases}$$



a) Amplitude and
b) Phase spectrum

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin(n\pi t), \quad n=2k-1$$



Truncating the series at N=11

Symmetry Considerations (1)

Three types of symmetry

1. Even Symmetry : a function *f(t)* if its plot is symmetrical about the vertical axis.

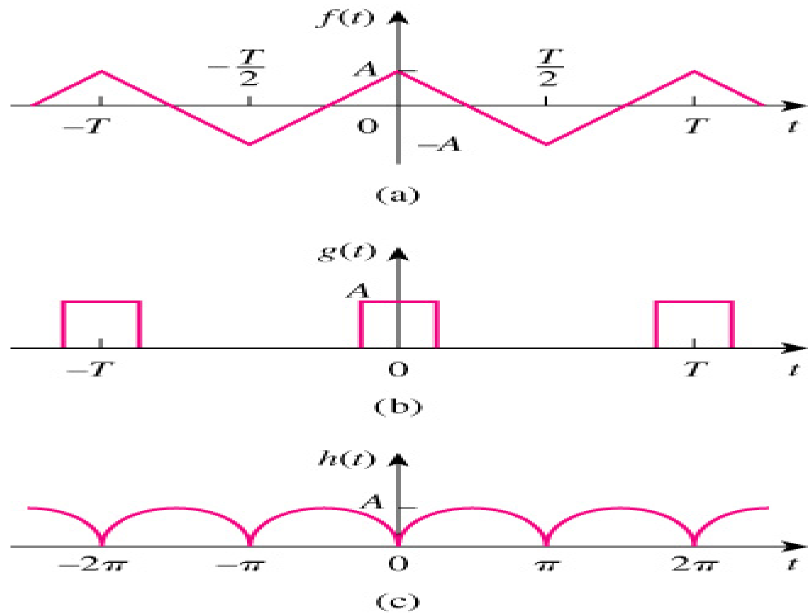
$$f(t) = f(-t)$$

In this case,

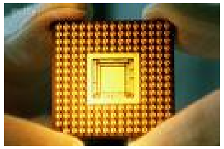
$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt$$

$$b_n = 0$$



Typical examples of even periodic function



Symmetry Considerations (2)

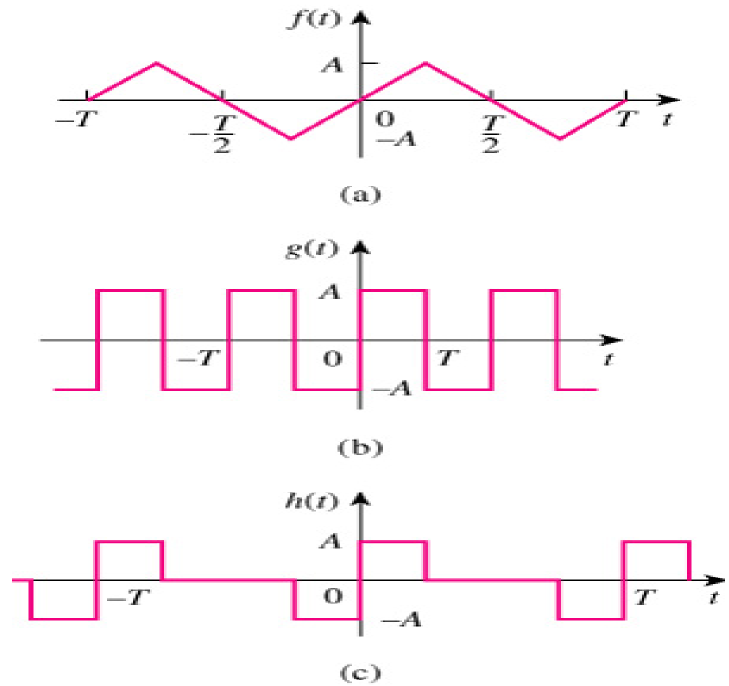
2. Odd Symmetry : a function *f(t)* if its plot is anti-symmetrical about the vertical axis.

$$\boxed{f(-t) = -f(t)}$$

In this case,

$$a_0 = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt$$



Typical examples of odd periodic function

Symmetry Considerations (3)

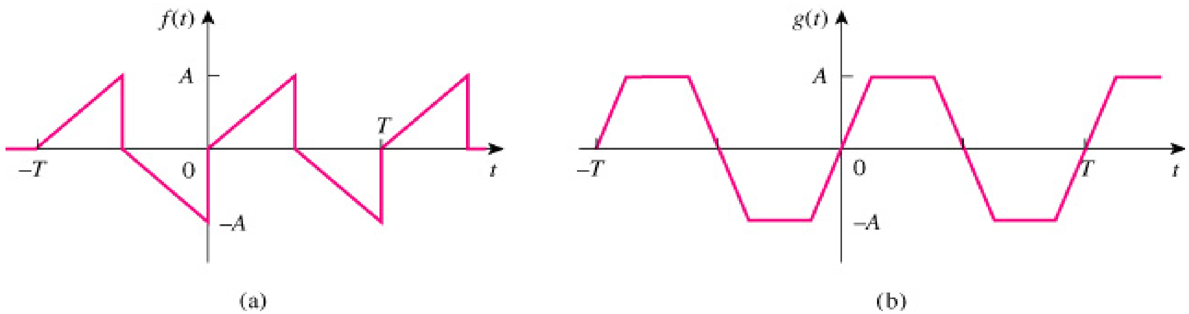
3. Half-wave Symmetry : a function *f(t)* if

$$f(t-\frac{T}{2}) = -f(t)$$

$$a_0 = 0$$

$$a_n = \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt & \text{, for n odd} \\ 0 & \text{, for an even} \end{cases}$$

$$b_n = \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt & \text{, for n odd} \\ 0 & \text{, for an even} \end{cases}$$

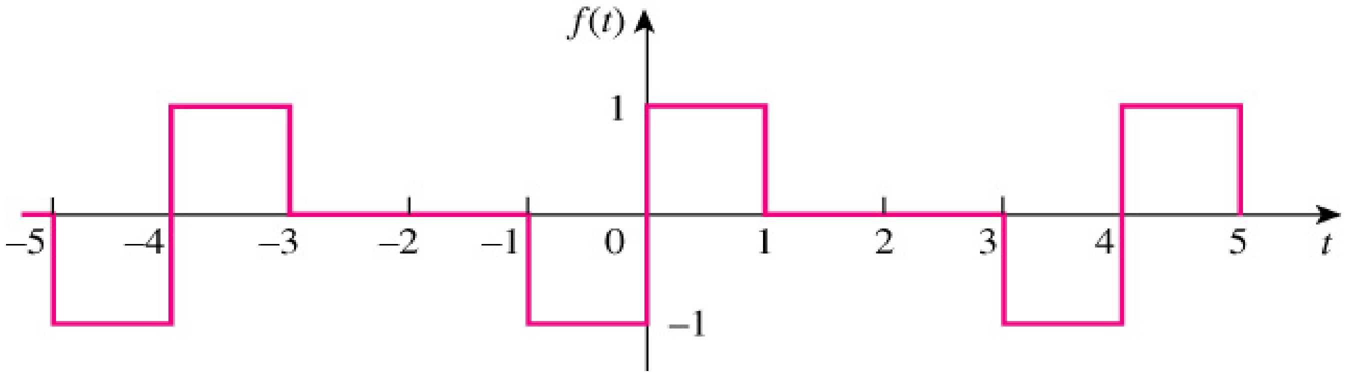


Typical examples of half-wave odd periodic functions

Symmetry Considerations (4)

Example 1

Find the Fourier series expansion of f(t) given below.



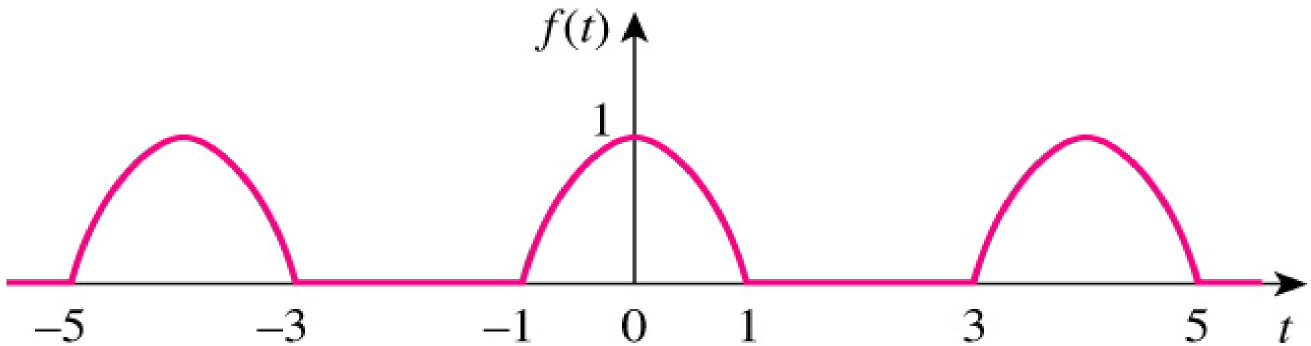
Ans:

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}t\right)$$

Symmetry Considerations (5)

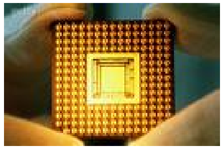
Example 2

Determine the Fourier series for the half-wave cosine function as shown below.



Ans:

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{n^2} \cos nt, \ n = 2k - 1$$



Circuit Applications (1)

Steps for Applying Fourier Series

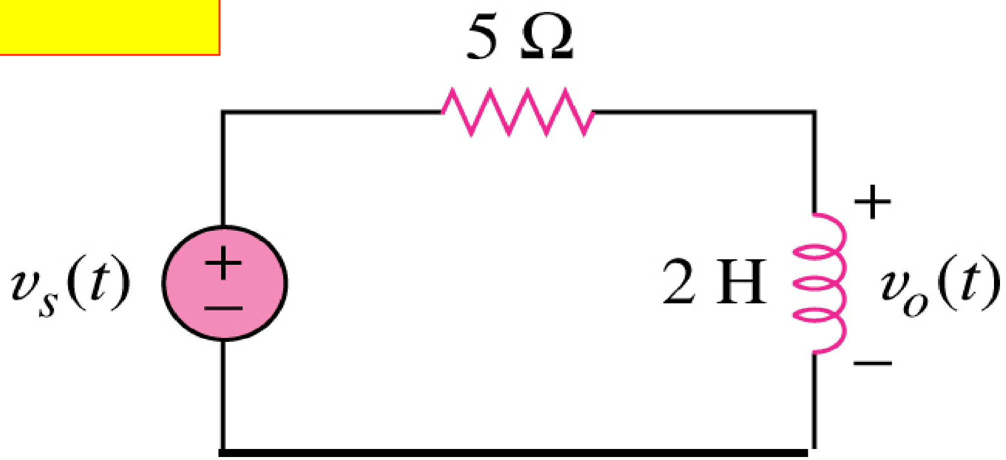
1. Express the excitation as a Fourier series.
2. Transform the circuit from the time domain to the frequency domain.
3. Find the response of the dc and ac components in the Fourier series.
4. Add the individual dc and ac responses using the superposition principle.

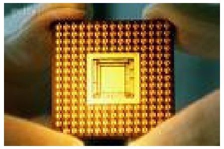
Circuit Applications (2)

Example

Find the response $v_0(t)$ of the circuit below when the voltage source $v_s(t)$ is given by

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi\omega t),\;\; n = 2k-1$$





Circuit Applications (3)

Solution

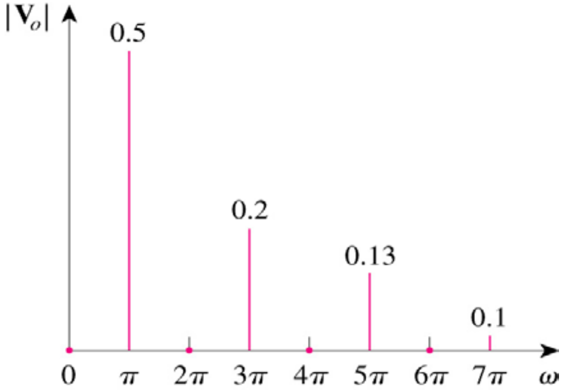
Phasor of the circuit $$\mathrm{V_0} = \frac{j2n\pi}{5 + j2n\pi}\mathrm{V_s}$$

For dc component, (ω_n=0 or n=0), V_s = ½ => V_o = 0

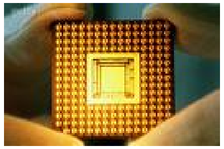
For nth harmonic, $$\mathrm{V}_S = \frac{2}{n\pi}\angle -90°,\quad \mathrm{V_0} = \frac{4\angle -\tan^{-1} 2n\pi/5}{\sqrt{25+4n^2\pi^2}}\mathrm{V_s}$$

In time domain,

$$v_0(t) = \sum_{k=1}^{\infty} \frac{4}{\sqrt{25+4n^2\pi^2}} cos(n\pi t - \tan^{-1}\frac{2n\pi}{5})$$



Amplitude spectrum of the output voltage



Average Power and RMS Values (1)

Given: $$v(t) = \mathrm{V}_{\mathrm{dc}} + \sum_{n=1}^{\infty} \mathrm{V}_{\mathrm{n}} \cos(n\omega_0 t - \phi_{Vn}) \quad \text{and} \quad i(t) = \mathrm{I}_{\mathrm{dc}} + \sum_{m=1}^{\infty} \mathrm{I}_{\mathrm{m}} \cos(m\omega_0 t - \phi_{\mathrm{Im}})$$

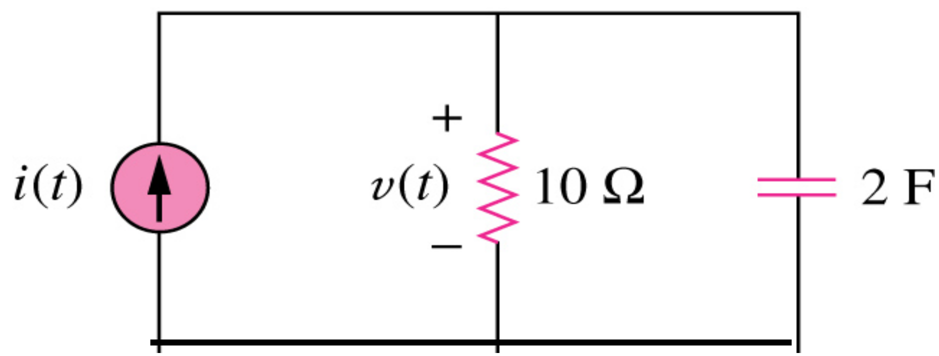
The average power is $$\mathrm{P} = \mathrm{V}_{\mathrm{dc}}\mathrm{I}_{\mathrm{dc}} + \frac{1}{2}\sum_{n=1}^{\infty} \mathrm{V}_{\mathrm{n}}\mathrm{I}_{\mathrm{n}} \cos(\theta_n - \phi_n)$$

The rms value is $$F_{rms} = \sqrt{a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

Average Power and RMS Values (2)

Example

Determine the average power supplied to the circuit shown below if
$i(t) = 2 + 10\cos(t + 10°) + 6\cos(3t + 35°)$ A



Answer: 41.5W

Exponential Fourier Series (1)

- The exponential Fourier series of a periodic function *f(t)* describes the spectrum of *f(t)* in terms of the amplitude and phase angle of ac components at positive and negative harmonic.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

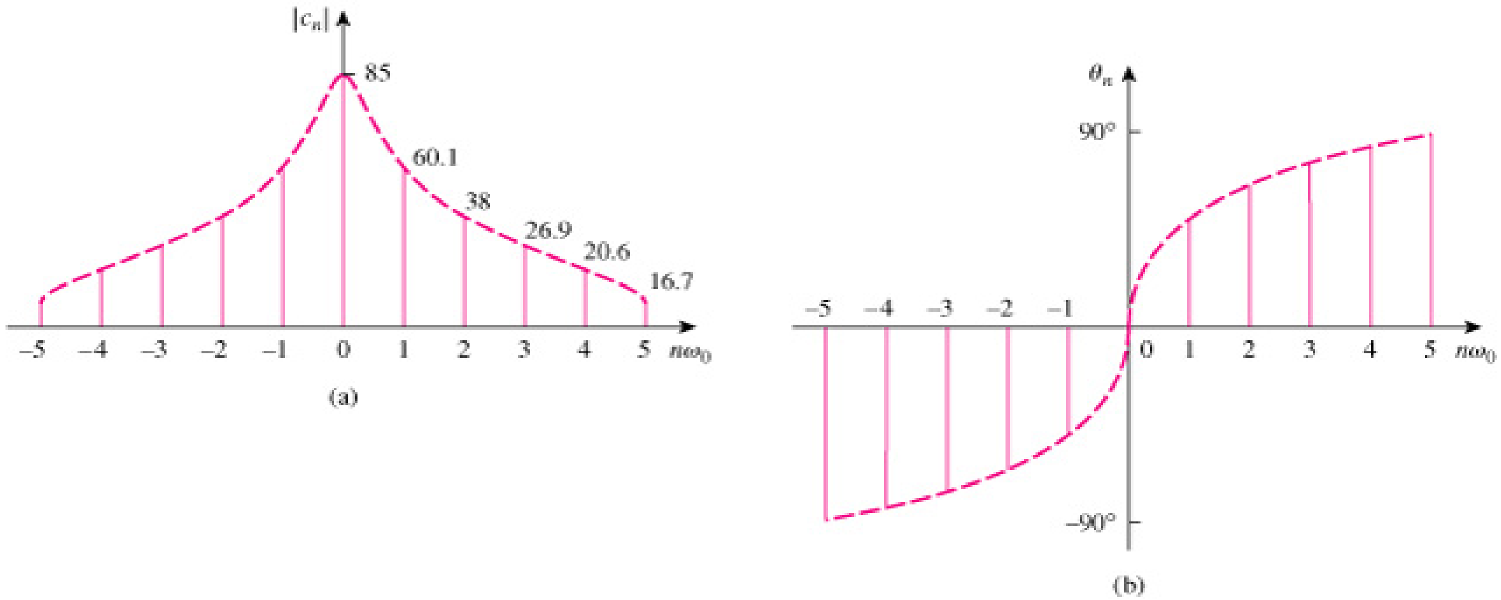
$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt, \text{ where } \omega_0 = 2\pi / T$$

- The plots of magnitude and phase of c_n versus $n\omega_0$ are called the complex amplitude spectrum and complex phase spectrum of *f(t)* respectively.

Exponential Fourier Series (2)

- The complex frequency spectrum of the function

  $f(t)=e^{t}$, $0<t<2\pi$ with $f(t+2\pi)=f(t)$



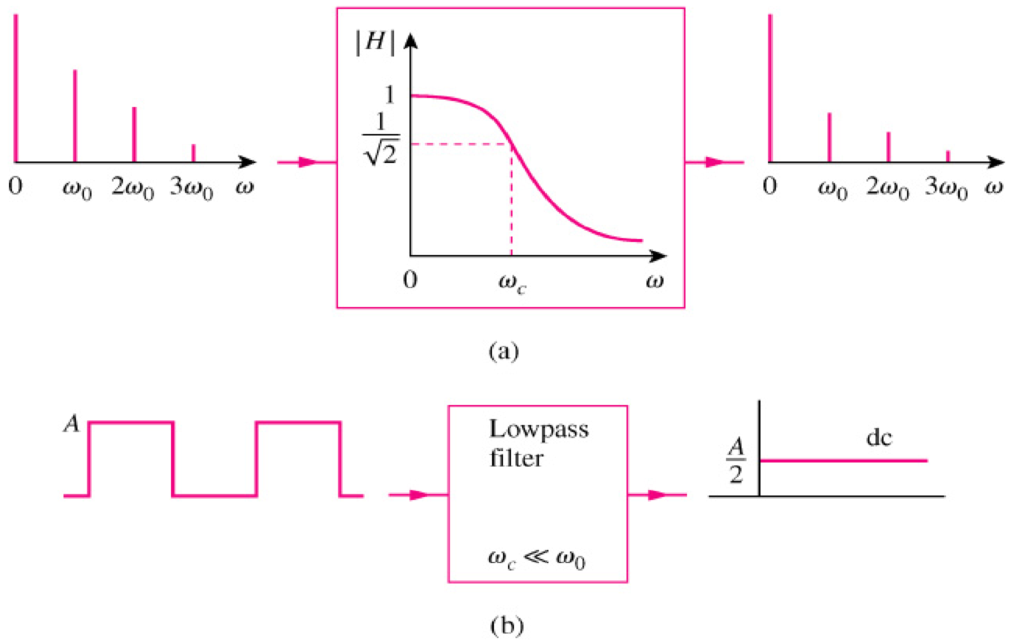
(a) Amplitude spectrum; (b) phase spectrum

Application – Filter (1)

•Filter are an important component of electronics and communications system.

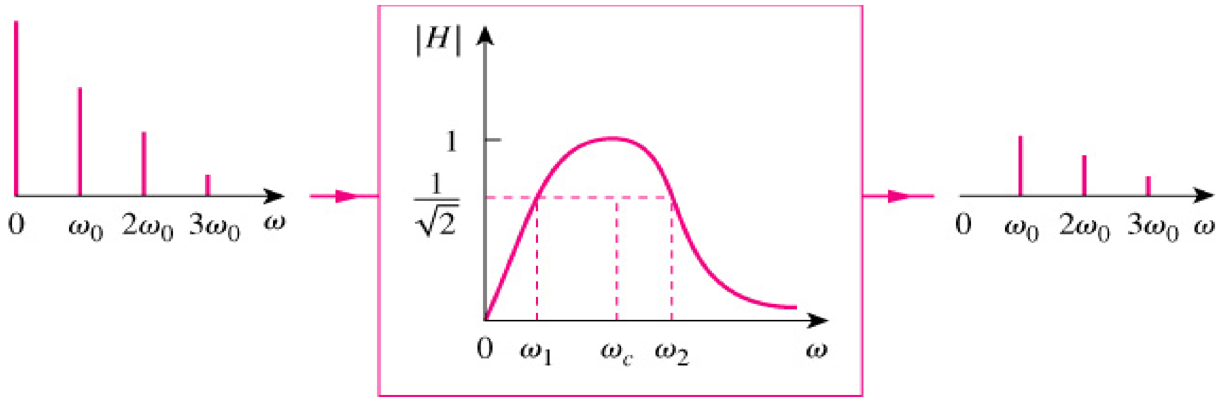
•This filtering process cannot be accomplished without the Fourier series expansion of the input signal.

•For example,

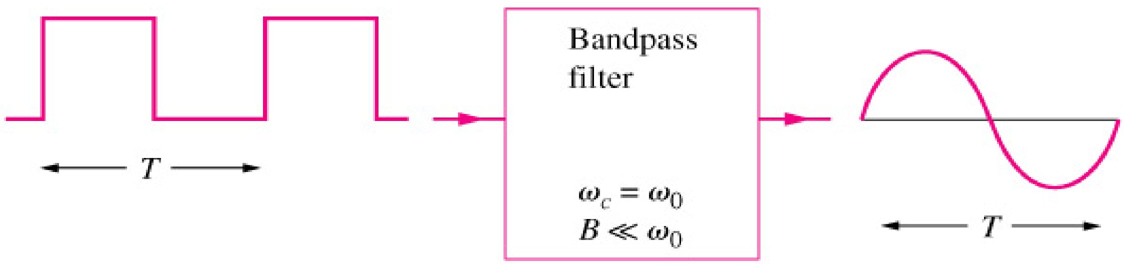


(a) Input and output spectra of a lowpass filter, (b) the lowpass filter passes only the dc component when $\omega_c << \omega_0$

Application – Filter (2)



(a)



(b)

(a) Input and output spectra of a bandpass filter, (b) the bandpass filter passes only the dc component when B << ω_0

BSC Modul 4: Advanced Circuit Analysis

- Fourier Series
- **Fourier Transform**
- Laplace Transform
- Applications of Laplace Transform
- Z-Transform

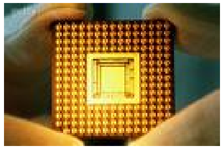
Definition of Fourier Transform (1)

- It is an integral transformation of *f(t)* from the time domain to the frequency domain $F(\omega)$

- $F(\omega)$ is a complex function; its magnitude is called the amplitude spectrum, while its phase is called the phase spectrum.

Given a function *f(t)*, its Fourier transform denoted by $F(\omega)$, is defined by

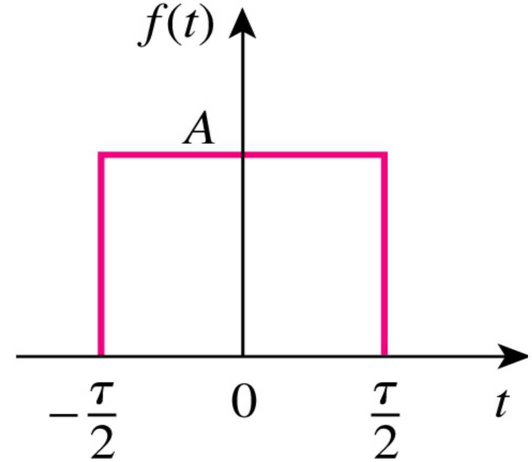
$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$



Definition of Fourier Transform (2)

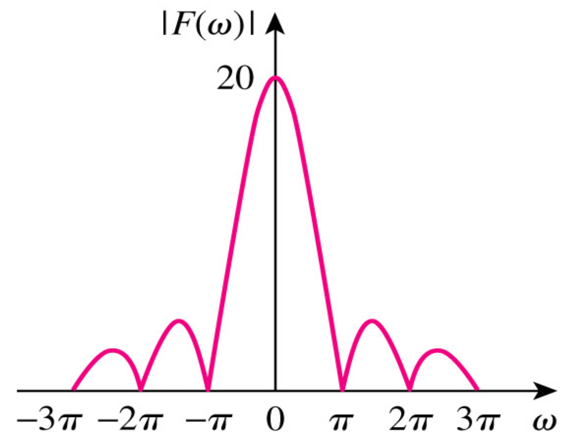
Example 1:

Determine the Fourier transform of a single rectangular pulse of wide τ and height A, as shown below.



Solution:

$$
\begin{aligned}
F(\omega) &= \int_{-\tau/2}^{\tau/2} A e^{j\omega t}\,dt \\
&= -\frac{A}{j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} \\
&= \frac{2A}{\omega}\left(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right) \\
&= A\tau \sin c \frac{\omega\tau}{2}
\end{aligned}
$$

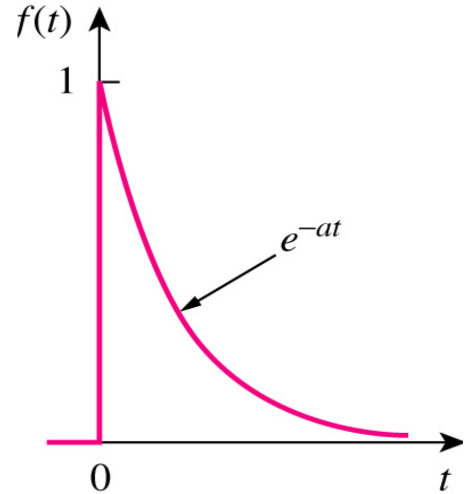


Amplitude spectrum of the rectangular pulse

Definition of Fourier Transform (3)

Example 2:

Obtain the Fourier transform of the "switched-on" exponential function as shown.



Solution:

$$f(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$$

Hence,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-jat}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{a+j\omega}$$

Properties of Fourier Transform (1)

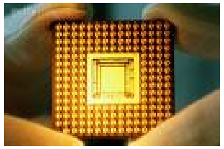
Linearity:

If $F_1(\omega)$ and $F_2(\omega)$ are, respectively, the Fourier Transforms of $f_1(t)$ and $f_2(t)$

$$F\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(\omega) + a_2 F_2(\omega)$$

Example:

$$F\left[\sin(\omega_0 t)\right] = \frac{1}{2j}\left[F\left(e^{j\omega_0 t}\right) - F\left(e^{-j\omega_0 t}\right)\right] = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$



Properties of Fourier Transform (2)

Time Scaling:

If $F(\omega)$ is the Fourier Transforms of $f(t)$, then

$$F\left[f(at)\right] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \quad a \text{ is a constant}$$

If |a|>1, frequency compression, or time expansion

If |a|<1, frequency expansion, or time compression

Properties of Fourier Transform (3)

Time Shifting:

If $F(\omega)$ is the Fourier Transforms of $f(t)$, then

$$\boxed{F\left[f(t-t_0)\right] = e^{-j\omega t_0} F(\omega)}$$

Example:

$$\boxed{F\left[e^{-(t-2)}u(t-2)\right] = \frac{e^{-j2\omega}}{1+j\omega}}$$

Properties of Fourier Transform (4)

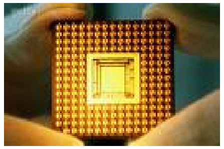
Frequency Shifting (Amplitude Modulation):

If $F(\omega)$ is the Fourier Transforms of $f(t)$, then

$$\boxed{F\left[f(t)e^{j\omega_0 t}\right] = F(\omega - \omega_0)}$$

Example:

$$\boxed{F\left[f(t)\cos(\omega_0 t)\right] = \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0)}$$



Properties of Fourier Transform (5)

Time Differentiation:

If $F(\omega)$ is the Fourier Transforms of $f(t)$, then the Fourier Transform of its derivative is

$$\boxed{F\left[\frac{df}{dt}u(t)\right] = j\omega F(s)}$$

Example:

$$\boxed{F\left[\frac{d}{dt}\left(e^{-at}u(t)\right)\right] = \frac{1}{a+j\omega}}$$

Properties of Fourier Transform (6)

<u>Time Integration</u>:

If $F(\omega)$ is the Fourier Transforms of $f(t)$, then the Fourier Transform of its integral is

$$\boxed{F\left[\int_{-\infty}^{t} f(t)dt\right] = \frac{F(\omega)}{j\omega}\pi F(0)\delta(\omega)}$$

Example:

$$\boxed{F[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)}$$

Circuit Application (1)

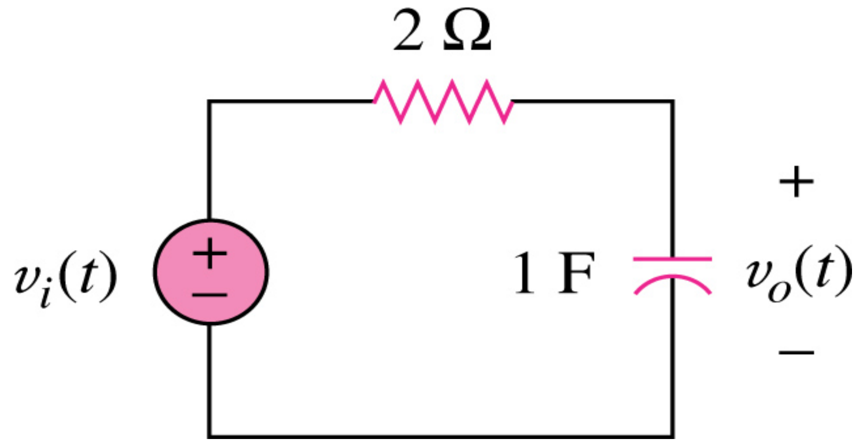
- Fourier transforms can be applied to circuits with non-sinusoidal excitation in exactly the same way as phasor techniques being applied to circuits with sinusoidal excitations.

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega)$$

$$Y(\omega) = H(\omega)X(\omega)$$

- By transforming the functions for the circuit elements into the frequency domain and take the Fourier transforms of the excitations, conventional circuit analysis techniques could be applied to determine unknown response in frequency domain.
- Finally, apply the inverse Fourier transform to obtain the response in the time domain.

Circuit Application (2)



Example:

Find $v_0(t)$ in the circuit shown below for

$v_i(t) = 2e^{-3t}u(t)$

Solution:

The Fourier transform of the input signal is $V_i(\omega) = \dfrac{2}{3 + j\omega}$

The transfer function of the circuit is $H(\omega) = \dfrac{V_0(\omega)}{V_i(\omega)} = \dfrac{1}{1 + j2\omega}$

Hence,

$$V_0(\omega) = \frac{1}{(3 + j\omega)(0.5 + j\omega)}$$

Taking the inverse Fourier transform gives $v_0(t) = 0.4(e^{-0.5t} - e^{-3t})u(t)$

BSC Modul 4: Advanced Circuit Analysis

- Fourier Series
- Fourier Transform
- **Laplace Transform**
- Applications of Laplace Transform
- Z-Transform

Definition of Laplace Transform

- It is an <u>integral transformation</u> of *f*(*t*) from the time domain to the complex frequency domain *F*(*s*)
- Given a function *f*(*t*), its Laplace transform denoted by *F*(*s*), is defined by

$$\boxed{F(s) = L\left[f(t)\right] = \int_0^{\infty} f(t) \cdot e^{-st} dt}$$

- Where the parameter s is a complex number

$$\boxed{s = \sigma + j\omega}$$

σ, ω – real numbers

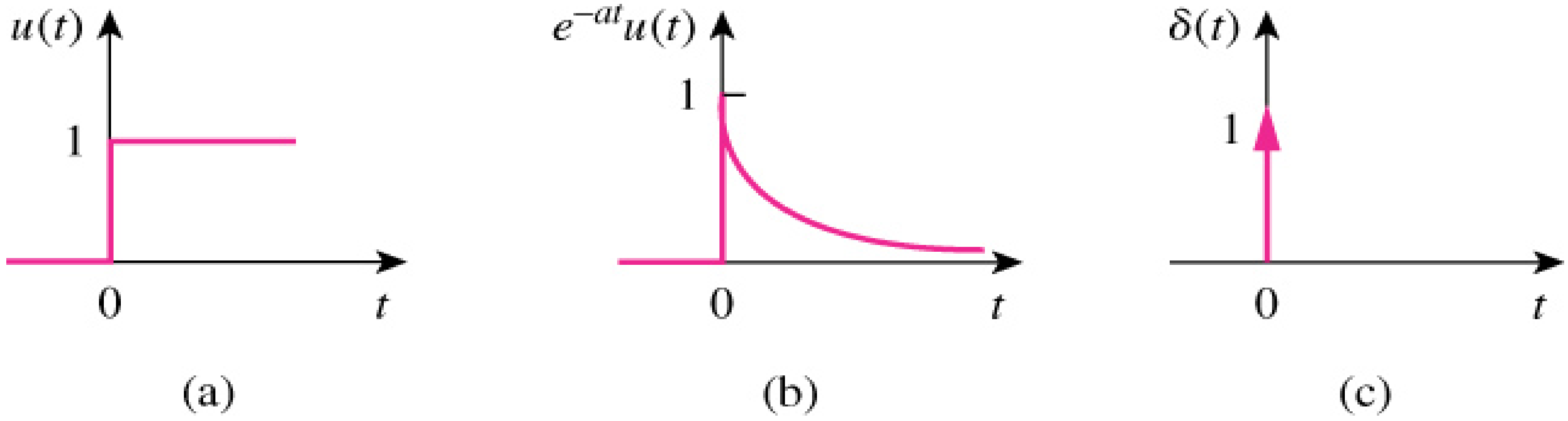
Bilateral Laplace Transform

- When one says "the Laplace transform" without qualification, the unilateral or one-sided transform is normally intended. The Laplace transform can be alternatively defined as the *bilateral Laplace transform* or two-sided Laplace transform by extending the limits of integration to be the entire real axis. If that is done the common unilateral transform simply becomes a special case of the bilateral transform.
- The bilateral Laplace transform is defined as follows:

$$\boxed{F(s) = L\left[f(t)\right] = \int_{-\infty}^{\infty} f(t) \cdot e^{-st} dt}$$

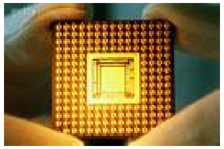
Examples of Laplace Transforms (1)

Determine the Laplace transform of each of the following functions shown:



a) The Laplace Transform of unit step, *u*(*t*) is given by

$$\boxed{L\left[u(t)\right] = F(s) = \int_0^{\infty} 1 e^{-st}\, dt = \frac{1}{s}}$$



Examples of Laplace Transforms (2)

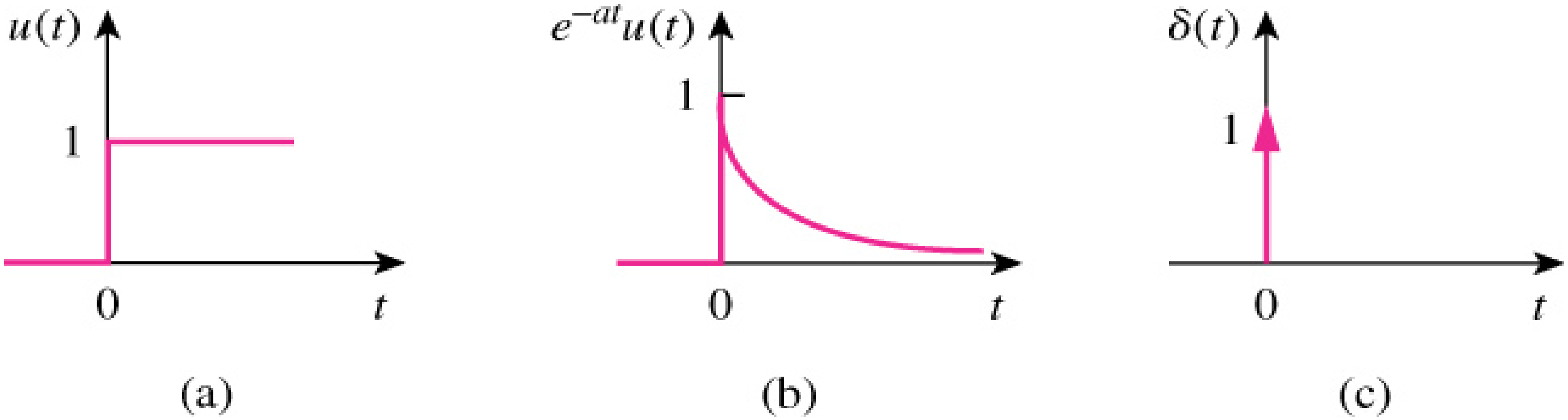
b) The Laplace Transform of exponential function, $e^{-at}u(t)$, $a>0$ is given by

$$L\left[u(t)\right] = F(s) = \int_0^{\infty} e^{\alpha t} e^{-st} dt = \frac{1}{s+\alpha}$$

c) The Laplace Transform of impulse function, $\delta(t)$ is given by

$$L\left[u(t)\right] = F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = 1$$

Examples of Laplace Transforms (3)



$$F(s) = \frac{1}{s}$$

$$F(s) = \frac{1}{s+\alpha}$$

$$F(s) = 1$$

Table of Selected Laplace Transforms (1)

1	$\frac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\frac{1}{s^2}$
$\frac{t^k}{k!}, k \geq 0$	$\frac{1}{s^{k+1}}$
e^{at}	$\frac{1}{s-a}$

$\cos \omega t$	$\frac{s}{s^2+\omega^2} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} = \frac{1/2j}{s-j\omega} - \frac{1/2j}{s+j\omega}$
$\cos(\omega t + \phi)$	$\frac{s \cos \phi - \omega \sin \phi}{s^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

Properties of Laplace Transform (1)

Linearity:

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$

$$L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

Example:

$$L\left[\cos(\omega t) u(t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) u(t)\right] = \frac{s}{s^2 + \omega^2}$$

Properties of Laplace Transform (2)

Scaling:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Example:

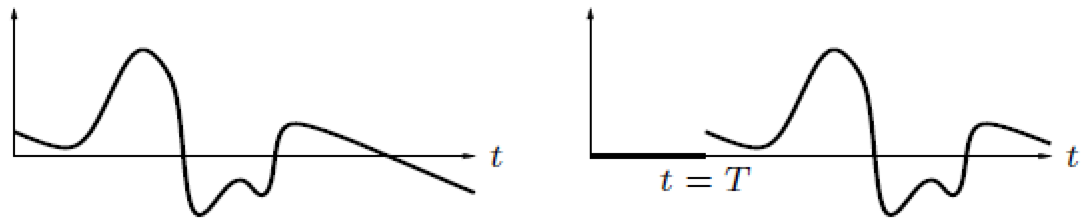
$$\boxed{L\left[\sin(2\omega t)u(t)\right] = \frac{2\omega}{s^2 + 4\omega^2}}$$

Properties of Laplace Transform (3)

Time Shift:

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$L\left[f(t-a)u(t-a)\right]=e^{-as}F(s)$$



Example:

$$L\left[\cos(\omega(t-a))u(t-a)\right]=e^{-as}\frac{s}{s^2+\omega^2}$$

Properties of Laplace Transform (4)

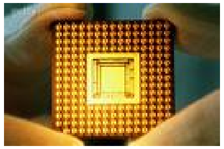
Frequency Shift:

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$\boxed{L\left[e^{-at} f(t)u(t)\right] = F(s+a)}$$

Example:

$$\boxed{L\left[e^{-at}\cos(\omega t)u(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}}$$



Properties of Laplace Transform (5)

Time Differentiation:

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its derivative is

$$L\left[\frac{df}{dt}u(t)\right] = sF(s) - f(0^-)$$

Time Integration:

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its integral is

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

Properties of Laplace Transform (6)

Initial and Final Values:

The initial-value and final-value properties allow us to find $f(0)$ and $f(\infty)$ of $f(t)$ directly from its Laplace transform $F(s)$.

$$f(0) = \lim_{s\to\infty} sF(s)$$

Initial-value theorem

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Final-value theorem

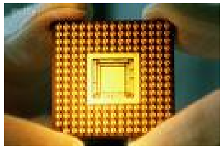
The Inverse Laplace Transform (1)

In principle we could recover $f(t)$ from $F(s)$ via

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) \cdot e^{st} \mathrm{d}s$$

$$F = \int \frac{1}{x}$$

But, this formula isn't really useful.



The Inverse Laplace Transform (2)

Suppose F(s) has the general form of

$$\boxed{F(s) = \frac{N(s)}{D(s)}}$$

numerator polynomial

denominator polynomial

The finding the inverse Laplace transform of *F(s)* involves two steps:

1. Decompose F(s) into simple terms using partial fraction expansion.
2. Find the inverse of each term by matching entries in Laplace Transform Table.

The Inverse Laplace Transform (3)

Example

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

$$f(t) = L^{-1}\left(\frac{3}{s}\right) - L^{-1}\left(\frac{5}{s+1}\right) + L^{-1}\left(\frac{6}{s^2+4}\right)$$

$$= (3 - 5e^{-t} + 3\sin(2t))u(t), \quad \mathrm{t} \geq 0$$

Application to Integro-differential Equations (1)

- The Laplace transform is useful in solving linear integro-differential equations.
- Each term in the integro-differential equation is transformed into s-domain.
- Initial conditions are automatically taken into account.
- The resulting algebraic equation in the s-domain can then be solved easily.
- The solution is then converted back to time domain.

Application to Integro-differential Equations (2)

Example:

Use the Laplace transform to solve the differential equation

$$\frac{d^2 v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

Given: $v(0) = 1$; $v'(0) = -2$

Application to Integro-differential Equations (3)

Solution:

Taking the Laplace transform of each term in the given differential equation and obtain

$$\left[s^2V(s) - sv(0) - v'(0)\right] + 6\left[sV(s) - v(0)\right] + 8V(s) = \frac{2}{s}$$

Substituting $v(0) = 1; \quad v'(0) = -2, \quad$ we have

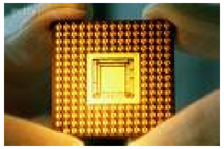
$$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s} = \frac{s^2 + 4s + 2}{s} \Rightarrow V(s) = \frac{\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{4}}{s+4}$$

By the inverse Laplace Transform,

$$v(t) = \frac{1}{4}(1 + 2e^{-2t} + e^{-4t})u(t)$$

BSC Modul 4: Advanced Circuit Analysis

- Fourier Series
- Fourier Transform
- Laplace Transform
- **Applications of Laplace Transform**
- Z-Transform

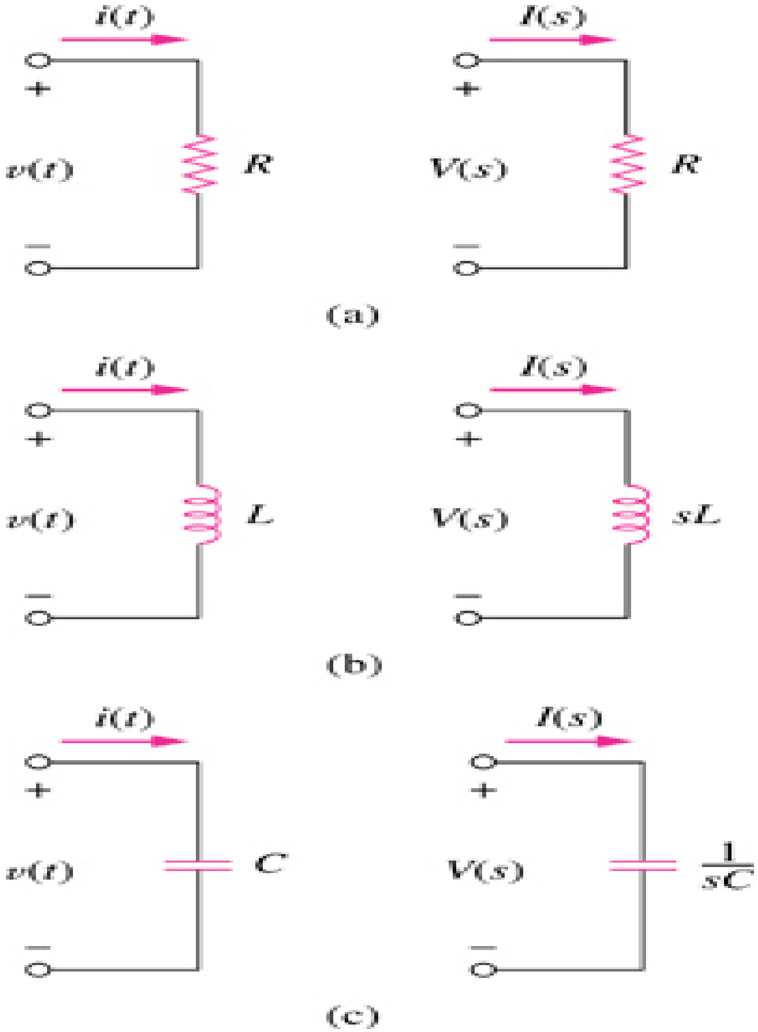


Circuit Element Models (1)

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the s-domain
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Circuit Element Models (2)



Assume zero initial condition for the inductor and capacitor,

Resistor : $V(s)=RI(s)$

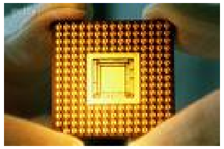
Inductor: $V(s)=sLI(s)$

Capacitor: $V(s) = I(s)/sC$

The impedance in the s-domain is defined as $Z(s) = V(s)/I(s)$

The admittance in the s-domain is defined as $Y(s) = I(s)/V(s)$

Time-domain and s-domain representations of passive elements under zero initial conditions.



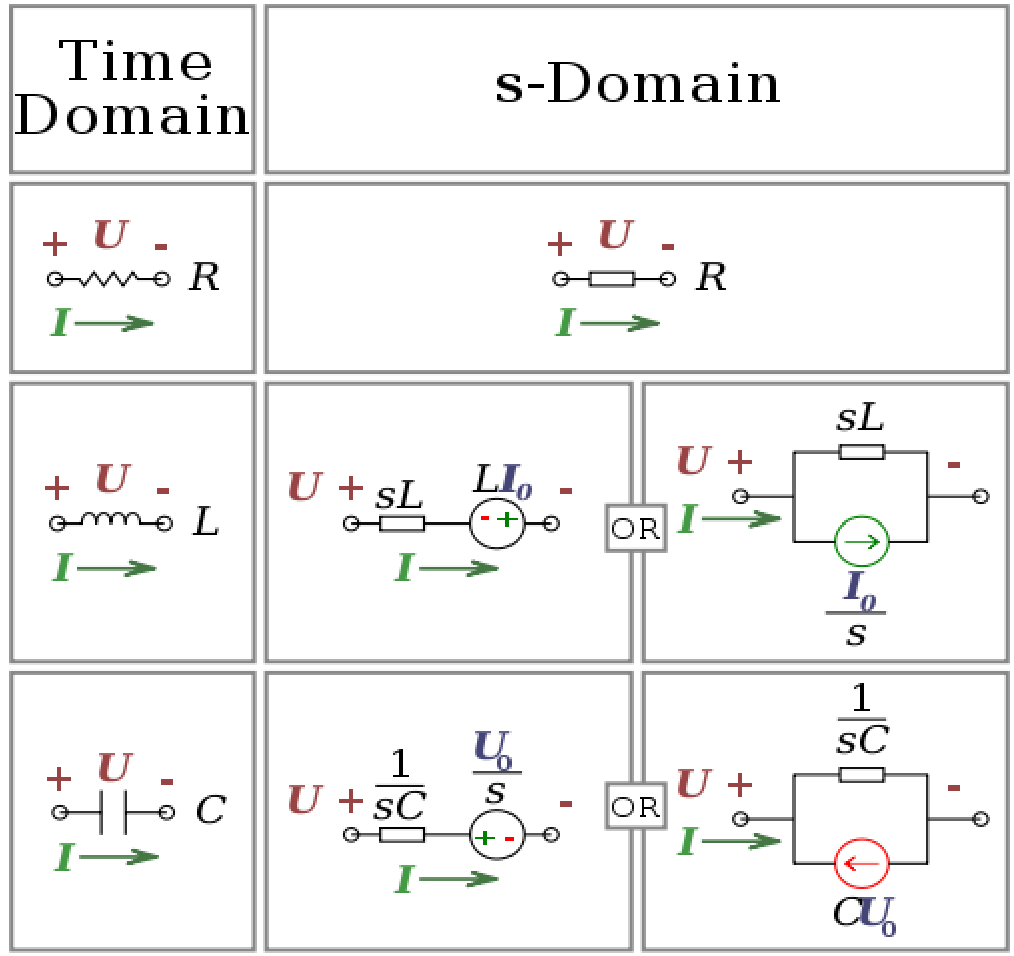
Circuit Element Models (3)

<u>Non-zero initial condition</u> for the inductor and capacitor,

Resistor : $V(s)=RI(s)$

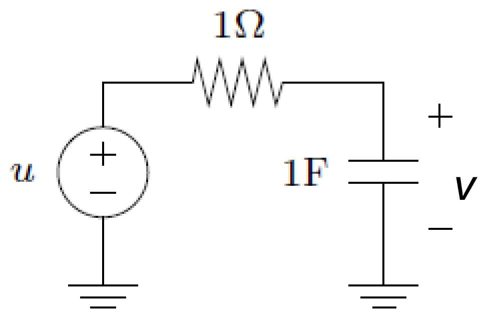
Inductor: $V(s)=sLI(s) + LI(0)$

Capacitor: $V(s) = I(s)/sC + v(0)/s$



Introductory Example

Charging of a capacitor

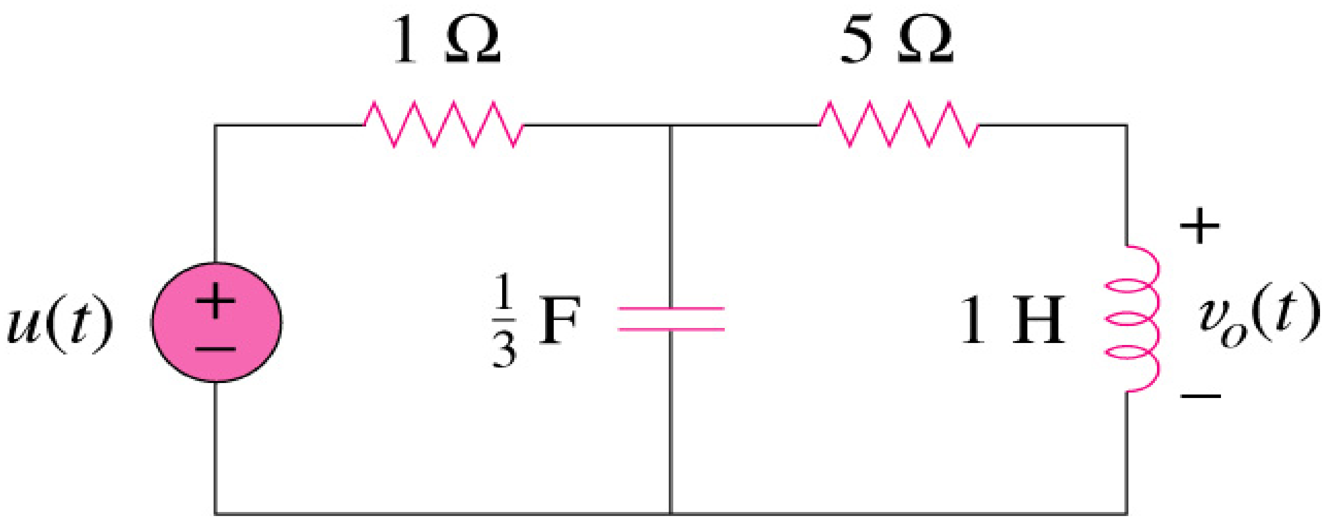


- capacitor is uncharged at $t = 0$, *i.e.*, $V(0) = 0$
- $u(t)$ is a unit step

Circuit Element Models Examples (1)

Example 1:

Find $v_0(t)$ in the circuit shown below, assuming zero initial conditions.



Circuit Element Models Examples (2)

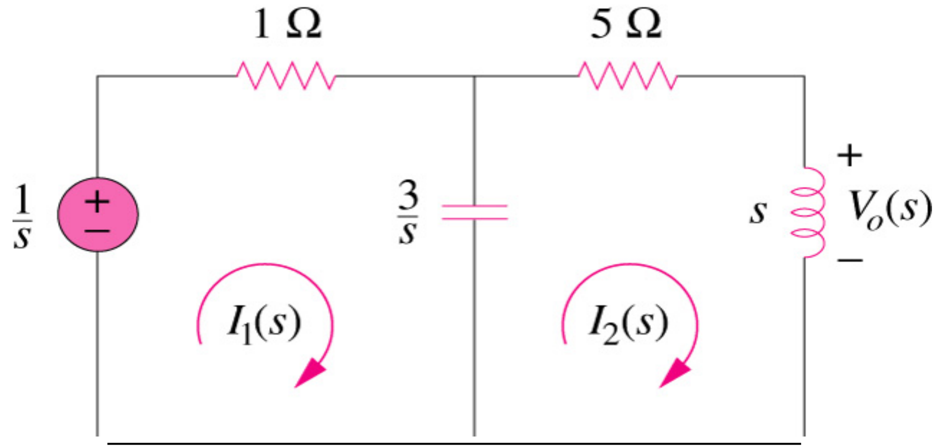
Solution:

Transform the circuit from the time domain to the s-domain:

$$u(t) \Rightarrow \frac{1}{s}$$

$$1\,\mathrm{H} \Rightarrow sL = s$$

$$\frac{1}{3}\mathrm{F} \Rightarrow \frac{1}{sC} = \frac{3}{s}$$



Apply mesh analysis, on solving for $V_0(s)$:

$$V_0(s) = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s+4)^2 + (\sqrt{2})^2}$$

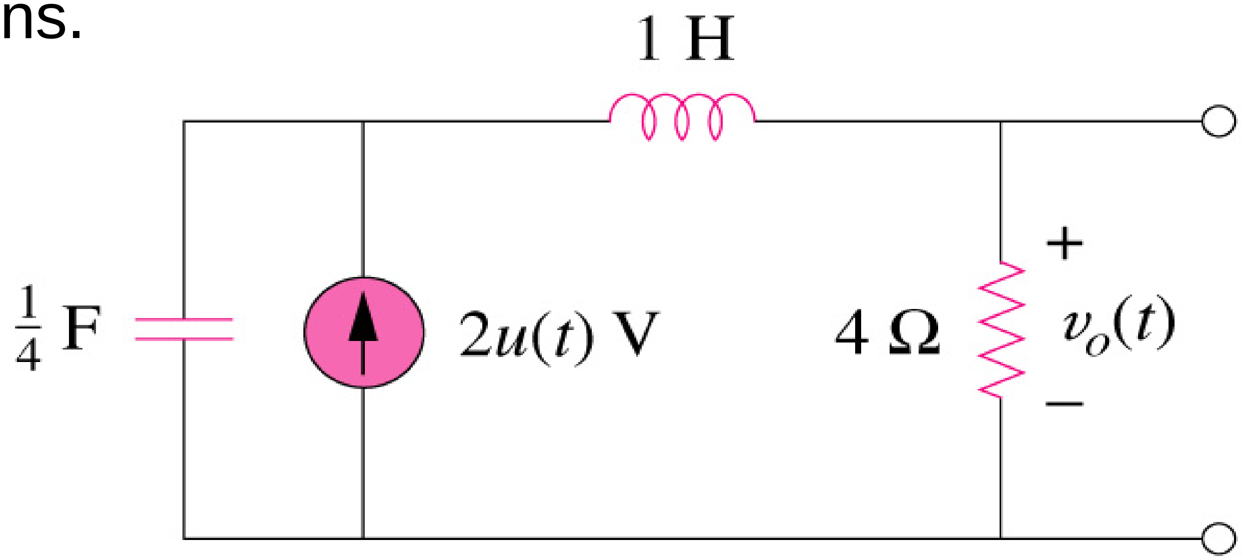
$$v_0(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin(\sqrt{2}t) \ \mathrm{V}, \quad t \geq 0$$

Inverse transform

Circuit Element Models Examples (3)

Example 2:

Determine $v_0(t)$ in the circuit shown below, assuming zero initial conditions.

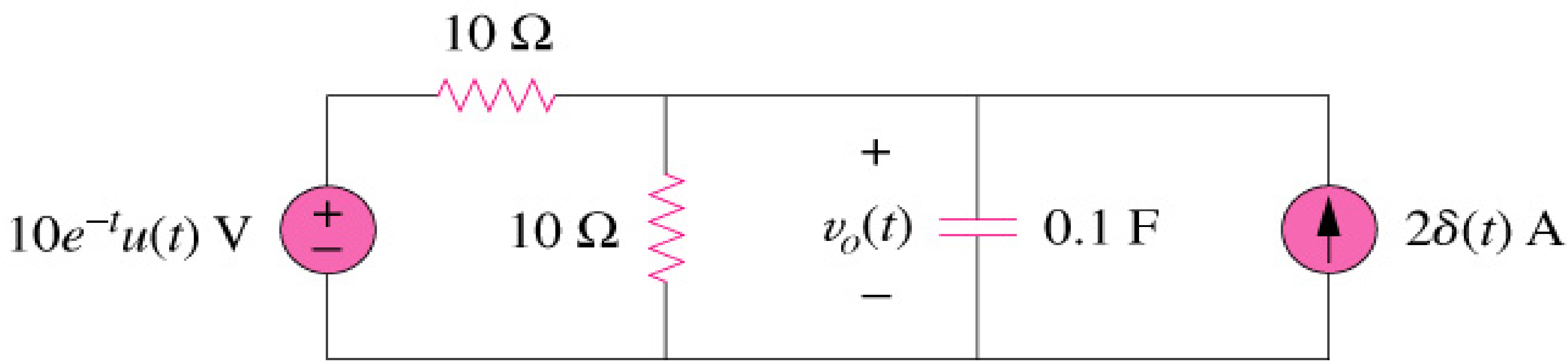


$$\text{Ans:}\quad 8(1-e^{-2t}-2te^{-2t})u(t)\quad \text{V}$$

Circuit Element Models Examples (4)

Example 3:

Find $v_0(t)$ in the circuit shown below. Assume $v_0(0)=5V$.

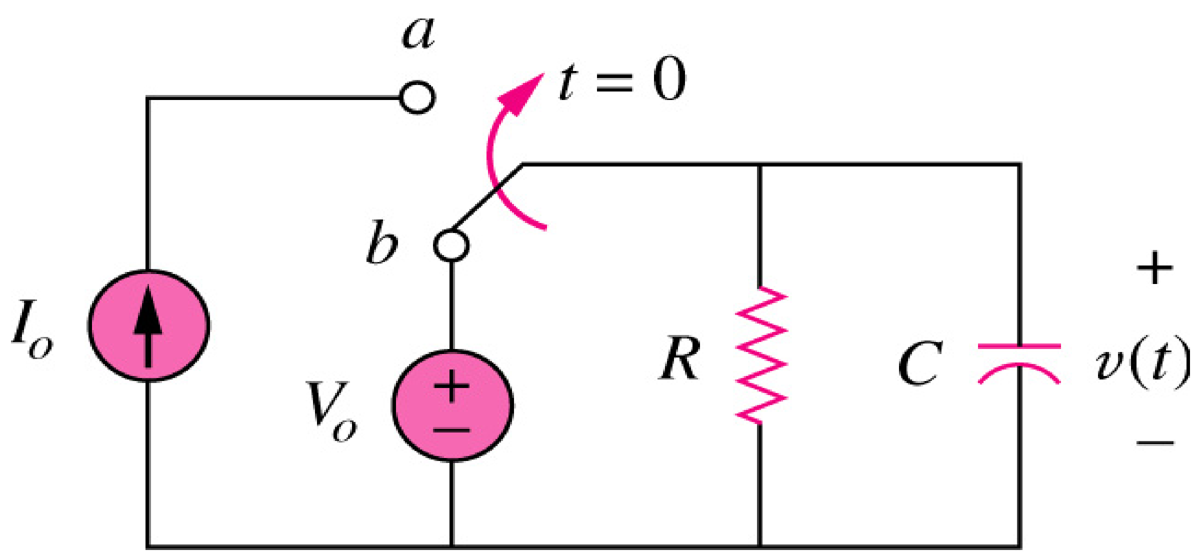


Ans: $\mathrm{v}_0(t) = (10e^{-t} + 15e^{-2t})u(t) \quad \mathrm{V}$

Circuit Element Models Examples (5)

Example 4:

The switch shown below has been in position *b* for a long time. It is moved to position *a* at *t=0*. Determine *v(t)* for t > 0.



$$\text{Ans}: \quad v(t) = (V_0 - I_0 R)e^{-t/\tau} + I_0 R, \ \ t > 0, \text{ where } \tau = RC$$

Circuit Analysis

- Circuit analysis is relatively easy to do in the s-domain.

- By transforming a complicated set of mathematical relationships in the time domain into the s-domain where we convert operators (derivatives and integrals) into simple multipliers of s and 1/s.

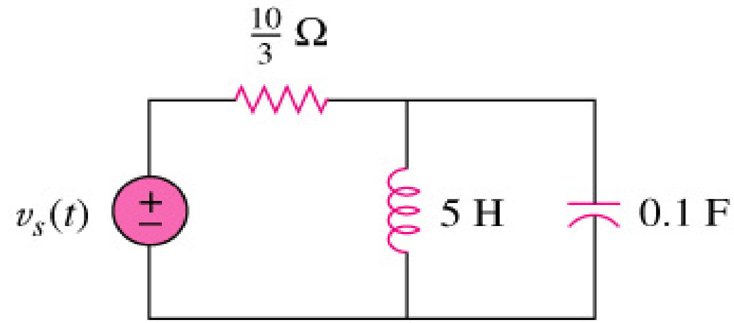
- This allow us to use algebra to set up and solve the circuit equations.

- In this case, all the circuit theorems and relationships developed for dc circuits are perfectly valid in the s-domain.

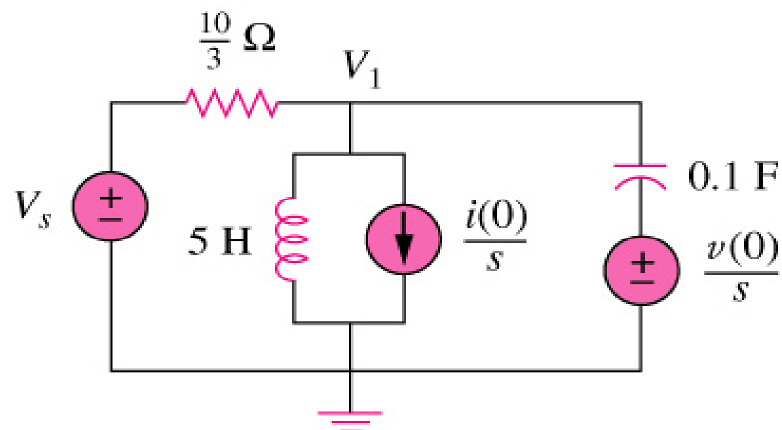
Circuit Analysis Example (1)

Example:

Consider the circuit below. Find the value of the voltage across the capacitor assuming that the value of $v_s(t)=10u(t)$ V and assume that at t=0, -1A flows through the inductor and +5V is across the capacitor.



(a)



(b)

Circuit Analysis Example (2)

Solution:

Transform the circuit from time-domain (a) into s-domain (b) using Laplace Transform. On rearranging the terms, we have

$$\boxed{\mathrm{V}_1 = \frac{35}{s+1} - \frac{30}{s+2}}$$

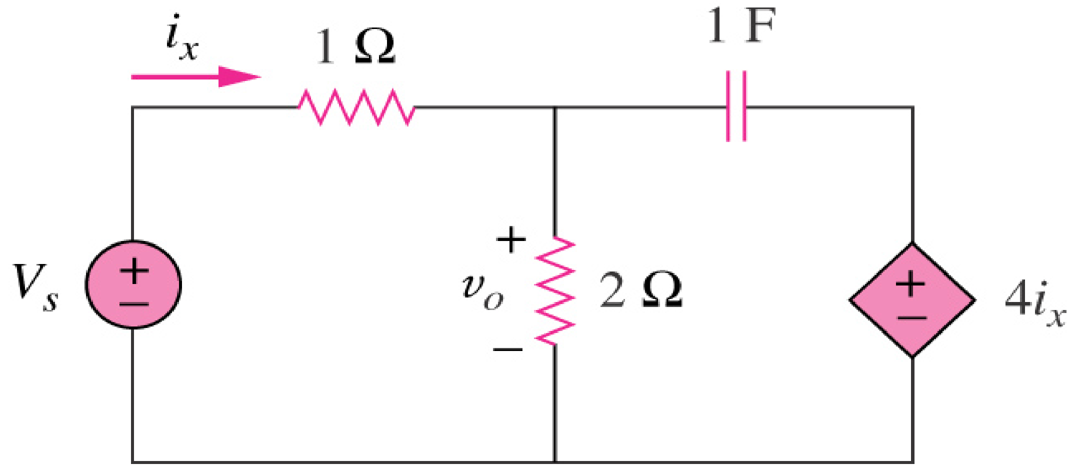
By taking the inverse transform, we get

$$\boxed{\mathrm{v}_1(t) = (35e^{-t} - 30e^{-2t})u(t) \quad \mathrm{V}}$$

Circuit Analysis Example (3)

Example:

The initial energy in the circuit below is zero at $t=0$. Assume that $v_s=5u(t)$ V. (a) Find $V_0(s)$ using the Thevenin theorem. (b) Apply the initial- and final-value theorem to find $v_0(0)$ an $v_0(\infty)$. (c) Obtain $v_0(t)$.



Ans: (a) $V_0(s) = 4(s+0.25)/(s(s+0.3))$ (b) 4,3.333V, (c) (3.333+0.6667e-0.3t)u(t) V.

Transfer Functions

- The transfer function H(s) is the ratio of the output response Y(s) to the input response X(s), assuming all the initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

h(t) is the impulse response function.

- Four types of gain:

1. H(s) = voltage gain = $V_0(s)/V_i(s)$
2. H(s) = Current gain = $I_0(s)/I_i(s)$
3. H(s) = Impedance = V(s)/I(s)
4. H(s) = Admittance = I(s)/V(s)

Transfer Functions Example (1)

Example:

The output of a linear system is $y(t)=10e^{-t}\cos 4t$ when the input is $x(t)=e^{-t}u(t)$. Find the transfer function of the system and its impulse response.

Solution:

Transform y(t) and x(t) into s-domain and apply H(s)=Y(s)/X(s), we get

$$\boxed{H(s)=\frac{Y(s)}{X(s)}=\frac{10(s+1)^2}{(s+1)^2+16}=10-40\frac{4}{(s+1)^2+16}}$$

Apply inverse transform for H(s), we get

$$\boxed{h(t)=10\delta(t)-40e^{-t}\sin(4t)u(t)}$$

Transfer Functions Example (2)

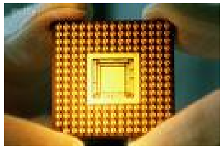
Example:

The transfer function of a linear system is

$$\boxed{H(s) = \frac{2s}{s+6}}$$

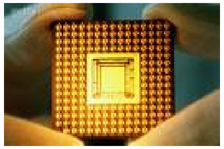
Find the output y(t) due to the input $e^{-3t} \cdot u(t)$ and its impulse response.

$$\boxed{\text{Ans}: \; -2e^{-3t} + 4e^{-6t}, \; \mathrm{t} \geq 0; \; 2\delta(\mathrm{t}) - 12\mathrm{e}^{-6\mathrm{t}} u(t)}$$



BSC Modul 4: Advanced Circuit Analysis

- Fourier Series
- Fourier Transform
- Laplace Transform
- Applications of Laplace Transform
- **Z-Transform**



Introduction

In continuous systems Laplace transforms play a unique role. They allow system and circuit designers to analyze systems and predict performance, and to think in different terms - like frequency responses - to help understand linear continuous systems.

Z-transforms play the role in sampled systems that Laplace transforms play in continuous systems.

In continuous systems, inputs and outputs are related by differential equations and Laplace transform techniques are used to solve those differential equations.

In sampled systems, inputs and outputs are related by difference equations and Z-transform techniques are used to solve those differential equations.

Fourier, Laplace and Z-Transforms

For right-sided signals (zero-valued for negative time index) the Laplace transform is a generalization of the Fourier transform of a continuous-time signal, and the z-transform is a generalization of the Fourier transform of a discrete-time signal.

Fourier Transform

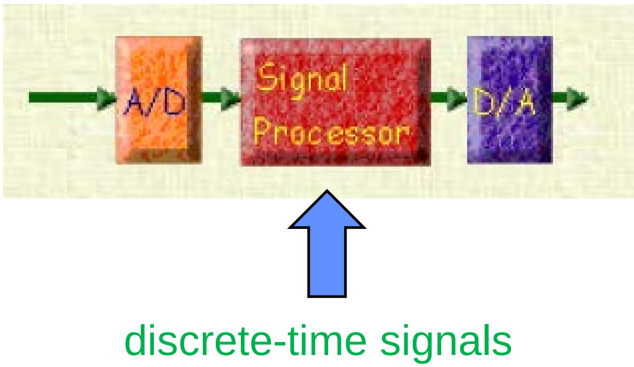
generalization

Laplace Transform | Z-Transform

continuous-time signals | discrete-time signals

The **Z-transform** converts a discrete time-domain signal, which is a sequence of real or complex numbers, into a complex frequency-domain representation.
It can be considered as a discrete-time equivalent of the Laplace transform.

There are numerous sampled systems that look like the one shown below.

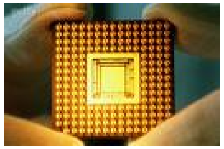


Definition of the Z-Transform

Let us assume that we have a sequence, y_k.
The subscript "k" indicates a sampled time interval and that y_k is the value of y(t) at the k^{th} sample instant.

y_k could be generated from a sample of a time function.
For example: $y_k = y(kT)$, where y(t) is a continuous time function, and T is the sampling interval.
We will focus on the index variable k, rather than the exact time kT, in all that we do in the following.

$$Z[y_k] = \sum_{k=0}^{\infty} y_k z^{-k}$$



Z-Transform Example

Given the following sampled signal:

$$y_k = y_0 \cdot a^k$$

We get the Z-Transform for $y_0 = 1$

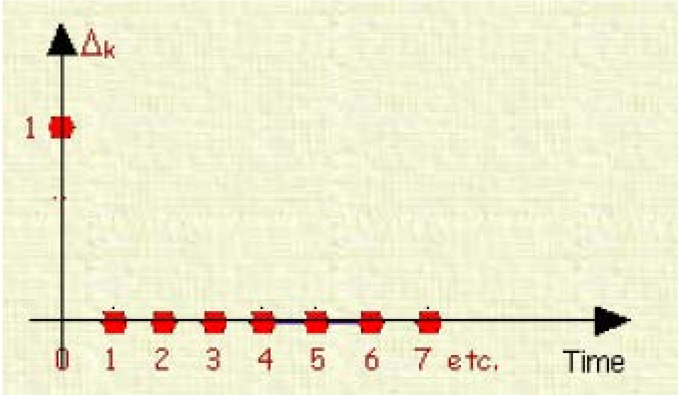
$$Z\left[1 \cdot a^k\right] = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

Z-Transform of Unit Impulse and Unit Step

Given the following sampled signal D_k:

D_k is zero for k>0, so all those terms are zero.
D_k is one for k = 0, so that

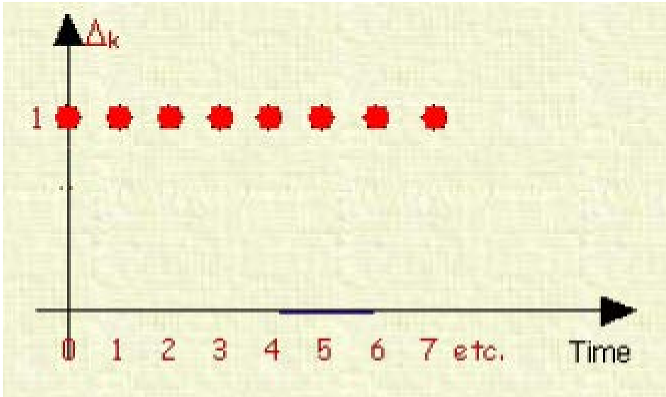
$$Z\left[D_k\right]=1$$

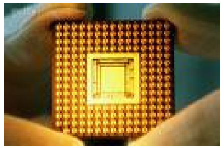


Given the following sampled signal u_k:

u_k is one for all k.

$$Z\left[u_k\right]=1+z^{-1}+z^{-2}+z^{-3}\ldots=\frac{z}{z-1}$$

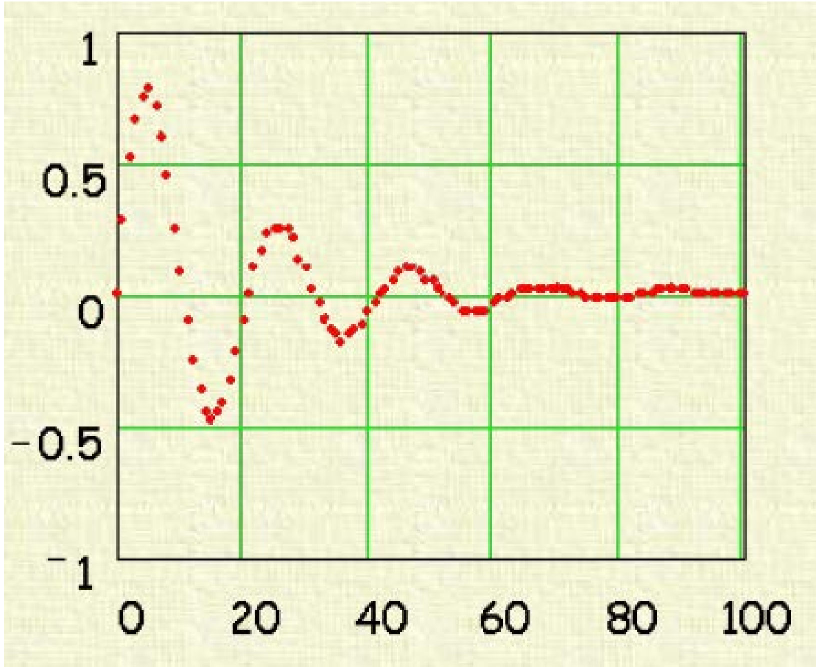




More Complex Example of Z-Transform

Given the following sampled signal f_k:

$$f_k = f(kT) = e^{-akT}\sin(bkT)$$



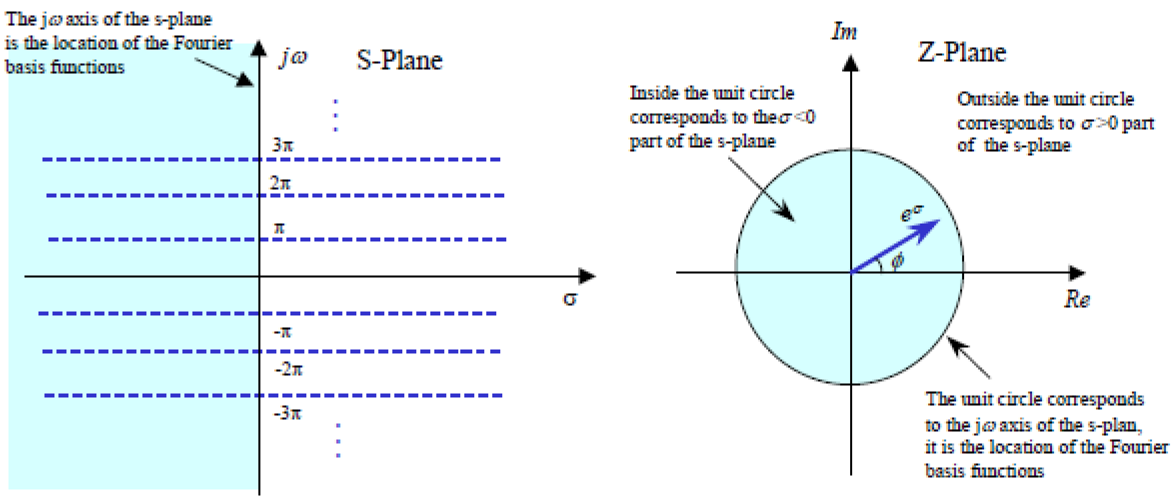
$$Z[f_k] = \sum_{k=0}^{\infty} f_k z^{-k} = \sum_{k=0}^{\infty} e^{-akT}\sin(bkT) z^{-k}$$

Finally:

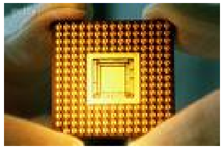
$$Z[f_k] = \frac{1}{2j}\left[\frac{z}{z-c} + \frac{z}{z-c^*}\right]$$

where $c = e^{-aT+jbT}$

S- and Z-Plane Presentation



(a) (b)



Inverse Z-Transform

The inverse *z*-transform can be obtained using one of two methods:

a) the inspection method,
b) the partial fraction method.

In the inspection method each simple term of a polynomial in z, $H(z)$, is substituted by its time-domain equivalent.

For the more complicated functions of *z*, the partial fraction method is used to describe the polynomial in terms of simpler terms, and then each simple term is substituted by its time-domain equivalent term.