Traveling Salesperson Problem: Exact Formulations

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1 Miller Tucker Zemlin Formulation

$$Min \sum_{(i,j)\in A} C_{ij}X_{ij} \tag{1}$$

$$S.T. \sum_{j\in V} X_{ij} = 1 \ \forall i \in V \tag{2}$$

$$\sum_{j\in V} X_{ji} = 1 \ \forall i \in V \tag{3}$$

$$u_i - u_j + nX_{ij} \leq n - 1 \ \forall i, j \in V \setminus \{1\} \tag{4}$$

$$u_i \geq 0 \ \forall i \in V \tag{5}$$

$$X_{ij} \in \{0, 1\} \ \forall (i, j) \in A \tag{6}$$

Instance size	Instance ID	Linear relaxation	Best integer objective value	Optimality gap	Running time (s)
15	1	55.35	65.13	0%	0.31
	2	51.45	71.82	0%	0.54
	3	50.82	60.94	0%	0.23
	4	43.20	57.79	0%	0.29
	5	42.56	59.33	0%	0.25
20	1	47.25	68.37	0%	0.51
	2	61.23	77.84	0%	1.52
	3	54.69	67.77	0%	0.44
	4	66.36	77.31	0%	0.81
	5	58.03	71.86	0%	0.10
25	1	57.36	79.48	0%	4.41
	2	64.26	83.08	0%	1.94
	3	62.31	72.54	0%	1.84
	4	60.32	82.21	0%	2.95
	5	66.72	79.09	0%	0.41
30	1	68.57	89.62	0%	11.14
	2	57.69	88.60	0%	5.53
	3	68.81	79.54	0%	3.5
	4	58.30	84.40	0%	10.07
	5	73.05	85.75	0%	2.73

Table 1: Summary of the results of MTZ formulation

2 Multi-commodity Flow Formulation

$$Min \sum_{(i,j)\in E} C_{ij}X_{ij} \tag{7}$$

$$S.T. \sum_{j\in V} X_{ij} = 1 \; \forall i \in V \tag{8}$$

$$\sum_{i\in V} X_{ij} = 1 \; \forall j \in V \tag{9}$$

$$\sum_{j:(i,j)\in E} y^k_{ij} - \sum_{j:(j,i)\in E} y^k_{ji} = \delta^k_i \; \forall i \in V, k = \{2, ..., n\} \tag{10}$$

$$y^k_{ij} \leq X_{ij} \; \forall (i,j) \in E, k = \{2, ..., n\} \tag{11}$$

$$y_{ij} \geq 0 \; \forall (i,j) \in E, k = \{2, ..., n\} \tag{12}$$

$$X_{ij} \in \{0,1\} \; \forall (i,j) \in E, k = \{2, ..., n\} \tag{13}$$

$$\delta^k_i = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Instance size	Instance ID	Linear relaxation	Best integer objective value	Optimality gap	Running time (s)
15	1	65.13	65.13	0%	0.32
	2	71.82	71.82	0%	0.30
	3	60.94	60.94	0%	0.29
	4	57.79	57.79	0%	0.30
	5	59.33	59.33	0%	0.31
20	1	68.37	68.37	0%	0.88
	2	77.84	77.84	0%	0.90
	3	67.77	67.77	0%	0.88
	4	77.31	77.31	0%	1.61
	5	71.86	71.86	0%	0.82
25	1	79.48	79.48	0%	2.08
	2	83.08	83.08	0%	2.31
	3	72.54	72.54	0%	1.31
	4	80.21	82.21	0%	2.06
	5	79.09	79.09	0%	2.05
30	1	89.62	89.62	0%	5.4
	2	88.60	88.60	0%	4.5
	3	79.54	79.54	0%	6.45
	4	84.40	84.40	0%	4.07
	5	85.75	85.75	0%	8.05

Table 2: Summary of the results of MCF formulation

3 Shortest Path with Constraints Formulation

$$Min \sum_{i \in V} \sum_{j \in V \setminus \{i\}} C_{ij} \sum_{t=1}^{n} X_{ij}^{t} \tag{15}$$

$$S.T. \quad \sum_{j \in V \setminus \{1\}} X_{1j}^{1} = 1 \tag{16}$$

$$\sum_{j \in V \setminus \{1,i\}} X_{ij}^{2} - X_{1i}^{1} = 0 \ \forall i \in V \setminus \{1\} \tag{17}$$

$$\sum_{j \in V \setminus \{1,i\}} X_{ij}^{t} - \sum_{j \in V \setminus \{1,i\}} X_{ji}^{t-1} = 0 \ \forall i \in V \setminus \{1\}, t = \{2, ..., n-1\} \tag{18}$$

$$X_{i1}^{n} - \sum_{j \in V \setminus \{1,i\}} X_{ji}^{n-1} = 0 \ \forall i \in V \setminus \{1\} \tag{19}$$

$$\sum_{i \in V \setminus \{1\}} X_{i1}^{n} = 1 \tag{20}$$

$$\sum_{t=2}^{n-1} \sum_{j \in V \setminus 1,i} X_{ij}^{t} + X_{i1}^{n} \leq 1 \ \forall i \in V \setminus \{1\} \tag{21}$$

$$X_{ij}^{t} \in \{0, 1\} \ \forall i \in V, j \in V, t = \{1, ..., n\} \tag{22}$$

Instance size	Instance ID	Linear relaxation	Best integer objective value	Optimality gap	Running time (s)
15	1	58.55	65.13	0%	1.05
	2	64.04	71.82	0%	0.97
	3	57.09	60.94	0%	0.38
	4	50.26	57.79	0%	0.34
	5	48.70	59.33	0%	0.68
20	1	56.49	68.37	0%	14.62
	2	69.81	77.84	0%	11.26
	3	60.25	67.77	0%	1.66
	4	69.94	77.31	0%	4.62
	5	64.47	71.86	0%	0.81
25	1	64.19	79.48	0%	70.1
	2	71.43	83.08	0%	59.11
	3	62.31	72.54	0%	71.84
	4	67.65	82.21	0%	32.75
	5	67.77	79.09	0%	4.41
30	1	71.66	89.62	0%	485
	2	57.69	88.60	0%	583
	3	68.81	79.54	0%	365
	4	58.30	84.40	0%	518
	5	73.05	85.75	0%	373

Table 3: Summary of the results of SPC formulation

4 Quadratic Formulation with Linearization

$$Min \sum_{i\in V}\sum_{j\in V\setminus\{i\}}\sum_{k=1}^{n-1} C_{ij}W_{ij}^{k} + \sum_{i\in V}\sum_{j\in V\setminus\{i\}} C_{ij}W_{ij}^{n} \tag{23}$$

$$S.T. \sum_{j=1}^{n} X_{ij} = 1 \; \forall i \in V \tag{24}$$

$$\sum_{i\in V} X_{ij} = 1 \forall j \in \{1,...,n\} \tag{25}$$

$$X_{ij} \in \{0,1\} \; \forall i \in V, j = \{1,...,n\} \tag{26}$$

$$W_{ij}^{k} \leq X_{ik} \; \forall i \in V, j \in V \setminus \{i\}, k = \{1,...,n-1\} \tag{27}$$

$$W_{ij}^{k} \leq X_{jk+1} \; \forall i \in V, j \in V \setminus \{i\}, k = \{1,...,n-1\} \tag{28}$$

$$W_{ij}^{k} \geq X_{ik} + X_{jk+1} - 1 \; \forall i \in V, j \in V \setminus \{i\}, k = \{1,...,n-1\} \tag{29}$$

$$W_{ij}^{n} \leq X_{in} \; \forall i \in V, j \in V \setminus \{i\}, k = n \tag{30}$$

$$W_{ij}^{n} \leq X_{j1} \; \forall i \in V, j \in V \setminus \{i\}, k = n \tag{31}$$

$$W_{ij}^{n} \geq X_{in} + X_{j1} - 1 \; \forall i \in V, j \in V \setminus \{i\}, k = n \tag{32}$$

Instance size	Instance ID	Linear relaxation	Best integer objective value	Optimality gap	Running time (s)
15	1	0	65.13	0%	6.64
	2	0	71.82	0%	5.68
	3	0	60.94	0%	12.01
	4	0	57.79	0%	57.64
	5	0	59.33	0%	14.41
20	1	0	68.37	0%	571
	2	0	77.84	0%	451
	3	0	67.77	0%	551
	4	0	77.31	0%	227
	5	0	71.86	0%	311
25	1	0	68.37	13% at $600s$	≥ 600
	2	0	77.84	21% at $600s$	≥ 600
	3	0	67.77	17% at $600s$	≥ 600
	4	0	77.31	24% at $600s$	≥ 600
	5	0	71.86	15% at $600s$	≥ 600
30	1	0	89.62	$\geq$ 50% at $600s$	≥ 600
	2	0	88.60	$\geq$ 50% at $600s$	≥ 600
	3	0	79.54	$\geq$ 50% at $600s$	≥ 600
	4	0	84.40	$\geq$ 50% at $600s$	≥ 600
	5	0	85.75	$\geq$ 50% at $600s$	≥ 600

Table 4: Summary of the results of Quad formulation

5 Dantzig Fulkerson Johnson Formulation: Subtour Elimination

$$Min \sum_{(i,j)\in A} C_{ij}X_{ij} \tag{33}$$

$$S.T. \sum_{j\in V} X_{ij} = 1 \; \forall i \in V \tag{34}$$

$$\sum_{j\in V} X_{ji} = 1 \; \forall i \in V \tag{35}$$

$$\sum_{j\in S} \sum_{i\in S\setminus\{j\}} X_{ij} \leq |S| - 1 \; \forall S \subset V, 2 \leq |S| \leq n-1 \tag{36}$$

$$X_{ij} \in \{0,1\} \; \forall (i,j) \in A \tag{37}$$

As the number of subsets S is exponential, the subtour elimination constraints are added sequentially after a solution violates such constraint.

Plain Loop

In this method, each time the model is solved to optimality. Then it is checked if the optimal solution violates any subtour elimination constraint or not. If any such constraint is violated, then the corresponding constraint is added to the model and again the model is solved to optimality.

Instance size	Instance ID	Linear relaxation	Best integer objective value	Optimality gap	Running time (s)
15	1	65.13	65.13	0%	0.12
	2	71.82	71.82	0%	0.15
	3	60.94	60.94	0%	0.10
	4	57.79	57.79	0%	0.17
	5	59.33	59.33	0%	0.18
20	1	68.37	68.37	0%	0.20
	2	77.84	77.84	0%	0.23
	3	67.77	67.77	0%	0.28
	4	77.31	77.31	0%	0.26
	5	71.86	71.86	0%	0.22
25	1	79.48	79.48	0%	0.28
	2	83.08	83.08	0%	0.35
	3	72.54	72.54	0%	0.31
	4	80.21	82.21	0%	0.26
	5	79.09	79.09	0%	0.31
30	1	89.62	89.62	0%	0.55
	2	88.60	88.60	0%	0.51
	3	79.54	79.54	0%	0.65
	4	84.40	84.40	0%	0.49
	5	85.75	85.75	0%	0.52

Table 5: Summary of the results of DFJ (plain loop) formulation

Callback or Lazy-cut

In this case, first a linear relaxation is solved. If the solution of the linear relaxation is an integer solution, then the separation algorithm is run to check if any subtour constraint is violated. If such constraint is violated, then the corresponding constraint is added to the formulation. If the solution of the linear relaxation is a fractional solution, then branch and bound is done to obtain an integer solution. If finally, a solution is obtained such that it is an integer and doesn't violate any subtour constraint, then this is a candidate solution, may not be optimal but can be used as an upper bound in branch and bound.

Instance size	Instance ID	Linear relaxation	Best integer objective value	Optimality gap	Running time (s)
15	1	59.65	65.13	0%	0.04
	2	62.45	71.82	0%	0.04
	3	52.82	60.94	0%	0.03
	4	49.17	57.79	0%	0.04
	5	52.46	59.33	0%	0.04
20	1	58.25	68.37	0%	0.05
	2	67.13	77.84	0%	0.06
	3	60.24	67.77	0%	0.04
	4	71.36	77.31	0%	0.07
	5	68.03	71.86	0%	0.03
25	1	68.16	79.48	0%	0.11
	2	70.34	83.08	0%	0.14
	3	66.61	72.54	0%	0.09
	4	73.50	82.21	0%	0.1
	5	71.72	79.09	0%	0.1
30	1	78.30	89.62	0%	.34
	2	80.68	88.60	0%	0.11
	3	73.83	79.54	0%	0.12
	4	72.80	84.40	0%	0.25
	5	80.57	85.75	0%	0.28

Table 6: Summary of the results of DFJ (callback) formulation

6 Discussion

First, we note that given sufficient time, all the above mentioned formulations are capable to yield the optimal solution. We identified that the DFJ formulations are the fastest among all the formulations. It can be observed that, how with increase in instance size, the running time increases significantly for the formulations. The running time of the SPC formulation is significantly high for larger instances. The linear relaxation of the quadratic formulation is 0 and the performance of this formulation is worst for larger instances.