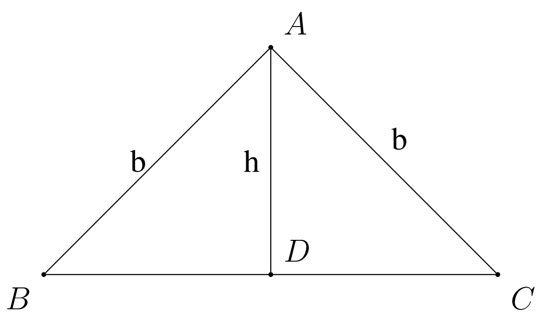
Assignment - 1

Prasanna Kumar R - SM21MTECH14001

Problem

An isoceles triangle has the extremities of its base at $\begin{pmatrix}2\\5\end{pmatrix}$ and $\begin{pmatrix}-2\\2\end{pmatrix}$. Find the two possible positions of the vertex if its area is 25 sq.units



Solution

Let the vertex B be $\begin{pmatrix}2\\5\end{pmatrix}$ and vertex C be $\begin{pmatrix}-2\\2\end{pmatrix}$.

Let the other vertex be A.

Since ABC is an Isoceles triangle,

$$\|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2$$

$$\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\mathbf{A}.\mathbf{B} = \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}.\mathbf{C}$$

$$\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2 = 2\mathbf{A}.\mathbf{B} - 2\mathbf{A}.\mathbf{C}$$

$$\|\mathbf{B}\|^2 - \|\mathbf{C}\|^2 = 2\mathbf{A}.(\mathbf{B}-\mathbf{C})$$

$$29 - 8 = 2.\mathbf{A}.\begin{pmatrix}4\\3\end{pmatrix}$$

$$\mathbf{A}.\begin{pmatrix}8\\6\end{pmatrix} = 21$$

If A is chosen as $\begin{pmatrix}x\\y\end{pmatrix}$,

$$8x + 6y = 21 \tag{1}$$

Given, the area of the triangle= 25 sq.units

The area of a triangle using vector product is obtained as

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\| = 25$$

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times\begin{pmatrix}4\\3\end{pmatrix}\right\| = 25$$

$$\left\|\begin{pmatrix}x-2\\y-5\end{pmatrix}\times\begin{pmatrix}4\\3\end{pmatrix}\right\| = 50$$

$$3x - 4y = 36 \tag{2}$$

Considering negative area for another equation,

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\| = -25$$

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times\begin{pmatrix}4\\3\end{pmatrix}\right\| = -25$$

$$\left\|\begin{pmatrix}x-2\\y-5\end{pmatrix}\times\begin{pmatrix}4\\3\end{pmatrix}\right\| = -50$$

$$-3x + 4y = 64 \tag{3}$$

Let us consider the matrices representation of equations (1) and (2),

$$\begin{bmatrix}3 & -4\\8 & 6\end{bmatrix}\mathbf{X_1} = \begin{bmatrix}36\\21\end{bmatrix}$$

$$\mathbf{AX_1} = \mathbf{B}$$

$$\mathbf{X_1} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{X_1} = \begin{bmatrix}3 & -4\\8 & 6\end{bmatrix}^{-1}\begin{bmatrix}36\\21\end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix}6 & 4\\-8 & 3\end{bmatrix}\begin{bmatrix}36\\21\end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix}300\\-225\end{bmatrix}$$

$$\mathbf{X_1} = \begin{bmatrix}6\\-4.5\end{bmatrix}$$

Let us consider the matrices representation of equations (1) and (3),

$$\begin{bmatrix} -3 & 4 \\ 8 & 6 \end{bmatrix} \mathbf{X_2} = \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$\mathbf{AX_2} = \mathbf{B}$$

$$\mathbf{X_2} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{X_2} = \begin{bmatrix} -3 & 4 \\ 8 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 64 \\ 21 \end{bmatrix}$$

$$= \frac{1}{-50} \begin{bmatrix} 300 \\ -575 \end{bmatrix}$$

$$\mathbf{X_2} = \begin{bmatrix} -6 \\ 11.5 \end{bmatrix}$$

$\therefore$ **The two possible vertex are** $(6, -4.5)$ **and** $(-6, 11.5)$