Assignment-2

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and latex-tikz codes from

Problem GATE EC 57

Let X and Y be jointly distributed random variables such that the conditional distribution of Y, given X $=x$, is uniform on the interval $(x-1, x+1)$. Suppose $E(X) = 1$ and $Var(X) = \frac{5}{3}$.
Variance of the random variable Y is

(A) $\frac{1}{2}$ (C) 1

(B) $\frac{2}{3}$ (D) 2

Solution

We know that

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} \tag{0.0.1}$$

Given, $f_{Y|X=x}(y)$ is uniform over the interval (x-1,x+1).

$$f_{Y|X=x}(y) = \begin{cases} \frac{1}{2} & y \in (x-1, x+1) \\ 0 & \text{otherwise} \end{cases} \tag{0.0.2}$$

Given $E(X) = 1$

$$\int_{-\infty}^{\infty} x f_X(x) dx = 1 \tag{0.0.3}$$

Now consider $E(Y|X = x)$,

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y) dy \tag{0.0.4}$$

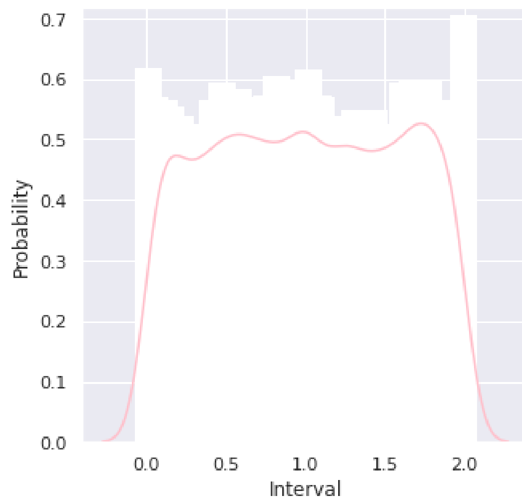

Fig. 4: Distribution of $f_{(Y|X=1)}(y)$

From (0.0.4)

$$E(Y|X = x) = \int_{-\infty}^{x-1} y f_{Y|X=x}(y) dy + \int_{x-1}^{x+1} y f_{Y|X=x}(y) dy + \int_{x+1}^{\infty} y f_{Y|X=x}(y) dy \tag{0.0.5}$$

$$E(Y|X = x) = \int_{x-1}^{x+1} y \left(\frac{1}{2}\right) dy \tag{0.0.6}$$

$$= x \tag{0.0.7}$$

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx \tag{0.0.8}$$

$$= \int_{-\infty}^{\infty} x f_X(x) dx \tag{0.0.9}$$

$$= E(X) \tag{0.0.10}$$

From (0.0.3) we get

$$E(Y) = 1 \tag{0.0.11}$$

$$Var(Y|X=x) = \int_{-\infty}^{\infty} (y - E(Y))^2 f_{Y|X=x}(y) dy \tag{0.0.12}$$

$$= \int_{x-1}^{x+1} (y-1)^2 \left(\frac{1}{2}\right) dy \tag{0.0.13}$$

$$Var(Y) = \int_{-\infty}^{\infty} Var(Y|X=x) f_X(x) dx \tag{0.0.14}$$

$$= \left(\frac{1}{2}\right) \int_{x-1}^{x+1} (y^2 - 2y + 1) dy \tag{0.0.15}$$

$$= \left(\frac{1}{2}\right) \left(\frac{6x^2+2}{3} + 2 - 4x\right) \tag{0.0.16}$$

$$= x^2 - 2x + \frac{4}{3} \tag{0.0.17}$$

$$Var(Y) = \int_{-\infty}^{\infty} (x^2 - 2x + \frac{4}{3}) f_X(x) dx \tag{0.0.18}$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x) dx - 2 \int_{-\infty}^{\infty} x f_X(x) dx + \frac{4}{3} \int_{-\infty}^{\infty} f_X(x) dx \tag{0.0.19}$$

$$f_X(x) dx = 1 \tag{0.0.20}$$

$$Var(X) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \frac{5}{3} \tag{0.0.21}$$

$$E(x) = \int_{-\infty}^{\infty} x f_X(x) dx = 1 \tag{0.0.22}$$

$$\tag{0.0.23}$$

From (0.0.19), (0.0.20), (0.0.21) and (0.0.22) we get

$$Var(Y) = \frac{5}{3} - 2 + \frac{4}{3} \tag{0.0.24}$$

$$= 1 \tag{0.0.25}$$

$\therefore$ **Option C is true**