Assignment-4

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Download all latex codes from:

https://github.com/PrasannaLanka/Assignment4/blob/main/Assignment4/codes/Assignment4.tex

and python codes from

https://github.com/PrasannaLanka/Assignment4/blob/main/Assignment4/codes/Assignment4.py

Problem: CSIR UGC NET EXAM (June 2013), Q.84

Let X_1, X_2, X_3, X_4, X_5 be independent and identically distributed random variables each following a uniform distribution on (0,1) and M denote their median. Then which of the following statements are true?

1) $\Pr\left(M < \frac{1}{3}\right) = \Pr\left(M > \frac{2}{3}\right)$
2) M is uniformly distributed on (0,1)
3) $E(M) = E(X_1)$
4) $V(M) = V(X_1)$

solution

Theorem 0.1 (Uniform distribution). *A random variable X is said to be uniformly distributed in $a \le x \le b$ if its density function is*

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

and the distribution is called uniform distribution. The mean and variance are respectively,

$$\mu = \frac{a+b}{2} \tag{0.0.2}$$

$$\sigma^2 = \frac{(b-a)^2}{12} \tag{0.0.3}$$

Theorem 0.2 (Beta distribution). *The Beta distribution is a continuous distribution defined on the range (0, 1) where the parameters are given by If $X \sim B(r,s)$, where $B(r,s)$ is a beta function*

$$f(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} \tag{0.0.4}$$

$$F(x) = \int_0^x \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} dx = \frac{B_x(r,s)}{B(r,s)} \tag{0.0.5}$$

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1} dx = \frac{(r-1)!(s-1)!}{(r+s-1)!} \tag{0.0.6}$$

$$B_x(r,s) = \int_0^x x^{r-1}(1-x)^{s-1} dx \tag{0.0.7}$$

$$E(X) = \frac{r}{r+s} \tag{0.0.8}$$

$$Var(X) = \frac{rs}{(r+s)^2(r+s+1)} \tag{0.0.9}$$

Definition 0.1 (Order statistics). *For given statistical sample $\{X_1, X_2, \cdots X_n\}$, the order statistics is obtained by sorting the sample in ascending order. It denoted as $\{X_{(1)}, X_{(2)}, \cdots X_{(n)}\}$.*

Definition 0.2 (Median of order statistics). *Median is defined as the middle number of a sorted sample. It is denoted by M and defined using order statistics of a sample as*

$$M = \begin{cases} X_{((n+1)/2)}, & \text{if } n \text{ is odd,} \\ \dfrac{X_{(n/2)} + X_{(n/2+1)}}{2}, & \text{if } n \text{ is even,} \end{cases} \tag{0.0.10}$$

Remark 0.3. *The order statistics of the uniform distribution on the unit interval have marginal distributions belonging to the Beta distribution family.*

$$X_{(k)} \sim B(k, n+1-k) \tag{0.0.11}$$

1) From definition (0.2) median M is given by

$$M = X_{((5+1)/2)} \tag{0.0.12}$$
$$= X_{(3)} \tag{0.0.13}$$

From remark (0.3)

$$X_{(3)} \sim B(3,3) \tag{0.0.14}$$

From (0.0.6)

$$B(3,3) = \frac{(3-1)!(3-1)!}{(3+3-1)!} = \frac{1}{30} \tag{0.0.15}$$

From (0.0.4)

$$f(x) = 30x^2(1-x)^2 \tag{0.0.16}$$

From (0.0.5)

$$F(x) = \int_0^x 30x^2(1-x)^2 dx \tag{0.0.17}$$
$$= 30x^3\left(\frac{1}{3} + \frac{x^2}{5} - \frac{x}{2}\right) \tag{0.0.18}$$

$$\Pr\left(M < \frac{1}{3}\right) = F\left(\frac{1}{3}\right) = 0.20987 \tag{0.0.19}$$
$$\Pr\left(M > \frac{2}{3}\right) = F(1) - F\left(\frac{2}{3}\right) = 0.20987 \tag{0.0.20}$$

$$\therefore \Pr\left(M < \frac{1}{3}\right) = \Pr\left(M > \frac{2}{3}\right) \tag{0.0.21}$$

Hence **Option 1 is true.**

2) From (0.0.13), median M is a third order statistic. Clearly from remark (0.3), M is not an uniform distribution.
Hence **Option 2 is false.**

3) From (0.0.1)

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.22}$$

From (0.0.2)

$$E(X_1) = \frac{1}{2} \tag{0.0.23}$$

From (0.0.8)

$$E(M) = \frac{3}{3+3} = \frac{1}{2} \tag{0.0.24}$$

$$\therefore E(M) = E(X_1) \tag{0.0.25}$$

Hence **Option 3 is true.**

4) From (0.0.3)

$$V(X_1) = \frac{1}{12} \tag{0.0.26}$$

From (0.0.9)

$$V(M) = \frac{1}{28} \tag{0.0.27}$$

$$\therefore V(M) \neq V(X_1) \tag{0.0.28}$$

Hence **Option 4 is false.**