

CSIR UGC NET EXAM (June 2013), Q.84

Lanka Prasanna

CS20BTECH11029

Uniform Distribution

A random variable X is said to be uniformly distributed in $a \leq x \leq b$ if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

and the distribution is called uniform distribution. The mean and variance are respectively,

$$\mu = \frac{a+b}{2} \quad (2)$$

$$\sigma^2 = \frac{(b-a)^2}{12} \quad (3)$$

Beta distribution

The Beta distribution is a continuous distribution defined on the range (0,1) where the parameters are given by

If $X \sim B(r, s)$, where $B(r, s)$ is a beta function

$$f(x) = \frac{1}{B(r, s)} x^{r-1} (1-x)^{s-1} \quad (4)$$

$$F(x) = \int_0^x \frac{1}{B(r, s)} x^{r-1} (1-x)^{s-1} dx = \frac{B_x(r, s)}{B(r, s)} \quad (5)$$

$$B(r, s) = \int_0^1 x^{r-1} (1-x)^{s-1} dx = \frac{(r-1)!(s-1)!}{(r+s-1)!} \quad (6)$$

$$B_x(r, s) = \int_0^x x^{r-1} (1-x)^{s-1} dx \quad (7)$$

$$E(X) = \frac{r}{r+s} \quad (8)$$

$$Var(X) = \frac{rs}{(r+s)^2(r+s+1)} \quad (9)$$

Order statistics

For given statistical sample $\{X_1, X_2, \dots, X_n\}$, the order statistics is obtained by sorting the sample in ascending order. It denoted as $\{X_{(1)}, X_{(2)}, \dots, X_{(n)}\}$.

Median of order statistics

Median is defined as the middle number of a sorted sample. It is denoted by M and defined using order statistics of a sample as

$$M = \begin{cases} X_{((n+1)/2)}, & \text{if } n \text{ is odd,} \\ \frac{X_{(n/2)} + X_{(n/2+1)}}{2}, & \text{if } n \text{ is even,} \end{cases} \quad (10)$$

Note

The order statistics of the uniform distribution on the unit interval have marginal distributions belonging to the Beta distribution family.

$$X_{(k)} \sim B(k, n + 1 - k) \quad (11)$$

Question

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Let X_1, X_2, X_3, X_4, X_5 be independent and identically distributed random variables each following a uniform distribution on $(0,1)$ and M denote their median. Then which of the following statements are true?

- ① $\Pr(M < \frac{1}{3}) = \Pr(M > \frac{2}{3})$
- ② M is uniformly distributed on $(0,1)$
- ③ $E(M) = E(X_1)$
- ④ $V(M) = V(X_1)$

Solution

The order statistics of given sample is $\{X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}\}$.
From definition (4) median M is given by

$$M = X_{((5+1)/2)} \quad (12)$$

$$= X_{(3)} \quad (13)$$

From note (5)

$$X_{(3)} \sim B(3, 3) \quad (14)$$

Option 1

From equation (6), Beta function of median is given by

$$B(3, 3) = \frac{(3-1)!(3-1)!}{(3+3-1)!} = \frac{1}{30} \quad (15)$$

From equation (4), PDF of median is

$$f(x) = 30x^2(1-x)^2 \quad (16)$$

From equation (5), CDF of median is

$$F(x) = \int_0^x 30x^2(1-x)^2 dx \quad (17)$$

$$= 30x^3 \left(\frac{1}{3} + \frac{x^2}{5} - \frac{x}{2} \right) \quad (18)$$

$$\Pr\left(M < \frac{1}{3}\right) = F\left(\frac{1}{3}\right) \quad (19)$$

$$= 0.20987 \quad (20)$$

$$\Pr\left(M > \frac{2}{3}\right) = \Pr(M < 1) - \Pr\left(M < \frac{2}{3}\right) \quad (21)$$

$$= F(1) - F\left(\frac{2}{3}\right) \quad (22)$$

$$= 0.20987 \quad (23)$$

$$\therefore \Pr\left(M < \frac{1}{3}\right) = \Pr\left(M > \frac{2}{3}\right) \quad (24)$$

Hence **Option 1 is true.**

Option 2

From (13), median M is a third order statistic. Clearly from note (5), M is a Beta distribution whose PDF is given by

$$f(x) = \begin{cases} 30x^2(1-x)^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

which is not a uniform distribution.

Hence **Option 2 is false.**

Option 3

From equation (1), density function for each of given random variables X_1, X_2, X_3, X_4, X_5 is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (26)$$

From equation (2)

$$E(X_1) = \frac{1}{2} \quad (27)$$

From equation (8)

$$E(M) = \frac{3}{3+3} = \frac{1}{2} \quad (28)$$

$$\therefore E(M) = E(X_1) \quad (29)$$

Hence **Option 3 is true.**

Option 4

From equation (3)

$$V(X_1) = \frac{1}{12} \quad (30)$$

From equation (9)

$$V(M) = \frac{1}{28} \quad (31)$$

$$\therefore V(M) \neq V(X_1) \quad (32)$$

Hence **Option 4 is false.**