CSIR UGC NET EXAM (June 2013), Q.84

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Topics covered

Prerequisites:

- Uniform Distribution and its properties
- Beta distribution and its properties
- Definition of order statistics
- Median of order statistics
- Important note

UGC NET problem

- Question
- Solution

Uniform distribution

A random variable X is said to be uniformly distributed in $a \leq x \leq b$ if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and the distribution is called uniform distribution. The mean and variance are respectively,

$$\mu = \frac{a+b}{2} \tag{2}$$

$$\sigma^2 = \frac{(b-a)^2}{12} \tag{3}$$

Beta distribution

The Beta distribution is a continuous distribution defined on the range (0,1) where the parameters are given by
If $X \sim B(r,s)$, where $B(r,s)$ is a beta function

$$f(x) = \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} \tag{4}$$

$$F(x) = \int_0^x \frac{1}{B(r,s)} x^{r-1}(1-x)^{s-1} dx = \frac{B_x(r,s)}{B(r,s)} \tag{5}$$

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1} dx = \frac{(r-1)!(s-1)!}{(r+s-1)!} \tag{6}$$

$$B_x(r,s) = \int_0^x x^{r-1}(1-x)^{s-1} dx \tag{7}$$

$$E(X) = \frac{r}{r+s} \tag{8}$$

$$Var(X) = \frac{rs}{(r+s)^2(r+s+1)} \tag{9}$$

Order statistics

For given statistical sample $\{X_1, X_2, \cdots X_n\}$, the order statistics is obtained by sorting the sample in ascending order. It denoted as $\{X_{(1)}, X_{(2)}, \cdots X_{(n)}\}$.

Median of order statistics

Median is defined as the middle number of a sorted sample. It is denoted by M and defined using order statistics of a sample as

$$M = \begin{cases} X_{((n+1)/2)}, & \text{if } n \text{ is odd,} \\ \dfrac{X_{(n/2)} + X_{(n/2+1)}}{2}, & \text{if } n \text{ is even,} \end{cases} \tag{10}$$

Note

The order statistics of the uniform distribution on the unit interval have marginal distributions belonging to the Beta distribution family.

$$X_{(k)} \sim B(k, n+1-k) \tag{11}$$

Question

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Let X_1, X_2, X_3, X_4, X_5 be independent and identically distributed random variables each following a uniform distribution on (0,1) and M denote their median. Then which of the following statements are true?

1. $\Pr\left(M < \frac{1}{3}\right) = \Pr\left(M > \frac{2}{3}\right)$
2. M is uniformly distributed on (0,1)
3. $E(M) = E(X_1)$
4. $V(M) = V(X_1)$

Solution

The order statistics of given sample is $\{X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}\}$.
From definition (5) median M is given by

$$M = X_{((5+1)/2)} \tag{12}$$

$$= X_{(3)} \tag{13}$$

From note (6)

$$X_{(3)} \sim B(3,3) \tag{14}$$

Option 1

From equation (6), Beta function of median is given by

$$B(3,3) = \frac{(3-1)!(3-1)!}{(3+3-1)!} = \frac{1}{30} \tag{15}$$

From equation (4), PDF of median is

$$f(x) = 30x^2(1-x)^2 \tag{16}$$

From equation (5), CDF of median is

$$F(x) = \int_0^x 30x^2(1-x)^2 dx \tag{17}$$

$$= 30x^3\left(\frac{1}{3} + \frac{x^2}{5} - \frac{x}{2}\right) \tag{18}$$

$$\Pr\left(M < \frac{1}{3}\right) = F\left(\frac{1}{3}\right) \tag{19}$$

$$= 0.20987 \tag{20}$$

$$\Pr\left(M > \frac{2}{3}\right) = \Pr\left(M < 1\right) - \Pr\left(M < \frac{2}{3}\right) \tag{21}$$

$$= F(1) - F\left(\frac{2}{3}\right) \tag{22}$$

$$= 0.20987 \tag{23}$$

$$\therefore \Pr\left(M < \frac{1}{3}\right) = \Pr\left(M > \frac{2}{3}\right) \tag{24}$$

Hence **Option 1 is true.**

Option 2

From (13), median M is a third order statistic. Clearly from note (6), M is a Beta distribution whose PDF is given by

$$f(x) = \begin{cases} 30x^2(1-x)^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

which is not a uniform distribution.
Hence **Option 2 is false.**

Option 3

From equation (1), density function for each of given random variables X_1, X_2, X_3, X_4, X_5 is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

From equation (2)

$$E(X_1) = \frac{1}{2} \tag{27}$$

From equation (8)

$$E(M) = \frac{3}{3+3} = \frac{1}{2} \tag{28}$$

$$\therefore E(M) = E(X_1) \tag{29}$$

Hence **Option 3 is true.**

Option 4

From equation (3)

$$V(X_1) = \frac{1}{12} \tag{30}$$

From equation (9)

$$V(M) = \frac{1}{28} \tag{31}$$

$$\therefore V(M) \neq V(X_1) \tag{32}$$

Hence **Option 4 is false.**