

Assignment-5

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Download all latex codes from:

<https://github.com/PrasannaLanka/Assignment5/blob/main/Assignment5/codes/Assignment5.tex>

PROBLEM UGC/MATH (2018 DEC-MATH SET-A) Q.104

Let X_1, X_2, \dots be i.i.d. $N(0, 1)$ random variables. Let $S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1$. Which of the following statements are correct?

- (A) $\frac{S_n - n}{\sqrt{2}} \sim N(0, 1)$ for all $n \geq 1$
- (B) For all $\epsilon > 0, \Pr(|\frac{S_n}{n} - 2| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$
- (C) $\frac{S_n}{n} \rightarrow 1$ with probability 1
- (D) $\Pr(S_n \leq n + \sqrt{n}x) \rightarrow \Pr(Y \leq x) \forall x \in R$, where $Y \sim N(0, 2)$

SOLUTION

Theorem 0.1 (Strong law of large numbers). Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$, Then as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \quad (0.0.1) \quad (A)$$

almost surely.

Theorem 0.2 (Weak law of large numbers). Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$, Then as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \quad (0.0.2)$$

in probability

Theorem 0.3 (Central limit theorem). The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger, given that all the samples are equal in size, regardless of the distribution of the individual samples.

Definition 0.1 (Convergence in probability). A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge in probability (denoted by i.p) to X if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \quad (0.0.3)$$

Definition 0.2 (Almost sure convergence). A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if

$$\Pr(\omega | X_n(\omega) \rightarrow X(\omega)) = 1 \quad (0.0.4)$$

Given X_1, X_2, \dots follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \dots\} \quad (0.0.5)$$

As X_1, X_2, \dots are i.i.d random variables therefore X_1^2, X_2^2, \dots are also identical and independent. We can write

$$E(X^2) = 0 \quad (0.0.6)$$

$$\text{Var}(X^2) = 1 \quad (0.0.7)$$

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \quad (0.0.8)$$

$$= \frac{\sum_i E(X_i^2 - 1)}{\sqrt{2}} \quad (0.0.9)$$

From (0.0.6) we can write

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \quad (0.0.10)$$

$$\text{Var}\left(\frac{S_n - n}{\sqrt{2}}\right) = \text{Var}\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \quad (0.0.11)$$

$$= \frac{\sum_i \text{Var}(X_i^2 - 1)}{\sqrt{2}} \quad (0.0.12)$$

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (X_i^2 - 1)^2 f_{X_i}(x) dx \quad (0.0.13)$$

$$= \int_{-\infty}^{\infty} (X_i^4 + 1 - 2X_i^2) f_{X_i}(x) dx \quad (0.0.14)$$

$$= \int_{-\infty}^{\infty} (X_i^4) f_{X_i}(x) dx + \int_{-\infty}^{\infty} f_{X_i}(x) dx - 2 \int_{-\infty}^{\infty} (X_i^2) f_{X_i}(x) dx \quad (0.0.15)$$

$$= Var(X^2) + 1 - 2(E(X^2)) \quad (0.0.16)$$

$$= 1 + 1 - 0 \quad (0.0.17)$$

$$= 2 \quad (0.0.18)$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = n\sqrt{2} \quad (0.0.19)$$

Hence from theorem 0.1 as $n \rightarrow \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n\sqrt{2}) \quad (0.0.20)$$

Hence **Option A is false.**

(B) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (0.0.21)$$

Assume that For all $\epsilon > 0, \Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \rightarrow 0$ as $n \rightarrow \infty$ is true

Hence from theorem 0.2 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X^2) \quad (0.0.22)$$

$$\Rightarrow \frac{S_n}{n} \xrightarrow{i.p} 1 \quad (0.0.23)$$

in probability.

From definition 0.1 we can write,

$$\Rightarrow \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \rightarrow 0, \forall \epsilon > 0 \quad (0.0.24)$$

But this is contradiction to our assumption.

Hence **Option B is false .**

(C) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (0.0.25)$$

Hence from theorem 0.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X^2) \quad (0.0.26)$$

$$\Rightarrow \frac{S_n}{n} \xrightarrow{a.s} 1 \quad (0.0.27)$$

almost surely.

$$\Pr(\lim_{n \rightarrow \infty} \frac{S_n}{n} = 1) = 1 \quad (0.0.28)$$

From definition 0.2 and (0.0.28) we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \quad (0.0.29)$$

with probability 1.

Hence **Option C is true.**

(D) Consider,

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \quad (0.0.30)$$

using (0.0.6) and (0.0.9).

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{2n}{\sqrt{n}} \quad (0.0.31)$$

$$= 2\sqrt{n}. \quad (0.0.32)$$

using (0.0.18).

From theorem 0.3 we can write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \quad (0.0.33)$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = \Pr(S_n \leq n + \sqrt{n}x) \quad (0.0.34)$$

Hence using (0.0.33), **Option D is false.**