

Assignment-5

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<https://github.com/PrasannaLanka/Assignment5/blob/main/Assignment5/codes/Assignment5.tex>

PROBLEM UGC/MATH (2018 DEC-MATH SET-A)
Q.104

Let X_1, X_2, \dots be i.i.d. $N(0, 1)$ random variables. Let $S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1$. Which of the following statements are correct?

- (A) $\frac{S_n - n}{\sqrt{2}} \sim N(0, 1)$ for all $n \geq 1$
- (B) For all $\epsilon > 0, \Pr(|\frac{S_n}{n} - 2| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$
- (C) $\frac{S_n}{n} \rightarrow 1$ with probability 1
- (D) $\Pr(S_n \leq n + \sqrt{n}x) \rightarrow \Pr(Y \leq x) \forall x \in R$, where $Y \sim N(0, 2)$

SOLUTION

Given X_1, X_2, \dots follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \dots\} \quad (0.0.1)$$

As X_1, X_2, \dots are i.i.d random variables therefore X_1^2, X_2^2, \dots are also identical and independent. We can write

$$E(X^2) = 0 \quad (0.0.2)$$

(A)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \quad (0.0.3)$$

$$= \frac{\sum_i E(X_i^2 - 1)}{\sqrt{2}} \quad (0.0.4)$$

From (0.0.2) we can write

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \quad (0.0.5)$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = Var\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \quad (0.0.6)$$

$$= \frac{\sum_i Var(X_i^2 - 1)}{\sqrt{2}} \quad (0.0.7)$$

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (X_i^2 - 1)^2 f_{X_i}(x) dx \quad (0.0.8)$$

$$= \int_{-\infty}^{\infty} (X_i^4 + 1 - 2X_i^2) f_{X_i}(x) dx \quad (0.0.9)$$

$$= 2 \quad (0.0.10)$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = n\sqrt{2} \quad (0.0.11)$$

Theorem 0.1 (Strong law of large numbers). Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$, Then as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \quad (0.0.12)$$

almost surely.

Hence from theorem 0.1 as $n \rightarrow \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n\sqrt{2}) \quad (0.0.13)$$

Hence **Option A is false.**

(B) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (0.0.14)$$

Assume that For all $\epsilon > 0, \Pr(|\frac{S_n}{n} - 2| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$ is true

Theorem 0.2 (Weak law of large numbers). Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$, Then as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \quad (0.0.15)$$

in probability

Hence from theorem 0.2 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} \text{Var}(X^2) \quad (0.0.16)$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} 1 \quad (0.0.17)$$

in probability.

Definition 0.1 (Convergence in probability). A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to converge in probability (denoted by i.p) to X if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \quad (0.0.18)$$

From definition 0.1 we can write,

$$\implies \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \rightarrow 0, \forall \epsilon > 0 \quad (0.0.19)$$

But this is contradiction to our assumption.

Hence **Option B is false**.

(C) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (0.0.20)$$

Hence from theorem 0.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} \text{Var}(X) \quad (0.0.21)$$

$$\implies \frac{S_n}{n} \xrightarrow{a.s} 1 \quad (0.0.22)$$

almost surely.

Definition 0.2 (Almost sure convergence). A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if

$$\Pr(\omega | X_n(\omega) \rightarrow X(\omega)) = 1 \quad (0.0.23)$$

From definition 0.2 we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \quad (0.0.24)$$

with probability 1.

Hence **Option C is true**.

(D) Consider,

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \quad (0.0.25)$$

using (0.0.2) and (0.0.4).

$$\text{Var}\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{2n}{\sqrt{n}} \quad (0.0.26)$$

$$= 2\sqrt{n}. \quad (0.0.27)$$

using (0.0.10).

Theorem 0.3 (Central limit theorem). The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger, given that all the samples are equal in size, regardless of the distribution of the individual samples.

From theorem 0.3 we can write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \quad (0.0.28)$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = \Pr(S_n \leq n + \sqrt{n}x) \quad (0.0.29)$$

Hence using (0.0.28), **Option D is false**.