Assignment-5

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Download all latex codes from:

https://github.com/PrasannaLanka/Assignment5/blob/main/Assignment5/codes/Assignment5.tex

PROBLEM UGC/MATH (2018 Dec-Math set-a) Q.104

Let $X_1, X_2, \cdots$ be i.i.d. $N(0,1)$ random variables. Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1$.Which of the following statements are correct?

(A) $\frac{S_n - n}{\sqrt{2}} \sim N(0,1)$ for all $n \geq 1$

(B) For all $\epsilon > 0$,$\Pr\left(|\frac{S_n}{n} - 2| > \epsilon\right) \to 0$ as $n \to \infty$

(C) $\frac{S_n}{n} \to 1$ with probability 1

(D) $\Pr(S_n \leq n + \sqrt{n}x) \to \Pr(Y \leq x) \forall x \in R$,where $Y \sim N(0,2)$

Solution

Theorem 0.1 (Strong law of large numbers)**.** *Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation(μ) and finite variance(σ^2).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$*

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \tag{0.0.1}$$

almost surely.

Theorem 0.2 (Weak law of large numbers)**.** *Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation(μ) and finite variance(σ^2).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$*

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \tag{0.0.2}$$

in probability

Theorem 0.3 (Central limit theorem)**.** *The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger,given that all the samples are equal in size,regardless of the distribution of the individual samples.*

Definition 0.1 (Convergence in probability)**.** *A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge in probability (denoted by i.p) to X if*

$$\lim_{n \to \infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \tag{0.0.3}$$

Definition 0.2 (Almost sure convergence)**.** *A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if*

$$\Pr(\omega | X_n(\omega) \to X(\omega)) = 1 \tag{0.0.4}$$

Definition 0.3 (chi-square distribution)**.** *If $X_1, X_2, \cdots$ are independent normally distributed random variables with mean 0 and variance 1. Then $S_n = \chi = X_1^2 + X_2^2 + \cdots + X_n^2$ is chi-square distributed with n degrees of freedom.*

$$E(\chi) = n \tag{0.0.5}$$

$$Var(\chi) = 2n \tag{0.0.6}$$

Given $X_1, X_2, \cdots$ follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \cdots\} \tag{0.0.7}$$

(A)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \tag{0.0.8}$$

$$= \frac{\sum_i E(X_i^2 - 1)}{\sqrt{2}} \tag{0.0.9}$$

From definition (0.3) and (0.0.5) we can write

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \tag{0.0.10}$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = Var\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \tag{0.0.11}$$

$$= \frac{\sum_i Var(X_i^2 - 1)}{\sqrt{2}} \quad (0.0.12)$$

Using definition 0.3

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (X_i^2 - 1)^2 f_{X_i}(x) dx \quad (0.0.13)$$

$$= \int_{-\infty}^{\infty} (X_i^4 + 1 - 2X_i^2) f_{X_i}(x) dx \quad (0.0.14)$$

$$= \int_{-\infty}^{\infty} (X_i^4) f_{X_i}(x) dx + \int_{-\infty}^{\infty} f_{X_i}(x) dx - 2\int_{-\infty}^{\infty} (X_i^2) f_{X_i}(x) dx \quad (0.0.15)$$

$$= Var(\chi) + 1 - 2(E(\chi)) \quad (0.0.16)$$

$$= 2n + 1 - 2n \quad (0.0.17)$$

$$= 1 \quad (0.0.18)$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = \frac{n}{\sqrt{2}} \quad (0.0.19)$$

Hence from theorem 0.1 as $n \to \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N\left(0, \frac{n}{\sqrt{2}}\right) \quad (0.0.20)$$

Hence **Option A is false.**

(B) Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \quad (0.0.21)$$

Assume that For all $\epsilon > 0$,$\Pr\left(|\frac{S_n}{n} - 2| > \epsilon\right) \to 0$ as $n \to \infty$ is true

Hence from theorem 0.2 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X^2) \quad (0.0.22)$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} 1 \quad (0.0.23)$$

in probability.

From definition 0.1 we can write,

$$\implies \Pr\left(|\frac{S_n}{n} - 1| > \epsilon\right) \to 0, \forall \epsilon > 0 \quad (0.0.24)$$

But this is contradiction to our assumption.

Hence **Option B is false .**

(C) Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \quad (0.0.25)$$

Hence from theorem 0.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X^2) \quad (0.0.26)$$

$$\implies \frac{S_n}{n} \xrightarrow{a.s} 1 \quad (0.0.27)$$

almost surely.

$$\Pr\left(\lim_{n\to\infty} \frac{S_n}{n} = 1\right) = 1 \quad (0.0.28)$$

From definition 0.2 and (0.0.28) we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \quad (0.0.29)$$

with probability 1.

Hence **Option C is true**.

(D) Consider,

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \quad (0.0.30)$$

using (0.0.5) and (0.0.9).

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{2n}{\sqrt{n}} \quad (0.0.31)$$

$$= 2\sqrt{n}. \quad (0.0.32)$$

using (0.0.18).

From theorem 0.3 we can write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \quad (0.0.33)$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = \Pr\left(S_n \leq n + \sqrt{n}x\right) \quad (0.0.34)$$

Hence using (0.0.33), **Option D is false.**