UGC/MATH (2018 Dec-Math set-a) Q.104

Lanka Prasanna

CS20BTECH11029

Topics covered

Prerequisites:

- Chi-square distribution
- Weak law of large numbers with its proof
- Strong law of large numbers
- Almost sure convergence
- Central limit theorem
- Properties of mean and variance

UGC NET problem

- Question
- Solution

Chi-square distribution

If $X_1, X_2, \cdots$ are independent normally distributed random variables with mean 0 and variance 1. Then $\chi = X_1^2 + X_2^2 + \cdots + X_n^2$ is chi-square distributed with n degrees of freedom.

$$E(\chi) = n \tag{1}$$

$$Var(\chi) = 2n \tag{2}$$

Weak law of large numbers

Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation(μ) and finite variance(σ^2).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \tag{3}$$

in probability

Proof:

Define a new variable

$$X \equiv \frac{X_1 + X_2 + \cdots X_n}{n} \tag{4}$$

Then, as $n \to \infty$

$$E(X) = E\left(\frac{X_1 + X_2 + \cdots X_n}{n}\right) \tag{5}$$

$$= \frac{E(X_1) + \cdots E(X_n)}{n} \tag{6}$$

$$= \frac{n\mu}{n} \tag{7}$$

$$= \mu \tag{8}$$

In addition,

$$Var(X) = Var\left(\frac{X_1 + X_2 + \cdots X_n}{n}\right) \tag{9}$$

$$= Var\left(\frac{X_1}{n}\right) + \cdots Var\left(\frac{X_n}{n}\right) \tag{10}$$

$$= \frac{n\sigma^2}{n^2} \tag{11}$$

$$= \frac{\sigma^2}{n} \tag{12}$$

Therefore, by Chebyshev inequality, for all $\epsilon > 0$,

$$\Pr\left(|X - \mu| \geq \epsilon\right) \leq \frac{Var(X)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \tag{13}$$

As $n \to \infty$, it follows that

$$\lim_{n\to\infty} \Pr\left(|X - \mu| \geq \epsilon\right) = 0 \tag{14}$$

Stated other way as $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \tag{15}$$

in probability

Strong law of large numbers

Let $X_1, X_2, \cdots$ be i.i.d random variables with same expectation(μ) and finite variance(σ^2).Let $S_n = X_1 + X_2 + \cdots X_n$,Then as $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \tag{16}$$

almost surely.

Almost sure convergence

A sequence of random variables X_n where $n \in N$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if

$$\Pr\left(\omega | X_n(\omega) \to X(\omega)\right) = 1 \tag{17}$$

Central limit theorem

The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger,given that all the samples are equal in size,regardless of the distribution of the individual samples.

Properties of mean and variance

If X is a random variable with a probability density function of $f(x)$. If a and b are constants.

$$E(X) = \int_R xf(x)dx \tag{18}$$

$$E(X+Y) = E(X) + E(Y) \tag{19}$$

$$E(aX+b) = aE(X) + b, \tag{20}$$

$$Var(X) = E(X^2) - E(X)^2 \tag{21}$$

$$Var(X+Y) = Var(X) + Var(Y) \tag{22}$$

$$Var(aX+b) = a^2 Var(X) \tag{23}$$

Question

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Let $X_1, X_2, \cdots$ be i.i.d. $N(0,1)$ random variables. Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1$.Which of the following statements are correct?

- A $\frac{S_n - n}{\sqrt{2}} \sim N(0,1)$ for all $n \geq 1$
- B For all $\epsilon > 0$,$\Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \to 0$ as $n \to \infty$
- C $\frac{S_n}{n} \to 1$ with probability 1
- D $\Pr\left(S_n \leq n + \sqrt{n}x\right) \to \Pr\left(Y \leq x\right) \forall x \in R$,where $Y \sim N(0,2)$

Solution

Given $X_1, X_2, \cdots$ follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in 1, 2, \cdots \tag{24}$$

Option A

From definition of chi-square distribution S_n is a chi-distributed function with n degrees of freedom.
From (1)

$$E(S_n) = n \tag{25}$$

From (25) and (20)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = \frac{E(S_n) - n}{\sqrt{2}} \tag{26}$$

$$= \frac{n - n}{\sqrt{2}} \tag{27}$$

$$= 0 \tag{28}$$

From (2)

$$Var(S_n) = 2n \tag{29}$$

Option A contd.

From (29) and (23)

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = Var\left(\frac{S_n}{\sqrt{2}}\right) \tag{30}$$
$$= \frac{Var(S_n)}{2} \tag{31}$$
$$= \frac{2n}{2} \tag{32}$$
$$= n \tag{33}$$

Hence, from central limit theorem

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n) \tag{34}$$

Hence **Option A is false.**

Option B

Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{35}$$

Assume that For all $\epsilon > 0$,Pr $\left(\left| \frac{S_n}{n} - 2 \right| > \epsilon \right) \to 0$ as $n \to \infty$ is true

Let

$$X \equiv \frac{X_1^2 + X_2^2 + \cdots X_n^2}{n} \tag{36}$$

From (25) and (20)

$$E(X) = E\left(\frac{S_n}{n}\right) \tag{37}$$

$$= \frac{E(S_n)}{n} \tag{38}$$

$$= \frac{n}{n} \tag{39}$$

$$= 1 \tag{40}$$

Option B contd.

From weak law of large numbers

$$\frac{S_n}{n} \xrightarrow{i.p} E\left(\frac{S_n}{n}\right) \tag{41}$$

$$\lim_{n\to\infty} \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) = 0 \tag{42}$$

This means for all $\epsilon > 0$,Pr $\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \to 0$ as $n \to \infty$
But this is contradiction to our assumption.
Hence **Option B is false .**

Option C

Given

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{43}$$

Hence from Strong law of large numbers we can write

$$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right) \tag{44}$$

From (40)

$$\frac{S_n}{n} \xrightarrow{a.s} 1 \tag{45}$$

almost surely.

$$\Pr\left(\lim_{n\to\infty} \frac{S_n}{n} = 1\right) = 1 \tag{46}$$

Option C contd.

From definition of Almost sure convergence and (46) we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \tag{47}$$

with probability 1.
Hence **Option C is true**.

Option D

Let

$$Y = \frac{S_n - n}{\sqrt{n}} \tag{48}$$

Using (25) and (20)

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{E(S_n) - n}{\sqrt{n}} \tag{49}$$

$$= \frac{n - n}{\sqrt{n}} \tag{50}$$

$$= 0 \tag{51}$$

using (25) and (20)

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{Var(S_n)}{n} \tag{52}$$

$$= \frac{2n}{n} \tag{53}$$

$$= 2 \tag{54}$$

Option D contd.

Hence, from central limit theorem

$$Y \sim N(0,2) \tag{55}$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \le x\right) = \Pr\left(S_n \le n + \sqrt{n}x\right) \tag{56}$$

Therefore,

$$\Pr\left(S_n \le n + \sqrt{n}x\right) \to \Pr\left(Y \le x\right) \forall x \in R \tag{57}$$

Hence, **Option D is true.**