

Assignment-5

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Download all latex codes from:

<https://github.com/PrasannaLanka/Assignment5/blob/main/Assignment5/codes/Assignment5.tex>

PROBLEM UGC/MATH (2018 DEC-MATH SET-A)

Q.104

Let X_1, X_2, \dots be i.i.d. $N(0, 1)$ random variables. Let $S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1$. Which of the following statements are correct?

- (A) $\frac{S_n - n}{\sqrt{2}} \sim N(0, 1)$ for all $n \geq 1$
- (B) For all $\epsilon > 0, \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \rightarrow 0$ as $n \rightarrow \infty$
- (C) $\frac{S_n}{n} \rightarrow 1$ with probability 1
- (D) $\Pr(S_n \leq n + \sqrt{nx}) \rightarrow \Pr(Y \leq x) \forall x \in R$, where $Y \sim N(0, 2)$

SOLUTION

Theorem 0.1 (chi-square distribution). *If X_1, X_2, \dots are independent normally distributed random variables with mean 0 and variance 1. Then $\chi = X_1^2 + X_2^2 + \dots + X_n^2$ is chi-square distributed with n degrees of freedom.*

$$E(\chi) = n \quad (0.0.1)$$

$$Var(\chi) = 2n \quad (0.0.2)$$

Theorem 0.2 (Weak law of large numbers). *Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$. Then as $n \rightarrow \infty$*

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \quad (0.0.3)$$

in probability

Proof. Define a new variable

$$X \equiv \frac{X_1 + X_2 + \dots + X_n}{n} \quad (0.0.4)$$

Then, as $n \rightarrow \infty$

$$E(X) = E\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) \quad (0.0.5)$$

$$= \frac{E(X_1) + \dots + E(X_n)}{n} \quad (0.0.6)$$

$$= \frac{n\mu}{n} \quad (0.0.7)$$

$$= \mu \quad (0.0.8)$$

In addition,

$$Var(X) = Var\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) \quad (0.0.9)$$

$$= Var\left(\frac{X_1}{n}\right) + \dots + Var\left(\frac{X_n}{n}\right) \quad (0.0.10)$$

$$= n \frac{\sigma^2}{n^2} \quad (0.0.11)$$

$$= \frac{\sigma^2}{n} \quad (0.0.12)$$

Therefore, by Chebyshev inequality, for all $\epsilon > 0$,

$$\Pr(|X - \mu| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \quad (0.0.13)$$

As $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} \Pr(|X - \mu| \geq \epsilon) = 0 \quad (0.0.14)$$

Stated other way as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \quad (0.0.15)$$

in probability □

Theorem 0.3 (Strong law of large numbers). *Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$. Then as $n \rightarrow \infty$*

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \quad (0.0.16)$$

almost surely.

Theorem 0.4 (Central limit theorem). *The Central limit theorem states that the distribution of the sample approximates a normal distribution as the*

sample size becomes larger, given that all the samples are equal in size, regardless of the distribution of the individual samples.

Definition 0.1 (Almost sure convergence). A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if

$$\Pr(\omega | X_n(\omega) \rightarrow X(\omega)) = 1 \quad (0.0.17)$$

Lemma 0.5 (Properties of mean and variance). If X is a random variable with a probability density function of $f(x)$. If a and b are constants.

$$E(X) = \int_{\mathbb{R}} x f(x) dx \quad (0.0.18)$$

$$E(X + Y) = E(X) + E(Y) \quad (0.0.19)$$

$$E(aX + b) = aE(X) + b, \quad (0.0.20)$$

$$\text{Var}(X) = E(X^2) - E(X)^2 \quad (0.0.21)$$

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \quad (0.0.22)$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X) \quad (0.0.23)$$

Given X_1, X_2, \dots follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \dots\} \quad (0.0.24)$$

(A) From theorem (0.1) S_n is a chi-distributed function with n degrees of freedom.

From (0.0.1)

$$E(S_n) = n \quad (0.0.25)$$

From (0.0.25) and (0.0.20)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = \frac{E(S_n) - n}{\sqrt{2}} \quad (0.0.26)$$

$$= \frac{n - n}{\sqrt{2}} \quad (0.0.27)$$

$$= 0 \quad (0.0.28)$$

From (0.0.2)

$$\text{Var}(S_n) = 2n \quad (0.0.29)$$

From (0.0.29) and (0.0.23)

$$\text{Var}\left(\frac{S_n - n}{\sqrt{2}}\right) = \text{Var}\left(\frac{S_n}{\sqrt{2}}\right) \quad (0.0.30)$$

$$= \frac{\text{Var}(S_n)}{2} \quad (0.0.31)$$

$$= \frac{2n}{2} \quad (0.0.32)$$

$$= n \quad (0.0.33)$$

Hence,

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n) \quad (0.0.34)$$

Hence **Option A is false.**

(B) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (0.0.35)$$

Assume that For all $\epsilon > 0, \Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \rightarrow 0$ as $n \rightarrow \infty$ is true let

$$X \equiv \frac{X_1^2 + X_2^2 + \dots + X_n^2}{n} \quad (0.0.36)$$

From (0.0.25) and (0.0.20)

$$E(X) = E\left(\frac{S_n}{n}\right) \quad (0.0.37)$$

$$= \frac{E(S_n)}{n} \quad (0.0.38)$$

$$= \frac{n}{n} \quad (0.0.39)$$

$$= 1 \quad (0.0.40)$$

From theorem (0.2)

$$\frac{S_n}{n} \xrightarrow{i.p} E\left(\frac{S_n}{n}\right) \quad (0.0.41)$$

$$\lim_{n \rightarrow \infty} \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) = 0 \quad (0.0.42)$$

This means for all $\epsilon > 0, \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \rightarrow 0$ as $n \rightarrow \infty$

But this is contradiction to our assumption.

Hence **Option B is false .**

(C) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (0.0.43)$$

Hence from theorem 0.3 we can write

$$\frac{S_n}{n} \xrightarrow{a.s} E\left(\frac{S_n}{n}\right) \quad (0.0.44)$$

From (0.0.40)

$$\frac{S_n}{n} \xrightarrow{a.s} 1 \quad (0.0.45)$$

almost surely.

$$\Pr\left(\lim_{n \rightarrow \infty} \frac{S_n}{n} = 1\right) = 1 \quad (0.0.46)$$

From definition 0.1 and (0.0.46) we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \quad (0.0.47)$$

with probability 1.

Hence **Option C is true.**

(D) Let

$$Y = \frac{S_n - n}{\sqrt{n}} \quad (0.0.48)$$

Using (0.0.25) and (0.0.20)

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{E(S_n) - n}{\sqrt{n}} \quad (0.0.49)$$

$$= \frac{n - n}{\sqrt{n}} \quad (0.0.50)$$

$$= 0 \quad (0.0.51)$$

using (0.0.25) and (0.0.20)

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{Var(S_n)}{n} \quad (0.0.52)$$

$$= \frac{2n}{n} \quad (0.0.53)$$

$$= 2 \quad (0.0.54)$$

Hence, from theorem (0.4)

$$Y \sim N(0, 2) \quad (0.0.55)$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = \Pr(S_n \leq n + \sqrt{nx}) \quad (0.0.56)$$

Therefore,

$$\Pr(S_n \leq n + \sqrt{nx}) \rightarrow \Pr(Y \leq x) \forall x \in R \quad (0.0.57)$$

Hence, **Option D is true.**