Discrete Structures (CS21201) Autumn 2025

Tutorial 6: Sets & Relations

15th September 2025

Question 1

Define two relations ρ and σ on $\mathbb{R}$ as follows.

1. $a \, \rho \, b$ if and only if $a - b \in \mathbb{Q}$.
2. $a \, \sigma \, b$ if and only if $a - b \in \mathbb{Z}$.

Prove that ρ and σ are equivalence relations on $\mathbb{R}$. Also, find the equivalence classes (with representatives).

Answer

[Reflexive:] For any $a \in \mathbb{R}$, $a - a = 0$. Since $0 \in \mathbb{Q}$ and $0 \in \mathbb{Z}$, both relations are reflexive.

[Symmetric:] If $a - b$ is rational (or integer), then $b - a = -(a - b)$ is also rational (or integer). Hence, both relations are symmetric.

[Transitive:] If $a - b$ and $b - c$ are rational (or integer), then

$$a - c = (a - b) + (b - c)$$

is again rational (or integer). Hence, both relations are transitive.

Thus, ρ and σ are equivalence relations.

Equivalence Classes:

1. Under ρ: The equivalence class of x is $[x]_\rho = \{\, x + r : r \in \mathbb{Q} \,\}$.
2. Under σ: The equivalence class of y is $[y]_\sigma = \{\, y + s : s \in \mathbb{Z} \,\}$.

We can choose a unique element in the interval [0,1) as a representative of each equivalence class of σ. For ρ however, identifying representatives in a mathematically rigorous way is not possible. If you assume the axiom of choice, then all you can say is that a unique representative from each equivalence class can be chosen. Such a choice of unique representatives from the equivalence classes gives us a set called a Vitali set.

Question 2

Let A be a poset under the relation ρ. Prove or disprove:

1. If ρ is a total order, then A is a lattice.
2. If A is a lattice, then ρ is a total order.

Answer

1. **True.** Suppose (A, ρ) is a total order. That means for every $x, y \in A$ either $x \leq y$ or $y \leq x$. For any two elements x, y define
$$x \wedge y := \min\{x, y\}, \qquad x \vee y := \max\{x, y\}.$$
Since x and y are comparable, both the minimum and maximum exist in A. Clearly, $x \wedge y$ is the greatest lower bound of $\{x, y\}$ and $x \vee y$ is the least upper bound of $\{x, y\}$. Hence every pair has a meet and a join, so A is a lattice.

2. **False.** Being a lattice only requires that every pair of elements has a meet and a join, not that all elements are comparable.

   Counterexample: Take $A = \mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$ with the order $\subseteq$. - Meets are intersections, joins are unions, so every pair has both. Thus A is a lattice. - But $\{1\}$ and $\{2\}$ are incomparable (neither is a subset of the other). Hence ρ is not a total order.

   Therefore, a lattice need not be totally ordered.

Question 3

Let $k \in \mathbb{N}$, $S = \{1, 2, \ldots, k\}$, and $A = \mathcal{P}(S) \setminus \{\varnothing\}$, where $\mathcal{P}(S)$ denotes the power set of S. In other words, the set A consists of all non-empty subsets of $\{1, 2, \ldots, k\}$. For each $a \in A$, denote by $\min(a)$ the smallest element of a (notice that here a is a set).

1. Define a relation ρ on A as follows: $a \,\rho\, b$ if and only if $\min(a) = \min(b)$. Prove that ρ is an equivalence relation on A.

   Answer:
   Reflexive: For any $a \in A$ we have $\min(a) = \min(a)$.
   Symmetric: For any $a, b \in A$, if $\min(a) = \min(b)$, then $\min(b) = \min(a)$.
   Transitive: For any $a, b, c \in A$, if $\min(a) = \min(b)$ and $\min(b) = \min(c)$, then $\min(a) = \min(c)$.

2. What is the size of the quotient set A/ρ?

   Answer:
   Any two non-empty subsets of S having the same minimum element are related. On the other hand, two subsets of S having different minimum elements are not related. Therefore, the equivalence classes of ρ have a one-to-one correspondence with elements of S (the minimum element of every member in the class). Since S contains k elements, there are exactly k equivalence classes, that is, the size of A/ρ is k.

3. Define a relation σ on A as follows: $a \,\sigma\, b$ if and only if either $a = b$ or $\min(a) < \min(b)$. Prove that σ is a partial order on A.

   Answer:
   Reflexive: By definition, every element is related to itself.
   Antisymmetric: Take two elements $a, b \in A$. Suppose that $a \,\sigma\, b$ and $b \,\sigma\, a$. If $a \neq b$, then by definition, $\min(a) < \min(b)$ and $\min(b) < \min(a)$, which is impossible. So we must have $a = b$.
   Transitive: Suppose $a \,\sigma\, b$ and $b \,\sigma\, c$ for some $a, b, c \in A$. If $a = b$ or $b = c$, then clearly $a \,\sigma\, c$. So suppose that $a \neq b$ and $b \neq c$. But then, $\min(a) < \min(b)$ and $\min(b) < \min(c)$. This implies that $\min(a) < \min(c)$, that is, $a \,\sigma\, c$.

4. Is σ also a total order on A?

   Answer:
   No! Take $k > 2$. The sets $\{1\}$ and $\{1, 2\}$ are distinct, but have the same minimum element, and are therefore not comparable.

Question 4

Give an example of a poset A and a non-empty subset S of A such that S has lower bounds in A, but glb(S) does not exist.

Answer

Take $A = \mathbb{Q}$ under the standard $<$ on rational numbers. Also take $S = \{x \in \mathbb{Q} \mid x^2 > 2\}$. Every rational number $< \sqrt{2}$ is a lower bound on S. Since $\sqrt{2}$ is irrational, glb(S) does not exist.
Another example: Take A to be the set of all irrational numbers between 1 and 5, and S to be the set of all irrational numbers between 2 and 3.
A simpler (but synthetic) example: Take $A = \{a, b, c, d\}$ and the relation on A as,

$$\rho = \{(a, a), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (d, d)\}$$

The subset $S = \{c, d\}$ of A has two lower bounds a and b, but these bounds are not comparable to one another.

Question 5

Let k be a fixed positive integer. Define a relation $\leq$ on $A = \mathbb{Z}^k$ by

$$(a_1, a_2, \ldots, a_k) \leq (b_1, b_2, \ldots, b_k) \quad \text{iff} \quad a_i \leq b_i \text{ for all } i = 1, 2, \ldots, k.$$

Prove that A is a lattice under this relation.

Answer

First note that $\leq$ is a partial order on $\mathbb{Z}^k$: it is reflexive, antisymmetric and transitive by checking those properties coordinate-wise.
For any two elements $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ in $\mathbb{Z}^k$ define

$$\mathbf{a} \wedge \mathbf{b} := (\min(a_1, b_1), \min(a_2, b_2), \ldots, \min(a_k, b_k)),$$

$$\mathbf{a} \vee \mathbf{b} := (\max(a_1, b_1), \max(a_2, b_2), \ldots, \max(a_k, b_k)).$$

We show $\mathbf{a} \wedge \mathbf{b}$ is the greatest lower bound of $\{\mathbf{a}, \mathbf{b}\}$.
- *Lower bound:* For each coordinate i we have $\min(a_i, b_i) \leq a_i$ and $\min(a_i, b_i) \leq b_i$. Hence $\mathbf{a} \wedge \mathbf{b} \leq \mathbf{a}$ and $\mathbf{a} \wedge \mathbf{b} \leq \mathbf{b}$.
- *Greatest:* If $\mathbf{z} = (z_1, \ldots, z_k)$ is any lower bound of $\{\mathbf{a}, \mathbf{b}\}$, then for every i we have $z_i \leq a_i$ and $z_i \leq b_i$, so $z_i \leq \min(a_i, b_i)$. Thus $\mathbf{z} \leq \mathbf{a} \wedge \mathbf{b}$. Therefore $\mathbf{a} \wedge \mathbf{b}$ is the greatest lower bound.

Similarly we show $\mathbf{a} \vee \mathbf{b}$ is the least upper bound of $\{\mathbf{a}, \mathbf{b}\}$.
- *Upper bound:* For each i, $a_i \leq \max(a_i, b_i)$ and $b_i \leq \max(a_i, b_i)$, so $\mathbf{a} \leq \mathbf{a} \vee \mathbf{b}$ and $\mathbf{b} \leq \mathbf{a} \vee \mathbf{b}$.
- *Least:* If $\mathbf{u} = (u_1, \ldots, u_k)$ is any upper bound of $\{\mathbf{a}, \mathbf{b}\}$, then for every i we have $a_i \leq u_i$ and $b_i \leq u_i$, hence $\max(a_i, b_i) \leq u_i$. Thus $\mathbf{a} \vee \mathbf{b} \leq \mathbf{u}$. Therefore $\mathbf{a} \vee \mathbf{b}$ is the least upper bound.

Since every pair in $\mathbb{Z}^k$ has both a meet and a join with respect to $\leq$, the poset $(\mathbb{Z}^k, \leq)$ is a lattice.