Q1.

a. Maximum Likelihood Estimation of θ (Binomial Distribution)

Let N be the Number of test Samples and $k_1, k_2, \ldots k_N \in D$ (Dataset) be samples obtained from a binomial Distribution

PDF of binomial Distribution is given by

$$f(k) = \frac{N!}{k!(N-k)!} \theta^{k} (1-\theta)^{N-k} = p(k|\theta)$$

The likelihood function $L(\theta)$ is given by

$$L(\theta) = \prod_{i=1}^{n} f(k_i) = \prod_{i=1}^{n} \frac{N!}{k_i!(N-k_i)!} \theta^{k_i} (1-\theta)^{N-k_i}$$

The Log-likelihood is given by

$$\ln L(\theta) = \ln \left(\prod_{i=1}^{n} \frac{N!}{k_i!(N-k_i)!} \theta^{k_i} (1-\theta)^{N-k_i} \right)$$

$$\Rightarrow \sum_{i=1}^{n} \ln(N!) - \sum_{i=1}^{n} \ln(k_i!) - \sum_{i=1}^{n} \ln(N-k_i)! + \sum_{i=1}^{n} k_i \ln(\theta)$$

$$- \sum_{i=1}^{n} (N-k_i) \ln(1-\theta)$$

②

taking the derivative w.r.t θ & equating it to zero

$$\frac{\partial \ln(L(\theta))}{\partial \theta} = \frac{\partial}{\partial \theta}\Big(\sum_{i=1}^{n} \ln(N!) - \sum_{i=1}^{n} \ln(k_i!) - \sum_{i=1}^{n} \ln(N-k_i)! + \sum_{i=1}^{n} k_i \ln(\theta) - \sum_{i=1}^{n} (N-k_i).\ln(1-\theta)\Big)$$

$$\Rightarrow 0 - 0 - 0 + \frac{1}{\theta}\sum_{i=1}^{n} k_i - \frac{1}{1-\theta}\sum_{i=1}^{n}(N-k_i)$$

As we know

$$\frac{\partial \ln(L(\theta))}{\partial \theta} = 0.$$

$$0 = \frac{1}{\theta}\sum_{i=1}^{n} k_i - \frac{1\left(\sum_{i=1}^{n} N - \sum_{i=1}^{n} k_i\right)}{(1-\theta)}$$

$$\frac{1}{\theta}\sum_{i=1}^{n} k_i = \frac{1\left(\sum_{i=1}^{n} N - \sum_{i=1}^{n} k_i\right)}{1-\theta}$$

$$(1-\theta)\sum_{i=1}^{n} k_i = \theta.N\sum_{i=1}^{n} 1 - \theta\sum_{i=1}^{n} k_i$$

$$\sum_{i=1}^{n} k_i - \cancel{\theta\sum_{i=1}^{n} k_i} = \theta N.n - \cancel{\theta\sum_{i=1}^{n} k_i}$$

$$\Rightarrow \theta = \frac{1}{N}\sum_{i=1}^{n}\frac{k_i}{n}.$$

$$\boxed{\Rightarrow \hat{\theta} = \frac{1}{N}\sum_{i=1}^{n}\frac{k_i}{n}.}$$

Q1.

b

Maximum Likelihood estimation found for the θ i.e..

$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{n} \frac{k_i}{n}$ in the first iteration is the best possible value of θ that supports the true error rate

Q2.

b. $m = [10 \ \ 0]^T$

$$S = \begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix}$$

To find the eigen values we have the following equation

$$\det(S - \lambda I) = 0$$

where λ is the eigen values

I is the Identity matrix i.e..

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Substituting the S & I in the above equation to find λ

$$|S - \lambda I| = \left| \begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0.$$

$$\Rightarrow \det\left(\begin{bmatrix} 16-\lambda & -12 \\ -12 & 34-\lambda \end{bmatrix} \right) = 0.$$

Solving the above eq^n using determinant

$$(16-\lambda)(34-\lambda) - (-12)(-12) = 0.$$

$$544 - 16\lambda - 34\lambda + \lambda^2 - 144 = 0$$

$$\lambda^2 + 400 - 50\lambda = 0.$$

$$\Rightarrow \begin{bmatrix} -24 & -12 \\ -12 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -24a - 12b \\ -12a - 6b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -24a - 12b = 0 \ldots (1)$$

$$-12a - 6b = 0 \ldots (2)$$

From either eq^n (1) or (2) we can inference

$$-24a = 12b.$$

$$a = -\frac{12}{24} b.$$

$$a = -\frac{1}{2} b.$$

$\therefore$ $a = -\frac{1}{2}$ $b = 1$ (by the ratio rule).

$$\boxed{\phi_2 \text{ or } V_2 = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \phi_2 \text{ or } V_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}$$

For $\lambda_1 = 10$.

$(S - \lambda I)\phi_1 = 0$ where $\phi_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

Where $S = \begin{bmatrix} 16 & -12 \\ -12 & 24 \end{bmatrix}$ $\lambda = 10$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

substituting above values in the above eq^n.

$$\left(\begin{bmatrix} 16 & -12 \\ -12 & 24 \end{bmatrix} - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Solving the above eqn for 'λ'

$$\Rightarrow \lambda^2 - 40\lambda - 10\lambda + 400$$

$$\Rightarrow \lambda(\lambda-40) - 10(\lambda-40) = 0$$

$$(\lambda-10)(\lambda-40) = 0 \quad \Rightarrow \lambda_1 - 10 = 0 / \lambda_2 - 40 = 0$$

$$\boxed{\lambda_2 = 40 \quad \lambda_1 = 10}$$

eigen values are 10, 40.

c. The eigenvectors of covariance matrix S can be calculated by substituting λ_1 & λ_2 values in the below equation

$$(S - \lambda I)\phi_2 = 0 \quad \text{where } \phi_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

Substituting the λ_2 & S, I values in the above equation

$$\lambda_2 = 40 \quad S = \begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix} - 40 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{bmatrix} 16-40 & -12 \\ -12 & 34-40 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \left(\begin{bmatrix} 6 & -12 \\ -12 & 14 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 6a - 12b \\ -12a + 14b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 6a - 12b = 0 \quad \text{--- (1)}$$

$$-12a + 14b = 0 \quad \text{--- (2)}$$

From eq^n (1) & (2) we can inferred that

$6a = 12b$

$\therefore a = 2b$

$\therefore a = 2 \quad b = 1$

$$\boxed{\phi_1 \text{ or } V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$\therefore$ eigen vector $= [\phi_1 \ \ \phi_2]$

$$\boxed{\text{eigen vector} = \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}}$$

e. From the above, we can tell that

$\phi_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$\therefore$ eigen vector $= [\phi_1 \ \ \phi_2]$

$$\boxed{\text{eigen vector} = \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}}$$

f. matrix $\phi = \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}$

The new covariance matrix is given by

$$\Sigma = \phi^T S \phi$$

we know that $\phi = \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}$

$\phi^T = \begin{bmatrix} 2 & 1 \\ -1/2 & 1 \end{bmatrix}$ $S = \begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix}$

$$\Sigma = \phi^T S \phi$$

Substituting above values in the above eq^n

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 16 & -12 \\ -12 & 34 \end{bmatrix} \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 32-12 & -24+34 \\ -\frac{1}{2}(16)-12 & -\frac{1}{2}(-12)+34 \end{bmatrix} \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 20 & +10 \\ -20 & 40 \end{bmatrix} \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 40+10 & -\frac{1}{2}(20)+10 \\ -40+40 & -\frac{1}{2}(-20)+40 \end{bmatrix}$$

$$\boxed{\Sigma \Rightarrow \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}}$$

When S is transformed to Σ, the covariance values become '0'. So in 'Σ', the features are un-related. In other words, Features are well seperated

g. The new value to plot contour are.

$$m' = \phi^T m$$

$$m = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \qquad \phi = \begin{bmatrix} 2 & -1/2 \\ 1 & 1 \end{bmatrix}$$

$$\phi^T = \begin{bmatrix} 2 & 1 \\ -1/2 & 1 \end{bmatrix}$$

$$m' = \begin{bmatrix} 2 & 1 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 - 0 \\ -\frac{1}{2}(10) - 0 \end{bmatrix}$$

$$m' = \begin{bmatrix} 20 \\ -5 \end{bmatrix}.$$

$$\boxed{m' = [20 \quad -5]^T}$$

From the Q2.F we know that

$$\Sigma = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}.$$

From the eq^n $S' = \Sigma$

$$\boxed{\therefore \quad S' = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}}$$