MASTERS PROJECT REPORT

QUANTUM ENTANGLEMENT MEASURE&DYNAMICAL EVOLUTION OF MANY QUBIT SYSTEM

Submitted by
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1. Declaration

(i) I hereby certify that I submit this material is for the partial fulfilment of the Master of Science degree in Physics. No portion of the work contained in this project report has been submitted in support of an application for another degree or qualification to this or any other institution by me.

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Prasanth Kumar A S
July 10,2021

(ii) I hereby certify that this material is done by Mr. Prasanth Kumar A S (19PHMP36), under my supervision, during the academic year 2020-21 for the fulfilment of his MSc. Physics Degree.

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Prof. V Subrahmanyam
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July 10,2021

2. Abstract

Ability to generate & control entanglement in a system with long-range interaction will be of great importance for future quantum technology. Heisenberg's spin chain provides an important and tractable theoretical example for applying density matrix formalism. A circular spin chain consists of even number of spin $\frac{1}{2}$ particles can be simulated by assuming that interaction between the spins is exchange interaction.

Ground state energy & ground state eigen state were found by using numerical computing with the help of Fortran programming, thereafter finding the bipartite entanglement by using Wooter's concurrence formula, also investigating the mutual information between different subsystems.

Now the ground state density matrix is mapped into some other density matrix by using Krauss-Sudarshan operators. Resultant state was undergone time evolution and corresponding variations of bipartite entanglement &mutual information were investigated.

Chapter 1

3. Spin Chain: An Overview

Understanding of the Problem &Background

Consider a spin chain of N (even) spin $\frac{1}{2}$ particles each one is connected to one another in the form of a chain. Our aim is to find the extent to which the nearest neighbors are entangled in the ground state of a composite system & also we need to find the dynamical evolution of the system. Ground state of this system is non degenerate.

In general Hamiltonian H of the given composite system is chosen to be Heisenberg exchange interaction.it is given by

$$H = J \ \sum_i S_i . S_{i+1} \ , \ (\ i + 1 = 1 \ when \ i = N \ ) \ \& \ \ J > 0 \ called \ coupling \ coefficient$$

Entanglement is a measure of how a subsystem of a composite system is related with rest of the system. Entanglement results from superposition of states. For a pure state of many qubits, entanglement is quantified by Von-Neumann entropy of reduced density matrix. Linear combination of two pure states, both of which are entangled can give no entanglement on superposition. Similarly, two pure states with no entanglement can give rise to entanglement on superposition. In pure state of many qubits system, subsystem of two qubit state is mixed in general. A pure state of composite system can be expressed as a linear combination of orthogonal states of individual systems (Schmidt decomposition), also a mixed state can be decomposed over pure states (Schmidt purification). The entanglement of a pair of qubits is the average entanglement of decomposition, minimized over all possible decompositions. Starting from the pure state $\rho = |\Psi\rangle\langle\Psi|$ &finding the reduced density matrix R_{ij} for pair of qubits, which is constructed by taking partial trace over the rest of the qubits $R_{ij} = tr\rho$. Von Neumann entropy for the subsystem can be calculated by finding the eigen values of R_{ij} as

$S(R_{ij}) = -\sum r_n log_2 \ r_n$, where r_n are the eigen values of R_{ij}.Concurrence measure will tell us how these qubits namely i&j are entangled among themselves, its value will be ranging from 0 to 1. If concurrence is 1 then it is completely entangled systems& if concurrence is zero the systems are unentangled. Bipartite Concurrence is calculated by Wooter's concurrence formula

$$C(\rho) = max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$$

where $\lambda_i's$ are the eigen values of $R_{ij}\widetilde{R_{ij}}$ written in descending order, $\widetilde{R_{ij}}$ is the time reversed density matrix and it is defined as

$$\widetilde{R_{ij}} = (\sigma_y \otimes \sigma_y) R_{ij}{}^{*} (\sigma_y \otimes \sigma_y)$$

Measure of Concurrence

Re expressing the Hamiltonian in terms of spin operators as

$$H = J \sum_{i\,j} \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z$$

This sum is over all nearest neighbor pairs on a given lattice. Working in a diagonal basis of S_i^z for every site, total Z- component of spin is a good quantum number, in addition to the total spin quantum number S. The above written Hamiltonian is invariant under time & rotation.

Elements of reduced density matrix (R_{ij}) is found as

$$\rho_{ij} = \begin{bmatrix} v_{ij} & 0 & 0 & 0 \\ 0 & w_{1ij} & z_{ij}^* & 0 \\ 0 & z_{ij} & w_{2ij} & 0 \\ 0 & 0 & 0 & u_{ij} \end{bmatrix}$$

Fig 1: Structure reduced density matrix for joint subsystems i&j

$$v_{ij} = \langle \left(\frac{1}{2} + S_i^z\right)\left(\frac{1}{2} + S_j^z\right) \rangle,$$
$$w_{1ij} = \langle \left(\frac{1}{2} + S_i^z\right)\left(\frac{1}{2} - S_j^z\right) \rangle,$$
$$w_{2ij} = \langle \left(\frac{1}{2} - S_i^z\right)\left(\frac{1}{2} + S_j^z\right) \rangle,$$
$$z_{ij} = \langle S_j^+ S_i^- \rangle,$$
$$z_{ij}^* = \langle S_j^- S_i^+ \rangle$$
$$u_{ij} = \langle \left(\frac{1}{2} - S_i^z\right)\left(\frac{1}{2} - S_j^z\right) \rangle$$

Here the diagonal matrix elements are related to diagonal correlation function& non diagonal matrix elements related to non-diagonal correlation.

Finding the eigen values of $R\tilde{R}$ matrix and found eigen values and substituted in the formula for concurrence and it becomes

$$C(R_{ij}) = 2 * max\{0, |z_{ij}| - \sqrt{u_{ij}v_{ij}}\}$$

By exploiting the time invariance & rotational invariance of the Hamiltonian we can say that

$$Cij = \left\{ \begin{array}{c} 0 \; for \; \Gamma_{ij} > 0 \\ 0 \; for \; \Gamma_{ij} < 0 \, , \, |\Gamma_{ij}| < 1/12 \\ 6(|\Gamma_{ij}| - 1/12) \; for \; \Gamma_{ij} < 0 \, , |\Gamma_{ij}| > 1/12 \end{array} \right\} \text{where } \Gamma_{ij} = \langle S_i^z S_j^z \rangle$$

So, we expressed the nearest neighbor concurrence only in terms of diagonal correlation function.

Now let's come to ground state energy of the system, which can also express in terms of nearest neighbor concurrence.

$$E_g = e_g N = 3N_n \Gamma_1$$

where e_g is the ground state energy per site& N_n is the number of nearest neighbor pairs of spins. Rewriting nearest neighbor concurrence in terms of ground state energy as

$$C_1 = 6 * (\frac{e_g N}{3N_n} - \frac{1}{12})$$

As $N \to \infty$ value of ground state energy goes to $ln2 - \frac{1}{4}$ & correspondingly concurrence will tend to $2ln2 - 1$.the above predictions can be made from the solution of Bethe-Ansatz.

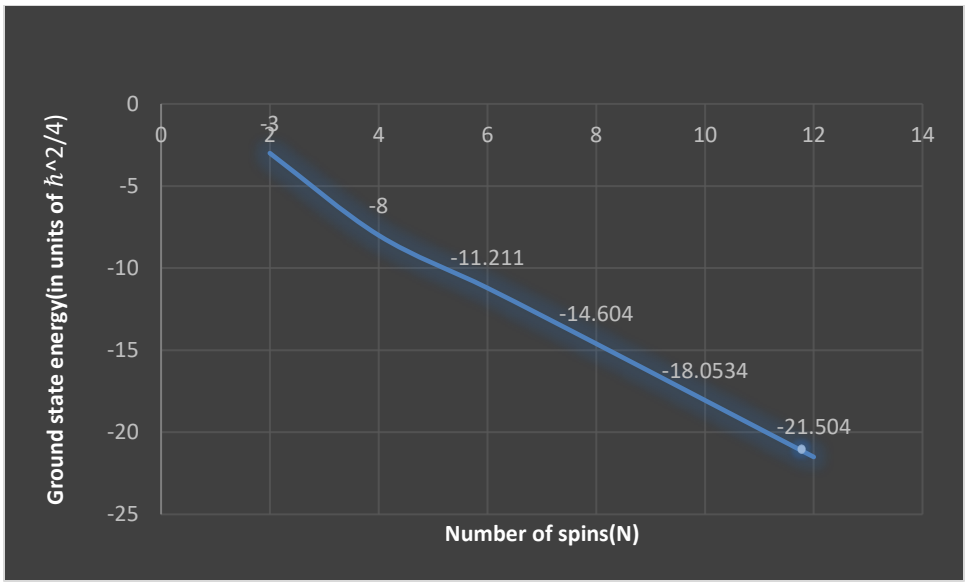

Fig 2: Variation of ground state energy with number of spins (Energy in units of $\frac{\hbar^2}{4}$)

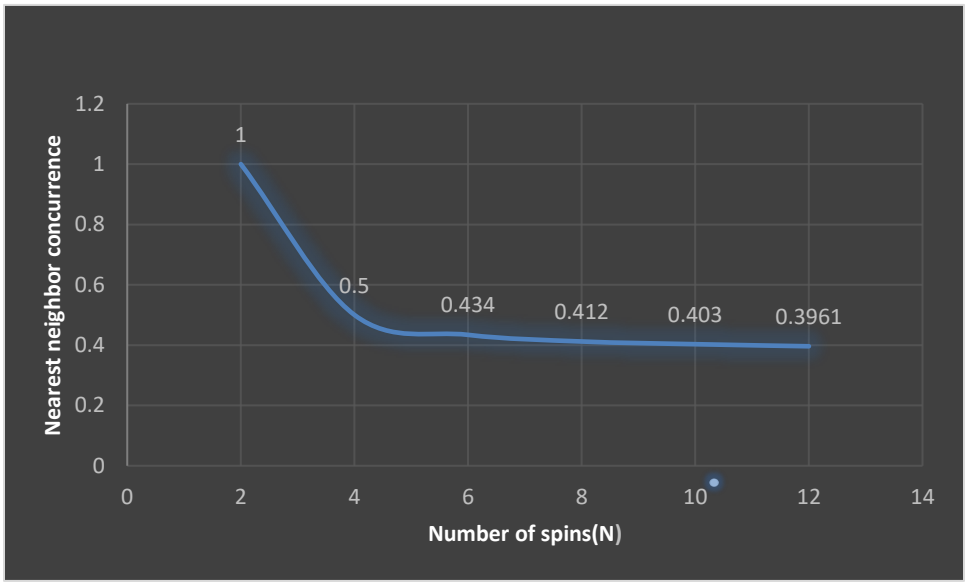

Fig 3 : variation of nearest neighbor concurrence with number of spins

Entropy & Mutual information

Shannon entropy quantifies how much information we gain on average, when we learnt the value. It is also known as classical entropy. Shannon entropy is defined by using probability distribution. Von Neumann entropy is the entropy related to quantum systems &states as described in a similar fashion, with density operators replacing probability distributions. Von-Neumann entropy is defined as

$S(\rho) = -tr(\rho ln_2 \rho) = -\sum_x \lambda_x ln\lambda_x$ where $\lambda_i{}'s$ are the eigen values of density matrix.

Measure of Mutual information

Mutual information is a quantitative measure of common information between two subsystems, which is defined as
$S(A{:}B) = S(A) + S(B) - S(A,B)$
Variation of bipartite mutual information with respect to the distance is shown in the figure below. which shows that mutual information decreases with increase in distance.

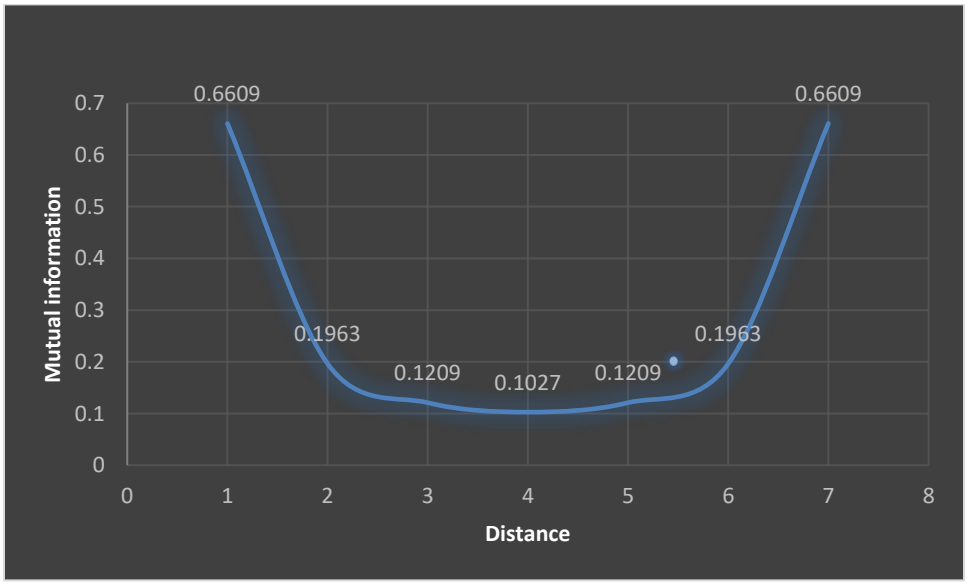

Fig 4: variation of mutual information with distance

Chapter 2

4. Quantum Channels& Dynamical Evolution

Evolution of many qubits system generally represented by Krauss-Sudarshan operators. Action of Krauss operators maps the given density operator into another as

$\rho_{out} = E_1\rho_{in}E_1^\dagger + E_2\rho_{in}E_2^\dagger + \ldots\ldots\ldots\ldots + E_n\rho_{in}E_n^\dagger$ where $\sum_{i=1}^{n} E_i^\dagger E_i = 1$

Where $E_1, E_2, \ldots . E_n$ *etc are Krauss* $-$ *Sudarshan operators*

Here we considered the action of the Krauss operator sets such as

1. $(\sqrt{p}\,\sigma_l^z, \sqrt{q}\,\sigma_m^z, \sqrt{1-p-q}\,I\,)$&
2. $(\sqrt{p}\,\sigma_l^z\sigma_m^z, \sqrt{1-p}\,I\,)$

ground state density matrix will transform as

1. $\rho_{out} = p\,\sigma_l^z\rho_{gs}\sigma_l^z + q\,\sigma_m^z\rho_{gs}\sigma_m^z + (1-p-q)\rho_{gs}$
2. $\rho_{out} = p\,\sigma_l^z\sigma_m^z\,\rho_{gs}\sigma_m^z\,\sigma_l^z + (1-p)\rho_{gs}$

where p&q are the channel parameters. m & l are the sites where the channel is applied. Maximizing the value of Von-Neumann entropy with respect to channel parameters and further analysis is done by fixing those parameters.

Now it is the time for time evolution! As we know that time evolution can be done by using unitary operator $U = e^{-\frac{iHt}{\hbar}}$ &

$\rho_{out}(t) = U\rho_{out}(0)\,U^\dagger = e^{-\frac{iHt}{\hbar}}\,\rho_{out}(0)\,e^{\frac{iHt}{\hbar}}$

Density matrices at different time are found out. Took the partial trace and reduced density matrix preserves its structure as in fig 1, but matrix elements are modified. Now we were able to find the concurrence & mutual information for bipartite systems at different times. Plot of variation of concurrence &mutual information are shown below.

5. Results & Plots

Case 1; $E_1 = \sqrt{p}\,\sigma_l^z, E_2 = \sqrt{q}\,\sigma_m^z\ , E_3 = \sqrt{1-p-q}\,I$

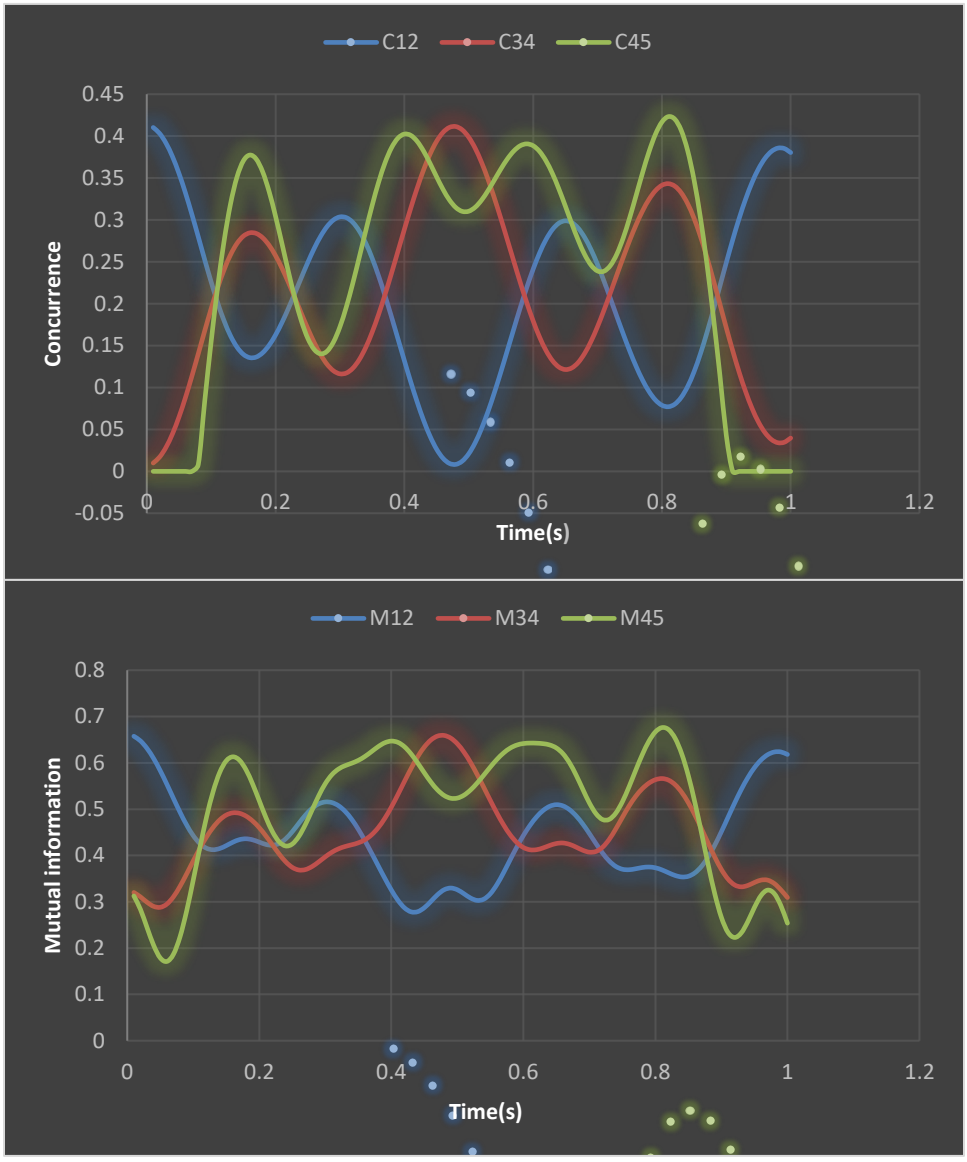

Fig 5: Variation of Concurrence & Mutual information with time for N=8
& Time varies from 0 to 1 s in the steps of 0.01s.
Channel is acted at site 4&5 and we studied the concurrence & mutual information for bipartite systems 1&2, 3&4 ,4&5 for the channel parameter p=0.33, q=0.33. Bipartite entanglement was less initially& from there it varies periodically for the pairs where the channel action is in either or both the sites. We can also see a similar behaviour of mutual information.

Case 2; $E_1 = \sqrt{p}\,\sigma_l^z \sigma_m^z$, $E_2 = \sqrt{1-p}\,I$

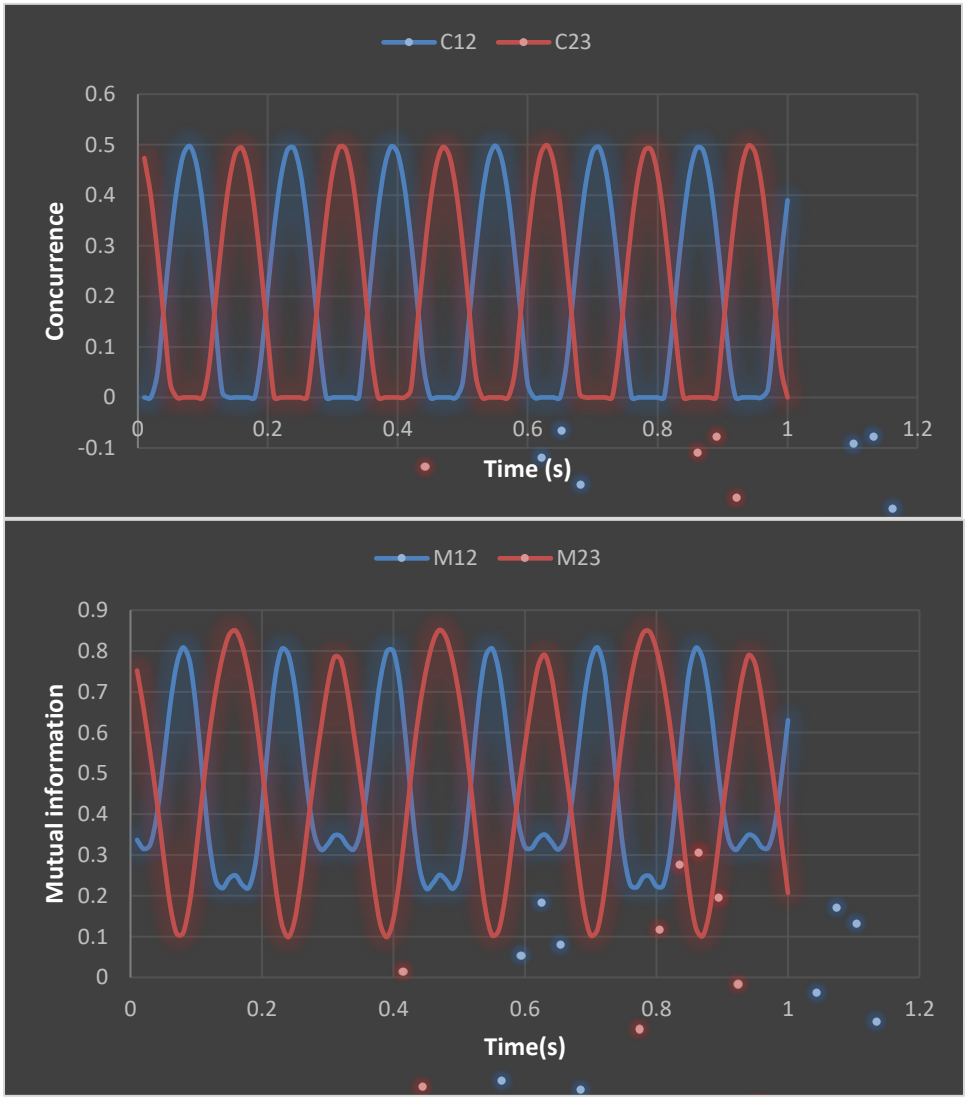

Fig 6: Variation of Concurrence & Mutual information with time for N=4
& Time varies from 0 to 1 s in the steps of 0.01s.
Channel is acted at site 2&3 and we studied the concurrence & mutual information for bipartite systems 1&2, 2&3 for the channel parameter p=0.5. Bipartite entanglement was less initially& from there it varies periodically for the pairs where the channel action is in only at any of the sites. We can also see a similar behaviour of mutual information. For smaller chains these variations are periodic.

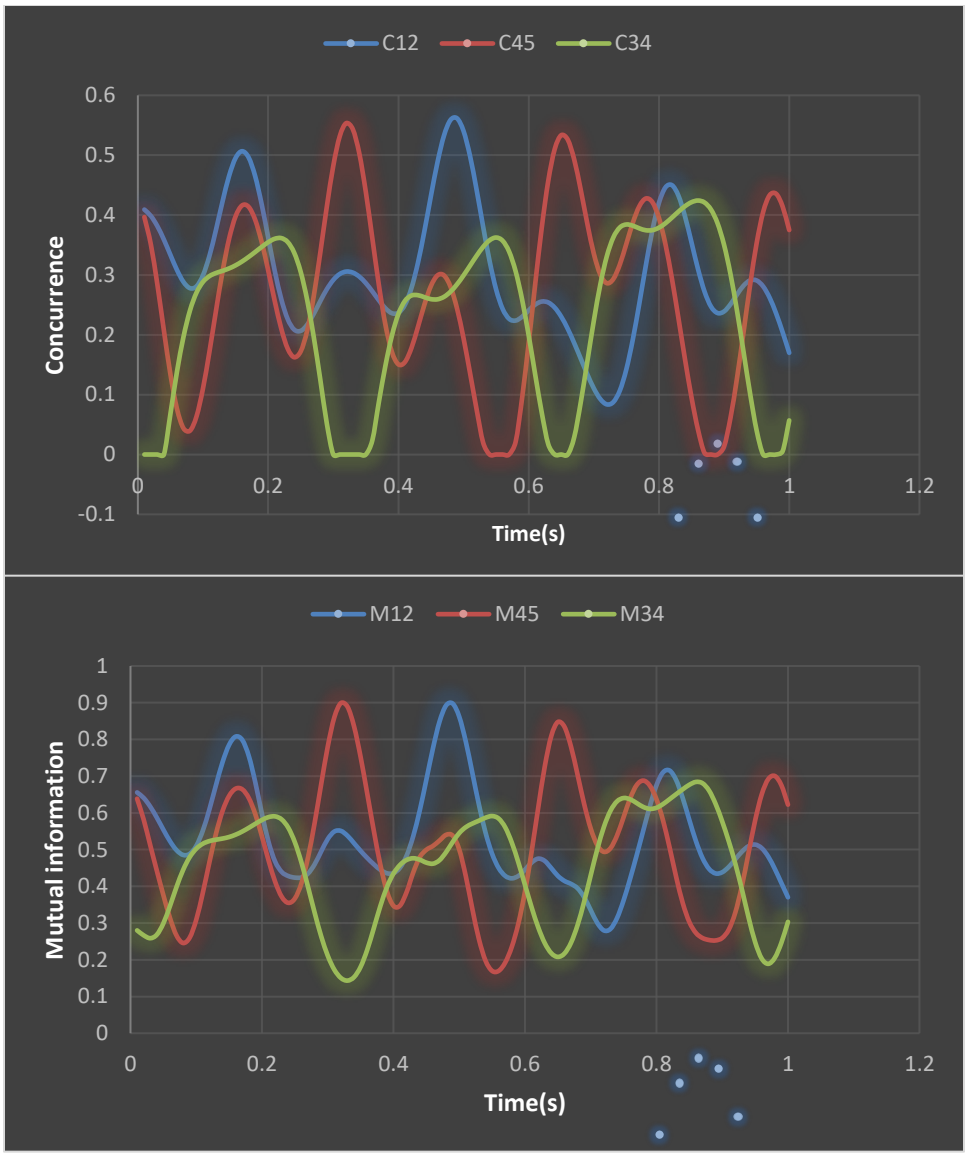

Fig 7: Variation of Concurrence & Mutual information with time for N=8 & Time varies from 0 to 1 s in the steps of 0.01s.

Channel is acted at site 4&5 and we studied the concurrence & mutual information for bipartite systems 1&2, 3&4,4&5 for the channel parameter p=0.5. Bipartite entanglement was less initially& from there it varies periodically for the pairs where the channel action is in only at any of the sites. We can also see a similar behaviour of mutual information. Periodic nature lost as we go for longer chain.

6. Concluding Remarks

Ground state of an antiferromagnetic ring with an even number of particles typically does not maximize the nearest neighbor concurrence over all states, but that it does achieve such a maximum over the set of translationally invariant states with no net spin in Z direction. This set of "balanced" states includes all the eigen states of total spin with eigen value zero.so we can also say that ground state maximizes concurrence relative to all the spin 0 or rotationally invariant states. A simple formula relates the concurrence between arbitrary pair of spins to their diagonal correlation function. This substantially simplifies the calculation of concurrence both numerically and analytically. Nearest neighbor concurrence is directly related to ground state energy. Nearest neighbor concurrence decreases as we go for higher number of particles system, finally it reaches a constant as $N \to \infty$.Correlation between non-nearest neighbor pair is zero. Mutual information between arbitrary pairs of spins depends on their separation. As separation increases mutual information decreases.

General evolution of ground state density matrix can be made by using Krauss-Sudarshan operators (CPTP-completely positive trace preserving map). Concurrence &mutual information varies periodically with time for shorter chains &periodicity lost as we go for longer chains. The set of quantum channels we chosen here does not bring any non-nearest neighbor correlation for a state which having maximum entropy. Krauss representations may bring long-range correlations, ability to generate and control such correlations will of great importance in future quantum technology.