Random Numbers

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CS21BTECH11047

***Abstract*—This is the solution manual of Class Assignment.**

1 Unifom Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
wge thttps://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/exrand.c
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/cdf_plot.py
```

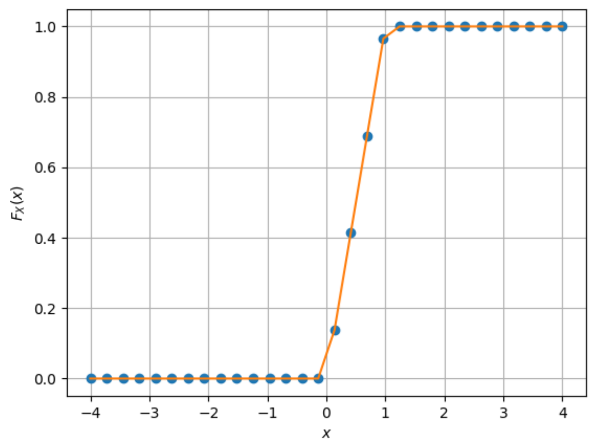

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{1.3}$$

Hence, If $x \leq 0$,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{1.4}$$

$$F_U(x) = \int_{-\infty}^{x} 0\, dx \tag{1.5}$$

$$= 0 \tag{1.6}$$

If $0 < x < 1$,

$$F_U(x) = \int_{0}^{x} f_U(x)\, dx \tag{1.7}$$

$$F_U(x) = \int_{0}^{x} 1\, dx \tag{1.8}$$

$$= x \tag{1.9}$$

If $x \geq 1$,

$$F_U(x) = \int_{1}^{x} f_U(x)\, dx \tag{1.10}$$

$$F_U(x) = \int_{1}^{x} 0\, dx \tag{1.11}$$

$$= 0 \tag{1.12}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.13}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.14}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/
    mean_variance.c
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/coeffs.h
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.15}$$

Solution:

$$F_U(x) = \begin{cases} x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.16}$$

$$dF_U(x) = \begin{cases} dx, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.17}$$

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.18}$$

$$E[U^k] = \int_{0}^{1} x^k dx \tag{1.19}$$

$$= \frac{1}{k+1} \tag{1.20}$$

To verify the result obtained for mean and variance from the C code and by the integral:
The result given by the C code was,
Mean = 0.50007 and Variance = 0.083301.
Now, using the formula

$$E[U^k] = \frac{1}{k+1} \tag{1.21}$$

$$E[U] = \frac{1}{1+1} = \frac{1}{2} \tag{1.22}$$

$$E[U^2] = \frac{1}{2+1} = \frac{1}{3} \tag{1.23}$$

Therefore, Mean = $E[U] = 0.50$
Variance =

$$E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} \tag{1.24}$$

$$= \frac{1}{12} \tag{1.25}$$

$$= 0.0833 \tag{1.26}$$

Hence the values of mean and variance are almost similar, and is hence theoretically verified.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/exrand.c
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/coeffs.h
```

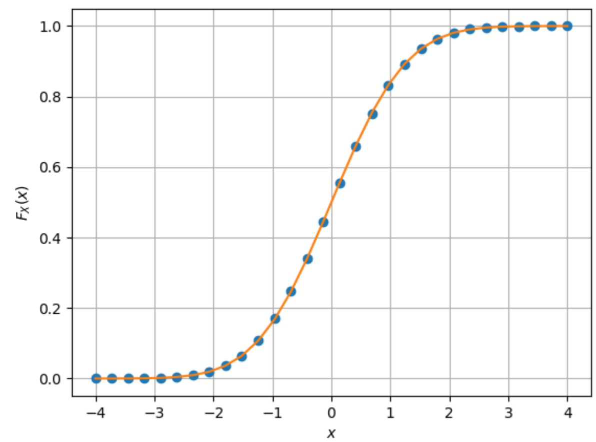

Fig. 2.2: The CDF of X

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2
The CDF of a random variable U has the

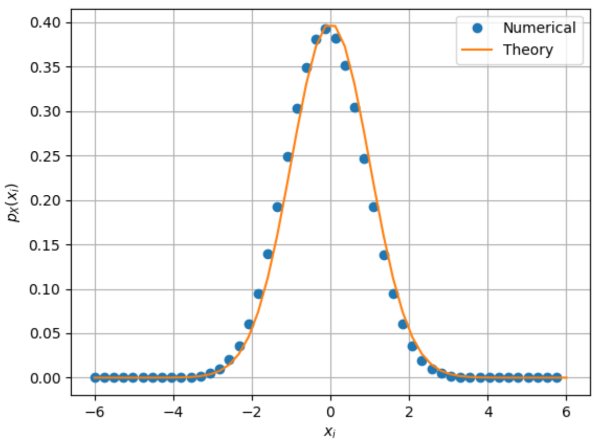

Fig. 2.3: The PDF of X

following properties:

a) $F_U(x)$ is a non decreasing function of x where $-\infty < x < \infty$
b) $F_U(x)$ ranges from 0 to 1
c) $F_U(x) = 0$ as $x \to -\infty$
d) $F_U(x) = 1$ as $x \to \infty$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/pdf_plot.py
```

The PDF of a random variable X has the following properties:

a) The probability density function is non-negative for all the possible values.

b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$
c) $f(x) = 0$ as $x \to -\infty$
d) $f(x) = 0$ as $x \to \infty$

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/
    mean_variance.c
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: By definition,

$$p_X(x)\, dx = dF_U(x) \tag{2.4}$$

Also, by definition,

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.5}$$

Hence,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x)\, dx \tag{2.6}$$

$$E[U] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$

$$= 0 \tag{2.9}$$

Since the function is an odd function its integral over real numbers would be 0 Also,

$$\text{variance} = E\left[U^2\right] - (E[U])^2 \tag{2.10}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

$$= \int_{-\infty}^{\infty} x x \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.13}$$

Using integration by parts,

$$\tag{2.14}$$

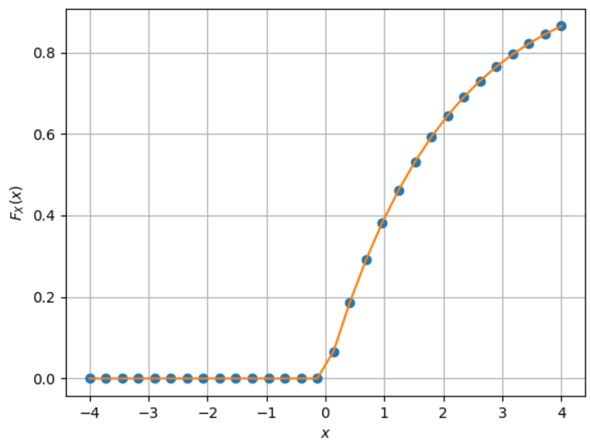

Fig. 3.1: The CDF of X

$$= \Big|_{-\infty}^{\infty} - x\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{x^2}{2}\right)} + \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}exp\left(-\frac{x^2}{2}\right) \tag{2.15}$$

$$= 0 + \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} \tag{2.16}$$

$$= 1 \tag{2.17}$$

$$\implies \text{variance} = 1 \tag{2.18}$$

Also,

$$F_U(x) = \int_{-\infty}^{x} p_X(x)\,dx \tag{2.19}$$

$$= \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\,dx \tag{2.20}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section3/cdf_plot.py
```

The CDF of X is plotted in Fig. 3.1

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$

$$= \Pr(-2ln(1-U) \le x) \tag{3.3}$$

$$= \Pr\left(ln(1-U) \ge -\frac{x}{2}\right) \tag{3.4}$$

$$= \Pr\left(1-U \ge e^{-\frac{x}{2}}\right) \tag{3.5}$$

$$= \Pr\left(U \le 1-e^{-\frac{x}{2}}\right) \tag{3.6}$$

Let $y = 1 - e^{-\frac{x}{2}}$

$$= \Pr(U \le y) \tag{3.7}$$

$$= F_U(y) \tag{3.8}$$

$$F_U(y) = \begin{cases} y, & y \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

$$0 \le 1 - e^{-\frac{x}{2}} \le 1 \tag{3.10}$$

$$0 \le e^{-\frac{x}{2}} \le 1 \tag{3.11}$$

$$\implies x > 0 \tag{3.12}$$

Therefore,

$$F_V(x) = F_U(y) = \begin{cases} 1 - e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.13}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section4/coeffs.h
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section4/exrand.c
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.3

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: The CDF of T is given by

$$F_T(t) = \Pr(T \le t) = \Pr(U_1 + U_2 \le t) \tag{4.2}$$

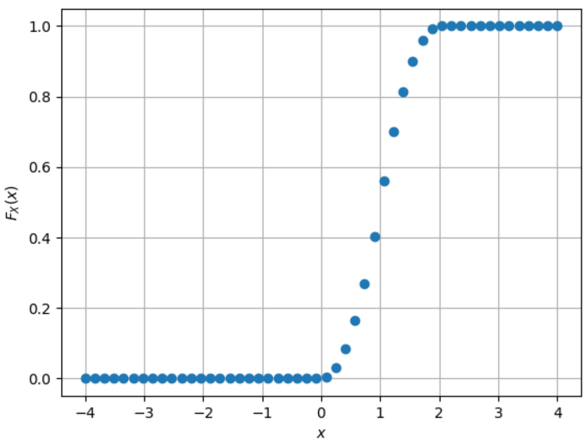

Fig. 4.2: The CDF of T

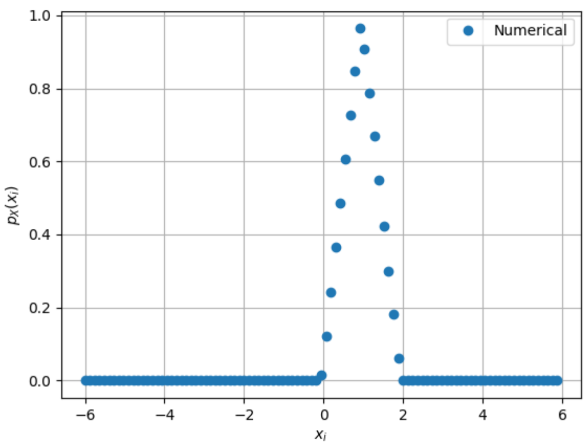

Fig. 4.3: The PDF of T

Since $U_1, U_2 \in [0,1] \implies U_1 + U_2 \in [0,2]$
Therefore, if $t \geq 2$, then $U_1 + U_2 \leq t$ is always true and if $t < 0$, then $U_1 + U_2 \leq t$ is always false.
Now, fix the value of U_1 to be some x

$$x + U_2 \leq t \implies U_2 \leq t - x \tag{4.3}$$

If $0 \leq t \leq 1$, then x can take all values in $[0, t]$

$$F_T(t) = \int_0^t \Pr\left(U_2 \leq t - x\right) p_{U_1}(x)\, dx \tag{4.4}$$
$$= \int_0^t F_{U_2}(t-x) p_{U_1}(x)\, dx \tag{4.5}$$

$$0 \leq x \leq t \implies 0 \leq t - x \leq t \leq 1 \tag{4.6}$$
$$\implies F_{U_2}(t-x) = t - x \tag{4.7}$$

$$F_T(t) = \int_0^t (t-x) \cdot 1 \cdot dx \tag{4.8}$$
$$= tx - \frac{x^2}{2}\bigg|_0^t \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$

If $1 < t < 2$, x can only take values in $[0, 1]$ as $U_1 \leq 1$

$$F_T(t) = \int_0^1 F_{U_2}(t-x) \cdot 1 \cdot dx \tag{4.11}$$

$$0 \leq x \leq t - 1 \implies 1 \leq t - x \leq t \tag{4.12}$$
$$t - 1 \leq x \leq 1 \implies 0 < t - 1 \leq t - x \leq 1 \tag{4.13}$$

$$F_T(t) = \int_0^{t-1} 1\, dx + \int_{t-1}^1 (t-x)\, dx \tag{4.14}$$
$$= t - 1 + t(1 - (t-1)) - \frac{1}{2} + \frac{(t-1)^2}{2} \tag{4.15}$$
$$= t - 1 + 2t - t^2 - \frac{1}{2} + \frac{t^2}{2} + \frac{1}{2} - t \tag{4.16}$$
$$= -\frac{t^2}{2} + 2t - 1 \tag{4.17}$$

Therefore,

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.18}$$

The PDF of T is given by

$$p_T(t) = \frac{d}{dt} F_T(t) \tag{4.19}$$
$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2 - t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.20}$$

4.5 Verify your results through a plot.
Solution: The following codes have been used to plot the CDF and PDF

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
```

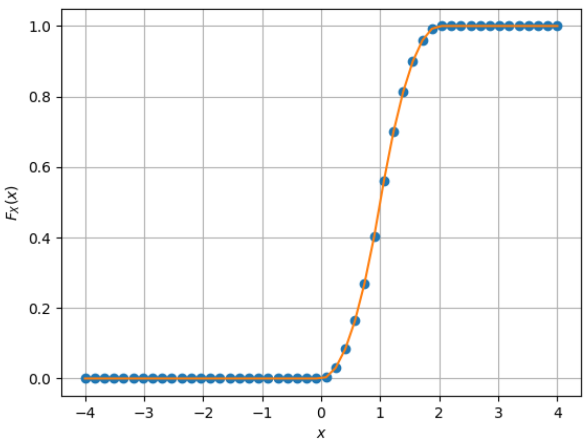

Fig. 4.5: The CDF of T

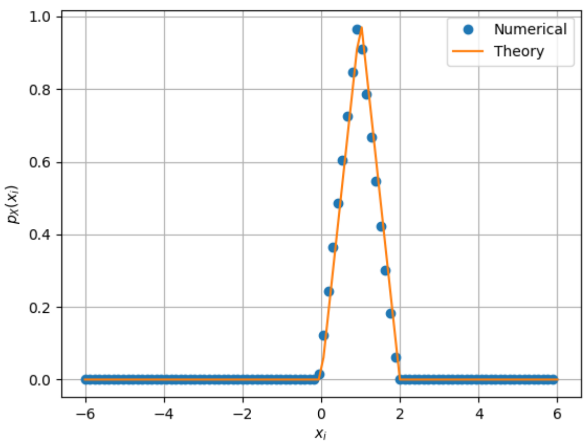

Fig. 4.5: The PDF of T

Class_Assignments/Section4/cdf_plot.py
wget https://github.com/PrashamW/AI1110–Assignments/blob/main/Class_Assignments/Section4/pdf_plot.py

The CDF of T is plotted in Fig. 4.5
The PDF of T is plotted in Fig. 4.5

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, and $N \sim 01$.

5.3 Plot Y using a scatter plot.

5.4 Guess how to estimate X from Y.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.6 Find P_e assuming that X has equiprobable symbols.

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.4}$$

5.10 Repeat the above exercise using the MAP criterion.

6 Gaussian to Other

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.