Random Numbers

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CS21BTECH11047

***Abstract*—This is the solution manual of Class Assignment.**

1 Unifom Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
wge thttps://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/exrand.c
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section1/cdf_plot.py
```

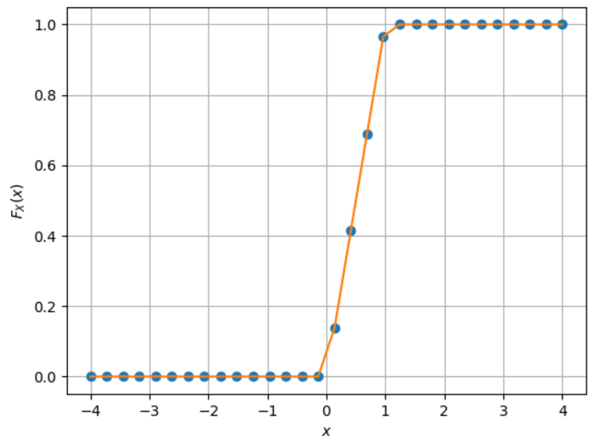

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$f_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{1.3}$$

Hence, If $x \leq 0$,

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{1.4}$$

$$F_U(x) = \int_{-\infty}^{x} 0\, dx \tag{1.5}$$

$$= 0 \tag{1.6}$$

If $0 < x < 1$,

$$F_U(x) = \int_{0}^{x} f_U(x)\, dx \tag{1.7}$$

$$F_U(x) = \int_{0}^{x} 1\, dx \tag{1.8}$$

$$= x \tag{1.9}$$

If $x \geq 1$,

$$F_U(x) = \int_{1}^{x} f_U(x)\, dx \tag{1.10}$$

$$F_U(x) = \int_{1}^{x} 0\, dx \tag{1.11}$$

$$= 0 \tag{1.12}$$

1.4 The mean of U is defined as

$$E\left[U\right] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.13}$$

and its variance as

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{1.14}$$

Write a C program to find the mean and variance of U.
Solution:

```
wget https://github.com/PrashamW/AI1110-
    Assignments/blob/main/
    Class_Assignments/Section1/
    mean_variance.c
wget https://github.com/PrashamW/AI1110-
    Assignments/blob/main/
    Class_Assignments/Section1/coeffs.h
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.15}$$

Solution:

$$F_U(x) = \begin{cases} x, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.16}$$

$$dF_U(x) = \begin{cases} dx, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{1.17}$$

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.18}$$

$$E[U^k] = \int_{0}^{1} x^k dx \tag{1.19}$$

$$= \frac{1}{k+1} \tag{1.20}$$

To verify the result obtained for mean and variance from the C code and by the integral:
The result given by the C code was,
Mean = 0.50007 and Variance = 0.083301.
Now, using the formula

$$E[U^k] = \frac{1}{k+1} \tag{1.21}$$

$$E[U] = \frac{1}{1+1} = \frac{1}{2} \tag{1.22}$$

$$E[U^2] = \frac{1}{2+1} = \frac{1}{3} \tag{1.23}$$

Therefore, Mean = $E[U] = 0.50$
Variance =

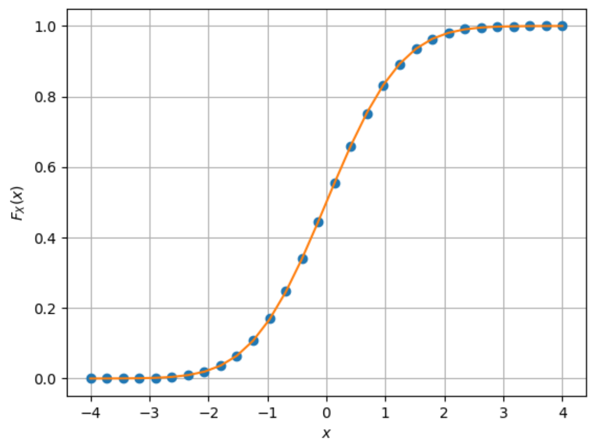

Fig. 2.2: The CDF of X

$$E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} \tag{1.24}$$

$$= \frac{1}{12} \tag{1.25}$$

$$= 0.0833 \tag{1.26}$$

Hence the values of mean and variance are almost similar, and is hence theoretically verified.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
wget https://github.com/PrashamW/AI1110-
    Assignments/blob/main/
    Class_Assignments/Section2/exrand.c
wget https://github.com/PrashamW/AI1110-
    Assignments/blob/main/
    Class_Assignments/Section2/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2
The CDF of a random variable U has the

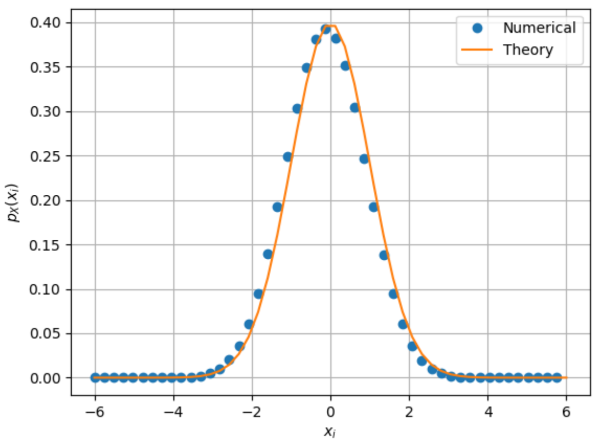

Fig. 2.3: The PDF of X

following properties:

a) $F_U(x)$ is a non decreasing function of x where $-\infty < x < \infty$
b) $F_U(x)$ ranges from 0 to 1
c) $F_U(x) = 0$ as $x \to -\infty$
d) $F_U(x) = 1$ as $x \to \infty$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/pdf_plot.py
```

The PDF of a random variable X has the following properties:

a) The probability density function is non-negative for all the possible values.
b) $\int_{-\infty}^{\infty} f(x)\, dx = 1$
c) $f(x) = 0$ as $x \to -\infty$
d) $f(x) = 0$ as $x \to \infty$

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/
    mean_variance.c
wget https://github.com/PrashamW/AI1110−
    Assignments/blob/main/
    Class_Assignments/Section2/coeffs.h
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: By definition,

$$p_X(x)\, dx = dF_U(x) \tag{2.4}$$

Also, by definition,

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.5}$$

Hence,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x)\, dx \tag{2.6}$$

$$E[U] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.7}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.8}$$

$$= 0 \tag{2.9}$$

Since the function is an odd function its integral over real numbers would be 0 Also,

$$\text{variance} = E\left[U^2\right] - (E[U])^2 \tag{2.10}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

$$= 1 \tag{2.13}$$

$$\implies \text{variance} = 1 \tag{2.14}$$

Also,

$$F_U(x) = \int_{-\infty}^{x} p_X(x)\, dx \tag{2.15}$$

$$= \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\, dx \tag{2.16}$$

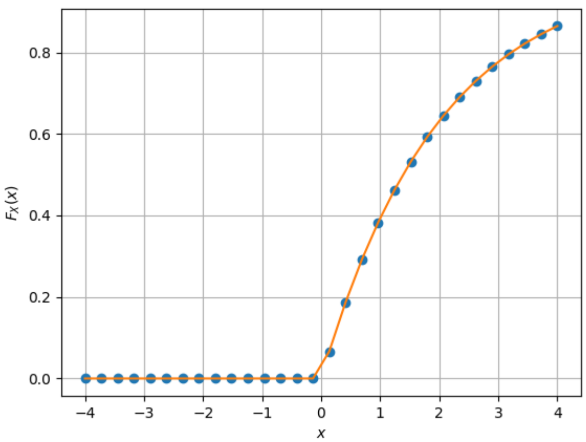

Fig. 3.1: The CDF of X

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:

```
wget https://github.com/PrashamW/AI1110-Assignments/blob/main/Class_Assignments/Section3/cdf_plot.py
```

The CDF of X is plotted in Fig. 3.1

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$\begin{aligned} F_V(x) &= \Pr(V \le x) && (3.2) \\ &= \Pr(-2ln(1-U) \le x) && (3.3) \\ &= \Pr\left(ln(1-U) \ge -\frac{x}{2}\right) && (3.4) \\ &= \Pr\left(1-U \ge e^{-\frac{x}{2}}\right) && (3.5) \\ &= \Pr\left(U \le 1-e^{-\frac{x}{2}}\right) && (3.6) \end{aligned}$$

Let $y = 1 - e^{-\frac{x}{2}}$

$$\begin{aligned} &= \Pr(U \le y) && (3.7) \\ &= F_U(y) && (3.8) \end{aligned}$$

$$F_U(y) = \begin{cases} y, & y \in (0,1) \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

$$\begin{aligned} 0 \le 1 - e^{-\frac{x}{2}} \le 1 && (3.10) \\ 0 \le e^{-\frac{x}{2}} \le 1 && (3.11) \\ \implies x > 0 && (3.12) \end{aligned}$$

Therefore,

$$F_V(x) = F_U(y) = \begin{cases} 1 - e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.13}$$