Linear Algebra: Review

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Linear Algebra: Review

(Vector Spaces)

Motivation of studying linear algebra

- **Linear Regression**: A supervised learning algorithm that uses linear algebra (LA) to model the relationship between dependent and independent variables. It can be solved using matrix algebra, which involves finding the inverse of a matrix and multiplying matrices.
- **Support Vector Machines (SVMs)**: A type of supervised learning algorithm that uses LA to separate data samples into different classes. SVMs find the hyperplane that maximally separates the data samples by solving a quadratic optimization problem using LA.
- **Neural Networks**: A type of ML algorithm that are inspired by the structure and function of the human brain. These use LA to represent the weights between neurons and to perform the calculations of forward and backward propagation through the network.

- **Principal Component Analysis (PCA)**: A method used for dimensionality reduction and feature extraction that uses LA to find the eigenvectors and eigenvalues of the covariance matrix of the data.

Learning Objectives

- In this section, we will study the concepts related to the vector spaces. In particular, we will study
  - Group
  - Field
  - Vector space
  - Metric
  - Norm
  - Inner product
  - Basis vectors
  - Linear independence of basis vectors
  - Span
  - Basis

Notations

- Scalars will be written as small case letters in italics: x
- Vectors will be written as small case letters in bold: $\mathbf{x}$
- Matrices will be written as upper case letters in bold: $\mathbf{X}$
- The $(i,k)^{\text{th}}$ component of a matrix $\mathbf{X}$ will be written as X_{ik}
- Single random variable will be written as upper case letter in italics: X
- Vector random variable will be written as upper case, italics, and bold: $\boldsymbol{X}$
- Complex conjugation of a variable x will be denoted as: $\bar{x}$
- Transpose of a vector will be represented using superscript T: $\mathbf{x}^{\mathrm{T}}$
- Complex conjugate transpose of a vector will be represented using superscript H: $\mathbf{x}^{\mathrm{H}}$

Group

A group $(G, *)$ consists of the following:

(i) A set G

(ii) A rule or binary operation '$*$' (set G is closed under this operation) which associates with each pair of elements x and y in G, an element $(x * y)$ in G such that

a. This binary operation is associative:

i.e., $x * (y * z) = (x * y) * z \quad \forall\, x, y, z \in G$.

b. There exists an element 'e' called identity of group G s.t. $e * x = x * e = x \quad \forall\, x \in G$

c. For every $x \in G$, there exists an element x^{-1} such that

$x * x^{-1} = e = x^{-1} * x \quad \forall\, x \in G$

d. If $x * y = y * x \quad \forall\, x, y \in G$

$\Rightarrow$ Commutative group or Abelian group

Examples

1. Set of n x n invertible matrices with matrix multiplication $\Rightarrow$ $G = \{n \times n$ invertible matrices$\}$, $* \equiv$ Matrix multip.
2. Set of periodic signals with time period T under binary operation '$*$' = '+'

Ex-1

1) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ ✓

2) $I_{n \times n} \cdot A = A \cdot I \quad \forall \; A_{n \times n} \in G$

$e \nearrow I$, $\uparrow e$

3) $A^{-1} \cdot A = A \cdot A^{-1} = I$

$AB \neq BA \Rightarrow n$

$G = \{$Set of periodic signal with $T\}$

$* = +$

Ex-2

1) $f_1 + (g_1 + g_2) = (f_1 + g_1) + g_2$ ✓

2) $f + 0 = 0 + f = f$

$\uparrow$ zero fn.

3) $f + (-f) = 0 = (-f) + f$

4) $f + g = g + f$

Field

A field $(F, +, \cdot)$ consists of the following:

(i) A set F

(ii) Two binary operations '+' and '·' such that

a. $(F, +)$ is an Abelian group

b. Define $F^*=F-\{0\}$. $(F^*, \cdot)$ is an Abelian group

c. The multiplication operation distributes over addition:

- Left distributive:

$$x\cdot(y+z)=x\cdot y + x\cdot z \qquad \forall\, x, y, z \in F$$

- Right distributive:

$$(x+y)\cdot z=x\cdot z + y\cdot z \qquad \forall\, x, y, z \in F$$

Examples: Check for F = R; F = C; F= Z; F=Q

Examples

F = R, C and Q form a field, while Z does not form a field.

set of real no.

C = set of complex nos ✓

Q = " " rational nos. ✓

Z = set of integers

Vector Space

A set V together with a map

$V \equiv$ consists of vector

$V_n(\mathbb{R}) \rightarrow$ field

dimension of this vector space

$$'+': V \times V \rightarrow V$$

$$(\mathbf{v}_1, \mathbf{v}_2) \rightarrow (\mathbf{v}_1 + \mathbf{v}_2) \text{ called vector addition}$$

And $'.' : F \times V \rightarrow V$

$$(\alpha, \mathbf{v}) \rightarrow (\alpha \cdot \mathbf{v}) \text{ called scalar multiplication}$$

is called a F-vector space or vector space over the field F if the following are satisfied:

(i) $(V, +)$ is an Abelian group

(ii) $\alpha \cdot (\mathbf{v}_1 + \mathbf{v}_2) = \alpha \cdot \mathbf{v}_1 + \alpha \cdot \mathbf{v}_2$ $\quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V$ and $\forall\, \alpha \in F$

(iii) $(\alpha_1 + \alpha_2) \cdot \mathbf{v} = \alpha_1 \cdot \mathbf{v} + \alpha_2 \cdot \mathbf{v}$ $\quad \forall\, \mathbf{v} \in V$ and $\forall\, \alpha_1, \alpha_2 \in F$

(iv) $(\alpha\beta) \cdot \mathbf{v} = \alpha(\beta\mathbf{v}) = \beta(\alpha\mathbf{v})$ $\quad \forall\, \mathbf{v} \in V$ and $\forall\, \alpha, \beta \in F$

(v) $1 \cdot \mathbf{v} = \mathbf{v}$ $\quad \forall\, \mathbf{v} \in V$

(vi) $\alpha \cdot \mathbf{0} = \mathbf{0}$ $\quad \forall\, \alpha \in F$

(vii) $0 \cdot \mathbf{v} = \mathbf{0}$ $\quad \forall\, \mathbf{v} \in V$

(viii) If $\mathbf{v} \neq \mathbf{0}$, then $\alpha \cdot \mathbf{v} = \mathbf{0}$ implies that $\alpha = 0$.

(ix) If V is a vector space over a field F, then any linear combination of vectors lying in V (with scalars from F) would again lie in V.

Vector Space

Examples:

1. Set of periodic signals with the same time period T over the field $\boxed{F = R}$ form a vector space
2. Set of finite energy signals over the field $F = R$ form a vector space
3. $V_n(F) = n$-dimensional vector space over the field $F = R$ or C

V = { set of periodic signal with period T }

+

$3 \cdot (f_1 + f_2) = 3f_1 + 3f_2$

Examples

$V_n(F)$ = n-dimensional vector space over the field $F = R$ or C

$\uparrow$

$n = 3$ $\quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad F = \mathbb{R}$

$+ :$ $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ $\quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \equiv$ additive identity

$\underline{0}$

$\underline{v} + \underline{w}$ $\quad \underline{v} \longrightarrow (-\underline{v})$

$(V, +)$

$3(\underline{v} + \underline{w}) = 3\underline{v} + 3\underline{w}$

Metric Space

X , d

Metric is a map

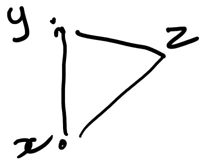
$$d: X \times X \rightarrow R \longrightarrow \text{real no.}$$

that satisfies the following:

(i) $d(x, y) \geq 0$ and $d(x, y)=0$ iff $x = y$ $\forall\, x, y \in X$

(ii) $d(x, y) = d(y, x)$ $\forall\, x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ $\forall\, x, y, z \in X$



This map is called a **metric** and a set equipped with this map is called a **metric space** and is denoted by $\underline{(X, d)}$.

Note that metric is a generalization of the notion of distance.

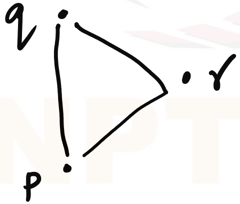
Examples

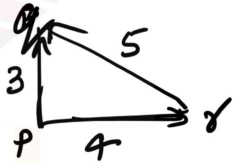
Euclidean Distance $\qquad p \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad q \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad d(p,q) = \sqrt{\underbrace{(x_1-y_1)^2}_{0} + \underbrace{(x_2-y_2)^2}_{0}}$

1) $d(p, q) \geq 0$ and $d(p, q) = 0$ iff $p = q$ ✓

$d(x, y) = d(y, x)$ $\qquad d(p,q) = \cancel{dp}\ d(q,p)$ ✓

$d(x, y) \leq d(x, z) + d(z, y)$





Norm

Let V be a F-vector space. A map

$\|\cdot\|: V \rightarrow R$ is called a **norm** if it satisfies the following:

(i) $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ iff $\mathbf{v}=0$ $\quad \forall\, \mathbf{v} \in V$

(ii) $\|\alpha \mathbf{v}\| = |\alpha|\, \|\mathbf{v}\|$ $\quad \forall\, \mathbf{v} \in V$ and $\forall\, \alpha \in F$

(iii) $\|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$ $\quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V$

A vector space equipped with a norm is called a **normed vector space**.

Example: Let V be a F-vector space equipped with a norm.

Prove that $d(\mathbf{v}_1, \mathbf{v}_2) = \|\mathbf{v}_1 - \mathbf{v}_2\|$ is a proper metric.

Examples

Let V be a F-vector space equipped with a norm. Prove that $d(\mathbf{v}_1, \mathbf{v}_2) = \|\mathbf{v}_1 - \mathbf{v}_2\|$ is a proper metric.

Proof:

1) $d(\mathbf{v}_1, \mathbf{v}_2) \geq 0$ and $d(\mathbf{v}_1, \mathbf{v}_2)=0$ iff $\mathbf{v}_1 = \mathbf{v}_2$

property -1 of norm

$d(v_1, v_2) = \|(v_1 - v_2)\| = \|v\| \geq 0$

$\|v\| = 0$ iff $v = 0$

$\Rightarrow v_1 - v_2 = 0$

$\Rightarrow v_1 = v_2$

2) $d(\mathbf{v}_1, \mathbf{v}_2) = d(\mathbf{v}_2, \mathbf{v}_1)$

$d(v_2, v_1) = \|v_2 - v_1\|$

$= \|(-1)(v_1 - v_2)\| = |(-1)| \; \|v_1 - v_2\|$

$= \|v_1 - v_2\| = d(v_1, v_2)$

3) $d(\mathbf{v}_1, \mathbf{v}_2) \leq d(\mathbf{v}_1, \mathbf{v}_3) + d(\mathbf{v}_3, \mathbf{v}_2)$

L.H.S.

$\|v_1 - v_2\|$ $\cancel{\leq}$

R.H.S

$\|v_1 - v_3\| + \|v_3 - v_2\|$ —— property -3

$\|v + w\| \leq \|v\| + \|w\|$

$d(v_1, v_2) = \|v_1 - v_2\| = \|v_1 - \cancel{v_3} + \cancel{v_3} - v_2\| \leq \|v\| + \|w\|$

Inner Product

Let V be a F-vector space.

A map

$$< , > : V \times V \rightarrow F$$

is called an inner product if it satisfies the following:

(i) $< \mathbf{v}, \mathbf{v} > \ \geq 0$ and $< \mathbf{v}, \mathbf{v} > \ = 0$ iff $\mathbf{v} = 0$ $\quad \forall\, \mathbf{v} \in V$

(ii) $< \mathbf{v}_1, \mathbf{v}_2 > \ = \ \overline{< \mathbf{v}_2, \mathbf{v}_1 >}$ $\quad \forall\, \mathbf{v}_1, \mathbf{v}_2 \in V$

'‾' denotes the complex conjugate operation.

(iii) It is linear in the first coordinate.

$$< \underbrace{\alpha_1\mathbf{v}_1 + \overset{\downarrow \mathrm{I}}{\alpha_2\mathbf{v}_2}}, \overset{\downarrow \mathrm{II}}{\mathbf{w}} > \ = \alpha_1 < \mathbf{v}_1, \mathbf{w} > + \ \alpha_2 < \mathbf{v}_2, \mathbf{w} > \quad \forall\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w} \in V \text{ and } \forall\, \alpha_1, \alpha_2 \in F$$

(iv) It is conjugate linear in the second coordinate.

$$< \overset{\downarrow \mathrm{I}}{\mathbf{v}}, \underbrace{\alpha_1\mathbf{w}_1 + \overset{\downarrow \mathrm{II}}{\alpha_2\mathbf{w}_2}} > \ = \overline{\alpha_1} < \mathbf{v}, \mathbf{w}_1 > + \ \overline{\alpha_2} < \mathbf{v}, \mathbf{w}_2 >$$

$$\forall\, \mathbf{w}_1, \mathbf{w}_2, \mathbf{v} \in V \text{ and } \forall\, \alpha_1, \alpha_2 \in F$$

Inner Product

Example 1: Define $\|\mathbf{v}\| = <\mathbf{v}, \mathbf{v}>^{1/2}$. Show that it is a proper norm.

Thus, if a space is an inner product space, we can define norm and then define metric.

Example 2: Consider $V_n(F)$, where vectors are n-tuples of scalars, i.e., $\mathbf{x} \in V$ and $x_i \in F$

i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} \leftarrow$$

Define inner product as: $\boxed{<\mathbf{x}, \mathbf{y}> = \sum_{i=1}^{n} x_i \bar{y}_i}$

Show that it satisfies all the properties of an inner product.

Linear Independence of Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots.., \mathbf{v}_n\}$ in $V_n(F)$ is called linearly independent (LI) if $\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.

Definition: The number of maximal LI vectors in a vector space V is called the **dimension** of the vector space and the maximal LI vectors is called a **basis** for V.

If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots.., \mathbf{e}_n\}$ is a basis for $V_n(F)$, then for any $\mathbf{v} \in V$ and for some $\alpha_i \in F$

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$$

$V_3(\mathbb{R})$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{e_1 \quad e_2 \quad e_3}$$

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3$$

$$= \underbrace{v_1}_{\alpha_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underbrace{v_2}_{\alpha_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \underbrace{v_3}_{\alpha_3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Orthogonal and Orthonormal Basis

Definition: A set of basis vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots.., \mathbf{v}_n\}$ spanning an inner product space V is called **orthogonal** basis if

(i) $\mathbf{v}_i \neq 0 \quad \forall \; i$

(ii) $< \mathbf{v}_i, \mathbf{v}_j > = 0 \quad \forall \; i \neq j$

$\langle v_1, v_1 \rangle = 1 = ||v_1||^2$

$\langle v_2, v_2 \rangle = 1$

$\langle v_3, v_3 \rangle = 1$

If additionally, the below condition is satisfied:

(iii) $< \mathbf{v}_i, \mathbf{v}_j > = 1 \quad \forall \; i = j$

Then, the set of basis vectors is called as the orthonormal basis. In this case, we can also write the above conditions as $< \mathbf{v}_i , \mathbf{v}_j > = \delta_{ij}$.

Summary

- In this section, we studied the concepts of
  - Group
  - Field
  - Vector space
  - Metric
  - Norm
  - Inner product
  - Basis vectors
  - Linear independence of basis vectors
  - Span
  - Basis

with examples.

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