Constraint Satisfaction Problems

Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs

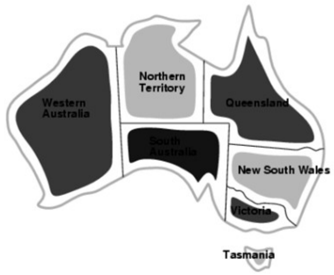
Constraint satisfaction problems (CSPs)

- Standard search problem:
  - state is a "black box" – any data structure that supports successor function, heuristic function, and goal test
- CSP:
  - state is defined by variables X_i with some values from domain D_i
  - a set of constraints C_i specifies allowable combinations of values for subsets of variables
  - a consistent state violates none of the constraints C
  - a complete assignment has values assigned to all variables.
  - A Solution is a complete, consistent assignment.
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring

- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}

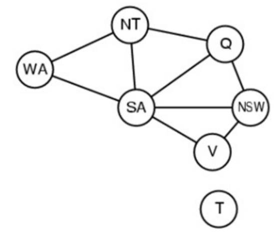
Example: Map-Coloring



- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

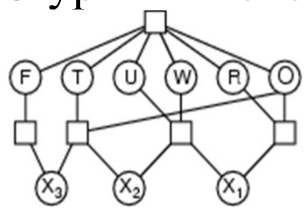
- Discrete variables
  - finite domains:
    - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming algorithms from operations research

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA $\neq$ green
- Binary constraints involve pairs of variables,
  - e.g., SA $\neq$ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints
- Global constraints: arbitrary # of constraints, not necessarily all the variables in a problem
  - e.g., *AllDiff*: all values must be different. Sudoku rows, cols, squares

Example: Cryptarithmetic

```
  T W O
+ T W O
-------
F O U R
```



- Variables: *F T U W R O* X_1 X_2 X_3
- Domains: *{0,1,2,3,4,5,6,7,8,9}*
- Constraints: *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F$, $T \neq 0$, $F \neq 0$

Real-world CSPs

- Common problems:
  - Assignment problems
    - e.g., who teaches what class
  - Timetabling problems
    - e.g., which class is offered when and where?
  - Transportation scheduling
  - Factory scheduling
- Notice that many real-world problems involve real-valued variables
- May also include preference constraints: constraint optimization

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it:

States are defined by the values assigned so far

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  - → fail if no legal assignments
- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   → use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. b = (n - ℓ)d at depth ℓ, hence n! · d^n leaves

Backtracking search

- Variable assignments are commutative}, i.e.,

[WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
  - → b = d and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called backtracking search
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve *n*-queens for $n \approx 25$

Backtracking search

function BACKTRACKING-SEARCH(*csp*) **returns** a solution, or failure
  return RECURSIVE-BACKTRACKING({}, *csp*)

function RECURSIVE-BACKTRACKING(*assignment,csp*) **returns** a solution, or failure
  if *assignment* is complete **then return** *assignment*
  var ← SELECT-UNASSIGNED-VARIABLE(*Variables[csp], assignment, csp*)
  for each *value* **in** ORDER-DOMAIN-VALUES(*var, assignment, csp*) **do**
    if *value* is consistent with *assignment* according to Constraints[*csp*] **then**
      add { *var* = *value* } to *assignment*
      result ← RECURSIVE-BACKTRACKING(*assignment, csp*)
      if *result* ≠ *failue* **then return** *result*
      remove { *var* = *value* } from *assignment*
  return *failure*

Backtracking example

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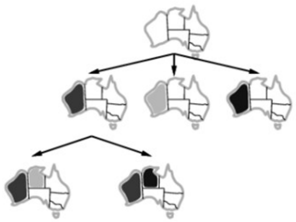
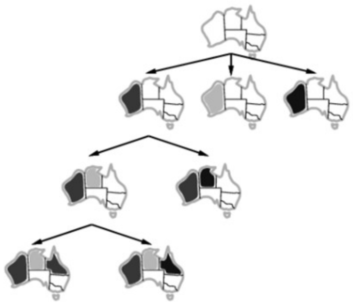


Backtracking example



Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Most constrained variable

- Most constrained variable:
  choose the variable with the fewest legal values

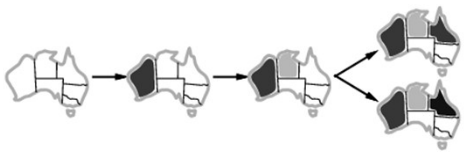
- a.k.a. minimum remaining values (MRV) heuristic

Most constraining variable

- Tie-breaker among most constrained variables
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables

Least constraining value

- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Combining these heuristics makes 1000 queens feasible

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly