EE2101-Control Systems
Ass-1,Problem 6

A PRASHANTH
EE19BTECH11003

September 9, 2020

Problem

Use MATLAB and the Symbolic Math Toolbox to find the inverse Laplace transform of the following frequency functions :

$$a. \quad G(s) = \frac{(s+5)(s^2+3s+10)}{(s+3)(s+4)(s^2+2s+100)} \tag{1.1}$$

$$b. \quad G(s) = \frac{(s^3+4s^2+2s+6)}{(s+8)(s^2+8s+3)(s^2+5s+7)} \tag{1.2}$$

P.S. Here we are using Python instead of MATLAB

a. Solution

Firstly let us take function a. i.e.

$$G(s) = \frac{(s+5)(s^2+3s+10)}{(s+3)(s+4)(s^2+2s+100)} \tag{2.1}$$

Inverse Laplace transform of G(s):

$$L^{-1}\left\{\frac{(s+5)\left(s^2+3s+10\right)}{(s+3)(s+4)\left(s^2+2s+100\right)}\right\} \tag{2.2}$$

Now let us take partial fraction of

$$\frac{(s+5)(s^2+3s+10)}{(s+3)(s+4)(s^2+2s+100)} \quad is \tag{2.3}$$

$$\implies \frac{s^3+8s^2+25s+50}{(s+3)(s+4)\left(s^2+2s+100\right)} \tag{2.4}$$

Solution

Now create partial fraction template using the denominator,

$$(s+3)(s+4)\left(s^2+2s+100\right) \tag{2.5}$$

i.e

$$\frac{s^3+8s^2+25s+50}{(s+3)(s+4)(s^2+2s+100)} = \frac{a_0}{s+3} + \frac{a_1}{s+4} + \frac{a_3 s + a_2}{s^2+2s+100}$$

Multiply equation by the denominator,

$$\implies \frac{\left(s^3+8s^2+25s+50\right)(s+3)(s+4)\left(s^2+2s+100\right)}{(s+3)(s+4)(s^2+2s+100)} =$$

$$\frac{a_0(s+3)(s+4)\left(s^2+2s+100\right)}{s+3} + \frac{a_1(s+3)(s+4)\left(s^2+2s+100\right)}{s+4} + \frac{(a_3 s+a_2)(s+3)(s+4)\left(s^2+2s+100\right)}{s^2+2s+100}$$

Solution

By simplifying above equation,

$$s^3 + 8s^2 + 25s + 50 = a_0\left(s+4\right)\left(s^2+2s+100\right)$$

$$+a_1\left(s+3\right)\left(s^2+2s+100\right) + \left(a_3 s + a_2\right)\left(s+3\right)\left(s+4\right)$$

Solve the unknown parameters by plugging the real roots of the denominator -3, -4

For the root denominator -3: $a_0 = \frac{20}{103}$

For the root denominator -4: $a_1 = -\frac{7}{54}$

$$\therefore a_0 = \frac{20}{103},\ a_1 = -\frac{7}{54}$$

Solution

Plug in the solutions to the known parameters,

$$s^3+8s^2+25s+50=\frac{20}{103}\left(s+4\right)\left(s^2+2s+100\right)$$

$$+\left(-\tfrac{7}{54}\right)\left(s+3\right)\left(s^2+2s+100\right)+\left(a_3s+a_2\right)\left(s+3\right)\left(s+4\right)$$

Expand

$$s^3+8s^2+25s+50=a_3s^3+\frac{359s^3}{5562}+a_2s^2+7a_3s^2+\frac{2875s^2}{5562}+7a_2s$$

$$+12a_3s+\tfrac{20107s}{2781}+12a_2-\tfrac{350}{9}+\tfrac{8000}{103}$$

Solution

Extract Variables from within fractions,

$$s^3+8s^2+25s+50 = a_3s^3+\frac{359}{5562}s^3+a_2s^2+7a_3s^2+\frac{2875}{5562}s^2+7a_2s$$

$+12a_3s+\frac{20107}{2781}s+12a_2-\frac{350}{9}+\frac{8000}{103}$

Group elements according to powers of s,

$$1\cdot s^3+8s^2+25s+50 = s^3\left(a_3+\frac{359}{5562}\right)+s^2\left(a_2+7a_3+\frac{2875}{5562}\right)$$

$+s\left(7a_2+12a_3+\frac{20107}{2781}\right)+\left(12a_2+\frac{8000}{103}-\frac{350}{9}\right)$

Solution

Equate the coefficients of similar terms on both sides to create a list of equations

$$\begin{bmatrix} 12a_2 - \frac{350}{9} + \frac{8000}{103} = 50 \\ 7a_2 + 12a_3 + \frac{20107}{2781} = 25 \\ a_2 + 7a_3 + \frac{2875}{5562} = 8 \\ a_3 + \frac{359}{5562} = 1 \end{bmatrix}$$

By solving above equation,

$$\therefore a_2 = \frac{2600}{2781},\ a_3 = \frac{5203}{5562}$$

Solution

Plug the solutions to the partial fraction parameters to obtain the final result

$\therefore a_0 = \frac{20}{103},\ a_1 = -\frac{7}{54}, a_2 = \frac{2600}{2781},\ a_3 = \frac{5203}{5562}$

i.e.

$$\frac{\frac{20}{103}}{s+3} + \frac{\left(-\frac{7}{54}\right)}{s+4} + \frac{\frac{5203}{5562}s + \frac{2600}{2781}}{s^2+2s+100}$$

Simplify above eqn.

$$\frac{20}{103\,(s+3)} - \frac{7}{54\,(s+4)} + \frac{5203s+5200}{5562\,(s^2+2s+100)}$$

Solution

$\therefore$ *the partial fractions of* $G(s)$ *are*

$$L^{-1}\left\{\frac{20}{103\,(s+3)}-\frac{7}{54\,(s+4)}+\frac{5203s+5200}{5562\,(s^2+2s+100)}\right\}$$

After expanding

$$=L^{-1}\left\{\frac{20}{103\,(s+3)}-\frac{7}{54\,(s+4)}+\frac{5203}{5562}\cdot\frac{s+1}{(s+1)^2+99}\right.$$

$$\left.-\frac{1}{1854}\cdot\frac{1}{(s+1)^2+99}\right\}$$

Solution

By using linearity property of Inverse Laplace Transform:
For functions f(s) , $g\left(s\right)$ and constants $a,\ b$:
$L^{-1}\left\{a\cdot f\left(s\right)+b\cdot g\left(s\right)\right\}=a\cdot L^{-1}\left\{f\left(s\right)\right\}+b\cdot L^{-1}\left\{g\left(s\right)\right\}$

$$= L^{-1}\left\{\frac{20}{103\left(s+3\right)}\right\}-L^{-1}\left\{\frac{7}{54\left(s+4\right)}\right\}+\frac{5203}{5562}L^{-1}\left\{\frac{s+1}{\left(s+1\right)^2+99}\right\}$$

$$-\frac{1}{1854}L^{-1}\left\{\frac{1}{\left(s+1\right)^2+99}\right\}$$

$$=\frac{20}{103}e^{-3t}-\frac{7}{54}e^{-4t}+\frac{5203}{5562}e^{-t}\cos\left(3\sqrt{11}t\right)$$

$$-\frac{1}{1854}e^{-t}\frac{1}{3\sqrt{11}}\sin\left(3\sqrt{11}t\right)$$

Solution

$$= \frac{20}{103}e^{-3t} - \frac{7}{54}e^{-4t} + \frac{5203}{5562}e^{-t}\cos\left(3\sqrt{11}t\right) - \frac{1}{5562\sqrt{11}}e^{-t}\sin\left(3\sqrt{11}t\right)$$

$\therefore$ **Inverse Laplace Transform of**

$$\frac{(s+5)(s^2+3s+10)}{(s+3)(s+4)(s^2+2s+100)}:$$

$$= \frac{20}{103}e^{-3t} - \frac{7}{54}e^{-4t} + \frac{5203}{5562}e^{-t}\cos\left(3\sqrt{11}t\right) - \frac{1}{5562\sqrt{11}}e^{-t}\sin\left(3\sqrt{11}t\right)$$

b. Solution

Now let us take function b. i.e.

$$G(s) = \frac{(s^3 + 4s^2 + 2s + 6)}{(s+8)(s^2+8s+3)(s^2+5s+7)} \tag{2.6}$$

Inverse Laplace transform of given equation is:

$$L^{-1}\left\{\frac{s^3 + 4s^2 + 2s + 6}{(s+8)\left(s^2+8s+3\right)\left(s^2+5s+7\right)}\right\} \tag{2.7}$$

Now by taking the partial fractions of above eqn, it is

$$= L^{-1}\left\{-\frac{266}{93\,(s+8)} + \frac{1199s + 534}{417\left(s^2+8s+3\right)} + \frac{-65s - 1014}{4309\left(s^2+5s+7\right)}\right\}$$

Solution

$$= L^{-1}\left\{-\frac{266}{93\left(s+8\right)}+\frac{1199}{417}\cdot\frac{s+4}{\left(s+4\right)^2-13}-\frac{4262}{417}\cdot\frac{1}{\left(s+4\right)^2-13}\right.$$
$$\left.+\frac{-65s-1014}{4309\left(s^2+5s+7\right)}\right\}$$
$$= L^{-1}\left\{-\frac{266}{93\left(s+8\right)}+\frac{1199}{417}\cdot\frac{s+4}{\left(s+4\right)^2-13}-\frac{4262}{417}\cdot\frac{1}{\left(s+4\right)^2-13}-\right.$$
$$\left.\frac{65}{4309}\cdot\frac{s+\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2+\frac{3}{4}}-\frac{1703}{8618}\cdot\frac{1}{\left(s+\frac{5}{2}\right)^2+\frac{3}{4}}\right\}$$

Using the linearity property of Inverse Laplace Transform:
For functions f(s) , $g\left(s\right)$ and constants $a,\ b$:
$L^{-1}\left\{a\cdot f\left(s\right)+b\cdot g\left(s\right)\right\}=a\cdot L^{-1}\left\{f\left(s\right)\right\}+b\cdot L^{-1}\left\{g\left(s\right)\right\}$

Solution

$$= -L^{-1}\left\{\frac{266}{93\left(s+8\right)}\right\} + \frac{1199}{417}L^{-1}\left\{\frac{s+4}{\left(s+4\right)^2-13}\right\}$$

$$-\frac{4262}{417}L^{-1}\left\{\frac{1}{\left(s+4\right)^2-13}\right\} - \frac{65}{4309}L^{-1}\left\{\frac{s+\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2+\frac{3}{4}}\right\}$$

$$-\frac{1703}{8618}L^{-1}\left\{\frac{1}{\left(s+\frac{5}{2}\right)^2+\frac{3}{4}}\right\}$$

$$= -\frac{266}{93}e^{-8t}+\frac{1199}{417}e^{-4t}\cosh\left(\sqrt{13}t\right)-\frac{4262}{417}e^{-4t}\frac{1}{\sqrt{13}}\sinh\left(\sqrt{13}t\right)$$

$$-\frac{65}{4309}e^{-\frac{5t}{2}}\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1703}{8618}\cdot\frac{2}{\sqrt{3}}e^{-\frac{5t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)$$

Solution

$$= -\frac{266}{93}e^{-8t} + \frac{1199}{417}e^{-4t}\cosh\left(\sqrt{13}t\right) - \frac{4262}{417\sqrt{13}}e^{-4t}\sinh\left(\sqrt{13}t\right)$$

$$-\frac{65}{4309}e^{-\frac{5t}{2}}\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1703e^{-\frac{5t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)}{4309\sqrt{3}}$$

$\therefore$ **Inverse Laplace Transform of** :

$$\frac{(s^3+4s^2+2s+6)}{(s+8)(s^2+8s+3)(s^2+5s+7)} \quad is$$

$$= -\frac{266}{93}e^{-8t} + \frac{1199}{417}e^{-4t}\cosh\left(\sqrt{13}t\right) - \frac{4262}{417\sqrt{13}}e^{-4t}\sinh\left(\sqrt{13}t\right)$$

$$-\frac{65}{4309}e^{-\frac{5t}{2}}\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1703e^{-\frac{5t}{2}}\sin\left(\frac{\sqrt{3}t}{2}\right)}{4309\sqrt{3}}$$

Python Code for Function a.

```
from sympy.integrals.transforms import inverse_laplace_transform
from sympy import exp, Symbol
from sympy.abc import s, t
from sympy import pprint
a = Symbol('a', positive=True)
x = inverse_laplace_transform(((s + 5) * (s ** 2 + 3 * s + 10)) / ((s +
    3) * (s + 4) * (s ** 2 + 2 * s + 100)), s, t)
pprint(x)
```

Terminal a :

Terminal

File Edit View Search Terminal Help

$$\frac{\left(-\sqrt{11}\cdot e^{3\cdot t}\cdot\sin(3\cdot\sqrt{11}\cdot t) + 57233\cdot e^{3\cdot t}\cdot\cos(3\cdot\sqrt{11}\cdot t) + 11880\cdot e^{t} - 7931\right)\cdot e^{-4\cdot t}\cdot\theta(t)}{61182}$$

```
------------------
(program exited with code: 0)
Press return to continue
```

Python Code for Function b.

```
import numpy as np
import sympy as sp
from sympy.integrals import inverse_laplace_transform

s = sp.symbols('s')
t = sp.symbols('t', positive = True)
tf = (-266) / (93 * (s + 8))

from sympy.integrals.transforms import inverse_laplace_transform
from sympy import exp, Symbol
from sympy.abc import s, t
from sympy import pprint
a = Symbol('a', positive=True)
x1 = inverse_laplace_transform(tf, s, t)
```

Python Code for Function b.

```
s = sp.symbols('s')
t = sp.symbols('t', positive = True)
tf = (1199 * s + 534) / (417 * (s ** 2 + 8 * s + 3))

from sympy.integrals.transforms import inverse_laplace_transform
from sympy import exp, Symbol
from sympy.abc import s, t
from sympy import pprint
a = Symbol('a', positive=True)
x2 = inverse_laplace_transform(tf,s,t)
```

Python Code for Function b.

```
s = sp.symbols('s')
t = sp.symbols('t', positive = True)
tf = (-65 * s - 1014) / (4309 * (s ** 2 + 5 * s + 7))

from sympy.integrals.transforms import inverse_laplace_transform
from sympy import exp, Symbol
from sympy.abc import s, t
from sympy import pprint
a = Symbol('a', positive=True)
x3 = inverse_laplace_transform(tf,s,t)

pprint(x1 + x2 + x3)
```

Terminal b :

Terminal

File Edit View Search Terminal Help

$$-\frac{13\cdot\left(131\cdot\sqrt{3}\cdot\sin\left(\frac{\sqrt{3}\cdot t}{2}\right)+15\cdot\cos\left(\frac{\sqrt{3}\cdot t}{2}\right)\right)\cdot e^{\frac{-5\cdot t}{2}}\cdot\theta(t)}{12927}+\frac{\sqrt{13}\cdot\left(-4262\cdot e^{\sqrt{13}\cdot t}+1199\cdot\sqrt{13}\cdot e^{\sqrt{13}\cdot t}+4262\cdot e^{-\sqrt{13}\cdot t}+1199\cdot\sqrt{13}\cdot e^{-\sqrt{13}\cdot t}\right)\cdot e^{-4\cdot t}\cdot\theta(t)}{10842}-\frac{266\cdot e^{-8\cdot t}\cdot\theta(t)}{93}$$

```
------------------
(program exited with code: 0)
Press return to continue
```