STUDY OF RICCI FLOW

Project Report

Submitted by

Mr. PRASHANTH P

Reg No: **M200438PH**

In partial fullfillment of the award of the degree of

MASTER OF SCIENCE
IN
PHYSICS

Under the guidance of

Dr. P N Bala Subramanian

Assistant Professor

Department Of Physics

National Institute of Technology Calicut

May 2022

DECLARATION

I hereby declare that this project work entitled "**Study of Ricci Flow**" is an original record of the work done by me under the guidance of **Dr P N Bala Subramanian**, Assistant Professor, Department of Physics, NIT Calicut and that, to the best of knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledge has been made in the text.

Prashanth P
Reg No:M200438PH
M.Sc Physics
Dept. of Physics
NIT Calicut

CERTIFICATE

This is to certify that the project report entitled **"Study of Ricci Flow"** submitted by **Mr.Prashanth P** (**Roll No: M200438PH**) to National Institute of Technology Calicut, towards partial fulfillment of requirements for the award of Degree of **Master of Science (Physics)** is a bonafide record of project work done by him under my supervision and guidance.

04/07/2022

Dr.P N Bala Subramanian
Assistant Professor
Supervisor and Guide
Dept. of Physics
NIT Calicut

Dr.Goutam Kumar Chandra
Assistant Professor
Course Coordinator
Dept. of Physics
NIT Calicut

Dr.Aji A Anappara
Associate Professor
Head of the Department
Dept. of Physics
NIT Calicut

Department of Physics ★ National Institute of Technology Calicut ★
NITC
Kozhikode
www.nitc.ac.in

ACKNOWLEDGEMENT

I would like to thank my guide Dr.P N Bala Subramanian, Assistant Professor, Department of Physics, National Institute of Technology Calicut for introducing me to this wonderful topic. I would like to thank him for showing the utmost involvement in the project by holding regular meetings and making sure that the work gets done in the right intervals of time. I would like to thank all the teachers in the Physics department who have imparted me with valuable knowledge and helped me gain an active interest in the domain. I would like to thank my parents and friends for understanding the requirements of the project and showering me with valuable uninterrupted time to help me spend more time on the project. I would like to thank our faculty advisor Dr.Goutam Kumar Chandra,Assistant Professor, Department of Physics, National Institute of Technology, Calicut, for handling the project in a well organised manner.

Prashanth P
MSc Physics

Abstract

Ricci Flow is a differential equation used to change the metric of a manifold based on the curvature. Ricci flow has proved to play a primary role in solving the Poincaré conjecture, an important application among its various theoretical applications, thus making it a very valuable topic to study. This thesis motivates the need to study Ricci flow and its significance in solving the Poincaré conjecture, Provides basic information on Riemannian Geometry, studies various properties of the Ricci flow equations, analyses the various modifications to Ricci Flow equations and its advantages, studies the different solutions to Ricci flow equations and finally makes an attempt to visualize solutions to Ricci flow.

Contents

Chapter 1

Poincaré conjecture

The most widely known and appreciated application of Ricci flow is its use in solving the Poincaré conjecture. Hence, Starting our discussion on Ricci flow with a brief introduction to Poincaré conjecture would be a right step.

1.1 Introduction to Poincaré conjecture

Henri Poincaré in an attempt to classify geometric objects published in 1895 a paper titled *Analysis Situs* [12] "analysis of position" in Latin) which became the foundation of Modern Topology. The book introduced a variety of problems that lead to to the development of research in the field. One such interesting problem was the Poincaré conjecture which as time passed became the holy grail of Mathematicians that stood unsolved for nearly a century. In the upcoming discussion we will discuss Poincaré conjecture formally providing Mathematical background wherever necessary.

1.2 Topology

Topology is the branch of Mathematics that deals with the properties of sets that are not changed by continuous deformations. Technically, it studies the properties that are preserved by homeomorphisms. As a trivial example consider a Sphere and a cube. We know that the sphere can be deformed continuously into a cube. Similarly we can also see that a doughnut surface can be deformed to a surface of a coffee mug as shown in Figure 1.1. We say that the doughnut and the coffee mug are topologically similar, but as you can see a cube or sphere cannot be converted to a doughnut or a coffee mug without introducing a hole and thus any map between the two shapes cannot be continuous. It seems that is intuitively easy to prove if two shapes are topologically equivalent and write a homeomorphism between them it might not be the case when shapes begin to get complicated. The way to prove that two Manifolds are not topologically equivalent is by looking at topological invariants(properties that are preserved by Homeomorphisms). If two manifolds have different invariants they cannot be homeomorphic. If the think about the possible list of properties that could be the invariants, we can immediately see that area and volume cannot be topological invariants because they are not preserved by homeomorphisms. We have a rough idea that the number of holes could be a good candidate for a topological invariant. In the following sections we will introduce a formal method to discuss the topic using the definition of fundamental groups.

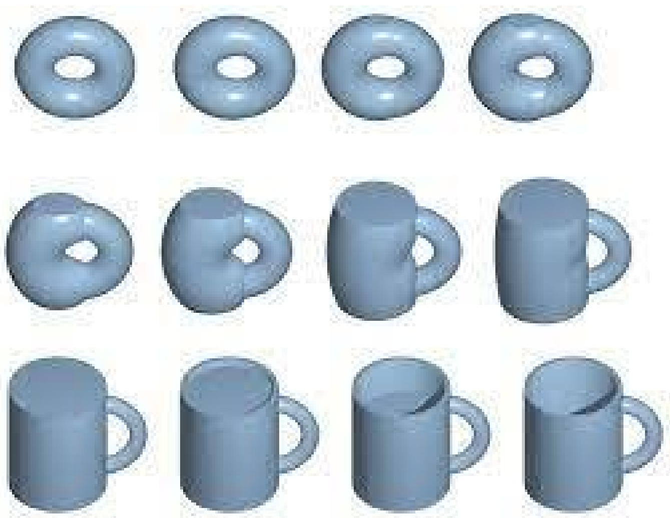

Figure 1.1: Converting a doughnut to a cup [5]

1.2.1 Mathematical Formalism

This sections introduces the basic notations and definitions that help us discuss the contents of the previous section in a formal way. Some of the The definitions mentioned in this chapter are adapted from the lecture notes of the course "The geometric Anatomy of Theoretical Physics" by Frederic P. Schuller. The notes of the lecture are made available for reference by various sources [14] we take the liberty of mentioning some definitions from the above mentioned source with the aim of presenting a very compact understanding for readers who are approaching the subject for the first time. However due credit is given to the source.

Definition 1.2.1 (Topology). *Let $\mathcal{M}$ be some set. A Choice $T \subseteq \mathscr{P}(\mathcal{M})$ is called a topology on $\mathcal{M}$ if*

$$\varnothing \in T \text{ and } \mathcal{M} \in T$$

$$U \text{ and } V \in T \rightarrow \bigcap U, V \in T$$

$$C \subseteq T \rightarrow \bigcup C \in T$$

Remark: There are many different topologies T that one can choose on the same set

Example 1.1. Let $\mathcal{M}$ be any set then $T = \{\varnothing, \mathcal{M}\}$is a topology called as the chaotic topology and the set $T = \{\mathscr{P}(\mathcal{M})\}$ is called a discrete topology. The pair $(\mathcal{M}, T)$ is called a topological space.

Definition 1.2.2 (Continuity). *[14] Let ($\mathcal{M}$,$T_{\mathcal{M}}$) and (N,T_N) be a topological space and let $\varphi : \mathcal{M} \rightarrow N$ be a map then φ is continuous if*

$$\forall\ V \in T_N : preim_{\varphi} \in T_{\mathcal{M}}$$

that is φ is continuous if preimages of open sets are open.

Definition 1.2.3 (Homeomorphism)**.** *Let $\varphi : \mathcal{M} \to N$ be a bijection, now equipping M and N with topology $(\mathcal{M}, T_{\mathcal{M}})$and (N, T_N) we call the bijection φ a homeomorphism if both*

$$\varphi : \mathcal{M} \to N$$

$$\varphi^{-1} : \mathcal{M} \to N$$

are continuous

Remark: if such a Homeomorphism exists we say that $(\mathcal{M}, T_{\mathcal{M}})$and (N, T_N) are Homeomorphic or isomorphic as topological spaces. Homeomorphisms are the structure preserving maps in a Topological level.

Definition 1.2.4 (Manifolds)**.** *[14] A topological space $(\mathcal{M}, T)$ is called a $d-$ dimensional (Topological) Manifold if for every point $p \in \mathcal{M}$ there exist $p \in U \in T$ and a homeomorphism x*

$$x : U \to x(U) \subseteq \mathbb{R}^d$$

Very Roughly a topological manifold is a topological space that locally looks like $\mathbb{R}^d$

1.3 Homotopic groups and Fundamental group

Definition 1.3.1 (Homotopic)**.** *[14] Let $(\mathcal{M}, T)$ be a topological space.Two curves $\gamma : [0,1] \to \mathcal{M}$ and $\delta : [0,1] \to \mathcal{M}$ are homotopic if there exists a continuous map $h : [0,1] \times [0,1] \to \mathcal{M}$ such that $h(0,\lambda) := \gamma(\lambda)$ and $h(1,\lambda) := \delta(\lambda)$ where $\lambda \in [0,1]$*

Intuitively two curves are homotopic if one curve can be continuously deformed into the other curve like shown below.

Definition 1.3.2 (Space of Loops)**.** *[14] Let $(\mathcal{M}, T)$ be a topological space then for every point in the Manifold we define a space of loops at the point p as*

$$\mathcal{L}_p := \{\gamma : [0,1] \to \mathcal{M} \mid \gamma \textit{ is continuous with } \gamma(0) = \gamma(1)$$

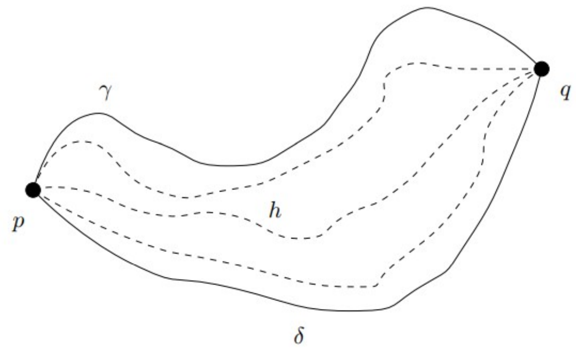

Figure 1.2: Homotopic curves [14]

Definition 1.3.3 (Concatenation operator). *We define the operator*

$$*_p : \mathcal{L} \times \mathcal{L} \to \mathcal{M}$$

Where $\lambda \in [0,1]$ *and* $(\gamma * \delta)(\lambda) = \gamma(2\lambda)$ *for* $0 \leq \lambda \geq 1/2$ *and* $(\gamma * \delta)(\lambda) = \gamma(2\lambda - 1)$ *for* $1/2 \leq \lambda \geq 1$

Definition 1.3.4 (Fundamental Group). *The Fundamental group*$(\pi_{1,p}, \cdot)$ *is the set*

$$\pi_{1,p} := \mathcal{L}/\sim$$

along with

$$\cdot : \pi_{1,p} \times \pi_{1,p} \to \pi_{1,p}$$

$$[\gamma] \cdot [\delta] := [\gamma * \delta]$$

where the Equivalence relation in the definition of the fundamental group is the Homotopy.

Generally speaking all the loops that can are homotopic to each other are put in a single equivalence class and the the fundamental group consists of all such different equivalence classes.

Definition 1.3.5 (Identity curve)**.** *at a point p is defined as*

$$\gamma_{id} : [0, 1] \to \mathcal{M} \mid \gamma_{id}(0) = p$$

Example 1.2. On a 2-sphere all loops on the surface are homotopic to each other hence the fundamental group (at every point) of the sphere is

$$\pi_{1,p} := \{\gamma_{id,\ p}\}$$

The same is true for a arbitrary potato shape with no holes. The loops on the surface of the potato shape are homotopic to each other hence the fundamental group is same as that of a sphere. Hence we can say that the two topologies are Homeomorphic to each other.

1.4 Poincaré conjecture

The most important problem in Topology is to classify manifolds. The aim is to have a set n-dimensional manifolds and a theorem that says that every n-dimensional manifold is homeomorphic to exactly one manifold on the list and a set of topological invariants that could be used to completely classify manifolds. We have seen in the above section that a 2- manifold whose fundamental group has just one element is homeomorphic to a two sphere. Poincare wanted to establish a similar kind of theorem for the three sphere.

Theorem 1.4.1 (Poincaré conjecture)**.** *[9]*

Any compact three manifold whose fundamental group is the trivial group (one element) must be Homeomorphic to the 3-sphere.

Poincaré conjecture was believed to be the first step in classifying 3-Manifolds.

1.4.1 Three manifolds

One may have a common misunderstanding that a sphere being a three dimensional object is a 3-manifold but that is not the case. A three dimensional sphere is indeed a 2-manifold. A three dimensional sphere is a three dimensional object but its surface which we see embedded in space is two dimensional and the location of any object on the surface can be completely specified by just two numbers and hence its called as 2-Manifold. Similarly a torus is also a 2-Manifold with holes. The surface of the earth is a 2-manifold because we can we can specify our position on earth using two numbers called the latitude and the longitude. From our previous example we can understand that it is not possible see a 3-sphere because 3-spheres cannot be visualized in three dimensional space, just like how 2-spheres cannot be visualized in a plane. The three sphere or in general any 3-manifold can be embedded in 4 dimensional space which we have no capabilities to imagine or see. Technically an n-dimensional manifold is an object that can be modeled locally on $\mathbb{R}^n$. It takes n numbers to specify a point on the manifold. We have no way of imagining a three manifold. But we have a feel for three manifolds using the analogy described below. Take two circular disks, bend both the disks so that it looks like a bowl or a hemisphere now glue both the disks so that the boundaries of both the disks touch but the insides do not touch, the object that results is a 2-sphere. In a similar way if you take two solid 3 dimensional balls and if you glue together both the balls in such a way that their boundaries touch but not allowing their interiors to touch we get the simplest of 3-manifolds. Poincaré was looking for properties of a three sphere and he conjectured that if we a loop and if we are able to continuously deform it and shrink it to a point we have a 3-sphere the Poincaré conjecture also established the same.

1.4.2 The path to solving the Poincaré conjecture

There have numerous attempts throughout history to solve the poincaré conjecture but none have been successful until recently where Grigori Perelman in 2002 proved it using methods of Ricci flow introduced by Richard Hamilton[10]. The poincaré conjecture was one of the seven problems listed by the clay institute of mathematics which had a prize of one million dollars for the person who solved it. Analogous conjectures have been made for higher dimensions but has been solved. Stephen Smale in 1961 solved a analogous conjecture in 4-dimensions[17] and in 1982 Michael Freedman solved it in 5-dimensions[7] but still solving the poincaré conjecture in 4- dimensions was still elusive.The biggest breakthrough in obtaining a classification for three manifolds was when William Thurston in 1970s proposed the *geometrization conjecture*. It said that every compact three manifold can be cut into finitely many pieces each of which admits one of eight geometric structures. Proving the *geometrization conjecture* would be our solution to obtaining a classification of all three manifolds and eventually solving poincaré conjecture. The solution to poincaré conjecture came not from topology but from Differential Geometry. Richard Hamilton had an insight that he choose pick up a metric for the manifold and let the metric evolve based on the curvature of the manifold[8]. He found that the metric should redistribute itself over the manifold. Hamilton's insights go a great way in proving poincaré conjecture but he ran into problems with singularities where the curvature blows up. The work of Grigori Perelman was to establish rules to deal with singularities and how to eliminate the singularities using surgery and how to let the flow continue.

1.4.3 Thurston geometrization conjecture

Two manifolds can be classified based on the number of genus or holes and we have discussed in detail the concepts necessary to talk about these concepts formally. The classification can be made in a much more elementary way by discussing how to construct these manifolds. A two manifold of genus one, a torus can build out of euclidean squares. We can start with a with a flat piece of the plane and wrap it up to form a cylinder and then bend the ends around to make it a torus and what you see. Eight squares can be neatly tiled to form a torus. In the same way a surface of genus 2 can be neatly tiled this by Pentagons and so on. This is called as geometrization theorem. It says how each manifold can be built from simple geometric pieces. William Thurston proposed a similar classification for the possible geometries of three manifolds. Thurston's geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structures[2].Thurston's geometrization conjecture implies the poincaré conjecture.

Chapter 2

Riemannian Geometry

2.1 Introduction

The definitions mentioned in this chapter are adapted from two sources, a research paper, [16] and the lecture notes of the course "The geometric Anatomy of Theoretical Physics" by Frederic P. Schuller. The notes of the lecture are made available for reference by various sources [14] .All the topics covered in this chapter are basics of Riemannian Geometry with well defined and globally same definitions with very little scope for reframing. The topics are the best understood if the concepts are defined using standard definitions. Hence we take the liberty of mentioning some definitions from the above mentioned source with the aim of presenting a very formal route to ricci flow for readers who are approaching the subject for the first time. However due credit is given to both the sources.

2.2 Bundles

Definition 2.2.1 (Bundles). *[14] A Bundle of a topological manifold is a triple(E, π, $\mathcal{M}$) where E is called as the total space, $\mathcal{M}$ is called as the base space, and π is a surjective map $\pi : E \to \mathcal{M}$ which is continuous called as projection.*

Definition 2.2.2 (fiber). *[14] Let $\pi : E \to \mathcal{M}$ be a bundle and $p \in \mathcal{M}$ then Then, $F_p := preim_\pi(\{p\})$ is called the fiber at the point p.*

The projection map of a bundle sends all the points in the fiber F_p to the point p in the Manifold. The fiber at the point $p \in \mathcal{M}$ is a set of points in E attached to the point p.

Definition 2.2.3 (Section)**.** *[3] Let Let $\pi : E \to \mathcal{M}$ be a bundle. A map $\sigma : \mathcal{M} \to E$ is called a Section of the bundle if $\pi \circ \sigma = id_{\mathcal{M}}$.*

A section is a map σ which sends each point $p \in \mathcal{M}$ to some point $\sigma(p)$ in its fiber F_p, so that the projection map takes it back to the point $p \in \mathcal{M}$.

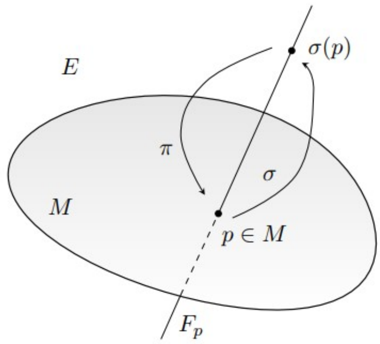

Figure 2.1: visual representation of a section [14]

2.2.1 Charts

Charts are projections that help map the real world to Real number spaces. Most of the operations that we are taught to perform in our high school are operations on charts. Charts are like maps of the real world that provides up with ways of doing calculus. The choice of chart is entirely up to the user and the Physics of the problem does not change with the choice of chart.

Definition 2.2.4 (Charts)**.** *[14] Let $(\mathcal{M}, T)$ be a d-dimensional manifold. Then, a pair (U, x) where $U \in T$ and $x : T \to x(U) \subseteq \mathbb{R}^d$ is a homeomorphism, is said to be a chart of the manifold.*

An Atlas of a manifold $\mathcal{M}$ is a collection $\mathscr{A} : (U_\alpha, x_\alpha)$ of charts such that the union of all the charts covers the entire Manifold.

Definition 2.2.5 (C^∞ atlas). *[3] An atlas $\mathscr{A}$ for a topological manifold is called a C^∞ atlas if any two charts $(U, x), (V, y) \in \mathscr{A}$ are C^∞ compatible.*

Either $U \cap V = \varnothing$(Two charts do not have any intersecting region) or if $U \cap V \neq \varnothing$(They have an intersecting region) then the transition map $y \circ x^{-1}$ from $x(U \cap V)$ to $y(U \cap V)$ must be smooth.

Definition 2.2.6 (Smooth function). *[16] A function $f : U \to \mathbb{R}^m$ where U is an open subset of $\mathbb{R}^n$ is called smooth or C^∞ function if all of its partial derivatives exist and are continuous on U.*

2.3 Vectors and Tensors

We begin by defining Vectors and Tensors in a very abstract sense. We would like to think of vectors as mathematically rigid as possible by just equipping a set with two additional operation using a field.

Definition 2.3.1 (Vector Space). *[14] Let $(K, +, \cdot)$ be a field. A K vector space , or vector space over K is a triple $(V, \oplus, \odot)$, where V is a set and*

$$\oplus : V \times V \to V$$

$$\odot : K \times V \to V$$

are maps satisfying the following axioms:

- *$(V, \oplus)$ is an abelian group;*
- *the map $\odot$ is an* action *of K on $(V, \oplus)$:*

$$\forall \lambda \in K : \forall f, g \in V : \lambda \odot (f \oplus g) = (\lambda \odot f) \oplus (\lambda \odot g);$$

$$\forall \lambda, \mu \in K : \forall f \in V : (\lambda + \mu) \odot f = (\lambda \odot f) \oplus (\mu \odot f);$$

$$\forall \lambda, \mu \in K : \forall f \in V : (\lambda \cdot \mu) \odot f = \lambda \odot (\mu \odot f);$$

$$\forall f \in V : 1 \odot f = f.$$

Elements of the set of the vector space are called as called *vectors*, while the elements of K are often called *scalars*.

Definition 2.3.2. *[14] Let F and G be vector spaces over the field K. Define the set*

$$Hom(F, G) := \{f \mid f : F \xrightarrow{\sim} G\}$$

where the notation $f : F \xrightarrow{\sim} G$ stands for ' a linear map from F to G.

The Hom of a vector space becomes very important while defining the Dual vector space.

The set $Hom(F, G)$ can itself be made into a vector space over K by defining:

$$\oplus : Hom(F, G) \times Hom(F, G) \to Hom(F, G)$$

$$(f, g) \mapsto f \oplus g$$

with

$$f \oplus g : F \xrightarrow{\sim} G$$

$$v \mapsto (f \oplus g)(v) := f(v) + g(v),$$

and

$$\odot : K \times Hom(F, G) \to Hom(F, G)$$

$$(\lambda, f) \mapsto \lambda \odot f$$

and

$$\lambda \odot f : F \xrightarrow{\sim} G$$

$$v \mapsto (\lambda \odot f)(v) := \lambda f(v).$$

Definition 2.3.3 (Dual vector space)**.** *[14] Let V be a vector space over K. The* dual *vector space to V is*

$$V^* := Hom(V, K)$$

where the Field K is in considered like a vector . the dual vector space is just another vector space with has as its elements all the linear functions from V to K

Definition 2.3.4 (Tensor)**.** *[14] Let V be a vector space over K. A (p, q)-tensor T on V is a multilinear map*

$$T : V^* \times \cdots \times V^* \times V \times \cdots \times V \to K.$$

and

$$T^p_q V := \underbrace{V \otimes \cdots \otimes V}_{p\ copies} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q\ copies} := \{T \mid T \text{ is a } (p, q)\text{-tensor on } V\}.$$

The set $T^p_q V$ can be made a vector space over a Field by defining

$$\odot : K \times T^p_q V \to T^p_q V$$

$$(\lambda, T) \mapsto \lambda \odot T,$$

Definition 2.3.5 (Components of a Tensor)**.** *Let V be a finite-dimensional vector space over K with basis $B = \{e_1, \ldots, e_{\dim V}\}$ and let $T \in T^p_q V$. We define the components of T in the basis B to be the numbers*

$$T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} := T(f^{a_1}, \ldots, f^{a_p}, e_{b_1}, \ldots, e_{b_q}) \in K$$

where $1 \le a_i, b_j \le \dim V$ and $\{f^1, \ldots, f^{\dim V}\}$ is the dual basis to B.

We can get back the tensor from its components by

$$T = \underbrace{\sum_{a_1=1}^{\dim V} \cdots \sum_{b_q=1}^{\dim V}}_{p+q \text{ sums}} T^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} e_{a_1} \otimes \cdots \otimes e_{a_p} \otimes f^{b_1} \otimes \cdots \otimes f^{b_q}$$

2.4 Tangent spaces

Tangent space is a vector space that contains all the tangent vectors at a point.

Definition 2.4.1 ($C^\infty(\mathcal{M})$ functions)**.** *[3] If $\mathcal{M}$ is a manifold. We define a vector space over R with the set*

$$C^\infty(\mathcal{M}) := \{f : \mathcal{M} \to R \mid f \textit{ is smooth}\}$$

and operations defined like for any $p \in \mathcal{M}$,

$$(u + w)(p) := u(p) + v(p)$$

$$(\lambda u)(p) := \lambda u(p).$$

Definition 2.4.2 (Tangent vector)**.** *[14] Let $\gamma : R \to \mathcal{M}$ be a smooth curve through $p \in \mathcal{M}$; let $\gamma(0) = p$. The* directional derivative operator *at p along γ is the linear map*

$$X_{\gamma,p} : C^\infty(\mathcal{M}) \xrightarrow{\sim} R$$

$$f \mapsto (f \circ \gamma)'(0),$$

$X_{\gamma,p}$ is called the *tangent vector* to the curve γ at the point p

Definition 2.4.3 (Tangent Space)**.** *[14] Let $\mathcal{M}$ be a manifold and $p \in \mathcal{M}$. The tangent space $T_p\mathcal{M}$ to $\mathcal{M}$ at p is the vector space over R with underlying set*

$$T_p\mathcal{M} := \{X_{\gamma,p} \mid \gamma \textit{ is a smooth curve through } p\},$$

With operations addition

$$\oplus : T_p\mathcal{M} \times T_p\mathcal{M} \to T_p\mathcal{M}$$

$$(X_{\gamma,p}, X_{\delta,p}) \mapsto X_{\gamma,p} \oplus X_{\delta,p}$$

and scalar multiplication

$$\odot : R \times T_p\mathcal{M} \to T_p\mathcal{M}$$

$$(\lambda, X_{\gamma,p}) \mapsto \lambda \odot X_{\gamma,p}$$

for any $f \in C^\infty(\mathcal{M})$,

$$(X_{\gamma,p} \oplus X_{\delta,p})(f) := X_{\gamma,p}(f) + X_{\delta,p}(f)$$

$$(\lambda \odot X_{\gamma,p})(f) := \lambda X_{\gamma,p}(f)$$

A basis can be induced on this Tangent space with the help of a chart we call this chart induced basis.

Definition 2.4.4 (Cotangent space)**.** *[14] Let $\mathcal{M}$ be a manifold and $p \in \mathcal{M}$. The cotangent space to $\mathcal{M}$ at p is*

$$T_p^*\mathcal{M} := (T_p\mathcal{M})^*$$

Definition 2.4.5. *[14] Let $\mathcal{M}$ be a manifold and $p \in \mathcal{M}$. The tensor space $(T_s^r)_p\mathcal{M}$ is defined as*

$$(T_s^r)_p\mathcal{M} := T_s^r(T_p\mathcal{M}) = \underbrace{T_p\mathcal{M} \otimes \cdots \otimes T_p\mathcal{M}}_{r \text{ copies}} \otimes \underbrace{T_p^*\mathcal{M} \otimes \cdots \otimes T_p^*\mathcal{M}}_{s \text{ copies}}$$

Definition 2.4.6 (Tangent bundle)**.** *[14] For a smooth manifold $\mathcal{M}$, the tangent bundle of $\mathcal{M}$ is defined as*

$$T\mathcal{M} := \coprod_{p \in \mathcal{M}} T_p\mathcal{M}$$

2.5 Push forward and Pull back

Definition 2.5.1 (Push forward)**.** *Let $\phi : \mathcal{M} \to N$ be a smooth map and $\mathcal{M}$ and N are smooth manifolds. The push-forward of ϕ at $p \in \mathcal{M}$ is the linear map:*

$$(\phi_*)_p : T_p\mathcal{M} \xrightarrow{\sim} T_{\phi(p)}N$$

$$X \mapsto (\phi_*)_p(X)$$

$$(\phi_*)_p(X)f := X(f \circ \phi)$$

Definition 2.5.2 (Pull back)**.** *[14] Let $\phi : \mathcal{M} \to N$ be a smooth map and $\mathcal{M}$ and N are smooth manifolds. The pull-back of ϕ at $p \in \mathcal{M}$ is the linear map:*

$$(\phi^*)_p : T^*_{\phi(p)}N \xrightarrow{\sim} T^*_p\mathcal{M}$$

$$v \mapsto (\phi^*)_p(v)$$

where $(\phi^*)_p(v)$ is

$$(\phi^*)_p(v) : T_p\mathcal{M} \xrightarrow{\sim} R$$

$$X \mapsto v((\phi_*)_p(X))$$

2.6 Metric

In a general sense a metric is a way to measure angles and lengths of paths on a Manifold. It acts as a map between the coordinate intervals and real distances in the ground. The metric gives the manifold its shape

Definition 2.6.1. *A Riemannian metric on a manifold $\mathcal{M}$ is a a (2,0) tensor field which is positive definite at each point of $\mathcal{M}$. It is generally denoted by g , and written as g_{ij} in its coordinate representation.*

A map $\phi : \mathcal{M} \to \mathcal{N}$ between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is an isometry if it is a diffeomorphism and $\phi^* h = g$. If this is true then the two Riemannian manifolds are isometric.

2.7 The Covariant Derivative

The need to use the covariant derivative arises because the basis changed from point to point and we must also consider the change in basis .

Definition 2.7.1. *[16] Given a vector bundle $\mathcal{E}$ over a manifold $\mathcal{M}$, a connection in $\mathcal{E}$ is a map*

$$\nabla : C^\infty(T\mathcal{M}) \times C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$$

which has the following properties:

1. *$\nabla_X Y$ is linear over $C^\infty(\mathcal{M})$ in X.*
2. *$\nabla_X Y$ is linear over $\mathbb{R}$ in Y.*
3. *∇ satisfies the product rule:*

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y$$

The torsion tensor of ∇,

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

is identically 0.

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

2.8 Laplacian

Definition 2.8.1. *The Laplacian is a family of operators*

$$\triangle : C^{\infty}\left(T^k_l\mathcal{M}\right) \to C^{\infty}\left(T^k_l\mathcal{M}\right)$$

2.9 The Lie Derivative

The Lie Derivative is a way of Differentiating Tensor fields.

Given a vector field X on a manifold $\mathcal{M}$, we define a time-dependent family of diffeomorphisms of $\mathcal{M}$ to itself, $\varphi_t : \mathcal{M} \to \mathcal{M}$ for $t \in (-\epsilon, \epsilon)$, such that φ_0 is the identity and

$$\frac{d}{dt}\varphi_t = X$$

at each point.[16] This can be thought of as the manifold flowing in the direction of the field X. We define the derivative of some (k,l)-tensor field F in the direction of X as a change in F when we move a small step in the direction of X. The value of change in F at a little step away versus at that point is gotten by pushing the value of F at the translated point back to the original point using the diffeomorphism φ_t. We define

$$\begin{aligned}(*\varphi_t)\, F_p\left(X_1,\ldots,X_k,\omega^1,\ldots,\omega^l\right) =& F_{\varphi_t(p)}\left(\varphi_{t*}\left(X_{1(p)}\right),\ldots,\varphi_{t*}\left(X_{k(p)}\right),\right.\\ &\left.\left(\varphi_t^{-1}\right)^*\left(\omega^1_{(p)}\right),\ldots,\left(\varphi_t^{-1}\right)^*\left(\omega^l_{(p)}\right)\right).\end{aligned}$$

The Lie derivative of F in the direction X is defined as

$$\mathcal{L}_X F = \left(\frac{d}{dt}\left(*\varphi_t\right)F\right)_{t=0}$$

2.10 Curvature Tensors

Curvature is the intrinsic property of the manifold. If you peel and orange and if you try to lay its skin on a flat table, we can realise that we wouldn't be able to do it without tearing the skin, this is because the surface of the orange had intrinsic curvature. On the other hand we could do the same for cylinder even though it looks curved it could be spread into a sheet. The curvature is intrinsic property of the manifold. The Riemann curvature tensor, It is defined to be the $(4,0)$-tensor with coordinates R_{ipqj} . This tensor carries information about the Second order Derivatives of g

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{ipqj}x^p x^q + ...$$

The Riemann curvature tensor has the explicit form

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}$$

1. $R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}$

2. The first Bianchi identity:

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

3. The second Bianchi identity:

$$\nabla_p R_{ijkl} + \nabla_i R_{jpkl} + \nabla_j R_{pikl} = 0$$

Definition 2.10.1. *[16] The Ricci curvature tensor, Rc, is the symmetric* $(2,0)$*tensor with coordinate expression*

$$R_{ij} := R^p_{pij}.$$

The scalar curvature is the trace of the Ricci tensor,

$$R := g^{ij}R_{ij}.$$

Definition 2.10.2. *[16] The Einstein tensor on a Riemannian n-manifold* $(\mathcal{M}^n, g)$ *is the tensor*

$$E_{ij} := R_{ij} - \frac{1}{n} R g_{ij}.$$

If two Riemannian metrics $\tilde{g} = Cg$ are related in the above means by a scaling factor C, then the various geometric quantities scale in the following way

1. $\tilde{g}^{ij} = C^{-1} g^{ij}$.

2. $\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$.

3. $\tilde{R}_{ijk}^l = R_{ijk}^l$.

4. $\tilde{R}_{ijkl} = C R_{ijkl}$.

5. $\tilde{R}_{ij} = R_{ij}$.

6. $\tilde{R} = C^{-1} R$.

7. The volume elements: $d\tilde{\mu} = C^{n/2} d\mu$.

Chapter 3

Connections Curvature and Variational formulas

3.1 Connections and Curvatures

In this section we will discuss the definitions briefly with respect to the basis.

Definition 3.1.1 (Levi civita connection)**.** *Let* $(\mathcal{M}^n, g)$ *be a Riemannian manifold with a metric and* $X, Y \in \Gamma(T\mathcal{M})$ *then* $\nabla_X Y$ *is called the Levi civita connection with the following conditions*

- *torsion free:* $[\nabla_X Y - \nabla_Y X] = [X, Y]$
- *metric compatibility:* $X\langle YZ\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

We can also find the levi civita connection by [3]

$X\langle \nabla_X Y, Z\rangle = 1/2\{X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle\} + \langle [X, Y]Z\rangle + \langle [X, Z]Y\rangle - \langle [Y, Z]X\rangle$

3.1.1 Christoffel symbols

The Christoffel symbol captures something very fundamental about the coordinate system. It captures the rate of change of metric in a way. It the derivative of the coordinate basis with respect to the coordinates represented with respect to the basis.

Definition 3.1.2 (Christoffel symbols)**.**

$$\nabla_{\partial i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k$$

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}$$

Definition 3.1.3 (Riemann tensor)**.** *The Riemann Tensor measures the commutativity of the Covariant differentiation.*

$$R_m(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_y(\nabla_X Z) - \nabla_{[X,Y]Z}$$

Definition 3.1.4 (Sectional curvature)**.** *Let* $\pi \subseteq T\mathcal{M}_p$ *then the sectional curvature is* $K(\pi) = \langle R_{iem}(e_1, e_2)e_2, e_1\rangle$

Definition 3.1.5 (Ricci Tensor)**.** *The Ricci Tensor is in a sense the average of the Sectional curvature. The ricci tensor is given by* $R_c(X,Y) = \langle R_{iem}(e_i, X)Y, e_i\rangle$*. Where* $\{e_i\}_{i=1}^n$ *an orthonormal basis for tangent space* $T\mathcal{M}_p$

Definition 3.1.6 (Scalar curvature)**.** *The scalar curvature is given by*

$$R(p) = \sum_{i=1}^{m} R_c(e_i, e_i)$$

3.1.2 Bianci Identities

The following are the Bianci Identities.

1. $Rijk^l + Riki^l + R^l_{kij} = 0$

2. $\nabla_i R^m_{jkl} + \nabla_j R^m_{kil} + \nabla_k R^m_{ijl} = 0$

3.2 Ricci Flow

Ricci flow takes a metric and deforms it in direction of negative Ricci Tensor.

Definition 3.2.1 (Ricci Flow). *Let g(t) be a metric and* $t \in$ *Interval on* $\mathcal{M}^n$*. Then the ricci flow equation is given by*

$$\frac{\partial}{\partial t} g(t) = -2R_c(g(t))$$

More generally we can have metrics g(s) where , $s \in \mathcal{I}$ *and let* $\frac{\partial}{\partial t}|_{s=0}\, g_{ij} = V_{ij}$

3.3 Variational Formulas

In this section let us discuss how the various associated parameters change when the metric varies based on the Ricci flow equation.

3.3.1 Variation of Christoffel Symbols

Theorem 3.3.1. *If the metric g(s) varies like* $\frac{\partial}{\partial t} g_{ij} = V_{ij}$ *then the Christoffel Symbols vary like*

$$\frac{\partial}{\partial s}\Gamma^k_{ij} = \frac{1}{2} g^{kl}((\nabla_i V_{jl}) + \nabla_j V_{il} - \nabla_l V_{ij})$$

Proof. Computing in normal coordinates at p. $\Gamma^k_{ij}(p) = 0$. This implies that $\partial_i g_{in}(p) = 0$ then

$$\frac{\partial}{\partial s}\Gamma^k_{ij(p)} = \frac{\partial}{\partial s}\frac{1}{2} g^{kl}(\partial i g_{il} + ...)_{(p)}$$

Multiplying inside and solving we get

$$\frac{\partial}{\partial s}\Gamma^k_{ij(p)} = \frac{1}{2} g^{kl}(\partial i V_{jl} + \partial j V_{il} - \partial l V_{ij})_{(p)}$$

Since the connections are zero we can change the derivative to to covariant derivative.Then we get

$$\frac{\partial}{\partial s}\Gamma^k_{ij(p)} = \frac{1}{2}g^{kl}(\Delta_i V_{jl} + \partial j V_{il} - \Delta_l V_{ij})_{(p)}$$

This is true in normal coordinates and both sides are tensorial and hence it is true in any coordinates.

For Ricci flow plug in V $= -R_c$. Plugging in we get

$$\frac{\partial}{\partial t}\Gamma_{ij} = -g^{kl}(\Delta_i R_{jl} + \Delta_{R_{il}} - \Delta_l R_{ij})$$

$\square$

3.3.2 Variation of Riemann Tensor

Theorem 3.3.2. *The Riemann Tensor varies like*

$$\frac{\partial}{\partial s}R^l_{ijk} = \nabla_i(\frac{\partial}{\partial s}\Gamma^l_{jk}) - \nabla_j(\frac{\partial}{\partial s}\Gamma^l_{ik})$$

Under Ricci flow we get

$$\frac{\partial}{\partial s}R^l_{ijk} = \Delta R^l_{ijk} + Quadradic$$

which resembles a Heat Equation.

3.3.3 Variation of Ricci Scalar

Theorem 3.3.3. *The Ricci Scalar varies like*

$$\frac{\partial R}{\partial t} = \Delta R + 2 \mid R_c \mid^2$$

Proof. We know

$$\frac{\partial}{\partial s}R = \frac{\partial}{\partial s}(g^{ij}R_{ij})$$

and hence

$$\frac{\partial}{\partial s}R = \frac{\partial}{\partial s}(g^{ij})R_{ij} + g^{ij}(\frac{\partial}{\partial s}R_{ij})$$

$$\frac{\partial}{\partial s}R = -g^{ij}\frac{\partial}{\partial s}g_{kl}g^{lj}R_{ij} + \frac{1}{2}\nabla_l(\nabla_l V_{il} - \nabla_l V_{ii}) - \frac{1}{2}\Delta\mathcal{V}$$

where $\mathcal{V} = g^{ij}V_{ij}$. This can be written as

$$\frac{\partial R}{\partial s} = -\langle V, R_c\rangle - \Delta\mathcal{V} + div(divV)$$

under Ricci flow $V = -2R_C$ and $divV = -2div(R_c) = -\nabla R$ and $div(divV) = \Delta R$ Whereas $V = -2R$ Plugging it all in we get

$$\frac{\partial}{\partial s} = -\langle -2R_c, R_c\rangle - \Delta(-2R) - \Delta R$$

□

$$\frac{\partial R}{\partial t} = \Delta R + 2 \mid R_c \mid^2$$

3.3.4 Variation of Volume form

Theorem 3.3.4. *The Volume form varies like*

$$\frac{\partial}{\partial s}d\mu = \frac{1}{2}V d\mu$$

3.3.5 Lichnerowicz Laplacian

Rough Laplacian is defined as if α is a tensor

$$\Delta\alpha = g^{ij}\nabla_i\nabla_j\alpha$$

$$\Delta\alpha = \sum_{l=1}^{n}\nabla\nabla\alpha(e_i, e_j)$$

Suppose V_{ij} is a symmetric 2 tensor the Lichnerowicz Laplacian is given by

$$(\Delta_L V)_{ij} = (\Delta v)_{ij} + 2R_{kijl}V_{kl} - R_{ik}V_{kj} - R_{jk}V_{kl}$$

$V = g^{ij}V_{ij}$ trace and $g^{ij}(\Delta_L V)_{ij} = \Delta\mathcal{V}$

lemma 3.3.1.

$$\frac{\partial}{\partial t}R_{ij} = \Delta_L R_{ij} = (\Delta_L R_c)_{ij})$$

Proof. A more general formula is

If $\frac{\partial}{\partial s}g_{ij} = V_{ij}$, then

$$\frac{\partial}{\partial s}R_{ij} = -\frac{1}{2}(\Delta_L V)_{ij} + (\mathcal{L}_X g)_{ij}$$

where

$$X = \frac{1}{2}\nabla\mathcal{V} - div(v)$$

If $V_{ij} = CR_{ij}$then,

$$X = C(\frac{1}{2}\nabla R - div(R_c)) = 0$$

By contracted 2^{nd} Bianci Identity under Ricci flow

$$\frac{\partial}{\partial t}R_{ij} = (\Delta_L R_c)_{ij}$$

□

Chapter 4

Maximal Principles

4.1 Heat Equation and maximum principles

We have seen that

$$\frac{\partial R}{\partial t} = \Delta R + 2 \mid R_c \mid^2$$

for lower bound we can replace $R_c by R$.

In general if a_{ij} is 2 tensor then

$\mid a_{ij} \mid^2 \geq \frac{1}{n}(\sum_i a_{ii})^2$

applying this to R we get

$$\frac{\partial R}{\partial t} \geq \Delta R + \frac{2}{n} R^2$$

This Equation gives us the Lower bound for scalar curvature

4.2 Maximum principle

Theorem 4.2.1. *[3] Let $\mathcal{M}^n$ be a closed Manifold and g(t) is a 1-parameter family of metrics. Let u be a super solution to heat equation*

$$\frac{\partial u}{\partial t} \geq \Delta u + X(t).\nabla_u$$

Then If $u(0) \geq c, u(t) \geq c \forall t \geq 0$

Proof. Let us try to prove this by contradiction. Let us say $\forall \epsilon > 0, U_\epsilon = U + \epsilon + \epsilon t$ we need to show $U_\epsilon > C, \quad \forall \epsilon > 0, \forall t \geq 0$. If not then $\exists \epsilon > 0$, such that there is a first time $t_0 > 0$, such that $\exists X_0 \in \mathcal{M}$ where $U_\epsilon(x_0, t_0) = C, \quad U_\epsilon(x, t) \geq C \ \forall x, t$

At (x_0, t_0) it hits C and before it was greater.

$$0 \geq \frac{\partial U_\epsilon}{\partial t} \geq U_{\epsilon t} + X \nabla U + \epsilon \geq \epsilon$$

This contradicts the fact that ϵ is positive

4.2.1 Applying maximum principle to heat equation

The maximum principle can be applied to the heat equation to get a lower bound of the solution.

$$\frac{\partial R}{\partial t} = \Delta R + 2 \mid R_c \mid^2 \geq \Delta R + \frac{2}{n} R^2$$

We make a generalization like

$$\frac{\partial U}{\partial t} \geq \Delta U + X . \nabla U + F(U)$$

where F is Lipschitz application.

4.2.2 Associated ODE

To get the Associated ODE we drop the space terms

$$\frac{d\mathcal{U}}{dt} = F(\mathcal{U})$$

comparing solutions to the PDE and ODE.

$$\mathcal{U}(0) = min_{t=0} U$$

□

Theorem 4.2.2. *[3] $\forall x \in \mathcal{M}, t \geq 0$ we have $u(x,t) \geq \mathcal{U}(t)$ From the original max principle then for ODE we get $F \equiv 0$ then solution to for the ODE is a constant function. Then for the ODE we get*

$$u(x,t) \geq \min_{t=0} u$$

4.3 Application to Maximum principle for scalar curvature

The maximum principle can be applied to the scalar curvature equation to get a lower bound of the scalar curvature. We have

$$\frac{d\rho}{dt} = \frac{2}{n}\rho^2$$

$$\rho(0) = min_{t=0}R$$

$$R(x,t) \geq \rho(t) = (\frac{1}{min_{t=0}R} - \frac{2}{n}t)^{-1}$$

This gives us a bound for the scalar curvature and also the Singularity time from that.

lemma 4.3.1. *If $R_c \geq 0$ at t=0 then $R_c \geq 0$ for all $t \geq 0$ It says that the R_c is preserved under Ricci flow*

Chapter 5

Existence of solutions and Curvature estimates

Ricci flow being a non parabolic differntial equation leads to doubts if we would be able to get a stable solution. In this section we prove short term existence of solutions using a modification to Ricci Flow called as Ricci DeTruck flow.

5.1 Short time existence and curvature estimates

If $\frac{\partial}{\partial s} g_{ij} = V_{ij}$then

$$\frac{\partial}{\partial s}(-2R_{ij}) = (\Delta_L V)_{ij} + (L_x g)_{ij}$$

where

$$(\Delta_L V)_{ij} = (\Delta_V)_{ij} + 2R_{kijl} - R_{ik}V_{kj} - R_{jk}V_{ki}$$

and

$$x = \frac{1}{2}\Delta\mathbf{V} - div(V)$$

The Ricci flow equations are not parabolic in nature.

5.2 Ricci - Deturck flow

Ricci - Deturck flow is a modification of Ricci flow to make the equations parabolic. The Ricci - Deturck flow[4] equations are given by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i$$

With $g(0) = g_0$ and where

$$W_j = g_{ik}\ g^{pq}(\Gamma^k_{pg} - \tilde{\Gamma}^k_{pq})$$

we can see that

$$\frac{\partial}{\partial s} W_j = g_{jk} g^{pq} \frac{\partial}{\partial s} \Gamma^k_{pq} + 0^{th}\text{order in V}$$

Let us take $g_{ij} = \delta_{ij}$

$$\frac{\partial}{\partial s} W_j = div(v)_j - \frac{1}{2} \nabla_j V + 0^{th} \text{ order in V}$$

now let us take differentiate the RHS of the Ricci - Deturck flow equations. we get

$$\frac{\partial}{\partial s} RHS = \frac{\partial}{\partial s} - 2R_{ij} + \nabla_i (\frac{\partial w}{\partial s})_j + \nabla_j (\frac{\partial w}{\partial s})_i + 1^{st} \text{ order in variation of V}$$

We are currently only interested in 2^{nd}order differentiation of V. Plugging in all known in the the above equation we get

$$= (\Delta_L V)_{ij} + 1^{st} \text{ order in V}$$

Which is a strictly parabolic equation and for strictly parabolic we get short term existence and uniqueness.

lemma 5.2.1. *If $\mathcal{M}^n$ is closed, g_o is the metric on $\mathcal{M}$, then$\exists!$ solution g(t) for*

$t \in [0, \delta)$ *some* $\delta > 0$ *to (*) with* $g(0) = g_0$

Theorem 5.2.1. *Using the Ricci Detruck flow we can have the existence of the solution for Ricci Flow*

Given g_0 there is a solution g(t) to Ricci Detruck equation with $g(0) = g_0$. We can define $W_{g(t)}$. We can define Diffeomorphisms $\phi : M \to M$ where $\phi_0 = i_d$ and

$$\frac{d}{dt}\phi_t(x) = -W_{g(t)} * (\phi_t(x))$$

At each $x \in \mathcal{M}$ this is a System of ODE (solvable).

Define $\tilde{g}(t) = (\phi_t) * g(t)$ we claim that with $\tilde{g}(0) = g(0)$ and $\frac{\partial}{\partial t}\tilde{g}(t) = -2R_c\,\tilde{g}(t)$. If we can prove this claim we can prove the existence of solution for Ricci flow.

Theorem 5.2.2 (Uniqueness)**.** *Suppose* $g_1(t)$ *and* $g_2(t)$ *are solutions to Ricci flow with* $g_1(0) = g_2(0)$ *then* $g_1(t) = g_2(t)$

Theorem 5.2.3 (Long time existence)**.** *Suppose* $\mathcal{M}^n$ *is closed, g(t),* $t \in [0, T)$, $T < \infty$ *solution to Ricci flow and suppose that*

$$Sup_{\mathbb{M}\times[0,T)} \mid R_m \mid < \infty$$

Then $\exists \in > 0$ *and an extension g(t), defined on* $[0, T + \epsilon)$ *which is a solution to Ricci flow. The Solution exists as long as the curvature is bounded.*

5.3 Hamilton's 1982 Theorem

If we look at normalised flow for $(3d, R_c > 0)$

$$\frac{\partial}{\partial t}g_{ij} = -R_{ij} + \frac{2}{3}r g_{ij}$$

with $g(0) = g_0$ then there exists a unique solution g(t) $t \in [o, \infty]\exists g_\infty$ having constant sectional curvature on $\mathcal{M}^3$ such that

$$||g(t) - g_\infty||C^R \leq C_{kl} - C_k t \quad k \in N$$

which means exponential convergence of the solution to a constant sectional curvature metric[8].

We can show that the curvature is actually getting better using the following estimates.

1. $R_c > 0$ - preserved
2. $R_c \geq \in R_g$ fro some $\epsilon > 0$
3. $\exists \delta > 0$ and a $C < \infty$ such that $\forall x, t \frac{|R_c - \frac{1}{3} R_g|^2}{R^2} \leq C R^{-\delta}$

If the metric becomes more and more like a cylinder then it violates 2.

Theorem 5.3.1 (Convergence of Ricci Flow for closed 3- manifolds)**.** *[3] If* $(\mathcal{M}^3, g_0)$ *is a closed 3 manifold with* $R_c > 0$ *then there exists a metric* g_∞ *on* $\mathcal{M}^3$ *with a constant positive sectional curvature.*

Chapter 6

Gradient Ricci Soliton

6.1 Gradient Ricci Soliton

Definition 6.1.1. *A Riemannian Manifold $\mathcal{M}^n, g$ is called a gradient Ricci soliton if there exists a f: $\mathcal{M} \to R$ and a constant $\epsilon \in R$ such that*

$$R_c + \nabla\nabla f + \frac{\epsilon}{2} g = 0$$

Now we have static metrics we need one parameter family of metrics solving Ricci flow.g is expanding , shrinking or steady if $\epsilon > 0, \epsilon < 0$ or $\epsilon = 0$ respectively.[3]

6.1.1 Cigar Soliton

The Cigar Soliton is an example gradient Ricci soliton. It is also called in Physics as Wittens Black hole

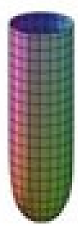

Figure 6.1: Hamilton's Cigar Soliton[13]

Given a manifold $\mathcal{M}^2 \equiv \mathcal{R}^2$ we have the metric for the cigar soliton as

$$g(t) = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$$

and the initial metric

$$g(0) = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

we define a function $\phi_t = \mathcal{R}^2 \to \mathcal{R}^2$ defined by

$$\phi(x, y) = (e^{-2t}x, e^{-2t}y),\ t \in (-\gamma, \gamma)$$

and we have $\phi_t^*(g(0)) = g(t)$

The cigar is a steady soliton.

Let $S = sinh^{-1}r$ and $g(0) = ds^2 + tan^2 s\ d\theta^2$ and $g = ds^2 + \phi(s)^2\ d\theta^2$. Generally the curvature is given by $R = \frac{-2\phi^1 1}{\phi}$. For the cigar we have $R = 4sech^2 s$ and equivalently

$$R = \frac{4}{1 + x^2 + y^2}$$

If the distance from the centre is close to ∞ we have $g(0) \approx ds^2 + d\theta^2$ which is equivalent to that of a cylinder.

6.1.2 Hamilton's Entropy

For a surface with positive curvature with (S^2, g) $R_g > 0$ we can define the Hamilton's Entropy as[8]

$$N(g) = \int_{s^2} log(R.A)Rd\mu$$

If $\frac{\partial}{\partial t}g = -2R_c = -R_g$

which is the Ricci flow in 2 dimensions we have

$$\frac{dN}{dt} = -\int_{s^2} \frac{|\nabla R|^2}{R} d\mu + \int_{s^2} (R-r)^2 d\mu$$

6.2 Canonical form for gradient solitons

A quadruple $(\mathcal{M}^n, g_0, t_0, t)$ where $(\mathcal{M}^n, g_0)$ is a Riemanian Manifold and $f_0 : \mathcal{M} \to R, t \in \mathcal{R}$ is a gradient Ricci soliton if

$$R_c(g_0) + Hess_{g_0}(f) + \frac{t}{2} g_0 = 0$$

6.2.1 Canonical Form Theorem

There exists a solution g(t) of Ricci flow on $\mathcal{M}^n$ with $g(0) = g_0$ and exists, parameters family of diffeos $\phi_t = \mathcal{M}$ with $\phi_0 = id$, $\exists f(t) : \mathcal{M} \to \mathcal{R}$ with $f(0) = f_0$ defined for all t such that $\tau(t) = \epsilon t + 1 > 0$ such that

1. $\frac{\partial}{\partial t} \phi_t(x) = \frac{1}{\tau(t)} (grad_{g_0} f_0) \ (\phi_t(x))$
2. $g(t) = \tau(t) \phi_t^* g_0$
3. $f(t) = f_0 . \phi_t$
4. $\frac{\partial}{\partial t}(t) = |grad_{g(t)} f(t)|^2_{g(t)}$
5. $R_c(g(t)) + Hess_{g(t)} f(t) + \frac{\epsilon}{2\tau(t)} g(t) = 0$

6.3 Rosenau Solution

The Rosenau solution is just two cigars on S^2. attached back to back. The Rosenau solution is an ancient solution and backwards in time it goes to a cigar.

We have $(\mathcal{R} \times \mathcal{S}^2, h)$ where $h = dz^2 + d\theta^2$ is a flat metric. Then the Rosenau solution is

$$g(t) = u(t)h, t < 0$$

with

$$u(z,t) = \frac{sinh\ (-t)}{cosh\ z + cosh\ t}$$

$$R(g(t)) = \frac{\Delta_n log\ u}{u}$$

$$R(g(t)) = \frac{cosh\ t\ \ cosh\ z + `1}{sinh(-t)(cosz + cosh\ t)} > 0$$

6.3.1 Getting Cigar Soliton as a limit

We can get the Cigar Soliton as a limit of the Rosenau solution. We have

$$u(z+t,t) = (-coshz\ cosht - sinhz - cosht)^{-1}$$

$$lim_{t\to-\infty} g(z+t,t) = lim_{t\to-\infty} u(z+t,t)h$$

$$lim_{t\to-\infty} g(z+t,t) = (coshz - sinhz + 1)^{-1} j$$

$$lim_{t\to-\infty} g(z+t,t) = (e^{-z}+1)^{-1} h$$

The above equation is just another version of the cigar soliton. Plugging in h we have

$$(e^{-2+1})(dz^2 + d\theta)$$

let us have $\tilde{\theta} = \frac{\theta}{2}, \tilde{Z} = \frac{Z}{2}$ then we get $4(e^{-\tilde{z}}+1)(d\tilde{z}^2 + d\tilde{\theta}^2)$, $\tilde{\theta} \in S^1$ which is another version of the cigar soliton.

Theorem 6.3.1. *[3] Suppose $(\mathcal{M}^2, g(t))$ is a complete ancient solution of Ricci flow with curvature bounded at each time. It means that $\to\to R(g(t)) \geq 0$. According to Strong maximum principle either $R(g(t)) \equiv 0$ or $R(g(t)) > 0$.*

The possible kinds of solutions are

1. Flat

2. Round S^2 or (RP^2) constant curvature

3. Cigar Soliton

4. Rosenau Solution

If not 1 to 3, and if $\mathcal{M}^2$ is compact and g(t) rotationally symmetric the the solution is a Rosenau solution.

6.4 Differential Harnak Inequalities

These are gradient estimates that should vanish or stay constant on Gradient Ricci solitons. Let the fundamental solution to heat equation [3].

$$\mathcal{R}^n, h(x,y,t) = (4\pi t)^{\frac{-n}{2}} e^{\frac{-d(x,y)^2}{4t}}$$

be h.

Now consider the solutions to the heat equation. If we want to compare the solutions at different points and time

$$\frac{\partial}{\partial t} log(u) = \frac{\frac{\partial}{\partial t} u}{u} = \frac{\Delta u}{u} = \Delta logu + \mid \triangledown logu \mid^2$$

$$Q = \Delta logu = \frac{\partial}{\partial t} logu - \mid \triangledown logu \mid^2$$

$$log(h) = fun(t) - \frac{d(x,y)^2}{4t}$$

$$\Delta log(h) = \frac{-n}{2t}$$

Theorem 6.4.1. *If $(\mathcal{M}^n, g)$ is a complete with $R_c > 0, u > 0$ solution to $\frac{\partial u}{\partial t} = \Delta u$*

then,

$$Q = \frac{\partial}{\partial t} logu - \mid \triangledown logu \mid_G^2 = \Delta_g logu \geq \frac{-n}{2t}$$

6.4.1 Hamiltons inequality for expanding gradient Ricci solitons

If $(\mathcal{M}^2, g(t))$ is a solution to Ricci flow with $R(g(t)) > 0$ the

$$-\frac{1}{t} \leq Q = \Delta log R + R$$

is the Harnacken type inequality for gradient Ricci soliton.

Chapter 7

Singularities

Generally The Ricci flow equations in 3 Manifolds are highly unstable and lead to Singularities. Hence it is necessary to study about singularity models. It was Perelman who used to surgery theory to find ways to evolve the metric even though Singularities occur.[11]

7.1 Dilated solutions

Dilated solutions provide us a way of modelling singularities. Let $(\mathcal{M}^2, g(t))$ be ancient Type I, let us choose (x_i, t_i) where $t_i \to -\infty$ and choose $K_i = R(x_i, t_i) = max_{\mathcal{M}^2} R(g(t_i))$. We define $g_i(t) = K_i g(t_i + K_i^{-1} t)$ as the Dilated solutions.

We have

$$g_i(0) = K_i g(t_i)$$

and

$$R_{max}(g_i(0)) = 1 = R_{g_i(0)}(x_i)$$

7.2 Ancient solutions with bounded curvature

Ancient solutions are solutions that are obtained by letting the evolution run back in time and letting time approach $-\infty$. There are two types of ancient solutions

7.2.1 Type I ancient

With $(\mathcal{M}^n_\infty, g^{(t)}_\infty)), t \in (-\infty, \omega), \omega >> 0$. Type 1 ancient solutions have

$$Sup_{\mathcal{M}\times(-\infty,0]}|t||R_m(x,t)| < \infty$$

Example: Expanding round sphere as we go backward in time. In 2d round S^2

7.2.2 Type II ancient

Type 2 ancient solutions have

$$Sup_{\mathcal{M}\times(-\infty,0]}|t||R_m(x,t)| = \infty$$

Example : Cigar and Rosenau solution

7.3 Singularities

Hamilton classified the singularities according to the speed of blow up of the curvature.

Type 1 singularity

$$Sup_{\mathcal{M}\times[0,T)}(t-T)|R_m| \leq \mathcal{C}$$

Type 2 singularity

$$Sup_{\mathcal{M}\times[0,T]}[t-T)|R_m| = \infty$$

7.4 3 Manifolds with positive Ricci curvature

Theorem 7.4.1. *[8] Let $\mathcal{M}^3$ be a closed 3-manifold which admits a smooth Riemannian metric with strictly positive Ricci curvature. Then $\mathcal{M}^3$ also admits a smooth metric of constant positive curvature.*

To prove the above Theorem we can start the Ricci flow on the closed 3-manifold $\mathcal{M}^3$, with a metric of strictly positive Ricci curvature. In this case the metric will become rounder as it evolves under the Ricci flow, but the metric will also become smaller as it evolves. As the evolution continues The manifold shrinks to a point in finite time, and its shape approaches that of a 3-sphere as we get closer to this time.

We need to take a limit as this finite time is approached, and show that the limit has constant positive sectional curvature, but we are disabled to do that because the manifold is shrinking to a point as the limit time is approached. To get around this problem, we rescale the manifold along with the time so that the volume of the manifold is constant. This rescaled metric will not longer shrink to a point. It will evolve so that so that the limit metric has constant positive sectional curvature. The volume preserving Ricci flow is called as the Normalised Ricci flow equations.

7.5 Ricci Flow evolution on a 3 Sphere

We have $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$. For a 3 sphere in first fundamental form $g == r^2\hat{g}$ and $R_{ij} = 2\hat{g}$. where $\hat{g}$ is a metric for unit sphere. Plugging in we have

$\partial t(r^2\hat{g}) = -4\hat{g}$

$2r\dot{r}\hat{g} + r^2\partial t\hat{g} = -4\hat{g}$ but we know that $\partial t\hat{g}$ is 0 Therefore

$r\dot{r} = -2$

$r\frac{dr}{dt} = -2$

$\int_{r_0}^{r} rdr = -2\int_0^t dt$

$[\frac{r^2}{2}]_{r_0}^{r} = -2t$

$r(t) = \sqrt{r_0^2 - 4t}$

at what t r=0 $t = \frac{r_0^2}{4}$

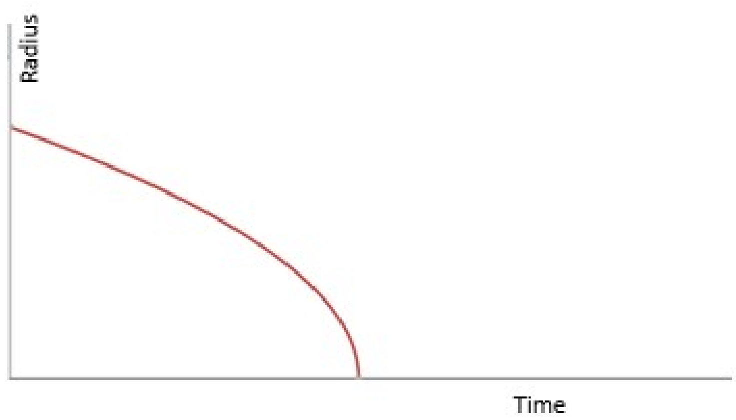

Figure 7.1: The variation of radius with time for a 3 sphere

Chapter 8

Visualization of Ricci Flow

8.1 Visualization of Ricci Flow

We have made use of an C program called as *Ricci_rot* [15] along with suitable visualization packages to visualize surfaces of revolution under ricci flow. We have chosen a very small subclass of metrics provided by the surface of revolution so that the metrics always remain embedded in 3- dimensions[6] thus making it possible to view them as they deform. The possible initial metrics given by Ricci rot is

$$g = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{\sin\rho + c_3 \sin 3\rho + c_5 \sin 5\rho}{1 + 3c_3 + 5c_5}\right)^2 \end{bmatrix}$$

We are provided with the option to vary c_3 and c_5 to get different possible initial metrics. A huge number of numerical instabilities arise and hence to minimize them spectral methods[1] , were used by first introducing a filter that transforms to a Fourier space by taking DFT and then using a filter to drop the shorter wavelength terms and then transforming back. In the below section we provide images in time of the surfaces flowing under Ricci flow to provide a visual idea of how surfaces deform under Ricci flow.

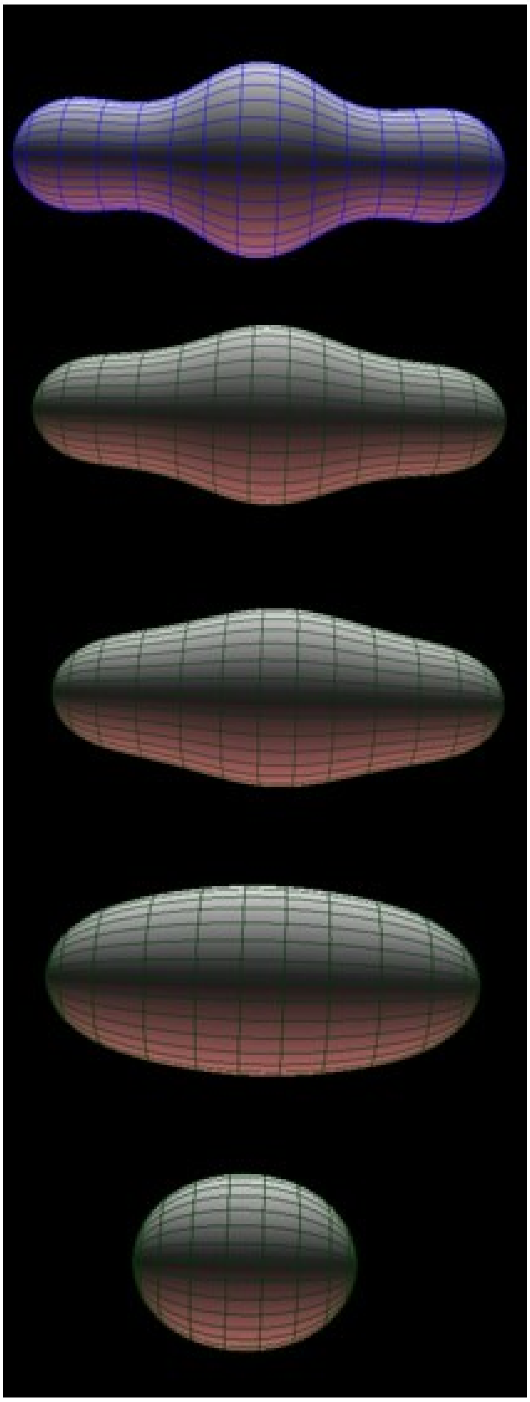

Figure 8.1: Ricci flow evolution of a Example surface 1 as time evolves

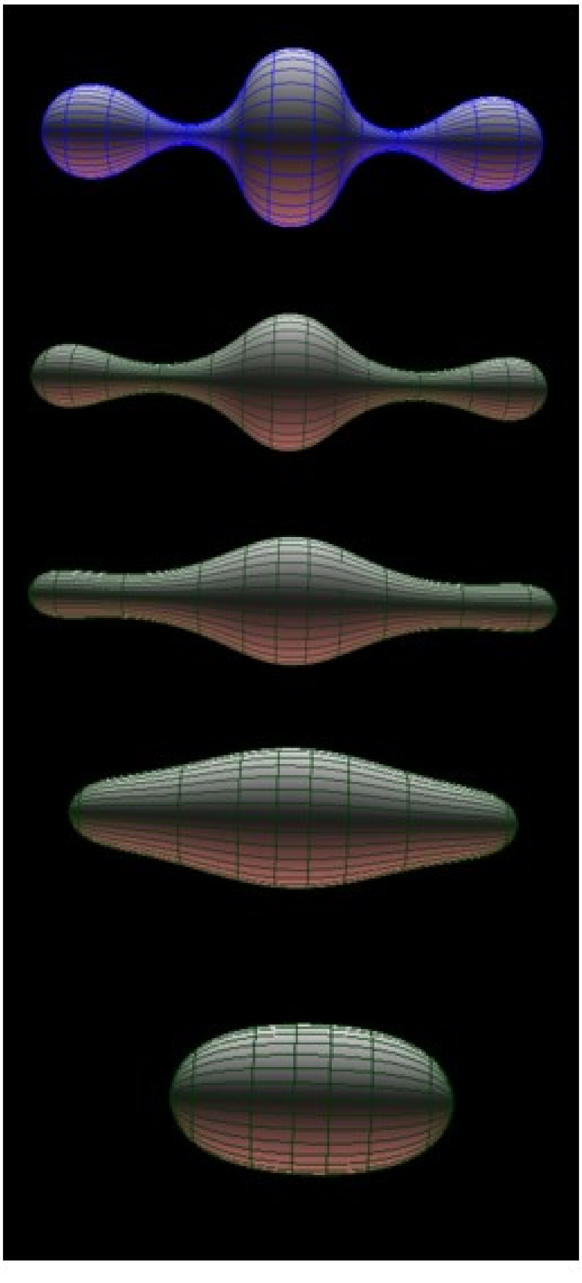

Figure 8.2: Ricci flow evolution of a Example surface 2 as time evolves

Chapter 9

Conclusion

In this Thesis we started by stating the Poincaré conjecture and discussed the mathematical formalism required to formally discuss the Poincaré conjecture. In Chapter 2 we introduce basics of Riemannian Geometry to help the reader recollect the basics required to understanding Ricci flow. In chapter 3 we discuss the various curvatures and variational formulas that show how various parameters change during Ricci flow. In the following chapters we discuss some of the properties of Ricci flow like maximal principles and existence of solutions. While discussing the existence of solutions we discuss the introduce a modification to Ricci flow called as Ricci- Detruck flow. In chapter 6 we discuss some common solutions to the Ricci Flow. In the following chapter we discuss about the singularities that arise while solving Ricci flow. Finally we conclude the thesis by giving visual examples of how the 2 manifolds evolve under Ricci flow.

Bibliography

[1] Gerald L Browning, James J Hack, and Paul N Swarztrauber. A comparison of three numerical methods for solving differential equations on the sphere. *Monthly Weather Review*, 117(5):1058–1075, 1989.

[2] Bang-Yen Chen. A comprehensive survey on parallel submanifolds in riemannian and pseudo-riemannian manifolds. *Axioms*, 8(4):120, 2019.

[3] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*, volume 77. American Mathematical Soc., 2006.

[4] Dennis M DeTurck. Deforming metrics in the direction of their ricci tensors. *Journal of Differential Geometry*, 18(1):157–162, 1983.

[5] Shireen Elhabian and Aly A. Farag. *A Flavor of Topology.* January 2010.

[6] Martin Engman. A note on isometric embeddings of surfaces of revolution. *The American Mathematical Monthly*, 111(3):251–255, 2004.

[7] Michael Hartley Freedman. The topology of four-dimensional manifolds. *Journal of Differential Geometry*, 17(3):357–453, 1982.

[8] Richard S Hamilton. Three-manifolds with positive ricci curvature. *Journal of Differential geometry*, 17(2):255–306, 1982.

[9] John Lee. *Introduction to topological manifolds*, volume 202. Springer Science and Business Media, 2010.

[10] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv preprint math/0211159*, 2002.

[11] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint math/0303109*, 2003.

[12] Henri Poincar et al. *Papers on topology: analysis situs and its five supplements*, volume 37. American Mathematical Soc., 2010.

[13] Daniel Ramos. Ricci flow on cone surfaces. *Portugaliae Mathematica*, 75(1):11–65, 2018.

[14] Simon Rea. *Frederic Schuller's lectures on the geometric anatomy of theoretical physics*. Jan 2016.

[15] J Hyam Rubinstein and Robert Sinclair. Visualizing ricci flow of manifolds of revolution. *Experimental Mathematics*, 14(3):285–298, 2005.

[16] Nick Sheridan and Hyam Rubinstein. Hamilton's ricci flow. *Honour thesis*, 2006.

[17] Stephen Smale. Generalized poincaré's conjecture in dimensions greater than four. *Annals of Mathematics*, pages 391–406, 1961.