BINOMIAL DISTRIBUTION

The Binomial distribution is also known as Bernoulli Distribution is associated with the name of a Swiss mathematician Jacob Bernoulli also known as Jacques or Jacob Bernoulli (1654-1705). Binomial Distribution is a probability distribution expressing the probability of one set of dichotomous alternatives. ie., success or failure.

Binomial Distribution

$$nC_x p^x q^{n-x}$$

Where

p= Probability of success in a single trial

q= 1-p

n= Number of trials

x= Number of successes in n trials

Constants of Binomial Distribution

Mean = np

Standard Deviation = $\sqrt{npq}$

Variance = npq

Example 1:

When a coin is tossed the probabilities of head and tail in case of an unbiased coin are equal. i.e., p=q=$\frac{1}{2}$

The various possibilities for all the events are the terms of the expansion $(q+p)^6$

$(q+p)^6$ =$q^6 + 6q^5p + 15q^4p^2 + 20q^3p^2 + 15q^4p^2 + 6qp^5 + p^6$

The probability of obtaining 4 heads is

$$15q^2p^4 = 15 * \left(\frac{1}{2}\right)^2 * \left(\frac{1}{2}\right)^4 = 0.234$$

The probability of obtaining 5 heads is

$6qp^5 = 6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^5 = 0.094$

The probability of obtaining 6 heads is

$$(p)^6 = \left(\frac{1}{2}\right)^6 = 0.016$$

The probability of obtaining 4 more heads is

$$0.234 + 0.094 + 0.016 = 0.344$$

Example 2: Assuming that half the population is vegetarian so that the chance of an individual being a vegetarian is $\frac{1}{2}$ and assuming that 100 investigators can take sample of 10 individuals to see whether they are vegetarians, how many investigators would you expect to report that three people or less were vegetarians?

Given $n = 1$; $p = q = \frac{1}{2}$

$$10C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$$

$$10C_x \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} 10C_x$$

The probabilities that in a sample of 10, three or less people are vegetarian is given by:

$$P(0) + P(1) + P(2) + P(3) = \frac{1}{1024}(10C_0 + 10C_1 + 10C_2 + 10C_3)$$

$$= \frac{1}{1024}(1 + 10 + 45 + 120)$$

$$= \frac{176}{1024} = \frac{11}{64}$$

Hence out of 100 investigators, the number of investigators who will report 3 or less vegetarians in a sample of 10 is

$$100\ X\ \frac{11}{64} = 17.2 \approx 17$$

Example 3: Check whether the following data follow a Binomial distribution or not. Mean = 3; variance =4.

Given Mean = np = 3 ……… (1)

Variance = npq =4 ……… (2)

$$\frac{(2)}{(1)} = \frac{npq}{np} = \frac{4}{3} = 1\frac{1}{3}$$

i.e. q = $1\frac{1}{3}$ which is >1.

Since q>1 which is not possible (0<q<1). Hence the given data does not follow Binomial distribution.

Example 4: The mean and variance of a binomial variate are 8 and 6. Find $P(X \geq 2)$.

Given Mean = np = 8 ……………. (1)

Variance = npq = 6 …………….. (2)

$$\frac{(2)}{(1)} = \frac{npq}{np} = \frac{6}{8} = \frac{3}{4}$$

$$q = \frac{3}{4}$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$p = \frac{1}{4} \qquad \ldots\ldots\ldots\ldots (3)$$

Substituting (3) in (1), we get

$$n\left(\frac{1}{4}\right) = 8 \quad ; \; n = 32$$

$$P(x = x) = nC_x p^x q^{n-x}$$

$$32C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}$$

$$P(x \geq 2) = 1 - P(x < 2)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - [32C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{32-0} + 32C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{32-1}]$$

$$= 1 - \left[\left(\frac{3}{4}\right)^{32} + 32\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{31}\right]$$

$$= 1 - \frac{35}{4} \times \left(\frac{3}{4}\right)^{31} \left(\frac{3}{4} + \frac{32}{4}\right)$$

$$= 1 - \frac{35}{4} \times \left(\frac{3}{4}\right)^{31} = 0.9988$$

Example 5: A machine manufacturing screws is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are (i) exactly three defectives (ii) not more than three defectives.

Given $p = 5\% = \frac{5}{100} = 0.05$

$$q = 1 - p = 1 - 0.05 = 0.95; \qquad n = 15$$

(i) P (exactly three defectives $= P(3)$

$$= nC_3 p^3 q^{n-3}$$

$$= 15C_3(0.05)^3(0.95)^{12}$$

$$= 455(0.000125)(0.54036)$$

$$= 0.0307$$

(ii) P (not more than 3 defectives $= P(X \leq 3)$

$$= P(0) + P(1) + P(2) + P(3)$$

$$= 15C_0(0.05)^0(0.95)^{15} + 15C_1(0.05)^1(0.95)^{14} + 15C_2(0.05)^2(0.95)^{13} + 15C_3(0.05)^3(0.95)^{12}$$

$$= 0.4632 + 0.3657 + 0.1347 + 0.030733$$

=0.994

Practice Problems:

1) In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that
   (i) All are good bulbs,
   (ii) At most there are 3 defective bulbs,
   (iii) Exactly there are three defective bulbs.

   (Solution: (i) 0.1216 (ii) 0.8666 (iii) 0.19)

2) 6 dice are thrown 729 times. How many times do you expect atleast three dice to show a five or six?

(Solution 233 times)

3) With the usual notation find 'p' for a binomial random variate 'X' if n=6 and if 9 P(X=4) = P (X=2)

(Solution p=0.25; q=0.75)

BINOMINAL DISTRIBUTION:

The binomial distribution can be derived if the following condition are satisfied:

(a) Each trial of a random experiment results in one of two outcomes success (S) or failure (F) so that it is a Bernoulli trial.
(b) Trials are independent
(c) Probability of success remains to be the same in each trial
(d) The number of trials is fixed, finite and known
Define X = Number of successes.

In n independent Bernoulli trials suppose x successes take place. Consequently, number of failures is n- x.

In one specified order probability of securing x successes and n – x failures is,

$p^x(1-p)^{n-x} = p^x q^{n-x}$, where q = 1 - p

Remark : SSFSFF ... SF

This is one specified order in which we obtain x successes (S) and (n - x) failures (F).

P[SSFSFF…S F] = P(S) P(S) P(F) P(S) P(F) P(F)…P(S) P(F)

$= p\ p\ q\ p\ q\ q \ldots pq = p^x q^{n-x}$.

There exists $\binom{n}{x}$ of such specified orders so that the probability of securing x successes (and n-x failures) is,

$$P[X = x] = \binom{n}{x} p^x q^{n-x}, x = 0,1,\ldots\ldots n$$

This specification is the **Binomial probability distribution**.

PROPERTIES:

Property 1. **Characteristic function:** $\phi_X(t) = \left[q + pe^{it}\right]^n$

Property 2. **Moment generating function:** $M_X(t) = \left[q + pe^{t}\right]^n$

Property 3. **Mean** $\mu_1^{'} = np$

Property 4. **Variance:** $\mu_2 = npq$

Property 5. **Recurrence relation for probabilities** $p(x+1) = \dfrac{n-x}{x+1}\left(\dfrac{p}{q}\right)p(x)$

Property 5. **If** $X \sim B(n_1, p), Y \sim B(n_2, p)$ then $X+Y \sim B(n_1+n_2, p)$

DERIVATION OF PROPERTIES:

Property 1. Characteristic Function:

$$\phi_X(t) = \sum_{x=0}^{n} e^{itx} P[X = x] = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} \left(e^{it} p\right)^x q^{n-x} = \left[q + pe^{it}\right]^n$$

$$\phi_x(t) = \left[q + pe^{it}\right]^n$$

Remark: The binomial expansion of $(q+p)^n = \binom{n}{0} p^0 q^n + \binom{n}{1} pq^{n-1} + ... + \binom{n}{n} p^n q^0$

Comparing the summation of $\phi_X(t)$ which this expansion we notice that the role of p is played by pe^t.

Property 2. Moment generating function:

MX(t) = $\left[q + t\,e^t\right]^n$

Using either characteristic function or the moment generating function mean and variance of the binomial distribution can be obtained.

$$\frac{dM_X(t)}{dt} = n(q + pe^t)^{n-1} pe^t$$

Property 3. Mean

$$\mu_1^{'} = \left[\frac{dM_X(t)}{dt}\right]_{t=0} = n(p+q)p = np, \text{ since p+q=1}$$

$$\mu_1^{'} = np$$

Property 4. Variance:

$$\frac{d^2M_X(t)}{dt^2} = \left[npe^t(q + pe^t)^{n-1}\right]$$

$$= np\left[e^t(q + pe^t)^{n-1} + (n-1)pe^{2t}(q + e^t)^{n-2}\right]$$

$$\mu_2^{'} = \left[\frac{d^2M_X(t)}{dt^2}\right]_{t=0} = np\left[(q+p)^{n-1} + (n-1)p(q+p)^{n-2}\right] = np[1 + (n-1)p]$$

V(X) = $\mu_2 = \mu_2' - (\mu_1')^2 = np[1+(n-1)p] - n^2p^2$

$$= np - np^2 \quad = np(1-p) = npq$$

V(X) = n p q

Variance of binomial random variate can not be greater than mean

$npq \le np,$ since q is a proper fraction

$\Rightarrow \quad V(X) \le \text{Mean}$

By further differentiating $\dfrac{d^2M_X(t)}{dt^2}$

successively, higher order moments about origin can be derived

Property 5. Recurrence Relation For The Probabilities:

$$p[X = x+1] = \binom{n}{x+1} p^{x+1} q^{n-x-1}$$

$$P[X = x] = \binom{n}{x} p^x q^{n-x}$$

Upon simplification we obtain

$$\frac{P[X = x+1]}{P[X = x]} = \frac{n-x}{x+1} \quad \frac{p}{q}$$

$$\Rightarrow P[X = x+1] = \frac{n-x}{x+1}\frac{p}{q}P[X = x]$$

Property 6. Additive property of binomial distribution:

If X_1 and X_2 are independent binomial distributed random variables whose parameters are (n_1, p_1) and (n_2, p_2), the sum X_1+X_2 does not follow binomial distribution unless $p_1 = p_2$.

MGF of X_1: $M_{X_2}(t) = \left(q_1 + p_1e^t\right)^{n_1}$

MGF of X_2: $M_{X_2}(t) = \left(q_2 + p_2 e^t\right)^{n_2}$

$M_{X_1+X_2}(t) = M_{X_1}.M_{X_2}$ since X_1 and X_2 are independently distributed

$$= \left(q_1 + p_1 e^t\right)^{n_1} \left(q_2 + p_2 e^t\right)^{n_2}$$

Two MGFs cannot exist for single random variable and $M_{X_1+X_2}(t)$ cannot be expressed in the form $(q + p e^t)^n$, X_1+X_2 can not follow binomial distribution.

However, if $p_1 = p_2 = p$,

$$M_{X_1+X_2}(t) = \left(q + p e^t\right)^{n_1+n_2} = \left(q + p e^t\right)^n$$

where n = $n_1+n_{2\backslash}$, which is nothing but the MGF of the Binomial distribution.

PROBLEMS ON BINOMIAL DISTRIBUTION

Problem: Derive mean and variance of binomial distribution directly.

Solution: **Mean =** $\mu_1^{'} = E(X) = \sum_{x=0}^{n} x\,P[X = x] = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$

$$= \sum_{x=1}^{n} x \binom{n}{x} P^x q^{n-x} = \sum_{x=1}^{n} x \frac{x!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np(q+p)^{n-1} = np$$

$$\mu_1^{'} = np$$

$$\mu_2^{1} = E(X^2) = E\left[X(X-1) + X\right] = E(X(X-1)) + E(X)\text{................(*)}$$

Consider $E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$

$$= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} n(n-1)p^2 \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-2-(x-2)}$$

$$= n(n-1)p^2 (q+p)^{n-2}$$

$$E[X(X-1)] = n(n-1)p^2 \qquad \text{...(**)}$$

Substitute (**) in (*),

$$\mu_2^1 = n(n-1)p^2 + np$$

Variance: V(X) = $\mu_2 = \mu_2^1 - (\mu_1^{'})^2 = n(n-1)p^2 + np - n^2p^2$

$$= np - np^2 = np(1-p) = npq$$

V(X) = npq

Problem : A coin is tossed 10 times, find the probability of getting 6 heads.

Solution: Getting a head is a success (S)

$$P(S) = \frac{1}{2} = p\,,\ q = \frac{1}{2}$$

x = Number of successes

Required event = [X = 6]

n = 10

Required probability: $P[X=5]=\binom{10}{6}\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{4}$

$$P[X=5]=\binom{10}{6}\left(\frac{1}{2}\right)^{10}$$

Problem : Use Chebyshev's inequality to verify that the probability is atleast $\frac{35}{36}$ that in

900 tosses of an unbiased coin the proportion of heads will be between 0.40 and 0.60.

Solution: Let X be the number of successes.

Proportion of successes: $Y=\frac{X}{n}$

$$E(Y)=\frac{E(X)}{n}=\frac{nP}{n}=P=\mu$$

$$V(Y)=V\left(\frac{X}{n}\right)=\frac{V(X)}{n^{2}}=\frac{npq}{n^{2}}=\frac{pq}{n}=\sigma^{2}$$

$$\sigma=\sqrt{\frac{pq}{n}}$$

$$\mu=p=\frac{1}{2}$$

$$\sigma = \sqrt{\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{900}} = \frac{1}{60}$$

$$\mu - k\sigma = \frac{1}{2} - \frac{k}{60} = 0.40 \left(By \quad hypothesis \right)$$

$\Rightarrow \quad k = 6$

$$\mu + k\sigma = \frac{1}{2} + \frac{k}{60} = 0.60 \quad \left(By \;\; hypothesis \right)$$

$\Rightarrow \quad k = 6$

Chebyshev inequality:

$$P\left[\left|Y - \mu\right| < K\sigma\right] \geq 1 - \frac{1}{k^2}$$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{36} = \frac{35}{36}$$

$$P\left[\left|Y - \frac{1}{2}\right| < \frac{1}{10}\right] \geq \frac{35}{36}$$

Problem : The mean are variance of a Binomial distribution are respectively. 24 and
8 find P[X ≥ 2]

Solution: Mean = np = 24
Variance = npq = 8

$$\frac{npq}{np} = q = \frac{8}{24} = \frac{1}{3}$$

$$q = \frac{1}{3} \Rightarrow P = \frac{2}{3}$$

$$np = 24 \Rightarrow n = \frac{24}{p} = \frac{24}{2/3} = 36$$

$$P[X \geq 2] = 1 - P[X < 2] \qquad \ldots(*)$$

$$P[X < 2] = P[X = 0] + P[X = 1] = \binom{n}{0} p^0 q^{n-0} + \binom{n}{1} pq^{n-1}$$

$$= \binom{36}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{36-0} + \binom{36}{1}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{36-1}$$

$$= \left(\frac{1}{3}\right)^{36} + 36\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{35}$$

$$= \left(\frac{1}{3}\right)^{36}\left[1 + 36\left(\frac{2}{3}\right)3\right]$$

$$= \left(\frac{1}{3}\right)^{36}[1 + 72] = 73\left(\frac{1}{3}\right)^{36} \qquad \ldots(**)$$

Substitute (**) in (*)

$$P[X \geq 2] = 1 - P[X < 2] = 1 - 73\left(\frac{1}{3}\right)^{36}$$

Problem: Five coins are tossed, determine binomial probabilities using recurrence relationship. Assume that the coin is unbiased.

Solution: $x = 5,\ p = \frac{1}{2},\ q = \frac{1}{2}$

x = 0, 1, 2, 3, 4, 5.

$$P[X = 0] = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{5-0} = \left(\frac{1}{2}\right)^5$$

$$P[X = x+1] = \frac{n+x}{x+1}\ \frac{p}{q}\ P[X = x]$$

Let X = 1

$$P[X = 1] = \frac{5-0}{0+1}\left(\frac{1/2}{1/2}\right)p[X = 0]$$

$$P[X = 1] = 5\left(\frac{1}{2}\right)^5$$

Let X = 2

$$P[X = 2] = \frac{5-1}{1+1}\,5\left(\frac{1}{2}\right)^5$$

$$= \frac{4}{2}5\left(\frac{1}{2}\right)^5 = 5\left(\frac{1}{2}\right)^4$$

$$P\left[X=2\right]=5\left(\frac{1}{2}\right)^4$$

Let X = 3,

$$P\left[X=3\right]=\frac{5-2}{2+1}\ P\left[X=2\right]=\frac{3}{3}\ \ 5\left(\frac{1}{2}\right)^4=5\left(\frac{1}{2}\right)^4$$

$$P\left[X=3\right]=5\left(\frac{1}{2}\right)^4$$

By symmetry the other probabilities are obtained

$$P\left[X=4\right]=5\left(\frac{1}{2}\right)^5$$

$$P\left[X=5\right]=\left(\frac{1}{2}\right)^5$$

Problem: Find the mode of the binomial distribution

Solution: The mode of any discrete distribution is defined as that value of *x* which maximizes,

P[X = x].

The recurrence relation for probabilities is

$$\frac{P\left[X=x\right]}{P\left[X=x-1\right]}=\frac{\binom{n}{x}p^x q^{n-x}}{\binom{n}{x-1}p^{x-1}q^{n-x+1}}$$

Upon simplification we obtain

$$\frac{P\left[X=x\right]}{P\left[X=x-1\right]}=1+\frac{(n+1)p-x}{x(1-p)}$$

Two cases will arise (n+1) p is (i) non- integer (ii) integer

(i) Decompose (n+1)p into the sum of integer and fractional parts such that the fractional part is non-negative

$$(n+1)p=a+f$$

a is non-negative integer

0 < f < 1

$$\frac{P\left[X=x\right]}{P\left[X=x-1\right]}=1+\frac{(a+f)-x}{x(1-p)}$$

$x \le a \Rightarrow$ the ratio is larger than unity

$x > a \Rightarrow$ the ratio is smaller than unity

$$x \le a \;\; \Rightarrow \;\; \frac{P[X = x]}{P[X = x-1]} > 1$$

$$x = 1 \;\; \Rightarrow \;\; P[X = 1] > P[X = 0]$$

$$x = 2 \;\; \Rightarrow \;\; P[X = 2] > P[x = 1]$$

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$$x = a \Rightarrow P[X = a] > P[x = a-1]$$

Thus, $\quad x \le a \Rightarrow P[X = 0] < P[X = 1] < \ldots\ldots\ldots\ldots < P[X = a]$

Similarly, $\quad x > a \Rightarrow P[X = a] > P[X = a+1] > \ldots\ldots\ldots\ldots > P[X = n]$

The probabilities tend to increase till x = a and from x = a+1 onwards they decrease. Therefore, x = a maximizes the Binomial probabilities.

Mode = a, where a is the integral part of (n+1)p.

(ii) Suppose (n+1)p is an integer

(n+1)p = a+0

$$\frac{P[X = x]}{P[X = x-1]} = 1 + \frac{a-x}{(1-p)x}$$

The ratio is larger than, equal or greater than one according to x < a, x = a, x > a respectively.

$$P[X = 0] < P[X = 1] < \ldots\ldots < P[X = a-1] = P[X = a] > P[X = a+1] > \ldots > P[X = n]$$

X = a-1 and X = a maximize the Binomial probabilities. Binomial distribution is bimodal,
Mode = a – 1 = a

Problem: Find the mode of the Binomial distribution for which

(a) $n=5, p=\frac{2}{3}$ **(b)** $n=6, p=\frac{1}{3}$

Solution: (a) $(n+1)\,p=(5+1)\frac{2}{3}=6\times\frac{2}{3}=4$

(n+1)p is an integer

(n+1)p = 4 + 0

The distribution is binomial having two modes at X = 3 and X = 4

(b) $(n+1)\,p=(6+1)\frac{1}{3}=\frac{7}{3}=2+\frac{1}{3}$

Integral part of (n+1)p = 2. The distribution has single mode occuring at the integral part of (n+1)p.

Mode = 2.

Problem: The random variable X follows binomial distribution with n and p as its

parameters. Find what distribution n-x follows?

Solution: Let Z = n – X

The MGF of Z is, $M_Z(t) = E(e^{tz})$

$=E\left[e^{t(n-X)}\right]=e^{tn}\,E\left(e^{-itX}\right)\;=e^{tn}\,E(e^{t'x})$, where t' = - t

$=e^{tn}\left(q+pe^{t'}\right)$, since X follows binomial distribution.

$$M_Z(t)=\left[e^{t}\left(q+pe^{t'}\right)\right]^n=\left[qe^{t}+e^{t}pe^{-t}\right]^n$$

$$\Rightarrow\; M_Z(t)=(p+qe^{t})^n$$

Z = n – X follows binomial distribution with n and q as its parameters.

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POISSON DISTRIBUTION

Poisson distribution is a discrete probability distribution and is very widely used in statistical work. It was developed by a French mathematician, 'Simeon Denis Poisson' (1781 -1840).

Poisson distribution may be expected in cases where the chance of any individual event being a success is small. The distribution is used to describe the behavior of rare events such as number of accidents on road, number of printing mistakes in a book, serious floods, accidental release of radiation from a nuclear reactor etc.

The poisson distribution is defined as :

$$P(x) = \frac{e^{-m}m^x}{x!}$$

Where x=0, 1, 2, 3, 4, …….

e=2.7183 (the base of natural logarithms)

m=the average of the Poisson distribution.

Constants of Poisson distribution:

Mean = variance = m

Note: Poisson distribution is a limiting case of binomial distribution under the following assumptions:

(i) The number of trials 'n' should be indefinitely large. i.e, $n \to \infty$

(ii) The probability of successes 'p' for each trial is indefinitely small.

(iii) $np = m$, should be finite where m is a constant.

Example 1: A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective. What is the appropriate probability that a box will fail to meet the guaranteed quality?

Given n=100

p=5% = $\frac{5}{100} = 0.05$

Mean m= np

= 100x0.05

= 5

The poisson distribution is $P(X = x) = \frac{e^{-m}m^x}{x!} = \frac{e^{-5}5^x}{x!}$

Now P(a box with fail to meet the guaranteed quality)

=P(X>10)

=1-P(X≤10)

=$1 - [P(0) + P(1) + \cdots + P(10)]$

$$= 1 - [\frac{e^{-5}5^0}{0!} + \frac{e^{-5}5^1}{1!} + \frac{e^{-5}5^2}{2!} + \frac{e^{-5}5^3}{3!} + \frac{e^{-5}5^4}{4!} + \frac{e^{-5}5^5}{5!} + \frac{e^{-5}5^6}{6!} + \frac{e^{-5}5^7}{7!} + \frac{e^{-5}5^8}{8!} + \frac{e^{-5}5^9}{9!} + \frac{e^{-5}5^{10}}{10!}]$$

$$= 1 - e^{-5}\left[1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \cdots + \frac{5^4}{10!}\right]$$

$$= 1 - e^{-5}[146.36]$$

$$= 1 - 0.986 \qquad = 0.014$$

Example 2: In a certain factory turning razar blades there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are in packets of 10. Use poisson distribution to calculate the appropriate number of packets containing (i) no defective, (ii) one defective (iii) 2 defective blades respectively in a consignment of 10,000 packets.

Given p=$\frac{1}{500}$, n=10, N=10,000

Mean m=np

m= $10 \times \frac{1}{500} = \frac{1}{50}$=0.02

The poisson distribution is

$$P(x) = \frac{e^{-m}m^x}{x!} = \frac{e^{-0.02}(0.02)^x}{x!}$$

(i) P(no defective) = P(0)

$$= \frac{e^{-0.02}m^0}{0!} = 0.980198$$

The total number of packets containing no defective blades in a consignment of 10,000 packets.

=N× P(no defective) $= 10{,}000 \times 0.98019$

= 9802 packets

(ii) P (one defective)=P(1)

$$=\frac{e^{-0.02}(0.02)^1}{1!}$$

=0.01960

Number of packets containing one defective

=N× P(one defective) $= 10{,}000 \times 0.1960$

=196 packets

(iii) P(two defective)=P(2)

$$= \frac{e^{-0.02}(0.02)^2}{2!}$$

=0.000196

Number of packets containing two defectives= N× P(two defective)

$= 10{,}000 \times 0.000196$

=2

Example 3: Using poisson distribution , find the probability that the ace of spades will be drawn from a pack of well shuffled cards atleast once in 104 consecutive trials.

Probability of the ace of spades = $\frac{1}{52}$

n=104 Mean m = np

$= 104 \times \frac{1}{52} = 2$

The poisson distribution is

$$P(X = x) = \frac{e^{-m}m^x}{x!} = \frac{e^{-2}2^x}{x!}$$

$$P(Atleast\ once) = P(1) + P(2) + \ldots + P(104)$$

$$= 1 - P(0) = 1 - \frac{e^{-2}2^0}{0!}$$

$$= 1 - e^{-2} = 1 - 0.136 = 0.864$$

Practice Problems:

1) Out of 1000 balls 50 are red and the rest white. If 60 balls are picked at random, what is the probability of picking up (i)3 red balls (ii) not more than 3 red balls in the sample.
Assume poisson distribution for the number of red balls picked up in the sample; where $e^{-3} = 0.0498$.
(Solution: (i) 0.2241 (ii)0.6474)

2) An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year?
(Solution 0.9998)

THE POISON DISTRIBUTION:

This distribution may be viewed as an approximation of the binomial distribution, which was invented by a French physicist Simon-De-Poisson (1781-1840) . It models several random phenomena such as the number of arrivals of customers at a railway booking counter in a time interval of width t, number of fatal accidents occurring on a road of a region in a finite time interval.

If the number of Bernoulli trials of a random experiment is fairly large and the probability of success is small it becomes increasingly difficult to compute the binomial probabilities. For values of n and p such that $n \geq 150$ and $p \leq 0.05$, the poisson distribution serves as an excellent approximation to the binominal distribution.

The random variable X is said to follow the Poisson distribution if and only if

$$p[X=x]=\frac{e^{-\lambda}\lambda^{x}}{\lfloor x}, x=0,1,2,.....$$

POISSON DISTRIBUTION AS A LIMITING CASE OF BINOMIAL DISTRIBUTION:

ASSUMPTIONS:

(i) Number of Bernoulli trials (n) is indefinitely large, (n $\to \infty$)
(ii) The trials are independent.
(iii) Probability of success (p) is very small, (p $\to$ 0)
(iv) λ = np is constant.

$$\lambda = np \Rightarrow p=\frac{\lambda}{n} --------(*)$$

Binomial probability:

$$p_n[X=x]=\binom{n}{x}p^{x}q^{n-x} ------(**)$$

$$\binom{n}{x}=\frac{\lfloor n}{\lfloor x \lfloor n-x}=\frac{n(n-1).......(n-x+1).(n-x)....3.2.1}{\lfloor x\ 1.2.3.....(n-x)}$$

$$=\frac{n(n-1)......(n-x+1)}{\lfloor x} --------(***)$$

Substitute (*) and (***) in (**),

$$P_n[X=x]=\frac{n(n-1).....[n-(x-1)]}{\lfloor x}\left(\frac{\lambda}{n}\right)^{x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{nx\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).......\left(1-\frac{(x-1)}{n}\right)}{\lfloor x}\frac{\lambda^{x}}{n^{x}}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$=\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)........\left(1-\frac{(x-1)}{n}\right)}{\lfloor x}\left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda}\left(1-\frac{\lambda}{n}\right)^{-x}\lambda^{x}$$

$$\underset{n\to\infty}{Lt}P_n[X=x]=\underset{n\to\infty}{Lt}\frac{\left\{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).........\left(1-\frac{(x-1)}{n}\right)\right\}}{\lfloor x}\left[\underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda}\left[\underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)\right]^{-x}\lambda^{x}$$

$$=\frac{1}{\lfloor x}(e)^{-\lambda}.1.\lambda^{x}$$

$$P[X=x]=\frac{e^{-\lambda}\lambda^{x}}{\lfloor x}, x=0,1,2,.......$$

PROPERTIES:

Property 1**:** Characteristic Function: $\phi_x(t) = e^{\lambda\left[e^{it}-1\right]}$

Property 2**:** Moment generating function : $M_X(t) = e^{\lambda\left[e^{t}-1\right]}$

Property 3**:** Mean : $\mu_1' = \lambda$

Property 4**:** Variance : $\mu_2 = \lambda$

Property 5**:** Recurrence relation for poisson probabilities:

$$p(x+1) = \frac{\lambda}{x+1} p(x)$$

Property 6**:** Poisson distribution satisfies the property of additivity

8.4.3 DERIVATION OF PROPERTIES

Property 1 Characteristic function of poisson distribution :

$$\phi_X(t) = E\left(e^{itX}\right)$$

$$= \sum_{x=0}^{\infty} e^{itx} P\left[X = x\right] = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\left[1 + \frac{\lambda e^{it}}{1!} + \frac{\left(\lambda e^{it}\right)^2}{2!} + \ldots\right]$$

$$= e^{-\lambda} e^{\lambda e^{it}} = \exp\left(-\lambda + \lambda e^{it}\right) = \exp\left(\lambda\left(e^{it}-1\right)\right) = e^{\lambda\left[e^{it}-1\right]}$$

Property 2 Moment generating function :

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{\lfloor x} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(e^t\lambda\right)^x}{\lfloor x} = e^{-\lambda} e^{\lambda e^t}$$

$$M_X(t) = e^{\lambda}\left[e^{t} - 1\right]$$

Property 3 Mean :

$$\mu_1^{'} = E(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{\lfloor x}$$

$$= e^{-\lambda}\sum_{x=1}^{\infty} \frac{x\lambda^{x}}{\lfloor x} = e^{-\lambda}\lambda\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{\lfloor x-1}$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{\lfloor 1} + \frac{\lambda^{2}}{\lfloor 2} + ...\right] \ \lambda e^{-\lambda}e^{\lambda} = \lambda e^{\circ} = \lambda$$

$$= \mu_1^{'} = \lambda$$

Property 4 Variance :

$$\mu_2^{1} = E\left(X^{2}\right) = \sum_{x=0}^{\infty}\left[x + x(x-1)\right]\frac{e^{-\lambda}\lambda^{x}}{\lfloor x}$$

$$= \sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^{x}}{\lfloor x} + \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^{x}}{\lfloor x}$$

$$= \lambda + e^{-\lambda}\lambda^{2}\left[1 + \frac{\lambda}{\lfloor 1} + \frac{\lambda^{2}}{\lfloor 2} + ...\right] = \lambda + e^{-\lambda}\lambda^{2}e^{\lambda} = \lambda + \lambda^{2}$$

$$V(X) = \mu_2^{1} - \left(\mu_1^{'}\right)^{2} = \lambda + \lambda^{2} - \lambda^{2} = \lambda$$

$$V(X) = \lambda$$

Remark : For Poisson distribution mean and variance are equal .

Property 5 Recurrence relation for poisson probabilities :

$$\frac{P[X=x+1]}{P[X=x]} = \frac{e^{-\lambda}\lambda^{x+1}}{\lfloor x+1} \Big/ \frac{e^{-\lambda}\lambda^{x}}{\lfloor x}$$

$$= \frac{\lambda}{x+1} \; P[X=x+1] = \frac{\lambda}{x+1} P[X=x]$$

$$P(x+1) = \frac{\lambda}{x+1} P(x)$$

Property 6 Additive property of Poisson distribution :

$$If\ X_1 \sim p(\lambda_1),\ X_2 \sim p(\lambda_2),\ then\ X_1 + X_2 \sim p(\lambda_1+\lambda_2)$$

Proof : Since X_1 and X_2 follow Poisson distribution with parameters λ_1 and λ_2, their

Moment generating functions are respectively,

$$\mathrm{M}_1(\mathrm{t}) = \mathrm{e}^{\lambda_1[\mathrm{e}^{\mathrm{it}}-1]},\ \mathrm{M}_2(\mathrm{t}) = \mathrm{e}^{\lambda_2[\mathrm{e}^{\mathrm{it}}-1]} \quad (\lambda_1+\lambda_2)\left[e^{it}-1\right]$$

$$M(t) = M_1(t) + M_2(t) = e^{(\lambda_1+\lambda_2)\left[e^{it}-1\right]}$$

Due to uniqueness of the moment generating function, M(t) is the moment generating function of a Poisson random variable. Thus, $X_1 + X_2$ follows poisson distribution with parameter $\lambda_1+\lambda_2$.

PROBLEMS ON POISSON DISTRIBUTION:

Problem : Using MGF derive mean and variance of poisson distribution.

Solution: Mean : $\frac{d}{dt}M_X(t) = e^{\lambda}\left[e^t - 1\right]\lambda e^t$

$$\mu_1^{'} = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \lambda$$

Variance: $\frac{d^2 M_X(t)}{dt^2} = \lambda\left[e^{\lambda\left[e^t-1\right]}e^t + e^t e^{\lambda\left[e^t-1\right]}\lambda e^t\right]$

$$\mu_2^1 = \left[\frac{d^2 M_X(t)}{dt^2}\right]_{t=0} = \lambda\left[1+\lambda\right] = \lambda + \lambda^2$$

$$V(X) = (\lambda+\lambda^2) - \lambda^2 = \lambda$$

$$V(X) = \lambda$$

Problem : Using the characteristic function derive the mean and variance of poisson

distribution

Solution : $\phi_X(t) = e^{\lambda\left[e^{it}-1\right]}$

Mean : $\frac{d\phi_X(t)}{dt} = \lambda i e^{it} e^{\lambda\left[e^{it}-1\right]}$

$$\left[\frac{d\phi_X(t)}{dt}\right]_{t=0} = i\lambda$$

$$\mu_1^{'} = i^{-1}\left[\frac{d\phi_X(t)}{dt}\right]_{t=0} = \lambda$$

Variance :

$$\frac{d^2\phi_X(t)}{dt^2} = \lambda i\left[ie^{it}e^{\lambda(eit-1)} + e^{it}e^{\lambda(eit-1)}\lambda ie^{it}\right]$$

$$\left[\frac{d^2\phi_X(t)}{dt^2}\right]_{t=0} = \lambda i^2\left[1+\lambda\right] = -\lambda\left[1+\lambda\right]$$

$$\mu_2^1 = (-i)^2\left[\frac{d^2\phi_X(t)}{dt^2}\right]_{t=0} = -\left[\frac{d^2\phi_X(t)}{dt^2}\right]_{t=0} = \lambda(1+\lambda)$$

V(X) = $\mu_2^1 - \left(\mu_1^{'}\right)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$

V(X) = λ

Problem : 1.4 percent of cases received by a switch board are wrong calls. Use

poisson approximation to the binomial probability that two out of 150 calls are wrong numbers.

Solution: Probability that a call received is a wrong one:

$$p = \frac{1.4}{100} = \frac{7}{500}$$

$$np = \frac{7}{500}\times 150 = \frac{21}{10} = \lambda$$

$$P\left[X=2\right] = \frac{e^{-21/10}\left(\frac{21}{10}\right)^2}{\lfloor 2} = \frac{e^{-2.1}\left(\frac{21}{10}\right)^2}{2} = \frac{0.1225}{2}\left(\frac{441}{100}\right)$$

$$P[X=2] = 0.2701$$

Problem : The probability of getting no misprint in a page of a book is e^{-3}. What is the

probability that a page contains more than 2 misprints?

Solution: Let X denote' number of misprints in a page' and follow poisson distribution.

$$P[X=0] = e^{-\lambda}$$

$$\lambda = 3$$

$$P[X=0] = e^{-3}$$

$$P[X > 2] = 1 - P[X = 0] - P[X = 1] - P[X = 2]$$

$$= 1 - e^{-3} - e^{-3}3 - \frac{e^{-3}3^2}{\underline{|2}}$$

$$= 1 - e^{-3}\left[4 + \frac{9}{2}\right] = 1 - e^{-3}\frac{17}{2}$$

$$= 1 - 0.4232 = 0.5768$$

$$P[X > 2] = 0.5768$$

Problem : The number of monthly breakdowns of a computer is a random variable that

follows poisson distribution with λ =1.5. Find the probabilities that the computer will function in a month (i) without a breakdown; (ii) with one break down.

Solution : Let X = Number of breakdowns in a month.

λ =1.5 (Given)

(i) $p[X=0]=e^{-\lambda}=e^{-1.5}=0.2231$

(ii) $p[X=1]=e^{-\lambda}\lambda=0.2231\times1.5=0.3347$

PROBLEM : Find the mode of the poisson distribution

Solution :A value assumed by the poisson random variable that maximizes the poisson

probabilities is mode, which is obtained solving the inequalities,

$$P[X=x-1]\le p[X=x]\ge p[X=x+1]$$

$$\Rightarrow P[X=x-1]\le P[X\le x]$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{x-1}}{\underline{|x-1}}\le\frac{e^{-\lambda}\lambda^{x}}{\underline{|x}}$$

$$\Rightarrow 1\le\frac{\lambda}{x}\Rightarrow x\le\lambda \qquad \text{...(*)}$$

$$P[X=x]\ge P[X=x+1]$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{x}}{\underline{|x}}\ge\frac{e^{-\lambda}\lambda^{x+1}}{\underline{|x+1}}\Rightarrow 1\ge\frac{\lambda}{x+1}\Rightarrow x+1\ge\lambda$$

$$\Rightarrow x\ge\lambda-1 \qquad \text{...(**)}$$

Combining (*) and (**),

$$\lambda-1\le x\le\lambda$$

Two cases will arise:

(i) λ is non-integer value. Then decompose λ into the sum of an integer (a) and a fraction (f) such that 0<f<1.
$\lambda=a+f$

Mode = a

(ii) λ is an integer. The distribution is bimodal.
Mode (1) = λ-1

Mode (2) = λ

PROBLEM : For the particulars of the problem (3.3.4) find mode

Solution: λ=1.5 = 1 + 0.5

Mode = 1

Problem : A poisson distribution is bimodal, having modes at x = 1 and x = 2. What is the

probability that x will have one or the other of these two values?

Solution : Mode occurs at x = λ-1 and x = λ, λ=2

Required event: [X = 1 or X = 2]

P[X = 1 or X = 2] = P[X = 1] +P[X = 2] = 4e^{-2}

Required probability : 4 e^{-2}

PROBLEM : X is a poisson variate such that, P[X = 1] = P[X = 2], find P[X = 4]

Solution : $p[X=1]=\dfrac{e^{-\lambda}\lambda}{\lfloor 1}$, $p[X=2]=\dfrac{e^{-\lambda}\lambda^2}{\lfloor 2}$, By equating, we obtain, $e^{-\lambda}\lambda = \dfrac{e^{-\lambda}\lambda^2}{2}$

$$\Rightarrow 2\lambda = \lambda^2 \Rightarrow \lambda = 2$$

$$p[X=4] = \frac{e^{-\lambda}\lambda^4}{\lfloor 4} = \frac{e^{-2}2^4}{24} = \frac{2}{3}e^{-2}$$

$$P[X=4] = \frac{2}{3}e^{-2}$$

PROBLEM : A car hire company has two cars, which it hires out day by day. The

number of demands for a car on each day is distributed as poisson with mean 1.5. Compute the proportion of days on which some demand is refused.

Solution : Number of demands : X

Proportion of days on which some demand id refused:

P[X>2] = 1 - P[X = 0] - P[X = 1] - P[X = 2]

where P[X > 2] = $\dfrac{\text{Number of days some demand is refused}}{\text{Total number of observed days}}$

$$= 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2}{2}e^{-\lambda}$$

$$= 1 - e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2}\right]$$ λ = 1.5 (Given)

P[X > 2] $= 1 - e^{-1.5}\left[1 + 1.5 + \dfrac{(1.5)^2}{2}\right]$

$$= 1 - 0.2231 \times 3.625$$

$$= 1 - 0.8087 = 0.1912$$

Problem : The average number of trucks arriving on any one day at a truck depot in a

town is 12. Find the chance that on any given day less than 9 trucks will arrive at the depot?

Solution : X = Number of trucks arrived

λ = 12

$p[X < 9] = \sum_{i=1}^{8} p[X = i]$ Apply recursive relation for Poisson probabilities

$$P[X = 0] = e^{-12}$$

$$P[X = 1] = \frac{12}{1} P[X = 0] = 12e^{-12}$$

$$P[X = 2] = \frac{12}{2} P[X = 1] = 6 \times 12e^{-12} = 72e^{-12}$$

$$P[X = 3] = \frac{12}{3} P[X = 2] = 4 \times 72e^{-12} = 288e^{-12}$$

$$P[X = 4] = \frac{12}{4} P[X = 3] = 3 \times 288e^{-12} = 864e^{-12}$$

$$P[X = 5] = \frac{12}{5} P[X = 4] = \frac{12}{5} \times 864e^{-12} = 2073.6e^{-12}$$

$$P[X = 6] = \frac{12}{6} P[X = 5] = 2 \times 2073.6e^{-12} = 4147.2e^{-12}$$

$$P[X = 7] = \frac{12}{7} P[X = 6] = \frac{12}{7} \times 4147.2e^{-12} = 7109.49e^{-12}$$

$$P[X=8]=\frac{12}{8}P[X=7]=\frac{12}{8}\times 7409.49e^{-12}=10664.23e^{-12}$$

$$P[X<9]=(1+12+72+288+864+2073.6+4147.2+7109.49+10664.23)e^{-12}$$
$$=25231.52e^{-12}=0.1550$$

$$P[X<9]=0.1550$$

Problem : If X_1 and X_2 are independently distributed Poisson random variables with parameters λ_1 and λ_2 respectively, what distribution $X_1 + X_2$ follows? Use characteristic function.

Solution : Characteristic function of X_1: $\phi_{X_1}(t)=e^{\lambda_1(e^{it}-1)}$

Characteristic function of X_2: $\phi_{X_2}(t)=e^{\lambda_2(e^{it}-1)}$

$$\phi_{X_1+X_2}(t)=e^{\lambda_1}(e^{it}-1)e^{\lambda_2}(e^{it}-1)\ =e^{(\lambda_1+\lambda_2)(e^{it-1})}\ =e^{\lambda(eit-1)}$$

$$\phi_{X_1+X_2}(t)=e^{\lambda(e^{it}-1)}\qquad \text{where } \lambda=\lambda_1+\lambda_2$$

The characteristic function of $X_1 + X_2$ is in the form of the c. f of Poisson random variable. By uniqueness of characteristic function $X_1 + X_2$ is also distributed Poisson with parameter $\lambda=\lambda_1+\lambda_2$

Problem : If X follows Poisson distribution find

$$P[X=\text{Even}]$$

$$P[X=\text{Even}]=e^{-\lambda}\left[1+\frac{\lambda^2}{\lfloor 2}+\frac{\lambda^4}{\lfloor 4}+\ldots\ldots\ldots\right] \qquad \ldots(*)$$

$$\Rightarrow \qquad e^{\lambda}=1+\frac{\lambda}{\lfloor 1}+\frac{\lambda^2}{\lfloor 2}+\frac{\lambda^3}{\lfloor 3}+\frac{\lambda^4}{\lfloor 4}+\ldots\ldots\ldots$$

$$\Rightarrow \qquad e^{-\lambda} = 1 - \frac{\lambda}{\lfloor 1} + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^3}{\lfloor 3} + \frac{\lambda^4}{\lfloor 4} + \ldots\ldots\ldots$$

$$\left[e^{\lambda} + e^{-\lambda}\right] = 2\left[1 + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^4}{\lfloor 4} + \ldots\ldots\ldots\right]$$

$$\Rightarrow \frac{1}{2}\left[e^{\lambda} + e^{-\lambda}\right] = 1 + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^4}{\lfloor 4} + \ldots\ldots\ldots(**)$$

Substitute (**) in (*) to obtain,

$$P[X = \text{Even}] = \frac{1}{2}e^{-\lambda}\left[e^{\lambda} + e^{-\lambda}\right]$$

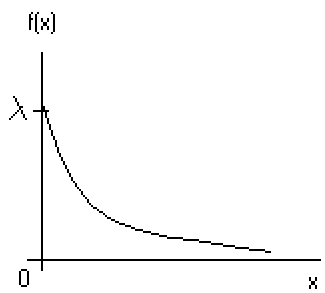
$$P[X = \text{Even}] = \frac{1}{2}\left[1 + e^{-2\lambda}\right]$$

****

THE EXPONENTIAL DISTRIBUTION :

A random variable X is said to posses exponential distribution if and only if its density can be expressed of the form

$$F(x) = \begin{cases} \lambda e^{-\lambda x}, & if \ x \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases}$$



The exponential distribution has several applications. In a queuing process if arrivals are random and follow Poisson distribution, the inter arrival time follows exponential distribution. For example, in a system if failure rate of a component is constant, then failure time follows exponential distribution.

One of the important properties of the exponential distribution is memoryless property.

PROPERTIES :

Property 1 Characteristic function : $\phi_X(t) = \left[1-\frac{it}{\lambda}\right]^{-1}$

Property 2 Moment generating function : $M_X(t) = \left[1-\frac{t}{\lambda}\right]^{-1}$

Property 3 Mean : $\mu_1' = \frac{1}{\lambda}$

Property 4 Variance : $\mu_2 = \frac{1}{\lambda^2}$

Property 5 Distribution function : $F(x) = \begin{cases} 0 & if \quad x \le 0 \\ 1 - e^{-\lambda x} & if \quad x > 0 \end{cases}$

Property 6 Exponential distribution satisfies the memoryless property.

DERIVATION OF PROPERTIES

Property 1 **Characteristic function :**

$$\phi_X(t) = E(e^{itx}) = \lambda\int_0^\infty e^{itx}e^{-\lambda x}dx = \lambda\int_0^\infty e^{-x(\lambda - it)}dx = \lambda[\lambda - it]^{-1} = \left[1-\frac{it}{\lambda}\right]^{-1}$$

Property 2 **Moment generating function :**

$$M_x(t) = \left[1-\frac{t}{\lambda}\right]^{-1}, t < \lambda$$

Property 3 **Mean :**

$$\frac{dM_X(t)}{dt} = \frac{1}{\lambda}\left[1-\frac{t}{\lambda}\right]^{-2}$$

$$\mu_1' = \left[\frac{dM_X(t)}{dt}\right]_{t=0} = \frac{1}{\lambda}$$

$$\mu_2' = \left[\frac{d^2M_X(t)}{dt^2}\right]_{t=0} = \frac{2}{\lambda^2}$$

Property 4 **Variance :**

V(X) = $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda}$

Property 6 Memoryless property :

Let X be the life time of a component. At time t its residual life time is, U= X-t. Consider the conditional probability, p[U ≤ u / X>t] which represents the probability that the residual life time of the component is less than or equal to u given that it already lived up to t.

$$p[U \le u / X > t] = p[X - t \le u / X > t]$$

$$= \frac{p[t < X \le u + t]}{p[X > t]}$$

$$p[t < X \le u + t] = \int_t^{u+t} \lambda\, e^{-\lambda v}\, dv$$

$$= \lambda\left[\frac{e^{-\lambda v}}{-\lambda}\right]_t^{u+t} = e^{-\lambda t} - e^{-\lambda(u+t)} = e^{-\lambda t}\left[1 - e^{-\lambda u}\right]$$

$$p[X > t] = \int_t^{\infty} \lambda\, e^{-\lambda v}\, dv = e^{-\lambda t}$$

$$p[U \le u / X > t] = e^{-\lambda t}\left[\frac{1 - e^{-\lambda u}}{e^{-\lambda t}}\right] = 1 - e^{-\lambda u}$$

The conditional probability is independent of t, which means that the residual life time of the component does not depend upon how long it already survived.

PROBLEMS ON EXPONENTIAL DISRTRIBUTION :

Problem : The random variable X follows exponential distribution with parameter λ. If $P[X \leq 1] = P[X>1]$, Find variance of X.

Solution: $P[X \leq 1] = \int_0^1 \lambda e^{-\lambda x} dx = 1 - e^{-\lambda}$ ... (*)

$P[X > 1] = \int_1^\infty \lambda e^{-\lambda x} dx = e^{-\lambda}$ ... (**)

Equating (*) and (**) we obtain,

$$1 - e^{-\lambda} = e^{-\lambda} \Rightarrow 2e^{-\lambda} = 1 \Rightarrow e^{-\lambda} = \frac{1}{2} \Rightarrow \lambda = 0.6931$$

$$V(X) = \frac{1}{\lambda^2} = 2.0817$$

Problem : X is distributed exponential with parameter λ, determine the Karl Pearson's

coefficient of skewness.

Solution: Let S_k denote the Karl Pearson's coefficient of skewness

$$S_k = \frac{\mu_3^2}{\mu_2^3}$$

where μ_2 , μ_3 are central moments of order two and three respectively.

$$\mu_3 = \mu_3' - 3\mu_2'(\mu_1') + (\mu_1')^3$$

$$\mu_3' = \frac{d^3 M_X(t)}{dt^3} = \frac{6}{\lambda^2}\left[1 - \frac{t}{\lambda}\right]^{-4} = \left[\frac{d^3 M_X(t)}{dt^3}\right]_{t=0} = \frac{6}{\lambda^3}$$

$$\mu_3 = \frac{6}{\lambda^3} - 3 \times \frac{1}{\lambda} \times \frac{2}{\lambda^2} + 2\left(\frac{1}{\lambda}\right)^3$$

$$= \frac{6}{\lambda^3} - \frac{6}{\lambda^3} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_2 = \frac{1}{\lambda^2}$$

$$S_k = \frac{\mu_3^2}{\mu_2^3} = \frac{4/\lambda^6}{1/\lambda^6} = 4$$

The exponential distribution is positively skewed.

Problem : Find an expression for r^{th} moment about origin, for the exponential distribution.

Solution: We can use the method of mathematical induction

Let k = 1, $\mu_1^{'} = \lambda\int_0^\infty xe^{-\lambda x}dx = \frac{1}{\lambda}$

Let k = 2, $\mu_2^{'} = \lambda\int_0^\infty x^2e^{-\lambda x}dx = \frac{1.2}{\lambda^2}$

Let k= r, $\mu_r^{'} = \frac{(1.2.3.........r)}{\lambda^r}$

Let k=r+1, $\mu_{r+1}^{'} = \lambda\int_0^\infty x^{r+1}e^{-\lambda x}dx = \frac{r+1}{\lambda}\lambda\int_0^\infty x^re^{-\lambda x}dx$ $= \left[\frac{r+1}{\lambda}\right]\frac{r!}{\lambda^r}$

$$\mu_{r+1} = \frac{(r+1)!}{\lambda^{r+1}}$$

Problem : The life of an electric bulb is exponentially distributed with failure rate λ = 1/3 (one failure in every 3000 hours on the average)

Find **(a)** the probability that the lamp will last between 2000 and 3000 hours

(b) P[1.5 ≤ x ≤ 3.0] **(c)** P[x > 3.5 / x > 2.5]

Solution : (a) **Let X denote the life time of an electric bulb**

$$P[X > 3] = \int_3^\infty \frac{1}{3}e^{-(x/3)}dx = \frac{1}{3}\left[-\frac{e^{-(x/3)}}{1/3}\right]_3^\infty = e^{-1} = 0.3679$$

(b) $P[1.5 \le X \le 3.0] = \int_{1.5}^{3.0}\frac{1}{3}e^{-(x/3)}dx = \left[-e^{-x/3}\right]_{1.5}^{3.0} = \left[e^{-1/2} - e^{-1}\right] = 0.2386$

(c) $P[X > 3.5/X > 2.5] = P[X > 1] = \int_1^\infty \frac{1}{3}e^{-\frac{x}{3}}dx = \left[e^{-\frac{1}{3}}\right] = 0.717$

(Consequence of memory less property)

****

THE GAMMA DISTRIBUTION :

The gamma distributed random variable is a continuous one. The gamma integral is defined as,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

It can be shown that (i) $\Gamma(n) = (n-1)\,\Gamma(n-1)$

(ii) $\Gamma(n) = (n-1)!$ where n is an integer

(iii) $\Gamma(1/2) = \sqrt{\pi}$

The gamma integral can be thought of a generalized factorial notion that is applicable to not only to positive integers but also to all positive real numbers.

The two parameter gamma distribution is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & if \quad x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

β and α are respectively called as the scale and shape parameters. If $\alpha = 1$ and $\frac{1}{\beta} = \theta$ we obtain

$f(x) = \theta e^{-\theta x}$, which is the exponential distribution.

PROPERTIES :

Property 1 Characteristic function : $\phi_X(t) = (1-i\beta t)^{-\alpha}$

Property 2 Moment generating function : $M_X(t) = (1-\beta t)^{-\alpha}$

Property 3 r^{th} raw moment : $\beta^r \dfrac{\Gamma(\alpha+r)}{\Gamma(\alpha)}$

Property 4 Mean : $\mu_1' = \alpha\beta$

Property 5 Variance : $\mu_2 = \alpha\beta^2$

Property 6 Distribution function :

$$F(x) = \begin{cases} 0 & if\ x<0 \\ \displaystyle\int_0^x \frac{1}{\beta^\alpha \Gamma(\alpha)} u^{\alpha-1} e^{-u/\beta} du\,, & if\ \ x>0 \end{cases}$$

DERIVATION OF PROPERTIES

Property 1 **Moment generating function :**

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1}\, e^{-\left(\frac{1}{\beta}-t\right)x}\, dx, \quad t<\frac{1}{\beta}$$

Let $\left(\frac{1}{\beta}-t\right)x = u \Rightarrow dx = \left(\frac{1}{\beta}-t\right)^{-1} du$

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \left(\frac{1}{\beta}-t\right)^{-(\alpha-1)} u^{\alpha-1} e^{-u} \left(\frac{1}{\beta}-t\right)^{-1} du$$

$$= \frac{\left(\frac{1}{\beta}-t\right)^{-\alpha}}{\beta^{\alpha}\Gamma(\alpha)}\int_0^{\infty} u^{\alpha-1}e^{-u}du \qquad = \frac{\left(\frac{1}{\beta}-t\right)^{-\alpha}}{\beta^{\alpha}\Gamma(\alpha)}\Gamma(\alpha) = (1-\beta t)^{-\alpha}$$

Property 2 **Characteristic function :**

In moment generating function $M_X(t)$ replace t with it , we obtain

CF : $\phi_X(t) = (1-i\beta t)^{-\alpha}$

Property 3 **r^{th} Moment about origin :**

$$\mu_r' = E(x^r) = \int_0^{\infty}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha+r-1}e^{-x/\beta}dx$$

Let $\frac{x}{\beta} = u \Rightarrow dx = \beta du$

$$\mu_r' = \int_0^{\infty}\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\beta u^{\alpha+r-1}e^{-u}\beta du$$

$$= \frac{\beta^{\alpha+r}}{\beta^{\alpha}}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}u^{\alpha+r-1}e^{-u}\,du \qquad = \frac{\beta^r}{\Gamma(\alpha)}\Gamma(\alpha+r) = \beta^r\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}$$

Property 4 **Mean :** $\mu_1' = \beta\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \quad = \alpha\beta\,\frac{\Gamma(\alpha)}{\Gamma(\alpha)} \quad = \alpha\beta$

Property 5 **Variance :** $\mu_2' = \beta^2 \dfrac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \dfrac{\beta^2(\alpha+1)\,\alpha\,\Gamma\alpha)}{\Gamma(\alpha)} = \beta^2\alpha(\alpha+1)$

$$V(X) = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \beta^2\alpha(\alpha+1) - (\alpha\,\beta)^2$$

$$= \beta^2\alpha^2 + \beta^2\alpha - \alpha^2\beta^2 = \alpha\beta^2$$

$$\mu_2 = \alpha\beta^2$$

Property 6 **Distribution fucntion :**

$$f(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ \displaystyle\int_0^x \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx & , \text{ if } 0 < x < +\infty \\ 1 & , \text{ if } x = \infty \end{cases}$$

PROBLEMS ON GAMMA DISTRIBUTION :

Problem : Find the mode of the gamma distribution.

Solution : To find stationary points, find $\dfrac{df(x)}{dx} = 0$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\right] = 0 \quad \Rightarrow \quad x^{\alpha-1} e^{-x/\beta}\left[\frac{\alpha-1}{x} - \frac{1}{\beta}\right] = 0$$

$$\Rightarrow \quad x = \beta(\alpha-1) \quad \Rightarrow \quad \text{Mode} = \beta(\alpha-1), \alpha > 1$$

ONE PARAMETER GAMMA DISTRIBUTION :

Let $\beta = 1$ in the two parameter gamma distribution,

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}, & if \quad x>0\\ 0 & \text{, otherwise}\end{cases}$$

PROPERTIES

Property 1 Characteristic function : $\phi_X(t) = (1-it)^{-\alpha}$

Property 2 Moment generating function : $M_X(t) = (1-t)^{-\alpha}$

Property 3 r^{th} raw moment : $\dfrac{\Gamma(\alpha+r)}{\Gamma(\alpha)}$

Property 4 Mean : $\mu_1' = \alpha$

Property 5 Variance : $\mu_2 = \alpha$

Property **6** Distribution function :

$$F(x)=\begin{cases}0 & if \ x<0\\ \displaystyle\int_0^x \frac{1}{\Gamma(\alpha)}u^{\alpha-1}e^{-u}\,du, & if \quad x>0\end{cases}$$

The properties of(9.9.1) are obtained from those of (9.7.1) putting $\beta = 1$

PROBLEMS ON GAMMA DISTRIBUTION :

Problem : Let $X_1, X_2 \ldots\ldots, X_n$ be independently distributed gamma random variables with respective parameters $\alpha_1, \alpha_2, \ldots., \alpha_n$. Examine if their sum is also gamma distributed.

Solution : Let $Y = X_1 + X_2 + \ldots.. + X_n$

$$M_Y(t) = E\left[e^{t(X_1+X_2+\ldots\ldots+X_n)}\right]$$

$$= E\left(e^{tx_1}\right) E\left(e^{tx_2}\right) \text{........} E\left(e^{tx_n}\right)$$

$$= (1-t)^{-\alpha_1} (1-t)^{-\alpha_2} \text{...........} (1-t)^{-\alpha_n}$$

$$= [1-t]^{-(\alpha_1+\alpha_2+\text{........}+\alpha_n)}$$

By uniqueness of the MGF, Y is gamma distributed with parameter

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

****

THE BETA DISTRIBUTION :

A beta distributed random variable assumes all values of the interval (0,1).

The integral denoted by $\beta(m,n)$ is given as,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx\,, \quad m>0\,,\ n>0$$

It can be shown that (i) $\beta(m,n) = \beta(n,m)$ (ii) $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

A random variable is said to follow beta distribution if its probability density function is expressed as follows :

$$f(x)=\begin{cases}\dfrac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}x^{m-1}(1-x)^{n-1} & ,0<\mathrm{x}<1;\alpha,\beta>0\\ 0 & ,\text{otherwise}\end{cases}$$

PROPERTIES

Property 1 r^th^ raw moment : $\mu_1^{'}=\dfrac{m(m+1)..........(m+r-1)}{(m+n)............(m+n+r-1)}$

Property 2 Mean : $\mu_1^{'}=\dfrac{m\,n}{(m+n)}$

Property 3 Variance : $\mu_2=\dfrac{m\,n}{(m+n)^2\,(m+n+1)}$

Property 4 Distribution function is

$$F(x)=\begin{cases}0 & ,\text{if } \mathrm{x}\le 0\\ \dfrac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\displaystyle\int_0^x u^{m-1}(1-u)^{n-1}du & ,\text{if } 0<\mathrm{x}<1\\ 1 & ,\text{if } \mathrm{x}\ge 1\end{cases}$$

DERIVATION OF PROPERTIES

Property 1 r^th^ Moment about origin :

$$\mu_{\mathrm{r}}^{'} = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\int_0^1 x^{\mathrm{r}}x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\int_0^1 x^{m+\mathrm{r}-1}(1-x)^{n-1}dx$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\beta(\mathrm{m+r,n}) \quad = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\frac{\Gamma(m+r)\Gamma(n)}{\Gamma(m+n+r)}$$

$$= \frac{\Gamma(m+n)}{\Gamma(m)}\ \frac{(m+r-1)\ldots\ldots\ldots\ldots m\,\Gamma(m)}{(m+n+r-1)\ldots\ldots\ldots(m+n)\,\Gamma(m+n)}$$

$$\mu_r' \quad = \frac{m(m+1)\ldots\ldots(m+r-1)}{(m+n)\ldots\ldots\ldots\ldots(m+n+r-1)}$$

Property 2 Mean : $\mu_1' = \dfrac{m}{m+n}$

Property 3 Variance : V(X)

$$\mu_2' \quad = \frac{m(m+1)}{(m+n)(m+n+1)}$$

$$V(X) \quad = \mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 \quad = \frac{m(m+1)}{(m+n)(m+n+1)} - \frac{m^2}{(m+n)^2}$$

$$= \frac{m}{(m+n)}\left[\frac{(m+1)}{(m+n+1)} - \frac{m}{m+n}\right]$$

$$= \frac{m}{(m+n)} \frac{n}{(m+n)(m+n+1)} = \frac{mn}{(m+n)^2 (m+n+1)}$$

$$\mu_2 = \frac{mn}{(m+n)^2 (m+n+1)}$$

Property 4 Distribution Function :

$$F(x) = \begin{cases} 0 & , \text{if } x \le 0 \\ \dfrac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \displaystyle\int_0^x u^{m-1}(1-u)^{n-1}\,du & , \text{if } 0 < x < 1 \end{cases}$$

The above integral is called incomplete beta integral for which a closed form solution can not be obtained. Such integrals can be solved by iterative procedures

$$F(x) = \begin{cases} 0 & , \text{if } x \le 0 \\ \dfrac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \displaystyle\int_0^x u^{m-1}(1-u)^{n-1}\,du & , \text{if } 0 < x < 1 \\ 1 & , \text{if } x \ge 1 \end{cases}$$

PROBLEMS ON BETA DISTRIBUTION :

Problem : Find the mode of beta distribution.

Solution : $$\frac{d}{dx}\left[\frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\left(x^{m-1}(1-x)^{n-1}\right)\right]$$

$$\Rightarrow \quad \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}\; x^{m-1}(1-x)^{n-1}\left[\frac{m-1}{x} - \frac{n-1}{1-x}\right] = 0$$

$$\Rightarrow \quad \frac{m-1}{x} - \frac{n-1}{1-x} = 0$$

$$\Rightarrow \quad (m-1)(1-x) = (n-1)x$$

$$\Rightarrow \quad (m-1) - (m-1)x = (n-1)x$$

$$\Rightarrow \quad (m-1) = (m+n-2)x$$

$$\Rightarrow \quad x = \frac{m-1}{m+n-2}$$

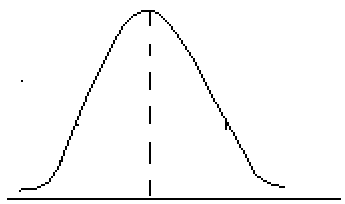
$$\Rightarrow \quad Mode = \frac{m-1}{m+n-2}$$

****

THE NORMAL DISTRIBUTION :

It is a continuous probability distribution that serves as a corner stone to the modern theory of statistics. Scientists of 18^{th} century identified this as a distribution of errors in measurement attributed to chance mechanism.

A random variable X is said to posses normal distribution with mean μ and variance σ^2 , if its probability density function can be expressed of the form,



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} , \quad -\infty < x < +\infty$$

$-\infty$ μ ∞

The standard notation used to denote a random variable to follow normal distribution with appropriate mean and variance is, X ~ N(μ , σ²)

PROPERTIES :

The following are some of the properties of normal distribution :

Property 1 Normal curve is symmetrically distributed around its mean since,

$$f(x-\mu) = f(\mu + x), \forall x$$

Property 2 The horizontal axis serves as an asymptotic, since

$$\underset{x\to\infty}{\mathrm{Lt}} f(x) = \underset{x\to-\infty}{\mathrm{Lt}} f(x) = 0$$

Property 3 The distribution function F(x) is given by,

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(v-\mu)^2}{2\sigma^2}} \, dv$$

Solution for F(x) cannot be obtained in closed form so that incomplete

integrals are to be worked out using iterative procedures. However, ready made tables exist to find the values of F(x) for given values of x upto two decimals. These tables are available for normal distribution whose mean and variance are respectively 0 and 1.

Property 5 The total area bound by the normal curve and horizontal axis is equal to

unity.

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(v-\mu)^2}{2\sigma^2}} \, dv = 1$$

STANDARD NORMAL DISTRIBUTION :

If a random variable X follows normal distribution with mean μ and variance σ^2 , its transformation $Z = \frac{X-\mu}{\sigma^2}$ follows standard normal distribution (mean 0 and unit variance)

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad , \quad -\infty < z < +\infty$$

The distribution function of the standard normal distribution

$$F(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx$$

PROPERTIES :

Property 1 Characteristic function : $\phi_X(t) = e^{-t^2/2}$

Property 2 Moment generating function : $M_X(t)=e^{t^2/2}$

Property 3 Moments : $\mu_{2n}=\frac{2}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)$

: $\mu_{2n+1}=0$

Property 4 Recurrence relationship : $\mu_{2n}=(2n-1)\mu_{2n-2}$

Property 5 Distribution Function : $F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\,du$

DERIVATION OF PROPERTIES

Property 1 **Characteristic function :**

$$\phi_X(t) = E(e^{itx}) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{itx}e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}+itx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-(x-it)^2}{2}-\frac{t^2}{2}}\,dx$$

$$= e^{-\frac{t^2}{2}}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-(x-it)^2}{2\sigma^2}}\,dx\right\}$$

$$= e^{-\frac{t^2}{2}}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-v^2}{2}}\,dv\right\},\ \text{where } x - it = v$$

$$\phi_X(t) = e^{-\frac{t^2}{2}}$$

Property 2 **Moment generating function :**

$$M_X(t) = e^{t^2/2}$$

By replacing t in MGF with 'it' the characteristic function can be obtained.

Property 3 **Moments :**

Since the normal distribution is symmetrically distributed all of its odd ordered moments vanish. Expression for a moment is obtained provided that it is even ordered.

$$\mu_{2n} = E(X^{2n}) = \int_{-\infty}^{\infty} x^{2n} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^{2n}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \qquad = 2\int_{0}^{\infty} x^{2n}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$$

Substituting u for $x^2/2$, we obtain,

$$\mu_{2n} = 2\int_{0}^{\infty}\frac{1}{\sqrt{2\pi}}u^{n-\frac{1}{2}}e^{-u}\,du$$

$$= \frac{2}{2\sqrt{\pi}} \int_0^\infty 2^n \, u^{\left(n+\frac{1}{2}\right)-1} e^{-u} \, du \quad = \frac{2^n}{\sqrt{\pi}} \int_0^\infty u^{\left(n+\frac{1}{2}\right)-1} e^{-u} \, du$$

$$= \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right)$$

The standard normal curve is symmetric for which *f(x) = f(-x)*

The expression $x^{2n+1} f(x)$ is an odd function

Since $(-x)^{2n+1} f(x) = -x^{2n+1} f(x)$

g(x) = -g(x)

Consequently, $$\frac{1}{\sqrt{2\pi}} \int_0^\infty x^{2n+1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = 0$$

$$\Leftrightarrow \qquad \mu_{2n+1} = 0$$

Property 4 **Recurrence relationship for moments :**

$$\mu_{2n} = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2} - 1\right) \Gamma\left(n + \frac{1}{2} - 1\right)$$

$$= \frac{2^n}{\sqrt{\pi}} \Gamma\left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right)$$

$$\mu_{2(n-1)} = \frac{2^{n-1}}{\sqrt{\pi}}\Gamma\left(n-1+\frac{1}{2}\right) = \frac{2^{n-1}}{\sqrt{\pi}}\Gamma\left(n-\frac{1}{2}\right)$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = \frac{\frac{2^n}{\sqrt{\pi}}\Gamma\left(n-\frac{1}{2}\right)\ \Gamma\left(n-\frac{1}{2}\right)}{\frac{2^{n-1}}{\sqrt{\pi}}\Gamma\left(n-\frac{1}{2}\right)} = 2\left(n-\frac{1}{2}\right) = 2\left(\frac{2n-1}{2}\right) = 2n-1$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = 2n-1$$

$$\Rightarrow \quad \mu_{2n} = (2n-1)\mu_{2n-2}$$

Property 5 **Distribution function :**

$$F(x) = \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

For a given value of x, F(x) can be evaluated by iterative procedure. However ,to find F(x) values readymade tables are available.

Property 2 **Mode of the normal distribution :**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow \log f(x) = \log e - \frac{(x-\mu)^2}{2\sigma^2}$$

Maximizing *f(x)* is same as maximizing log *f(x)*

$$\Rightarrow \quad \frac{d \log f(x)}{dx} = \frac{-(x-\mu)^2}{2\sigma^2} = 0$$

$$\Rightarrow \quad x - \mu = 0$$

$$\Rightarrow \quad x = \text{Mode} = \mu$$

$$\Rightarrow \quad \frac{d^2 \log f(x)}{dx^2} = -\frac{1}{\sigma^2} < 0$$

$$\Rightarrow \quad \text{Mode} = \mu$$

For standard normal distribution Mode = 0

MEDIAN OF THE NORMAL DISTRIBUTION :

$$\int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2} = \int_{M}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

The above equation is satisfied if and only if Median is equal to μ.

Median = μ

For the normal distribution mean, median and mode coincide with μ.

AREA PROPERTY :

$$P[\mu-\sigma \le X \le \mu+\sigma] = 0.6826 \quad \text{(One sigma limit)}$$

$$P[\mu-2\sigma \le X \le \mu+2\sigma] = 0.9544 \quad \text{(Two sigma limit)}$$

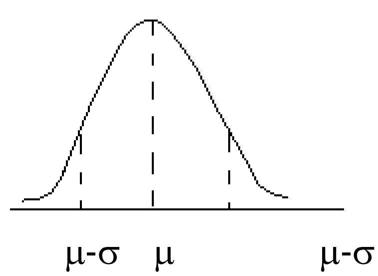
$$P[\mu-3\sigma \le X \le \mu+3\sigma] = 0.9973 \quad \text{(Three sigma limit)}$$

The limits $\mu \pm \sigma, \mu \pm 2\sigma$ and $\mu \pm 3\sigma$ are respectively called the one, two and three sigma limits.

PROBLEMS ON NORMAL DISTRIBUTION :

Problem : Obtain the points of inflexion for normal distribution

Solution :



The points of inflexion are stationary points of the function under optimization for which the first two derivatives vanish. These are the points at which curve switches from convexity to concavity or from concavity to convexity.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{d\ f(x)}{dx} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[\frac{-2}{2\sigma^2}(x-\mu)\right]$$

$$\frac{d\ f(x)}{dx} = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)$$

$$\frac{d^2 f(x)}{dx^2} = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \left(\frac{x-\mu}{\sigma}\right)^2 \frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 0$$

$$-1+\left(\frac{x-\mu}{\sigma}\right)^2 = 0 \quad \Rightarrow \quad \left(\frac{x-\mu}{\sigma}\right)^2 = 1 \quad \Rightarrow \quad \frac{x-\mu}{\sigma} = \pm 1$$

$$\Rightarrow \quad x-\mu = \pm\sigma \quad \Rightarrow \quad x = \mu \pm \sigma$$

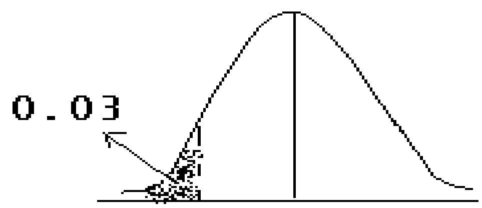
$x = \mu \pm \sigma$ are the points of inflexion of normal distribution

Problem : A machine fills instant coffee into jars of 6-ounce capacity. The actual quantity filled in is a random variable that is normally distributed with a standard deviation of 0.05 ounce. 3 percent of the jars are to contain less than 6 ounces of coffee, what is the mean fill of these jars ?

Solution : $\dfrac{\text{Number of jars with a content of less than 6 ounces}}{\text{Number of jars filled}} = 0.030(\text{given})$

$$P[X \leq 6] = 0.03$$

$$P\left[\frac{X-\mu}{\sigma} \leq \frac{6-\mu}{\sigma}\right] = P\left[\frac{X-\mu}{0.05} \leq \frac{X-\mu}{0.05}\right] = P[Z \leq z] = 0.03$$



where Z follows standard normal distribution

$-\infty$ z = -1.88 0

From standard normal area tables given in the appendix we obtain,

$$z = \frac{6-\mu}{\sigma} - (-1.88) \quad \Rightarrow \quad z = \frac{6-\mu}{0.05} - (-1.88)$$

$$\mu = 6 + (1.88)(0.05) = 6 + 0.094$$

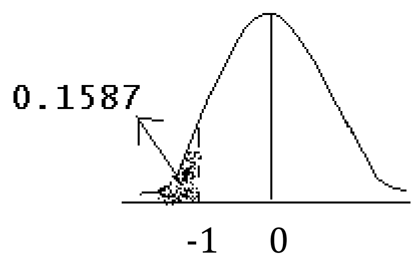
$$\mu = 6.094$$

Problem : The time required to load a cargo ship is normally distributed with mean 12 and variance 4. What is the probability that the ship will be loaded in 10 hours ?

Solution : T ~ N (12, 4)

Required event: E = [T ≤ 10]
Required probability: p[T ≤ 10]

$$P\left[\frac{T-\mu}{\sigma} \le \frac{10-\mu}{\sigma}\right] = P\left[Z \le \frac{10-12}{2}\right] = P[Z \le -1] = 0.1587$$



Remark :

The required probability is derived using symmetry property of the normal Distribution. The probabilities given in appendix cover positive Z-values ($0 \le z \le \infty$). The required probability is a tail probability(shaded area).

The areas of the intervals $(-\infty,-1)$, $(1,\infty)$ are one and the same. The area of the interval $(0,\infty)$ is ½ . The area of interval (0,1) can be found in the area tables of standard normal distribution which is subtracted from ½ to obtain the required probability.

Problem : In a photographic process the developing times of prints may be looked upon as

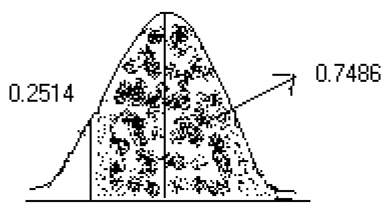
a random variable having the normal distribution with a mean of 16.28 seconds and a standard deviation 0.12 second. Find the probability that it will take (i) Atleast 16.20 seconds to develop one of the prints; (ii) atmost 16.35 seconds to develop one of the prints

Solution : Print developing time : X ~ N(16.28 , $(0.12)^2$)

(i) Required event : [X ≥ 16.20]

$$P[X \geq 16.20] = P\left[\frac{X - 16.28}{0.12} \geq \frac{16.20 - 16.28}{0.12}\right] = P[Z \geq -0.6667]$$

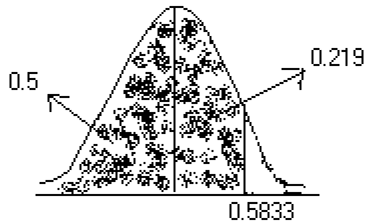
$$P[X \geq 16.20] = 0.7486 (\text{Dotted area})$$



(ii) Required event : [X ≤ 16.35]

$$P[X \leq 16.20] = P\left[Z \leq \frac{16.35 - 16.28}{0.12}\right]$$

$$= P[X \leq 0.5833] (\text{dotted area})$$



= 0.5 + 0.2190 = 0.7190

$P[X \le 16.35] = 0.7190$

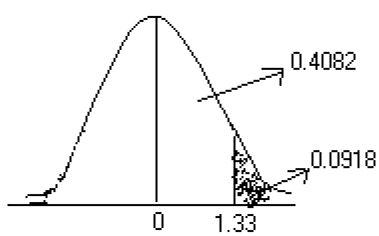
Problem : The mean height and variance of soldiers in a regiment are respectively 68

inches and 9 inches square. How many soldiers in a regiment of 10,000 would you expect to be over 6 feet tall.

Solution : Height of a soldier : X

Required event : [X ≥ 72]

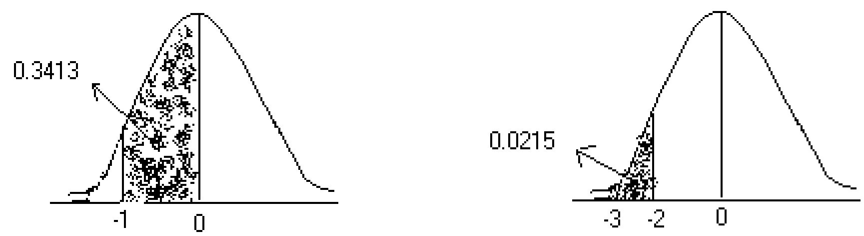
$$P[X \ge 72] = P\left[Z \ge \frac{72-68}{3}\right] = P\left[Z \ge \frac{4}{3}\right] = P[Z \ge 1.33] = 0.0918$$



Number of soldiers over 72 inches = (10,000) (0.0918) = 918

Problem : The random variable X follows normal distribution with mean 3 and variance 4. Find $P[1 \le x^2 \ge 9]$

Solution : $[1 \le X^2 \le 9] = [-3 \le X \le -1] \cup [1 \le X \le 3]$



$P[1 \le X^2 \le 9] = P[-3 \le X \le -1] + P[1 \le X \le 3]$ ----(*)

Consider $P[1 \le X \le 3] = P\left[\frac{1-3}{2} \le Z \le \frac{3-3}{2}\right] = P[-1 \le Z \le 0] = 0.3413$ ...(**)

$P[-3 \le X \le -1] = P\left[\frac{-3-3}{2} \le Z \le \frac{-1-3}{2}\right]$

$= P[-3 \le Z \le -2] = 0.0215$ ...(***)

From (*), (**), (***) it follows that,

$P[1 \le X^2 \le 9] = 0.3413 + 0.0215 = 0.3628$

Problem : The monthly income of a group of 10,000 persons were found to be

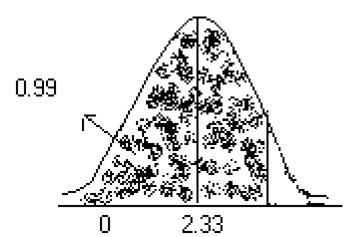
normally distributed with mean Rs. 750 and s.d Rs.50. What is the lowest income among the richest 100 ?

Solution : Monthly income : X ~ N(750,250)

$$P[X \geq x] = \frac{100}{10000} = 0.01$$

$$P\left[\frac{X-750}{50} \geq \frac{x-750}{50}\right] = 0.01$$

$$P[Z \geq z] = 0.01 \Rightarrow 1 - P[Z \leq z] = 0.01 \Rightarrow \qquad P[Z \leq z] = 0.99$$



For the given probability 0.99 we have to find z from standard normal tables.

z = 2.33

$$\Rightarrow \qquad \frac{x-750}{50} = 2.33$$

$$\Rightarrow \qquad x = 750 + (50)(2.33) = 866.5$$

The lowest income of the richest 100 is Rs.866.5

Problem : X is distributed normally with mean 25 and variance 25. Find (a) the limits which include the middle 50% of the area under normal curve, and (b) the values of x corresponding to the points of inflexion of the normal curve.

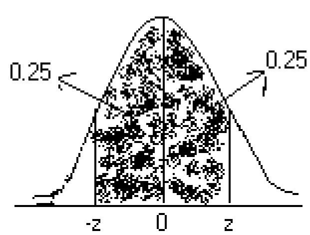
Solution : Let $Z = \frac{X-\mu}{\sigma}$ follow standard normal distribution P[-z ≤ Z ≤ z] = 0.50

(a) The limits which include the 50% area under the normal curve are μ-x and μ+x, where μ = 25

P[x-μ ≤ X ≤ x+μ] = 0.50

$$P\left[-\frac{x}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{x}{\sigma}\right] = 0.50$$

$$P\left[-\frac{x}{\sigma} \leq Z \leq \frac{x}{\sigma}\right] = P\left[-z \leq Z \leq z\right] = 0.50$$



$$P\left[0 \leq Z \leq z\right] = 0.25$$

From standard normal tables we obtain z=0.67

$$\frac{x}{\sigma} = \frac{x}{5} = 0.67$$

$x = 3.35$

$x\text{-}\mu \;=\; 3.35 - 25 \;= -21.65$

$x + \mu = \; 3.35 + 25 \;= 25.35$

$P[-21.6 \qquad 5 \leq X \leq 25.35] = 0.50$

(b) The points of inflexion of normal distribution are,

$x = \mu \pm \sigma$

$x = \mu + \sigma = 25+5 = 30$

$x = \mu - \sigma = 25\text{-}5 = 20$

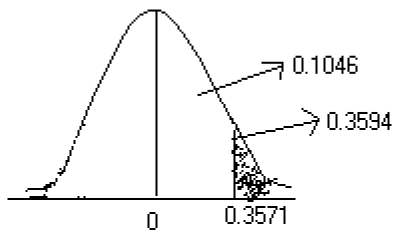
Problem : In a normal population with mean 15 and s.d 3.5, it is known that 647 observations exceed 16.25. What is the total number of observations in the population ?

Solution : $X \sim N\,[15, (3.5)^2]$

Let the size of the normal population be : N

$$P[X > 16.25] = \frac{\text{Number of observations each exceeded 16.25}}{\text{Total number of observations}}$$

$$P[X > 16.25] = \frac{625}{N}$$



$$= P[X > 16.25] = P\left[\frac{16.25 - 15}{3.5}\right] \quad = P[Z \geq 0.3571]$$

$$P[X > 16.25] = 0.3594 = \frac{647}{N}$$

$$N = \frac{647}{0.3594} = 1800$$

Problem : The moment generating function of a random variable is e^{3t+8t^2}

Find P[-1 ≤ X ≤ 9].

Solution : $M_X(t) = e^{3t+8t^2}$ is the moment generating function of a normal distribution.

The MGF of an arbitrary normal variate is given by,

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

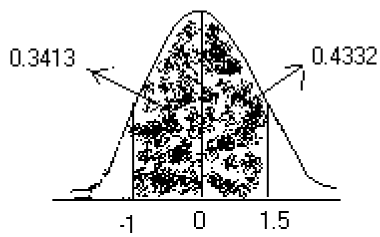
Comparing the coefficients, we obtain,

$$\mu = 3; \quad \frac{1}{2}\sigma^2 = 8$$

$$\Rightarrow \quad \sigma^2 = 16 \Rightarrow \sigma = 4$$

$X \sim N(3, 4)$

$$P[-1 \leq X \leq 9] = P\left[\frac{-1-3}{4} \leq Z \leq \frac{9-3}{4}\right] = P[-1 \leq Z \leq 1.5]$$



$p[-1 \leq X \leq 9] = 0.3413 + 0.4332 = 0.7745$

THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION :

(i) Let X be a random variable that follows binomial distribution, p being the probability of success in n independent Bernoulli trials.

Let $p = \frac{1}{2}$ and n be large

Define $Z = \frac{X - np}{\sqrt{npq}}$

Then, Z follows the standard normal distribution.

(ii) Suppose it is desired to find P[X = x], using normal approximation.

Consequently, a continuity correction is made and the binomial probability

is approximated by normal probability.

$$P\left[x-\frac{1}{2}\le X\le x+\frac{1}{2}\right]$$

Problem : Using normal approximation find the probability of getting 10 heads

in 16 trials of tossing a balanced coin.

Solution : Required event : $P\left[10-\frac{1}{2}\le X\le 10+\frac{1}{2}\right]$

Required probability: $P[9.5\le X\le 10.5]$

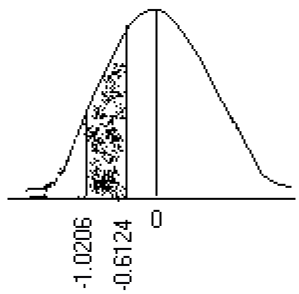
Mean : np = $24\times\frac{1}{2}=12$

Variance : npq = $24\times\frac{1}{2}\times\frac{1}{2}=6$

Standard deviation : $S.D=\sqrt{6}$

$$P[9.5\le X\le 10.5] = P\left[\frac{9.5-12}{\sqrt{6}}\le Z\le\frac{10.5-12}{\sqrt{6}}\right]$$

$$= P\left[\frac{-2.5}{\sqrt{6}} \le Z \le \frac{-1.5}{\sqrt{6}}\right] \qquad = P[-1.0206 \le Z \le -0.6124]$$



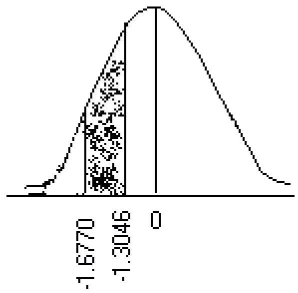
P[9.5 ≤ X ≤ 10.5] = 0.3222

Problem : A random variable follows binomial distribution with n= 30, p=0.60.

Using the normal approximation evaluate the probabilities that it will take on
(a) the value 14, (b) a value less than 17

Solution : p = 0.6 ; np = (30) (0.6) = 18; npq = (18) (0.4) = 7.2 $\sqrt{npq} = 2.6833$

(a) Required event : [13.5 ≤ X ≤ 14.5]

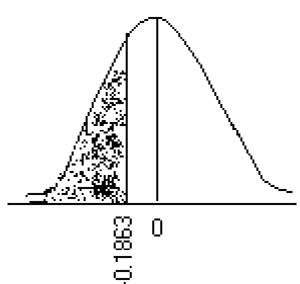


Required probability : $P[13.5 \leq X \leq 14.5] = P\left[\frac{13.5-18}{2.6833} \leq Z \leq \frac{14.5-18}{2.6833}\right]$

$P[-1.6770 \leq Z \leq -1.3046] = 0.0493$

$P[13.5 \leq X \leq 14.5]$ = 0.0493

(b) Required event : [X ≤ 17.5]



Required probability : $P[X \leq 17.5] = P\left[Z \leq \frac{17.5-18}{2.6833}\right] =$

$$P[Z \leq -0.1863] = 0.4250$$

$P[X \leq 17.5] =$ 0.4250

NORMAL DISTRIBUTION

Example 1: An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hrs and a S.D. of 40 hrs. Find

(i) The probability that a bulb burns more than 834 hrs.

(ii) The probability that a bulb burns between 778 and 834 hrs.

Solution:

Given $\mu = 800\ hrs$

$\sigma = 40\ hrs$

$z = \frac{X-800}{40}$

(i) P(a bulb burns more than 834 hrs)
=P(X>834)

When X=834, $z = \frac{834-800}{40} = 0.85$

P(X>834)=P(Z>0.85)

=1-P(Z<0.85)

=1-0.323

=0.6977

(ii) To find P(778<X<834)

When X=778, $z_1 = \frac{778-800}{40} = -0.55$

X=834, $z_2 = \frac{834-800}{40} = 0.85$

P(778<X<834)=P(-0.55<Z<0.85)

= P(0<Z<0.55)+0.3023

=0.2088+0.3023

=0.5111

Example 2: A normal distribution has mean μ=20 and S.D. σ =10. Find P(15≤ X ≤40).

Given μ=20 $\sigma = 10$

When X=15, $z = \frac{X-\mu}{\sigma} = \frac{15-20}{10} = -0.5$

X=40, $z = \frac{40-20}{10} = 2$

$P(15 \le X \le 40) = P(-0.5 \le z \le 2)$

$= P(-0.5 \le z \le 0) + P(0 \le z \le 2)$

$= P(0 \le z \le 0.5) + P(0 \le z \le 2)$

=0.1915+0.4772

=0.6687

Example 3: The savings bank account of a customer showed an average balance of Rs.150 and a S.D. of Rs.50. Assuming that the account balances are normally distributed.

i. What percentage of account is over Rs.200?
ii. What percentage of account is between Rs.120 and Rs.170?
iii. What percentage of account is less than Rs.75?

Given mean $\mu = 150 \quad S.D.\sigma = 50$

i). P(percentage of account is over Rs.200)

=P(X≥200)

We know that $z = \frac{X-\mu}{\sigma}$

When X=200, $z = \frac{200-150}{50} = 1$

P(X≥200) =P(z>1)

=0.5-P(z<1)

=0.5-0.3413

=0.1587

Percentage of account over Rs.200 is 15.87.

ii). P(percentage of account between Rs.120 and Rs.170)

=P(120≤X≤170)

When X=120, $z = \frac{120-150}{50} = \frac{-30}{50} = -0.6$

X=170, $z = \frac{170-150}{50} = \frac{20}{50} = 0.4$

$P(120 \leq X \leq 170) = P(-0.6 < z < 0.4)$

$= P(0 < z < 0.6) + P(0 < z < 0.4)$

$= 0.2257 + 0.1554$

$= 0.3811$

Percentage of account between Rs.120 and Rs.170 is 38.11.

iii). P(percentage of account less than Rs.75)

$= P(X < 75)$

When X=75, $z = \frac{75-150}{50} = \frac{-75}{50} = -1.5$

$P(X < 75) = P(z < -1.5)$

$= 0.5 - P(0 < z < 1.5)$

$= 0. - 0.4332$

$= 0.0668$

Percentage of account less than Rs.75 is 6.68%.

Practice Problems:

1) In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for (i) more than 2150 hours, (ii) less than 1950 hours and (iii) more than 1920 hours but less than 2160 hours.

(Solution: (i) 67 (ii) 134 (iii) 1909)

2) In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and the standard deviation.
   (Solution Mean $\mu = 50 \quad S.D.\sigma = 10$)
3) In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution.
   (Solution Mean $\mu = 50.288 \quad S.D.\sigma = 10.36$)

WEIBULL DISTRIBUTION

A random variable 'X' is said to follow Weibull distribution with two parameters β>0, α>0 if the probability density function is given by

$$f(x) =\propto \beta x^{\beta-1} e^{-\alpha x^{\beta}}, x > 0$$

Mean and Variance of the Weibull Distribution

$$E(X) = Mean =\propto^{-\frac{1}{\beta}}.\gamma\left(\frac{1}{\beta}+1\right)$$

$$Var(X) =\propto^{-\frac{2}{\beta}}\left[\gamma\left(\frac{2}{\beta}+1\right)-\left[\gamma\left(\frac{1}{\beta}+1\right)\right]^2\right]$$

Example 1: The life time of a component measured in hours follows Weibull distribution with parameter $\propto = o.2, \beta = 0.5$. Find the mean life time of the component.

For a weibull distribution, the mean is given by

$$E(X) = Mean = \propto^{-\frac{1}{\beta}}.\gamma\left(\frac{1}{\beta}+1\right)$$

Given, $\propto = 0.2, \beta = 0.5$

Hence, $E(X) = (0.2)^{\frac{-1}{0.5}}\gamma\left(1+\frac{1}{0.5}\right)$=50 hours

Example 2: If the life X (in years) of a certain type of car has a weibull distribution with the parameter $\beta = 2$, find the value of the parameter α, given that probability that the life of the car exceeds 5 years is $e^{-0.25}$. Find these values of $\propto$ *and* β, find the mean and variance of X.

The pdf of a weibull distribution is given by

$$f(x) = \propto \beta x^{\beta-1}e^{-\alpha x^{\beta}}, x > 0$$

Given β=2

Hence, $f(x) = 2 \propto xe^{-\propto x^2}$

Now, $P(X > 5) = \int_5^{\infty} 2 \propto xe^{-\propto x^2}dx$

$$=\int_{25}^{\infty} \propto e^{-\propto t}dt,$$

Take t=x^2

$$dt = 2x\,dx$$

$$= [-e^{-\propto t}]_{25}^{\infty} \quad .....(1)$$

Given that $P(X > 5) = e^{-0.25}$

$$e^{-25\propto} = e^{-0.25} \qquad \ldots\ldots [\text{by } (1)]$$

$$25 \propto = 0.25$$

$$\propto = \frac{1}{100}$$

$$E(X) = Mean = \propto^{-\frac{1}{\beta}} . \gamma\left(\frac{1}{\beta} + 1\right)$$

$$= \left(\frac{1}{100}\right)^{\frac{-1}{2}} \gamma \frac{3}{2}$$

$$= 10.\frac{1}{2}\gamma\left(\frac{1}{2}\right) \qquad [note: \gamma(n+1) = n\gamma n]$$

$$=5\sqrt{\pi}$$

$$Var(X) = \propto^{-\frac{2}{\beta}}\left[\gamma\left(\frac{2}{\beta} + 1\right) - \left[\gamma\left(\frac{1}{\beta} + 1\right)\right]^2\right]$$

$$= \left(\frac{1}{100}\right)^{-1}\left[\gamma 2 - \left(\gamma \frac{3}{2}\right)^2\right]$$

$$= 100\left[1 - \frac{\pi}{4}\right] \qquad \left[note: \left(\gamma \frac{3}{2}\right)^2 = \left(\frac{1}{2}\sqrt{\pi}\right)^2\right]$$

WEIBULL DISTRIBUTION :

The Weibull distribution based on the two parameters p and σ is given by,

$$f\left(\frac{x}{\sigma}, p\right) = \frac{p}{\sigma} x^{p-1} \exp\left(\frac{-x^p}{\sigma}\right) \qquad x > 0,\ p > 0,\ \sigma > 0$$

p and σ are respectively called as the shape and the scale parameters.

Problem : Obtain the exponential distribution from the Weibull distribution with

an appropriate transformation of Weibull random variable.

Solution : let $x^p = u$

$p\, x^{p-1}\, dx = du$

$$g\left(\frac{u}{\sigma}\right) = \frac{1}{\sigma}\, e^{\frac{-u}{\sigma}} \quad , \quad 0 < u < \infty$$

Moments : The rth moment about origin is,

$$\mu_r' = E(x^r) = \frac{p}{\sigma}\int_0^\infty x^r\, x^{p-1}\, e^{\left(\frac{-x^p}{\sigma}\right)}\, dx$$

$$x^p = \sigma v \qquad \Rightarrow \quad x = (\sigma v)^{1/p} \qquad \Rightarrow \quad p x^{p-1} dx = \sigma\, dv$$

$$\mu_r' = \frac{1}{\sigma}\int_0^\infty (\sigma v)^{r/p} e^{-v}\, \sigma\, dv$$

$$= (\sigma)^{r/p}\, \frac{1}{\sigma}\int_0^\infty (v)^{\left(\frac{r}{p}+1\right)-1} e^{-v}\, dv = (\sigma)^{r/p}\, \Gamma\left(\frac{r}{p}+1\right)$$

$$\mu_r' = (\sigma)^{r/p}\, \Gamma\left(\frac{r+p}{p}\right)$$

Mean : $\mu_1' = (\sigma)^{1/p}\, \Gamma\left(\frac{1+p}{p}\right)$

$$\mu_2' = (\sigma)^{2/p}\,\Gamma\left(\frac{2+p}{p}\right)$$

Variance : $\mu_2 = \mu_2' - (\mu_1')^2$

$$= \sigma^{\frac{2}{p}}\left[\Gamma\left(\frac{p+2}{p}\right) - \Gamma^2\left(\frac{p+1}{p}\right)\right]$$

Weibull distribution has applicability in reliability and life testing in which it is viewed as a failure time distribution .Exponential distribution arises from Weibull distribution [vide problem(2.6.1)]

For Weibull distribution the reliability function is given by,

R(t) = $e^{t^p/\sigma}$

The hazard rate can be obtained as,

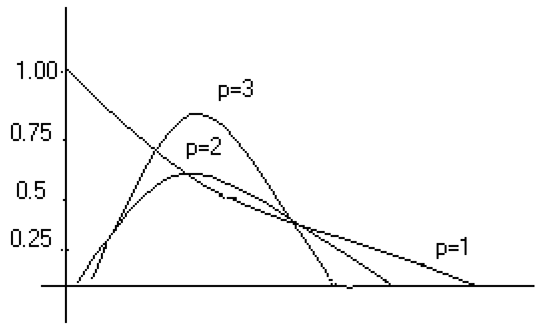
R(t) = $\frac{p}{\sigma} t^{p-1}$

p > 1 $\Rightarrow \mu(t)$ increases

p < 1 $\Rightarrow \mu(t)$ decreases

p = 1 $\Rightarrow \mu(t)$ remains to be constant (exponential failure rate)

p is viewed as the shape parameter since as p varies the probability density function's shape varies.


1.00
0.75
0.5
0.25
p=3
p=2
p=1