Statistics and Reliability
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Chapter 9

# Reliability Engineering

In this chapter we shall consider a very important and growing area of application of some of the probability concepts discussed in the previous chapters, namely Reliability Engineering. Reliability considerations are playing an increasing role in almost all engineering disciplines. Generally the term 'reliability' of a component (or a system, that is, a set of components assembled to perform a certain function) is understood as its capability to function without breakdown. The better the component performs its intended function, the more reliable it is. In the broader sense, reliability is associated with dependability, with successful performance and with the absence of breakdown or failures. However, from the engineering analysis point of view, it is necessary to define reliability quantitatively as a probability. Technically, reliability may be defined as the probability that a component (or a system) will perform properly for a specified period of time '$t$' under a given set of operating conditions. In other words, it is the probability that the component does not fail during the interval $(0, t)$. We give below the formal mathematical definition of reliability and discuss the related concepts.

## CONCEPTS OF RELIABILITY

The definitions given below with respect to a component hold good for a system or a device also.

If a component is put into operation at some specified time, say $t = 0$, and if $T$ is the time until it fails or ceases to function properly, $T$ is called the *life length* or *time to failure* of the component. Obviously $T (\leq 0)$ is a continuous random variable with some probability density function $f(t)$. Then the *reliability* or *reliability function* of the component at time '$t$', denoted by $R(t)$, is defined as

$$R(t) = P(T > t) \text{ or } 1 - P(T \leq t) \tag{1}$$

$$= 1 - F(t),$$

where $F(t)$ is the cumulative distribution function of $T$, given by

$$F(t) = \int_0^t f(t)\,dt$$

Thus $$R(t) = 1 - \int_0^t f(t)\,dt = \int_t^\infty f(t)\,dt \qquad (2)$$

Since $F(0) = 0$ and $F(\infty) = 1$ by the property of *cdf*, $R(0) = 1$ and $R(\infty) = 0$ *i.e.*, $0 \le R(t) \le 1$. Also since $\frac{d}{dt}F(t) = f(t)$,

we get $$f(t) = -\frac{dR(t)}{dt} \qquad (3)$$

Now the conditional probability of failure in the interval $(t, t+\Delta t)$, given that the component has survived upto time $t$, viz.,

$$P\{t \le T \le t + \Delta t / T \ge t\} = \frac{f(t)\Delta t}{1 - F(t)}, \text{ by the definitions of } pdf \text{ and } cdf$$

$$= f(t)\,\Delta t / R(t)$$

$\therefore$ The conditional probability of failure per unit time is given by $\frac{f(t)}{R(t)}$ and is called the *instantaneous failure rate* or *hazard function* of the component, denoted by $\lambda(t)$.

Thus $$\lambda(t) = \frac{f(t)}{R(t)} \qquad (4)$$

Now, using (3) in (4), we have

$$-\frac{R'(t)}{R(t)} = \lambda(t) \qquad (5)$$

Integrating both sides of (5) with respect to $t$ between 0 and $t$, we have

$$\int_0^t \frac{R'(t)}{R(t)}\,dt = -\int_0^t \lambda(t)\,dt$$

i.e., $$\{\log R(t)\}_0^t = -\int_0^t \lambda(t)\,dt$$

i.e., $$\log R(t) - \log R(0) = -\int_0^t \lambda(t)\,dt$$



i.e., $$\log_e [R(t)] = -\int_0^t \lambda(t)\,dt \quad [\because R(0) = 1]$$

$\therefore$ $$R(t) = e^{-\int_0^t \lambda(t)\,dt} \qquad (6)$$

Using (6) in (4), we have

$$f(t) = \lambda(t)\, e^{-\int_0^t \lambda(t)\,dt} \qquad (7)$$

The expected value of the time to failure $T$, denoted by $E(T)$ and variance denoted by $\sigma_T^2$ are two important parameters frequently used to characterise reliability. $E(T)$ is called *mean time to failure* and denoted by MTTF.

Now MTTF $= E(T) = \int_0^\infty t f(t)\,dt$ $= \int_0^\infty R(t)\,dt$

$$= -\int_0^\infty t R'(t)\,dt, \text{ by (3)}$$

$$= -[tR(t)]_0^\infty + \int_0^\infty R(t)\,dt \qquad (8)$$

on integration by parts

Now $$[tR(t)]_{t=\infty} = \left[t e^{-\int_0^t \lambda(t)dt}\right]_{t=\infty} = \left[\frac{t}{e^{\int_0^t \lambda(t)dt}}\right]_{t=\infty} = 0 \qquad (9)$$

and $[tR(t)]_{t=0} = 0 \times R(0) = 0 \times 1 = 0 \qquad (10)$

Using (9) and (10) in (8), we get

$$\text{MTTF} = \int_0^\infty R(t)\,dt \qquad (11)$$

$$\text{Var}(T) = \sigma_T^2 = E\{T - E(T)\}^2 \text{ or } E(T^2) - \{E(T)\}^2$$

$$= \int_0^\infty t^2 f(t)dt - (MTTF)^2 \qquad (12)$$

*Conditional reliability* is another concept useful to describe the reliability of a component or system following a *wear-in period (burn-in period)* or after a *warranty period*.

It is defined as

$$R(t/T_0) = P\{T > T_0 + t / T > T_0\}$$

$$= \frac{P\{T > T_0 + t\}}{P\{T > T_0\}} = \frac{R(T_0 + t)}{R(T_0)} \tag{13}$$

$$= \frac{e^{-\int_0^{T_0+t} \lambda(t)dt}}{e^{-\int_0^{T_0} \lambda(t)dt}} = e^{-\left[\int_0^{T_0+t} \lambda(t)dt - \int_0^{T_0} \lambda(t)dt\right]}$$

$$= e^{-\int_{T_0}^{T_0+t} \lambda(t)dt} \tag{14}$$

## Some Special Failure Distributions

Some of the probability distributions describe the failure process and reliability of a component or a system more satisfactorily than others. They are the exponential, Weibull, normal and lognormal distributions. We shall now derive the reliability characteristics relating to these failure distributions using the formulas derived above.

### (1) The exponential distribution

If the time to failure $T$ follows an exponential distribution with parameter $\lambda$, then its *pdf* is given by

$$f(t) = \lambda e^{-\lambda t}, \; t \geq 0 \tag{15}$$

Then, from (2),

$$R(t) = \int_t^{\infty} \lambda e^{-\lambda t} \, dt = [-e^{-\lambda t}]_t^{\infty} = e^{-\lambda t} \tag{16}$$

From (4)
$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \tag{17}$$

This means that when the failure distribution is an exponential distribution with parameter $\lambda$, the failure rate at any time is a constant, equal to $\lambda$. Conversely when $\lambda(t)$ = a constant $\lambda$, we get from (7),

$$f(t) = \lambda \cdot e^{-\int_0^t \lambda dt} = \lambda e^{-\lambda t}, \; t \geq 0$$

Due to this property, the exponential distribution is often referred to as *constant failure rate distribution* in reliability contexts. We have already derived in the earlier chapters, that



$$\text{MTTF} = E(T) = \frac{1}{\lambda} \tag{18}$$

and
$$\text{Var}(T) = \sigma_T^2 = \frac{1}{\lambda^2} \tag{19}$$

Also
$$R(t/T_0) = \frac{R(T_0 + t)}{R(T_0)} = \frac{e^{-\lambda(T_0+t)}}{e^{-\lambda T_0}} \text{ by (16)}$$

$$= e^{-\lambda t} \tag{20}$$

This means that the time to failure of a component is not dependent on how long the component has been functioning. In other words the reliability of the component for the next 1000 hours, say, is the same regardless of whether the component is brand new or has been operating for several hours. This property is known as the *memoryless property* of the constant failure rate distribution.

### (2) The Weibull distribution

The *pdf* of the Weibull distribution was defined as

$$f(t) = \alpha\beta t^{\beta-1} e^{-\alpha t^{\beta}}, \; t \geq 0 \tag{21}$$

An alternative form of Weibull's *pdf* is

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \theta > 0, \beta > 0, t \geq 0 \tag{22}$$

(22) is obtained from (21) by putting $\alpha = \frac{1}{\theta^{\beta}}$. $\beta$ is called the *shape parameter* and $\theta$ is called the *characteristic life* or *scale parameter* of the Weibull's distribution (22). If $T$ follows Weibull's distribution (22),

then
$$R(t) = \int_t^{\infty} \frac{\beta}{\theta} \cdot \left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right] dt$$

$$= \int_x^{\infty} e^{-x} \, dx, \text{ on putting } \left(\frac{t}{\theta}\right)^{\beta} = x$$

$$= e^{-x} = \exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right] \tag{23}$$

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\beta}{\theta} \cdot \left(\frac{t}{\theta}\right)^{\beta-1} \tag{24}$$

As derived in the chapter 'Some special probability distributions', we have

$$\text{MTTF} = E(T) = \theta\,\Gamma\left(1 + \frac{1}{\beta}\right) \tag{25}$$

and $$\text{Var}(T) = \sigma^2_T = \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\} \quad (26)$$

Now $$R(t/T_0) = \frac{R(t + T_0)}{R(T_0)}$$

$$= \frac{\exp\left[-\left(\frac{t + T_0}{\theta}\right)^\beta\right]}{\exp\left[-\left(\frac{T_0}{\theta}\right)^\beta\right]}$$

$$= \exp\left[-\left(\frac{t + T_0}{\theta}\right)^\beta + \left(\frac{T_0}{\theta}\right)^\beta\right] \quad (27)$$

### (3) The normal distribution

If the time to failure $T$ follows a normal distribution $N(\mu, \sigma)$ its *pdf* is given by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(t-\mu)^2}{2\sigma^2}\right], -\infty < t < \infty$$

In this case, $MTTF = E(T) = \mu$ and

$\text{Var}(T) = \sigma^2_T = \sigma^2$.

$R(t) = \int_t^\infty f(t)\,dt$ is found out by expressing the integral in terms of standard normal integral and using the normal tables.

$\lambda(t)$ is then found out by using (4).

### (4) The lognormal distribution

If $X = \log T$ follows a normal distribution $N(\mu, \sigma)$, then $T$ follows a lognormal distribution whose *pdf* is given by

$$f(t) = \frac{1}{st\sqrt{2\pi}} \exp\left[-\frac{1}{2s^2}\left\{\log\left(\frac{t}{t_M}\right)\right\}^2\right], t \geq 0 \quad (28)$$

where $s = \sigma$ is a *shape parameter* and $t_M$, the *median time to failure* is the location parameter, given by $\log t_M = \mu$.

It can be proved that

$$\text{MTTF} = E(T) = t_M \exp\left(\frac{s^2}{2}\right) \quad (29)$$

and $$\text{Var}(T) = \sigma^2_T = t^2_M \exp(s^2)\,[\exp(s^2) - 1] \quad (30)$$



In this case,

$$F(t) = P(T \leq t) = P\{\log T \leq \log t\}$$

$$= P\left\{\frac{\log T - \mu}{\sigma} \leq \frac{\log t - \mu}{\sigma}\right\}, \text{ since } \log T \text{ follows a } N(\mu, \sigma$$

$$= P\left\{\frac{\log T - \log t_M}{s} \leq \frac{1}{s}\log\left(\frac{t}{t_M}\right)\right\}$$

$$= P\left\{Z \leq \frac{1}{s}\log\left(\frac{t}{t_M}\right)\right\}, \text{ where } Z \text{ follows the standard}$$

normal distribution $N(0, 1)$.

Then $R(t)$ and $\lambda(t)$ are computed using (1) and (4).

**Worked Example 9(A)**

### Example 1

The density function of the time to failure in years of the gizmos (for use widgets) manufactured by a certain company is given by $f(t) = \frac{200}{(t+10)^3}, t \geq$

(a) Derive the reliability function and determine the reliability for the fi year of operation.
(b) Compute the MTTF.
(c) What is the design life for a reliability 0.95?
(d) Will a one-year burn-in period improve the reliability in part (*a*)? If what is the new reliability?

(a) $$f(t) = \frac{200}{(t+10)^3}, t \geq 0$$

$$R(t) = \int_t^\infty f(t)\,dt = \left[\frac{-100}{(t+10)^2}\right]_t^\infty = \frac{100}{(t+10)^2}$$

$\therefore$ $$R(1) = \frac{100}{(1+10)^2} = 0.8264.$$

(b) $$\text{MTTF} = \int_0^\infty R(t)\,dt = \int_0^\infty \frac{100}{(t+10)^2}\,dt$$

$$= \left(\frac{-100}{t+10}\right)_0^\infty = 10 \text{ years.}$$

(c) *Design life* is the time to failure ($t_D$) that corresponds to a specified reliability. Now it is required to find $t_D$ corresponding to $R = 0.95$

$$\therefore \quad \frac{100}{(t_D+10)^2} = 0.95$$

*i.e.*, $(t_D + 10)^2 = 100.2632$

$\therefore \quad t_D = 0.2598$ year or 95 days

(d) $$R(t/1) = \frac{R(t+1)}{R(1)} = \frac{100}{(t+11)^2} \div \frac{100}{11^2} = \frac{121}{(t+11)^2}$$

Now $R(t/1) > R(t)$, if $\dfrac{121}{(t+11)^2} > \dfrac{100}{(t+10)^2}$

*i.e.*, if $\dfrac{(t+10)^2}{(t+11)^2} > \dfrac{100}{121}$

*i.e.*, if $\dfrac{t+10}{t+11} > \dfrac{10}{11}$

*i.e.*, $11t > 10t$, which is true, as $t \geq 0$

$\therefore$ One year burn-in period will improve the reliability.

Now $R(1/1) = \dfrac{121}{(1+11)^2} = 0.8403 > 0.8264.$

## Example 2

The time to failure in operating hours of a critical solid-state power unit has the hazard rate function $\lambda(t) = 0.003\left(\dfrac{t}{500}\right)^{0.5}$, for $t \geq 0$.

(a) What is the reliability if the power unit must operate continuously for 50 hours?

(b) Determine the design life if a reliability of 0.90 is desired.

(c) Compute the MTTF.

(d) Given that the unit has operated for 50 hours, what is the probability that it will survive a second 50 hours of operation?



(a) $$R(t) = \exp\left[-\int_0^t \lambda(t)\,dt\right]$$

$$\therefore \quad R(50) = \exp\left[-\int_0^{50} 0.003\left(\frac{t}{500}\right)^{0.5} dt\right]$$

$$= \exp\left[-\frac{0.003}{\sqrt{500}}\cdot\frac{2}{3}t^{3/2}\right]_0^{50}$$

$$= \exp\left[-\frac{0.003}{\sqrt{500}}\times\frac{2}{3}\times 50\sqrt{50}\right]$$

$$= \exp[-0.03162]$$

$$= 0.9689.$$

(b) $R(t_D) = 0.90$

i.e., $$\exp\left[-\int_0^{t_D} 0.003\left(\frac{t}{500}\right)^{0.5} dt\right] = 0.90$$

i.e., $$-\int_0^{t_D} \frac{0.003}{\sqrt{500}}\, t^{1/2}\,dt = -0.10536$$

i.e., $$\frac{0.003}{\sqrt{500}}\times\frac{2}{3}\,t_D^{3/2} = 0.10536$$

$$t_D = \left\{\frac{3\times\sqrt{500}\times 0.10536}{2\times 0.003}\right\}^{2/3} = 111.54 \text{ hours.}$$

(c) $$\text{MTTF} = \int_0^\infty R(t)\,dt$$

$$= \int_0^\infty e^{-\left(\frac{0.003}{\sqrt{500}}\times\frac{2}{3}\times t^{3/2}\right)}\,dt$$

$$= \int_0^\infty e^{-at^{3/2}}\,dt, \text{ where } a = \frac{0.003\times 2}{3\times\sqrt{500}}$$

$$= \int_0^\infty e^{-x}\cdot\frac{2}{3a^{2/3}}\,x^{-1/3}\,dx, \text{ on putting } x = at^{3/2}$$

$$= \frac{2}{3a^{2/3}}\Gamma(2/3) = \frac{2}{3a^{2/3}}\frac{3}{2}\Gamma(5/3)$$

$$= \frac{0.9033}{a^{2/3}}, \text{ from the table of values of Gamma function.}$$

$= 451.65$ hours.

(d) $$P(T \geq 100/T \geq 50) = \frac{P(T \geq 100)}{P(T \geq 50)} = \frac{R(100)}{R(50)}$$

$$= \exp\left[-\int_{50}^{100} \lambda(t)\,dt\right]$$

$$= \exp\left[\left\{-\frac{0.002}{\sqrt{500}} \times 100^{3/2}\right\} - \left\{\frac{-0.002}{\sqrt{500}} \times 50^{3/2}\right\}\right]$$

$$= \exp\left[\{-0.08944\} - \{-0.03162\}\right]$$

$$= 0.9438$$

## Example 3

The reliability of a turbine blade is given by $R(t) = \left(1 - \frac{t}{t_0}\right)^2$, $0 \leq t \leq t_0$, where $t_0$ is the maximum life of the blade.

(a) Show that the blades are experiencing wear out.
(b) Compute MTTF as a function of the maximum life.
(c) If the maximum life is 2000 operating hours, what is the design life for a reliability of 0.90?

(a) $$R(t) = \left(1 - \frac{t}{t_0}\right)^2, \; 0 \leq t \leq t_0$$

Now $$\lambda(t) = -\frac{R'(t)}{R(t)}$$

$$= -\left(1 - \frac{t}{t_0}\right)^{-2}\left\{-\frac{2}{t_0}\left(1 - \frac{t}{t_0}\right)\right\}$$

$$= \frac{2}{t_0 - t}$$



$$\lambda'(t) = \frac{2}{(t_0 - t)^2} > 0 \text{ and so } \lambda(t) \text{ is an increasing function of } t.$$

When the failure rate increases with time, it indicates that the blades are experiencing wear out.

(b) $$\text{MTTF} = \int_0^{\infty} R(t)\,dt = \int_0^{t_0}\left(1 - \frac{t}{t_0}\right)^2 dt$$

$$= \left[-\frac{t_0}{3}\left(1 - \frac{t}{t_0}\right)^3\right]_0^{t_0} = \frac{t_0}{3}$$

(c) When $t_0 = 2000$, $R(t_D) = 0.90$

i.e., $$\left(1 - \frac{t_D}{2000}\right)^2 = 0.90$$

$$\therefore \quad 1 - \frac{t_D}{2000} = 0.9487$$

$$\therefore \quad t_D = 102.63 \text{ hours.}$$

## Example 4

Given that $R(t) = e^{-\sqrt{0.001t}}$, $t \geq 0$

(a) Compute the reliability for a 50 hours mission.
(b) Show that the hazard rate is decreasing.
(c) Given a 10-hour wear-in period, compute the reliability for a 50-hour mission.
(d) What is the design life for a reliability of 0.95, given a 10-hour wear-in period?

(a) $$R(t) = e^{-\sqrt{0.001t}}, \; t \geq 0$$

$$\therefore \quad R(50) = e^{-\sqrt{0.001 \times 50}} = 0.9512$$

(b) $$\lambda(t) = \frac{-R'(t)}{R(t)} = -e^{\sqrt{0.001t}} \times e^{-\sqrt{0.001t}} \times -\sqrt{0.001} \times \frac{1}{2\sqrt{t}}$$

$$= \frac{\sqrt{0.001}}{2\sqrt{t}}, \text{ which is a decreasing function of } t.$$

(c) $$R(t/T_0) = \frac{R(t + T_0)}{R(T_0)}$$

$$\therefore \quad R(50/10) = \frac{R(60)}{R(10)} = \frac{e^{-\sqrt{0.001 \times 60}}}{e^{-\sqrt{0.001 \times 10}}} = 0.8651$$

(d) $R(t_D/10) = 0.95$

i.e., $\dfrac{R(t_D+10)}{R(10)} = 0.95$

i.e., $e^{-\sqrt{0.001\times(t_D+10)}} = 0.95 \times e^{-\sqrt{0.001\times 10}}$

$\therefore \quad \sqrt{0.001\times(t_D+10)} = 0.15129$

$\therefore \quad t_D = 12.89$ hours.

## Example 5

A one-year guarantee is given based on the assumption that no more than 10% of the items will be returned. Assuming an exponential distribution, what is the maximum failure rate that can be tolerated?

If $T$ is the time to failure of the item,

then $P(T \geq 1) \geq 0.9$ ($\because$ no more than 10% will be returned)

i.e., $R(1) \geq 0.9$

i.e., $\int_1^{\infty} \lambda e^{-\lambda t}\, dt \geq 0.9$ ($\because T$ follows an exponential distribution)

i.e., $(-e^{-\lambda t})_1^{\infty} \geq 0.9$

i.e., $e^{-\lambda} \geq 0.9$

$\therefore -\lambda \geq -0.1054$

$\therefore \lambda \leq 0.1054$/year

## Example 6

A manufacturer determines that, on the average, a television set is used 1.8 hours per day. A one-year warranty is offered on the picture tube having a MTTF of 2000 hours. If the distribution is exponential, what percentage of the tubes will fail during the warranty period?

Since the distribution of the time to failure of the picture tube is exponential,

$R(t) = e^{-\lambda t}$, where $\lambda$ is the failure rate

Given that MTTF = 2000 hours

i.e., $\int_0^{\infty} e^{-\lambda t}\, dt = 2000$

i.e., $\dfrac{1}{\lambda} = 2000$ or $\lambda = 0.0005$/hour



$P(T \leq 1 \text{ year}) = P(T \leq 365 \times 1.8 \text{ hours})$ [$\because$ the T.V. is operated for 1.8 hours/day]

$= 1 - P\{T > 657\}$

$= 1 - R(657)$

$= 1 - e^{-0.0005 \times 657}$

$= 0.28$

i.e., 28% of the tubes will fail during the warranty period.

## Example 7

Night watchmen carry an industrial flashlight 8 hours per night, 7 nights per week. It is estimated that on the average the flashlight is turned on about 20 minutes per 8-hour shift. The flashlight is assumed to have a constant failure rate of 0.08/hour while it is turned on and of 0.005/hour when it is turned off but being carried.

(a) Estimate of MTTF of the light in working hours.

(b) What is the probability of the light's failing during one 8-hour shift?

(c) What is the probability of its failing during one month (30 days) of 8-hour shifts?

**Note** *In the earlier problems, we have not taken into account the possibility that an item may fail while it is not operating. Often such failure rates are small enough to be neglected. However, for items that are operated only a small fraction of time, failure during non-operation may be quite significant and so should be taken into account. In this situation, the composite failure rate $\lambda_c$ is computed as*

*$\lambda_c = f\lambda_0 + (1-f)\lambda_N$, where $f$ is the fraction of time the unit is operating and $\lambda_0$ and $\lambda_N$ are the failure rates during operating and non-operating times*]

Since the flashlight is used only a small fraction of time (20 minutes out of 8 hours), we compute the composite failure rate $\lambda_c$ and use it for other calculations.

$$\lambda_c = f\lambda_0 + (1-f)\lambda_N$$

$$= \frac{1}{24} \times 0.08 + \frac{23}{24} \times 0.005 = 0.008125/\text{hour}$$

(a) $\text{MTTF} = \dfrac{1}{\lambda_c} = \dfrac{1}{0.008125} = 123$ hours

(b) $P(T \leq 8) = \int_0^8 \lambda_c e^{-\lambda_c t}\, dt = 1 - e^{-8\lambda_c} = 0.0629$

(c) $p$ = probability of failure in one day (night) = 0.0629

$q = 1 - p = 0.9371$

$n = 30$

$\therefore$ Probability of failure in 30 days

$= 1 -$ Probability of no failure in 30 days

$= 1 - nC_0\, p^0\, q^n$, by binomial law

$= 1 - (0.9371)^{30}$

$= 0.8576$

## Example 8

A relay circuit has an MTBF of 0.8 year. Assuming random failures,

(a) Calculate the probability that the circuit will survive 1 year without failure.

(b) What is the probability that there will be more than 2 failures in the first year?

(c) What is the expected number of failures per year?

Since the failures are random events, the number of failures in an interval of length $t$ follows a Poisson process, given by $P\{N(t) = n\} = \dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0, 1, 2, \cdots, \infty$, where $\lambda$ = failure rate.

Then the time between failures follows an exponential distribution with mean $\dfrac{1}{\lambda}$.

Now MTBF = Mean time between failures $= \dfrac{1}{\lambda}$

$= 0.8$ year

$\therefore \quad \lambda = \dfrac{1}{0.8}$ per year = 1.25 per year

(a) $P\{N(1) = 0\} = \dfrac{e^{-\lambda}\lambda^0}{0!} = e^{-1.25} = 0.2865$

(b) $P\{N(1) > 2\} = 1 - e^{-\lambda}\left\{\dfrac{\lambda^0}{0!} + \dfrac{\lambda^1}{1!} + \dfrac{\lambda^2}{2!}\right\}$

$= 1 - 0.2865\left\{1 + 1.25 + \dfrac{(1.25)^2}{2}\right\}$

$= 0.1315$



(c) $E\{N(t)\} = \lambda t$

$\therefore E$ {Number of failures per year} $= \lambda = 1.25$

## Example 9

For a system having a Weibull failure distribution with a shape parameter of 1.4 and a scale parameter of 550 days, find (a) $R$ (100 days); (b) the $B1$ life; (c) MTTF; (d) the standard deviation, (e) the design life for a reliability of 0.90.

The *pdf* of the Weibull distribution is given by $f(t) = \dfrac{\beta}{\theta}\left(\dfrac{t}{\theta}\right)^{\beta-1} \exp\left\{-\left(\dfrac{t}{\theta}\right)^{\beta}\right\}$, $t \geq 0$. Now $\beta = 1.4$ and $\theta = 550$ days

(a) $\quad R(t) = \exp\left\{-\left(\dfrac{t}{\theta}\right)^{\beta}\right\}$

$\therefore \ R(100) = \exp\left\{-\left(\dfrac{100}{550}\right)^{1.4}\right\} = 0.9122$

(b)

**Note** *The life time corresponding to a reliability of 0.99 is called the B1 life. Similarly that corresponding to $R = 0.999$ is called the B.1 life.*

Let $t_R$ be the $B1$ life of the system

Then $R(t_R) = 0.99$

i.e., $\exp\left\{-\left(\dfrac{t_R}{550}\right)^{1.4}\right\} = 0.99$

$\therefore \quad \left(\dfrac{t_R}{550}\right)^{1.4} = 0.01005$

$\therefore t_R = 550 \times (0.01005)^{\frac{1}{1.4}} = 20.6$ days

(c) MTTF $= \theta \cdot \Gamma\left(1 + \dfrac{1}{\beta}\right) = 550 \times \Gamma\left(1 + \dfrac{1}{1.4}\right) = 550 \times \Gamma(1.71)$

$= 550 \times 0.91057$, from the Gamma tables

$= 500.8$ days

(d) $(\text{S.D.})^2 = \theta^2 \left\{\left|\left(1+\frac{2}{\beta}\right) - \left[\left|\left(1+\frac{1}{\beta}\right)\right]^2\right.\right\}$

$$= 550^2 \left\{\left|\left(1+\frac{2}{1.4}\right) - \left[\left|\left(1+\frac{1}{1.4}\right)\right]^2\right.\right\}$$

$$= 550^2 \left\{|(2.43) - [|(1.71)]^2\right\}$$

$$= 550^2\, [1.43 \times |(1.43) - \{|(1.71)\}^2]$$

$$= 550^2\, [1.43 \times 0.88604 \times (0.91057)^2]$$

$\therefore$ S.D. $= 550 \times 0.66174 = 363.96$ days

(e) Let $t_D$ be the required design life for $R = 0.90$

$$\therefore \quad R(t_D) = \exp\left\{-\left(\frac{t_D}{550}\right)^{1.4}\right\} = 0.90$$

$$\therefore \quad \left(\frac{t_D}{550}\right)^{1.4} = 0.10536$$

$$\therefore \quad t_D = 550 \times (0.10536)^{\frac{1}{1.4}} = 110.2 \text{ days}$$

## Example 10

A device has a decreasing failure rate characterised by a two-parameter Weibull distribution with $\theta = 180$ years and $\beta = 0.5$. The device is required to have a design life reliability of 0.90.

(a) What is the design life, if there is no wear-in period?
(b) What is the design life, if there is a wear-in period of 1 month in the beginning?

(a) Let $t_D$ be the required design life for $R = 0.90$.

$$\therefore \quad R(t_D) = \exp\left\{-\left(\frac{t_D}{180}\right)^{0.5}\right\} = 0.90$$

$$\therefore \quad \left(\frac{t_D}{180}\right)^{0.5} = 0.10536$$

$$\therefore \quad t_D = 180 \times (0.10536)^2 = 1.998 \simeq 2 \text{ years.}$$

(b) If $T_0$ is the wear-in period, then

$$R(t/T_0) = \exp\left\{-\left(\frac{t+T_0}{\theta}\right)^{\beta} + \left(\frac{T_0}{\theta}\right)^{\beta}\right\}$$

Let $t_W$ be the required design life with wear-in.

$$\text{Then } \exp\left\{-\left(\frac{t_W + \frac{1}{12}}{180}\right)^{0.5} + \left(\frac{\frac{1}{12}}{180}\right)^{0.5}\right\} = 0.90$$

$$\text{i.e., } -\left(\frac{12t_W + 1}{12 \times 180}\right)^{0.5} + \left(\frac{1}{12 \times 180}\right)^{0.5} = -0.10536$$

$$\text{i.e., } \left(\frac{12t_W + 1}{12 \times 180}\right)^{0.5} = 0.12688$$

$\therefore\ 12t_W + 1 = 12 \times 180 \times 0.01610 = 34.776$

i.e., $t_W = 2.18$ years

Thus a wear-in period of 1 month increases the design life by nearly 0.81 year or 10 months.

## Example 11

Two components have the same MTTF; the first has a constant failure rate $\lambda_0$ and the second follows a Weibull distribution with $\beta = 2$.

(a) Find $\theta$ in terms of $\lambda_0$.
(b) If for each component the design life reliability must be 0.9, how much longer (in percentage) is the design life of the second (Weibull) component?

(a) MTTF is the same for both the components.

For the constant failure rate distribution (viz., exponential distribution),

$$\text{MTTF} = \frac{1}{\lambda_0} \tag{1}$$

For the Weibull distribution with $\beta = 2$ (viz., Rayleigh distribution), $R(t) = e^{-t^2/\theta^2}$

$$\text{MTTF} = \int_0^{\infty} R(t)\, dt = \int_0^{\infty} e^{-t^2/\theta^2}\, dt$$

$$= \theta \int_0^{\infty} e^{-x^2}\, dx, \text{ on putting } \frac{t}{\theta} = x$$

$$= \theta \frac{\sqrt{\pi}}{2} \tag{2}$$

From (1) and (2), we get

$$\frac{\theta\sqrt{\pi}}{2} = \frac{1}{\lambda_0}$$

$$\therefore \quad \theta = \frac{2}{\lambda_0\sqrt{\pi}} \tag{3}$$

(b) Let $t_1$ and $t_2$ be the required design lives for the two components. Then $R(t_1) = e^{-\lambda_0 t_1} = 0.90$, for the first.

$$\therefore \quad \lambda_0 t_1 = 0.10536$$

$$\therefore \quad t_1 = \frac{0.10536}{\lambda_0} \tag{4}$$

$$R(t_2) = e^{-t_2^2/\theta^2} = 0.90, \text{ for the second.}$$

$$\therefore \quad t_2^2 = 0.10536\,\theta^2$$

or $\quad t_2 = 0.32459\,\theta$

$$= 0.32459 \times \frac{2}{\lambda_0\sqrt{\pi}}, \text{ using (3)}$$

$$= \frac{0.36626}{\lambda_0} \tag{5}$$

Increase in design life $= t_2 - t_1 = 0.26090/\lambda_0$

$$\therefore \% \text{ increase in design life} = \frac{(t_2 - t_1)}{t_1} \times 100$$

$$= \frac{0.26090}{0.10536} \times 100$$

$$= 247.6.$$

## Example 12

The wearout (time to failure) of a machine part is normally distributed with 90% of the failures occurring symmetrically between 200 and 270 hours of use.

(a) Find the MTTF and standard deviation of failure times.
(b) What is the reliability if the part is to be used for 210 hours and then replaced?
(c) Determine the design life if no more than a 1% probability of failure prior to the replacement is to be tolerated.
(d) Compute the reliability for a 10-hour use if the part has been operating for 200 hours.



Let the time to failure $T$ follow $N(\mu, \sigma)$, where $\mu$ is the MTTF and $\sigma$ is the S.D.

(a) 90% of the failures lie symmetrically between 200 and 270

$$\therefore \int_{200}^{235} f(t)\,dt = \int_{235}^{270} f(t)\,dt = 0.45, \text{ where } \mu = 235 \text{ hours}$$

Putting $z = \frac{t - 235}{\sigma}$, we get $\int_0^{35/\sigma} \phi(z)\,dz = 0.45$

From the normal tables, $\frac{35}{\sigma} = 1.645$

$$\therefore \quad \sigma = \frac{35}{1.645} = 21.28 \text{ hours}$$

(b) $R(t) = \int_t^{\infty} f(t)\,dt$

$$\therefore \quad R(210) = \int_{210}^{\infty} f(t)\,dt = \int_{-1.17}^{\infty} \phi(z)\,dz$$

$$= 0.5 + \int_0^{1.17} \phi(z)\,dz = 0.5 + 0.379 = 0.879$$

(c) If $t_D$ is the required design life, then

$$\int_{-\infty}^{t_D} f(t)\,dt = 0.01 \quad \text{or} \quad \int_{-\infty}^{(t_D-235)/21.28} \phi(z)\,dz = 0.01$$

From the normal tables, $\frac{t_D - 235}{21.28} = -2.32$

$$\therefore \quad t_D = 185.6 \text{ hours}$$

(d) $$\frac{R(t+T_0)}{R(T_0)} = \frac{R(210)}{R(200)} = \frac{\int_{210}^{\infty} f(t)\,dt}{\int_{200}^{\infty} f(t)\,dt}$$

$$= \frac{\int_{-1.17}^{\infty}\phi(z)\,dz}{\int_{-1.64}^{\infty}\phi(z)\,dz} = \frac{0.5+0.3790}{0.5+0.4495} = 0.93$$

## Example 13

A cutting tool wears out with a time to failure that is normally distributed. It is known that about 34.5% of the tools fail before 9 working days and about 78.8% fail before 12 working days.

(a) Compute the MTTF

(b) Determine its design life for a reliability of 0.99.

(c) Determine the probability that the cutting tool will last one more day given that it has been in use for 5 days.

(a) Let $T$ follow a $N(\mu, \sigma)$

Given $\int_{-\infty}^{9} f(t)\,dt = 0.345$

i.e., $\int_{-\infty}^{\frac{9-\mu}{\sigma}} \phi(z)\,dz = 0.345$, on putting $z = \frac{t-\mu}{\sigma}$

i.e., $\int_{0}^{\frac{\mu-9}{\sigma}} \phi(z)\,dz = 0.155$

$\therefore \quad \frac{\mu-9}{\sigma} = 0.4$ (1)

using the normal tables.

Also $\int_{-\infty}^{12} f(t)\,dt = 0.788$

i.e., $\int_{-\infty}^{\frac{12-\mu}{\sigma}} \phi(z)\,dz = 0.788$

i.e., $\int_{0}^{\frac{12-\mu}{\sigma}} \phi(z)\,dz = 0.288$



$\therefore \quad \frac{12-\mu}{\sigma} = 0.8$ (2)

using the normal tables

Solving equations (1) and (2), we get

$\mu = 10$ and $\sigma = 2.5$

i.e., M.T.T.F = 10 days.

(b) Let $t_R$ be the required design life for $R = 0.99$

$\therefore \int_{t_R}^{\infty} f(t)\,dt = 0.99$ or $\int_{\frac{t_R-10}{2.5}}^{\infty} \phi(z)\,dz = 0.99$

$\therefore \int_{0}^{\frac{10-t_R}{2.5}} \phi(z)\,dz = 0.49$

$\therefore \quad \frac{10-t_R}{2.5} = 2.32$, using the normal tables.

$\therefore \quad t_R = 4.2$ days

(c) $$P(T \ge 6/T > 5) = \frac{P(T\ge 6)}{P(T>5)} = \frac{\int_{6}^{\infty} f(t)\,dt}{\int_{5}^{\infty} f(t)\,dt}$$

$$= \frac{\int_{-1.6}^{\infty}\phi(z)\,dz}{\int_{-2}^{\infty}\phi(z)\,dz} = \frac{0.5+\int_{0}^{1.6}\phi(z)\,dz}{0.5+\int_{0}^{2}\phi(z)\,dz}$$

$$= \frac{0.94520}{0.97725} = 0.9672.$$

## Example 14

A complex machine has a high number of failures. The time to failure was found to be log-normal with $s = 1.25$. Specifications call for a reliability of 0.95 at 1000 cycles.

(a) Determine the median time to failure that must be achieved by engineering modifications to meet the specifications. Assume that design changes do not affect the shape parametre $s$.

(b) What are the corresponding MTTF and standard deviation?
(c) If the desired median time to failure is obtained, what is the reliability during the next 1000 cycles given that it has operated for 1000 cycles?

(a) $R(t) = \int_t^{\infty} f(t)\,dt$, where $f(t)$ is the *pdf* of the lognormal distribution.

$$= \int_{\left(\frac{\log t - \log t_M}{s}\right)}^{\infty} \phi(z)\,dz, \text{ where } \phi(z) \text{ is the } pdf \text{ of the standard normal distribution}$$

Now $R(1000) = 0.95$

$$\therefore \int_{\frac{1}{s}\log\left(\frac{1000}{t_M}\right)}^{\infty} \phi(z)\,dz = 0.95$$

From the normal tables, $\frac{-1}{1.25}\log\left(\frac{1000}{t_M}\right) = 1.645$

$$\therefore \quad \log\left(\frac{t_M}{1000}\right) = 1.645 \times 1.25$$

$$\therefore \quad t_M = \text{Median} = 1000 \times \log(1.645 \times 1.25)$$

$$= 7817 \text{ cycles.}$$

(b) $\text{MTTF} = t_M \times e^{s^2/2} = 7817 \times e^{0.78125}$

$$= 17074 \text{ cycles}$$

$$\sigma^2 = t_M^2\, e^{s^2}(e^{s^2} - 1)$$

$$= (7817)^2 \times e^{1.5625} \times (e^{1.5625} - 1)$$

$$\therefore \quad \sigma = 33155 \text{ cycles.}$$

(c) $R(t/T_0) = \frac{R(t + T_0)}{R(T_0)}$

$$= \frac{R(2000)}{R(1000)} = \frac{\int_{2000}^{\infty} f(t)\,dt}{\int_{1000}^{\infty} f(t)\,dt}$$



$$= \int_{\frac{1}{1.25}\log\left(\frac{2000}{7817}\right)}^{\infty} \phi(z)\,dz \div \int_{\frac{1}{1.25}\log\left(\frac{1000}{7817}\right)}^{\infty} \phi(z)\,dz$$

$$= \int_{-0.47}^{\infty} \phi(z)\,dz \div \int_{-0.71}^{\infty} \phi(z)\,dz$$

$$= \frac{0.6808}{0.7611} = 0.89$$

## Example 15

Fatigue wearout of a component has a log normal distribution with $t_M = 5000$ hours and $s = 0.20$.

(a) Compute the MTTF and SD.
(b) Find the reliability of the component for 3000 hours.
(c) Find the design life of the component for a reliability of 0.95.

(a) $\text{MTTF} = t_M \times e^{s^2/2} = 5000 \times e^{0.02}$

$$= 5101 \text{ hours}$$

$$\sigma^2 = t_M^2 \times e^{s^2}(e^{s^2} - 1)$$

$$= 5000^2 \times e^{0.04} \times (e^{0.04} - 1)$$

$$\therefore \quad \sigma = 1030 \text{ hours}$$

(b) $R(3000) = \int_{3000}^{\infty} f(t)\,dt$, where $f(t)$ is the *pdf* of the lognormal distribution

$$= \int_{\frac{1}{0.2}\log\left(\frac{3000}{5000}\right)}^{\infty} \phi(z)\,dz = \int_{-2.55}^{\infty} \phi(z)\,dz = 0.9946$$

(c) If $t_D$ is the required design life,
$R(t_D) = 0.95$

i.e., $\int_{\frac{1}{0.2}\log\left(\frac{t_D}{5000}\right)}^{\infty} \phi(z)dz = 0.95$

From the normal tables, $\frac{1}{0.2}\log\left(\frac{t_D}{5000}\right) = -1.645$

$\therefore \quad t_D = 3598.2$ hours

## Exercise 9(A)

**Part A** (Short answer questions)

1. What do you understand by reliability of a device?
2. Write down the mathematical definition of reliability.
3. Prove that $0 \le R(t) \le 1$, where $R(t)$ is the reliability function.
4. What is hazard function? How is it related to reliability function?
5. Derive the relation that expresses $R(t)$ in terms of $\lambda(t)$.
6. Derive the relation that expresses the *pdf* of the time to failure in terms of hazard function.
7. What is MTTF? How is it related to reliability function?
8. Express the conditional reliability $R(t/T_0)$ in terms of the hazard function.
9. Prove that the hazard function is a constant for the exponential failure distribution.
10. When the hazard function is a constant, prove that the failure time distribution is an exponential distribution.
11. Write down the *pdf* of the constant failure rate distribution.
12. When the hazard function is a constant $\lambda$, what are the MTTF and variance of the time to failure?
13. Prove that, for an exponential distribution, $R(t/T_0) = R(t)$.
14. What is memoryless property of the constant failure rate distribution?
15. Write down the *pdf* of Weibull distribution with shape parameter $m$ and scale parameter $n$.
16. Write down the formulas for $R(t)$ and $\lambda(t)$ for a Weibull distribution with shape parameter $\beta$ and characteristic life $\theta$.
17. When the time to failure $T$ follows a Weibull distribution with parameters $\beta$ and $\theta$, what are MTTF and Var($T$)?
18. Derive the formula for $R(t/T_0)$ when $T$ follows a Weibull distribution $(\beta, \theta)$.
19. Write down the density function of a lognormal distribution in terms of its shape parameter and median time to failure.
20. Write down the values of MTTF and $\sigma_T^2$ for a lognormal distribution $(s, t_M)$.

**Part B**

21. The density function of the time to failure of an appliance is

$$f(t) = \frac{32}{(t+4)^3};\ t\ (>0) \text{ is in years.}$$

   (a) Find the reliability function $R(t)$.
   (b) Find the failure rate $\lambda(t)$.
   (c) Find the MTTF.

22. A household appliance is advertised as having more than a 10-year life. If its *pdf* is given by

$$f(t) = 0.1(1 + 0.05t)^{-3};\ t \ge 0,$$



determine its reliability for the next 10 years, if it has survived a 1-year warranty period.
What is its MTTF before the warranty period?
What is its MTTF after the warranty period assuming that it has still survived?

23. A logic circuit is known to have a decreasing failure rate of the form

$$\lambda(t) = \frac{1}{20\sqrt{t}} \Big/ \text{year, where } t \text{ is in years.}$$

   (a) If the design life is one year, what is the reliability?
   (b) If the component undergoes wearin for one month before being put into operation, what will the reliability be for a one-year design life?

24. A component has the following hazard rate, where $t$ is in years:

$$\lambda(t) = 0.4\,t,\ t \ge 0$$

   (a) Find $R(t)$.
   (b) Determine the probability of the component failing within the first month of its operation.
   (c) What is the design life is a reliability of 0.95 is desired?

25. The *pdf* of the time to failure of a system is given by $f(t) = 0.01$, $0 \le t \le 100$ days. Find (a) $R(t)$; (*b*) the hazard rate function; (c) the MTTF; (d) the standard deviation.
26. The failure distribution is defined by

$$f(t) = \frac{3t^2}{10^9},\ 0 \le t \le 1000 \text{ hours.}$$

   (a) What is the probability of failure within a 100-hour warranty period?
   (b) Compute the MTTF.
   (c) Find the design life for a reliability of 0.99.
   (d) Compute the average failure rate over the first 500 hours.

   [**Hint:** Average failure rate of a component over the interval $(t_1, t_2)$ is defined as

$$\text{AFR}(t_1, t_2) = \frac{1}{t_2 - t_1} \times \int_{t_1}^{t_2} \lambda(t)\,dt = \frac{1}{t_2 - t_1} \log\left\{\frac{R(t_1)}{R(t_2)}\right\}$$

27. The *pdf* of the time to failure of a new fuel injection system which experiences high failure rate is given by
   $f(t) = 1.5/(t+1)^{2.5}$ $t \ge 0$, where $t$ is measured in years. The reliability over its intended life of 2 years is unacceptable. Will a wear-in period of 6 months significantly improve upon this reliability? If so, by how much?
28. A home computer manufacturer determines that his machine has a constant failure rate of $\lambda = 0.4$ per year in normal use. For how long should the warranty be set, if no more than 5% of the computers are to be returned to the manufacturer for repair?
   [**Hint:** Find to such that $R(T_0) \ge 0.95$]

29. A more general exponential reliability model is defined by $R(t) = a^{...}$; $a > 1$; $b > 0$, where $a$ and $b$ are parameters. Find the hazard rate function and show how this model is equivalent to $R(t) = e^{-\lambda t}$.

30. Show that, for an exponential distribution ($\lambda$), the residual mean life is $\frac{1}{\lambda}$, regardless of the length of the time the system has been operating.

**[Hint:** MTTF $(t/T_0) = \int_{T_0}^{\infty} R(t/T_0)\, dt$]

31. The one-month reliability on an indicator lamp is 0.95 with the failure rate specified as constant. What is the probability that more than two spare bulbs will be needed during the first year of operation (Ignore replacement time).

32. A turbine blade has demonstrated a Weibull failure pattern with a decreasing failure rate characterised by a shape parameter of 0.6 and a scale parameter of 800 hours.
   (a) Compute the reliability for a 100-hour mission.
   (b) If there is a 200-hour burn-in of the blades, what is the reliability for a 100-hour mission?

33. Two components have the same MTTF; the first has a constant failure rate $\lambda = 0.141$ and the second follows a Weibull distribution with shape parameter equal to 2.
   (a) Find the characteristic life of the second component.
   (b) If for each component the design life reliability must be 0.8, how much longer (in percentage) is the design life of the second component than the first?

34. A pressure gauge has a Weibull failure distribution with a shape parameter of 2.1 and a characteristic life of 12,000 hours. Find (a) $R(5000)$; (b) the $B1$ and $B.1$ lives; (c) the MTTF and (d) the probability of failure in the first year of continuous operation.

35. A component has a Weibull failure distribution with $\beta = 0.86$ and $\theta = 2450$ days. By how many days will the design life for a 0.90 reliability specification be extended as a result of a 30-day burn-in period?

36. For a three-parameter Weibull distribution with $\beta = 1.54$, $\theta = 8500$ and $t_0 = 50$ hours, find (a) $R$ (150 hours); (b) MTTF; (c) SD and (d) the design life for a reliability of 0.98.
**[Hint:** *pdf* of a 3-parameter Weibull distribution is $f(t) = \frac{\beta}{\theta}\left(\frac{t-t_0}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t-t_0}{\theta}\right)^{\beta}\right]$; $t \geq t_0$. All formulas for this distribution can be had from the corresponding formulas for the 2-parameter Weibull distribution by replacing $t$ by $(t - t_0)$.

$$\text{MTTF} = t_0 + \theta\left[\Gamma\left(1+\frac{1}{\beta}\right)\right].$$

37. A failure *PDF* for an appliance is assumed to be a normal distribution with a mean of 5 years and an S.D. of 0.8 year. Find the design life of the appliance for (a) a reliability of 90%, (b) a reliability of 99%.

38. A lathe cutting tool has a life time that is normally distributed with an S.D. of 12.0 (cutting) hours. If a reliability of 0.99 is desired over 100 hours of use, find the corresponding MTTF. If the reliability has to lie in (0.8, 0.9) find the range within which the tool has to be used.

39. A rotor used in A.C. motor has a time to failure that is lognormal with an MTTF of 3600 operating hours and a shape parameter $s$ equal to 2.
   (a) Determine the probability that the rotor will survive the first 10[illegible] hours.
   (b) If it has not failed till the first 100 hours, what is the probability that it will survive another 100 hours?

40. The life time of a mechanical valve is known to be lognormal with median = 2236 hours and $s = 0.41$.
   (a) Find the design life for a reliability of 0.98.
   (b) Compute the MTTF and S.D. of the time to failure.
   (c) The component is used in pumping device that will require 5 weeks of continuous use. What is the probability of a failure occurring because of the valve?

## RELIABILITY OF SYSTEMS

As was pointed out, a system is generally understood as a set of components assembled to perform a certain function. In the previous section we have considered a number of important failure models (distributions) for the individual components. To evaluate the reliability of a complex system, we may apply a particular failure law to the entire system. But it will be proper if we determine an appropriate reliability model for each component and then compute the reliability of the system by applying the relevant rules of probability according to the configuration of the components within the system. In this section, we shall discuss the system reliability in respect of a few simple but relatively important cases.

### Serial Configuration

*Series* or *nonredundant* configuration is one in which the components of the system are connected in series (or serially) as shown in the following reliability block diagram (Fig. 9.1). Each block represents a component.

Scanned by CamScanner

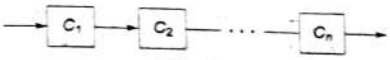


**Fig. 9.1**

In series configuration, all components must function for the system to function. In other words the failure of any component causes system failure.

Let $R_1(t)$, $R_2(t)$ and $R_s(t)$ be the reliabilities of the components $C_1$ and $C_2$ and the system (assuming that there are only 2 components in series),

Then $R_1 = P(C_1)$ = probability that $C_1$ functions

and $R_2 = P(C_2)$ = probability that $C_2$ functions

Now $R_s$ = probability that both $C_1$ and $C_2$ function

$= P(C_1 \cap C_2) = P(C_1) \cdot P(C_2)$, assuming that $C_1$ and $C_2$ function independently.

$= R_1 \times R_2$

This result may be extended. If $C_1, C_2, \cdots, C_n$ be a set of $n$ independent components in series with reliabilities $R_1(t), R_2(t), \cdots, R_n(t)$, then

$$R_s(t) = R_1(t) \times R_2(t) \times \cdots \times R_n(t)$$

$$\leq \min\{R_1(t), R_2(t), \cdots, R_n(t)\} \qquad [\because 0 < R_i(t) < 1]$$

i.e., the system reliability will not be greater than the smallest of the component reliabilities.

## Deductions

1. If each component has a constant failure rate $\lambda_i$, then

$$R_s(t) = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdots e^{-\lambda_n t}$$

$$= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t} = e^{-\lambda_s t}$$

This means that the system also has a constant failure rate $\lambda_s = \sum_{i=1}^{n} \lambda_i$

2. If the components follow the Weibull failure law the parameters $\beta_i$ and $\theta_i$, then

$$R_s(t) = \exp\left[-\left(\frac{t}{\theta_1}\right)^{\beta_1}\right] \times \exp\left[-\left(\frac{t}{\theta_2}\right)^{\beta_2}\right] \cdots \times \exp\left[-\left(\frac{t}{\theta_n}\right)^{\beta_n}\right]$$

$$= \exp\left[-\sum_{i=1}^{n}\left(\frac{t}{\theta_i}\right)^{\beta_i}\right]$$

This means that the system does not follow Weibull failure law, even though every component follows a Weibull failure distribution.



## Parallel Configuration

*Parallel* or *redundant* configuration is one in which the components of the system are connected in parallel as shown in the following reliability block diagram. (Fig. 9.2).

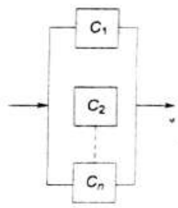


**Fig. 9.2**

In parallel configuration, all components must fail for the system to fail. This means that if one or more components function, the system continues to function.

Taking $n = 2$ and denoting the system reliability by $R_p$ ('$p$' for parallel configuration), we have

$$R_p = P(C_1 \text{ or } C_2 \text{ or both function})$$
$$= P(C_1 \cup C_2)$$
$$= P(C_1) + P(C_2) - P(C_1 \cap C_2)$$
$$= P(C_1) + P(C_2) - P(C_1)\,P(C_2), \text{ since } C_1 \text{ and } C_2 \text{ are independent}$$
$$= R_1 + R_2 - R_1 R_2 = 1 - (1 - R_1)(1 - R_2)$$

Extending to $n$ components, we have

$$R_p = 1 - (1 - R_1)(1 - R_2) \cdots (1 - R_n)$$
$$\geq \text{Max}\{R_1, R_2, \cdots, R_n\}$$

**Note** *For a two component system in parallel having constant failure rate,*

$$R_i(t) = 1 - e^{-\lambda_i t};\ i = 1, 2.$$

Hence $$R_p(t) = 1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$
$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$$

$$\text{MTTF} = \int_0^\infty R_p(t)\,dt$$
$$= \int_0^\infty e^{-\lambda_1 t}\,dt + \int_0^\infty e^{-\lambda_2 t}\,dt - \int_0^\infty e^{-(\lambda_1 + \lambda_2)t}\,dt$$
$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

## Parallel Series Configuration

A system, in which $m$ subsystems are connected in series where each subsystem has $n$ components connected in parallel as in shown Fig. 9.3 is said to be in parallel series configuration or *low-level redundancy*.

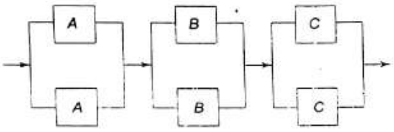


Fig. 9.3

If $R$ is the reliability of the individual component, the reliability of each of the subsystems $= 1 - (1 - R)^n$ (In the diagram $n = 2$).

Since $m$ subsystems are connected in series [In the Fig. 9.3 $m = 3$], the system reliability for the low-level redundancy is given by

$$R_{\text{Low}} = \{1 - (1 - R)^n\}^m$$

## Series-Parallel Configuration

A system, in which $m$ subsystems are connected in parallel where each subsystem has $n$ components connected in series as in Fig. 9.4, is said to be in series-parallel configuration or *high level redundancy*.

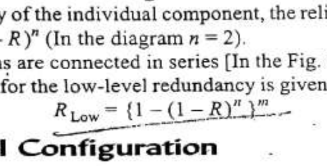


Fig. 9.4

If $R$ is the reliability of each component, the reliability of each of the subsystems

$$= R^n \text{ (In Fig. 9.4, } n = 2)$$

Since $m$ subsystems are connected in parallel (In the diagram $m = 3$), the system reliability for the high-level redundancy is given by

$$R_{\text{High}} = 1 - (1 - R^n)^m$$

**Note**

When $m = n = 2$, $R_{Low} \geq R_{High}$, since

$$\begin{aligned} R_{Low} - R_{High} &= \{1 - (1 - R)^2\}^2 - \{1 - (1 - R^2)^2\} \\ &= (2R - R^2)^2 - (2R^2 - R^4) \\ &= 2R^2 - 4R^3 + 2R^4 \\ &= 2R^2 (1 - R)^2 \geq 0. \end{aligned}$$



## Two Component-System Reliability by Markov Analysis

Markov analysis can be applied to compute system reliability. For simplicity, we consider a system containing two components. To use Markov analysis, the system is considered to be in one of the four states at any time as detailed in Table 9.1:

Table 9.1

| *State* | *Component 1* | *Component 2* |
|---|---|---|
| 1 | operating | operating |
| 2 | failed | operating |
| 3 | operating | failed |
| 4 | failed | failed |

Since the probability that the system undergoes a transition from one state to another depends only on the present state but not on any of its previous states, we can use Markov analysis to compute $P_i(t)$, the probability that the system is in state $i$ at time $t$ and hence the system reliability.

If we assume that the components have constant failure rates $\lambda_1$ and $\lambda_2$, the state transition diagram of the system will be as shown in Fig. 9.5. The nodes in the diagram represent the states of the system and the branches represent the transition rates from one node to another which will be the same as instantaneous failure rates.

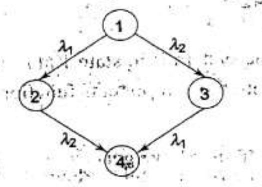


Fig. 9.5

Now $P_1(t + \Delta t) = P\{$the system is in state 1 at time $t$ and neither Component nor Component 2 fails during $(t, t + \Delta t)\}$

$= P_1(t) \times P\{$Component 1 does not fail in $\Delta t$ interval$\} \times$

$P\{$Component 2 does not fail in $\Delta t$ interval$\}$

$= P_1(t)\, \{1 - \lambda_1 \Delta t\}\, \{1 - \lambda_2 \Delta t\}$, [since $P\{$Component $i$ fails in $\Delta t$ interval$\} = \lambda_i \Delta t$ and $P\{$Component $i$ does not fail in $\Delta t$ interval$\} = 1 - \lambda_i \Delta t$]

$= P_1(t)\, \{1 - (\lambda_1 + \lambda_2)\, \Delta t\}$, omitting $(\Delta t)^2$ term.

$$\frac{P_1(t + \Delta t) - P_1(t)}{\Delta t} = -(\lambda_1 + \lambda_2)\, P_1(t)$$

Taking limits on both sides as $\Delta t \to 0$, we get

$$\frac{dP_1(t)}{dt} = -(\lambda_1 + \lambda_2)\, P_1(t) \quad (1)$$

**Note** *Equation (1) and similar equations corresponding to the other nodes may be obtained mechanically as follows:*

*L.H.S. of the equation is $P'_i(t)$. To obtain the R.H.S. of the equation, we note down the arrows entering and/or leaving the node i. If it is an entering arrow, we multiply the transition rate corresponding to that arrow with the probability corresponding to the node from which it emanates. If it is a leaving arrow, we multiply the negative of the transition rate corresponding to that arrow with the probability corresponding to the node from which it emanates (viz., the current node). Then the R.H.S. is obtained by adding these products.*

For the other states, the corresponding equations are

$$\frac{dP_2(t)}{dt} = \lambda_1 \cdot P_1(t) - \lambda_2 \cdot P_2(t) \quad (2)$$

$$\frac{dP_3(t)}{dt} = \lambda_2\, P_1(t) - \lambda_1\, P_3(t) \quad (3)$$

$$\frac{dP_4(t)}{dt} = \lambda_2\, P_2(t) + \lambda_1\, P_3(t) \quad (4)$$

Solving equation (1), we get

$$P_1(t) = c_1\, e^{-(\lambda_1 + \lambda_2)t} \quad (5)$$

Initially, $P_1(0) = P$ (the system is in state 1 at $t = 0$)

$= P$ (both the components function at $t = 0$)

$= 1$

Using this initial condition in (5), we get $c_1 = 0$

$$\therefore \quad P_1(t) = e^{-(\lambda_1 + \lambda_2)t} \quad (6)$$

Using (6) in (2), we have

$$P'_2(t) + \lambda_2\, P_2(t) = \lambda_1\, e^{-(\lambda_1 + \lambda_2)t} \quad (7)$$

I.F. equation (7) $= e^{\lambda_2 t}$

$\therefore$ Solution of equation (7) is

$$e^{\lambda_2 t} \cdot P_2(t) = c_2 - e^{-\lambda_1(t)} \quad (8)$$

Using the initial condition $P_2(0) = 0$ in (8), we get $c_2 = 1$

$\therefore$ Solution of equation (2) becomes

$$P_2(t) = e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$$

Similarly, the solution of equation (3) is

$$P_3(t) = e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$$

To get $P_4(t)$, we need not solve equation (4), but we may use the relation

$$P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1,$$



as the system has to be in any one of the four mutually exclusive and exhaustive states.

Now the system reliability depends on the configuration.

For series configuration,

$R_s(t) = P$ (both components are functioning)

$= P$ (the system is in state 1)

$= P_1(t) = e^{-(\lambda_1 + \lambda_2)t}$, which agrees with the result derived already.

For parallel configuration,

$R_p(t) = P$ (at least one of the components function)

$= P$ (the system is in state 1, 2 or 3)

$= P_1(t) + P_2(t) + P_3(t)$

$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$, which agrees with the result derived already.

**Reliability of a Standby System by Markov Analysis**

In parallel configuration with two components, we have assumed so far that both the components are operating from the start. Such a system is called an *active parallel (redundant) system*. Now we shall consider the parallel configuration with two components in which only one component will be operating from the start and the other, called the *standby* or *backup component*, will be kept in reserve and brought into operation only when the first fails. Such a system is called a *passive parallel system* or *standby redundant system*.

Let us now compute the reliability of such a standby system with the simplifying assumption that the standby unit does not fail while it is kept in standby mode (in reserve)

Let $\lambda_1$ and $\lambda_2$ be the constant failure rates of the active unit and the standby unit (when operative) respectively. This system will be in any of three states described below:

State 1 Main unit is functioning and the other in standby mode

State 2 Main unit has failed and the standby unit is functioning

State 3 Both units have failed.

The state transition diagram for the standby system is given in Fig. 9.6:

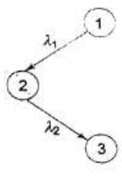


**Fig. 9.6**

From Fig. 9.6, we get the following Markov equation easily as explained earlier.

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$$P_1'(t) = -\lambda_1 P_1(t)$$

$$P_2'(t) = \lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$P_3'(t) = \lambda_2 P_2(t) \text{ or } P_1(t) + P_2(t) + P_3(t) = 1$$

Solving equation, we get $P_1(t) = c_1 e^{-\lambda_1 t}$

Using the initial condition $P_1(0) = 1$, we get $C_1 = 1$

$\therefore$ Solution of equation (1) in $P_1(t) = e^{-\lambda_1 t}$

Using (5) in (2), it becomes

$$P_2'(t) + \lambda_2 P_2(t) = \lambda_1 e^{-\lambda_1 t}$$

I.F. of equation (6) $= e^{\lambda_2 t}$

$\therefore$ Solution of equation (2) is

$$e^{\lambda_2 t} P_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + c_2$$

Using the initial condition $P_2(0) = 0$ in (7),

we get $$c_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1}$$

Using this value in (7), the solution of (2) is got as

$$P_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

Using (5) and (8) in (3), we can get the value of $P_3(t)$.

Now the reliability of the standby system is given by

$$\begin{aligned} R(t) &= P(\text{main unit or standby unit is functioning}) \\ &= P(\text{system is in state 1 or 2}) \\ &= P_1(t) + P_2(t) \\ &= e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \\ &= \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\right) \end{aligned}$$

**Note** *1. When the failure rates of the main and standby units are equal, viz., when* $\lambda_1 = \lambda_2 = \lambda$,

$$\begin{aligned} R(t) &= \lim_{h \to 0}\left[\frac{1}{h}\left\{(\lambda + h)e^{-\lambda t} - \lambda e^{-(\lambda + h)t}\right\}\right] \\ &= e^{-\lambda t} \lim_{h \to 0}\left[1 + \lambda\left(\frac{1 - e^{-ht}}{h}\right)\right] \\ &= e^{-\lambda t} \lim_{h \to 0}\left[1 + \frac{\lambda}{h}\left\{ht - \frac{h^2 t^2}{2!} + \cdots\right\}\right] \\ &= (1 + \lambda t)e^{-\lambda t} \end{aligned}$$

*2. When* $\lambda_1 = \lambda_2 = \lambda$,



*(i) MTTF (for active redundant system)*

$$= \int_0^{\infty} \left(2e^{-\lambda t} - e^{-2\lambda t}\right) dt = \frac{3}{2\lambda}$$

*(ii) MTTF (for standby redundant system)*

$$= \int_0^{\infty} (1 + \lambda t)e^{-\lambda t} dt = \frac{2}{\lambda}.$$

## Worked Example 9(C)

### Example 1

An electronic circuit consists of 5 silicon transistors, 3 silicon diodes, 10 composition resistors and 2 ceramic capacitors connected in series configuration. The hourly failure rate of each component is given below:

| | | |
|---|---|---|
| Silicon transistor | : | $\lambda_t = 4 \times 10^{-5}$ |
| Silicon diode | : | $\lambda_d = 3 \times 10^{-5}$ |
| Composition resistor | : | $\lambda_r = 2 \times 10^{-4}$ |
| Ceramic capacitor | : | $\lambda_c = 2 \times 10^{-4}$ |

Calculate the reliability of the circuit for 10 hours, when the components follow exponential distribution.

Since the components are connected in series, the system (circuit) reliability is given by

$$\begin{aligned} R_s(t) &= R_1(t) \cdot R_2(t) \cdot R_3(t) \cdot R_4(t) \\ &= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 t} \cdot e^{-\lambda_4 t} \\ &= e^{-(5\lambda_t + 3\lambda_d + 10\lambda_r + 2\lambda_c)t} \end{aligned}$$

$$\begin{aligned} \therefore \quad R_s(10) &= e^{-(20 \times 10^{-5} + 9 \times 10^{-5} + 20 \times 10^{-4} + 4 \times 10^{-4}) \times 10} \\ &= e^{-(20 + 9 + 200 + 40) \times 10^{-4}} \\ &= e^{-0.0269} = 0.9735 \end{aligned}$$

### Example 2

There are 16 components in a non-redundant system. The average reliability of each component is 0.99. In order to achieve at least this system reliability by using a redundant system with 4 identical new components, what should be the least reliability of each new component?

For the non-redundant system,

$$R_s = R^{16} = (0.99)^{16} = 0.85$$

Let the new components have a reliability of $R'$ each.

Then for the redundant system with 4 components, $R_p \geq 0.85$

i.e., $1-(1-R')^4 \geq 0.85$

i.e., $(1-R')^4 \leq 0.15$

$\therefore$ $1-R' \leq (0.15)^{\frac{1}{4}}$ or 0.62

$\therefore$ $R' \geq 0.38$

i.e., the reliability of each of the new components should be at least 0.38.

## Example 3

Thermocouples of a particular design have a failure rate of 0.008 per hour. How many thermocouples must be placed in parallel if the system is to run for 100 hours with a system failure probability of no more than 0.05? Assume that all failures are independent.

If $T$ is the time to failure of the system, it is required that

$P(T \leq 100) \leq 0.05$ (1)

i.e., $1-R_p(100) \leq 0.05$

Let the number of thermocouples to be connected in parallel be $n$.

Then $R_p(t) = 1-(1-R)^n$ (2)

where $R$ is the reliability of each couple.

The failure rate of each couple = 0.008 (constant)

$\therefore$ $R = e^{-0.008t}$ (3)

Using (3) in (2), we have

$1-R_p(t) = (1-e^{-0.008\,t})^n$

$\therefore$ $1-R_p(100) = (1-e^{-0.8})^n$ (4)

Using (4) in (1), we have

$(1-e^{-0.8})^n \leq 0.05$

i.e., $(0.55067)^n \leq 0.05$ (5)

By trials, we find that (5) is not satisfied when $n = 0, 1, 2, 3, 4$ and 5.

When $n = 6$, $(0.55067)^n = 0.02788 < 0.05$

Hence 6 thermocouples must be used in the parallel configuration.

## Example 4

Two parallel, identical and independent components have constant failure rates. If it is desired that $R_s(1000) = 0.95$, find the component and system MTTF. In addition to the independent components if a common mode component with a constant failure rate of 0.00001 is connected to the system, find the new system MTTF.



Let $R$ and $\lambda$ be the reliability and failure rate of each of the two independent components. Then $R(t) = e^{-\lambda t}$

Since the two components are connected in parallel,

$$R_s(t) = 1-\{1-R(t)\}^2$$

$$= 1-\{1-e^{-\lambda t}\}^2$$

i.e., $2e^{-\lambda t} - e^{-2\lambda t} = R_s(t)$

$\therefore$ $2e^{-1000\,\lambda} - e^{-2000\,\lambda} = R_s(1000) = 0.95$

i.e., $x^2 - 2x + 0.95 = 0$, on putting $x = e^{-1000\,\lambda}$

$$x = \frac{2 \pm \sqrt{4-3.8}}{2} = 1.22361 \text{ or } 0.77639$$

i.e., $e^{-1000\,\lambda} = 1.22361$ or $0.77639$

$-1000\,\lambda = 0.20181$ or $-0.25310$

Rejecting the value 0.20181, we get

$\lambda = 0.0002531$

$\therefore$ Component MTTF $= \dfrac{1}{\lambda} = \dfrac{1}{0.0002531} = 3951$

and system MTTF $= \displaystyle\int_0^\infty R_s(t)\,dt$

$$= \int_0^\infty (2e^{-\lambda t} - e^{-2\lambda t})\,dt$$

$$= \frac{3}{2\lambda} = \frac{3}{0.0005062} = 5926.5$$

**Note** *Common Mode Failure Component*

*We have assumed that the components connected in series and parallel configurations are independent. This assumption of independence of failures among the components within a system may be easily violated. For example, the components in the system may share the same power source, environmental dust, humidity, vibration etc., resulting in what is known as 'common mode failure'. This common mode failure is represented by including an extra (common mode) component in series with those components that share the failure mode.*

Let $\lambda'$ be the failure rate of common mode component connected in series with the already existing redundant system. Then the system reliability becomes

$R'_s(t) = R_s(t) \cdot R'(t)$, where $R'(t)$ is the reliability of the common mode component.

$$= (2e^{-\lambda t} - e^{-2\lambda t})\, e^{-\lambda' t}$$

$$\therefore \quad R'_s(1000) = (2e^{-0.2531} - e^{-0.5062}) \times e^{-0.01}$$

($\because \lambda = 0.0002531$ and $\lambda' = 0.0000$

$$= 0.94$$

Now the new system MTTF is given by

$$(\text{MTTF})' = \int_0^\infty [2\, e^{-(\lambda + \lambda')t} - e^{-(2\lambda + \lambda')t}]\, dt$$

$$= \frac{2}{\lambda + \lambda'} - \frac{1}{2\lambda + \lambda'}$$

$$= \frac{2}{0.0002631} - \frac{1}{0.0005162} = 5664.4$$

## Example 5

It is known that the reliability function for a critical solid state power unit for u in a communication satellite is $R(t) = \dfrac{10}{10+t}$, $t \geq 0$ and $t$ measured in years.

(a) How many units must be placed in parallel in order to achieve a reliabili of 0.98 for 5 year operation?

(b) If there is an additional common mode with constant failure rate of 0.00 as a result of environmental factors, how many units should be placed in parallel?

(a) Let $n$ units be placed in parallel.

Then $R_s(t) = 1 - \left(1 - \dfrac{10}{10+t}\right)^n$

As $\quad R_s(5) \geq 0.98$, we have

$$1 - \left(1 - \frac{2}{3}\right)^n \geq 0.98$$

i.e., $\dfrac{1}{3^n}$ 0.02 or $3^n \geq 50 \quad \therefore n = 4$

(b) When the additional common mode unit is also used,

$$R_s(t) = \left[1 - \left(1 - \frac{10}{10+t}\right)^n\right] \times e^{-0.002t}$$

As $\quad R_s(5) \geq 0.98$, we have

$$\left\{1 - \left(\frac{1}{3}\right)^n\right\} e^{-0.01} \geq 0.98$$



$$1 - \frac{1}{3^n} \geq \frac{0.98}{0.99}$$

$$\frac{1}{3^n} \leq \left(1 - \frac{0.98}{0.99}\right)$$

$3^n \geq 99$ (nearly) $\therefore n = 5$.

## Example 7

If two identical components having a guaranteed life of 2 months and a consta failure rate of 0.15 per year are connected in parallel, what is the system reliabi y for 10,000 hours of continuous operation?

If $R_c(t)$ is the reliability of each component, then $R_c(t) = e^{-0.15t}$ ($t$ in years

Each component has a guaranteed life (wear-in period) of 2 months or $\frac{1}{6}$ yea

$$= R_c(t/t_0) = \frac{R_c(t + t_0)}{R_c(t_0)} = \frac{e^{-0.15\left(t + \frac{1}{6}\right)}}{e^{-0.15 \times \frac{1}{6}}} = e^{-0.15t}$$

If $R_s(t)$ is the system reliability, then

$$R_s(t) = 2R_c(t/t_0) - \{R_c(t/t_0)\}^2$$

$$= 2e^{-0.15t} - e^{-0.30t}$$

$\therefore \; R_s(10000 \text{ hours}) = R_s(1.14 \text{ years})$

$$= 2e^{-0.15 \times 1.14} - e^{-0.30 \times 1.14}$$

$$= 0.975$$

## Example 7

For a redundant system with $n$ independent identical components with con failure rate $\lambda$, show that the MTTF is equal to

$$\frac{1}{\lambda} \sum_{i=1}^{n} {}_nC_i \frac{(-1)^{i-1}}{i}$$

If $\lambda = 0.01$/hour, what is the minimum value of $n$ so that the MTTF is at leas hours?

If $R_s(t)$ is the system reliability,

$R_s(t) = 1 - (1 - e^{-\lambda t})^n$, since the component reliability $= e^{-\lambda t}$

$$= 1 - \sum_{i=0}^{n} (-1)^i \cdot {}_nC_i\, e^{-i\lambda t}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot nC_i\, e^{-i\lambda t}$$

$$\therefore \text{MTTF} = \int_0^\infty R_s(t)\, dt = \sum_{i=1}^{n} (-1)^{i-1}\, nC_i \int_0^\infty e^{-i\lambda t}\, dt$$

$$= \frac{1}{\lambda} \sum_{i=1}^{n} nC_i \frac{(-1)^{i-1}}{i}$$

When $n = 2$, $\text{MTTF} = 100 \sum_{i=1}^{2} 2C_i \frac{(-1)^{i-1}}{i}$

$$= 100\left[2 - \frac{1}{2}\right] = 150$$

When $n = 3$, $\text{MTTF} = 100 \sum_{i=1}^{3} 3C_i \frac{(-1)^{i-1}}{i}$

$$= 100\left(3 - \frac{3}{2} + \frac{1}{3}\right) = 100 \times \frac{11}{6} = 183$$

When $n = 4$, $\text{MTTF} = 100 \sum_{i=1}^{4} 4C_i \frac{(-1)^{i-1}}{i}$

$$= 100 \times \frac{25}{12} = 208$$

Hence the required minimum value of $n = 4$.

## Example 8

A jet engine consists of 5 modules (connected in series) each of which was found to have a Weibull failure distribution with a shape parameter of 1.5. Their characteristic lives are (in operating cycles) 3600, 7200, 5850, 4780 and 9300. Find the reliability function of the engine and the MTTF.

If $R(t)$ is the reliability function of the engine (system), then

$$R(t) = R_1(t) \cdot R_2(t) \ldots R_5(t)$$

$$= e^{-(t/\theta_1)^{1.5}} \cdot e^{-(t/\theta_2)^{1.5}} \ldots e^{-(t/\theta_5)^{1.5}}$$

$$= e^{-(\theta_1^{-1.5} + \theta_2^{-1.5} + \cdots + \theta_5^{-1.5}) t^{1.5}}$$

$$= e^{-0.000012642\, t^{1.5}}$$

$$= e^{-\left(\frac{t}{1842.7}\right)^{1.5}}$$



This shows that the engine failure time also follows a Weibull's distribution with

$$\beta = 1.5 \text{ and } \theta = 1842.7.$$

$$\text{MTTF of the engine} = \theta \,\Gamma\!\left(1 + \frac{1}{\beta}\right)$$

$$= 1842.7 \times \Gamma\!\left(1 + \frac{1}{1.5}\right)$$

$$= 1842.7 \times 0.9033$$

$$= 1664.5 \text{ cycles.}$$

## Example 9

A signal processor has a reliability of 0.90. Because of the lower reliability a redundant signal processor is to be added. However, a signal splitter must be added before the processors and a comparator must be added after the signal processors. Each of the new components has a reliability of 0.95. Does adding a redundant signal processor increase the system reliability?

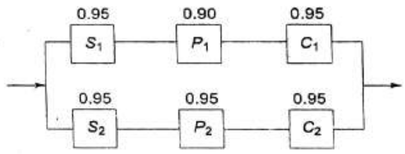



Fig. 9.7

$$R_1 = R\,(\text{first subsystem}) = R(S_1) \cdot R(P_1) \cdot R(C_1)$$

$$= 0.95 \times 0.90 \times 0.95$$

$$= 0.81225$$

$$R_2 = R\,(\text{second subsystem}) = R(S_2) \cdot R(P_2) \cdot R(C_2)$$

$$= (0.95)^3$$

$$= 0.857375$$

$$R(\text{system}) = 1 - (1 - R_1)(1 - R_2)$$

$$= 1 - 0.0268 = 0.9732$$

The addition of a redundant signal processor increases the reliability.

## Example 10

Six identical components with constant failure rates are connected in (a) high level redundancy with 3 components in each subsystem (b) low level redundancy with 2 components in each subsystem. Determine the component MTTF in each case, necessary to provide a system reliability of 0.90 after 100 hours of operation.

Let $\lambda$ be the constant failure rate of each component. Then $R = e^{-\lambda t}$, for each component. For high level redundancy,

$$R_s(t) = 1 - [1 - \{R(t)\}^3]^2$$
$$= 1 - (1 - e^{-3\lambda t})^2$$

$$\therefore \quad R_s(100) = 1 - (1 - e^{-300\lambda})^2 = 0.90$$

i.e., $\quad (1 - e^{-300\lambda})^2 = 0.1$

i.e., $\quad 1 - e^{-300\lambda} = 0.31623$

$$\therefore \quad e^{-300\lambda} = 0.68377$$

$$\therefore \quad 300\lambda = 0.38013$$

$$\therefore \text{ MTTF of each component} = \frac{1}{\lambda} = \frac{300}{0.38013} = 789.2 \text{ hours.}$$

For low level redundancy

$$R_s(t) = [1 - \{1 - R(t)\}^2]^3$$
$$= [1 - \{1 - e^{-\lambda t}\}^2]^3$$

$$\therefore \quad R_s(100) = [1 - \{1 - e^{-100\lambda}\}^2]^3 = 0.90$$

i.e., $\quad 1 - (1 - e^{-100\lambda})^2 = 0.96549$

$$\therefore \quad (1 - e^{-100\lambda})^2 = 0.03451$$

$$\therefore \quad 1 - e^{-100\lambda} = 0.18577$$

$$\therefore \quad e^{-100\lambda} = 0.81423$$

$$\therefore \quad 100\lambda = 0.20551$$

$$\therefore \text{ MTTF of each component} = \frac{1}{\lambda} = \frac{100}{0.20551} = 486.6 \text{ hours.}$$

## Example 11

A system consists of two sub-systems in parallel. The reliability of each sub-system is given by (Weibull failure) $R(t) = e^{-\left(\frac{t}{\theta}\right)^2}$. Assuming that common mode failure may be neglected, determine the system MTTF.



The system reliability is given by

$$R_s(t) = 1 - \left[1 - e^{-\left(\frac{t}{\theta}\right)^2}\right]^2$$
$$= 1 - [1 - 2e^{-t^2/\theta^2} + e^{-2t^2/\theta^2}]$$
$$= 2e^{-t^2/\theta^2} - e^{-2t^2/\theta^2}$$

$$\therefore \text{ System MTTF} = \int_0^\infty R_s(t)\, dt$$
$$= 2\int_0^\infty e^{-t^2/\theta^2}\, dt - \int_0^\infty e^{-2t^2/\theta^2}\, dt$$
$$= 2\int_0^\infty e^{-x^2}\cdot \theta\, dx - \int_0^\infty e^{-y^2}\cdot \frac{\theta}{\sqrt{2}}\, dy,$$

(on putting $t = \theta x$ in the first integral and $t = \frac{1}{\sqrt{2}}\theta y$ in the second integral)

$$= 2\theta \times \frac{\sqrt{\pi}}{2} - \frac{\theta}{2} \times \frac{\sqrt{\pi}}{2}$$
$$= 1.15\,\theta.$$

## Example 12

A system is designed to operate for 100 days. The system consists of three components in series. Their failure distributions are (1) Weibull with shape parameter 1.2 and scale parameter 840 days; (2) lognormal with shape parameter 0.7 and median 435 days; (3) constant failure rate of 0.0001.

(a) Compute the system reliability.

(b) If two units each of components 1 and 2 are available, determine the high level redundancy reliability. Assume that the components 1 and 2 are configured as a sub-assembly.

(c) If two units each of components 1 and 2 are available, determine the low level redundancy reliability.

(a) For the first (Weibull) component,

$$R_1(t) = \exp\left\{-\left(\frac{t}{\theta}\right)^\beta\right\} = \exp\left\{-\left(\frac{t}{840}\right)^{1.2}\right\}$$

$$\therefore \quad R_1(100) = \exp\left\{-\left(\frac{100}{840}\right)^{1.2}\right\} = 0.9252$$

For the second (lognormal) component,

$$R_2(t) = \int_t^\infty f(t)\,dt, \text{ where } f(t) \text{ is the lognormal } pdf.$$

$$\therefore R_2(100) = \int_{100}^\infty f(t)\,dt$$

$$= \int_{\frac{1}{0.7}\log\left(\frac{100}{435}\right)}^\infty \phi(z)\,dy, \text{ where } \phi(z) \text{ is the standard normal density}$$

$$= \int_{-2.10}^\infty \phi(z)\,dz = 0.9821$$

r the third (constant failure rate) component,

$$R_3(t) = e^{-\lambda t} = e^{-0.0001t}$$

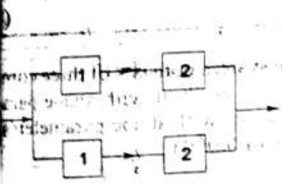
$$\therefore R_3(100) = e^{-0.01} = 0.9900$$

ce the three components are connected in series,

$$R(100) = R_1(100) \times R_2(100) \times R_3(100)$$

$$= 0.9252 \times 0.9821 \times 0.9900$$

$$= 0.8996$$

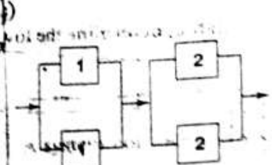



$$R_{HL} = 1 - (1 - R_1 R_2)^2$$

$$= 1 - \{1 - 0.9252 \times 0.9821\}^2$$

$$= 0.9917$$

Fig. 9.8




$$R_{LL} = \{1 - (1 - R_1)^2\} \times \{1 - (1 - R_2)^2\}$$

$$= 0.9944 \times 0.9997$$

$$= 0.9941$$

Fig. 9.9



## Example 13

Find the variance of the time to failure for two identical units, each with a failure rate $\lambda$, placed in standby parallel configuration. Compare the result with the variance of the same two units placed in active parallel configuration. Ignore switching failures and failures in the standby mode.

*For standby parallel configuration:*

$$R(t) = (1 + \lambda t)\, e^{-\lambda t}$$

$$f(t) = -R'(t) = -[\lambda e^{-\lambda t} - \lambda(1 + \lambda t)e^{-\lambda t}]$$

$$= \lambda^2 t\, e^{-\lambda t}$$

MTTF = $E(T)$, where $T$ is the time to failure of the system

$$= \int_0^\infty t f(t)\,dt \left[\text{or} \int_0^\infty R(t)\,dt\right]$$

$$= \lambda^2 \left[t^2\left(\frac{e^{-\lambda t}}{-\lambda}\right) - 2t\left(\frac{e^{-\lambda t}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda t}}{-\lambda^3}\right)\right]_0^\infty$$

$$= \frac{2}{\lambda}$$

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda^2 \int_0^\infty t^3 e^{-\lambda t}\,dt$$

$$= \lambda^2 \left[t^3\left(\frac{e^{-\lambda t}}{-\lambda}\right) - 3t^2\left(\frac{e^{-\lambda t}}{\lambda^2}\right) + 6t\left(\frac{e^{-\lambda t}}{-\lambda^3}\right) - 6\left(\frac{e^{-\lambda t}}{\lambda^4}\right)\right]_0^\infty$$

$$= \frac{6}{\lambda^2}$$

Variance of $T = E(T^2) - E^2(T)$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2} \qquad (1)$$

*For the active parallel configuration*

$$R(t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

$$\therefore \quad f(t) = -R'(t) = 2\lambda e^{-\lambda t} - 2\lambda e^{-2\lambda t}$$

$$\text{MTTF} = E(T) = \int_0^\infty R(t)\,dt \left[\text{or} \int_0^\infty t\, f(t)\,dt\right]$$

$$= \left(\frac{2e^{-\lambda t}}{-\lambda} + \frac{e^{-2\lambda t}}{2\lambda}\right)_0^\infty = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$$

$$E(T^2) = \int_0^\infty t^2 f(t)\, dt$$

$$= 2\lambda\left[\int_0^\infty t^2 e^{-\lambda t}\, dt - \int_0^\infty t^2 e^{-2\lambda t}\, dt\right]$$

$$= 2\lambda\left[\left\{t^2\left(\frac{e^{-\lambda t}}{-\lambda}\right) - 2t\left(\frac{e^{-\lambda t}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda t}}{-\lambda^3}\right)\right\}_0^\infty\right.$$

$$\left. - \left\{t^2\left(\frac{e^{-2\lambda t}}{-2\lambda}\right) - 2t\left(\frac{e^{-2\lambda t}}{4\lambda^2}\right) + 2\left(\frac{e^{-2\lambda t}}{-8\lambda^3}\right)\right\}_0^\infty\right]$$

$$= 2\lambda\left[\frac{2}{\lambda^3} - \frac{1}{4\lambda^3}\right] = \frac{7}{2\lambda^2}$$

$$\text{Var}(T) = E(T^2) - E^2(T)$$

$$= \frac{7}{2\lambda^2} - \frac{9}{4\lambda^2} = \frac{5}{4\lambda^2} \qquad (2)$$

Comparing (1) and (2), we see that the variance is greater for standby parallel system.

## Example 14

A computerised airline reservation system has a main computer online and a secondary standby computer. The online computer fails at a constant rate of 0.001 failure per hour and the standby unit fails when on-line at the constant rate of 0.005 failure per hour. There are no failures while the unit is in the standby mode.

(a) Determine the system reliability over a 72 hours period.

(b) The airline desires to have a system MTTF of 2000 hours. Determine the minimum MTTF of the main computer to achieve this goal, assuming that the standby computer MTTF does not change.

(a) $\lambda_1 = 0.001$/hour; $\lambda_2 = 0.005$/hour

$$R_s(t) = \frac{1}{\lambda_2 - \lambda_1}\{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\}$$

$$= \frac{1}{0.004}\{0.005\, e^{-0.001t} - 0.001\, e^{-0.005t}\}$$



$$\therefore \quad R_s(72) = \frac{1}{4}\{5\, e^{-0.072} - e^{-0.360}\}$$

$$= 0.9887$$

(b) $$\text{MTTF} = \int_0^\infty R(t)\, dt$$

$$= \frac{1}{\lambda_2 - \lambda_1}\int_0^\infty (\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t})\, dt$$

$$= \frac{1}{\lambda_2 - \lambda_1}\left(\frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{\lambda_2}\right) = \frac{\lambda_2 + \lambda_1}{\lambda_1\lambda_2} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

The requirement is $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 2000$

$$\therefore \quad \frac{1}{\lambda_1} = 2000 - \frac{1}{0.005}$$ ($\because$ MTTF and hence $\lambda_2$ do not change for standby unit)

$$= 1800$$

i.e., MTTF of the main computer should be increased to 1800 hours.

## Example 15

A fuel pump with an MTTF of 3000 hours is to operate continuously on a 500 hour mission.

(a) What is the mission reliability?

(b) Two such pumps are put in standby parallel configuration. If there are no failures of the back up pump while in standby mode, what are the system MTTF and the mission reliability?

(c) If the standby failure rate is 75% of that of the main pump (when operational), what are the system MTTF and the mission reliability?

(a) $\text{MTTF} = \frac{1}{\lambda} = 3000 \qquad \therefore \lambda = \frac{1}{3000}$

$R(t) = e^{-\lambda t} \qquad \therefore R(500) = e^{-(500/3000)} = 0.8465$

(b) $\lambda_1 = \lambda_2 = \lambda = \frac{1}{3000}$

$$R_s(t) = (1 + \lambda t)e^{-\lambda t}$$

$$\therefore R_s(500) = \left(1 + \frac{500}{3000}\right)e^{-(500/300)} = 0.9876$$

$$\text{MTTF} = \frac{2}{\lambda} = 6000 \text{ hours}$$

(c) $\lambda_1 = \frac{1}{3000}$; $\lambda_2 = \frac{3}{4} \times \frac{1}{3000} = \frac{1}{4000}$

$$R_s(t) = \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}\right)$$

$$= 12{,}000 \times \left(\frac{1}{3000} e^{-\frac{1}{4000}t} - \frac{1}{4000} e^{-\frac{1}{3000}t}\right)$$

$$\therefore R_s(500) = \left(4\, e^{-1/8} - 3\, e^{-1/6}\right)$$

$$= 0.9905$$

$$\text{MTTF} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 7000 \text{ hours.}$$

## Exercise [illegible]

**Part A** (Short answer questions)

1. What do you mean by non-redundant and redundant configurations?
2. Derive the reliability of a system consisting of two components in series in terms of reliabilities of the components.
3. Derive the reliability of a system consisting of two components in parallel in terms of the reliabilities of the components.
4. Prove that the reliability of a non-redundant system cannot exceed the least component reliability.
5. Prove that the reliability of a redundant system is not less than the maximum component reliability.
6. Find the MTTF of (a) a non-redundant system (b) a redundant system consisting of two constant failure rate components.
7. Explain low level redundancy with a diagram.
8. Obtain the reliability of a low level redundancy with $m$ subsystems each of which consists of $n$ components.
9. Explain high level redundancy with a diagram.
10. Obtain the reliability of a high level redundancy with $m$ subsystems each of which consists of $n$ components.
11. What is the difference between active and standby redundant systems?
12. Write down the formulas for the reliability of a standby redundant system when the main and standby components have
    (i) unequal constant failure rates and
    (ii) equal constant failure rates.



13. What are the MTTF's for active and standby redundant systems when the main and the standby components have unequal constant failure rates.
14. What are the MTTF's for active and standby redundant systems when the main and standby components have equal constant failure rates.
15. Find the reliability of an aircraft electronic system consisting of sensor, guidance, computer and fire control subsystems with respective reliabilities 0.80, 0.90, 0.95 and 0.65 are connected in series.
16. In a lead there are 16 glow bulbs connected in series and the average reliability of each bulb is 0.99. Compute the reliability of the load.
17. An equipment consists of 3 components $A$, $B$ and $C$ in parallel and the respective reliabilities are $R_A = 0.92$, $R_B = 0.95$ and $R_C = 0.96$. Calculate the equipment reliability.
18. If the reliability of each of 3 components connected in parallel is 0.9, calculate the reliability of the system.
19. If $R_A = 0.8$, $R_B = 0.9$ and $R_C = 0.85$ and if $A$, $B$, $C$ are connected in series to form a subsystem, what is the reliability of the high level redundant system with two such subsystems.
20. If two $A$'s, two $B$'s and two $C$'s in Question. (19) are connected in parallel and the three subsets are connected to form a low level redundant system, find the reliability of the system.

**Part B**

21. The time to failure (in years) of a certain brand of computer has the *pdf*
    $$f(t) = 1/(t+1)^2;\ t > 0$$
    (a) If three of these computers are placed in parallel aboard the proposed space station, what is the system reliability for the first 6 months of operation?
    (b) What is the system design life in days if a reliability of 0.999 required?
22. The Weibull components, each having a shape parameter of 0.80 must operate in series. Determine a common characteristic life in order that they have a design life of 1 year with a reliability of 0.99.
23. If three components, each with constant failure rate are placed in parallel, determine the system reliability for 0.1 year and the MTTF. Their failure rates are 5 per year, 10 year and 15 per year.
24. Specifications for a power unit consisting of three independent and serially connected components require a design life of 5 years with a 0.95 reliability.
    (a) If the constant failure rates $\lambda_1, \lambda_2, \lambda_3$ are such $\lambda_1 = 2\lambda_2$ and $\lambda_3 = 3\lambda_2$, what should be the MTTF of each component of the system?
    (b) If two identical power units are placed in parallel, what is the system reliability at 5 years and what is the system MTTF?
25. A power supply consists of three rectifiers in series. Each rectifier has a Weibull failure distribution with $\beta = 2.1$. However they have different characteristic lifetimes given by 12,000 hours, 18,500 hours and 21,500

hours. Find the MTTF and the design life of the power supply corresponding to a reliability of 0.90.

26. What is the maximum number of identical and independent Weibull components having a scale parameter of 10000 operating hours and a shape parameter of 1.3 that can be put in series if a reliability of 0.95 at 100 operating hours is desired? What is the resulting MTTF?

27. Which of the following systems has the higher reliability at the end of 100 operating hours?
   (i) Two constant failure rate components in parallel each having an MTTF of 100 hours.
   (ii) A Weibull component with a shape parameter of 2 and a characteristic life of 10,000 hours in series with a constant failure rate component with a failure rate of 0.00005.

28. Find the minimum number of redundant components, each having reliability of 0.4, necessary to achieve a system reliability of 0.95. There is a common mode failure probability of 0.03.

29. Three communication channels in parallel have independent failure modes of 0.1 failure per hour. These components must share a common transceiver. Determine the MTTF of the transceiver in order that the system has a reliability of 0.85 to support a 5 hour mission. Assume constant failure rates.

30. The reliability of a communication channel is 0.40. How many channels should be placed in parallel redundancy so as to achieve the reliability of receiving the information is 0.80. If these channels are used to configure high level and low level redundant systems, what are the corresponding system reliabilities?

31. $2n$ identical constant failure rate components are used to configure redundant systems either with two subsystems in parallel or with $n$ subsystems in series. Which system will give higher reliability?

32. Compute the reliability of the system for the connection given in Fig. 9.10, if the reliability of $A$, $B$, $C$, $D$ are 0.95, 0.99, 0.90 and 0.96 respectively.

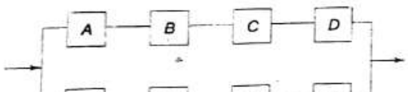



Fig. 9.10

33. Compute the reliability of the system for the connection given in Fig. 9.11, if the reliabilities of $A$, $B$, $C$, $D$ are 0.90, 0.95, 0.96 and 0.98 respectively.



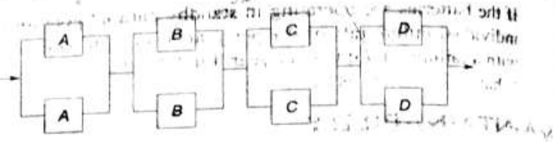


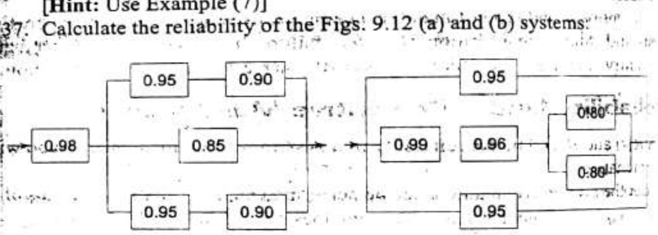

Fig. 9.11

34. A non-redundant system with 100 components has a design life ($L$) reliability of 0.90. The system is redesigned so that it has only 70 components. Estimate the design life of the redesigned system, assuming that all the components have constant failure rates of the same value.

35. Find the variance of the time to failure of the following systems, assuming a constant failure rate $\lambda$ for each component:
   (a) a system with two components in series
   (b) a system with two components in parallel.

36. How many identical components with constant failure rate must be connected in parallel to at least double the MTTF of a single component?
**[Hint:** Use Example (7)**]**

37. Calculate the reliability of the Figs. 9.12 (a) and (b) systems.




(a) (b)

Fig. 9.12

38. Nine components each with a reliability 0.9 are used to configure (a) high level redundancy with 3 subsystems and (b) a low level redundancy with 3 subsystems. In order to equalise the reliabilities of two systems, to which system a common mode failure component is to be added. What is its reliability?

39. Two stamping machines with constant failure rate operate in passive parallel positions on an assembly line. If one fails the other takes up the load by doubling its operating speed. When this happens, however, the failure rate also doubles. Compare the design lives of the main machine and the standby system for a reliability of 0.99 and 0.95.

40. Two nickel-cadmium batteries provide electrical power to operate a satellite transceiver.

If the batteries are operating in standby parallel positions, they have an individual failure rate of 0.1 per year. If one fails the other can operate with a failure rate of 0.3 per year. Find the system reliability for 2 years. What is the system MTTF?

## MAINTAINABILITY

No equipment (system) can be perfectly reliable in spite of the utmost care and best effort on the part of the designer and manufacturer. In fact, very few systems are designed to operate without maintenance of any kind. For a large number of systems, maintenance is a must, as it is one of the effective ways of increasing the reliability of the system.

Usually two kinds of maintenance are adopted. They are *preventive maintenance* and *corrective or repair maintenance*. Preventive maintenance is maintenance done periodically before the failure of the system, so as to increase the reliability of the system by removing the ageing effects of wear, corrosion, fatigue and related phenomena. On the other hand, repair maintenance is performed after the failure has occurred so as to return the system to operation as soon as possible.

The amount and type of maintenance that is used depends on the respective costs and safety consideration of system failure. It is generally assumed that a preventive maintenance action is less costly than a repair maintenance action.

## Reliability Under Preventive Maintenance

Let $R(t)$ and $R_M(t)$ be the reliability of a system without maintenance and with maintenance.

Let the preventive maintenance be performed on the system at intervals of $T$.

Since $R_M(t) = P$ {the maintained system does not fail before $t$}, we have

$$R_M(t) = R(t), \text{ for } 0 \le t < T$$
$$= R(T), \text{ for } t = T.$$

After performing the first maintenance operation at $T$, the system becomes as good as new.

Hence, if $T \le t < 2T$,

$R_M(t) = P$ {the system does not fail upto $T$ and it survives for a time $(t - T)$ without failure}

$$= R(T) \cdot R(t - T), \text{ for } T \le t < 2T$$

Similarly after two maintenance operations,

$$R_M(t) = \{R(T)\}^2 \cdot R(t - 2T), \text{ for } 2T \le t < 3T$$

Proceeding like this, we get in general,

$$R_M(t) = \{R(T)\}^n \cdot R(t - nT), \text{ for } nT \le t < (n + 1)T \qquad (n = 0, 1, 2, \ldots)$$

MTTF of a system with preventive maintenance is given by



$$\text{MTTF} = \int_0^\infty R_M(t)\, dt$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} R_M(t)\, dt, \text{ by dividing the range into intervals of length } T$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} \{R(T)\}^n R(t - nT)\, dt$$

$$= \sum_{n=0}^{\infty} \{R(T)\}^n \int_0^T R(t')\, dt', \text{ on putting } t - nT = t'$$

$$= \frac{\int_0^T R(t)\, dt}{1 - R(T)}$$

**Note**

*1. If the system has a constant failure rate $\lambda$, then $R(t) = e^{-\lambda t}$ and*

$$R_M(t) = (e^{-\lambda T})^n \cdot e^{-\lambda(t - nT)}, \text{ for } nT \le t < (n + 1)T, \quad \text{where } n = 0, 1, 2, \cdots$$

$$= e^{-n\lambda T} \cdot e^{-\lambda t + n\lambda T}$$

$$= e^{-\lambda t} = R(t), \text{ for } 0 \le t < \infty.$$

*This means that, in the case of constant failure rate, preventive maintenance does not improve the reliability of the system.*

*2. If the failure distribution of the system is Weibull with parameters $\beta$ and $\theta$, then*

$$R(t) = \exp\left\{-\left(\frac{t}{\theta}\right)^\beta\right\}, \text{ for non-maintained system.}$$

*For the maintained system,*

$$R_M(t) = \left[\exp\left\{-\left(\frac{T}{\theta}\right)^\beta\right\}\right]^n \times \exp\left[-\left(\frac{t - nT}{\theta}\right)^\beta\right],$$

*for $nT \le t \le (n + 1)T$ and $n = 0, 1, 2, \cdots$*

$$= \exp\left\{-n\left(\frac{T}{\theta}\right)^\beta\right\} \times \exp\left\{-n\left(\frac{t - nT}{\theta}\right)^\beta\right\}$$

*To examine the effects of preventive maintenance, we find $R_M(t)/R(t)$ at the time of preventive maintenance $t = nT$, where $n = 1, 2, 3, \cdots$*

$$\frac{R_M(nT)}{R(nT)} = \frac{\exp\left\{-n\left(\frac{T}{\theta}\right)^{\beta}\right\}}{\exp\left\{-\left(\frac{nT}{\theta}\right)^{\beta}\right\}}$$

$$= \exp\left\{-n\left(\frac{T}{\theta}\right)^{\beta} + \left(\frac{nT}{\theta}\right)^{\beta}\right\} > 1,$$

*if* $-\frac{nT^{\beta}}{\theta^{\beta}} + \frac{n^{\beta} \cdot T^{\beta}}{\theta^{\beta}} > 0$

*i.e., if* $n^{\beta-1} - 1 > 0$

*i.e., if* $\beta > 1$

*This means that $R_M(t) > R(t)$, viz., preventive maintenance will improve the reliability of the Weibull system, only when the shape parameter $\beta > 1$.*

## Repair Maintenance-Maintainability

A measure of how fast a component (system) may be repaired following failure is known as *maintainability*. Repairs require different lengths of time and even the time to perform a given repair is uncertain (random), because circumstances, skill level, experience of maintenance personnel and such other factors vary. Hence the time $T$ required to repair a failed component (system) is a continuous R.V.

Maintainability is mathematically defined as the cumulative distribution function (*cdf*) of the R.V.T. representing the time to repair and denoted as $M(t)$.

i.e., $$M(t) = P\{T \le t\} = \int_0^t m(t)\, dt \qquad (1)$$

where $m(t)$ is the *pdf* of $T$.

The expected value of repair time $T$ is called *the mean time to repair* (MTTR) and is given by

$$\text{MTTR} = E(T) = \int_0^t t \times m(t)\, dt \qquad (2)$$

If the conditional probability that the (component) system will be repaired (made operational) between $t$ and $t + \Delta t$, given that it has failed at $t$ and the repair starts immediately, is $\mu(t)\,\Delta t$, then $\mu(t)$ is called the *instantaneous repair rate* or *simply the repair rate* and denotes the number of repairs in unit time.

i.e., $$\mu(t)\,\Delta t = \frac{P\{t \le T \le t + \Delta t\}}{P(T > t)}$$



$$= \frac{m(t)\,\Delta t}{1 - M(t)}$$

$$\therefore \quad \mu(t) = \frac{m(t)}{1 - M(t)} \qquad (3)$$

From (1), on differentiation, we get

$$m(t) = \frac{d}{dt} M(t) \qquad (4)$$

Using (4) in (3), we have

$$\mu(t) = \frac{M'(t)}{1 - M(t)} \qquad (5)$$

Integrating both sides of (5) with respect to $t$ between 0 and $t$, we get

$$\int_0^t \mu(t)\, dt = \int_0^t \frac{M'(t)}{1 - M(t)}\, dt$$

i.e., $$[-\log\{1 - M(t)\}]_0^t = \int_0^t \mu(t)\, dt$$

i.e., $$1 - M(t) = e^{-\int_0^t \mu(t)\, dt} \qquad [\because M(0) = \ldots$$

or $$M(t) = 1 - e^{-\int_0^t \mu(t)\, dt}$$

Using (5) and (6) in (4), we get

$$m(t) = \mu(t) \cdot e^{-\int_0^t \mu(t)\, dt}$$

**Note** *If $\mu(t) = \mu$ (a constant), then from (7),*

$$m(t) = \mu e^{-\mu t}, t > 0$$

*i.e., the time to repair $T$ follows an exponential distribution with parameter $\mu$.*

*Conversely if $m(t) = \mu e^{-\mu t}, t > 0$, then*

$$M(t) = 1 - e^{-\mu t}, \text{ using (6)}$$

$$\mu(t) = \frac{\mu e^{-\mu t}}{e^{-\mu t}}, \text{ using (3)}$$

$$= \mu$$

*For the constant repair rate distribution,*

$$MTTR = \int_0^{\infty} t\,m(t)\, dt = \int_0^{\infty} t \cdot \mu e^{-\mu t}\, dt$$

$$= \mu\left[t\left(\frac{e^{-\mu t}}{-\mu}\right) - \left(\frac{e^{-\mu t}}{\mu^2}\right)\right]_0^{\infty} = \frac{1}{\mu}$$

## Reliability of a Two Component Redundant System with Repair by Markov Analysis

Let us consider a two component redundant system in which both the failure rate and repair rate are constant.

If we assume that repair can be completed for a failed unit before the other unit has failed, system failure is ruled out and the system reliability is improved. Otherwise both units may fail resulting in less reliability.

The corresponding Markov state-transition diagram is given in Fig. 9.13, in which $\lambda_1$ and $\lambda_2$ represent the transition rates from states 1 and 2 to states 2 and 3 respectively

**Note** *They do not necessarily represent the failure rates of the two components.*

$\mu_2$ represents the transition rate from state 2 to state 1.

State 1 represents the situation in which both the components are operating, state 2 represents the situation in which one component is operating and the other is being repaired and state 3 represents the situation in which both components have failed.

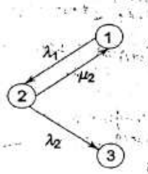



Fig. 9.13

Proceeding as in the discussion of system reliability without repair using Markov analysis, we get the following differential equations for the state probabilities:

$$P_1'(t) = -\lambda_1 P_1(t) + \mu_2 P_2(t) \quad (1)$$

$$P_2'(t) = \lambda_1 P_1(t) - (\lambda_2 + \mu_2) P_2(t) \quad (2)$$

$$P_3'(t) = \lambda_2 P_2(t) \quad (3)$$

Equation (1) is $(D + \lambda_1) P_1(t) - \mu_2 P_2(t) = 0$ (1)′

Equation (2) is $\lambda_1 P_1(t) - (D + \lambda_2 + \mu_2) P_2(t) = 0$ (2)′

where $D = \frac{d}{dt}$. Equations (1)′ and (2)′ are simultaneous differential equations in $P_1(t)$ and $P_2(t)$.

Eliminating $P_2(t)$ from (1)′ and (2)′, we get

$$\{(D + \lambda_1)(D + \lambda_2 + \mu_2) - \lambda_1 \mu_2\} P_1(t) = 0$$

i.e., $\{D^2 + (\lambda_1 + \lambda_2 + \mu_2)D + \lambda_2 \lambda_2\} P_1(t) = 0$ (3)

A.E. corresponding to equation (3) is



$$m^2 + (\lambda_1 + \lambda_2 + \mu_2)m + \lambda_1 \lambda_2 = 0 \quad (4)$$

Solving equation (4), we get

$$m = \frac{-(\lambda_1 + \lambda_2 + \mu_2) \pm \sqrt{(\lambda_1 + \lambda_2 + \mu_2)^2 - 4\lambda_1\lambda_2}}{2}$$

$$= m_1, m_2 \text{ (say)}$$

$\therefore$ Solution of equation (3) is

$$P_1(t) = Ae^{m_1 t} + Be^{m_2 t} \quad (5)$$

Initially both the components are operating, viz., the system is in state 1.

$$P_1(0) = 1$$

$\therefore$ we get $A + B = 1$ (6) [from (5)]

Using (5) in (1), we have

$$P_2(t) = \frac{1}{\mu_2} [m_1 Ae^{m_1 t} + m_2 Be^{m_2 t} + \lambda_1 Ae^{m_1 t} + \lambda_1 Be^{m_2 t}] \quad (7)$$

Initial condition is $P_2(0) = 0$

$$m_1 A + m_2 B + \lambda_1 A + \lambda_1 B = 0 \quad \text{[from (7)]}$$

i.e., $(m_1 + \lambda_1)A + (m_2 + \lambda_1)B = 0$ (8)

Solving equations (6) and (8), we get

$$A = -\frac{(m_2 + \lambda_1)}{m_1 - m_2} \text{ and } B = \frac{m_1 + \lambda_1}{m_1 - m_2} \quad (9)$$

Using (9) in (5), we have

$$P_1(t) = -\frac{(m_2 + \lambda_1)}{m_1 - m_2} e^{m_1 t} + \frac{m_1 + \lambda_1}{m_1 - m_2} e^{m_2 t} \quad (10)$$

Using (9) in (7), we have

$$P_2(t) = \frac{1}{\mu_2}\left[\frac{(m_1 + \lambda_1)(m_2 + \lambda_1)}{m_1 - m_2}\left(e^{m_2 t} - e^{m_1 t}\right)\right]$$

$$= \frac{1}{\mu_2(m_1 - m_2)} [\{m_1 m_2 + \lambda_1(m_1 + m_2) + \lambda_1^2\} (e^{m_2 t} - e^{m_1 t})]$$

$$= \frac{1}{\mu_2(m_1 - m_2)} [(\lambda_1 \lambda_2 - \lambda_1(\lambda_1 + \lambda_2 + \mu_2) + \lambda_1^2\} (e^{m_2 t} - e^{m_1 t})]$$

[$\because$ $m_1, m_2$ are roots of (4)]

$$= \frac{\lambda_1}{m_1 - m_2} (e^{m_1 t} - e^{m_2 t}) \quad (11)$$

If $R(t)$ is the system reliability, then

$$R(t) = P\{\text{one or both components are operating}\}$$
$$= P\{\text{system is in state 1 or 2}\}$$
$$= P_1(t) + P_2(t)$$
$$= \frac{m_1}{m_1 - m_2} e^{m_2 t} - \frac{m_2}{m_1 - m_2} e^{m_1 t} (), \text{ on using (10) and (11)} \quad (12)$$

### Deductions

1. Let the system be a two-component active redundant system under repair. In this case, both the components may operate simultaneously.
Let us make a simplifying assumption that the rate of failure for each component is $\lambda$ and the rate of repair for each component is $\mu$.
$\therefore$ $\lambda_1$ = Rate of transition of the system from state 1 to state 2.
= Rate of failure of either component 1 or component 2 ($\because$ state 2 corresponds to either component operating)
$= \lambda + \lambda$ ($\because$ simultaneous failures of components are ruled out)
$= 2\lambda$
Also $\lambda_2 = \lambda$ and $\mu_2$ = Rate of repair of one of the components
$= \lambda$
Hence $R(t)$, the reliability of the system is given by (12), where $m_1, m_2$ are the roots of the equation

$$m^2 + (3\lambda + \mu)m + 2\lambda^2 = 0 \quad (13)$$

$$\text{i.e., } m_1, m_2 = \frac{1}{2}\left\{-(3\lambda + \mu) \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}\right\} \quad (14)$$

In this case,

$$\text{MTTF} = \int_0^\infty R(t)\,dt = \frac{m_1}{m_1 - m_2}\int_0^\infty e^{m_2 t}\,dt - \frac{m_2}{m_1 - m_2}\int_0^\infty e^{m_1 t}\,dt$$

$$= \frac{1}{m_1 - m_2}\left\{-\frac{m_1}{m_2} + \frac{m_2}{m_1}\right\} \quad (\because m_1, m_2 < 0)$$

$$= -\frac{(m_1 + m_2)}{m_1 m_2} = \frac{3\lambda + \mu}{2\lambda^2} \quad (15)$$

**Note** *If we put $\mu = 0$, we get $MTTF = \frac{3}{2\lambda}$, which we have derived already for a 2 component active redundant system without repair.*



2. Let the system be a two-component standby redundant system under repair.
System changes from state 1 to state 2 due to the failure of the main component. As before the standby component is assumed not to fail in the standby mode. Hence the rate of transition $\lambda_1$ from state 1 to state 2 is the same as the rate of failure of the main component viz; $\lambda$.
Similarly $\lambda_2$ can be considered as the rate of failure of the standby component viz., $\lambda$
The rate of transition from state 2 to state 1 is the same as the rate of repair of the failed main component. viz., $\mu_2 = \mu$, say.
In this case $R(t)$ is given by (12), where $\mu_1, \mu_2$ are given by the equation (4), in which $\mu_2$ is replaced by $\mu$.

$$\text{Also MTTF} = \frac{(m_1 + m_2)}{m_1 m_2} = \frac{2\lambda + \mu}{\lambda^2} = \frac{2}{\lambda} + \frac{\mu}{\lambda^2} \quad (16)$$

**Note** *If we put $\mu = 0$, we get $MTTF = \frac{2}{\lambda}$, which we have derived already for a 2 component standby redundant system without repair.*

## AVAILABILITY

Closely associated with the reliability of repairable (maintained) systems is concept of *availability*. Like reliability and maintainability, availability is also a probability.

*Availability* is defined as the probability that a component (or system) is performing its intended function at a given time '$t$' on the assumption that it is operated and maintained as per the prescribed conditions. This is referred to as *point availability* and denoted by $A(t)$.

It is to be observed that reliability is concerned with failure-free operation up to time $t$, whereas availability is concerned with the capability to operate at the point of time $t$.

If $A(t)$ is the point availability of a component (or system), then

$$A(t_2 - t_1) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} A(t)\,dt$$

is called the *interval availability* or *mission availability*.

In particular, the interval availability over the interval $(0, T)$ is given by

$$A(T) = \frac{1}{T}\int_0^T A(t)\,dt$$

Now $\lim_{T\to\infty} A(t)$ is called the *steady-state* or *asymptotic* or *long-run availability* and denoted by $A$ or $A(\infty)$.

## Availability Function of a Single Component (or System)

Let us assume that the component will be in one of two possible state: operating (state 1) or under repair (state 2). Let the component have constant failure rate $\lambda$ and constant repair rate $\mu$. The corresponding Markov state transition diagram is given in Fig. 9.14:

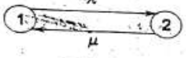


**Fig. 9.14**

The differential equation for the state probability of the component is

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + \mu P_2(t) \quad (1)$$

$$P_1(t) + P_2(t) = 1 \quad (2)$$

Using (2) in (1), we have

$$P_1'(t) + (\lambda + \mu) P_1(t) = \mu \quad (3)$$

I.F. of equation (3) $= e^{(\lambda+\mu)t}$

Solution of equation (3) is

$$e^{(\lambda+\mu)t} \cdot P_1(t) = \mu \int e^{(\lambda+\mu)t}\, dt + c$$

$$= \frac{\mu}{\lambda+\mu} e^{(\lambda+\mu)t} + c \quad (4)$$

Since the component is in state 1 initially, $P_1(0) = 1$.

Using this in (4), we get $c = 1 - \frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}$.

Using this value of $c$ in (4), we get

$$P_1(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad (5)$$

Since state 1 is the available state,

$$A(t) = P_1(t)$$

i.e.,
$$\boxed{A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$

The interval availability over $(0, T)$ is given by

$$A(T) = \frac{1}{T}\int_0^T A(t)\, dt$$



$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{(\lambda+\mu)^2 T}\{1 - e^{-(\lambda+\mu)T}\} \quad (7)$$

The steady-state availability is then given by

$$A(\infty) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{(\lambda+\mu)^2 T} \lim_{T\to\infty}\left\{\frac{1 - e^{-(\lambda+\mu)T}}{T}\right\}$$

$$= \frac{\mu}{\lambda+\mu} \quad (8)$$

$$= \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}} \quad (9)$$

**Note** *Since failures of the component are random events, the number $N(t)$ of failures in interval 't' follows a Poisson process given by*

$$P\{N(t) = r\} = e^{-\lambda t} \cdot \frac{(\lambda t)^r}{r!} \quad r = 0, 1, 2, \cdots,$$

*where $\lambda$ represents the constant failure rate of the component. Then the time between failures is a continuous R. V. that follows an exponential distribution with parameter $\lambda$. The expected value of the time between failures is given by $\frac{1}{\lambda}$ and is called mean time between failures and denoted as MTBF.)*

Hence equation (9) is also given as

$$A(\infty) = \frac{\text{MTBF}}{\text{MTBF} + \text{MTTR}} \quad (10)$$

Formulas (9) or (10) may be used even if failure and repair distributions are not exponential.

**Note** *In most situations, repair rates are much larger than failure rates. Hence $\frac{\lambda}{\mu}$ is very small.*

*From (8), we have* $A(\infty) = \frac{1}{1 + \frac{\lambda}{\mu}} = \left(1 + \frac{\lambda}{\mu}\right)^{-1}$

$$= 1 - \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 - \cdots\infty$$

*i.e.,* $A(\infty) \cong 1 - \frac{\lambda}{\mu}$, *omitting higher power of* $\frac{\lambda}{\mu}$.

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Now the component unavailability is given by

$$\bar{A}(t) = 1 - A(t) \text{ or } P_2(t)$$

$$= 1 - \frac{\mu}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$= \frac{\lambda}{\lambda+\mu} \{1 - e^{-(\lambda+\mu)t}\}$$

$$\therefore \bar{A}(\infty) = \frac{\lambda}{\lambda+\mu}$$

## System Availability

If we consider a non-redundant system consisting of $n$ independent components connected in series, then the system reliability $A_s(t)$ is given by

$$A_s(t) = A_1(t) \cdot A_2(t) \ldots A_n(t)$$

[$\because$ all the components must be available for the system to be available]

For a standby system consisting of $n$ independent components connected in parallel, the system availability $A_s(t)$ is given by

$$A_s(t) = 1 - \{1 - A_1(t)\}\{1 - A_2(t)\} \ldots \{1 - A_n(t)\}$$

[$\because$ all the components must be unavailable for the system to be unavailable]

where $A_i(t)$ is the availability of the $i$th component.

## Availability of a Two Component Stand by System with Repair by Markov Analysis

Let us consider a two component standby system in which the failure rates $\lambda_1$ and $\lambda_2$ and common repair rate $\mu$ are constant.

As before we assume that the standby component does not fail in standby mode, but the repair of the standby unit is also permitted. Also we assume that only one repair, viz., the repair of the main unit or the standby unit is possible. In other words we assume that there is a single repair person, who can repair the main unit before the standby unit fails.

The corresponding Markov state transition diagram is given in Fig. 9.15, where in state 1 main unit is operating while the standby unit is not, in state 2 main unit has failed while the standby unit has become operative and in state 3 standby also has failed.



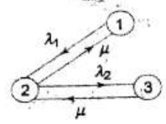


Fig. 9.15

The differential equations for the state probabilities are

$$P_1'(t) = -\lambda_1 P_1(t) + \mu P_2(t) \quad (1)$$

$$P_2'(t) = \lambda_1 P_1(t) + \mu P_3(t) - (\lambda_2 + \mu) P_2(t) \quad (2)$$

and $$P_1(t) + P_2(t) + P_3(t) = 1 \quad (3)$$

The availability of the system $A_s(t)$ is given by $A_s(t) = P_1(t) + P_2(t)$.

If we solve the equations (1), (2) and (3) simultaneously either directly or by using Laplace transforms, we can get

$$A_s(t) = \left\{1 - \frac{\lambda_1\lambda_2}{m_1 m_2}\right\} - \frac{\lambda_1\lambda_2}{m_1 - m_2}\left\{\frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2}\right\},$$

where $m_1, m_2$ are the roots of the equation

$$m^2 + (\lambda_1 + \lambda_2 + 2\mu)m + (\lambda_1\lambda_2 + \lambda_1\mu + \mu^2) = 0$$

The initial conditions for solving the above equations are $P_1(0) = 1$ and $P_2(0) = P_3(0) = 0$.

**Note** *The solution of the above equations has been avoided, as the steady-state availability of the standby system is frequently required.*

Now in the steady-state $P_i(t)$ does not depend on $t$ and $P_i'(t) = 0$. Hence the above equations become

$$-\lambda_1 P_1 + \mu P_2 = 0 \quad (4)$$

$$\lambda_1 P_1 + \mu P_3 - (\lambda_2 + \mu) P_2 = 0 \quad (5)$$

and $$P_1 + P_2 + P_3 = 1 \quad (6)$$

Using (6) in (5), we have

$$(\mu - \lambda_1)P_1 + (2\mu + \lambda_2)P_2 = \mu \quad (7)$$

Using (4) in (7), we have

$$(\mu^2 + \lambda_1\mu + \lambda_1\lambda_2)P_1 = \mu^2$$

$$\therefore \quad P_1 = \frac{\mu^2}{\mu^2 + \lambda_1\mu + \lambda_1\lambda_2} \text{ or } \left(1 + \frac{\lambda_1}{\mu} + \frac{\lambda_1\lambda_2}{\mu^2}\right)^{-1} \quad (8)$$

$$P_2 = \frac{\lambda_1}{\mu} P_1 \quad (9)$$

and $$P_3 = 1 - P_1 - \frac{\lambda_1}{\mu} P_1$$

$$= \frac{1}{\mu^2}[\mu^2 - \mu^2 P_1 - \lambda_1 \mu P_1]$$

$$= \frac{1}{\mu^2}[(\mu^2 + \lambda_1\mu + \lambda_1\lambda_2)P_1 - \mu^2 P_1 - \lambda_1 \mu P_1]$$

$$= \frac{\lambda_1\lambda_2}{\mu^2} P_1 \quad (10)$$

The steady-state availability of the standby system is then given by

$$A_s(\infty) = P_1 + P_2$$

$$= \frac{\lambda_1\mu + \mu^2}{\mu^2 + \lambda_1\mu + \lambda_1\lambda_2}$$

We can alternatively obtain this value of $A_s(\infty)$ from that of $A_s(t)$ by letting $t \to \infty$ and using $m_1 m_2 = \mu^2 + \lambda_1 \mu + \lambda_1 \lambda_2$

## Worked Example 9(C)

### Example 1

If a device has a failure rate of

$\lambda(t) = (0.015 + 0.02t)$/year, where $t$ is in years,

(a) Calculate the reliability for a 5 year design life, assuming that no maintenance is performed.

(b) Calculate the reliability for a 5 year design life, assuming that annual preventive maintenance restores the device to an as-good as new condition.

(c) Repeat part (b) assuming that there is a 5% chance that the preventive maintenance will cause immediate failure.

(a) $$R(t) = e^{-\int_0^t \lambda(t)\,dt}$$

$\therefore$ $$R(5) = e^{-\int_0^5 (0.015 + 0.02t)\,dt}$$

$$= e^{-(0.015 \times 5 + 0.01 \times 25)}$$

$$= e^{-0.325} = 0.7225$$

(b) Since annual preventive maintenance is performed, there will be 4 preventive maintenances in the first 5 years.

$$R_M(t) = \{R(T)\}^n \times R(t - nT), \text{ after } n \text{ maintenances}$$

Here $t = 5$, $T = 1$ and $n = 4$

$\therefore$ $$R_M(5) = \{R(1)\}^4 \times R(5-4)$$

$$= \{R(1)\}^5$$

$$= \{e^{-0.025}\}^5, \text{ using (1)}$$

$$= 0.8825.$$

(c) $P$ {preventive maintenance causes immediate failure} = 0.05

$\therefore$ $P$ {the device survives after each preventive maintenance} = 0.95

As there are 4 maintenances,

$R_M(5) = R_M(5)$ without breakdown $\times$ probability of no breakdown in 5 years

$$= 0.8825 \times (0.95)^4$$

$$= 0.7188.$$

### Example 2

The time to failure (in hours) of a piece of equipment is uniformly distributed over (0, 1000) hours.

(a) Determine the MTTF.

(b) Determine the MTTF if preventive maintenance will restore the system to as good as new condition and is performed every 100 operating hours.

(c) Compare the reliability with and without preventive maintenance at 225 operating hours. Assume the 100 hour maintenance interval and a maintenance-induced failure probability of 0.01 each time preventive maintenance is performed.

(d) Is there a significant improvement in reliability if a 50 hour preventive maintenance interval is assumed?

The *pdf* of the time to failure is given by

$$f(t) = \frac{1}{1000}, \text{ in } 0 \le t \le 1000.$$

(a) $$\text{MTTF} = \int_0^{1000} t\frac{1}{1000}\,dt = \frac{1}{1000} \times \left(\frac{t^2}{2}\right)_0^{1000} = 500 \text{ hours.}$$

(b) $$(\text{MTTF})_M = \int_0^{100} R(t)\,dt \div \{1 - R(100)\}, \text{ where}$$

$$R(t) = \int_t^{1000} \frac{1}{1000}\,dt$$

$$= 1 - 0.001\,t$$

i.e., $$(\text{MTTF})_M = \frac{\int_0^{100}(1-0.001t)\,dt}{1-(1-0.001\times 100)}$$

$$= \frac{1}{0.1}\left\{t - 0.001\frac{t^2}{2}\right\}_0^{100} = 950 \text{ hours.}$$

(c) $R(225) = 1 - 0.001 \times 225 = 0.775$

$R_M(225) = \{R(T)\}^2 \times R(t-2T) \times (1-p)^2$ [see (c) in the previous example], where $p$ is the probability of maintenance induced failure.

**Note** *There will be $n = 2$ preventive maintenances in (0, 225)*

$$= \{R(100)\}^2 \times R(25) \times (1-0.01)^2$$

$$= (0.9)^2 \times (0.975) \times (0.99)^2$$

$$= 0.774$$

$$R(225) \cong R_M(225)$$

(d) If $T = 50$, $n$ = the number of preventive maintenance = 4.

$$\therefore R'_M(225) = \{R(50)\}^4 \times R(225-200)$$

$$= (0.95)^4 \times 0.975 = 0.794$$

If there is no maintenance induced failure probability, the reliability is improved when $T = 50$.

## Example 3

A reliability engineer has determined that the hazard rate function for a milling machine is $\lambda(t) = 0.0004521t^{0.8}$, $t \geq 0$, where $t$ is measured in years. Determine which of the following options will provide the greatest reliability over the machine's 20 years operating life.

*Option A* : Do nothing-operate the machine until in fails.

*Option B*: An annual preventive maintenance program (with no maintenance-induced failures)

*Option C*: Operate a second machine in parallel with the first (active redundant).

**Option A**

$$R_A(t) = \exp\left[-\int_0^t 0.0004521\,t^{0.8}\,dt\right]$$

$$= \exp\left[-\frac{0.0004521}{1.8} \times t^{1.8}\right]$$



$$\therefore R_A(20) = \exp\left[-\frac{0.0004521}{1.8} \times 20^{1.8}\right] = 0.9463$$

**Option B**

$$R_B(t) = \{R(T)\}^n \cdot R(t - nT)$$

$$= \{R(1)\}^{19} \times R(20-19) \quad [\because \text{there will be 19 maintenances in } (0, 20)]$$

$$= \left\{\exp\left[\frac{-0.0004521}{1.8}\right]\right\}^{20} = 0.9950$$

**Option C**

$$R_C(20) = 1 - \{1 - R(20)\}^2$$

$$= 1 - \{1 - 0.9463\}^2 = 0.9971$$

Option C will give the greatest reliability.

## Example 4

The time to repair a power generator is best described by its *pdf*

$$m(t) = \frac{t^2}{333},\ 1 \leq t \leq 10 \text{ hours}$$

(a) Find the probability that a repair will be completed in 6 hours.
(b) What is the MTTR?
(c) Find the repair rate.

(a) $P(T < 6) = P(1 \leq T < 6)$, where $T$ is the time to repair

$$= \int_1^6 m(t)\,dt$$

$$= \int_1^6 \frac{t^2}{333}\,dt = \left(\frac{t^3}{999}\right)_1^6 = 0.2152$$

(b) $$\text{MTTR} = \int_0^\infty t\,m(t)\,dt = \int_1^{10} \frac{t^3}{333}\,dt = \left(\frac{t^4}{4\times 333}\right)_1^{10}$$

$$= 7.5 \text{ hours}$$

(c) Repair rate $= \mu(t) = \dfrac{m(t)}{1 - M(t)}$

$$= \frac{t^2/333}{\int_t^{10} \frac{t^2}{333}\,dt} = \frac{t^2/333}{\frac{1}{999}(10^3 - t^3)}$$

$$= \frac{3t^2}{1000 - t^3} \text{ per hour.}$$

## Example 5

The time to repair of an equipment follows a lognormal distribution with a MTTR of 2 hours and a shape parameter of 0.2.

(a) Find the median time to repair.

(b) Determine the repair time such that 95% of the repairs will be accomplished within the specified time.

(c) Determine the probability that a repair will be completed within 100 minutes.

(a) MTTR of a lognormal distribution is given by

MTTR = $t_M \exp(s^2/2)$, where $t_M$ is the median and $s$ is the shape parameters

$$\therefore t_M = \text{MTTR} \times \exp\left(-\frac{s^2}{2}\right)$$

$$= 2 \times e^{-0.02} = 1.96 \text{ hours.}$$

(b) Let $T$ represent the time to repair.

$$P\{T \le T_R\} = 0.95$$

i.e., $\int_0^{T_R} f(t)\,dt = 0.95$, where $f(t)$ is the *pdf* of the lognormal distribution

i.e., $\int_{\frac{1}{s}\log\left(\frac{0}{t_M}\right)}^{\frac{1}{s}\log\left(\frac{T_R}{t_M}\right)} \phi(z)\,dz = 0.95$, where $\phi(z)$ is the standard normal density function

i.e., $\int_{-\infty}^{5\log\left(\frac{T_R}{1.96}\right)} \phi(z)\,dz = 0.95$

$$\therefore \quad 5\log\left(\frac{T_R}{1.96}\right) = 1.645, \text{ from the normal tables}$$

$$\therefore \quad \frac{T_R}{1.96} = e^{0.329}$$

$$\therefore \quad T_R = 2.72 \text{ hours.}$$

(c) $P\left\{T \le \frac{100}{60}\right\} = \int_0^{5/3} f(t)\,dt$



$$= \int_{-\infty}^{5\log\left(\frac{1.67}{1.96}\right)} \phi(z)\,dz$$

$$= \int_{-\infty}^{-0.8} \phi(z)\,dz = 0.071.$$

## Example 6

A mechanical pumping device has a constant failure rate of 0.023 failure per hour and an exponential repair time with a mean of 10 hours. If two pumps operate in an active redundant configuration, determine the system MTTF and the system reliability for 72 hours.

Refer to deduction (1) under the discussion of reliability of a two component redundant system with repair (using Markov analysis).

For the active redundant configuration,

$$R(t) = \frac{m_1}{m_1 - m_2} e^{m_2 t} - \frac{m_2}{m_1 - m_2} e^{m_1 t} \quad (1)$$

where $m_1, m_2 = \frac{1}{2}\left\{-(3\lambda + \mu) \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}\right\}$

Here $\lambda = 0.023$ and $\mu = \frac{1}{10} = 0.1$

$$\therefore \quad m_1, m_2 = \frac{1}{2}\left\{-(0.069 + 0.1) \pm \sqrt{(0.023)^2 + 6 \times 0.023 \times 0.1 + (0.1)^2}\right\}$$

$$= -0.0065, -0.1625$$

Using these values in (1), we have

$$R(t) = \frac{-0.0065}{0.156} e^{-0.1625t} + \frac{0.1625}{0.156} e^{-0.0065t}$$

$$R(72) = -0.0417 \times e^{-11.7} + 1.0417 \times e^{-0.468}$$

$$= 0.6524$$

$$\text{MTTF} = \frac{3\lambda + \mu}{2\lambda^2} = \frac{3 \times 0.023 + 0.1}{2 \times (0.023)^2} = 159.7 \text{ hours}$$

## Example 7

An engine health monitoring system consists of a primary unit and a standby unit. The MTTF of the primary unit is 1000 operating hours and the MTTF of the standby unit is 333 hours when in operation. There are no failures while the backup unit is in standby.

If the primary unit may be repaired at a repair rate of 0.01 per hour, while the standby unit is operating, estimate the design life for a reliability of 0.90.

Refer to the discussion of reliability of a two component redundant system with repair (using Markov analysis).

For the standby randundant system,

$$R(t) = \frac{m_1}{m_1 - m_2} e^{m_2 t} - \frac{m_2}{m_1 - m_2} e^{m_1 t} \quad (1)$$

where $m_1$ and $m_2$ are the roots of the equation

$$m^2 + (\lambda_1 + \lambda_2 + \mu)m + \lambda_1 \lambda_2 = 0 \quad (2)$$

Here $\lambda_1 = \frac{1}{1000} = 0.001$/hour; $\lambda_2 = \frac{1}{333} = 0.003$/hour and $\mu = 0.01$/hour.

Using these values in (2), we have

$$m_2 + 0.014\, m + 0.000003 = 0$$

$$\therefore \quad m_1, m_2 = \frac{-0.014 \pm \sqrt{(0.014)^2 - 4 \times 0.000003}}{2}$$

$$= -0.00022, -0.01378$$

Using these values in (1), we get

$$R(t) = -0.01622 \times e^{-0.01378t} + 1.01622 \times e^{-0.00022t}$$

When the reliability is 0.90, the design life $D$ is given by

$$1.01622 \times e^{-0.00022D} - 0.01622 \times e^{-0.01378D} = 0.90$$

Solving this equation by trials, we get

$$D = 550 \text{ hours.}$$

## Example 8

Reliability testing has indicated that a voltage inverter has a 6 month reliability of 0.87 without repair facility. If repair facility is made available with an MTTR of 2.2 months, compute the availability over the 6-month period. (Assume constant failure and repair rates)

For constant failure rate $\lambda$, reliability is given by $R(t) = e^{-\lambda t}$.

As $R(6) = 0.87$, $e^{-6\lambda} = 0.87$

$\therefore \quad \lambda = 0.0232$/month

$$\text{MTTR} = \frac{1}{\mu} = 2.2 \therefore \mu = 0.4545/\text{month}$$

Interval availability over $(0, T)$ is given by



$$A(T) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{(\lambda + \mu)^2 T}\{1 - e^{-(\lambda + \mu)T}\}$$

$$\therefore \quad A(6) = \frac{0.4545}{0.4777} + \frac{0.0232}{(0.4777)^2 \times 6}\{1 - e^{-0.4777 \times 6}\}$$

$$= 0.967$$

## Example 9

A critical communications relay has a constant failure rate of 0.1 per day. Once it has failed, the mean time to repair is 2.5 days (the repair rate is constant).

(a) What are the point availability at the end of 2 days, the interval availability over a 2-day mission, starting from zero and the steady-state availability?

(b) If two communication relays operate in series, compute the availability at the end of 2 days.

(c) If they operate in parallel, compute the steady-state availability of the system.

(d) If one communication relay operates in a standby mode with no failure in standby, what is the steady-state availability?

$\lambda = 0.1$ per day; $\frac{1}{\mu} = 2.5 \therefore \mu = 0.4$ per day

(a) $$A_P(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

$$\therefore \quad A_P(2) = \frac{0.4}{0.5} + \frac{0.1}{0.5} \cdot e^{-0.5 \times 2} = 0.8736$$

$$A_I(T) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{(\lambda + \mu)^2 \times T}\{1 - e^{-(\lambda + \mu)T}\}$$

$$\therefore \quad A_I(2) = \frac{0.4}{0.5} + \frac{0.1}{(0.5)^2 \times 2}\{1 - e^{-0.5 \times 2}\} = 0.9264$$

$$A(\infty) = \frac{\mu}{\lambda + \mu} = \frac{0.4}{0.5} = 0.8$$

(b) $A_s(2) = \{A_P(2)\}^2 = (0.8736)^2 = 0.7632$

(c) $A_s(\infty) = 1 - \{1 - A(\infty)\}^2$

$= 1 - \{1 - 0.8\}^2 = 0.96$

(d) For the standby redundant system,

$$A_s(\infty) = \frac{\lambda_1 \mu + \mu^2}{\mu^2 + \lambda_1 \mu + \lambda_1 \lambda_2}; \text{ Here } \lambda_1 = \lambda_2 = 0.1 \text{ and } \mu = 0.4$$

i.e., $A_s(\infty) = \dfrac{0.04 + 0.16}{0.16 + 0.04 + 0.01} = \dfrac{0.20}{0.21} = 0.9524$

## Example 10

A new computer has a constant failure rate of 0.02 per day (assuming continuous use) and a constant repair rate of 0.1 per day.

(a) Compute the interval availability for the first 30 days and the steady-state availability.

(b) Determine the steady-state availability if a standby unit is purchased. Assume no failures in standby.

(c) If both units are active, what is the steady-state availability?

$\lambda = 0.02/\text{day}; \ \mu = 0.1/\text{day}.$

(a) $$A_I(T) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{(\lambda+\mu)^2 \times T}\{1 - e^{-(\lambda+\mu)T}\}$$

$$\therefore A_I(30) = \frac{0.1}{0.12} + \frac{0.02}{(0.12)^2 \times 30}\{1 - e^{-0.12 \times 30}\}$$

$$= 0.8784$$

$$A(\infty) = \frac{\mu}{\lambda+\mu} = \frac{0.1}{0.12} = 0.8333$$

(b) For the standby redundant system,

$$A_s(\infty) = \frac{\lambda\mu + \mu^2}{\mu^2 + \lambda\mu + \lambda^2} = \frac{0.002 + 0.01}{0.01 + 0.002 + 0.0004}$$

$$= 0.9677$$

For the active redundant system,

$$A_s(\infty) = 1 - \{1 - A(\infty)\}^2$$

$$= 1 - \{1 - 0.8333\}^2$$

$$= 0.9722.$$

## Example 11

**An office machine has a time to failure distribution that is lognormal with shape parameter $s = 0.86$ and a scale parameter $t_M = 40$ operating hours. The repair distribution is normal with a mean of 3.5 hours and a standard deviation of 1.8 hours. Find the steady-state availability of the machine.**

For the lognormal failure distribution, mean = MTTF = $t_M \exp(s^2/2)$

$$= 40 \exp\{(0.86)^2/2\} = 40 \times e^{0.3698} = 57.9 \text{ hours}$$



Mean of the normal repair distribution

$$= \text{MTTR} = 3.5 \text{ hours}$$

Now $$A(\infty) = \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}} = \frac{57.9}{57.9 + 3.5}$$

$$= 0.943.$$

## Example 12

The distribution of the time to failure of a component is Weibull with $\beta = 2.4$ and $\theta = 400$ hours and the repair distribution is lognormal with $t_M = 4.8$ hours and $s = 1.2$. Find the steady-state availability.

For the Weibull failure distribution,

$$\text{Mean} = \text{MTTF} = \theta \left\lceil\left(1 + \frac{1}{\beta}\right)\right.$$

$$= 400 \left\lceil\left(1 + \frac{1}{2.4}\right)\right.$$

$$= 400 \times \lceil(1.42)$$

$$= 400 \times 0.88636$$

$$= 354.5 \text{ hours}$$

For the lognormal repair distribution,

$$\text{Mean} = \text{MTTR} = t_M \exp(s^2/2)$$

$$= 4.8 \times \exp\{(1.2)^2/2\}$$

$$= 9.86 \text{ hours}$$

$$A(\infty) = \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}}$$

$$= \frac{345.5}{354.5 + 9.86} = 0.9729$$

## Example 13

A system may be found in one of the three states: operating, degraded or failed. When operating, it fails at the constant rate of 1 per day and becomes degraded at the rate of 1 per day. If degraded, its failure rate increases to 2 per day. Repair occurs only in the failed mode and restores the system to the operating state with a repair rate of 4 per day. If the operating and degraded states are considered the available states, determine the steady-state availability.

**Note** *A system is said to be in degraded state, if it continues to perform its function but at a less than specified operating level.*
*For example, a copying machine may not be able to automatically feed originals and may require manual operation or a computer system may not be able to access all of its direct access storage devices.)*

The situation of this problem may be represented by the Markov state transition diagram as in Fig. 9.16, in which states 1, 2 and 3 represent respectively the operating, degraded and failed states of the system:

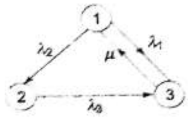


Fig. 9.16

The differential equations for the state probabilities are

$$P_1'(t) = -(\lambda_1 + \lambda_2) P_1(t) + \mu \cdot P_3(t) \quad (1)$$

$$P_2'(t) = \lambda_2 P_1(t) - \lambda_3 P_2(t) \quad (2)$$

$$P_1(t) + P_2(t) + P_3(t) = 1 \quad (3)$$

When the system is in steady-state,

$$P_i'(t) = 0 \text{ and } P_i(t) = P_i \text{(constant)}; \; i = 1, 2, 3$$

∴ The above equations become

$$-(\lambda_1 + \lambda_2) P_1 + \mu \cdot P_3 = 0 \quad (4)$$

$$\lambda_2 P_1 - \lambda_3 P_2 = 0 \quad (5)$$

$$P_1 + P_2 + P_3 = 1 \quad (6)$$

Solving the equations (4), (5) and (6), we get

$$P_1 = \frac{\lambda_3 \mu}{(\lambda_2 + \lambda_3)\mu + \lambda_3(\lambda_1 + \lambda_2)} \text{ and } P_2 = \frac{\lambda_2 \mu}{(\lambda_2 + \lambda_3)\mu + \lambda_3(\lambda_1 + \lambda_2)}$$

Now $A(\infty) = P_1 + P_2$

$$= \frac{(\lambda_2 + \lambda_3)\mu}{(\lambda_2 + \lambda_3)\mu + \lambda_3(\lambda_1 + \lambda_2)}$$

$$= \frac{(1+2) \times 4}{(1+2) \times 4 + 2 \times (1+1)}, \text{ using the given values}$$

$$= 0.75$$



## Example 14

For a two component standby system with repair permitted for either component with a single repair crew and no failures in standby, the failure and repair rates are given by

$\lambda_1$ = rate of failure of main unit = 0.002
$\lambda_2$ = rate of failure of standby unit = 0.001
$\mu$ = repair rate for either unit = 0.01

Compute the steady-state availability of the system.

The steady-state probabilities are given by

$$P_1 = \left(1 + \frac{\lambda_1}{\mu} + \frac{\lambda_1 \lambda_2}{\mu^2}\right)^{-1} \text{ and } P_2 = \frac{\lambda_1}{\mu} P_1$$

[Refer to the discussion of availability of a two component redundant system with repair by Markov analysis]

$$\therefore \quad P_1 = \left(1 + \frac{0.002}{0.01} + \frac{0.000002}{0.0001}\right)^{-1} = 0.8197$$

$$P_2 = \frac{0.002}{0.01} \times 0.8197 = 0.1639$$

Now $A_s(\infty) = P_1 + P_2 = 0.9836.$

## Example 15

A generator system consists of a primary unit and a standby unit. The primary fails at a constant rate of 2 per month and the standby unit fails only when online at a constant rate of 4 per month. Repair can begin only when both units have failed. Both units are repaired at the same time with an MTTR of $\frac{2}{3}$ month. Derive the steady-state equations for the state probabilities and solve for the system availability.

The Markov state transition diagram for this problem is given in Fig. 9.17:

In state 1, main unit is operating and the other is in standby mode.
In state 2, main unit has failed and the standby unit operates.
In state 3, standby unit has also failed in addition to the main unit.

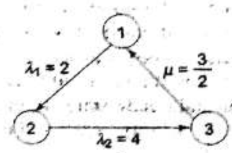


Fig. 9.17

The steady-state probabilities of the three states are

$$-\lambda_1 P_1 + \mu P_3 = 0$$

$$\lambda_1 P_1 - \lambda_2 P_2 = 0 \quad (1)$$

$$\lambda_2 P_2 - \mu P_3 = 0 \quad (2)$$

or

$$P_1 + P_2 + P_3 = 1 \quad (3)$$

Solving equations (1), (2) and (4), we get (4)

$$P_1 = \frac{\mu\lambda_2}{\mu(\lambda_1+\lambda_2)+\lambda_1\lambda_2} \text{ and } P_2 = \frac{\mu\lambda_1}{\mu(\lambda_1+\lambda_2)+\lambda_1\lambda_2}$$

The system steady-state availability is given by

$$A_s(\infty) = P_1 + P_2$$

$$= \frac{\mu(\lambda_1+\lambda_2)}{\mu(\lambda_1+\lambda_2)+\lambda_1\lambda_2}$$

$$= \frac{\frac{3}{2}\times(2+4)}{\frac{3}{2}\times(2+4)+8} = \frac{9}{17} = 0.5294.$$

## Exercise 9(C)

**Part A** (Short answer questions)

1. What do you mean by preventive maintenance and corrective maintenance?
2. How are the reliabilities with and without maintenance related?
3. Derive the formula for MTTF of a system with preventive maintenance.
4. Show that preventive maintenance does not improve the reliability of a system with constant failure rate.
5. When will the reliability of a Weibull system improve due to preventive maintenance?
6. What do you mean by maintainability?
7. How are the maintainability and repair rate related?
8. Find the MTTR of a system with constant repair rate.
9. What is the value of MTTF of a two component redundant system with constant failure rate $\lambda$ and constant repair rate $\mu$, when the system is (i) active redundant (ii) standby redundant?
10. What is availability? How is it different from reliability?
11. Define point, mission and steady-state availabilities.
12. Express steady-state availability in terms of mean times to failure and repair.



13. Express the system availability in terms of component availabilities, when the system is (i) non-redundant (ii) redundant.
14. Write down the formula for the steady-state availability of a two component standby redundant system with repair. State the conditions under which this formula may be used.

**Part B**

15. A flange bolt wears out because of fatigue in accordance with the lognormal distribution with MTTF = 10,000 hours and $s = 2$. If preventive maintenance consists of periodically replacing the bolt, what is the reliability at 550 hours with and without preventive maintenance? Assume that the bolts are replaced every 100 operating hours.
16. Preventive maintenance is to be performed every 5 days on a system having a rectangular failure distribution in (0, 100) days. Derive the reliability function for the system under preventive maintenance. Compare the MTTF and the reliability at 17 days with and without preventive maintenance.
17. A compressor has a Weibull failure process with $\beta = 2$ and $\theta = 100$ days. If preventive maintenance is resorted to every 20 days, find the reliability for 90 days. Find also the design lives at 0.90 reliability without and with preventive maintenance.
18. For a component having a lognormal failure distribution with $s = 1$ and median 5000 hours, prove that the reliability at 5000 hours increases by about 70%, if we perform preventive maintenance at intervals of 500 hours.
19. The *pdf* of the time to failure in years of the drive train on a Rapit Transit Authority bus is given by

$$f(t) = 0.2 - 0.02\,t,\ 0 \le t \le 10 \text{ years.}$$

   (a) If the bus undergoes preventive maintenance every 6 months that restores it to as-good-as-new condition, determine its reliability at the end of a 15 month warranty.
   (b) Compute the MTTF under the preventive maintenance plan given above.
20. Find the MTTF for an active two-component redundant system with constant failure rate $\lambda$ under preventive maintenance done at regular intervals of $T$. Deduce the same when there is no preventive maintenance.
[**Hint:** $R(t) = 2e^{-\lambda t} - e^{-2\lambda t}$]
21. The time to repair an engine module is lognormal with $s = 1.21$. Specifications require 90% of the repairs to be accomplished within 10 hours. Determine the necessary median and mean time to repair.
22. If the failure time of an equipment is lognormal with the shape parameter 0.7 and if it is designed to have a 90% chance of operating for 5 years, find its MTTF.
When it fails, the time to repair follows a lognormal distribution with a

mean of 2 hours and a shape parameter of 1, what is the probability that it is repaired within 4 hours?

23. A computer system consists of two active parallel processors each having a constant failure rate of 0.5 failure per day. Repair of a failed processor requires an average of $\frac{1}{2}$ a day (exponential distribution). Find the system reliability for a single day and the system MTTF. Find the same if there is no repair facility.
24. An on-board computer system has, through the use of built-in test equipment, the capability of being restored when a failure occurs. A standby computer is available for use whenever the primary fails. Assuming that $\lambda_1 = 0.0005$ per hour, $\lambda_2 = 0.002$ per hour and $\mu = 0.1$ per hour, determine the system reliability at 1000 hours and at 2000 hours.
25. An airlines maintains an on-line reservation system with a standby computer available if the primary fails. The one-line system fails at the constant rate of once per day while the standby fails (only when on-line) at the constant rate of twice per day. If the primary unit may be repaired at a constant rate with an MTTR of 0.5 of a day, what is the single day reliability?
26. A computer has an MTTF = 34 hours and an MTTR = 2.5 hours. What is the steady-state availability? If the MTTR is reduced to 1.5 hours, what MTTF can be tolerated without decreasing the steady-state availability of the computer?
27. A component has MTBF = 200 hours and MTTR = 10 hours with both failure and repair distributions exponential. Find the availability of the component in the interval (0, 10) hours, at the end of 10 hours and after a long time.
28. Given two components, each having a constant failure rate of 0.10 failure per hour and a constant repair rate of 0.20 repair per hour, compute point and interval availability for a 10 hour mission and steady-state availability for both series and parallel configurations.
29. For the standby redundant system given in problem (25) above, find the availability of the system at the end of one day.
30. In example (9), determine how many days the relay must be operating in order for the point availability to be within 0.001 of the steady-state availability.
31. If the system availability for a standby system with repair allowed for either component with a single repair crew and no failure while in standby mode is 0.9, what is the maximum acceptable value of the failure to repair rate ratio $\frac{\lambda}{\mu}$? (Assume that $\lambda_1 = \lambda_2 = \lambda$)
32. A system has a steady-state availability of 0.90. Two such systems, each with its own repair crew, are placed in parallel. What is the steady-state availability (a) for a standby parallel configuration with perfect switching and no failure of the unit in standby; (b) for an active parallel configuration?
33. In Example (13), if $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ and $\mu = 10$, find the steady-state availability.
34. A two-component active redundant system can be repaired only after both units have failed. Only one unit can then be repaired at a time. The unit have constant failure rates of $\lambda_1$ and $\lambda_2$ respectively and the mean repair time to complete both repairs is $\frac{1}{\mu}$. Find an expression for steady-state availability of the system.
35. A radar unit consists of 2 active redundant primary components: a power supply and a transceiver. The power supply has a constant failure rate of 0.0031 failure per operating hour and takes an average of 5 hours to repair (constant repair rate). The transceiver fails on the average every 100 operating hours (constant failure rate) and takes 3 hours to repair (constant repair rate). What is the point availability of the radar unit after 10 hours of use? How much does it differ from the steady-state availability?



## ANSWERS

### Exercise 9(A)

21. (a) $16/(t+4)^2$; (b) $2/(t+4)$; (c) 64
22. (a) 0.46; (b) 0.045 yr; (c) 19.955 yrs.
23. 0.905; 0.928 24. $e^{-t^2/5}$; 0.0014; 3.32 months
25. $\frac{1}{100}(100-t)$; $\frac{1}{100-t}$; 50 ; 50
26. 0.001; 750 hours; 215.4 hours; 0.00027/hour.
27. yes; by 0.09 28. 47 days 29. $b \log a$
31. 0.0246 32. 0.75; 0.89 33. 8; 138.8%
34. 0.85; 1342 hours; 447 hours; 10629 hours; 0.3945
35. 15.9 days 36. 99.9; 7701; 5075 37. 4 yrs; 3.16 years
38. 128 years; (112.6, 117.9) years
39. 0.7852; 0.8579
40. 964.8 hours; 2432; 1604.5; 0.0084

### Exercise 9(B)

15. 0.4446 16. 0.85 17. 0.9998
18. 0.999 19. 0.8495 20. 0.9290
21. 0.9630; 40.6 days 22. 5588 years
23. 0.8068; 0.24 year
24. 585, 292, 195 years; 0.9975; 147 years

25. 8837.5 hours; 3417 hours
26. $n = 20$; 922 hours
27. $R_1 = 0.9909$; $R_2 = 0.9949$; system (ii) has higher reliability
28. 8
29. 50 hours
30. 4; 0.2944; 0.4096
31. $n$ subsystems in series
32. 0.9649
33. 0.9856
34. 0.7$L$
35. (a) $\frac{1}{4\lambda^2}$; (b) $\frac{5}{4\lambda^2}$
36. 4
37. (a) 0.9769; (b) 0.998
38. To be added to the low level redundancy system; 0.9830
39. (i) system design life = 10 × machine design life;
    (ii) system design life = 5 × machine design life
40. 0.9537; 13.3 years

## Exercise 9(C)

15. 0.6736; 0.5707
16. $R_M(t) = (0.95)^n \times \{(1 + 0.05\, n) - 0.01t\}$; $R_M(17) = 0.8402$;
    $R(17) = 0.83$; $(\text{MTTF})_M = 97.5$ days; MTTF = 50 days
17. 0.8437; 32.5 days; 55.9 days
19. 0.7743; 6.3 months
20. $\frac{3 - e^{-\lambda T}}{2\lambda}$; $\frac{3}{2\lambda}$
21. Median = 0.6731 hours; MTTR = 1.3996 hours
22. MTTF = 2.5984 years; 0.9995
23. $R = 0.90$; MTTF = 7 days, with repair
    $R = 0.845$; MTTF = 3 days, without repair
24. 0.9904; 0.9808
25. 0.7125
26. 0.9315; 20.4 hours
27. 0.981; 0.969; 0.952
28. 0.468; 0.596 and 0.445 for series configuration 0.900; 0.948 and 0.889 for parallel configuration
29. 0.7125
30. 11 days
31. 0.393
32. 0.989; 0.990
33. 0.8475
34. $A(\infty) = \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right) P_1$ where $P_1 = \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1 + \lambda_2}{\mu}\right)^{-1}$
35. $A_s(10) = 0.9966$; $A_s(\infty) = 0.9996$