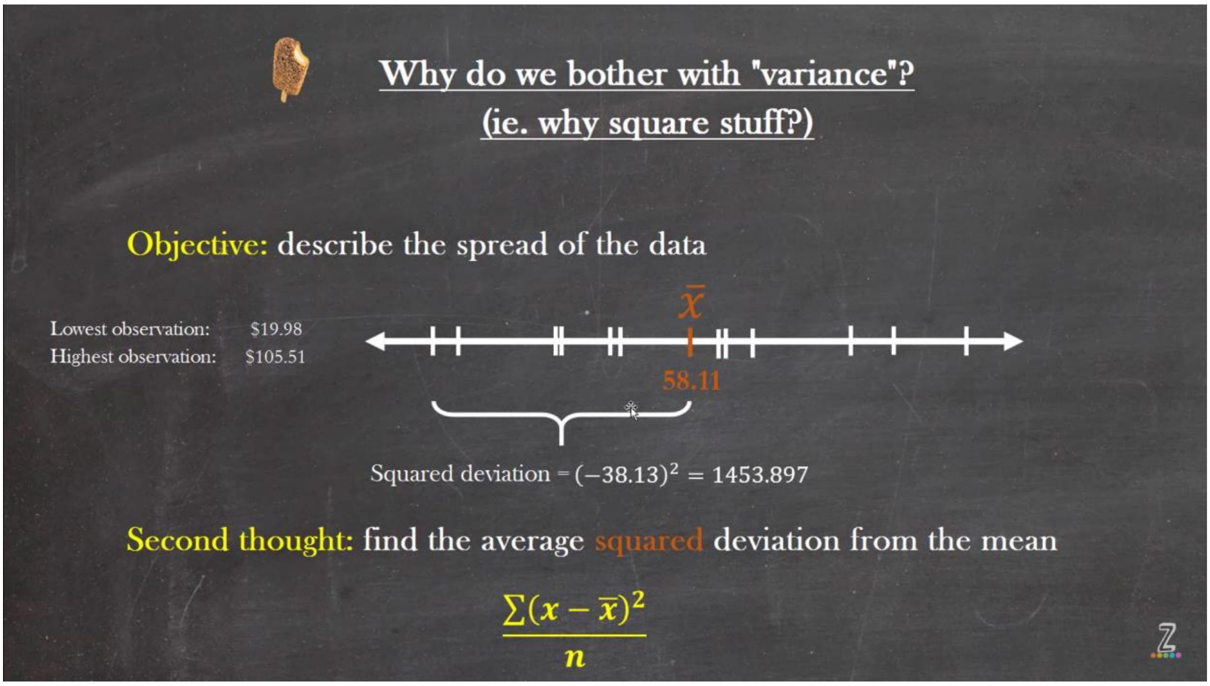
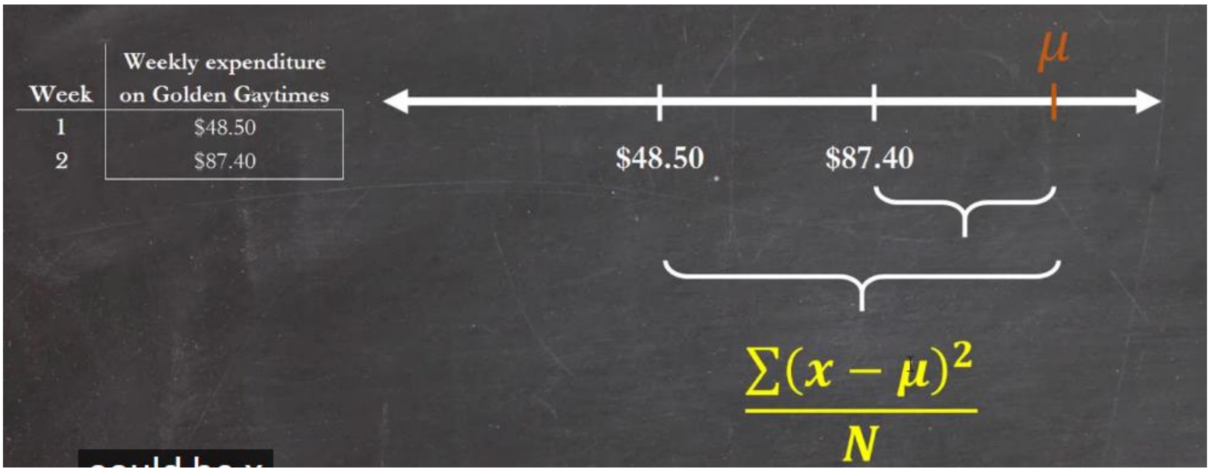
Variance and Standard Deviation

Variance : Average squared deviation from the population mean.

Why don't we take the absolute value to avoid negating it???
Because of moments, variance is actually a form of second moment.



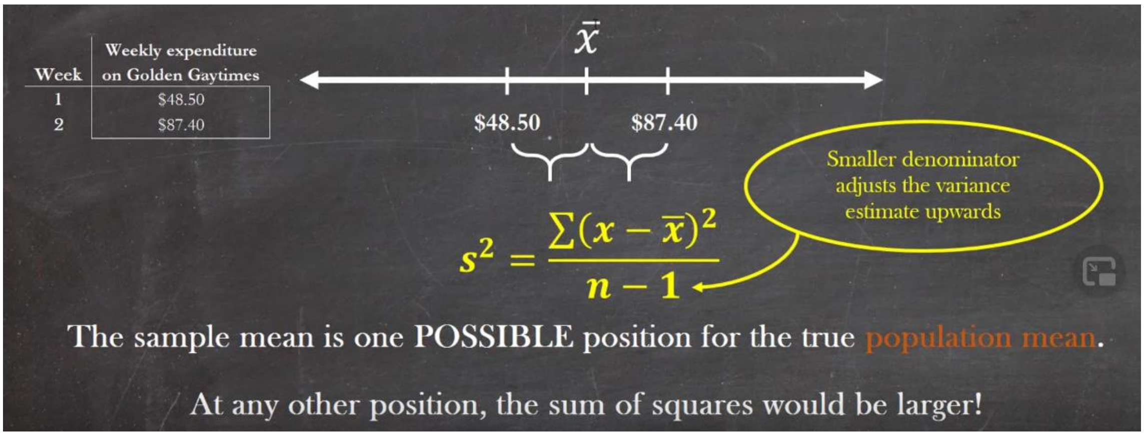
Variance : Average squared deviation from the population mean.



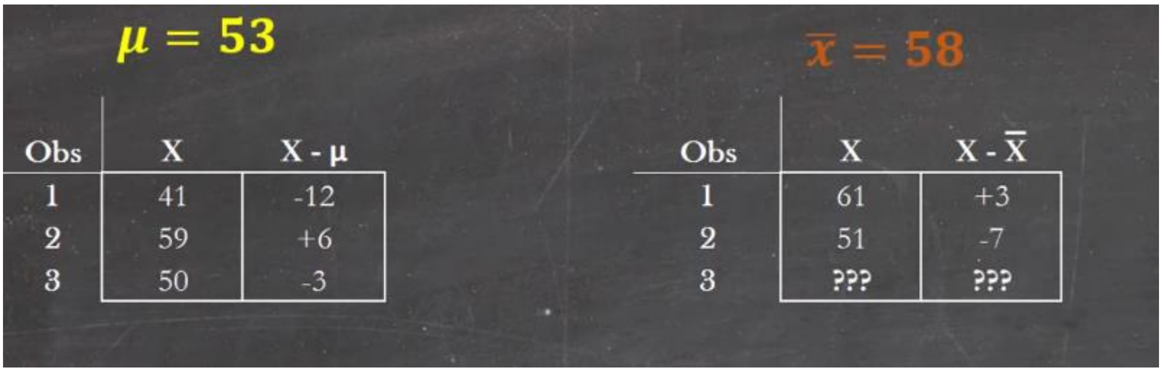
The mean in the image is a theoretical example, but in a real life scenario we don't know where the population mean is??

So we can use sample mean as an estimate to approximate for the population mean.

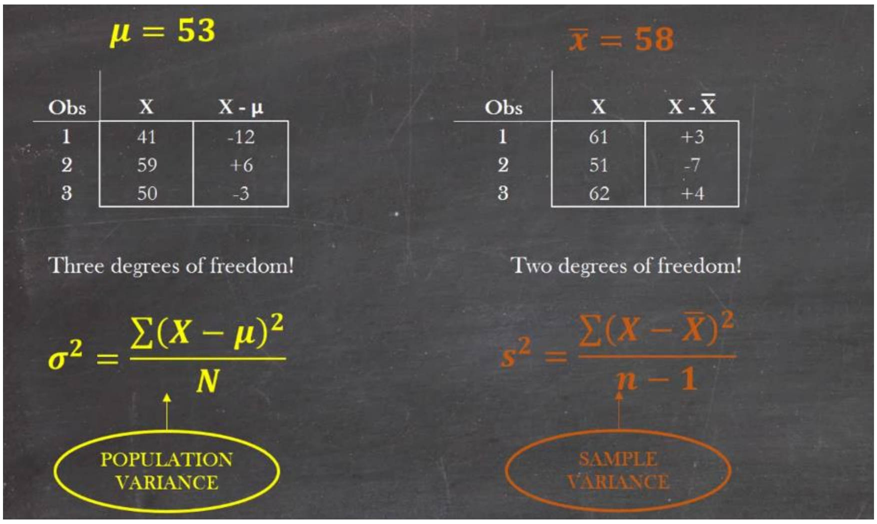
No matter where the population mean is, the squared distances between the population mean will be way larger than the squared distances between the sample means.
So we have to take this into account, otherwise the estimate of the true variance will less than it should be

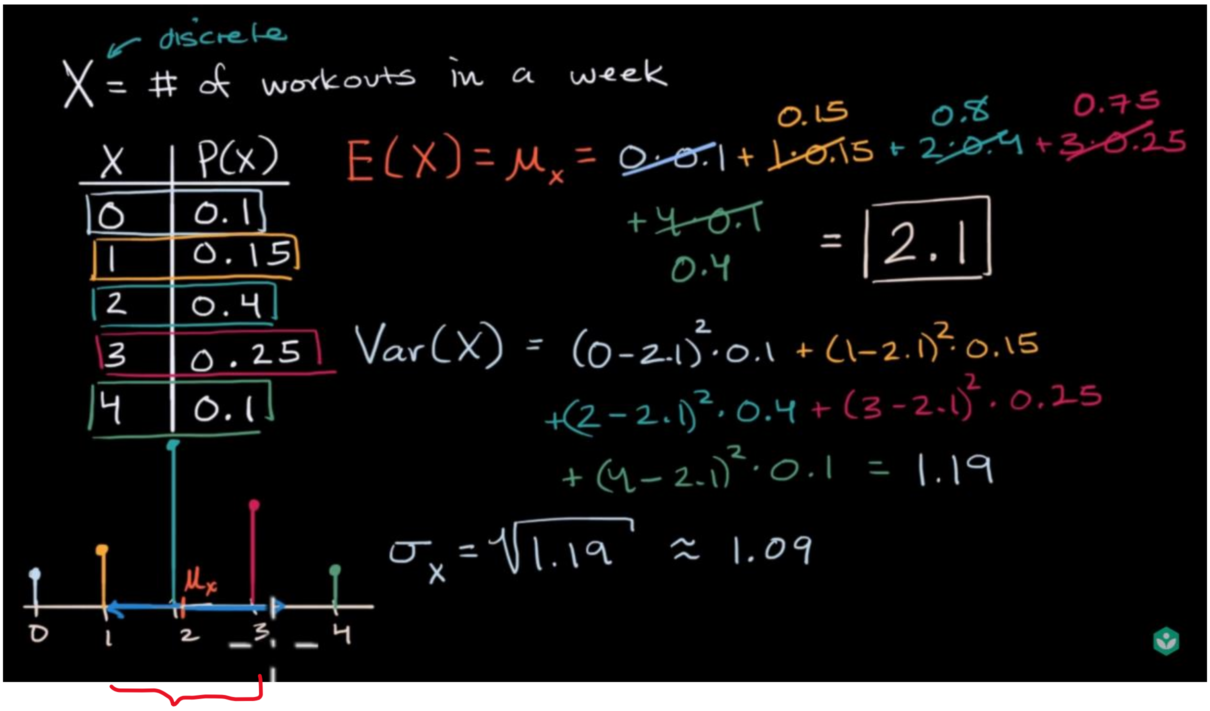


Degrees of freedom



What will the 3rd deviation in the sample ????
As already known, the sample mean always lies among the sample.
So +3 -7 **+4** → 0.





Standard deviation

The expected value(mean) , variance of Discrete Random variable

The *expected value* (or *expectation*) of a random variable is the theoretical mean of the random variable.

$$E(X) = \mu$$

To calculate the expected value of a discrete random variable X:

$$E(X) = \sum_{\text{all } x} x \cdot p(x)$$

Examples of g(X): X^3 or $\sqrt{X}$

The expectation of a function $g(X)$:

$$E[g(X)] = \sum_{\text{all } x} g(x) \cdot p(x)$$

The variance of X:

$$\sigma^2 = E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 \cdot p(x)$$

The expectation of
the squared distance of
X from its mean

A handy relationship:

$$E[(X-\mu)^2] = E(X^2) - [E(X)]^2$$

$$= E(X^2) - \mu^2$$

Problem question

Suppose you bought a novelty coin that has a probability of 0.6 of coming up heads when flipped.

Let X represent the number of heads when this coin is tossed twice.

The probability distribution of X:

x	0	1	2
$p(x)$	0.16	0.48	0.36

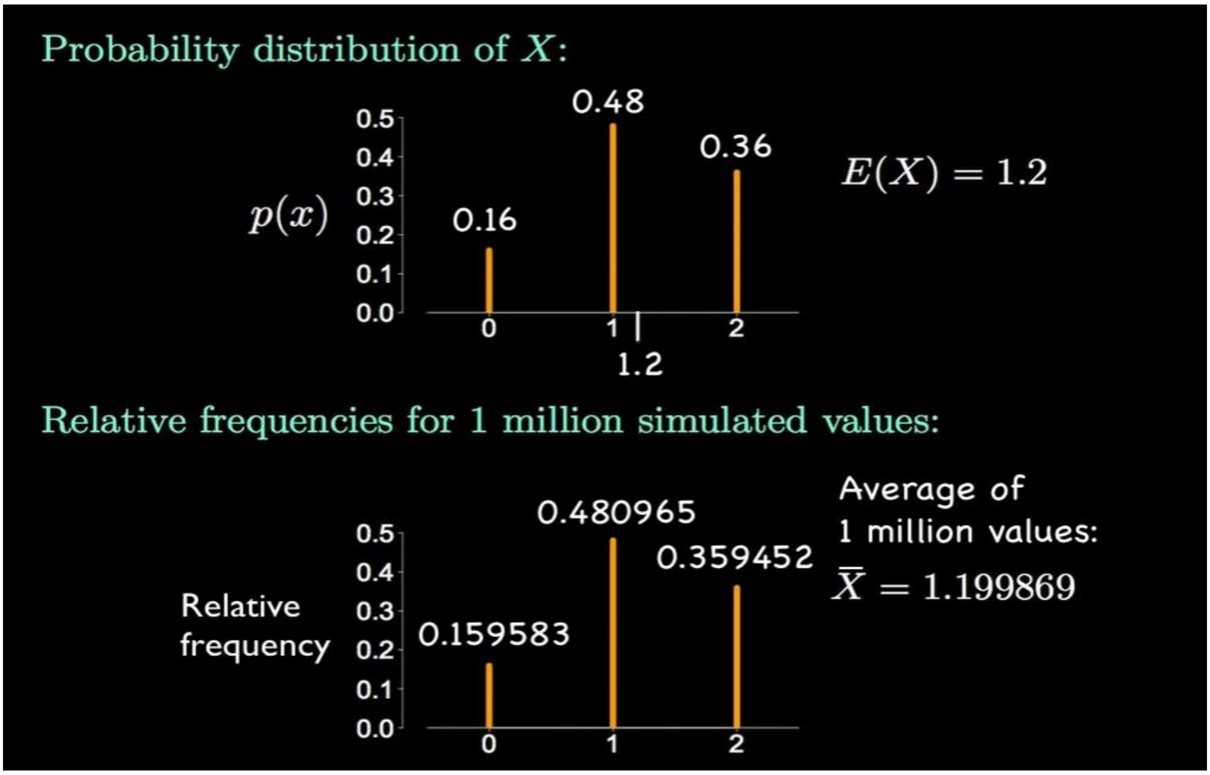
What is $E(X)$?

x	0	1	2
$p(x)$	0.16	0.48	0.36

$$E(X) = \sum x \cdot p(x)$$

$$= 0 \cdot 0.16 + 1 \cdot 0.48 + 2 \cdot 0.36$$

$$= 1.2$$



Law of large numbers tells us that if were to sample more and more values from this distribution, their average would converge to the expected value as the number of observations increases.

Lets find the expected value of f(x)

x	0^2	1^2	2^2
$p(x)$	0.16	0.48	0.36

$$E(X^2) = \sum x^2 \cdot p(x)$$

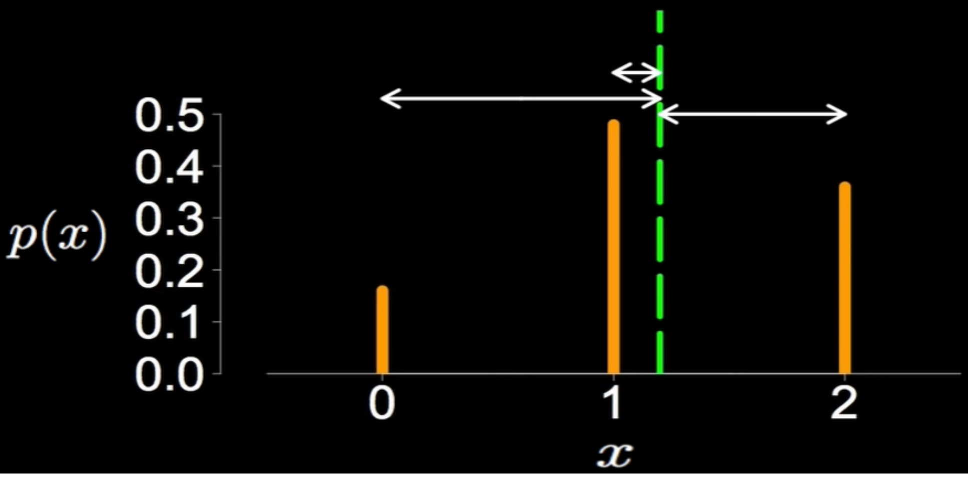
$$= 0^2 \cdot 0.16 + 1^2 \cdot 0.48 + 2^2 \cdot 0.36$$

$$= 1.92$$

What is the variance of X?

The expectation of the squared distance from the mean

$$\sigma^2 = E[(X-\mu)^2]$$



$\mu = 1.2$

$(0-1.2)^2 \quad (1-1.2)^2 \quad (2-1.2)^2$

x	0	1	2
$p(x)$	0.16	0.48	0.36

$$E[(X-\mu)^2] = \sum (x-\mu)^2 \cdot p(x)$$
$$= (0-1.2)^2 \cdot 0.16$$
$$+ (1-1.2)^2 \cdot 0.48$$
$$+ (2-1.2)^2 \cdot 0.36$$
$$= 0.48$$

$\sigma^2 = 0.48$

$\sigma = \sqrt{0.48}$

Another formula to calculate the variance

x	0	1	2
$p(x)$	0.16	0.48	0.36

$E(X) = 1.2$

$E(X^2) = 1.92$

$E[(X-\mu)^2] = 0.48$

It's often easier to carry out the calculations this way

$$E[(X-\mu)^2] = E(X^2) - [E(X)]^2 \checkmark$$

The definition of the variance

$1.92 - 1.2^2 = 0.48$

Important formulae

FORMULA

1. $\sum_i p(x_i) = 1$

2. $F(x) = P[X \leq x]$

   e.g., $P[X \leq 4] = F[4]$

   $F[1] = P[0] + P[1]$

   $F[2] = P[0] + P[1] + P[2] = F[1] + P[2]$

   $F[3] = P[0] + P[1] + P[2] + P[3] = F[2] + P[3]$

   ... ...

3. $P[1] = F[1] - F[0]$

   $P[2] = F[2] - F[1]$

   $P[3] = F[3] - F[2]$

4. Mean $= E[X] = \Sigma x_i p(x_i) =$ Expected value

5. $E[X^2] = \Sigma x_i^2 p(x_i)$

6. Variance $= \mathrm{Var}[X] = E[X^2] - [E[X]]^2$

7. $E[aX + b] = a\,E[X] + b$

8. $\mathrm{Var}[aX \pm b] = a^2\,\mathrm{Var}\,X$

9. Probability mass function $p(x) = P[X = x]$

10. Standard deviation $= \sqrt{\mathrm{Var}(X)}$

Problems

Question

Determine the mean, variance, E[(2X+1)], Var(2X+1) of a discrete random variable X given its distribution as follows :

$X = x_i$	1	2	3	4	5	6
$F(X = x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Solution :

Mean $= E[X] = \Sigma x_i p(x_i)$; $E[X^2] = \Sigma x_i^2 p(x_i)$

Var(X) $= E[X^2] - [E(X)]^2$; $E[aX + b] = a E[X] + b$

$\text{Var}[aX \pm b] = a^2 \text{ Var } X$

x_i	$F(x)$	$P(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$
1	$F[1] = \frac{1}{6}$	$P(1) = F(1) = \frac{1}{6}$	$(1)\left(\frac{1}{6}\right) = \frac{1}{6}$	1	$(1)\left(\frac{1}{6}\right) = \frac{1}{6}$
2	$F[2] = \frac{2}{6}$	$P(2) = F(2) - F(1) = \frac{2}{6} - \frac{1}{6} = \frac{1}{6}$	$(2)\left(\frac{1}{6}\right) = \frac{2}{6}$	4	$(4)\left(\frac{1}{6}\right) = \frac{4}{6}$
3	$F[3] = \frac{3}{6}$	$P(3) = F(3) - F(2) = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$	$(3)\left(\frac{1}{6}\right) = \frac{3}{6}$	9	$(9)\left(\frac{1}{6}\right) = \frac{9}{6}$
4	$F[4] = \frac{4}{6}$	$P(4) = F(4) - F(3) = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$	$(4)\left(\frac{1}{6}\right) = \frac{4}{6}$	16	$(16)\left(\frac{1}{6}\right) = \frac{16}{6}$
5	$F[5] = \frac{5}{6}$	$P(5) = F(5) - F(4) = \frac{5}{6} - \frac{4}{6} = \frac{1}{6}$	$(5)\left(\frac{1}{6}\right) = \frac{5}{6}$	25	$(25)\left(\frac{1}{6}\right) = \frac{25}{6}$
6	$F[6] = \frac{6}{6}$	$P(6) = F(6) - F(5) = \frac{6}{6} - \frac{5}{6} = \frac{1}{6}$	$(6)\left(\frac{1}{6}\right) = \frac{6}{6}$	36	$(36)\left(\frac{1}{6}\right) = \frac{36}{6}$
			$E[X] = \frac{21}{6} = \frac{7}{2}$		$E[X^2] = \frac{91}{6}$

$\text{Var}(X) = E[X^2] - [E(X)]^2$	$E[2X+1] = 2E[X] + 1$	$\text{Var}[2X+1]$
$= \frac{91}{6} - \left(\frac{7}{2}\right)^2$	$= 2\left[\frac{7}{2}\right] + 1$	$= 2^2 \text{Var}[X]$
$= \frac{91}{6} - \frac{49}{4}$	$= 7 + 1$	$= 4\left(\frac{35}{12}\right)$
$= \frac{35}{12}$	$= 8$	$= \frac{35}{3}$

Question

The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Determine the expected demand for watches. Also compute the variance. **[A.U. N/D 2006]**

Solution : Let X denote the demand

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$	
1	0.08	0.08	1	0.08	$\text{Var}[X]$ $= E[X^2] - [E(X)]^2$ $= 19.70 - (4.06)^2$ $= 19.70 - 16.4836$ $= 3.2164$
2	0.12	0.24	4	0.48	
3	0.19	0.57	9	1.71	
4	0.24	0.96	16	3.84	
5	0.16	0.80	25	4.00	
6	0.10	0.60	36	3.60	
7	0.07	0.49	49	3.43	
8	0.04	0.32	64	2.56	
		$E[X] = \Sigma x_i p(x_i)$ $= 4.06$		$E[X^2] = \Sigma x_i^2 p(x_i)$ $= 19.70$	

Question

When a die is thrown, X denotes the number that turns up. Find E (X), E (X^2) and var (X). **[A.U. M/J 2006]**

Solution : X is a discrete random variable taking values, 1, 2, 3, 4, 5, 6 and with probability $\frac{1}{6}$ for each

$E[X] = \Sigma x_i p(x_i)$	$E[X^2] = \Sigma x_i^2 p(x_i)$	$\text{Var}(X) = E[X^2] - [E(x)]^2$

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$	$\text{Var}[X]$
1	1/6	1/6	1	1/6	$= E[X^2] - [E(X)]^2$
2	1/6	2/6	4	4/6	$= \frac{91}{6} - \left(\frac{7}{2}\right)^2$
3	1/6	3/6	9	9/6	
4	1/6	4/6	16	16/6	$= \frac{91}{6} - \frac{49}{4}$
5	1/6	5/6	25	25/6	
6	1/6	6/6	36	36/6	$= \frac{35}{12}$
		$E[x] = \frac{21}{6} = \frac{7}{2}$		$E[X^2] = \frac{91}{6}$	

Question

Determine the constant K given the following probability distribution of discrete random variable X. Also find mean and Variance of X.

$X = x_i$	1	2	3	4	5	Total
$P(X = x_i)$	0.1	0.2	K	2K	0.1	1.0

Solution : We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=1}^{5} P(x_i) = 1$

we have, $0.1 + 0.2 + K + 2K + 0.1 = 1$

i.e., $3K = 1 - 0.4 = 0.6$

$K = 0.2$

Mean $= E[X] = \Sigma x_i p(x_i)$	$E[X^2] = \Sigma x_i^2 p(x_i)$

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$
1	0.1	0.1	1	0.1
2	0.2	0.4	4	0.8
3	0.2	0.6	9	1.8
4	0.4	1.6	16	6.4
5	0.1	0.5	25	2.5
		$E[X] = 3.2$		$E[X^2] = 11.6$

$$\text{Var}[X] = E[X^2] - [E(x)]^2$$

$$= 11.6 - (3.2)^2 = 11.6 - 10.24 = 1.36$$

Question

If X has the distribution function

$$F[x] = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{3} & \text{for } 1 \le x < 4 \\ \frac{1}{2} & \text{for } 4 \le x < 6 \\ \frac{5}{6} & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

Find (1) The probability distribution of X.

(2) $P(2 < X < 6)$

(3) Mean of X

(4) Variance of X. **[A.U. A/M. 2008]**

Solution : (1) The probability distribution of X.

Mean $= E[X] = \Sigma x_i p(x_i)$	$E[X^2] = \Sigma x_i^2 p(x_i)$

$$\boxed{\text{Var}(X) = E[X^2] - [E(x)]^2}$$

$$P[x_1 < X < x_2] = F(x_2) - F(x_1) - P(X = x_2)$$

x_i	$F(x)$	$p(x_i)$	$x_i\, p(x_i)$	x_i^2	$x_i^2\, p(x_i)$
0	$F(0) = 0$	$p(0) = F(0) = 0$	0	0	0
1	$F(1) = \frac{1}{3}$	$p(1) = F(1) - F(0)$ $= \frac{1}{3} - 0 = \frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$
2	$F(2) = \frac{1}{3}$	$p(2) = F(2) - F(1)$ $= \frac{1}{3} - \frac{1}{3} = 0$	0	4	0
3	$F(3) = \frac{1}{3}$	$p(3) = F(3) - F(2)$ $= \frac{1}{3} - \frac{1}{3} = 0$	0	9	0
4	$F(4) = \frac{1}{2}$	$p(4) = F(4) - F(3)$ $= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$	$\frac{4}{6}$	16	$\frac{16}{6}$
5	$F(5) = \frac{1}{2}$	$p(5) = F(5) - F(4)$ $= \frac{1}{2} - \frac{1}{2} = 0$	0	25	0
6	$F(6) = \frac{5}{6}$	$p(6) = F(6) - F(5)$ $= \frac{5}{6} - \frac{1}{2} = \frac{2}{6}$	$\frac{12}{6}$	36	$\frac{72}{6}$
7	$F(7) = \frac{5}{6}$	$p(7) = F(7) - F(6)$ $= \frac{5}{6} - \frac{5}{6} = 0$	0	49	0
8	$F(8) = \frac{5}{6}$	$p(8) = F(8) - F(7)$ $= \frac{5}{6} - \frac{5}{6} = 0$	0	64	0
9	$F(9) = \frac{5}{6}$	$p(9) = F(9) - F(8)$ $= \frac{5}{6} - \frac{5}{6} = 0$	0	81	0
10	$F(10) = 1$	$p(10) = F(10) - F(9)$ $= 1 - \frac{5}{6} = \frac{1}{6}$	$\frac{10}{6}$	100	$\frac{100}{6}$
			$E(X) = \frac{28}{6}$		$E[X^2] = \frac{190}{6}$

Question

A random variable X has the following probability distribution.

$X = x_i$:	-2	-1	0	1	2	3
$P(X = x_i)$:	0.1	k	0.2	$2k$	0.3	$3k$

Find [A.U. N/D 2007, M/J 2009, A.U CBT M/J 2010]

(1) The value of k, [A.U N/D 2011]

(2) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$

(3) Find the cumulative distribution of X and

(4) Evaluate the mean of X.

Solution : (1) We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=1}^{3} P(x_i) = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$6k + 0.6 = 1 \Rightarrow 6k = 1 - 0.6 \Rightarrow 6k = 0.4$$

$$\Rightarrow k = \frac{0.4}{6} = \frac{1}{15}$$

(2) (i) $P[X < 2]$

$$= P[X = -2] + P[X = -1] + P[X = 0] + P[X = 1]$$

$$= 0.1 + \frac{1}{15} + 0.2 + \frac{2}{15} = 0.3 + \frac{3}{15} = \frac{1}{2}$$

(ii) $P[-2 < X < 2]$

$$= P[X = -1] + P[X = 0] + P[X = 1]$$

$$= \frac{1}{15} + 0.2 + \frac{2}{15} = 0.2 + \frac{3}{15} = \frac{2}{5}$$

(3) & (4)

x_i	$p(x_i)$	Cumulative distribution $F(x_i)$	E[X] $x_i p(x_i)$
-2	$p(-2) = 0.1$	$F(-2) = p(-2) = 0.1$	-0.2
-1	$p(-1) = \frac{1}{15}$	$F(-1) = F(-2) + P(-1) = 0.1 + \frac{1}{15} = 0.17$	$-\frac{1}{15}$
0	$p(0) = 0.2$	$F(0) = F(-1) + p(0) = 0.17 + 0.2 = 0.37$	0
1	$p(1) = \frac{2}{15}$	$F(1) = F(0) + p(1) = 0.37 + \frac{2}{15} = 0.50$	$\frac{2}{15}$
2	$p(2) = 0.3$	$F(2) = F(1) + p(2) = 0.50 + 0.3 = 0.80$	0.6
3	$p(3) = \frac{3}{15}$	$F(3) = F(2) + p(3) = 0.80 + \frac{3}{15} = 1.00$	$\frac{9}{15}$

(4) Mean $= E[X] = \Sigma x_i p(x_i) = 16/15$

Question

A discrete random variable X has the following probability distribution.

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a. **[A.U CBT N/D 2011, Trichy M/J 2011]**

(ii) Find $P(X < 3)$, $P(0 < X < 3)$, $P(X \geq 3)$

(iii) Find the distribution function of X. **[AU Tvli. A/M 2009]**

Solution : (i) We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=1}^{8} P(x_i) = 1$

$$a + 3a + 5a + 7a + 11a + 13a + 15a + 17a = 1$$

$$\text{(i.e.,) } 81a = 1 \quad \therefore \boxed{a = \frac{1}{81}}$$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= a + 3a + 5a = 9a = 9 \times \frac{1}{81} = \frac{1}{9} \qquad \text{... (1)}$$

$$P(0<X<3) = P(X = 1) + P(X = 2)$$

$$= 3a + 5a = 8a = 8 \times \frac{1}{81} = \frac{8}{81}$$

$$P[X \geq 3] = 1 - P[X < 3]$$

$$= 1 - \frac{1}{9} = \frac{8}{9} \quad \text{by (1)}$$

Distribution function $F(x)$ of X.

Distribution function F(x) of X.

x_i	$p(x_i)$	$F(x_i)$
0	$p(0) = \frac{1}{81}$	$F(0) = p(0) = \frac{1}{81}$
1	$p(1) = \frac{3}{81}$	$F(1) = F(0) + p(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	$p(2) = \frac{5}{81}$	$F(2) = F(1) + p(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	$p(3) = \frac{7}{81}$	$F(3) = F(2) + p(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	$p(4) = \frac{9}{81}$	$F(4) = F(3) + p(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	$p(5) = \frac{11}{81}$	$F(5) = F(4) + p(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	$p(6) = \frac{13}{81}$	$F(6) = F(5) + p(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	$p(7) = \frac{15}{81}$	$F(7) = F(6) + p(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	$p(8) = \frac{17}{81}$	$F(8) = F(7) + p(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81} = 1$

Question

A random variable X has the following probability function :

$X = x_i$:	0	1	2	3	4	5	6	7
$P(X = x_i)$:	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

(a) Find K **[A.U N/D 2010, M/J 2012, M//J 2014]**

(b) Evaluate P[X < 6], P[X ≥ 6]

(c) If $P[X \leq C] > \frac{1}{2}$, then find the minimum value of C.

(d) Evaluate P [1.5 < X < 4.5/X > 2]

(e) Find P [X < 2], P[X > 3], P[1<X<5]

Solution : (a) We know that, $\sum_{i} P(x_i) = 1$. Here, $\sum_{i=0}^{7} P(x_i) = 1$

i.e., $0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

i.e., $10K^2 + 9K - 1 = 0$

$\Rightarrow K = -1$ or $K = 1/10$

Since, $P(X) \geq 0$ the value $K = -1$ is not permissible

Hence, we have $K = \frac{1}{10}$

$\therefore$

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(b) (i) $P[X \geq 6] = P[X = 6] + P[X = 7]$

$$= \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$$

(ii) $P[X < 6] = 1 - P[X \geq 6]$

$$= 1 - \frac{19}{100} = \frac{81}{100}$$

(c)

x_i	$p(x_i)$	$F(x_i)$
0	0	$F(0) = p(0) = 0$
1	$\frac{1}{10}$	$F(1) = F(0) + p(1) = 0 + \frac{1}{10} = \frac{1}{10} < \frac{1}{2}$
2	$\frac{2}{10}$	$F(2) = F(1) + p(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$
3	$\frac{2}{10}$	$F(3) = F(2) + p(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$

4	$\frac{3}{10}$	$F(4) = F(3) + p(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$
5	$\frac{1}{100}$	$F(5) = F(4) + p(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100} > \frac{1}{2}$
6	$\frac{2}{100}$	$F(6) = F(5) + p(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100} > \frac{1}{2}$
7	$\frac{7}{100} + \frac{1}{10} = \frac{17}{100}$	$F(7) = F(6) + p(7) = \frac{83}{100} + \frac{17}{100} = \frac{100}{100} = 1 > \frac{1}{2}$

$\therefore$ The minimum value of $c = 4$. $[\because P[X \le C] > 1/2]$

(d) $P[1.5 < X < 4.5/X > 2]$

$$= \frac{P[(1.5 < X < 4.5) \cap X > 2]}{P(X > 2)}$$

$$P(W_1/W_2) = \frac{p(w_1 \cap w_2)}{p(w_2)}$$

conditional probability

$$= \frac{P[2 < X < 4.5]}{1 - P[X \le 2]}$$

$$= \frac{P(3) + P(4)}{1 - [P(0) + P(1) + P(2)]}$$

$$= \frac{\frac{2}{10} + \frac{3}{10}}{1 - \left[0 + \frac{1}{10} + \frac{2}{10}\right]} = \frac{\frac{5}{10}}{1 - \frac{3}{10}} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(e) (i) $P(X < 2) = P[X = 0] + P[X = 1] = 0 + k = k = \frac{1}{10}$

(ii) $P(X > 3) = 1 - P(X \le 3)$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - [0 + k + 2k + 2k]$$

$$= 1 - 5k = 1 - \frac{5}{10} = 1 - \frac{1}{2} = \frac{1}{2}$$

(iii) $P(1 < X < 5) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= 2k + 2k + 3k = 7k = \frac{7}{10}$$

Question

A random variable 'X' has the following probability function :

$X = x_i$:	0	1	2	3	4
$P(X = x_i)$:	k	$3k$	$5k$	$7k$	$9k$

Find k, P [X ≥ 3] and P (0 < X < 4) [A.U. Tvli. A/M 2009]

Solution : (i) We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=0}^{4} P(x_i) = 1$

(i.e.,) $k + 3k + 5k + 7k + 9k = 1$

$$25k = 1 \Rightarrow k = \frac{1}{25}$$

(ii) $P(X \geq 3) = P(X = 3) + P(X = 4)$

$$= 7k + 9k = 16k = 16\left(\frac{1}{25}\right) = \frac{16}{25}$$

(iii) $P(0 < X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$

$$= 3k + 5k + 7k = 15k$$

$$= 15\left(\frac{1}{25}\right) = \frac{3}{5}$$

Question **(Important)**

Let X be a random variable such that $P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2)$ and $P(X < 0) = P(X = 0) = P(X>0)$. Determine the probability mass function and the distribution of X.

[AU N/D 2006]

Solution :

Given : $P[X < 0] = P[X = 0] = P[X > 0]$ ... (1)

$P[X = -2] = P[X - 1] = P[X = 1] = P[X = 2]$ ... (2)

Let $P[X = -2] = P[X = -1] = P[X = 1] = P[X = 2] = a$

We know that,

$$P[X < 0] = P[X = -2] + P[X = -1] = a + a = 2a$$

$\therefore$ (1) $\Rightarrow$ $P[X < 0] = P[X = 0] = P[X > 0] = 2a$

$X = x_i$	-2	-1	0	1	2
$P[X = x_i]$	a	a	$2a$	a	a

We know that, $\sum_i P(x_i) = 1$.

Here, $\sum_{i=-2}^{2} P(x_i) = 1$

i.e., $a + a + 2a + a + a = 1 \Rightarrow 6a = 1 \Rightarrow a = \frac{1}{6}$

x_i	$p(x_i)$	$F(x_i)$
-2	$p(-2) = \frac{1}{6}$	$F(-2) = p(-2) = \frac{1}{6}$
-1	$p(-1) = \frac{1}{6}$	$F(-1) = F(-2)+p(-1) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$
0	$p(0) = \frac{1}{3} = \frac{2}{6}$	$F(0) = F(-1) + p(0) = \frac{2}{6} + \frac{2}{6} = \frac{4}{6}$
1	$p(1) = \frac{1}{6}$	$F(1) = F(0) + p(1) = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$
2	$p(2) = \frac{1}{6}$	$F(2) = F(1) + p(2) = \frac{5}{6} + \frac{1}{6} = \frac{6}{6} = 1$

Question (Important)

If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ find the probability distribution and cumulative distribution function of X.

Solution : **[A.U CBT A/M 2011] [A.U N/D 2012]**

X is a discrete random variable.

Given : $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4]$

Let $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4] = k$ ... (1)

(1) ⇒

$X = x_i$	1	2	3	4
$P[X = x_i]$	$\frac{k}{2}$	$\frac{k}{3}$	k	$\frac{k}{5}$

We know that, $\sum_i P[x_i] = 1$, Here, $\sum_{i=1}^{4} P(x_i) = 1$

$$\text{i.e., } \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{61}{30}k = 1 \Rightarrow k = \frac{30}{61}$$

x_i	$p(x_i)$	$F(x_i)$
1	$p(1) = \frac{k}{2} = \frac{15}{61}$	$F(1) = p(1) = \frac{15}{61}$
2	$p(2) = \frac{k}{3} = \frac{10}{61}$	$F(2) = F(1) + p(2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$p(3) = k = \frac{30}{61}$	$F(3) = F(2) + p(3) = \frac{25}{61} + \frac{30}{61} = \frac{55}{61}$
4	$p(4) = \frac{k}{5} = \frac{6}{61}$	$F(4) = F(3) + p(4) = \frac{55}{61} + \frac{6}{61} = \frac{61}{61} = 1$

Question **Very very important**

The probability function of an infinite discrete distribution is given by $P[X = j] = \frac{1}{2^j}$, $j = 1, 2, ..., \infty$. Find the mean and variance of the distribution. Also find P[X is even], $P[X \geq 5]$ and P[X is divisible by 3]. [A.U. N/D 2006] [A.U N/D 2011]

Solution : Given : $P[X = j] = \frac{1}{2^j}$

$$\text{Mean} = E[X] = \sum_{j=1}^{\infty} x_j\, p(x_j)$$

$$= (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right)^2 + (3)\left(\frac{1}{2}\right)^3 + \ldots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots\right]$$

$$= \frac{1}{2}[1-x]^{-2} \qquad \text{Here, } x = \frac{1}{2}$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right)^{-2} = \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = \frac{1}{2}\, 4 = 2$$

$$E[X^2] = \sum_{j=1}^{\infty} x_j^2\, p(x_j) = \sum_{j=1}^{\infty} (x_j)(x_j+1)\, p(x_j) - \sum_{j=1}^{\infty} x_j\, p(x_j)$$

$$= \left[(1)(2)\frac{1}{2} + (2)(3)\left(\frac{1}{2}\right)^2 + (3)(4)\left(\frac{1}{2}\right)^3 + \ldots\right] - 2$$

$$= \frac{1}{2}\left[1.2 + 2.3\left(\frac{1}{2}\right) + 3.4\left(\frac{1}{2}\right)^2 + \ldots\right] - 2$$

$$= \frac{1}{2}\left[2\,[1-x]^{-3}\right] - 2 \qquad \text{where } x = \frac{1}{2}$$

$$= \frac{1}{2}2\left(1-\frac{1}{2}\right)^{-3} - 2 = \frac{1}{2}\ 2\left(\frac{1}{2}\right)^{-3} - 2 = (8) - 2 = 6$$

FORMULA	$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$
	$(1-x)^{-3} = \frac{1}{2}[1.2 + 2.3x + 3.4x^2 + \ldots]$

$$\text{Variance of X} = \text{Var(X)} = E[X^2] - \left[E[X]\right]^2$$

$$= 6 - (2)^2 = 6 - 4 = 2$$

(1) P[X is even] $= P[X = 2] + P[X = 4] + \ldots$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \ldots$$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots$$

$$= (1-x)^{-1} - 1 \text{ Here, } x = \frac{1}{4} \qquad [\because (1-x)^{-1} = 1 + x + x^2 + \ldots]$$

$$= \left(1-\frac{1}{4}\right)^{-1} - 1 = \left(\frac{3}{4}\right)^{-1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

(2) P [X ≥ 5] = P[X = 5] + P [X = 6] + ...

$$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \ldots$$

$$= \left(\frac{1}{2}\right)^5 \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \ldots\right]$$

$$= \left(\frac{1}{2}\right)^5 [1-x]^{-1} \qquad \text{Here, } x = \frac{1}{2}$$

$$= \left(\frac{1}{2}\right)^5 \left[1-\frac{1}{2}\right]^{-1} = \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{-1} = \frac{1}{2^4} = \frac{1}{16}$$

(3) P[X is divisible by 3] (or) P[X is multiple of 3]

$$= P[X = 3] + P[X = 6] + \ldots$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \ldots$$

$$= \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \ldots$$

$$= \left(1 - x\right)^{-1} - 1. \qquad \text{Here, } x = \frac{1}{8}$$

$$= \left(1 - \frac{1}{8}\right)^{-1} - 1 = \left(\frac{7}{8}\right)^{-1} - 1 = \frac{8}{7} - 1 = \frac{1}{7}$$

Binomial formulae

n is positive

(i) $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r + \ldots\ldots$$

(ii) $$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}(-x)^r + \ldots\ldots$$

n is negative

(iii) $$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots\ldots(n+r-1)}{r!}x^r + \ldots\ldots$$

(iv) $$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots\ldots(n+r-1)}{r!}(-x)^r + \ldots\ldots$$

(v) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots. \infty$

(vi) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots. \infty$

(vii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots. \infty$

(viii) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots. \infty$

(ix) $(1+x)^{-3} = 1 - 3x + 6x^2 - \ldots\ldots\ldots. \ \ldots.. \infty$

(x) $(1-x)^{-3} = 1 + 3x + 6x^2 + \ldots\ldots\ldots. \ \ldots\ldots. \infty$

Question **Very very important**

Given the probability function of a random variable X as

x :	0	1	2	3	4
P (x) :	0.1	0.2	0.3	0.2	0.2

Find the probability function, means and variance of $Y=3X+X^2$.

Solution : The random variable $Y = X^2 + 3X$ takes the values 0, 4, 10, 18 and 28 respectively for X = 0, 1, 2, 3 and 4. From the probability values for X, we have $P(Y = 0) = P(X = 0) = 0.1$

$$P(Y = 4) = P(X = 1) = 0.2\ ;$$

$$P(Y = 10) = P(X = 2) = 0.3\ ;$$

$$P(Y = 18) = P(X = 3) = 0.2\ ;$$

$$P(Y = 28) = P(X = 4) = 0.2$$

Hence, the probability function of Y is as follows :

Y	0	4	10	18	28
P(Y)	0.1	0.2	0.3	0.2	0.2

$$E(Y) = \Sigma\,[y_i\,P\,(Y_i)] = y_1\,P(Y_1) + y_2\,P(Y_2) + \ldots + y_5\,P(y_5)$$

$$= 0 \times 0.1 + 4 \times 0.2 + 10 \times 0.3 + 18 \times 0.2 + 28 \times 0.2 = 13$$

$$E(Y^2) = \Sigma_i\,[y_i^2\,P\,(Y_i)] = 0\times0.1+16\times0.2+100\times0.3+324\times0.2+78.4\times0.2$$

$$= 254.8$$

$$\text{Var}(Y) = E\,(Y^2) - [\,E\,(Y)\,]^2$$

$$= 254.8 - 169 = 85.8$$

Question **Very very important**

If Var(X) = 4, then find Var(3X + 8), where X is a random variable.

[A.U, Model] [A.U Tvli M/J 2011]

Solution : We know that, $\text{Var}(aX + b) = a^2 \text{ Var}(X)$

$$\text{Var}(3X + 8) = 3^2 \text{ Var}(X)$$

$$= 3^2 [4] = 36$$

Question **Very very important**

Let X be a Random Variable with E(X) = 1 and E[X (X − 1)] = 4. Find Var X and Var (2 − 3X), Var $\left[\frac{X}{2}\right]$

[A.U. May, 2000], [A.U. A/M. 2008]

Solution : Given : $E(X) = 1$, $E[X(X-1)] = 4$

$$E[X(X-1)] = E[X^2 - X] = E[X^2] - E[X]$$

$$4 = E(X^2) - 1$$

$$E(X^2) = 4 + 1 = 5$$

$$\text{Var } X = E(X^2) - [E(X)]^2 = 5 - 1 = 4$$

$$\text{Var}(2 - 3X) = (-3)^2 \text{ Var } X \quad [\because \text{Var}(aX \pm b) = a^2 \text{ Var } X]$$

$$= 9(4) = 36$$

$$\text{Var}\left[\frac{X}{2}\right] = \left(\frac{1}{2}\right)^2 \text{Var } X = \frac{1}{4}[4] = 1$$

Question **Very very important**

If the probability mass function of a r.v. X is given by $P(X = r) = k r^3$, $r = 1, 2, 3, 4$ find

(i) the value of k, (ii) $P\left(\frac{1}{2} < X < \frac{5}{2} / X > 1\right)$

(iii) the mean and variance of X and

(iv) the distribution function of X

Solution : [A.U Tvli. M/J 2010]

We know that, $\sum_i p(x_i) = 1 \Rightarrow 100k = 1 \Rightarrow k = \frac{1}{100}$

Given : $P(X = r) = kr^3$, $r = 1, 2, 3, 4$

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$	$F[X]$
1	$k = \frac{1}{100}$	$\frac{1}{100}$	1	$\frac{1}{100}$	$F[1] = p(1) = \frac{1}{100}$
2	$8k = \frac{8}{100}$	$\frac{16}{100}$	4	$\frac{32}{100}$	$F[2] = F[1] + p(2) = \frac{1}{100} + \frac{8}{100} = \frac{9}{100}$
3	$27k = \frac{27}{100}$	$\frac{81}{100}$	9	$\frac{243}{100}$	$F[3] = F[2] + p(3) = \frac{9}{100} + \frac{27}{100} = \frac{36}{100}$
4	$64k = \frac{64}{100}$	$\frac{256}{100}$	16	$\frac{1024}{100}$	$F[4] = F[3] + p(4) = \frac{36}{100} + \frac{64}{100} = 1$
	$100k = 1$	$\frac{354}{100}$		$\frac{1300}{100}$	

$$E[X] = \Sigma x_i p(x_i) = \frac{354}{100} = 3.54$$

$$E[X^2] = \Sigma x_i^2 p(x_i) = (1300)\left(\frac{1}{100}\right) = 13$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 13 - \left(\frac{354}{100}\right)^2 = 13 - [3.54]^2 = 13 - 12.5316 = 0.4684$$

$$\text{P}\left[\frac{1}{2} < X < \frac{5}{2} \Big/ X > 1\right] = \frac{\text{P}\left[\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)\right]}{\text{P}[X > 1]}$$

$$= \frac{\text{P}[X = 2]}{\text{P}[X > 1]} = \frac{\text{P}[X = 2]}{\text{P}[X = 2] + \text{P}[X = 3] + \text{P}[X = 4]}$$

$$= \frac{\frac{8}{100}}{\frac{99}{100}} = \frac{8}{99}$$

Joint probability mass function

For the bivariate probability distribution of (X, Y) given below, find $P(X \le 1)$, $P(Y \le 3)$, $P(X \le 1, Y \le 3)$, $P(X \le 1/Y \le 3)$, $P(Y \le 3/X \le 1)$ and $P(X + Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Solution:

$$P(X \le 1) = P(X = 0) + P(X = 1)$$

$$= \sum_{j=1}^{6} P(X = 0, Y = j) + \sum_{j=1}^{6} P(X = 1, Y = j)$$

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \right.$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$$

$$= \sum_{i=0}^{2} P(X = i, Y = 1) + \sum_{i=0}^{2} P(X = i, Y = 2)$$

$$+ \sum_{i=0}^{2} P(X = i, Y = 3)$$

$$= \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{32} + \frac{1}{8} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

P

$$= \left(0 + 0 + \frac{1}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) = \frac{9}{32}$$

$$P(X \le 1/Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(Y \le 3/X \le 1) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{9/32}{7/8} = \frac{9}{28}$$

$$P(X + Y \le 4) = \sum_{j=1}^{4} P(X = 0, Y = j) + \sum_{j=1}^{3} P(X = 1, Y = j) + \sum_{j=1}^{2} P(X = 2, Y = j)$$

$$= \frac{3}{32} + \frac{1}{4} + \frac{1}{16} = \frac{13}{32}$$

Question (Very very important)

The joint probability mass function of (X, Y) is given by $P(x, y) = K(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also, find the probability distribution of $(X + Y)$ and $P[X + Y > 3]$. [A.U. 2004]

X \ Y	1	2	3	Total
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
Total	15K	24K	33K	72K

Total Probability = 1

$\therefore$ 72K = 1

$\Rightarrow K = \frac{1}{72}$

Solution :

X \ Y	1	2	3	$P_X(x) = P_{i*}$
0	$\frac{3}{72}$ **P(0,1)**	$\frac{6}{72}$ **P(0,2)**	$\frac{9}{72}$ **P(0,3)**	$P(X=0) = \frac{18}{72}$ **sum of the I row**
1	$\frac{5}{72}$ **P(1,1)**	$\frac{8}{72}$ **P(1,2)**	$\frac{11}{72}$ **P(1,3)**	$P(X=1) = \frac{24}{72}$ **sum of the II row**
2	$\frac{7}{72}$ **P(2,1)**	$\frac{10}{72}$ **P(2,2)**	$\frac{13}{72}$ **P(2,3)**	$P(X=2) = \frac{30}{72}$ **sum of the III row**
$P_Y(y) = P_{*j}$	$P(Y=1) = \frac{15}{72}$ **sum of the I column**	$P(Y=2) = \frac{24}{72}$ **sum of the II column**	$P(Y=3) = \frac{33}{72}$ **sum of the III column**	1

The marginal distribution of X :

$P(X = 0) = \frac{18}{72}$; $P(X = 1) = \frac{24}{72}$; $P(X = 2) = \frac{30}{72}$

The marginal distributions of Y :

$P(Y = 1) = \frac{15}{72}$; $P(Y = 2) = \frac{24}{72}$; $P(Y = 3) = \frac{33}{72}$

The conditional distribution of X, given Y is $P\{X = x_i/Y = y_j\}$

$$P(X = 0/Y = 1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{3}{72}}{\frac{15}{72}} = \frac{3}{15} = \frac{1}{5}$$

$$P(X = 1/Y = 1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{\frac{5}{72}}{\frac{15}{72}} = \frac{5}{15} = \frac{1}{3}$$

$$P(X = 2/Y = 1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{\frac{7}{72}}{\frac{15}{72}} = \frac{7}{15}$$

$$P(X = 0/Y = 2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{\frac{6}{72}}{\frac{24}{72}} = \frac{6}{24} = \frac{1}{4}$$

$$P(X = 1/Y = 2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{\frac{8}{72}}{\frac{24}{72}} = \frac{8}{24} = \frac{1}{3}$$

$$P(X = 2/Y = 2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{\frac{10}{72}}{\frac{24}{72}} = \frac{10}{24} = \frac{5}{12}$$

$$P(X = 0/Y = 3) = \frac{P(X=0, Y=3)}{P(Y=3)} = \frac{\frac{9}{72}}{\frac{33}{72}} = \frac{9}{33}$$

$$P(X = 1/Y = 3) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{\frac{11}{72}}{\frac{33}{72}} = \frac{11}{33} = \frac{1}{3}$$

$$P(X = 2/Y = 3) = \frac{P(X=2, Y=3)}{P(Y=3)} = \frac{\frac{13}{72}}{\frac{33}{72}} = \frac{13}{33}$$

The conditional distribution of Y, given X is $P\{Y = y_j/X = x_i\}$

$$P\{Y = 1/X = 0\} = \frac{P(X=0; Y=1)}{P(X=0)} = \frac{\frac{3}{72}}{\frac{18}{72}} = \frac{3}{18} = \frac{1}{6}$$

$$P\{Y = 2/X = 0\} = \frac{P(X=0; Y=2)}{P(X=0)} = \frac{\frac{6}{72}}{\frac{18}{72}} = \frac{6}{18} = \frac{1}{3}$$

$$P\{Y = 3/X = 0\} = \frac{P(X=0; Y=3)}{P(X=0)} = \frac{\frac{9}{72}}{\frac{18}{72}} = \frac{9}{18} = \frac{1}{2}$$

$$P\{Y = 1/X = 1\} = \frac{P(X=1; Y=1)}{P(X=1)} = \frac{\frac{5}{72}}{\frac{24}{72}} = \frac{5}{24}$$

$$P\{Y = 2/X = 1\} = \frac{P(X=1; Y=2)}{P(X=1)} = \frac{\frac{8}{72}}{\frac{24}{72}} = \frac{1}{3}$$

$$P\{Y = 3/X = 1\} = \frac{P(X=1; Y=3)}{P(X=1)} = \frac{\frac{11}{72}}{\frac{24}{72}} = \frac{11}{24}$$

$$P\{Y = 1/X = 2\} = \frac{P(X=2; Y=1)}{P(X=2)} = \frac{\frac{7}{72}}{\frac{30}{72}} = \frac{7}{30}$$

$$P\{Y = 2/X = 2\} = \frac{P(X=2; Y=2)}{P(X=2)} = \frac{\frac{10}{72}}{\frac{30}{72}} = \frac{1}{3}$$

$$P\{Y = 3/X = 2\} = \frac{P(X=2; Y=3)}{P(X=2)} = \frac{\frac{13}{72}}{\frac{30}{72}} = \frac{13}{30}$$

Probability distribution of (X + Y)	
(X + Y)	P
1 **P(0,1)**	$\frac{3}{72}$
2 **P(0,2)+P(1,1)**	$\frac{6}{72}+\frac{5}{72}=\frac{11}{72}$
3 **P(0,3)+P(1,2)+P(2,1)**	$\frac{9}{72}+\frac{8}{72}+\frac{7}{72}=\frac{24}{72}$
4 **P[1,3]+P[2,2]**	$\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$
5 **P(2,3)**	$\frac{13}{72}$
Total	1

$$P[X + Y > 3] = P[X+Y=4] + P[X+Y=5] = \frac{21}{72}+\frac{13}{72}=\frac{34}{72}$$

Question Very very important

Consider two random variables X and Y with joint PMF given in Table 5.3.

a. Find $P(X \leq 2, Y \leq 4)$.
b. Find the marginal PMFs of X and Y.
c. Find $P(Y = 2|X = 1)$.
d. Are X and Y independent?

	$Y = 2$	$Y = 4$	$Y = 5$
$X = 1$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$
$X = 2$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{8}$
$X = 3$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$

a. To find $P(X \leq 2, Y \leq 4)$, we can write

$$\begin{aligned} P(X \leq 2, Y \leq 4) &= P_{XY}(1,2) + P_{XY}(1,4) + P_{XY}(2,2) + P_{XY}(2,4) \\ &= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{3}{8}. \end{aligned}$$

b. Note from the table that

$$R_X = \{1, 2, 3\} \text{ and } R_Y = \{2, 4, 5\}.$$

Now we can use Equation 5.1 to find the marginal PMFs:

$$P_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{11}{24} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 2 \\ \frac{1}{4} & y = 4 \\ \frac{1}{4} & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

c. Using the formula for conditional probability, we have

$$\begin{aligned} P(Y = 2|X = 1) &= \frac{P(X = 1, Y = 2)}{P(X = 1)} \\ &= \frac{P_{XY}(1,2)}{P_X(1)} \\ &= \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}. \end{aligned}$$

d. Are X and Y independent? To check whether X and Y are independent, we need to check that $P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$, for all $x_i \in R_X$ and all $y_j \in R_Y$. Looking at the table and the results from previous parts, we find

$$P(X = 2, Y = 2) = \frac{1}{6} \neq P(X = 2)P(Y = 2) = \frac{3}{16}.$$

Thus, we conclude that X and Y are not independent.

The joint probability distribution of a two-dimensional discrete random variable (X, Y) is given below :

Y	X: 0	1	2	3	4	5
0	0	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05

(i) Find, P(X > Y) and P{Max (X, Y) = 3} and

(ii) Find, the probability distribution of the random variable Z = Min (X, Y)

There's not really any probability or algebra involved here, it's just a question of logic. Saying that $\min(X_1, X_2) > t$ is just the same as $X_1 > t$ **and** $X_2 > t$, because if the smaller one is greater than t then the other one must be too.

Hence

$$P(\min(X_1, X_2) > t) = P(X_1 > t \text{ and } X_2 > t)\ ,$$

and if you have suitable **independence** properties this is

$$P(X_1 > t)\, P(X_2 > t)\ .$$

Link →

https://math.stackexchange.com/questions/661438/probability-of-a-min-max

Question **Very very important**

The two dimensional random variable (X, Y) has the joint density function $f(x, y) = \frac{x+2y}{27}$, x = 0, 1, 2 ; y = 0, 1, 2. Find the conditional distribution of Y given X = x. Also, find the conditional distribution of X given Y = 1. [A.U Tvli. A/M 2009]

Solution : Given : $f(x,y) = \frac{x+2y}{27}$, $x = 0, 1, 2, ; y = 0, 1, 2$

Y \ X	0	1	2	$P_Y(y) = P_{*j}$
0	0 P(0,0)	$\frac{1}{27}$ P(1,0)	$\frac{2}{27}$ P(2,0)	$P[Y=0]=\frac{3}{27}$ sum of the I row
1	$\frac{2}{27}$ P(0,1)	$\frac{3}{27}$ P(1,1)	$\frac{4}{27}$ P(2,1)	$P[Y=1]=\frac{9}{27}$ sum of the II row
2	$\frac{4}{27}$ P(0,2)	$\frac{5}{27}$ P(1,2)	$\frac{6}{27}$ P(2,2)	$P[Y=2]=\frac{15}{27}$ sum of the III row
$P_X(x) = P_{i*}$	$P[X=0]=\frac{6}{27}$ sum of the I column	$P[X=1]=\frac{9}{27}$ sum of the II column	$P[X=2]=\frac{12}{27}$ sum of the III column	1

The conditional distribution of Y given X = x.

$$P(Y = 0/X = 0) = \frac{P(X=0; Y=0)}{P(X=0)} = \frac{P(0,0)}{P[X=0]} = \frac{0}{\frac{6}{27}} = 0$$

$$P(Y = 1/X = 0) = \frac{P(X=0; Y=1)}{P(X=0)} = \frac{P(0,1)}{P[X=0]} = \frac{\frac{2}{27}}{\frac{6}{27}} = \frac{2}{6} = \frac{1}{3}$$

$$P(Y = 2/X = 0) = \frac{P(X=0; Y=2)}{P(X=0)} = \frac{P(0,2)}{P[X=0]} = \frac{\frac{4}{27}}{\frac{6}{27}} = \frac{4}{6} = \frac{2}{3}$$

$$P(Y = 0/X = 1) = \frac{P(X=1; Y=0)}{P(X=1)} = \frac{P(1,0)}{P[X=1]} = \frac{\frac{1}{27}}{\frac{9}{27}} = \frac{1}{9}$$

$$P(Y = 1/X = 1) = \frac{P(X=1; Y=1)}{P(X=1)} = \frac{P(1,1)}{P[X=1]} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{3}{9} = \frac{1}{3}$$

$$P(Y = 2/X = 1) = \frac{P(X=1; Y=2)}{P(X=1)} = \frac{P(1,2)}{P[X=1]} = \frac{\frac{5}{27}}{\frac{9}{27}} = \frac{5}{9}$$

$$P(Y = 0/X = 2) = \frac{P(X=2; Y=0)}{P(X=2)} = \frac{P(2,0)}{P[X=2]} = \frac{\frac{2}{27}}{\frac{12}{27}} = \frac{2}{12} = \frac{1}{6}$$

$$P(Y = 1/X = 2) = \frac{P(X=2; Y=1)}{P(X=2)} = \frac{P(2,1)}{P[X=2]} = \frac{4/27}{12/27} = \frac{1}{3}$$

$$P(Y = 2/X = 2) = \frac{P(X=2; Y=2)}{P(X=2)} = \frac{P(2,2)}{P[X=2]} = \frac{6/27}{12/27} = \frac{6}{12} = \frac{1}{2}$$

The conditional distribution of X given Y = 1

$$P(X = 0/Y = 1) = \frac{P(X=0; Y=1)}{P(Y=1)} = \frac{P(0,1)}{P[Y=1]} = \frac{\frac{2}{27}}{\frac{9}{27}} = \frac{2}{9}$$

$$P(X = 1/Y = 1) = \frac{P(X=1; Y=1)}{P(Y=1)} = \frac{P(1,1)}{P[Y=1]} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{3}{9}$$

$$P(X = 2/Y = 1) = \frac{P(X=2; Y=1)}{P(Y=1)} = \frac{P(2,1)}{P[Y=1]} = \frac{\frac{4}{27}}{\frac{9}{27}} = \frac{4}{9}$$

Question Very very important

Two discrete r.v.'s X and Y have the joint probability density function;

$$P(x, y) = \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!\,(x-y)!} \; ; \; y = 0, 1, 2, \ldots x,$$

$x = 0, 1, 2, \ldots$ where m, p are constants with $m > 0$ and $0 < p < 1$. Find (i) the marginal probability density function X and Y, (ii) the conditional distribution of Y for a given X and of X for a given Y.

[A.U. N/D. 2005] [A.U Tvli. M/J 2010]

Solution :

Given the joint probability density function of the two discrete random variables X and Y is $p(x,y) = \dfrac{e^{-m} m^x p^y (1-p)^{x-y}}{y!\,(x-y)!}$, $y = 0, 1, 2, \ldots x$; $x = 0, 1, 2, \ldots$

(i) Then the marginal probability density function of X is,

$$P(x) = \sum_{y=0}^{\infty} p(x,y) = \sum_{y=0}^{\infty} \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= e^{-m} m^x \sum_{y=0}^{\infty} \frac{p^y (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= \frac{e^{-m}\, m^x}{x\,!} \sum_{y=0}^{\infty} \frac{x\,!\, p^y\, (1-p)^{x-y}}{y\,!\, (x-y)\,!}$$

$$= \frac{e^{-m}\, m^x}{x\,!} \sum_{y=0}^{\infty} {}^{x}C_y\, p^y\, q^{x-y}$$

$$= \frac{e^{-m}\, m^x}{x\,!} [p+q]^x$$

$$= \frac{e^{-m}\, m^x}{x\,!} \qquad [\because p+q=1],\ x=0,1,2,...$$

which is a probability function of a Poisson distribution with parameter m.

And $\quad P(y) = \sum_{x=0}^{\infty} p(x,y)$

$$= \sum_{x=0}^{y-1} p_{XY}(x,y) + \sum_{x=y}^{\infty} \frac{e^{-m}\, m^x\, p^y\, (1-p)^{x-y}}{y\,!\, (x-y)\,!}$$

$$= 0 + \frac{e^{-m}\, p^y}{y!} \sum_{x=y}^{\infty} \frac{m^x\, (1-p)^{x-y}}{(x-y)\,!}$$

$$= \frac{e^{-m}\, p^y\, m^y}{y\,!} \sum_{x=y}^{\infty} \frac{m^x\, m^{-y}\, (1-p)^{(x-y)}}{(x-y)\,!}$$

$$= \frac{e^{-m}\, (mp)^y}{y\,!} \sum_{x=y}^{\infty} \frac{[m\,(1-p)]^{(x-y)}}{(x-y)\,!}$$

$$= \frac{e^{-m}\, (mp)^y}{y\,!} \times e^{m\,(1-p)}$$

$$= \frac{e^{-(mp)}\, (mp)^y}{y!},\ y = 0,\ 1,\ 2,\ ...\ x$$

which is the probability function of a Poisson distribution with parameter (mp)

(ii) The conditional distribution of Y for given X is,

$$P(y/x) = \frac{P(x,y)}{P(x)} = \frac{e^{-m} m^x p^y (1-p)^{x-y} x!}{y!(x-y)!\, m^x e^{-m}}$$

$$= \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y}$$

$$= xC_y p^y (1-p)^{x-y}, \; x > y$$

And the conditional probability distribution of X given Y is,

$$P(x/y) = \frac{P(x,y)}{P(y)} = \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!(x-y)!} \cdot \frac{y!}{e^{-mp}(mp)^y}$$

$$= \frac{e^{(-mp)} (mq)^{x-y}}{(x-y)!} \quad ; \qquad q = 1-p, x > y.$$