$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 13 - \left(\frac{354}{100}\right)^2 = 13 - [3.54]^2 = 13 - 12.5316 = 0.4684$$

$$P\left[\frac{1}{2} < X < \frac{5}{2} / X > 1\right] = \frac{P\left[\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)\right]}{P[X > 1]}$$

$$= \frac{P[X=2]}{P[X>1]} = \frac{P[X=2]}{P[X=2]+P[X=3]+P[X=4]}$$

$$= \frac{\frac{8}{100}}{\frac{99}{100}} = \frac{8}{99}$$

1.2 Continuous Random Variables

(i) Definition : Continuous Random Variable

A random variable X is said to be continuous if it takes all possible values between certain limits say from real number 'a' to real number 'b'.

Example : The length of time during which a vacuum tube installed in a circuit functions is a continuous random variable.

Note : If X is a continuous random variable for any x_1 and x_2

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2)$$

(ii) Probability density function :

For a continous random variable X, a probability density function is a function such that

(1) $f(x) \ge 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \le X \le b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b

for any a and b.

Note : A probability density function is zero for the values of X which do not occur and it is assumed to be zero wherever it is not specifically defined.

(iii) Cumulative distribution function

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx \text{ for } -\infty < x < \infty.$$

Note : The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating.

$$\frac{d}{dx}\int_{-\infty}^{x} f(t)\, dt = f(x) \qquad [\because \text{ fundamental theorem of calculus}]$$

$$f(x) = \frac{d}{dx} F[x] \text{ as long as the derivative exists.}$$

(iv) The mean or expected value of a continuous random variable X.

Suppose X is a continuous random variable with probability density function $f(x)$. The mean or expected value of X, denoted as μ or E(X) is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

A useful identity is that for any function g,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

(v) The variance of a continuous random variable X.

The variance of X, denoted as V(X) or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= E[X^2] - [E(X)]^2$$

Note : The standard deviation of X is $\sigma = \sqrt{\text{Var}(X)}$.

(vi) FORMULA

1. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

2. $F[x] = P[X \le x] = \int_{-\infty}^{x} f(t)\, dt, \quad -\infty < x < \infty$

3. $f(x) = \frac{d}{dx} F[x]$

4. Mean $= E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

5. $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

6. Variance $= \text{Var}[X] = E[X^2] - [E[X]]^2$

7. $P[a \le X \le b] = F(b) - F(a)$

8. $P(a \le X \le b) = P(a \le X < b) = P[a < X \le b]$

   $= P[a < X < b]$, X being a continuous random variable.

9. $0 \le F(x) \le 1$

10. F(x) is a non-decreasing function of X.

    i.e., if $x_1 < x_2$ then $F(x_1) < F(x_2)$

11. $F[-\infty] = \lim_{x \to -\infty} F(x) = 0$

    $F[\infty] = \lim_{x \to \infty} F(x) = 1$

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Problems based on $\int_{-\infty}^{\infty} f(x)\,dx = 1$, $F(x) = P(X \le x] = \int_{-\infty}^{x} f(t)\,dt$

Example 1.2.1

A continuous random variable X has p.d.f.

$f(x) = k$, $0 \le x \le 1$. **Determine the constant** k. **Find** $P\left[X \le \frac{1}{4}\right]$.

Solution :

(i) Formula : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here, $\int_0^1 f(x)\,dx = 1$

$$\int_0^1 k\,dx = 1$$

$$k\int_0^1 dx = 1$$

$$k\left[x\right]_0^1 = 1$$

$$k[1-0] = 1$$

$$k = 1$$

(ii)

Here, $P[X \le 1/4] = \int_0^{1/4} f(x)\,dx$

$$= \int_0^{1/4} k\,dx$$

$$= \int_0^{1/4} dx \quad [\because k=1]$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

Example 1.2.2

Given that the p.d.f of a R.V. X is f(x) = kx, 0 < x < 1, find K and P(X > 0.5) **[A.U. Dec, 96]**

Solution :

(i)Formula : $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^1 Kx\, dx = 1$$

$$k \int_0^1 x\, dx = 1$$

$$K \left[\frac{x^2}{2}\right]_0^1 = 1$$

$$K \left[\frac{1}{2} - 0\right] = 1$$

$$K = 2$$

(ii) $P(X > 0.5) = \int_{0.5}^{\infty} f(x)\, dx$

$$= \int_{1/2}^{1} 2x\, dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \left[x^2\right]_{1/2}^{1}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Example 1.2.3

If $f(x) = \begin{cases} Kxe^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$ is the p.d.f. of a random variable X. Find K. **[A.U CBT M/J 2010]**

Solution : For a p.d.f., $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Here, $\int_0^{\infty} Kxe^{-x}\, dx = 1$ $[\because x > 0]$

$$K \left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left[\frac{e^{-x}}{(-1)^2}\right]\right]_0^{\infty} = 1$$

$$K\, [-x\,e^{-x} - e^{-x}]_0^{\infty} = 1$$

$$K\, [(-0 - 0) - (-0 - 1)] = 1 \Rightarrow K = 1 \qquad [e^{-\infty} = 0]$$

Example 1.2.4

A continuous random variable X has probability density function given by $f(x) = 3x^2, 0 \leq x \leq 1$. Find K such that $P(X > K) = 0.5$

Solution : **[A.U. Model Q. Paper] [A.U N/D 2010]**

Given : $P(X > K) = 1 - P[X \leq K] = 1 - 0.5 = 0.5$

i.e., $\int_0^K f(x)\,dx = 0.5 \Rightarrow \int_0^K 3x^2\,dx = 0.5$

$$\Rightarrow 3\int_0^K x^2\,dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_0^K = \frac{1}{2} \Rightarrow \left[x^3\right]_0^K = \frac{1}{2}$$

$$\Rightarrow K^3 - 0 = \frac{1}{2} \Rightarrow K^3 = \frac{1}{2} \Rightarrow K = \left(\frac{1}{2}\right)^{\frac{1}{3}} = 0.7937$$

[Note : $P[X \leq K] = 1 - P[X > K] = 1 - 0.5 = 0.5$]

Example 1.2.5

Let X be a continuous R.V with probability density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P[X \leq 0.4]$ (ii) $P[X > 3/4]$ (iii) $P[X > \frac{1}{2}]$

(iv) $P[1/2 < X < 3/4]$ (v) $P[X > 3/4 / X > 1/2]$

(vi) $P[X < 3/4 / X > 1/2]$

Solution : $P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

(i)
$$P(X \leq 0.4) = \int_0^{0.4} f(x)\,dx = \int_0^{0.4} 2x\,dx = 2\left[\frac{x^2}{2}\right]_0^{0.4} = [x^2]_0^{0.4} = (0.4)^2 - 0 = 0.16$$

(ii)
$$P[X > \frac{3}{4}] = \int_{3/4}^{1} 2x\,dx = 2\int_{3/4}^{1} x\,dx = 2\left[\frac{x^2}{2}\right]_{3/4}^{1} = \left[x^2\right]_{3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

(iii) $P\left[X>\frac{1}{2}\right] = \int_{1/2}^{1} 2x\,dx$

$$= 2\int_{1/2}^{1} x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \left[x^2\right]_{1/2}^{1}$$

$$= 1-\frac{1}{4}$$

$$= \frac{3}{4}$$

(iv) $P\left[\frac{1}{2}<X<\frac{3}{4}\right] = \int_{1/2}^{3/4} 2x\,dx$

$$= 2\int_{1/2}^{3/4} x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/2}^{3/4}$$

$$= \left[x^2\right]_{1/2}^{3/4}$$

$$= \frac{9}{16}-\frac{1}{4}$$

$$= \frac{9-4}{16} = \frac{5}{16}$$

(v) $P\left[X>\frac{3}{4}/X>\frac{1}{2}\right]$

$$= \frac{P\left[(X>\frac{3}{4})\cap(X>\frac{1}{2})\right]}{P\left[X>\frac{1}{2}\right]}$$

$$= \frac{P\left[X>\frac{3}{4}\right]}{P\left[X>\frac{1}{2}\right]}$$

$$= \frac{\frac{7}{16}}{\frac{3}{4}} \text{ by (ii) \& (iii)}$$

$$= \frac{7}{16}\times\frac{4}{3}$$

$$= \frac{7}{12}$$

(vi) $P\left[X<\frac{3}{4}/X>\frac{1}{2}\right]$

$$= \frac{P\left[(X<\frac{3}{4})\cap(X>\frac{1}{2})\right]}{P\left[X>\frac{1}{2}\right]}$$

$$= \frac{P\left[\frac{1}{2}<X<\frac{3}{4}\right]}{P\left[X>\frac{1}{2}\right]}$$

$$= \frac{\frac{5}{16}}{\frac{3}{4}} \text{ by (iv) \& (iii)}$$

$$= \frac{5}{16}\times\frac{4}{3}$$

$$= \frac{5}{12}$$

Example 1.2.6

In a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Verify $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) Find $P(0<X\le 1)$

(c) Find F(x) [cumulative distribution]

Solution :

(a)
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{2} \frac{x^2}{3}\,dx$$
$$= \frac{1}{3}\int_{-1}^{2} x^2\,dx$$
$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2}$$
$$= \frac{1}{3}\left[\left(\frac{8}{3}\right) - \left(\frac{-1}{3}\right)\right]$$
$$= \frac{1}{3}\left[\frac{8}{3} + \frac{1}{3}\right] = \frac{1}{3}\left[\frac{9}{3}\right] = 1$$

(b)
$$P(0<X\le 1) = \int_{0}^{1} \frac{x^2}{3}\,dx$$
$$= \frac{1}{3}\int_{0}^{1} x^2\,dx$$
$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{0}^{1}$$
$$= \frac{1}{3}\left[\frac{1}{3} - 0\right]$$
$$= \frac{1}{9}$$

(c)
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{x} f(t)\,dt = 0 + \int_{-1}^{x} \left(\frac{t^2}{3}\right) dt$$
$$= \frac{1}{3}\int_{-1}^{x} t^2\,dt = \frac{1}{3}\left[\frac{t^3}{3}\right]_{-1}^{x} = \frac{1}{3}\left[\frac{x^3}{3} + \frac{1}{3}\right] = \frac{1}{9}[x^3 + 1]$$

Therefore,
$$F(x) = \begin{cases} 0, & x \le -1 \\ \dfrac{x^3+1}{9}, & -1 < x < 2 \\ 1, & x \ge 2 \end{cases}$$

$p(x \le 2)$

$\Rightarrow p(x < 2)$

$\int f(x)\,dx = \frac{x^3+1}{9}$

Example 1.2.7

A continuous random variable X has the density function $f(x) = \frac{K}{1+x^2}, -\infty < x < \infty$. Find the value of K and the distribution function. **[A.U N/D 2011, N/D 2014]**

Solution : Given $f(x)$ is a p.d.f

$$\therefore \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2}\, dx = 1$$

$$K \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1$$

$$2K \int_{0}^{\infty} \frac{1}{1+x^2}\, dx = 1$$

$$2K \left[\tan^{-1} x\right]_0^{\infty} = 1$$

$$2K \left[\frac{\pi}{2} - 0\right] = 1$$

$$\left[\because \tan^{-1} \infty = \frac{\pi}{2}\right]$$

$$\pi K = 1 \Rightarrow K = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(\frac{-\pi}{2}\right)\right]$$

$$\left[\because \tan^{-1}(-\infty) = -\frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$$

Example 1.2.8

The p.d.f. of a continuous R.V. X is $f(x) = Ke^{-|x|}$. Find K and the F[x]. **[A.U. 2005] [A.U Trichy M/J 2011] [A.U A/M 2010]**

Solution : Given : $f(x)$ in a p.d.f.

$$\therefore \int_{-\infty}^{\infty} f(x)\, dx = 1$$

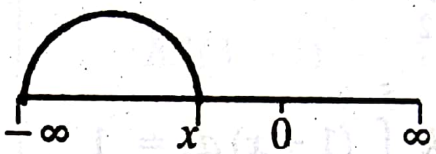
$$\int_{-\infty}^{\infty} Ke^{-|x|}\,dx = 1 \quad \text{i.e.,} \quad 2\int_{0}^{\infty} Ke^{-x}\,dx = 1$$

$$2K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \qquad \because |x| = x \text{ in the interval } (0, \infty)$$

$$2K\left[\left(\frac{0}{-1}\right) - \left(\frac{1}{-1}\right)\right] = 1$$

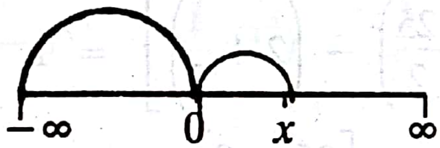
$$2K\,[0 + 1] = 1\,; \quad 2K = 1\,; \quad K = \frac{1}{2}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$



Given : $f(x) = Ke^{-|x|} = \begin{cases} Ke^{x}\,; & -\infty < x < 0 \\ Ke^{-x}\,; & 0 < x < \infty \end{cases} = \begin{cases} \frac{1}{2}e^{x}\,; & -\infty < x < 0 \\ \frac{1}{2}e^{-x}\,; & 0 < x < \infty \end{cases}$

For $x \le 0$, $\quad F(x) = \int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx$



$$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{x} = \frac{1}{2}[e^{x} - 0] = \frac{1}{2}e^{x}$$

For $x > 0$, $\quad F(x) = \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_{0}^{x} \frac{1}{2}e^{-x}\,dx$

$$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{0} + \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^{x}$$

$$= \frac{1}{2}[1 - 0] + \frac{1}{2}\left[\left(\frac{e^{-x}}{-1}\right) - \left(\frac{1}{-1}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}[-e^{-x} + 1]$$

$$= \frac{1}{2} + \frac{1}{2}[1 - e^{-x}] = \frac{1}{2}[2 - e^{-x}]$$

Example 1.2.9

A continuous random variable X that can assume any value between x = 2 and x = 5 has a density function given by f (x) = k (1 + x). Find P [X < 4] [A.U M/J 2006, M/J 2007, N/D 2011, N/D 2012]

[A.U CBT N/D 2008, CBT Dec. 2009]

Solution :

(i) Formula : $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Here, $\int_{2}^{5} k(1+x)\, dx = 1$

$$k \int_{2}^{5} (1+x)\, dx = 1$$

$$k \left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k \left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1$$

$$k \left[\frac{35}{2} - \frac{8}{2}\right] = 1$$

$$k \left[\frac{27}{2}\right] = 1, \; k = \frac{2}{27}$$

(ii) $P[X < 4] = \int_{2}^{4} f(x)\, dx$

$$= \int_{2}^{4} k(1+x)\, dx$$

$$= \int_{2}^{4} \left(\frac{2}{27}\right)(1+x)\, dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{2}{27}[(4+8) - (2+2)]$$

$$= \frac{2}{27}[12 - 4]$$

$$= \frac{16}{27}$$

Example 1.2.10

A continuous random variable X has the distribution function

$$F[x] = \begin{cases} 0 & , \; x < 1 \\ K(x-1)^4 , & 1 \le x \le 3 \\ 0 & , \; x > 3 \end{cases}$$

find K, probability density function f(x), P[X<2] [A.U. A/M. 2008]

Solution :

We know that,

$$P[X \le x] = F[x]$$

$$P[X<2] = F[2] = K(x-1)^4$$

$$f(x) = \frac{d}{dx} F[x]$$

$$= \frac{d}{dx}[K(x-1)^4]$$

$$= K\,4(x-1)^3$$

$$\therefore\; f(x) = \begin{cases} 0 & , \quad x \le 1 \\ 4K(x-1)^3, & 1<x\le 3 \\ 0 & , \quad x > 3 \end{cases}$$

We know that,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_1^3 4K\,(x-1)^3\,dx = 1$$

$$4K\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$K\left[(x-1)^4\right]_1^3 = 1$$

$$K[(3-1)^4 - (1-1)^4] = 1$$

$$K\,[2^4 - 0] = 1$$

$$16K = 1 \Rightarrow K = \frac{1}{16}$$

$$P[X<2] = F[2] = K(2-1)^4 = \frac{1}{16}(1^4) = \frac{1}{16}$$

Example 1.2.11

Is the function defined as follows, a density function ?

$$f(x) = \begin{cases} 0 & \text{for } x<2 \\ \frac{1}{18}(3+2x) & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

[A.U N/D 2006]

Solution : Condition for probability density function is $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Given : $f(x) = \begin{cases} 0 & \text{for } x<2 \\ \frac{1}{18}(3+2x) & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x>4 \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{2} 0\, dx + \int_{2}^{4} \frac{1}{18}(3+2x)\, dx + \int_{4}^{\infty} 0\, dx$$

$$= 0 + \frac{1}{18}\int_{2}^{4}(3+2x)\, dx + 0 = \frac{1}{18}\left[3x + \frac{2x^2}{2}\right]_2^4$$

$$= \frac{1}{18}[3x + x^2]_2^4 = \frac{1}{18}[(12+16) - (6+4)]$$

$$= \frac{1}{18}[28 - 10] = \frac{1}{18}(18) = 1$$

Hence, the given function is density function.

Example 1.2.12

If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & , 0 \le x \le 1 \\ a & , 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0 & , \text{otherwise} \end{cases}$$

[A.U. N/D 2007, N/D 2008]

(1) Find the value of a.

(2) The cumulative distribution function of X.

(3) If x_1, x_2 and x_3 are 3 independent observations of X. What is the probability that exactly one of these 3 is greater than 1.5?

Solution : (1) Since, $f(x)$ is a p.d.f, then

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \quad \text{i.e.,} \int_{0}^{3} f(x)\, dx = 1$$

$$\text{i.e.,} \int_{0}^{1} ax\, dx + \int_{1}^{2} a\, dx + \int_{2}^{3}(3a - ax)\, dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$a\left[\frac{1}{2} - 0\right] + a\,[2 - 1] + \left(9a - \frac{9a}{2}\right) - (6a - 2a) = 1$$

$$\frac{a}{2} + a + \frac{9a}{2} - 4a = 1$$

$$6a - 4a = 1$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

(2) (i) If $x < 0$, then $F(x) = 0$

(ii) If $0 \le x \le 1$, then $F[x] = \int_0^x ax\,dx = \int_0^x \frac{x}{2}\,dx$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{1}{4}\left[x^2\right]_0^x = \frac{x^2}{4}$$

(iii) If $1 \le x \le 2$, then $F[x] = \int_{-\infty}^x f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^x a\,dx = a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^x$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[x\right]_1^x \qquad [\because a = \frac{1}{2}]$$

$$= \frac{1}{4}\left[x^2\right]_0^1 + \frac{1}{2}(x - 1) = \frac{1}{4} + \frac{1}{2}(x - 1)$$

$$= \frac{1}{4} + \frac{x}{2} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4}$$

(iv) If $2 \le x \le 3$, then $F(x) = \int_{-\infty}^x f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a - ax)\,dx$$

(3) $P(X > 1.5) = \int_{1.5}^{3} f(x)\,dx$

$$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left[x\right]_{1.5}^{2} + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_{2}^{3}$$

$$= \frac{1}{2}(2 - 1.5) + \frac{1}{4}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

Choosing an X and observing its value can be considered as a trial and $X > 1.5$ can be considered as a success.

i.e., $p = P[X > 1.5] = \frac{1}{2}$

$\therefore p = \frac{1}{2}, q = \frac{1}{2}$ $\quad [\because q = 1 - p]$

As we choose 3 independent observation of X, $n = 3$.

By Bernoulli's theorem. $\quad \left[P(x) = n\,C_x p^x q^{n-x}\right]$

P(exactly one value > 1.5)

$$= P(1 \text{ success}) = 3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{3-1}$$

$$= 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

Example 1.2.13

Experience has shown that while walking in a certain park, the time X (in mins.), between seeing two people smoking has a density function of the form

$$f(x) = \begin{cases} \lambda x e^{-x}; & x > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

[A.U. N/D 2007]

(1) Calculate the value of λ.
(2) Find the distribution function of X.
(3) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes ? In atleast 7 minutes ?

Solution : Given : $f(x) = \begin{cases} \lambda x e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

(1) Formula : $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^{\infty} \lambda x e^{-x}\, dx = 1$$

$$\lambda \left[x \frac{e^{-\lambda}}{-1} - (1) \frac{e^{-x}}{(-1)^2} \right]_0^{\infty} = 1$$

$$\lambda \left[-x e^{-x} - e^{-x} \right]_0^{\infty} = 1$$

$$\lambda \left[(0 - 0) - (-0 - 1) \right] = 1$$

$$\lambda = 1$$

(2) $F[X] = \int_{-\infty}^{x} f(x)\, dx$ for $x \geq 0$

$$= \int_0^{x} x e^{-x}\, dx$$

$$= \left[x \frac{e^{-x}}{-1} - (1) \frac{e^{-x}}{(-1)^2} \right]_0^{x}$$

$$= (-x e^{-x} - e^{-x}) - (0 - 1)$$

$$= -e^{-x} [x + 1] + 1$$

$$= 1 - (x + 1) e^{-x}$$

(3)(a) $P(2 < X < 5) = F(5) - F(2)$

$$= \left[1 - (5+1)e^{-5}\right] - \left[1 - (2+1)e^{-2}\right]$$

$$= 1 - 6e^{-5} - 1 + 3e^{-2}$$

$$= 3e^{-2} - 6e^{-5}$$

$$= 0.37$$

(3)(b) $P(X \geq 7) = 1 - P[X < 7] = 1 - F($

$$= 1 - [1 - (7 + 1) e^{-7}]$$

$$= 1 - 1 + 8e^{-7}$$

$$= 8e^{-7}$$

$$= 0.007$$

Example 1.2.14

The diameter of an electric cable X is a continuous r.v. with pdf $f(x) = kx(1-x)$, $0 \le x \le 1$

(i) Find the value of k

(ii) c.d.f of X

(iii) the value of a such that $P(X < a) = 2P(X > a)$

(iv) $P\left[X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right]$

Solution :

(i) Formula : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here, $\int_0^1 kx(1-x)\,dx = 1$

$$k\int_0^1 (x-x^2)\,dx = 1$$

$$k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[\left(\frac{1}{2}-\frac{1}{3}\right)-0\right] = 1$$

$$k\left[\frac{1}{6}\right] = 1$$

$$k = 6$$

(ii) Formula : $F(x) = P[X \le x]$

$$= \int_{-\infty}^{x} f(t)\,dt$$

Here, $F[X=x] = \int_0^x k(x-x^2)\,dx$

$$= \int_0^x 6(x-x^2)\,dx \quad [\because k=6]$$

$$= 6\int_0^x (x-x^2)\,dx$$

$$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^x$$

$$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]$$

$$= 3x^2 - 2x^3 \text{ for } 0 \le x \le 1$$

(iii) $P[X < a] = 2P[X > a]$

W.K.T, $P[X < a] = P[X > a] = \frac{1}{2}$

$$\therefore\ P[X < a] = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a k(x - x^2)\,dx = \frac{1}{2}$$

$$\int_0^a 6(x - x^2)\,dx = \frac{1}{2}$$

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

$$6\left[\frac{a^2}{2} - \frac{a^3}{3}\right] = \frac{1}{2}$$

$$3a^2 - 2a^3 = \frac{1}{2}$$

$$6a^2 - 4a^3 = 1$$

$$4a^3 - 6a^2 + 1 = 0$$

$$\therefore\ a = \frac{1}{2}$$

(iv) $P\left[X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right]$

$$= \frac{P\left[\left(X \le \frac{1}{2}\right) \cap \left(\frac{1}{3} < X < \frac{2}{3}\right)\right]}{P\left[\frac{1}{3} < X < \frac{2}{3}\right]}$$

$$= \frac{P\left[\frac{1}{3} < X < \frac{1}{2}\right]}{P\left[\frac{1}{3} < X < \frac{2}{3}\right]} \quad \dots (1)$$

$$P\left[\frac{1}{3} < X < \frac{1}{2}\right] = \int_{1/3}^{1/2} 6(x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2}$$

$$= 6\left[\left(\frac{1}{8} - \frac{1}{24}\right) - \left(\frac{1}{18} - \frac{1}{81}\right)\right]$$

$$= 6\left[\frac{1}{12} - \frac{7}{162}\right] = 6\left[\frac{13}{324}\right] = \frac{13}{54}$$

$$P\left[\frac{1}{3} < X < \frac{2}{3}\right] = \int_{1/3}^{2/3} 6(x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{2/3}$$

$$= 6\left[\left(\frac{4}{18} - \frac{8}{81}\right) - \left(\frac{1}{18} - \frac{1}{81}\right)\right]$$

$$= 6\left[\frac{10}{81} - \frac{7}{162}\right] = \frac{13}{27}$$

$$(1) \Rightarrow \frac{\frac{13}{54}}{\frac{13}{27}} = \frac{13}{54} \times \frac{27}{13} = \frac{1}{2}$$

Example 1.2.17

A continuous random variable X has p.d.f $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such (i) $P[X \le a] = P[X > a]$ (ii) $P[X > b] = 0.05$

Solution :

(i) $P[X \le a] = P[X > a]$

$$\Rightarrow P[X \le a] = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^a 3x^2\,dx = \frac{1}{2}$$

$$\Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

$$\Rightarrow \left[x^3\right]_0^a = \frac{1}{2}$$

$$\Rightarrow a^3 - 0 = \frac{1}{2}$$

$$\Rightarrow a^3 = \frac{1}{2}$$

$$\Rightarrow a = \left(\frac{1}{2}\right)^{1/3}$$

$$\Rightarrow a = 0.7937$$

(ii) $P[X > b] = 0.05$

$$\Rightarrow \int_b^1 f(x)\,dx = 0.05$$

$$\Rightarrow \int_b^1 3x^2\,dx = 0.05$$

$$\Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05$$

$$\Rightarrow \left[x^3\right]_b^1 = 0.05$$

$$\Rightarrow 1 - b^3 = 0.05$$

$$\Rightarrow b^3 = 1 - 0.05$$

$$\Rightarrow b^3 = 0.95$$

$$\Rightarrow b = (0.95)^{1/3}$$

$$\Rightarrow b = 0.9830$$

Problems based on $f(x) = \frac{d}{dx} E[x]$, mean, variance

Example 1.2.18

The cumulative distribution function (cdf) of a random variable X is $F(x) = 1 - (1 + x)e^{-x}$, $x > 0$. Find the probability density function of X. Mean and variance of X. **[AU M/J 2006, AU N/D 2010]**

Solution : Given : $F(x) = 1 - (1 + x)e^{-x}$, $x > 0$

$$= 1 - e^{-x} - xe^{-x},\ x > 0$$

$$\text{p.d.f,} \quad f(x) = \frac{d}{dx}[F(x)]$$

$$= \frac{d}{dx}[1 - e^{-x} - xe^{-x}]$$

$$= 0 + e^{-x} - [x(-e^{-x}) + e^{-x}(1)]$$

$$= e^{-x} + xe^{-x} - e^{-x}$$

$$= xe^{-x}, \quad x > 0$$

$$E[X] = \int_0^\infty x f(x)\, dx = \int_0^\infty x\, x e^{-x}\, dx = \int_0^\infty x^2 e^{-x}\, dx$$

$$= \left[x^2\left[\frac{e^{-x}}{-1}\right] - (2x)\left[\frac{e^{-x}}{(-1)^2}\right] + (2)\frac{e^{-x}}{(-1)^3}\right]_0^\infty$$

$$= (0 - 0 + 0) - (0 - 0 - 2)$$

$$= 2 \qquad [\because e^{-\infty}=0,\ e^{-0}=1]$$

$$E[X^2] = \int_0^\infty x^2 f(x)\, dx = \int_0^\infty x^2 (x e^{-x})\, dx = \int_0^\infty x^3 e^{-x}\, dx$$

$$= \left[x^3\left[\frac{e^{-x}}{-1}\right] - (3x^2)\left[\frac{e^{-x}}{(-1)^2}\right] + (6x)\left[\frac{e^{-x}}{(-1)^3}\right] - (6)\left[\frac{e^{-x}}{(-1)^4}\right]\right]_0^\infty$$

$$= (0 - 0 + 0 - 0) - (0 - 0 + 0 - 6)$$

$$= 6$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 6 - (2)^2$$

$$= 6 - 4$$

$$= 2$$

Example 1.2.22

If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) Show that f(x) is a pdf of a continuous random variable X.

(b) Find its distribution function F(x) **[A.U CBT A/M 2011]**

Solution : To prove : $\int_{-\infty}^{\infty} f(x)\,dx = 1,\ f(x) \geq 0$

Here, $\int_{0}^{\infty} x e^{-x^2/2}\,dx = 1$

$$\text{L.H.S} = \int_{0}^{\infty} x e^{-x^2/2}\,dx = \int_{0}^{\infty} e^{-t}\,dt = \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= 0 - \left(\frac{1}{-1}\right) = 1 = \text{R.H.S}$$

put $t = \frac{x^2}{2}$

$dt = \frac{2x}{2}\,dx$

$x \to 0 \Rightarrow t \to 0$

$x \to \infty \Rightarrow t \to \infty$

$$\text{(b) } F(x) = P(X \leq x) = \int_{0}^{x} f(x)\,dx$$

$$= \int_{0}^{x} x e^{-x^2/2}\,dx = \int_{0}^{x^2/2} e^{-t}\,dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^{x^2/2} = -\left[e^{-t}\right]_0^{x^2/2} = -\left[e^{-x^2/2} - 1\right]$$

$$= 1 - e^{-x^2/2},\ x \geq 0$$