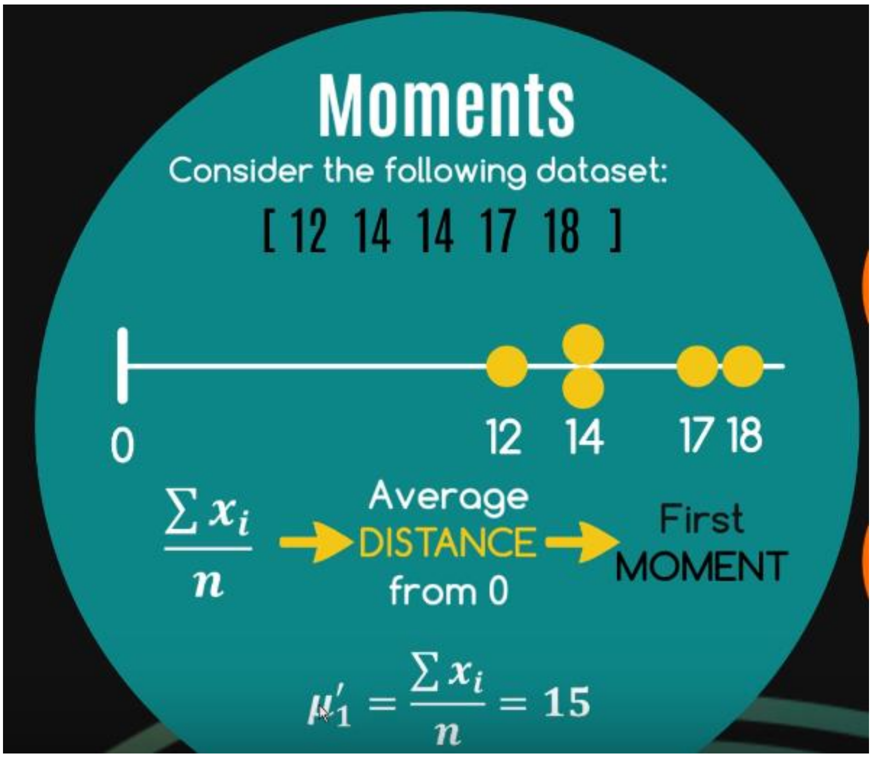
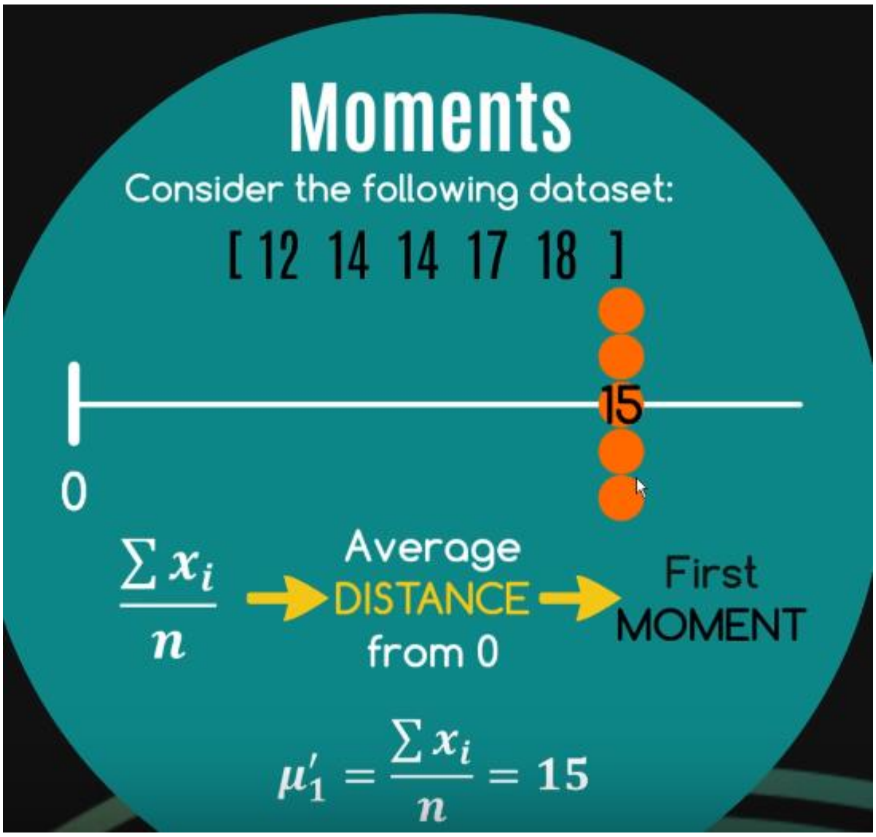
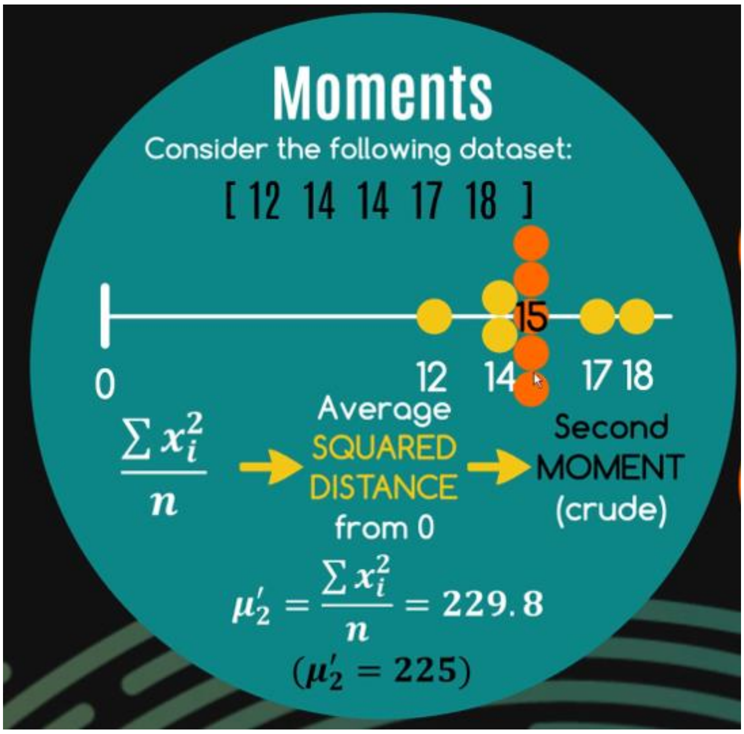
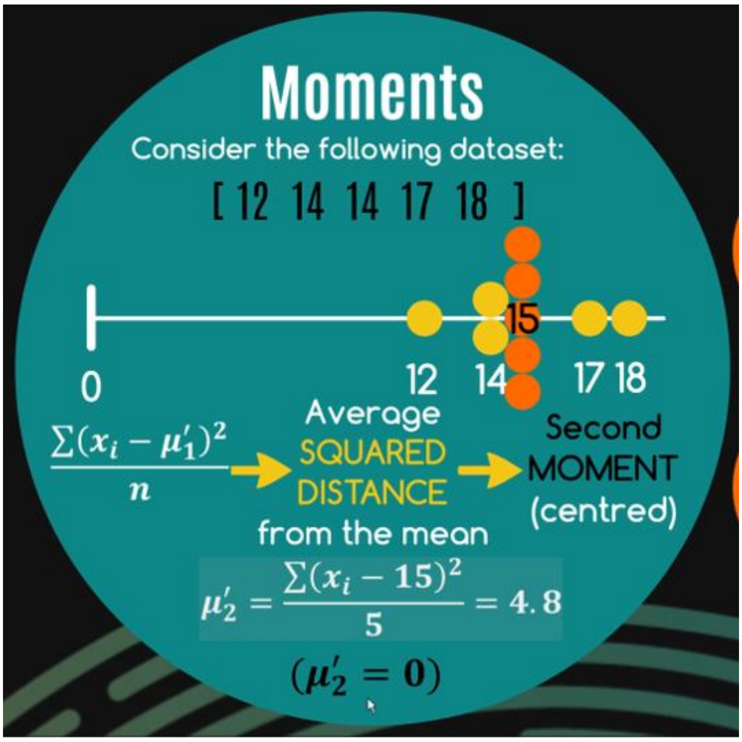




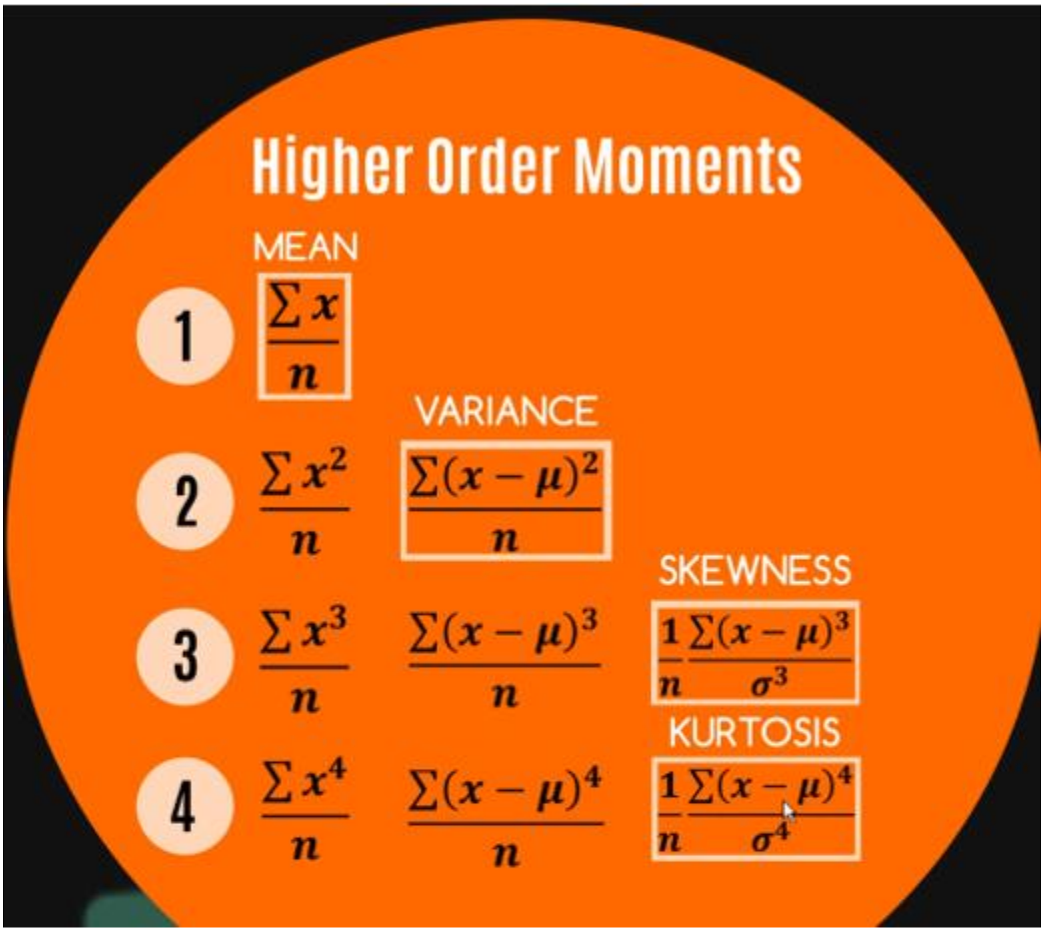
Both the average(1^{st} moment) is 15, then what is the difference between 1^{st} and 2^{nd} moment???



Greater the spread, then greater the average squared distances from 0.
But the difference between two moments is less and not specifying that much.



Now it clearly tells the difference between two date-sets. It is not necessary to start from zero, the point can be anywhere, it can be mean also.



The concept of variance requires mean.
The concept of skewness requires variance.
The concept of kurtosis doesn’t requires skewness.

Moments

Moments [Discrete case]

Let X be discrete R.V. taking the values $x_1, x_2, \ldots x_n$ with probability mass function $p_1, p_2, \ldots p_n$ respectively then the r^{th} moment about the origin is

$$\mu_r' \text{ (about the origin) } = \sum_{i=1}^{n} x_i^r p_i \quad \ldots (1)$$

and

$$\mu_r' \text{ (about any point } x = A) = \sum_{i=1}^{n} (x_i - A)^r p_i \quad \ldots (2)$$

and

$$\mu_r \text{ (about mean) } = \sum_{i=1}^{n} (x_i - \text{Mean})^r p_i \quad \ldots (3)$$

In particular from (1)

$$\mu_1' = \sum_{i=1}^{n} x_i p_i = \text{Mean } (\bar{x})$$

$$\mu_2' = \sum_{i=1}^{n} x_i^2 p_i = \text{Mean square value.}$$

$$\mu_2 = \sum_{i=1}^{n} (x_i - \text{mean})^2 p_i = \text{variance}$$

$$= \mu_2' - (\mu_1')^2 \qquad [\because \bar{x} = \mu_1']$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Moments [Continuous case]

If X is a continuous R.V. with probability density function $f(x)$ defined in the interval (a, b) then

$$\mu_r' = \int_a^b x^r f(x)\, dx$$

$$\mu_r' \text{ (about a point A) } = \int_a^b (x - A)^r f(x)\, dx$$

$$\mu_r \text{ (about the mean) } = \int_a^b (x - \bar{x})^r f(x)\, dx$$

Moments

Let X be a discrete random variable and X takes the values $x_1, x_2, x_3, \ldots, x_n$ with probabilities $p_1, p_2, p_3, \ldots, p_n$ Then r^{th} moment is

- μ_r'(about origin) $= \sum_{i=1}^{n} x_i^r p_r$
- μ_r'(about any point x=A) $= \sum_{i=1}^{n} (x_i - A)^r p_i$
- μ_r'(about mean $\bar{x}$) $= \sum_{i=1}^{n} (x_i - \bar{x})^r p_i$

Similarly, for continuous case on the interval (a, b), the r^{th} moment is

- μ_r'(about origin) $= \int_a^b x^r f(x)dx$
- μ_r'(about any point x=A) $= \int_a^b (x - A)^r f(x)dx$
- μ_r'(about mean $\bar{x}$) $= \int_a^b (x - \bar{x})^r f(x)dx$

Moments

mean $= E(X) = \mu_1'$ and

Variance $= \sigma^2 = E(X^2) - (E(X))^2 = \mu_2{}' - (\mu_1')^2$.

If X is discrete, then $M_X(t) = E(e^{tx}) = \sum_{i=1}^{n} e^{tx_i} p(x_i)$,
where X takes the values $x_1, x_2, x_3, \ldots, x_n$ with probabilities $p(x_1), p(x_2), p(x_3), \ldots p(x_n)$

If X is continuous, then $M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$.

Moment generating function

mean $= E(X) = \mu_1'$ and

Variance $= \sigma^2 = E(X^2) - (E(X))^2 = \mu_2{}' - (\mu_1')^2$.

If X is discrete, then $M_X(t) = E(e^{tx}) = \sum_{i=1}^{n} e^{tx_i} p(x_i)$, where X takes the values $x_1, x_2, x_3, \ldots, x_n$ with probabilities $p(x_1), p(x_2), p(x_3), \ldots p(x_n)$

If X is continuous, then $M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$.

Properties of MGF

- The r^{th} moment μ_r' is the coefficient of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$ in series of powers of t.

$$
\begin{aligned}
M_X(t) = E(e^{tx}) = E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \cdots + \frac{(tx)^r}{r!} + \ldots\right] \\
= 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \cdots + \frac{t^r}{r!}E(x^r) + \ldots \\
= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \ldots
\end{aligned}
$$

Therefore, μ_r' is the coefficient of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$ in series of powers of t.

- $$\mu'_r = E(X^r) = \left[\frac{d^r}{dt^r}(M_X(t))\right]_{t=0}.$$

  In particular, if $r = 1$ then $\mu'_1 = \left[\frac{d(M_X(t))}{dt}\right]_{t=0}$ is the mean of X.
- $M_{cX}(t) = M_X(ct)$
- $M_{aX+b}(t) = e^{bt} M_X(at)$
- $M_{X+Y}(t) = M_X(t)M_Y(t)$

Characteristic function

Let X be a discrete random variable and X takes the values $x_1, x_2, x_3, \dots$ with probabilities $p_1, p_2, p_3, \dots$ Then the characteristic function is defined as $\phi_X(t) = E(e^{itx}) = \sum_r e^{itx_r} p(x_r)$.

Similarly, for continuous case $\phi_X(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$.

Properties of Characteristic function

- $\mu'_r = E(X^r)$=the coefficient of $\frac{i^r t^r}{r!}$ in the expansion of $\phi_X(t)$ in series of ascending powers of it.
- $$\mu'_r = E(X^r) = \frac{1}{i^r}\left[\frac{d^r}{dt^r}(\phi_X(t))\right]_{t=0}.$$

  In particular, if $r = 1$ then $\mu'_1 = \frac{1}{i}\left[\frac{d(\phi_X(t))}{dt}\right]_{t=0}$ is the mean of X.
- $\phi_{aX+b}(t) = e^{ibt}\phi(at)$
- $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$

Moment generating functions and their properties

Moments Generating Function : (M.G.F)

An important device that can be used to calculate the higher moments is the moment generating function.

Moment generating function of a random variable X about the origin is defined as

$$M_X(t) = E[e^{tX}] = \begin{cases} \Sigma_x e^{tx} p(x), \text{ if X is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx, \text{ if X is continuous} \end{cases}$$

Question

Find the moment generating function of the random variable X whose probability mass function $P(X = x) = \frac{1}{2^x}, x = 1, 2, 3, \ldots,$ Deduce the mean and variance from moment generating function.

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \frac{(e^t)^2}{2^2} + \ldots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \ldots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2 - e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2}{2 - e^t}\right]$$

$$= \frac{e^t}{2 - e^t}$$

$$\text{Mean} = m_1 = \left[\frac{d}{dt}\left[\frac{e^t}{2 - e^t}\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}[e^t(2 - e^t)^{-1}]\right]_{t=0}$$

$$= [e^t(-1)(2-e^t)^{-2}(-e^t) + (2-e^t)^{-1}e^t]_{t=0}$$

$$= [e^{2t}(2 - e^t)^{-2} + (2 - e^t)^{-1} e^t]_{t=0}$$

$$= (2 - 1)^{-2} + (2 - 1)^{-1}$$

$$= (1)^{-2} + (1)^{-1}$$

$$= \frac{1}{(1)^2} + \frac{1}{1}$$

$$= 2$$

Question

If X represents the outcome, when a fair die is tossed, find the moment generating function (MGF) of X and hence find E(X) and Var (X).

Moment generating function:

$$M_x(t) \Rightarrow E[e^{tx}]$$

$$\Rightarrow \sum_{x=0}^{6} \left(e^{tx} p(x)\right)$$

$$\Rightarrow e^{t}\left(\tfrac{1}{6}\right) + e^{2t}\left(\tfrac{1}{6}\right) + e^{3t}\left(\tfrac{1}{6}\right) + \dots + e^{6t}\left(\tfrac{1}{6}\right)$$

$$M_x(t) \Rightarrow \left(\tfrac{1}{6}\right)\left[e^{t} + e^{2t} + e^{3t} + \dots + e^{6t}\right]$$

$$\text{Mean} \Rightarrow \frac{d}{dt}\left[M_x(t)\right]_{t=0}$$

$$\Rightarrow \frac{d}{dt}\left[\tfrac{1}{6}\left(e^{t} + e^{2t} + e^{3t} + \dots + e^{6t}\right)\right]_{t=0}$$

$$\Rightarrow \frac{1}{6}\left[e^{t} + e^{2t}(2) + e^{3t}(3) + \cdots + e^{6t}(6)\right]_{t=0}$$

$$\Rightarrow \frac{1}{6}\left[1 + 2 + 3 + \cdots + 6\right] \Rightarrow \frac{1}{6}(21) \Rightarrow 7/2$$

$\boxed{E(x) \Rightarrow \text{Mean} \Rightarrow 7/2}$ 1st moment

$$E(x^2) \Rightarrow \frac{d^2}{dt^2}\left(M_x(t)\right)_{t=0}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{d}{dt}\left(M_x(t)\right)\right]_{t=0}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{6}\left(e^{t} + 2e^{2t} + \cdots + 6e^{6t}\right)\right]_{t=0}$$

$$\Rightarrow \frac{1}{6}\left[e^{t} + 4e^{2t} + 9e^{3t} + \cdots + 36e^{6t}\right]_{t=0}$$

$$\Rightarrow \frac{1}{6}\left[1 + 4(1) + 9 + 16 + \cdots + 36\right]$$

$$\Rightarrow \frac{1}{6}(91) \Rightarrow 91/6$$

$\boxed{E(x^2) \Rightarrow 91/6}$ 2nd moment

$$\text{Variance}(X) \Rightarrow E(x^2) - \left(E(x)\right)^2$$

$$\Rightarrow 91/6 - \left(7/2\right)^2 \Rightarrow 91/6 - 49/4 \Rightarrow \frac{35}{12}$$

$\boxed{\text{Variance}(X) \Rightarrow 35/12}$

Balanced coin

↳ tossed 3 times

(HHH) ; (HHT) ; (HTH) ; (THH) } 8 possible outcomes

(TTT) ; (TTH) ; (THT) ; (HTT) } $(2^3 \Rightarrow 8)$

Balanced coin

↳ tossed 4 times $(2^4 \Rightarrow 16$ possible outcomes$)$

(HHHH) ; (HTHH) ; (HHTH) ; (THHH) ; (HHHT)

(TTTT) ; (THTT) ; (TTHT) ; (HTTT) ; (TTTH)

(HTTH) ; (THHT) ; (HTHT) ; (THTH) ; (HHTT) ; (THTT)

(THHT) ; (HTTH)

*) Exactly 0 heads ⇒ 1 $\left[4C_0 \Rightarrow \frac{4!}{0!\,(4!)} \Rightarrow 1 \right]$

*) Exactly 1 head ⇒ 4 $\left[4C_1 \Rightarrow \frac{4!}{1!\,(3!)} \Rightarrow \frac{4 * 3!}{3!} \Rightarrow 4 \right]$

*) Exactly 2 heads ⇒ 6 $\left[4C_2 \Rightarrow \frac{4!}{2! * 2!} \Rightarrow \frac{\overset{2}{\cancel{4}} * 3 * \cancel{2!}}{(2*1) * \cancel{2!}} \Rightarrow 6 \right]$

*) Exactly 3 heads ⇒ 4 $\left[4C_3 \Rightarrow \frac{4!}{3! * 1!} \Rightarrow \frac{4 * \cancel{3!}}{\cancel{3!}} \Rightarrow 4 \right]$

*) Exactly 4 heads ⇒ $\left[4C_4 \Rightarrow \frac{4!}{0! * 4!} \Rightarrow 1 \right]$

Question

Find the probability distribution of the total number of heads obtained in four tosses of a balanced coin. Hence obtain the MGF of X, mean of X and variance of X. **[AU A/M 2008]**

Solution :

X :	Number of heads obtained in 4 tosses of a coin				
x :	0	1	2	3	4
$p(x)$:	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

x:	0	1	2	3	4
$p(x)$:	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

i) Moment generating function

$$M_x(t) \Rightarrow E[e^{tx}]$$

$$\Rightarrow \sum_{x=0}^{4}\left(e^{tx}p(x)\right)$$

$$\Rightarrow \cancel{e^{0t}(1/16) + e^{2t}(4/16) + e^{3t}(6/16) + e}$$

$$\Rightarrow e^{0t}(1/16) + e^{1t}(4/16) + e^{2t}(6/16) + e^{3t}(4/16) + e^{4t}(1/16)$$

$$\Rightarrow (1/16) + e^{t}(4/16) + e^{2t}(6/16) + e^{3t}(4/16) + e^{4t}(1/16)$$

$$M_x(t) \Rightarrow \frac{1}{16}\left[1 + 4e^{t} + 6e^{2t} + 4e^{3t} + 1e^{4t}\right]$$

ii) <u>Mean</u>

$$\hookrightarrow \frac{d}{dt}\left(M_x(t)\right)_{t=0}$$

$$\Rightarrow \frac{d}{dt}\left[1/16\left(1 + 4e^{t} + 6e^{2t} + 4e^{3t} + e^{4t}\right)\right]_{t=0}$$

$$\Rightarrow 1/16\left(0 + 4e^{t} + 12e^{2t} + 12e^{3t} + e^{4t}(4)\right)_{t=0}$$

$$\Rightarrow \frac{1}{16}\left[0+4(1)+12(1)+12(1)+4(1)\right] \Rightarrow \frac{32}{16} \Rightarrow 2$$

$$\boxed{E(x) \Rightarrow 2}$$

ii) Variance

$$\hookrightarrow \frac{d^2}{dt^2}\left(M_x(t)\right)_{t=0}$$

$$\Rightarrow \frac{d}{dt}\left[4e^{t}+12e^{2t}+12e^{3t}+4e^{4t}\right]_{t=0}\left(\frac{1}{16}\right)$$

$$\Rightarrow \left[4e^{t}+24e^{2t}+36e^{3t}+16e^{4t}\right]_{t=0}\left(\frac{1}{16}\right)$$

$$\Rightarrow (4+24+36+16)\left(\frac{1}{16}\right) \Rightarrow \frac{80}{16} \Rightarrow 5$$

$$\boxed{E(x^2) \Rightarrow 5}$$

$$\text{Variance }(x) \Rightarrow E(x^2) - (E(x))^2$$

$$\Rightarrow 5 + (2)^2$$

$$\boxed{\text{Variance }(x) \Rightarrow 1}$$

Question

For the triangular distribution

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

[A.U. M/J 2006, N/D 2013]

find the mean, variance and the moment generating function (MGF) also find cdf of F(x). **[A.U CBT M/J 2010, CBT N/D 2011]**

[A.U N/D 2013]

Solution : Given : $f(x) = \begin{cases} x\ , & 0 < x \le 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \text{... (1)}$$

$$= \int_0^1 (x)(x)\, dx + \int_1^2 (x)(2-x)\, dx$$

$$= \int_0^1 x^2\, dx + \int_1^2 (2x - x^2)\, dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \left[\frac{1}{3} - 0\right] + \left[x^2 - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = \frac{4 + 1 - 2}{3} = 1$$

$$\text{Variance, } V(X) = E(X^2) - [E(X)]^2 \qquad \text{... (2)}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 x\, dx + \int_1^2 x^2 (2-x)\, dx$$

$$= \int_0^1 x^3\, dx + \int_1^2 (2x^2 - x^3)\, dx = \left[\frac{x^4}{4}\right]_0^1 + \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{4} - 0\right) + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} +$$

$$= -\frac{14}{4} + \frac{14}{3} = \frac{-42 + 56}{12} = \frac{14}{12} = \frac{7}{6}$$

$$\therefore (2) \Rightarrow \text{Var}(X) = E[X^2] - [E(X)]^2$$

$$= \frac{7}{6} - (1)^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

Moment generating function $\Big\}$ $M_x(t) \Rightarrow E(e^{tx}) \Rightarrow \int_{-\infty}^{\infty} \left(e^{tx} f(x)\right) dx$

$$\Rightarrow \int_0^1 e^{tx}(x)\,dx + \int_1^2 e^{tx}(2-x)\,dx$$

$$\Rightarrow \int_0^1 (x e^{tx})dx + 2\int_1^2 e^{tx}dx - \int_1^2 (x e^{tx})dx$$

$\int dv = \int e^{tx}$

$u = x$ | $v \Rightarrow e^{tx}/t$

$u' = 1$ | $v_1 \Rightarrow e^{tx}/t^2$

$$\Rightarrow \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + 2\left[\frac{e^{tx}}{t}\right]_1^2$$

$$- \left[\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_1^2$$

$$\Rightarrow \left[\left(0 - \frac{e^{(0)}}{t^2}\right) + \left(e^t/t - e^t/t^2\right)\right] + 2\left(\frac{e^{2t}}{t} - \frac{e^t}{t}\right)$$

$$- \left[\left(\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2}\right) - \left(\frac{e^t}{t} - \frac{e^t}{t^2}\right)\right]$$

$$\Rightarrow \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} + \frac{1}{t^2} + \cancel{\frac{2e^{2t}}{t}} - \cancel{\frac{2e^t}{t}} - \cancel{\frac{2e^{2t}}{t}} + \frac{e^{2t}}{t^2} + \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}$$

$$\Rightarrow 1/t^2 + e^{2t}/t^2 - 2e^t/t^2 \Rightarrow (1 + e^{2t} - 2e^t)/t^2$$

$$\Rightarrow (1 - 2e^t + e^{2t})/t^2 \Rightarrow (1-e^t)^2/t^2$$

$$\boxed{M_x(t) \Rightarrow (1-e^t)^2/t^2}$$

To find the cdf of F (x)

$$F(x) = P[X \le x] = \int_0^x f(x)\,dx$$

(i) If $x \le 0$, then $F(x) = 0$

(ii) If $0 < x \le 1$, then

$$F(x) = \int_0^x x\,dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

(iii) If $1 \le x < 2$, then

$$F[x] = \int_0^1 x\,dx + \int_1^x (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2} = 2x - \frac{x^2}{2} - 1$$

(iv) If $x > 2$, then

$$F(x) = \int_{-\infty}^x f(x)\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 (2-x)\,dx + \int_2^x 0\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + (4-2) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2 - 2 + \frac{1}{2} = 1$$

Question

Let the random variable X have the p.d.f

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the moment generating function, mean and variance of X.

[A.U. A/M. 2005, N/D 2012]

Solution : The m.g.f is given by

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} e^{tx} \frac{1}{2} e^{-x/2}\, dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{(t-\frac{1}{2})x}\, dx = \frac{1}{2}\int_0^{\infty} e^{-(\frac{1}{2}-t)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(\frac{1}{2}-t)x}}{-\left(\frac{1}{2}-t\right)}\right]_0^{\infty} = -\frac{1}{2}\left[\frac{e^{-(\frac{1}{2}-t)x}}{\frac{1}{2}-t}\right]_0^{\infty}$$

$$= -\frac{1}{2}\left[0 - \frac{1}{\frac{1}{2}-t}\right] = -\frac{1}{2}\left[-\frac{1}{\frac{1-2t}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-2t}\right] = \frac{1}{1-2t}$$

$$E(X) = \text{Mean} = M_X{}'(0) = \frac{d}{dt}\left[\frac{1}{1-2t}\right]_{t=0}$$

$$= \left[\frac{-1}{(1-2t)^2}(-2)\right]_{t=0} = 2$$

$$E(X^2) = M_X{}''(0) = \frac{d}{dt}\left[M_X{}'(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{2}{(1-2t)^2}\right]_{t=0} = \left[\frac{-4}{(1-2t)^3}(-2)\right]_{t=0}$$

$$= \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$\text{Varience} = E(X^2) - (E(X))^2$$

$$= 8 - (2)^2 = 8 - 4 = 4$$

Question

The density function of a random variable x is given by $f(x) = Kx(2-x)$, $0 \le x \le 2$. Find K, mean, variance and r^{th} moment.

[A.U. N/D 2006] [A.U. M/J 2007] [A.U Trichy A/M 2010]

Given : $f(x) = Kx(2-x)$, $0 \le x \le 2$ is a p.d.f.

We know that, if $f(x)$ is a p.d.f then,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad \int_0^2 Kx(2-x)\,dx = 1$$

$$K\int_0^2 (2x - x^2)\,dx = 1 \quad \Rightarrow \quad K\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$K\left[\left(4 - \frac{8}{3}\right) - (0-0)\right] = 1 \quad \Rightarrow \quad K\left[\frac{4}{3}\right] = 1 \Rightarrow K = \frac{3}{4}$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^2 x\, Kx\,(2-x)\, dx$$

$$= \int_0^2 \frac{3}{4}(2x^2 - x^3)\, dx \qquad [\because K = \tfrac{3}{4}]$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\left(\frac{16}{3} - \frac{16}{4}\right) - (0-0)\right]$$

$$= \frac{3}{4}(16)\left[\frac{1}{3} - \frac{1}{4}\right] = 12\left[\frac{1}{12}\right] = 1$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 x^2\, Kx\,(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\, dx \qquad [\because K = \tfrac{3}{4}]$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left[\left(8 - \frac{32}{5}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{40-32}{5}\right] = \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\mu_r{}' = E[X^r] = \int x^r f(x)\, dx = \int_0^2 x^r\, kx\,(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\, dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\left(\frac{2^{r+2}}{r+2}\right) - \frac{2^{r+3}}{r+3}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{(3)\,(2^{r+3})}{4}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= \frac{(3)\,(2^{r+3})}{4}\left[\frac{r+3-r-2}{(r+2)\,(r+3)}\right]$$

$$= \frac{(3)\;(2^{r+1})}{(r+2)\,(r+3)}$$

Question

A continuous R.V. X has the p.d.f f(x) given by $f(x) = c\,e^{-|x|}$, $-\infty < x < \infty$. Find the value of c and moment generating function of X. **[A.U. M/J 2007]**

Solution : Given : $f(x) = c\,e^{-|x|}$

Given $f(x)$ is a p.d.f.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} c\,e^{-|x|}\,dx = 1$$

$$\Rightarrow \quad 2\int_{0}^{\infty} ce^{-x}\,dx = 1 \quad \Rightarrow \quad 2c\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow \quad -2c\left[e^{-x}\right]_0^{\infty} = 1 \quad \Rightarrow \quad -2c\,[0-1] = 1$$

$$\Rightarrow \quad 2c = 1 \quad \Rightarrow \quad c = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{2}e^{-|x|}$$

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx}\,\frac{1}{2}e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}\,e^{-|x|}\,dx = \frac{1}{2}\,2\int_{0}^{\infty} e^{tx}\,e^{-x}\,dx$$