Continuous Random variable

1.2 Continuous Random Variables

(i) Definition : Continuous Random Variable

A random variable X is said to be continuous if it takes all possible values between certain limits say from real number 'a' to real number 'b'.

Example : The length of time during which a vacuum tube installed in a circuit functions is a continuous random variable.

Note : If X is a continuous random variable for any x_1 and x_2

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

(ii) Probability density function :

For a continous random variable X, a probability density function is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)\, dx$ = area under $f(x)$ from a to b

for any a and b.

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

A useful identity is that for any function g,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

(v) The variance of a continuous random variable X.

The variance of X, denoted as V(X) or σ^2, is

Note : A probability density function is zero for the values of X which do not occur and it is assumed to be zero wherever it is not specifically defined.

(iii) Cumulative distribution function

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx \text{ for } -\infty < x < \infty.$$

Note : The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating.

$$\frac{d}{dx}\int_{-\infty}^{x} f(t)\, dt = f(x) \qquad [\because \text{ fundamental theorem of calculus}]$$

$$f(x) = \frac{d}{dx} F[x] \text{ as long as the derivative exists.}$$

(iv) The mean or expected value of a continuous random variable X.

Suppose X is a continuous random variable with probability density function $f(x)$. The mean or expected value of X, denoted as μ or E(X) is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= E[X^2] - [E(X)]^2$$

Note : The standard deviation of X is $\sigma = \sqrt{\text{Var}(X)}$.

(vi) FORMULA

1. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

2. $F[x] = P[X \le x] = \int_{-\infty}^{x} f(t)\, dt, \quad -\infty < x < \infty$

3. $f(x) = \frac{d}{dx} F[x]$

4. $\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$

5. $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

6. $\text{Variance} = \text{Var}[X] = E[X^2] - [E[X]]^2$

7. $P[a \le X \le b] = F(b) - F(a)$

8. $P(a \le X \le b) = P(a \le X < b) = P[a < X \le b]$

$= P[a < X < b]$, X being a continuous random variable.

9. $0 \le F(x) \le 1$

10. $F(x)$ is a non-decreasing function of X.

i.e., if $x_1 < x_2$ then $F(x_1) < F(x_2)$

11. $F[-\infty] = \lim_{x \to -\infty} F(x) = 0$

$F[\infty] = \lim_{x \to \infty} F(x) = 1$

Question

Given that the p.d.f of a R.V. X is f(x) = kx, 0 < x < 1, find K and P(X > 0.5) **[A.U. Dec, 96]**

Solution :

(i) Formula : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Kx\,dx = 1$$

$$k\int_0^1 x\,dx = 1$$

$$K\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$K\left[\frac{1}{2} - 0\right] = 1$$

$$K = 2$$

(ii) $P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$

$$= \int_{1/2}^{1} 2x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \left[x^2\right]_{1/2}^{1}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

Question

If $f(x) = \begin{cases} Kxe^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$ is the p.d.f. of a random variable X.

Find K. [A.U CBT M/J 2010]

Solution : For a p.d.f., $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Here, $\int_{0}^{\infty} Kxe^{-x}\, dx = 1$ $[\because x > 0]$

$$K\left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left[\frac{e^{-x}}{(-1)^2}\right]\right]_0^{\infty} = 1$$

$$K\left[-xe^{-x} - e^{-x}\right]_0^{\infty} = 1$$

$$K\left[(-0-0) - (-0-1)\right] = 1 \Rightarrow K = 1 \qquad [e^{-\infty} = 0]$$

Question

A continuous random variable X has probability density function given by $f(x) = 3x^2, 0 \le x \le 1$. Find K such that $P(X > K) = 0.5$

Solution : [A.U. Model Q. Paper] [A.U N/D 2010]

Given : $P(X > K) = 1 - P[X \le K] = 1 - 0.5 = 0.5$

$$\text{i.e., } \int_0^K f(x)\, dx = 0.5 \Rightarrow \int_0^K 3x^2\, dx = 0.5$$

$$\Rightarrow 3\int_0^K x^2\, dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_0^K = \frac{1}{2} \Rightarrow \left[x^3\right]_0^K = \frac{1}{2}$$

$$\Rightarrow K^3 - 0 = \frac{1}{2} \Rightarrow K^3 = \frac{1}{2} \Rightarrow K = \left(\frac{1}{2}\right)^{\frac{1}{3}} = 0.7937$$

[Note : $P[X \le K] = 1 - P[X > K] = 1 - 0.5 = 0.5$]

Question

Let X be a continuous R.V with probability density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P[X \le 0.4]$ (ii) $P[X > 3/4]$ (iii) $P\left[X > \frac{1}{2}\right]$

(iv) $P[1/2 < X < 3/4]$ (v) $P[X > 3/4 / X > 1/2]$

(vi) $P[X < 3/4 / X > 1/2]$

Solution : $P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

(i)
$$P(X \le 0.4) = \int_0^{0.4} f(x)\,dx = \int_0^{0.4} 2x\,dx = 2\left[\frac{x^2}{2}\right]_0^{0.4} = [x^2]_0^{0.4} = (0.4)^2 - 0 = 0.16$$

(ii)
$$P\left[X > \frac{3}{4}\right] = \int_{3/4}^{1} 2x\,dx = 2\int_{3/4}^{1} x\,dx = 2\left[\frac{x^2}{2}\right]_{3/4}^{1} = \left[x^2\right]_{3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

(iii)
$$P\left[X > \frac{1}{2}\right] = \int_{1/2}^{1} 2x\,dx = 2\int_{1/2}^{1} x\,dx = 2\left[\frac{x^2}{2}\right]_{1/2}^{1} = \left[x^2\right]_{1/2}^{1} = 1 - \frac{1}{4}$$

(iv)
$$P\left[\frac{1}{2} < X < \frac{3}{4}\right] = \int_{1/2}^{3/4} 2x\,dx = 2\int_{1/2}^{3/4} x\,dx = 2\left[\frac{x^2}{2}\right]_{1/2}^{3/4} = \left[x^2\right]_{1/2}^{3/4} = \frac{9}{16} - \frac{1}{4}$$

$$= \frac{3}{4}$$

$$= \frac{9-4}{16} = \frac{5}{16}$$

(v) $P\left[X > \frac{3}{4} / X > \frac{1}{2}\right]$

$$= \frac{P\left[(X > \frac{3}{4}) \cap (X > \frac{1}{2})\right]}{P\left[X > \frac{1}{2}\right]}$$

$$= \frac{P\left[X > \frac{3}{4}\right]}{P\left[X > \frac{1}{2}\right]}$$

$$= \frac{\frac{7}{16}}{\frac{3}{4}} \text{ by (ii) \& (iii)}$$

$$= \frac{7}{16} \times \frac{4}{3}$$

$$= \frac{7}{12}$$

(vi) $P\left[X < \frac{3}{4} / X > \frac{1}{2}\right]$

$$= \frac{P\left[(X < \frac{3}{4}) \cap (X > \frac{1}{2})\right]}{P\left[X > \frac{1}{2}\right]}$$

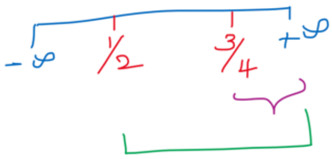
$$= \frac{P\left[\frac{1}{2} < X < \frac{3}{4}\right]}{P\left[X > \frac{1}{2}\right]}$$

$$= \frac{\frac{5}{16}}{\frac{3}{4}} \text{ by (iv) \& (iii)}$$

$$= \frac{5}{16} \times \frac{4}{3}$$

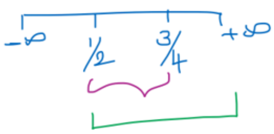
$$= \frac{5}{12}$$

$P(X > 3/4 \mid X > 1/2)$



$$\frac{P(X > 3/4)}{P(X > 1/2)}$$

$P(X < 3/4 \mid X > 1/2)$



$$\frac{P(1/2 < X < 3/4)}{P(X > 1/2)}$$

Question

In a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Verify $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (b) Find $P(0 < X \leq 1)$

(c) Find $F(x)$ [cumulative distribution]

Solution :

(a)
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{2} \frac{x^2}{3}\,dx$$
$$= \frac{1}{3}\int_{-1}^{2} x^2\,dx$$
$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2}$$
$$= \frac{1}{3}\left[\left(\frac{8}{3}\right) - \left(\frac{-1}{3}\right)\right]$$
$$= \frac{1}{3}\left[\frac{8}{3} + \frac{1}{3}\right] = \frac{1}{3}\left[\frac{9}{3}\right] = 1$$

(b)
$$P(0 < X \leq 1) = \int_{0}^{1} \frac{x^2}{3}\,dx$$
$$= \frac{1}{3}\int_{0}^{1} x^2\,dx$$
$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{0}^{1}$$
$$= \frac{1}{3}\left[\frac{1}{3} - 0\right]$$
$$= \frac{1}{9}$$

(c)
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{x} f(t)\,dt = 0 + \int_{-1}^{x} \left(\frac{t^2}{3}\right) dt$$
$$= \frac{1}{3}\int_{-1}^{x} t^2\,dt = \frac{1}{3}\left[\frac{t^3}{3}\right]_{-1}^{x} = \frac{1}{3}\left[\frac{x^3}{3} + \frac{1}{3}\right] = \frac{1}{9}[x^3 + 1]$$

Therefore,
$$F(x) = \begin{cases} 0, & x \leq -1 \\ \frac{x^3 + 1}{9}, & -1 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

Question

A continuous random variable X has the density function $f(x) = \frac{K}{1+x^2}, -\infty < x < \infty$. Find the value of K and the distribution function. [A.U N/D 2011, N/D 2014]

Solution : Given $f(x)$ is a p.d.f

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2}\,dx = 1$$

$$K \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 1$$

$$2K \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 1$$

$$2K \left[\tan^{-1} x\right]_0^{\infty} = 1$$

$$2K \left[\frac{\pi}{2} - 0\right] = 1$$

$$[\because \tan^{-1} \infty = \frac{\pi}{2}]$$

$$\pi K = 1 \Rightarrow K = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(\frac{-\pi}{2}\right)\right]$$

$$[\because \tan^{-1}(-\infty) = -\frac{\pi}{2}]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$$

Question

The p.d.f. of a continuous R.V. X is $f(x) = Ke^{-|x|}$. Find K and the F[x].

[A.U. 2005] [A.U Trichy M/J 2011][A.U A/M 2010]

Solution : Given : $f(x)$ in a p.d.f.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

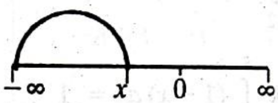
$$\int_{-\infty}^{\infty} Ke^{-|x|}\,dx = 1 \qquad \text{i.e.,} \quad 2\int_{0}^{\infty} Ke^{-x}\,dx = 1$$

$$2K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \qquad \because |x| = x \text{ in the interval } (0, \infty)$$

$$2K\left[\left(\frac{0}{-1}\right) - \left(\frac{1}{-1}\right)\right] = 1$$

$$2K\,[0 + 1] = 1 \;; \quad 2K = 1 \;; \quad K = \frac{1}{2}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

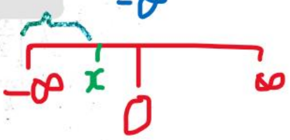


$$\text{Given : } f(x) = Ke^{-|x|} = \begin{cases} Ke^{x}\,; & -\infty < x < 0 \\ Ke^{-x}\,; & 0 < x < \infty \end{cases} = \begin{cases} \frac{1}{2}e^{x}\,; & -\infty < x < 0 \\ \frac{1}{2}e^{-x}\,; & 0 < x < \infty \end{cases}$$

$\int_{-\infty}^{x}$

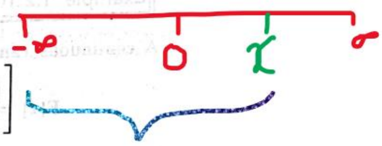
For $x \le 0$, $\quad F(x) = \int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx$

$$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{x} = \frac{1}{2}[e^{x} - 0] = \frac{1}{2}e^{x}$$



For $x > 0$, $\quad F(x) = \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_{0}^{x} \frac{1}{2}e^{-x}\,dx$

$$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{0} + \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^{x}$$



$$= \frac{1}{2}[1 - 0] + \frac{1}{2}\left[\left(\frac{e^{-x}}{-1}\right) - \left(\frac{1}{-1}\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}[-e^{-x} + 1]$$

$\int_{-\infty}^{0} + \int_{0}^{x}$

$$= \frac{1}{2} + \frac{1}{2}[1 - e^{-x}] = \frac{1}{2}[2 - e^{-x}]$$

Question

A continuous random variable X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P[X < 4]$ [A.U M/J 2006, M/J 2007, N/D 2011, N/D 2012]

[A.U CBT N/D 2008, CBT Dec. 2009]

Solution :

(i) Formula : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Here, $\int_{2}^{5} k(1+x)\,dx = 1$

$$k \int_{2}^{5} (1+x)\,dx = 1$$

$$k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1$$

$$k\left[\frac{35}{2} - \frac{8}{2}\right] = 1$$

$$k\left[\frac{27}{2}\right] = 1,\ k = \frac{2}{27}$$

(ii) $P[X < 4] = \int_{2}^{4} f(x)\,dx$

$$= \int_{2}^{4} k(1+x)\,dx$$

$$= \int_{2}^{4} \left(\frac{2}{27}\right)(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{2}{27}[(4+8) - (2+2)]$$

$$= \frac{2}{27}[12 - 4]$$

$$= \frac{16}{27}$$

Question

A continuous random variable X has the distribution function

$$F[x] = \begin{cases} 0 & , \quad x < 1 \\ K(x-1)^4 , & 1 \le x \le 3 \\ 0 & , \quad x > 3 \end{cases}$$

find K, probability density function f(x), P[X<2] [A.U. A/M. 2008]

We know that,

$$P[X \le x] = F[x]$$

$$P[X<2] = F[2] = K(x-1)^4$$

$$f(x) = \frac{d}{dx} F[x]$$

$$= \frac{d}{dx}[K(x-1)^4]$$

$$= K\,4(x-1)^3$$

$$\therefore f(x) = \begin{cases} 0 & , \quad x \le 1 \\ 4K(x-1)^3 , & 1<x\le 3 \\ 0 & , \quad x > 3 \end{cases}$$

We know that,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_1^3 4K(x-1)^3\,dx = 1$$

$$4K\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$K\left[(x-1)^4\right]_1^3 = 1$$

$$K[(3-1)^4 - (1-1)^4] = 1$$

$$K[2^4 - 0] = 1$$

$$16K = 1 \Rightarrow K = \frac{1}{16}$$

$$P[X<2] = F[2] = K(2-1)^4 = \frac{1}{16}(1^4) = \frac{1}{16}$$

Question

Is the function defined as follows, a density function ?

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{1}{18}(3+2x) & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

[A.U N/D 2006]

Solution : Condition for probability density function is $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Given : $$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{1}{18}(3+2x) & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} 0\,dx + \int_{2}^{4} \frac{1}{18}(3+2x)\,dx + \int_{4}^{\infty} 0\,dx$$

$$= 0 + \frac{1}{18}\int_{2}^{4}(3+2x)\,dx + 0 = \frac{1}{18}\left[3x + \frac{2x^2}{2}\right]_2^4$$

$$= \frac{1}{18}[3x + x^2]_2^4 = \frac{1}{18}[(12+16) - (6+4)]$$

$$= \frac{1}{18}[28 - 10] = \frac{1}{18}(18) = 1$$

Hence, the given function is density function.

Question (Important)

If the density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & , 0 \le x \le 1 \\ a & , 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0 & , \text{otherwise} \end{cases}$$

[A.U. N/D 2007, N/D 2008]

(1) Find the value of a.

(2) The cumulative distribution function of X.

(3) If x_1, x_2 and x_3 are 3 independent observations of X. What is the probability that exactly one of these 3 is greater than 1.5?

Solution : (1) Since, $f(x)$ is a p.d.f, then

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \qquad \text{i.e., } \int_{0}^{3} f(x)\, dx = 1$$

$$\text{i.e., } \int_0^1 ax\, dx + \int_1^2 a\, dx + \int_2^3 (3a - ax)\, dx = 1$$

$$a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$a\left[\frac{1}{2}-0\right]+a\,[2-1]+\left(9a-\frac{9a}{2}\right)-(6a-2a) = 1$$

$$\frac{a}{2}+a+\frac{9a}{2}-4a = 1$$

$$6a-4a = 1$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

(2) (i) If $x < 0$, then $F(x) = 0$

(ii) If $0 \le x \le 1$, then $F[x] = \int_0^x ax\,dx = \int_0^x \frac{x}{2}\,dx$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^x = \frac{1}{4}\left[x^2\right]_0^x = \frac{x^2}{4}$$

(iii) If $1 \le x \le 2$, then $F[x] = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^x a\,dx = a\left[\frac{x^2}{2}\right]_0^1 + a\left[x\right]_1^x$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[x\right]_1^x \qquad [\because a = \frac{1}{2}]$$

$$= \frac{1}{4}\left[x^2\right]_0^1 + \frac{1}{2}(x-1) = \frac{1}{4}+\frac{1}{2}(x-1)$$

$$= \frac{1}{4}+\frac{x}{2}-\frac{1}{2} = \frac{x}{2}-\frac{1}{4}$$

(iv) If $2 \le x \le 3$, then $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a-ax)\,dx$$

(3) $P(X > 1.5) = \int_{1.5}^{3} f(x)\,dx$

$$= \int_{1.5}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \left(\frac{3}{2} - \frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left[x\right]_{1.5}^{2} + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_{2}^{3}$$

$$= \frac{1}{2}(2 - 1.5) + \frac{1}{4}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

Choosing an X and observing its value can be considered as a trial and $X > 1.5$ can be considered as a success.

i.e., $p = P[X > 1.5] = \frac{1}{2}$

$\therefore p = \frac{1}{2}, q = \frac{1}{2}$ $\quad [\because q = 1 - p]$

As we choose 3 independent observation of X, $n = 3$.

By Bernoulli's theorem. $\left[P(x) = n\,C_x\,p^x\,q^{n-x}\right]$

P(exactly one value > 1.5)

$$= P(1 \text{ success}) = 3\,C_1 \left(\frac{1}{2}\right)^{1} \left(\frac{1}{2}\right)^{3-1}$$

$$= 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{2} = \frac{3}{8}$$

Question

Experience has shown that while walking in a certain park, the time X (in mins.), between seeing two people smoking has a density function of the form

$$f(x) = \begin{cases} \lambda x e^{-x}; & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

[A.U. N/D 2007]

(1) Calculate the value of λ.

(2) Find the distribution function of X. Cumulative distribution

(3) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes ? In atleast 7 minutes ?

Solution : Given : $f(x) = \begin{cases} \lambda x e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

(1) Formula : $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} \lambda x e^{-x}\,dx = 1$$

$$\lambda \left[x\frac{e^{-\lambda}}{-1} - (1)\frac{e^{-x}}{(-1)^2}\right]_0^{\infty} = 1$$

$$\lambda\left[-xe^{-x} - e^{-x}\right]_0^{\infty} = 1$$

$$\lambda\,[(0-0) - (-0-1)] = 1$$

$$\lambda = 1$$

(2) $F[X] = \int_{-\infty}^{x} f(x)\,dx$ for $x \geq 0$

$$= \int_0^x xe^{-x}\,dx$$

$$= \left[x\frac{e^{-x}}{-1} - (1)\frac{e^{-x}}{(-1)^2}\right]_0^x$$

$$= (-xe^{-x} - e^{-x}) - (0-1)$$

$$= -e^{-x}[x+1] + 1$$

$$= 1 - (x+1)e^{-x}$$

(3)(a) $P(2<X<5) = F(5) - F(2)$

$$= \left[1-(5+1)e^{-5}\right] - \left[1-(2+1)e^{-2}\right]$$

$$= 1 - 6e^{-5} - 1 + 3e^{-2}$$

$$= 3e^{-2} - 6e^{-5}$$

$$= 0.37$$

$$\int_2^5 f(x)\,dx$$

(3)(b) $P(X \geq 7) = 1 - P[X<7] = 1-F($

$$= 1 - [1-(7+1)e^{-7}]$$

$$= 1 - 1 + 8e^{-7}$$

$$= 8e^{-7}$$

$$= 0.007$$

$$\int_7^{\infty} f(x)\,dx$$

Question

The diameter of an electric cable X is a continuous r.v. with pdf $f(x) = kx(1-x)$, $0 \le x \le 1$

(i) Find the value of k

(ii) c.d.f of X

(iii) the value of a such that $P(X < a) = 2P(X > a)$

(iv) $P\left[X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right]$

Solution :

(i) Formula : $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Here, $\int_0^1 kx(1-x)\, dx = 1$

$$k \int_0^1 (x - x^2)\, dx = 1$$

$$k \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k \left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = 1$$

$$k \left[\frac{1}{6}\right] = 1$$

$$k = 6$$

(ii) Formula : $F(x) = P[X \le x]$

$$= \int_{-\infty}^{x} f(t)\, dt$$

Here, $F[X{=}x] = \int_0^x k(x - x^2)\, dx$

$$= \int_0^x 6(x - x^2)\, dx \quad [\because k = 6]$$

$$= 6 \int_0^x (x - x^2)\, dx$$

$$= 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^x$$

$$= 6 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]$$

$$= 3x^2 - 2x^3 \text{ for } 0 \le x \le 1$$

(iii) $P[X < a] = 2P[X > a]$

W.K.T, $P[X < a] = P[X > a] = \frac{1}{2}$

$\therefore P[X < a] = \frac{1}{2}$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a k(x - x^2)\,dx = \frac{1}{2}$$

$$\int_0^a 6(x - x^2)\,dx = \frac{1}{2}$$

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

$$6\left[\frac{a^2}{2} - \frac{a^3}{3}\right] = \frac{1}{2}$$

$$3a^2 - 2a^3 = \frac{1}{2}$$

$$6a^2 - 4a^3 = 1$$

$$4a^3 - 6a^2 + 1 = 0$$

$$\therefore a = \frac{1}{2}$$

(iv) $P\left[X \le \frac{1}{2} / \frac{1}{3} < X < \frac{2}{3}\right]$

$$= \frac{P\left[\left(X \le \frac{1}{2}\right) \cap \left(\frac{1}{3} < X < \frac{2}{3}\right)\right]}{P\left[\frac{1}{3} < X < \frac{2}{3}\right]}$$

$$= \frac{P\left[\frac{1}{3} < X < \frac{1}{2}\right]}{P\left[\frac{1}{3} < X < \frac{2}{3}\right]} \quad \ldots (1)$$

$$P\left[\frac{1}{3} < X < \frac{1}{2}\right] = \int_{1/3}^{1/2} 6(x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2}$$

$$= 6\left[\left(\frac{1}{8} - \frac{1}{24}\right) - \left(\frac{1}{18} - \frac{1}{81}\right)\right]$$

$$= 6\left[\frac{1}{12} - \frac{7}{162}\right] = 6\left[\frac{13}{324}\right] = \frac{13}{54}$$

$$P\left[\frac{1}{3} < X < \frac{2}{3}\right] = \int_{1/3}^{2/3} 6(x - x^2)\,dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{2/3}$$

$$= 6\left[\left(\frac{4}{18} - \frac{8}{81}\right) - \left(\frac{1}{18} - \frac{1}{81}\right)\right]$$

$$= 6\left[\frac{10}{81} - \frac{7}{162}\right] = \frac{13}{27}$$

$$(1) \Rightarrow \frac{\frac{13}{54}}{\frac{13}{27}} = \frac{13}{54} \times \frac{27}{13} = \frac{1}{2}$$

Question

A continuous random variable X has p.d.f $f(x) = 3x^2$, $0 \le x \le 1$.

Find a and b such (i) $P[X \le a] = P[X > a]$ (ii) $P[X > b] = 0.05$

Solution :

(i) $P[X \le a] = P[X > a]$

$$\Rightarrow \quad P[X \le a] = \frac{1}{2}$$

$$\Rightarrow \quad \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \quad \int_0^a 3x^2\,dx = \frac{1}{2}$$

$$\Rightarrow \quad 3\left[\frac{x^3}{3}\right]_0^a = \frac{1}{2}$$

$$\Rightarrow \quad \left[x^3\right]_0^a = \frac{1}{2}$$

$$\Rightarrow \quad a^3 - 0 = \frac{1}{2}$$

$$\Rightarrow \quad a^3 = \frac{1}{2}$$

$$\Rightarrow \quad a = \left(\frac{1}{2}\right)^{1/3}$$

$$\Rightarrow \quad a = 0.7937$$

(ii) $P[X > b] = 0.05$

$$\Rightarrow \quad \int_b^1 f(x)\,dx = 0.05$$

$$\Rightarrow \quad \int_b^1 3x^2\,dx = 0.05$$

$$\Rightarrow \quad 3\left[\frac{x^3}{3}\right]_b^1 = 0.05$$

$$\Rightarrow \quad \left[x^3\right]_b^1 = 0.05$$

$$\Rightarrow \quad 1 - b^3 = 0.05$$

$$\Rightarrow \quad b^3 = 1 - 0.05$$

$$\Rightarrow \quad b^3 = 0.95$$

$$\Rightarrow \quad b = (0.95)^{1/3}$$

$$\Rightarrow \quad b = 0.9830$$

Problems based on $f(x) = \frac{d}{dx} E[x]$, mean, variance

Question

The cumulative distribution function (cdf) of a random variable X is $F(x) = 1 - (1 + x) e^{-x}, x > 0$. Find the probability density function of X. Mean and variance of X. [AU M/J 2006, AU N/D 2010]

Solution : Given :

$$F(x) = 1 - (1 + x) e^{-x}, \; x > 0$$
$$= 1 - e^{-x} - xe^{-x}, \; x > 0$$

p.d.f,
$$f(x) = \frac{d}{dx}[F(x)]$$
$$= \frac{d}{dx}[1 - e^{-x} - xe^{-x}]$$
$$= 0 + e^{-x} - [x(-e^{-x}) + e^{-x}(1)]$$
$$= e^{-x} + xe^{-x} - e^{-x}$$
$$= xe^{-x}, \quad x > 0$$

$$E[X] = \int_0^\infty x f(x)\, dx = \int_0^\infty x\, x\, e^{-x}\, dx = \int_0^\infty x^2 e^{-x}\, dx$$
$$= \left[x^2\left[\frac{e^{-x}}{-1}\right] - (2x)\left[\frac{e^{-x}}{(-1)^2}\right] + (2)\,\frac{e^{-x}}{(-1)^3}\right]_0^\infty$$
$$= (0 - 0 + 0) - (0 - 0 - 2)$$
$$= 2 \qquad [\because e^{-\infty}=0,\; e^{-0}=1]$$

$$E[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 (x e^{-x})\,dx = \int_0^\infty x^3 e^{-x}\,dx$$

$$= \left[x^3\left[\frac{e^{-x}}{-1}\right] - (3x^2)\left[\frac{e^{-x}}{(-1)^2}\right] + (6x)\left[\frac{e^{-x}}{(-1)^3}\right] - (6)\left[\frac{e^{-x}}{(-1)^4}\right]\right]_0^\infty$$

$$= (0 - 0 + 0 - 0) - (0 - 0 + 0 - 6)$$

$$= 6$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 6 - (2)^2$$

$$= 6 - 4$$

$$= 2$$

Question(Important)

If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) Show that f(x) is a pdf of a continuous random variable X.

(b) Find its distribution function F(x) [A.U CBT A/M 2011]

Solution : To prove : $\int_{-\infty}^{\infty} f(x)\,dx = 1,\ f(x) \geq 0$

Here, $\int_0^{\infty} x e^{-x^2/2}\,dx = 1$

$$\text{L.H.S} = \int_0^{\infty} x e^{-x^2/2}\,dx = \int_0^{\infty} e^{-t}\,dt = \left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= 0 - \left(\frac{1}{-1}\right) = 1 = \text{R.H.S}$$

put $t = \dfrac{x^2}{2}$

$dt = \dfrac{2x}{2}\,dx$

$x \to 0 \Rightarrow t \to 0$

$x \to \infty \Rightarrow t \to \infty$

(b) $$F(x) = P(X \leq x) = \int_0^{x} f(x)\,dx$$

$$= \int_0^{x} x e^{-x^2/2}\,dx = \int_0^{x^2/2} e^{-t}\,dt$$

$$= \left[\frac{e^{-t}}{-1}\right]_0^{x^2/2} = -\left[e^{-t}\right]_0^{x^2/2} = -\left[e^{-x^2/2} - 1\right]$$

$$= 1 - e^{-x^2/2},\ x \geq 0$$

$\int dv = \int e \, dx$

$\int_0^\infty (x^3 e^{-x}) dx$

$u = x^3 \qquad v = \frac{e^{-x}}{-1} \Rightarrow -e^{-x}$

$u' = 3x^2 \qquad v_1 = e^{-x}$

$u'' = 6x \qquad v_2 = -e^{-x}$

$u''' = 6 \qquad v_3 = e^{-x}$

$$uv - u'v_1 + u''v_2 - u'''v_3$$

$$-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}$$

$$e^{-x}\left[-x^3 - 3x^2 - 6x - 6\right]_0^\infty$$

$$e^{-\infty}[\infty] - 1[-6] \Rightarrow 6$$

Question **Very very important**

Let X be a continuous random variable with probability density function (pdf) is

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0. & \text{elsewhere.} \end{cases}$$

1. Find $P(X \leq 0.4)$, $P(X \geq \frac{3}{4})$ and $P(X \geq \frac{1}{2}.)$
2. Evaluate $P(\frac{1}{2} < X \leq \frac{3}{4})$ and $P(X > \frac{3}{4}/X > \frac{1}{2})$.
3. Find the cumulative distribution function.

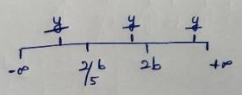
Question Very very important

The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b. The DOE has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \le y \le 2b \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F(y)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate b.

*) From $-\infty$ to y $[-\infty \text{ to } \frac{2}{5}b]$



$$F(y) \Rightarrow \int_{-\infty}^{y} f(y)\, dy$$

$$\boxed{F(y) \Rightarrow 0}$$

*) From $-\infty$ to y $[-\infty \text{ to } 2b]$ $\quad \frac{2}{5}b \le y \le 2b$

$$F(y) \Rightarrow \int_{-\infty}^{\frac{2}{5}b} + \int_{\frac{2}{5}b}^{y} \frac{5}{8b}\, dy$$

$$\Rightarrow \frac{5}{8b}\left[y\right]_{\frac{2}{5}b}^{y} \Rightarrow \frac{5}{8b}\left[y - \frac{2}{5}b\right]$$

$$\Rightarrow \frac{5y}{8b} - \left[\frac{10b}{40b}\right] \Rightarrow \frac{5y}{8b} - \frac{1}{4}$$

$$\boxed{F(y) \Rightarrow \frac{5y}{8b} - \frac{1}{4}}$$

*) From $-\infty$ to y $[-\infty \text{ to } +\infty]$ $\quad \boxed{y \ge 2b}$

$$F(y) \Rightarrow \frac{5(2b)}{8b} - \frac{1}{4} \Rightarrow \frac{10}{8} - \frac{1}{4} \Rightarrow \frac{5}{4} - \frac{1}{4} \Rightarrow \frac{4}{4} \Rightarrow 1$$

$$\boxed{F(y) \Rightarrow 1}$$

To determine the probability that the winning bid is less than the preliminary bid estimate b, we have

$$P(Y \le b) = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}.$$

Joint Probability function (2 variables)

Two dimensional random Variables

Our study of random variables and their probability distributions in the preceding sections was restricted to one-dimensional sample spaces, in that we recorded outcomes of an experiment as values assumed by a single random variable.

There will be situations, however, where we may find it desirable to record the simultaneous outcomes of several random variables.

For example, we might measure the amount of precipitate P and volume V of gas released from a controlled chemical experiment, giving rise to a two-dimensional sample space consisting of the outcomes (p, v), or we might be interested in the hardness H and tensile strength T of cold-drawn copper, resulting in the outcomes (h, t).

The probability density function of Y, denoted by $f_Y(y)$, is called the *marginal* probability density function of Y.

Integrating with respect to y to remove all the y

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

"Integrate away the y-variable"

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

Summing all the possible values of y keeping the x-variable fixed

$$f_X(x_i) = \sum_{j=1}^{m} f(x_i, y_j) \quad \text{for } i = 1, \dots, n$$

Two dimensional random Variables

In a study to determine the likelihood of success in college based on high school data, we might use a threedimensional sample space and record for each individual his or her aptitude test score, high school class rank, and grade-point average at the end of freshman year in college.

For example, if an 18-wheeler is to have its tires serviced and X represents the number of miles these tires have been driven and Y represents the number of tires that need to be replaced, then $p(30000, 5)$ is the probability that the tires are used over 30,000 miles and the truck needs 5 new tires.

Joint probability distribution

Definition

The function $p(x, y)$ is a joint probability distribution or probability mass function of the discrete random variables X and Y if

1. $p(x, y) \geq 0$, *for all* (x, y),
2. $\sum_x \sum_y p(x, y) = 1$,

   where $p(x, y) = P(X = x, Y = y)$.

Joint (vs) Marginal (vs) Conditional probability

Joint probability is the **probability** of two events occurring simultaneously. **Marginal probability** is the **probability** of an event irrespective of the outcome of another variable. Conditional **probability** is the **probability** of one event occurring **in the** presence of a second event. 27 sept. 2019

Question **Very very important**

Continuous case

Suppose the point Probability Density Function (PDF) is given by

$$f(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & ; 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Obtain the marginal PDF of X and that of Y. Hence, otherwise fin

P $\left[\frac{1}{4} \le y \le \frac{3}{4}\right]$.

[A.U. N/D 2004, N/D 2005, N/D 2012

Solution : Given that $f(x, y) = \begin{cases} \frac{6}{5}(x + y^2), & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

The marginal p.d.f of X is,

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \frac{6}{5}\int_0^1 (x + y^2)\, dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_{y=0}^{y=1}$$

$$= \frac{6}{5}\left(x + \frac{1}{3}\right), \; 0 \le x \le 1$$

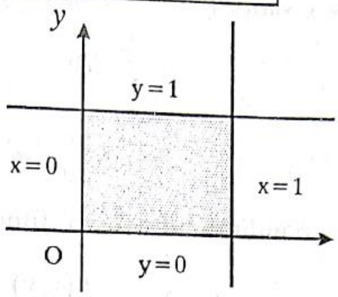
The marginal p.d.f of Y is,

Given : $0 \le x \le 1$;
$0 \le y \le 1$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= \frac{6}{5}\int_0^1 (x + y^2)\, dx = \frac{6}{5}\left[\frac{x^2}{2} + xy^2\right]_0^1$$

$$= \frac{6}{5}\left(y^2 + \frac{1}{2}\right), \; 0 \le y \le 1.$$



$$P\left[\frac{1}{4} \le y \le \frac{3}{4}\right] = \int_{1/4}^{3/4} f(y)\, dy = \int_{1/4}^{3/4} \frac{6}{5}\left(y^2 + \frac{1}{2}\right) dy$$

$$= \frac{6}{5}\left[\frac{y^3}{3} + \frac{y}{2}\right]_{1/4}^{3/4} = \frac{6}{5}\left[\left(\frac{(3/4)^3}{3} + \frac{(3/4)}{2}\right) - \left(\frac{(1/4)^3}{3} + \frac{(1/4)}{2}\right)\right]$$

$$= \frac{6}{5}\left[\left(\frac{9}{64} + \frac{3}{8}\right) - \left(\frac{1}{192} + \frac{1}{8}\right)\right] = \frac{6}{5}\left[\frac{37}{96}\right] = \frac{37}{80} = 0.4625$$

Question **Very very important**

Let X and Y have j.p.d. $f(x, y) = 2, 0 < x < y < 1$. Find the m.d.f. find the conditional density function of Y given $X = x$. [A.U. A/M 2003]

[A.U CBT A/M 2011]

Given : $0 < x < y < 1$;   i.e., choose, $x = 0, x = 1$ $y = 0, y = 1, x < y$

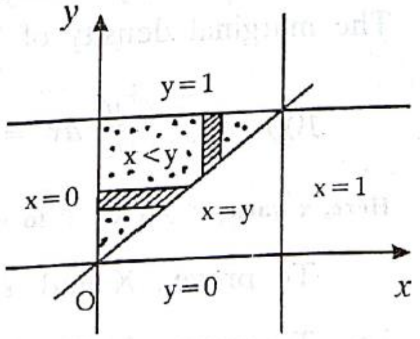
Solution : The marginal density function of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

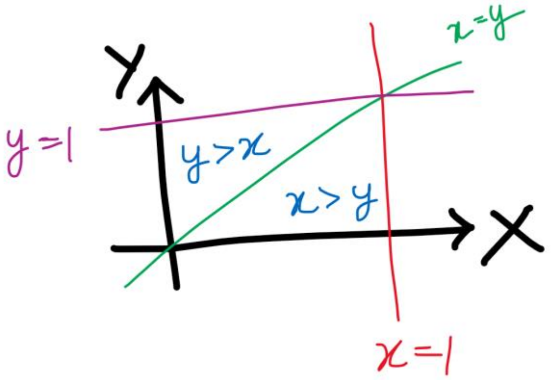
Here y varies from $y = x$ to $y = 1$ [Vertical strip]

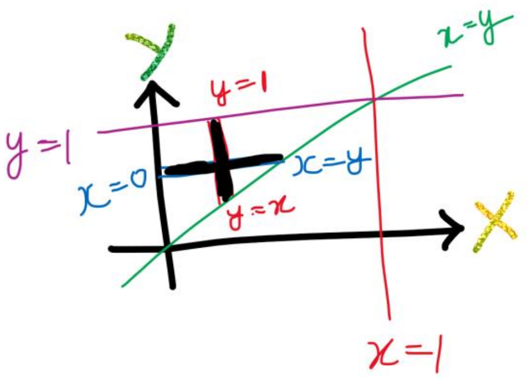
$$= \int_{x}^{1} 2dy \qquad [\because x < y < 1]$$

$$= [2y]_x^1 = 2(1 - x),\ 0 < x < 1$$



The marginal density function of Y is given by,





x varies from $x=0$ to $x=y$

Mariginal probability density function of Y $\Big\}$ $\int_0^y 2 \, dx$

$2[x]_0^y \Rightarrow 2y$

Question **Very very important**

Let X and Y have the joint densities function $f(x,y) = k(x-y)$, $0 \le y \le x \le 1$ and 0 elsewhere.

(a)Find k. (b)Find the marginal densities of X and Y.

Solution:

(a)

$$f(x,y) = k(x-y) \quad , \quad 0 \le y \le x \le 1$$

$$\int_0^1 \int_0^x k(x-y)\,dy\,dx$$

$$= \int_0^1 (kxy - \frac{1}{2}ky^2)\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 kx^2 - \frac{1}{2}kx^2\,dx$$

$$= \frac{k}{2}(\frac{x^3}{3})\Big|_0^1 = \frac{k}{6} = 1, \therefore k = 6$$

(b) The marginal densities of X and Y is

$$f_X(x) = \int_0^x 6(x-y)\,dy$$

$$f_Y(y) = \int_y^1 6(x-y)\,dx$$

Question **Very very important**

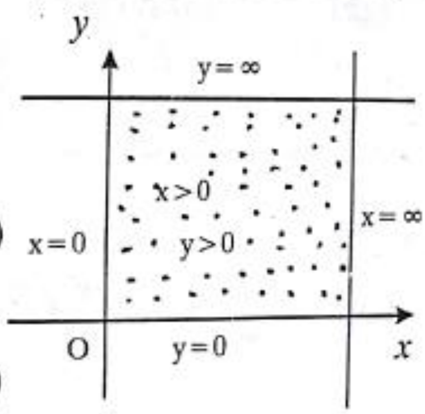
The joint p.d.f of the random variable (X, Y) is given by $f(x, y) = Kxy\, e^{-(x^2 + y^2)}$, $x > 0, y > 0$. Find the value of K and also prove that X and Y are independent.

[A.U. May, 2000, 2004, N/D 2006, N/D 2011, M/J 2012]
[N/D 2007, M/J 2009, Tvli A/M 2009, N/D 2013]

Solution : Here, the range space is the entire first quadrant of the XY-plane.

$$\int_0^\infty \int_0^\infty Kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1 \qquad \ldots (1)$$

Given : $x > 0$,
$y > 0$



$$K \int_0^\infty \int_0^\infty xy\, e^{-x^2}\, e^{-y^2}\, dx\, dy = 1$$

$$K \left[\int_0^\infty x\, e^{-x^2}\, dx\right] \left[\int_0^\infty y\, e^{-y^2}\, dy\right] = 1 \qquad \ldots (2)$$

Take, $$\int_0^\infty x\, e^{-x^2}\, dx \qquad \ldots (3)$$

Put $x^2 = t$ | $x \to 0 \Rightarrow t \to 0$

$2x\, dx = dt$ | $x \to \infty \Rightarrow t \to \infty$

$x\, dx = \frac{1}{2} dt$

$$(3) \Rightarrow \; = \int_0^\infty e^{-t}\, \frac{1}{2}\, dt = \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_0^\infty = -\frac{1}{2}\left[e^{-t}\right]_0^\infty$$

$$= -\frac{1}{2}[0 - 1] = \frac{1}{2} \qquad [\because e^{-\infty} = 0]$$

$$\therefore \int_0^\infty x e^{-x^2}\, dx = \int_0^\infty y e^{-y^2}\, dy = \frac{1}{2} \quad \ldots (4)$$

[x and y are dummy variables]

$$\therefore (2) \Rightarrow k \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = 1$$

$$\Rightarrow \frac{k}{4} = 1 \quad \Rightarrow k = 4$$

To prove : X and Y are independent

i.e., To prove : $\boxed{f(x)\, f(y) = f(x,y)}$

Proof : The marginal density of X is given by

$$f(x) = \int_0^\infty f(x,y)\, dy$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dy$$

$$= \int_0^\infty 4xy\, e^{-x^2} e^{-y^2}\, dy$$

$$= 4x e^{-x^2} \int_0^\infty y e^{-y^2}\, dy$$

$$= 4x e^{-x^2} \left[\frac{1}{2}\right] \quad \text{by } (4)$$

$$= 2x e^{-x^2}, \quad x > 0$$

The marginal density of Y is given by

$$f(y) = \int_0^\infty f(x,y)\, dx$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dx$$

$$= \int_0^\infty 4xy\, e^{-x^2} e^{-y^2}\, dx$$

$$= 4y\, e^{-y^2} \int_0^\infty x e^{-x^2}\, dx$$

$$= 4y e^{-y^2} \left[\frac{1}{2}\right] \quad \text{by } (4)$$

$$= 2y e^{-y^2}, \; y > 0$$

$$f(x) f(y) = (2x e^{-x^2})(2y e^{-y^2}) = 4xy\, e^{-(x^2+y^2)} = f(x,y)$$

$\therefore$ X and Y are independent.

Question Very very important

Given $f_{xy}(x, y) = Cx(x-y), 0 < x < 2, -x < y < x$ and 0, elsewhere

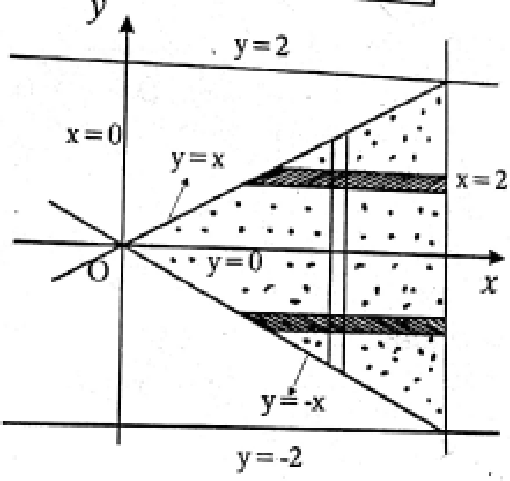
(a) Evaluate C ; (b) Find $f_X(x)$; (c) $f_{Y/X}\left(\frac{y}{x}\right)$. and (d) $f_Y(y)$

[A.U. N/D 2004, M/J 2006, N/D 2010, M/J 2013]

Solution : By the property of j.p.d.f, we have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Given : $0 < x < 2,$
$-x < y < x$



$$\int_0^2\int_{-x}^{x} Cx(x-y)\,dy\,dx = 1$$

$$C\int_0^2\int_{-x}^{x}\left[x^2 - xy\right] dy\,dx = 1$$

$$C\int_0^2\left[x^2y - \frac{xy^2}{2}\right]_{y=-x}^{y=x} dx = 1$$

$$C\int_0^2\left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right] dx = 1$$

$$C\int_0^2\left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right] dx = 1$$

$$C\int_0^2 2x^3\,dx = 1$$

$$2C\left[\frac{x^4}{4}\right]_0^2 = 1 \Rightarrow 2C\left[\frac{16}{4} - 0\right] = 1 \Rightarrow 8C = 1 \Rightarrow C = \frac{1}{8}$$

(b) The marginal density of X is given by,

$$f(x) = \int_{-x}^{x} f(x,y)\,dy = \frac{1}{8}\int_{-x}^{x} x(x-y)\,dy$$

Conditional probability of Joint probability density function

$$f_{Y/X}\left(\frac{y}{x}\right). \qquad f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)}$$

gives

Joint probability density function(x,y) in the marginal density function(x)

$$= \frac{1}{8}\int_{-x}^{x}(x^2 - xy)\,dy = \frac{1}{8}\left[x^2 y - \frac{xy^2}{2}\right]_{y=-x}^{y=x}$$

$$= \frac{1}{8}\left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right] = \frac{1}{8}\left[2x^3\right] = \frac{x^3}{4},\ 0 < x < 2$$

$$\text{(c) } f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}} = \frac{1}{2x^2}(x-y),\ -x < y < x$$

$$\text{(d) } f(y) = \begin{cases} \int_{-y}^{2}\frac{1}{8}x(x-y)\,dx, & \text{if } -2 \le y \le 0 \\ \int_{y}^{2}\frac{1}{8}x(x-y)\,dx & \text{if } 0 \le y \le 2 \end{cases}$$

$$= \begin{cases} \frac{1}{8}\int_{-y}^{2}(x^2 - xy)\,dx \\ \frac{1}{8}\int_{y}^{2}(x^2 - xy)\,dx \end{cases} = \begin{cases} \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{x=-y}^{x=2} \\ \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{x=y}^{x=2} \end{cases}$$

$$= \begin{cases} \frac{1}{8}\left[\left(\frac{8}{3} - 2y\right) - \left(\frac{-y^3}{3} - \frac{y^3}{2}\right)\right] \\ \frac{1}{8}\left[\left(\frac{8}{3} - 2y\right) - \left(\frac{y^3}{3} - \frac{y^3}{2}\right)\right] \end{cases} = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5}{48}y^3 & \text{if } -2 \le y \le \\ \frac{1}{3} - \frac{y}{4} + \frac{1}{48}y^3 & \text{if } 0 \le y \le 2 \end{cases}$$

Question

For random variables X and Y, the joint probability density function is given by

$$f_{X,Y}(x,y) = \frac{1+xy}{4} \quad |x| \le 1,\ |y| \le 1$$

$$= 0 \quad \text{otherwise}$$

Find the marginal density $f_X(x), f_Y(y)$ and $f_{Y/X}(y/x)$. Are X and Y independent?

$$f_X(x) = \int_{-1}^{1} \frac{1+xy}{4} dy$$

$$= \frac{1}{2}$$

Similarly

$$f_Y(y) = \frac{1}{2} \qquad -1 \le y \le 1$$

and

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{1+xy}{2} \qquad \ne f_Y(y)$$

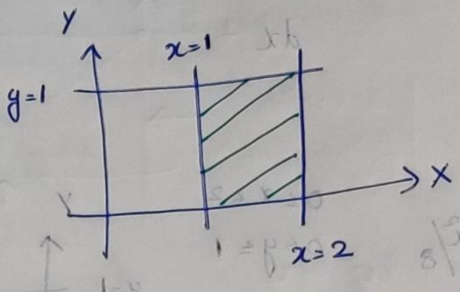
Hence, X and Y are not independent.

Question

Question

$$f(x,y) \Rightarrow xy^2 + x^2/8 \qquad 0 \le x \le 2, \quad 0 \le y \le 1$$

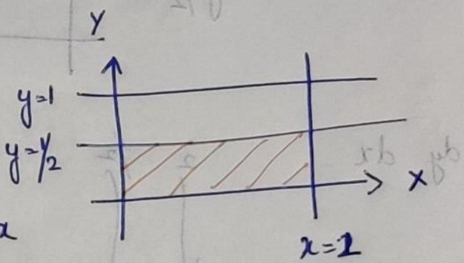
i) $P(X>1)$



$$\Rightarrow \int_1^2 \int_0^1 \left(xy^2 + x^2/8\right) dy\, dx$$

$$\Rightarrow 19/24$$

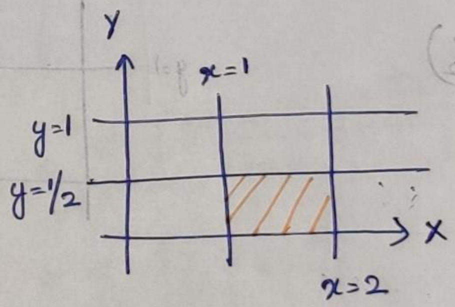
ii) $P(Y<1/2)$



$$\Rightarrow \int_0^2 \int_0^{1/2} \left(xy^2 + x^2/8\right) dy\, dx$$

$$\Rightarrow 1/4$$

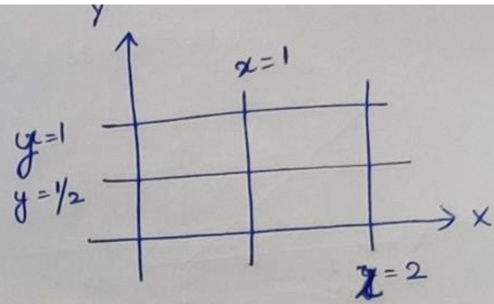
iii) $P(X>1, Y<1/2)$



$$\int_1^2 \int_0^{1/2} \left(xy^2 + x^2/8\right) dy\, dx$$

$$\Rightarrow 5/24$$

iv) $P(X > 1 \mid Y < 1/2)$



$\hookrightarrow \dfrac{P(X > 1, Y < 1/2)}{P(Y < 1/2)} \Rightarrow \dfrac{5/24}{1/4}$

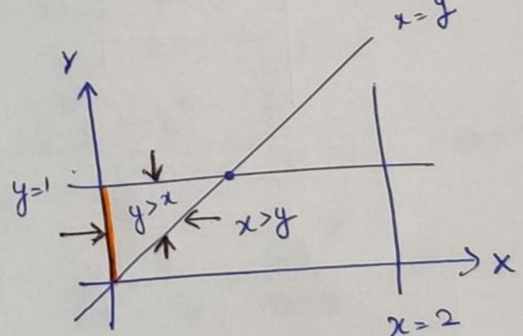
$\Rightarrow 5/6$

v) $P(Y < 1/2 \mid X > 1)$

$\hookrightarrow \dfrac{P(Y < 1/2, X > 1)}{P(X > 1)} \Rightarrow \dfrac{P(X > 1, Y < 1/2)}{P(X > 1)} \Rightarrow \dfrac{5/24}{19/24} \Rightarrow 5/19$

vi) $P(X < Y)$

$$\int_0^y \int_0^1 \left(xy^2 + x^2/8\right) dy\, dx \Rightarrow 53/480$$



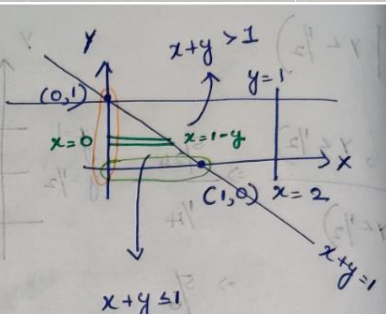
Here $(0.2, 0.9)$ $y > x$

Here $(0.9), 0.2)$ $x > y$

vii) $P(X + Y \le 1)$

$$\int_0^1 \int_0^{1-y} \left(xy^2 + x^2/8\right) dx\, dy$$

$\Rightarrow 13/480$



(or)

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \left(xy^2 + x^2/8\right) dy\, dx$$

$\hookrightarrow 13/480$

Question

If the joint pdf of a two-dimensional random variable (X, Y) is given by,

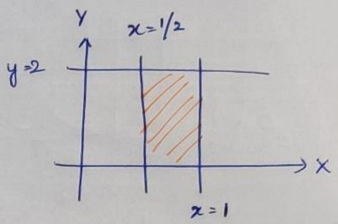
$$f(x, y) = x^2 + \frac{xy}{3},\ 0 < x < 1;\ 0 < y < 2$$
$$= 0, \quad \text{elsewhere}$$

[A.U. M/J 2006] [A.U. N/D 2006]

Find (i) $P\left(X > \frac{1}{2}\right)$; (ii) P (Y < X) and (iii) $P\left(P(Y < \frac{1}{2} / X < \frac{1}{2}\right)$

Check whether the conditional density functions are valid.

$f(x,y) \Rightarrow x^2 + xy/3 \qquad 0<x<1,\ 0<y<2$



i) $P(X > 1/2)$

$$\int_{y=0}^{y=2}\int_{x=1/2}^{x=1}\left(x^2 + xy/3\right) dx\, dy$$

$$\Rightarrow \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_{1/2}^{1} dy$$

$$\Rightarrow \int_0^2 \left[\left(1/3 + y/6\right) - \left(\frac{1/8}{3} + \frac{1/4\, y}{6}\right)\right] dy$$

$$\Rightarrow \int_0^2 \left(1/3 + y/6 - 1/24 - y/24\right) dy$$

$$\Rightarrow \left[y/3 + y^2/12 - y/24 - y^2/48\right]_0^2$$

$$\Rightarrow 2/3 + 4/12 - 2/24 - 4/48$$

$$\Rightarrow \cancel{2/3 + 1/3 - 1/12 - 1/12}$$

$$\Rightarrow \cancel{3/3}\quad 2/3 + 1/3 - 1/12 - 1/12$$

$$\Rightarrow 3/3 - 2/12 \Rightarrow 1 - 1/6 \Rightarrow 5/6$$

$$\boxed{P(X > 1/2) \Rightarrow 5/6}$$

3	3, 12, 24, 48
4	1, 4, 8, 16
2	1, 1, 2, 4
2	1, 1, 1, 2
	1, 1, 1, 1

Method-1

i) $P(X > 1/2)$

*) Find the marginal density function of X

*) Substitute the limit

Marginal density function of X $\Rightarrow f_X(x)$

$$f_X(x) = \int_{y=0}^{2} \left(x^2 + \frac{xy}{3}\right) dy$$

$$\Rightarrow \left[x^2 y + \frac{xy^2}{6}\right]_0^2$$

$$\Rightarrow \cancel{2x^2 + \frac{2y^2}{6}} \Rightarrow 2x^2 + \frac{4x}{6} \Rightarrow 2x^2 + \frac{2}{3}x$$

$$\boxed{f_X(x) \Rightarrow 2x^2 + \frac{2}{3}x \quad (0 < x < 1)}$$

Marginal density function of $Y \Rightarrow f_Y(y)$

$$f_Y(y) \Rightarrow \int_{x=0}^{x=1} \left(x^2 + \frac{xy}{3}\right) dx$$

$$\Rightarrow \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^1$$

$$\Rightarrow \left(\frac{1}{3} + \frac{y}{6}\right) \Rightarrow \frac{6+3y}{18} \Rightarrow \frac{\cancel{3}(2+y)}{\cancel{18}\,6}$$

$$\boxed{f_Y(y) \Rightarrow \frac{2+y}{6} \quad (0 < y < 2)}$$

Method-2

$P(X > 1/2)$

$$\hookrightarrow \int_{x=1/2}^{x=1} f_X(x)\, dx$$

$$\Rightarrow \int_{x=1/2}^{x=1} \left(2x^2 + \frac{2}{3}x\right) dx$$

$$\Rightarrow \left[\frac{2x^3}{3} + \frac{\cancel{2}x^2}{3(\cancel{2})}\right]_{x=1/2}^{x=1}$$

$$\Rightarrow \frac{2}{3}(1) + \frac{1}{3}(1) - \left(\frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3}\left(\frac{1}{2}\right)^2\right)$$

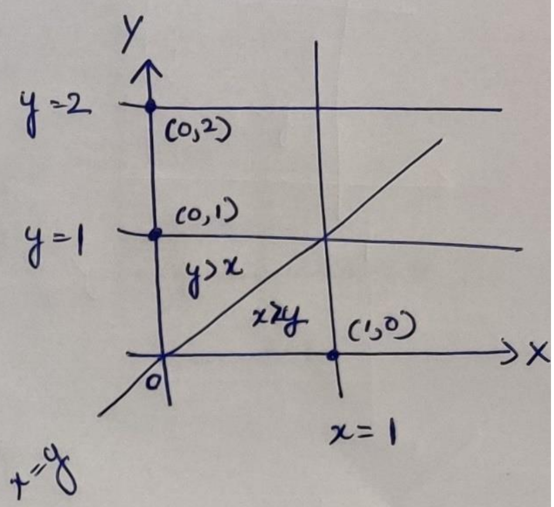
$$\Rightarrow \frac{3}{3} - \left(\frac{2}{3}\left(\frac{1}{\cancel{8}_{4}}\right) + \frac{1}{3}\left(\frac{1}{4}\right)\right)$$

$$\Rightarrow 1 - \left(\frac{1}{12} + \frac{1}{12}\right) \Rightarrow \left(1 - \frac{2}{12}\right) \Rightarrow \left(1 - \frac{1}{6}\right) \Rightarrow \frac{5}{6}$$

$$\boxed{P(X > 1/2) \Rightarrow 5/6}$$

ii) $P(Y < x)$

$$\Rightarrow \int_{x=0}^{1} \int_{y=0}^{y=x} \left(x^2 + xy/3\right) dy\, dx$$



$$\Rightarrow \int_0^1 \left(\left[x^2 y + \frac{xy^2}{6}\right]_0^x\right) dx$$

$$\Rightarrow \int_0^1 \left(x^3 + \frac{x^3}{6}\right) dx \Rightarrow \cancel{\left[\frac{x^5}{5} + \frac{x^4}{24}\right]_0^1}$$

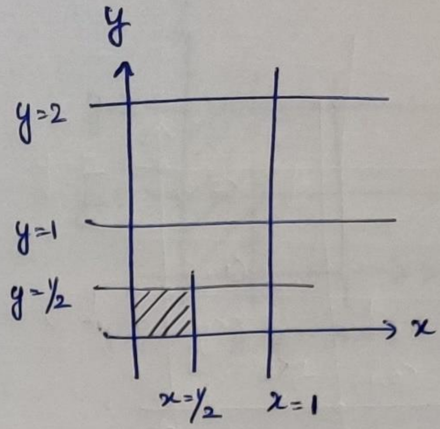
$$\Rightarrow \cancel{1/5 + 1/24 \Rightarrow \frac{24+5}{120} \Rightarrow \frac{29}{120}}$$

$$\Rightarrow \left[\frac{x^4}{4} + \frac{x^4}{24}\right]_0^1 \Rightarrow 1/4 + 1/24 \Rightarrow \frac{24+4}{96} \Rightarrow \frac{\cancel{28}^{\,7}}{\cancel{96}_{\,24}} \Rightarrow 7/24$$

$$\boxed{P(Y < X) \Rightarrow 7/24}$$

iii) $P\left(Y < \frac{1}{2} \mid X < \frac{1}{2}\right)$

$$\hookrightarrow \frac{P(Y < \frac{1}{2},\ X < \frac{1}{2})}{P(X < \frac{1}{2})}$$



$$P(Y < \tfrac{1}{2},\ X < \tfrac{1}{2}) \Rightarrow \int_0^{1/2}\int_0^{1/2} f(x,y)\, dy\, dx$$

$$\Rightarrow \int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$\Rightarrow \int_0^{1/2} \left[x^2 y + \frac{xy^2}{6}\right]_0^{1/2} dx$$

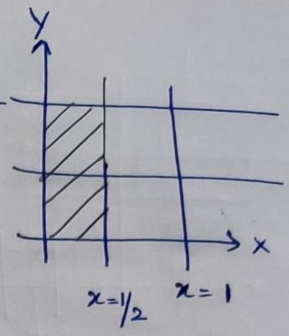
$$\Rightarrow \cancel{\int_0^{1/2} \frac{x^3 y}{3} + \frac{x^2 y^2}{12}} \Rightarrow \int_0^{1/2} \left(\tfrac{1}{2}x^2 + \frac{\frac{1}{4}x}{6}\right) dx$$

$$\Rightarrow \int_0^{1/2} \left(\tfrac{1}{2}x^2 + \tfrac{1}{24}x\right) dx \Rightarrow \left[\tfrac{1}{2}\, x^3/3 + \tfrac{1}{24}\, x^2/2\right]_0^{1/2}$$

$$\Rightarrow \left(\tfrac{1}{2}\,\frac{1/8}{3} + \tfrac{1}{24}\,\frac{1/4}{2}\right) \Rightarrow \left(\frac{1}{48} + \frac{1}{192}\right) \Rightarrow 5/192$$

$$\boxed{P(Y < \tfrac{1}{2},\ X < \tfrac{1}{2}) \Rightarrow 5/192}$$

$P(X < 1/2) \Rightarrow$



$$\int_{y=0}^{2} \int_{x=0}^{1/2} f(x,y)\, dx\, dy$$

$$\Rightarrow \int_{y=0}^{2} \int_{x=0}^{1/2} \left(x^2 + xy/3\right) dx\, dy$$

$$\Rightarrow \int_{y=0}^{2} \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^{1/2} dy$$

$$\Rightarrow \int_{y=0}^{2} \left(\frac{1}{24} + \frac{y}{24}\right) dy$$

$$\Rightarrow \int_{y=0} \left[y/24 + y^2/48\right]_0^2 \Rightarrow 2/24 + 4/48 \Rightarrow 2/24 + 2/24$$

$$\Rightarrow 4/24 \Rightarrow 1/6$$

$$\boxed{P(X < 1/2) \Rightarrow 1/6}$$

$$P(Y < 1/2 \mid X < 1/2) \Rightarrow \frac{P(Y < 1/2,\ X < 1/2)}{P(X < 1/2)} \Rightarrow \frac{5/192}{1/6} \Rightarrow 5/32$$

$$\boxed{P(Y < 1/2 \mid X < 1/2) \Rightarrow 5/32}$$

Validating Conditional Density functions

Conditional density functions

*) Conditional density function of x with given y } $\int_{x=0}^{1} \frac{f_{x,y}(x,y)}{\boxed{f_Y(y)}}\, dx$

*) Conditional density function of Y with given X } $\int_{y=0}^{2} \frac{f_{x,y}(x,y)}{\boxed{f_X(x)}}\, dy$

$$\int_0^1 f(x/y)\, dx$$

$$= \int_0^1 \frac{f(x,y)}{f(y)}\, dx$$

$$= \int_0^1 \left(\frac{6x^2 + 2xy}{2+y}\right) dx$$

$$= \frac{1}{2+y}\left[6\frac{x^3}{3} + 2y\frac{x^2}{2}\right]_{x=0}^{x=1}$$

$$= \frac{1}{2+y}\left[(2+y) - (0+0)\right]$$

$$= \frac{1}{2+y}(2+y)$$

$$= 1$$

$$\int_0^2 f(y/x)\, dy$$

$$= \int_0^2 \frac{f(y,x)}{f(x)}\, dx$$

$$= \int_0^2 \frac{f(x,y)}{f(x)}\, dx$$

$$= \int_0^2 \left(\frac{3x+y}{6x+2}\right) dy$$

$$= \frac{1}{6x+2}\int_0^2 (3x+y)\, dy$$

$$= \frac{1}{6x+2}\left[3xy + \frac{y^2}{2}\right]_{y=0}^{y=2}$$

$$= \frac{1}{6x+2}\left[(6x+2) - (0+0)\right]$$

$$= 1$$