**Example 2.1.3**

The joint probability mass function of (X, Y) is given by $P(x, y) = K(2x + 3y)$, $x = 0, 1, 2$ ; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also, find the probability distribution of $(X + Y)$ and $P[X + Y > 3]$. [A.U. 2004]

[A.U N/D 2007] [A.U N/D 2008] [A.U. Tvli. A/M 2009]

[A.U CBT A/M 2011, N/D 2011, N/D 2013] [A.U N/D 2014]

| Y \ X | 1 | 2 | 3 | Total |
|---|---|---|---|---|
| 0 | 3K | 6K | 9K | 18K |
| 1 | 5K | 8K | 11K | 24K |
| 2 | 7K | 10K | 13K | 30K |
| Total | 15K | 24K | 33K | 72K |

Total Probability = 1

$\therefore$ 72K = 1

$\Rightarrow K = \dfrac{1}{72}$

*Solution :*

| Y \ X | 1 | 2 | 3 | $P_X(x) = P_{i*}$ |
|---|---|---|---|---|
| 0 | $\frac{3}{72}$ P(0,1) | $\frac{6}{72}$ P(0,2) | $\frac{9}{72}$ P(0,3) | $P(X=0)=\frac{18}{72}$ sum of the I row |
| 1 | $\frac{5}{72}$ P(1,1) | $\frac{8}{72}$ P(1,2) | $\frac{11}{72}$ P(1,3) | $P(X=1)=\frac{24}{72}$ sum of the II row |
| 2 | $\frac{7}{72}$ P(2,1) | $\frac{10}{72}$ P(2,2) | $\frac{13}{72}$ P(2,3) | $P(X=2)=\frac{30}{72}$ sum of the III row |
| $P_Y(y)=P_{*j}$ | $P(Y=1)=\frac{15}{72}$ sum of the I column | $P(Y=2)=\frac{24}{72}$ sum of the II column | $P(Y=3)=\frac{33}{72}$ sum of the III column | 1 |

The marginal distribution of X :

$$P(X = 0) = \frac{18}{72}\ ;\ P(X = 1) = \frac{24}{72}\ ;\ P(X = 2) = \frac{30}{72}$$

The marginal distributions of Y :

$$P(Y = 1) = \frac{15}{72}\ ;\ P(Y = 2) = \frac{24}{72}\ ;\ P(Y = 3) = \frac{33}{72}$$

The conditional distribution of X, given Y is $P\{X = x_i/Y = y_j\}$

$$P(X = 0/Y = 1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{3}{72}}{\frac{15}{72}} = \frac{3}{15} = \frac{1}{5}$$

$$P(X = 1/Y = 1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{\frac{5}{72}}{\frac{15}{72}} = \frac{5}{15} = \frac{1}{3}$$

$$P(X = 2/Y = 1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{\frac{7}{72}}{\frac{15}{72}} = \frac{7}{15}$$

$$P(X = 0/Y = 2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{\frac{6}{72}}{\frac{24}{72}} = \frac{6}{24} = \frac{1}{4}$$

$$P(X = 1/Y = 2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{\frac{8}{72}}{\frac{24}{72}} = \frac{8}{24} = \frac{1}{3}$$

$$P(X = 2/Y = 2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{\frac{10}{72}}{\frac{24}{72}} = \frac{10}{24} = \frac{5}{12}$$

$$P(X = 0/Y = 3) = \frac{P(X=0, Y=3)}{P(Y=3)} = \frac{\frac{9}{72}}{\frac{33}{72}} = \frac{9}{33}$$

$$P(X = 1/Y = 3) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{\frac{11}{72}}{\frac{33}{72}} = \frac{11}{33} = \frac{1}{3}$$

$$P(X = 2/Y = 3) = \frac{P(X=2, Y=3)}{P(Y=3)} = \frac{\frac{13}{72}}{\frac{33}{72}} = \frac{13}{33}$$

The conditional distribution of Y, given X is $P\{Y = y_j/X = x_i\}$

$$P\{Y = 1/X = 0\} = \frac{P(X=0;\, Y=1)}{P(X=0)} = \frac{\frac{3}{72}}{\frac{18}{72}} = \frac{3}{18} = \frac{1}{6}$$

$$P\{Y = 2/X = 0\} = \frac{P(X=0;\, Y=2)}{P(X=0)} = \frac{\frac{6}{72}}{\frac{18}{72}} = \frac{6}{18} = \frac{1}{3}$$

$$P\{Y = 3/X = 0\} = \frac{P(X=0;\, Y=3)}{P(X=0)} = \frac{\frac{9}{72}}{\frac{18}{72}} = \frac{9}{18} = \frac{1}{2}$$

$$P\{Y = 1/X = 1\} = \frac{P(X=1;\, Y=1)}{P(X=1)} = \frac{\frac{5}{72}}{\frac{24}{72}} = \frac{5}{24}$$

$$P\{Y = 2/X = 1\} = \frac{P(X=1;\, Y=2)}{P(X=1)} = \frac{\frac{8}{72}}{\frac{24}{72}} = \frac{1}{3}$$

$$P\{Y = 3/X = 1\} = \frac{P(X=1;\, Y=3)}{P(X=1)} = \frac{\frac{11}{72}}{\frac{24}{72}} = \frac{11}{24}$$

$$P\{Y = 1/X = 2\} = \frac{P(X=2;\, Y=1)}{P(X=2)} = \frac{\frac{7}{72}}{\frac{30}{72}} = \frac{7}{30}$$

$$P\{Y = 2/X = 2\} = \frac{P(X=2;\, Y=2)}{P(X=2)} = \frac{\frac{10}{72}}{\frac{30}{72}} = \frac{1}{3}$$

$$P\{Y = 3/X = 2\} = \frac{P(X=2;\, Y=3)}{P(X=2)} = \frac{\frac{13}{72}}{\frac{30}{72}} = \frac{13}{30}$$

Probability distribution of (X + Y)

| (X + Y) | P |
|---|---|
| 1<br>P(0,1) | $\frac{3}{72}$ |
| 2<br>P(0,2) + P(1,1) | $\frac{6}{72}+\frac{5}{72}=\frac{11}{72}$ |
| 3<br>P(0,3) + P(1,2) + P(2,1) | $\frac{9}{72}+\frac{8}{72}+\frac{7}{72}=\frac{24}{72}$ |
| 4<br>P[1,3] + P[2,2] | $\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$ |
| 5<br>P(2,3) | $\frac{13}{72}$ |
| Total | 1 |

$$P[X + Y > 3] = P[X+Y=4] + P[X+Y=5] = \frac{21}{72}+\frac{13}{72}=\frac{34}{72}$$

**Example 2.1.4**

**The joint probability distribution of a two-dimensional discrete random variable (X, Y) is given below :**

| Y | X | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 |
| 1 | 0.01 | 0.02 | 0.04 | 0.05 | 0.06 | 0.08 |
| 2 | 0.01 | 0.03 | 0.05 | 0.05 | 0.05 | 0.06 |
| 3 | 0.01 | 0.02 | 0.04 | 0.06 | 0.06 | 0.05 |

**(i) Find, P(X > Y) and P{Max (X, Y) = 3} and**

**(ii) Find, the probability distribution of the random variable.**
**Z = Min (X, Y)**

**Example 2.1.6**

**The two dimensional random variable (X, Y) has the joint density function $f(x, y) = \dfrac{x + 2y}{27}$, $x = 0, 1, 2$ ; $y = 0, 1, 2$. Find the conditional distribution of Y given X = x. Also, find the conditional distribution of X given Y = 1.**

**[A.U Tvli. A/M 2009]**

*Solution :* Given : $f(x,y) = \dfrac{x+2y}{27}$, $x = 0, 1, 2,$ ; $y = 0, 1, 2$

| Y \ X | 0 | 1 | 2 | $P_Y(y) = P_{*j}$ |
|---|---|---|---|---|
| 0 | 0 <br> P(0,0) | $\frac{1}{27}$ <br> P(1,0) | $\frac{2}{27}$ <br> P(2,0) | $P[Y=0]=\frac{3}{27}$ <br> sum of the I row |
| 1 | $\frac{2}{27}$ <br> P(0,1) | $\frac{3}{27}$ <br> P(1,1) | $\frac{4}{27}$ <br> P(2,1) | $P[Y=1]=\frac{9}{27}$ <br> sum of the II row |
| 2 | $\frac{4}{27}$ <br> P(0,2) | $\frac{5}{27}$ <br> P(1,2) | $\frac{6}{27}$ <br> P(2,2) | $P[Y=2]=\frac{15}{27}$ <br> sum of the III row |
| $P_X(x) = P_{i*}$ | $P[X=0]=\frac{6}{27}$ <br> sum of the I column | $P[X=1]=\frac{9}{27}$ <br> sum of the II column | $P[X=2]=\frac{12}{27}$ <br> sum of the III column | 1 |

The conditional distribution of Y given X = $x$.

$$P(Y = 0/X = 0) = \frac{P(X=0; Y=0)}{P(X=0)} = \frac{P(0,0)}{P[X=0]} = \frac{0}{\frac{6}{27}} = 0$$

$$P(Y = 1/X = 0) = \frac{P(X=0; Y=1)}{P(X=0)} = \frac{P(0,1)}{P[X=0]} = \frac{\frac{2}{27}}{\frac{6}{27}} = \frac{2}{6} = \frac{1}{3}$$

$$P(Y = 2/X = 0) = \frac{P(X=0; Y=2)}{P(X=0)} = \frac{P(0,2)}{P[X=0]} = \frac{\frac{4}{27}}{\frac{6}{27}} = \frac{4}{6} = \frac{2}{3}$$

$$P(Y = 0/X = 1) = \frac{P(X=1; Y=0)}{P(X=1)} = \frac{P(1,0)}{P[X=1]} = \frac{\frac{1}{27}}{\frac{9}{27}} = \frac{1}{9}$$

$$P(Y = 1/X = 1) = \frac{P(X=1; Y=1)}{P(X=1)} = \frac{P(1,1)}{P[X=1]} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{3}{9} = \frac{1}{3}$$

$$P(Y = 2/X = 1) = \frac{P(X=1; Y=2)}{P(X=1)} = \frac{P(1,2)}{P[X=1]} = \frac{\frac{5}{27}}{\frac{9}{27}} = \frac{5}{9}$$

$$P(Y = 0/X = 2) = \frac{P(X=2; Y=0)}{P(X=2)} = \frac{P(2,0)}{P[X=2]} = \frac{\frac{2}{27}}{\frac{12}{27}} = \frac{2}{12} = \frac{1}{6}$$

$$P(Y = 1/X = 2) = \frac{P(X=2; Y=1)}{P(X=2)} = \frac{P(2,1)}{P[X=2]} = \frac{4/27}{12/27} = \frac{1}{3}$$

$$P(Y = 2/X = 2) = \frac{P(X=2; Y=2)}{P(X=2)} = \frac{P(2,2)}{P[X=2]} = \frac{6/27}{12/27} = \frac{6}{12} = \frac{1}{2}$$

The conditional distribution of X given Y = 1

$$P(X = 0/Y = 1) = \frac{P(X=0; Y=1)}{P(Y=1)} = \frac{P(0,1)}{P[Y=1]} = \frac{\frac{2}{27}}{\frac{9}{27}} = \frac{2}{9}$$

$$P(X = 1/Y = 1) = \frac{P(X=1; Y=1)}{P(Y=1)} = \frac{P(1,1)}{P[Y=1]} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{3}{9}$$

$$P(X = 2/Y = 1) = \frac{P(X=2; Y=1)}{P(Y=1)} = \frac{P(2,1)}{P[Y=1]} = \frac{\frac{4}{27}}{\frac{9}{27}} = \frac{4}{9}$$

**Example 2.1.8**

**Two discrete r.v.'s X and Y have the joint probability density function;**

$$P(x, y) = \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!(x-y)!} \; ; y = 0, 1, 2, \ldots x,$$

**$x = 0, 1, 2, \ldots$ where m, p are constants with $m > 0$ and $0 < p < 1$. Find (i) the marginal probability density function X and Y, (ii) the conditional distribution of Y for a given X and of X for a given Y.**

**[A.U. N/D. 2005] [A.U Tvli. M/J 2010]**

**Solution :**

Given the joint probability density function of the two discrete random variables X and Y is $p(x,y) = \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!(x-y)!}$,

$y = 0, 1, 2, \ldots x \; ; \quad x = 0, 1, 2, \ldots$

(i) Then the marginal probability density function of X is,

$$P(x) = \sum_{y=0}^{\infty} p(x,y) = \sum_{y=0}^{\infty} \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!(x-y)!}$$

$$= e^{-m} m^x \sum_{y=0}^{\infty} \frac{p^y (1-p)^{x-y}}{y!(x-y)!}$$

$$= \frac{e^{-m} m^x}{x!} \sum_{y=0}^{\infty} \frac{x!\, p^y (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= \frac{e^{-m} m^x}{x!} \sum_{y=0}^{\infty} x C_y p^y q^{x-y}$$

$$= \frac{e^{-m} m^x}{x!} [p+q]^x$$

$$= \frac{e^{-m} m^x}{x!} \qquad [\because p+q=1],\ x=0,1,2,\ldots$$

which is a probability function of a Poisson distribution with parameter $m$.

And $$P(y) = \sum_{x=0}^{\infty} p(x,y)$$

$$= \sum_{x=0}^{y-1} p_{XY}(x,y) + \sum_{x=y}^{\infty} \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= 0 + \frac{e^{-m} p^y}{y!} \sum_{x=y}^{\infty} \frac{m^x (1-p)^{x-y}}{(x-y)!}$$

$$= \frac{e^{-m} p^y m^y}{y!} \sum_{x=y}^{\infty} \frac{m^x m^{-y} (1-p)^{(x-y)}}{(x-y)!}$$

$$= \frac{e^{-m} (mp)^y}{y!} \sum_{x=y}^{\infty} \frac{[m(1-p)]^{(x-y)}}{(x-y)!}$$

$$= \frac{e^{-m} (mp)^y}{y!} \times e^{m(1-p)}$$

$$= \frac{e^{-(mp)} (mp)^y}{y!},\ y = 0,\ 1,\ 2,\ \ldots\ x$$

which is the probability function of a Poisson distribution with parameter $(mp)$.

(ii) The conditional distribution of Y for given X is,

$$P(y/x) = \frac{P(x,y)}{P(x)} = \frac{e^{-m} m^x p^y (1-p)^{x-y} x!}{y!(x-y)!m^x e^{-m}}$$

$$= \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y}$$

$$= xC_y p^y (1-p)^{x-y}, \; x > y$$

And the conditional probability distribution of X given Y is,

$$P(x/y) = \frac{P(x,y)}{P(y)} = \frac{e^{-m} m^x p^y (1-p)^{x-y}}{y!(x-y)!} \cdot \frac{y!}{e^{-mp}(mp)^y}$$

$$= \frac{e^{(-mp)} (mq)^{x-y}}{(x-y)!} \quad ; \quad q = 1-p, \; x > y.$$

## PROBLEMS UNDER CONTINUOUS RANDOM VARIABLES

**Example 2.1.9**

**Suppose the point Probability Density Function (PDF) is given by**

$$f(x, y) = \begin{cases} \frac{6}{5}(x+y^2) & ; 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

**Obtain the marginal PDF of X and that of Y. Hence, otherwise find $P\left[\frac{1}{4} \le y \le \frac{3}{4}\right]$.** **[A.U. N/D 2004, N/D 2005, N/D 2012**

*Solution :* Given that $f(x, y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \textit{otherwise} \end{cases}$

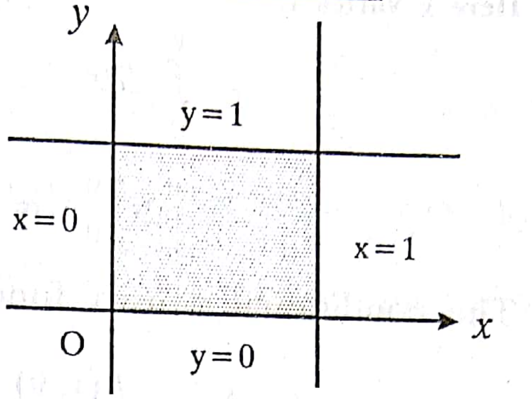
The marginal p.d.f of X is,

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \frac{6}{5}\int_0^1 (x+y^2)\, dy = \frac{6}{5}\left[xy + \frac{y^3}{3}\right]_{y=0}^{y=1}$$

$$= \frac{6}{5}\left(x + \frac{1}{3}\right), \; 0 \le x \le 1$$

The marginal p.d.f of Y is,

Given : $0 \le x \le 1;$
$0 \le y \le 1$




$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= \frac{6}{5}\int_{0}^{1} (x + y^2)\, dx = \frac{6}{5}\left[\frac{x^2}{2}+xy^2\right]_0^1$$

$$= \frac{6}{5}\left(y^2 + \frac{1}{2}\right),\ 0 \le y \le 1.$$

$$P\left[\frac{1}{4} \le y \le \frac{3}{4}\right] = \int_{1/4}^{3/4} f(y)\, dy = \int_{1/4}^{3/4} \frac{6}{5}\left(y^2 + \frac{1}{2}\right) dy$$

$$= \frac{6}{5}\left[\frac{y^3}{3}+\frac{y}{2}\right]_{1/4}^{3/4} = \frac{6}{5}\left[\left(\frac{(3/4)^3}{3}+\frac{(3/4)}{2}\right)-\left(\frac{(1/4)^3}{3}+\frac{(1/4)}{2}\right)\right]$$

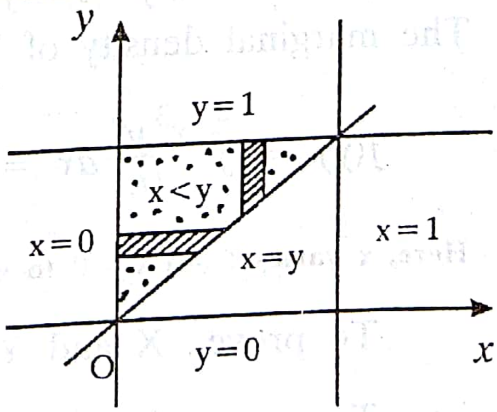
$$= \frac{6}{5}\left[\left(\frac{9}{64}+\frac{3}{8}\right)-\left(\frac{1}{192}+\frac{1}{8}\right)\right] = \frac{6}{5}\left[\frac{37}{96}\right] = \frac{37}{80} = 0.4625$$

**Example 2.1.10**

**Let X and Y have j.p.d. f(x, y) = 2, 0 < x < y < 1. Find the m.d.f. find the conditional density function of Y given X=x. [A.U. A/M 2003] [A.U CBT A/M 2011]**

*Solution* : The marginal density function of X is given by

Given : $0 < x < y < 1;$
i.e., choose, $x = 0, x = 1$
$y = 0, y = 1, x < y$




$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

Here y varies from y = x to y = 1 **[Vertical strip]**

$$= \int_{x}^{1} 2dy \qquad [\because x < y < 1]$$

$$= [2y]_x^1 = 2(1-x),\ 0 < x < 1$$

The marginal density function of Y is given by,

**Example 2.1.12**

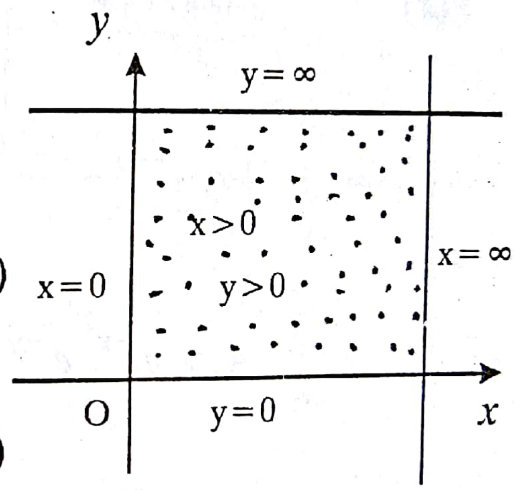
**The joint p.d.f of the random variable (X, Y) is given by $f(x, y) = Kxy\,e^{-(x^2+y^2)}$, $x > 0, y > 0$. Find the value of K and also prove that X and Y are independent.**

**[A.U. May, 2000, 2004, N/D 2006, N/D 2011, M/J 2012]**
**[N/D 2007, M/J 2009, Tvli A/M 2009, N/D 2013]**

*Solution* : Here, the range space is the entire first quadrant of the XY-plane.

$$\int_0^\infty \int_0^\infty Kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1 \qquad \dots (1)$$

| Given : $x > 0$, $y > 0$ |
|---|




$$K\int_0^\infty \int_0^\infty xy\,e^{-x^2}\,e^{-y^2}\,dx\,dy = 1$$

$$K\left[\int_0^\infty x\,e^{-x^2}\,dx\right]\left[\int_0^\infty y\,e^{-y^2}\,dy\right] = 1 \qquad \dots (2)$$

Take, $$\int_0^\infty x\,e^{-x^2}\,dx \qquad \dots (3)$$

Put $x^2 = t$ | $x \to 0 \Rightarrow t \to 0$

$2x\,dx = dt$ | $x \to \infty \Rightarrow t \to \infty$

$x\,dx = \frac{1}{2}\,dt$

$$(3) \Rightarrow \quad = \int_0^\infty e^{-t}\,\frac{1}{2}\,dt = \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_0^\infty = -\frac{1}{2}\left[e^{-t}\right]_0^\infty$$

$$= -\frac{1}{2}\,[0-1] = \frac{1}{2} \qquad [\because e^{-\infty} = 0]$$

$$\therefore \int_0^\infty x e^{-x^2} dx = \int_0^\infty y e^{-y^2} dy = \frac{1}{2} \quad \dots (4)$$

[$x$ and $y$ are dummy variables]

$$\therefore (2) \Rightarrow k \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = 1$$

$$\Rightarrow \frac{k}{4} = 1 \quad \Rightarrow k = 4$$

To prove : X and Y are independent

i.e., To prove : $f(x)\, f(y) = f(x, y)$

**Proof :** The marginal density of X is given by

$$f(x) = \int_0^\infty f(x, y)\, dy$$

$$= \int_0^\infty 4xy\, e^{-(x^2 + y^2)} dy$$

$$= \int_0^\infty 4xy\, e^{-x^2} e^{-y^2} dy$$

$$= 4x e^{-x^2} \int_0^\infty y e^{-y^2} dy$$

$$= 4x e^{-x^2} \left[\frac{1}{2}\right] \quad \text{by } (4)$$

$$= 2x e^{-x^2}, \quad x > 0$$

The marginal density of Y is given by

$$f(y) = \int_0^\infty f(x, y)\, dx$$

$$= \int_0^\infty 4xy\, e^{-(x^2 + y^2)} dx$$

$$= \int_0^\infty 4xy\, e^{-x^2} e^{-y^2} dx$$

$$= 4y\, e^{-y^2} \int_0^\infty x e^{-x^2} dx$$

$$= 4y e^{-y^2} \left[\frac{1}{2}\right] \quad \text{by } (4)$$

$$= 2y e^{-y^2}, \ y > 0$$

$$f(x) f(y) = (2x e^{-x^2})(2y e^{-y^2}) = 4xy\, e^{-(x^2 + y^2)} = f(x, y)$$

$\therefore$ X and Y are independent.

Example 2.1.13

Given $f_{xy}(x, y) = Cx(x-y), 0 < x < 2, -x < y < x$ and 0, elsewhere

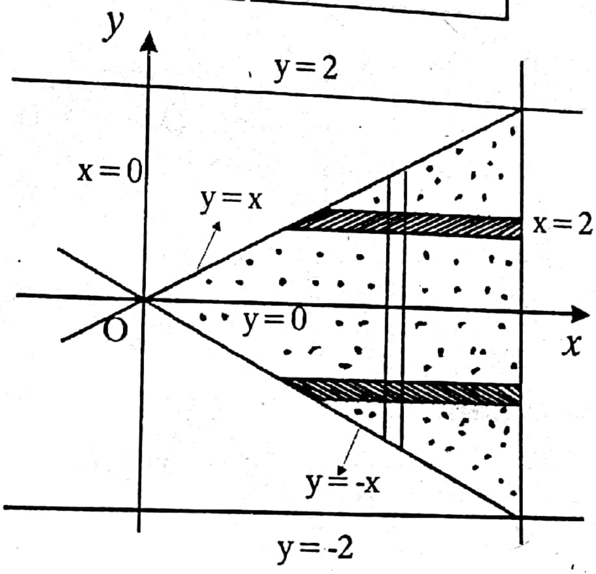
(a) Evaluate C ; (b) Find $f_X(x)$ ; (c) $f_{Y/X}\left(\frac{y}{x}\right)$. and (d) $f_Y(y)$

**[A.U. N/D 2004, M/J 2006, N/D 2010, M/J 2013]**

*Solution* : By the property of j.p.d.f, we have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Given : $0 < x < 2,$
$-x < y < x$




$$\int_0^2\int_{-x}^{x} Cx(x-y)\,dy\,dx = 1$$

$$C\int_0^2\int_{-x}^{x}\left[x^2 - xy\right]dy\,dx = 1$$

$$C\int_0^2\left[x^2y - \frac{xy^2}{2}\right]_{y=-x}^{y=x} dx = 1$$

$$C\int_0^2\left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right]dx = 1$$

$$C\int_0^2\left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right]dx = 1$$

$$C\int_0^2 2x^3\,dx = 1$$

$$2C\left[\frac{x^4}{4}\right]_0^2 = 1 \Rightarrow 2C\left[\frac{16}{4} - 0\right] = 1 \Rightarrow 8C = 1 \Rightarrow C = \frac{1}{8}$$

(b) The marginal density of X is given by,

$$f(x) = \int_{-x}^{x} f(x,y)\,dy = \frac{1}{8}\int_{-x}^{x} x(x-y)\,dy$$

$$= \frac{1}{8}\int_{-x}^{x} (x^2 - xy)\,dy = \frac{1}{8}\left[x^2 y - \frac{xy^2}{2}\right]_{y=-x}^{y=x}$$

$$= \frac{1}{8}\left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right] = \frac{1}{8}\left[2x^3\right] = \frac{x^3}{4},\ 0 < x < 2$$

(c) $$f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}} = \frac{1}{2x^2}(x-y),\ -x < y < x$$

(d) $$f(y) = \begin{cases} \int_{-y}^{2} \frac{1}{8}\,x(x-y)\,dx, & \text{if } -2 \le y \le 0 \\ \int_{y}^{2} \frac{1}{8}x(x-y)\,dx & \text{if } 0 \le y \le 2 \end{cases}$$

$$= \begin{cases} \frac{1}{8}\int_{-y}^{2} (x^2 - xy)\,dx \\ \frac{1}{8}\int_{y}^{2} (x^2 - xy)\,dx \end{cases} = \begin{cases} \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{x=-y}^{x=2} \\ \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{x=y}^{x=2} \end{cases}$$

$$= \begin{cases} \frac{1}{8}\left[\left(\frac{8}{3} - 2y\right) - \left(\frac{-y^3}{3} - \frac{y^3}{2}\right)\right] \\ \frac{1}{8}\left[\left(\frac{8}{3} - 2y\right) - \left(\frac{y^3}{3} - \frac{y^3}{2}\right)\right] \end{cases} = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5}{48}y^3 & \text{if } -2 \le y \le 0 \\ \frac{1}{3} - \frac{y}{4} + \frac{1}{48}y^3 & \text{if } 0 \le y \le 2 \end{cases}$$

**Example 2.1.14**

**Suppose that X and Y are independent and that these are the distribution tables for X and Y.**

| $x$ : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f(x)$ : | 0.1 | 0.1 | 0.3 | 0.2 | 0.3 |

| $y$ : | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f(y)$ : | 0.2 | 0.3 | 0.2 | 0.1 | 0.2 |

**What is the joint probability space ?**

**Example 2.1.18**

If the joint pdf of a two-dimensional random variable (X, Y) is given by,

$$f(x, y) = x^2 + \frac{xy}{3}, \quad 0 < x < 1; 0 < y < 2$$

$$= 0, \qquad \text{elsewhere}$$

[A.U. M/J 2006] [A.U. N/D 2006]

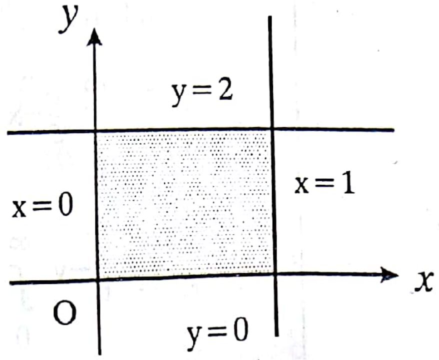
Find (i) $P\left(X > \frac{1}{2}\right)$ ; (ii) $P(Y < X)$ and (iii) $P\left(P\left(Y < \frac{1}{2} / X < \frac{1}{2}\right)\right)$

Check whether the conditional density functions are valid.

[A.U Trichy A/M 2010] [A.U A/M 2011, M/J 2009, M/J 2014]

*Solution :*

Given : $0 < x < 1,\ 0 < y < 2$




| The marginal density of X is given by, | The marginal density function of Y is |
|---|---|
| $f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$ | $f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$ |
| $= \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$ | $= \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx$ |
| $= \left[x^2y + \frac{xy^2}{6}\right]_{y=0}^{y=2}$ | $= \left[\frac{x^3}{3} + \frac{y}{3}\frac{x^2}{2}\right]_{x=0}^{x=1}$ |
| $= 2x^2 + \frac{2x}{3},\ 0 < x < 1$ | $= \left(\frac{1}{3} + \frac{y}{6}\right) - (0 + 0)$ |
| | $= \frac{1}{3} + \frac{y}{6}$ |
| | $= \frac{2+y}{6},\ 0 < y < 2$ |

(i) $P\left(X > \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} f(x)\, dx = \int_{1/2}^{1} \left(2x^2 + \frac{2x}{3}\right) dx = \left[\frac{2x^3}{3} + \frac{x^2}{3}\right]_{1/2}^{1}$

$$= \frac{2}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{12}$$

$$= 1 - \frac{2}{12} = 1 - \frac{1}{6} = \frac{5}{6}$$

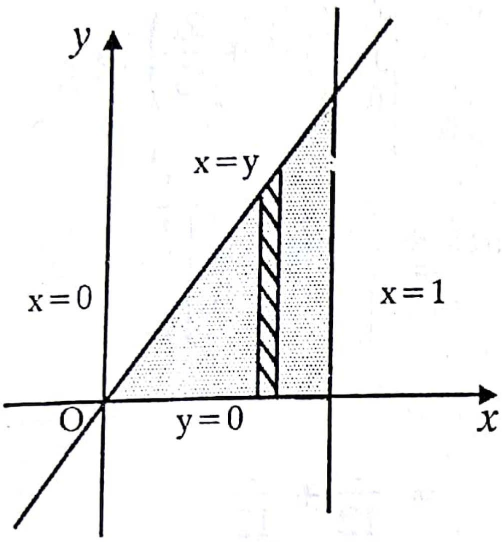
(ii) To find : $P[Y < X]$

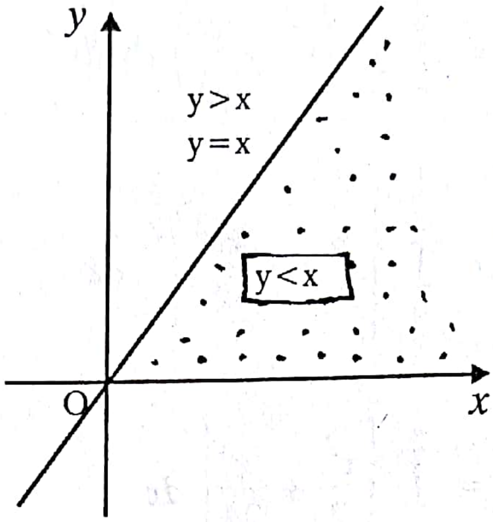
Let $P[Y < X] = \iint\limits_R \left(x^3 + \frac{xy}{3}\right) dy\, dx$ ... (1)

Here, outer limit $x$ is $0 < x < 1$

i.e., $x$ limit varies from $x = 0$ to $x = 1$;

inner limit $y$. Here, $Y < X$, $0 < y < 2$




$$(1) \Rightarrow P(Y < X) = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{6}\right)_0^x dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{6}\right) dx$$

$$= \frac{7}{6} \int_0^1 x^3\, dx$$

$$= \frac{7}{6} \left[\frac{x^4}{4}\right]_0^1 = \frac{7}{24}$$

$$\text{iii) } P\left(Y < \frac{1}{2} / X < \frac{1}{2}\right) = \frac{P\left(X < \frac{1}{2};\ Y < \frac{1}{2}\right)}{P\left(X < \frac{1}{2}\right)} \quad \text{... (2)}$$

$$P\left(X < \frac{1}{2};\ Y < \frac{1}{2}\right)$$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f(x, y)\, dy\, dx$$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_0^{\frac{1}{2}} \left[x^2 y + \frac{xy^2}{6}\right]_0^{\frac{1}{2}} dx$$

$$= \int_0^{\frac{1}{2}} \left[\frac{x^2}{2} + \frac{x}{24}\right] dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{48}\right]_0^{\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{\left(\frac{1}{2}\right)^2}{48}$$

$$= \frac{5}{192}$$

$$P\left(X < \frac{1}{2}\right)$$

$$= \int_0^{\frac{1}{2}} f(x)\, dx$$

$$= \int_0^{\frac{1}{2}} \left(2x^2 + \frac{2x}{3}\right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{x^2}{3}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{12} + \frac{1}{12}$$

$$= \frac{2}{12}$$

$$= \frac{1}{6}$$

$$\therefore (2) \Rightarrow P\left(Y < \frac{1}{2} / X < \frac{1}{2}\right) = \frac{5/192}{1/6}$$

$$= \frac{5}{192} \times 6 = \frac{5}{32}$$

Checking the conditional density functions are valid

$$\int_0^1 f(x/y)\,dx$$

$$= \int_0^1 \frac{f(x,y)}{f(y)}\,dx$$

$$= \int_0^1 \left(\frac{6x^2+2xy}{2+y}\right) dx$$

$$= \frac{1}{2+y}\left[6\frac{x^3}{3} + 2y\frac{x^2}{2}\right]_{x=0}^{x=1}$$

$$= \frac{1}{2+y}\left[(2+y)-(0+0)\right]$$

$$= \frac{1}{2+y}(2+y)$$

$$= 1$$

$$\int_0^2 f(y/x)\,dy$$

$$= \int_0^2 \frac{f(y,x)}{f(x)}\,dx$$

$$= \int_0^2 \frac{f(x,y)}{f(x)}\,dx$$

$$= \int_0^2 \left(\frac{3x+y}{6x+2}\right) dy$$

$$= \frac{1}{6x+2}\int_0^2 (3x+y)\,dy$$

$$= \frac{1}{6x+2}\left[3xy + \frac{y^2}{2}\right]_{y=0}^{y=2}$$

$$= \frac{1}{6x+2}\left[(6x+2)-(0+0)\right]$$

$$= 1$$

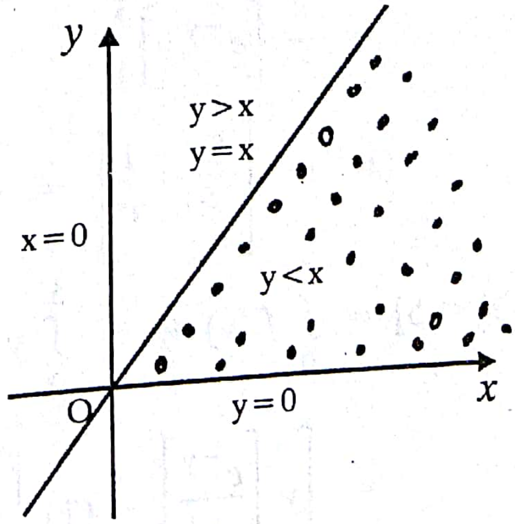
**Example 2.1.19**

**Given that the joint p.d.f of (X, Y) is**

$$f(x,y) = e^{-y}, \quad x>0,\ y>x$$
$$= 0, \quad \text{elsewhere.}$$

**Find (i) P (X > 1 / Y < 5) and (ii) the marginal distributions of X and Y**

*Solution* : Given : $x > 0,\ y > x$



**Example 2.1.22**

If the joint density function of the two random variables 'X' and 'Y' be

$$f(x, y) = e^{-(x+y)}, \; x \geq 0, \; y \geq 0$$
$$= 0, \quad \text{otherwise}$$

[A.U N/D. 2003]

Find (i) $P(X < 1)$ and (ii) $P(X + Y < 1)$

[A.U N/D. 2009] [A.U CBT M/J 2010]

*Solution :* To find the marginal density function of X.

Let the marginal density function of X be $g(x)$ and it is defined as

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{0}^{\infty} e^{-(x+y)}\, dy$$

$$= e^{-x} \int_{0}^{\infty} e^{-y}\, dy = e^{-x}\left[-e^{-y}\right]_{0}^{\infty} = e^{-x}, x \geq 0$$

$$\therefore g(x) = e^{-x}, x \geq 0$$

(i) $$P(X < 1) = \int_{0}^{1} g(x)\, dx = \int_{0}^{1} e^{-x}\, dx = \left[-e^{-x}\right]_{0}^{1}$$

$$= -\left[e^{-1} - 1\right] = 1 - e^{-1}$$

(ii) $$P(X + Y < 1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy$$

'$x$' varies from '0' to $(1 - y)$ and '$y$' varies from '0' to '1'.

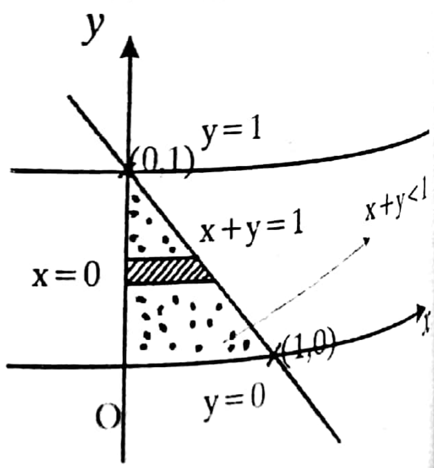
$$\therefore P(X + Y < 1) = \int_{0}^{1} \int_{0}^{1-y} e^{-(x+y)}\, dx\, dy$$

Given : $x \geq 0, y \geq 0$
$x + y < 1$

$$= \int_{0}^{1} e^{-y} \left[-e^{x}\right]_{0}^{1-y} dy$$

$$= \int_{0}^{1} e^{-y} \left(1 - e^{-(1-y)}\right) dy$$

$$= \int_{0}^{1} \left[e^{-y} - e^{-1}\right] dy$$



The joint p.d.f of two random variables X and Y is given by

$$f(x, y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}, \quad 0 \le x < \infty, \; 0 \le y < \infty$$

Find the marginal distributions of X and Y, the conditional distribution of Y for X = $x$ and the expected value of this conditional distribution.

[A.U Trichy M/J 2009] [A.U. A/M 2004, A/M 2011]

Solution : (i) The marginal distribution of X is

$$g(x) = \int_0^\infty \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4} \int_0^\infty \frac{1+y+x}{(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4} \int_0^\infty \left\{(1+y)^{-3} + x(1+y)^{-4}\right\} dy$$

$$= \frac{9}{2(1+x)^4} \left[\frac{(1+y)^{-2}}{2} - \frac{x(1+y)^{-3}}{3}\right]_0^\infty$$

$$= \frac{9}{2(1+x)^4} \left[\frac{1}{2} + \frac{x}{3}\right] = \frac{3}{4}\, \frac{2x+3}{(1+x)^4}, \quad 0 \le x \le \infty$$

From the form of the joint p.d.f of (X, Y).

The marginal distribution of Y is

$$h(y) = \int_0^\infty \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\,dx = \frac{3}{4}\;\frac{2y+3}{4(1+y)^4},\quad 0 \le y < \infty$$

The conditional p.d.f of Y for X = $x$ is

$$f(y/x) = \frac{f(x,y)}{g(x)} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}\cdot\frac{4(1+x)^4}{3(2x+3)}$$

$$= \frac{6(1+x+y)}{(2x+3)(1+y)^4},\quad 0 \le x \le \infty,\ 0 \le y \le \infty$$

Conditional Expectation

$$E[Y/X=x] = \int_0^\infty y f(y/x)\,dy$$

$$\therefore\ E[Y/X=x] = \int_0^\infty y\,\frac{6(1+x+y)}{(2x+3)(1+y)^4}\,dy$$

$$= \frac{6}{2x+3}\int_0^\infty \frac{(1+y-1)(1+x+y)}{(1+y)^4}\,dy$$

$$= \frac{6}{2x+3}\int_0^\infty \frac{[(1+y)^2+(x-1)(1+y)-x]}{(1+y)^4}\,dy$$

$$= \frac{6}{2x+3}\int_0^\infty \left\{(1+y)^{-2}+(x-1)(1+y)^{-3}-x(1+y)^{-4}\right\}dy$$

$$= \frac{6}{2x+3}\left[-\frac{1}{1+y}+(x-1)\left\{-\frac{1}{2(1+y)^2}\right\}+\frac{x}{3(1+y)^3}\right]_0^\infty$$

$$= \frac{6}{2x+3}\left[1+\frac{x-1}{2}-\frac{x}{3}\right]$$

$$\therefore\ E[Y/X=x] = \frac{6}{2x+3}\left[\frac{6+3x-3-2x}{6}\right] = \frac{x+3}{2x+3}$$

## 1. COVARIANCE

If X and Y are random variables, then co-variance between them is defined as,

$$\begin{aligned}
\text{Cov}(X, Y) &= E[(X - E(X))(Y - E(Y))] \\
&= E[XY - XE(Y) - YE(X) + E(X)E(Y)] \\
&= E[XY] - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

**Note :** If X and Y are independent, then cov $(X, Y) = 0$

If X and Y are independent, then $E[XY] = E(X) \cdot E(Y)$

$\Rightarrow \text{Cov}(X, Y) = 0$

**(i) Cov (aX, bY) = ab Cov (X, Y)**

$$\begin{aligned}
\text{Cov}(aX, bY) &= E[(aX)(bY)] - E(aX)E(bY) \\
&= ab\,E(XY) - ab\,E(X)E(Y) \\
&= ab\,[E(XY) - E(X)E(Y)] \\
&= ab\,\text{Cov}(X, Y)
\end{aligned}$$

**(ii) Cov (X + a, Y + b) = Cov (X, Y)**

$$\text{Cov}(X + a, Y + b) = E[(X + a)(Y + b)] - E(X + a)E(Y + b)$$

$$= E[XY + bX + aY + ab] - [E(X) + a][E(Y) + b]$$

$$= E(XY) + b\,E(X) + a\,E(Y) + ab - E(X)E(Y) - aE(Y)$$
$$- b\,E(X) - ab$$

$$= E(XY) - E(X)E(Y) = \text{Cov}(X, Y)$$

**(iii) $Cov(aX + b, cY + d) = ac\, Cov(X, Y)$**

$$Cov(aX + b, cY + d) = E[(aX + b)(cY + d)] - E(aX + b)\, E(cY + d)$$

$$= E[ac\,XY + ad\,X + bc\,Y + bd] - [aE(X) + b][cE(Y) + d]$$

$$= ac\,E(XY) + ad\,E(X) + bc\,E(Y) + bd - ac\,E(X)\,E(Y) - ad\,E(X) - bc\,E(Y) - bd$$

$$= ac\,[E(XY) - E(X)\,E(Y)] = ac\,Cov(X, Y)$$

**(iv) $V(X_1 + X_2) = V(X_1) + V(X_2) + 2Cov(X_1, X_2)$**

$$V(X_1 + X_2) = E[(X_1 + X_2)^2] - [E(X_1 + X_2)]^2$$

$$= E(X_1^2 + 2X_1 X_2 + X_2^2) - [E(X_1) + E(X_2)]^2$$

$$= E(X_1^2) + 2E(X_1 X_2) + E(X_2^2) - [E(X_1)]^2 - [E(X_2)]^2 - 2E(X_1)\,E(X_2)$$

$$= E(X_1^2) - [E(X_1)]^2 + E(X_2^2) - [E(X_2)]^2 + 2[E(X_1 X_2) - E(X_1)\,E(X_2)]$$

$$= V(X_1) + V(X_2) + 2\,Cov(X_1, X_2)$$

**(v) $V(X_1 - X_2) = V(X_1) + V(X_2) - 2\,Cov(X_1, X_2)$**

If $X_1$ and $X_2$ are independent then

$$V(X_1 \pm X_2) = V(X_1) \pm V(X_2)$$