1.3 MOMENTS - MOMENT GENERATING FUNCTIONS AND THEIR PROPERTIES :

Moments [Discrete case]

Let X be discrete R.V. taking the values $x_1, x_2, \dots x_n$ with probability mass function $p_1, p_2, \dots p_n$ respectively then the r^{th} moment about the origin is

$$\mu_r' \text{ (about the origin) } = \sum_{i=1}^{n} x_i^r p_i \quad \dots (1)$$

and $$\mu_r' \text{ (about any point } x = \text{A) } = \sum_{i=1}^{n} (x_i - \text{A})^r p_i \quad \dots (2)$$

and $$\mu_r \text{ (about mean) } = \sum_{i=1}^{n} (x_i - \text{Mean})^r p_i \quad \dots (3)$$

In particular from (1)

$$\mu_1' = \sum_{i=1}^{n} x_i p_i = \text{Mean } (\bar{x})$$

$$\mu_2' = \sum_{i=1}^{n} x_i^2 p_i = \text{Mean square value.}$$

$$\mu_2 = \sum_{i=1}^{n} (x_i - \text{mean})^2 p_i = \text{variance}$$

$$= \mu_2' - (\mu_1')^2 \qquad [\because \bar{x} = \mu_1']$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Moments [Continuous case]

If X is a continuous R.V. with probability density function $f(x)$ defined in the interval (a, b) then

$$\mu_r' = \int_a^b x^r f(x)\, dx$$

$$\mu_r' \text{ (about a point A)} = \int_a^b (x - A)^r f(x)\, dx$$

$$\mu_r \text{ (about the mean)} = \int_a^b (x - \bar{x})^r f(x)\, dx$$

Moments Generating Function : (M.G.F)

An important device that can be used to calculate the higher moments is the moment generating function.

Moment generating function of a random variable X about the origin is defined as

$$M_X(t) = E[e^{tX}] = \begin{cases} \Sigma_x e^{tx} p(x), & \text{if X is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx, & \text{if X is continuous} \end{cases}$$

where t being a real parameter assuming that the integration or summation is absolutely convergent for some positive number h such that $|t| < h$

$$\therefore M_X(t) = E[e^{tX}] = E\left[1 + \frac{tX}{\lfloor 1} + \frac{(tX)^2}{\lfloor 2} + \ldots + \frac{(tX)^r}{\lfloor r} + \ldots\right]$$

$$= 1 + t\,E(X) + \frac{t^2}{\lfloor 2} E(X^2) + \ldots + \frac{t^r}{\lfloor r} E(X^r) + \ldots$$

$$= 1 + t\mu_1' + \frac{t^2}{\lfloor 2}\mu_2' + \ldots + \frac{t^r}{\lfloor r}\mu_r' + \ldots$$

where $\mu_r' = r^{th}$ moment about the origin.

$= E(X^r) = \int x^r f(x)\, dx$ or

$= \Sigma x^r p(x)$ depending upon X is continuous or discrete

The coefficient of $\frac{t^r}{\lfloor r}$ in E (e^{tX}) gives μ_r'

$\therefore M_X(t)$ generates moments about the origin and hence we call it as moment generating function.

Note 1 : $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$

2 : $M_X(t)$ (about X $= a$) $= E[e^{t(X-a)}]$

where $\mu_r' = E[(X-a)^r]$, r^{th} moment about the point $x = a$

Note 3 : Mean $= \overline{X}$

$$M_X(t)\ [\text{about } X = \overline{X}] = 1 + \frac{t}{\lfloor 1}\mu_1 + \frac{t^2}{\lfloor 2}\mu_2 + \ldots + \frac{t^r}{r!}\mu_r + \ldots$$

when $\mu_r = [(X - \overline{X})^r]$

$= r^{th}$ central moment

Limitations of m.g.f

1. A random variable X may have no moment although its m.g.f exists.

2. A random variable X can have its moment generating function and some (or all) moments, yet the moment generating function does not generate the moments.

3. A random variable X can have all or some moments, but moment generating function do not exist except perhaps at one point.

Properties of moment Generating function [A.U Tvli. A/M 2009]

1. Let Y = $aX + b$, where X is a R.V with moment generating function $M_X(t)$. Then

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}] = E[e^{taX} e^{bt}]$$

$$= e^{bt} E[e^{Xat}] = e^{bt} M_X(at)$$

2. $M_{cX}(t) = E[e^{cXt}] = E[e^{X(ct)}] = M_X(ct)$

where c is a constant.

3. If X and Y are two independent random variables, then

$$M_{X+Y}(t) = M_X(t) \,.\, M_Y(t)$$

Proof : $M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX+tY}] = E[e^{tX} e^{tY}]$

$$= E[e^{tX}]\, E[e^{tY}] \quad [\because \text{X and Y are independent}]$$

$$= M_X(t)\, M_Y(t)$$

Example 1.3.1

Find the moment generating function of the RV X whose probability function $P(X = x) = \frac{1}{2^x}, x = 1, 2, \ldots$ Hence find its mean.

[A.U Tvli A/M 2009] [A.U CBT A/M 2011]

Solution :

$$M_X(t) = E[e^{tX}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \frac{(e^t)^2}{2^2} + \ldots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \ldots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2 - e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2}{2 - e^t}\right]$$

$$= \frac{e^t}{2 - e^t}$$

$$\text{Mean} = m_1 = \left[\frac{d}{dt}\left[\frac{e^t}{2 - e^t}\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}[e^t(2 - e^t)^{-1}]\right]_{t=0}$$

$$= [e^t(-1)(2 - e^t)^{-2}(-e^t) + (2 - e^t)^{-1}e^t]_{t=0}$$

$$= [e^{2t}(2 - e^t)^{-2} + (2 - e^t)^{-1}e^t]_{t=0}$$

$$= (2 - 1)^{-2} + (2 - 1)^{-1}$$

$$= (1)^{-2} + (1)^{-1}$$

$$= \frac{1}{(1)^2} + \frac{1}{1}$$

$$= 2$$

Scanned by CamScanner

Example 1.3.2

If X represents the outcome, when a fair die is tossed, find the moment generating function (MGF) of X and hence find E(X) and Var (X).

Solution : The probability distribution of X is given by

$$P_i = P(X = i) = \frac{1}{6}, \; i = 1, 2, \ldots 6$$

$$M_X(t) = \sum_{i=1}^{6} e^{txi} P_i = \frac{1}{6}[e^t + e^{2t} + \ldots + e^{6t}]$$

$$E(X) = [M_X'(t)]_{t=0} = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$$

$$= \frac{1}{6}[1+2+3+4+5+6] = \frac{1}{6}[21] = \frac{7}{2}$$

$$E(X^2) = [M_X''(t)]_{t=0}$$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0}$$

$$= \frac{1}{6}[1+4+9+16+25+36] = \frac{1}{6}[91]$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Example 1.3.3

Find the probability distribution of the total number of heads obtained in four tosses of a balanced coin. Hence obtain the MGF of X, mean of X and variance of X. **[AU A/M 2008]**

Solution :

X :	Number of heads obtained in 4 tosses of a coin				
x :	0	1	2	3	4
$p(x)$:	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(i) MGF

$$M_X(t) = \sum_{x=0}^{4} e^{tx} p(x)$$

$$= p(0) + e^t p(1) + e^{2t} p(2) + e^{3t} p(3) + e^{4t} p(4)$$

$$= \frac{1}{16} + e^t \left(\frac{4}{16}\right) + e^{2t} \left(\frac{6}{16}\right) + e^{3t} \left(\frac{4}{16}\right) + e^{4t} \left(\frac{1}{16}\right)$$

$$= \frac{1}{16} [1 + 4e^t + 6e^{2t} + 4e^{3t} + e^{4t}]$$

$$E[X] = [M_X'(t)]_{t=0}$$

$$= \left[\frac{1}{16} [0 + 4e^t + 12e^{2t} + 12e^{3t} + 4e^{4t}]\right]_{t=0}$$

$$= \frac{1}{16} [4 + 12 + 12 + 4] = \frac{1}{16} [32] = 2$$

$$E[X^2] = [M_X''(t)]_{t=0}$$

$$= \left[\frac{1}{16} [4e^t + 24e^{2t} + 36e^{3t} + 16e^{4t}]\right]_{t=0}$$

$$= \frac{1}{16} [4 + 24 + 36 + 16] = \frac{1}{16} [80] = 5$$

Variance $[X] = E(X^2) - [E(X)]^2 = 5 - (2)^2 = 5 - 4 = 1$

Example 1.3.4

For a discrete random variable. X with probability function

$$f(x) = \begin{cases} \dfrac{1}{x(x+1)}, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Show that E(X) does not exist eventhough m.g.f exists.

[A.U N/D 2012]

Solution : Given : $P(x) = f(x) = \begin{cases} \dfrac{1}{x(x+1)}, & x = 1, 2, 3, \ldots \\ 0, & \textit{otherwise} \end{cases}$

$$\therefore E[X] = \sum x\,P(x)$$

$$= \sum_{x=1}^{\infty} x \frac{1}{x(1+x)} = \sum_{x=1}^{\infty} \frac{1}{(1+x)} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ \ldots$$

$$= \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right) - 1$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - 1, \text{ it is a divergent series}$$

Hence, E(X) does not exist.

Now, we have, by definition the m.g.f as

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} P(x) \qquad [\because \text{X is a discrete r.v}$$

$$= \sum_{x=1}^{\infty} \frac{e^{tx}}{x(1+x)} = \sum_{x=1}^{\infty} \frac{y^x}{x(1+x)} [\text{put } y = e^t]$$

$$= \frac{y}{1.2}+\frac{y^2}{2.3}+\frac{y^3}{3.4}+\ldots$$

$$= y\left(1-\frac{1}{2}\right)+y^2\left(\frac{1}{2}-\frac{1}{3}\right)+y^3\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots$$

$$= \left(y+\frac{y^2}{2}+\frac{y^3}{3}+\ldots\right)-\frac{y}{2}-\frac{y^2}{3}-\frac{y^3}{4}\ldots$$

$$= -\left(-y+\frac{y^2}{2}-\frac{y^3}{3}-\ldots\right)+\left(1-\frac{y}{y}\right)-\frac{y}{2}-\frac{y^2}{3}-\frac{y^3}{4}-\ldots$$

$$\left[\because \log(1-z) = -z-\frac{z^2}{2}-\frac{z^3}{3}\ldots, |z|<1\right]$$

$$= -\log(1-y)+1+\frac{1}{y}\left(-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}\ldots\right) \text{ if } |y|<1$$

$$= -\log(1-y)+1+\frac{1}{y}\log(1-y)$$

$$= 1+\left(\frac{1}{y}-1\right)\log(1-y), \quad |y|<1$$

$$\therefore M_X(t) = 1 + \left(\frac{1}{e^t} - 1\right) \log(1 - e^t), \ |e^t| < 1$$

$$\text{Now, } |e^t| < 1 \quad \Rightarrow |e^t| < 1 \qquad \because e^t \geq 0 \, i.e., \ +ve \ \forall \ t$$

$$\Rightarrow t < \log e \text{ i.e., } t < 0$$

$$\therefore \quad M_X(t) = 1 + (e^{-t} - 1) \log(1 - e^t), \ t < 0$$

$M_X(t) = 0, \ t = 0$. Here we have $\lim_{t \to 0} \left(\frac{1}{e^t} - 1\right) \log(1 - e^t) = 0$

by using L'Hospital rule for indetermine form $(0 \times \infty)$ and $M_X(t)$ does not exist for $t > 0$

Example 1.3.5

For the triangular distribution

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

[A.U. M/J 2006, N/D 2013]

find the mean, variance and the moment generating function (MGF) also find cdf of F(x). **[A.U CBT M/J 2010, CBT N/D 2011]**

[A.U N/D 2013]

Solution : Given : $f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx \qquad \ldots (1)$$

$$= \int_0^1 (x)(x) \, dx + \int_1^2 (x)(2 - x) \, dx$$

$$= \int_0^1 x^2 \, dx + \int_1^2 (2x - x^2) \, dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \left[\frac{1}{3} - 0\right] + \left[x^2 - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = \frac{4 + 1 - 2}{3} = 1$$

Variance, $V(X) = E(X^2) - [E(X)]^2$ ... (2)

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 x\, dx + \int_1^2 x^2 (2 - x)\, dx$$

$$= \int_0^1 x^3\, dx + \int_1^2 (2x^2 - x^3)\, dx = \left[\frac{x^4}{4}\right]_0^1 + \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{4} - 0\right) + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}$$

$$= -\frac{14}{4} + \frac{14}{3} = \frac{-42 + 56}{12} = \frac{14}{12} = \frac{7}{6}$$

$$\therefore (2) \Rightarrow \text{Var}(X) = E[X^2] - [E(X)]^2$$

$$= \frac{7}{6} - (1)^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

The moment generating function of the Random variable X is

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^1 xe^{tx}\, dx + \int_1^2 (2 - x)\, e^{tx}\, dx$$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2 - x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[\left(\frac{e^t}{t} - \frac{e^t}{t^2}\right) - \left(0 - \frac{1}{t^2}\right)\right] + \left[\left(0 + \frac{e^{2t}}{t^2}\right) - \left(\frac{e^t}{t} + \frac{e^t}{t^2}\right)\right]$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2}$$

$$= \frac{1}{t^2}[e^{2t} - 2e^t + 1] = \frac{1}{t^2}[e^t - 1]^2$$

To find the cdf of F (x)

$$F(x) = P[X \le x] = \int_0^x f(x)\,dx$$

(i) If $x \le 0$, then $F(x) = 0$

(ii) If $0 < x \le 1$, then

$$F(x) = \int_0^x x\,dx = \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

(iii) If $1 \le x < 2$, then

$$F[x] = \int_0^1 x\,dx + \int_1^x (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2} = 2x - \frac{x^2}{2} - 1$$

(iv) If $x > 2$, then

$$F(x) = \int_{-\infty}^x f(x)\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 (2-x)\,dx + \int_2^x 0\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + (4-2) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2 - 2 + \frac{1}{2} = 1$$

Example 1.3.6

Let the random variable X have the p.d.f

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the moment generating function, mean and variance of X.

[A.U. A/M. 2005, N/D 2012]

Solution : The m.g.f is given by

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} e^{tx} \frac{1}{2} e^{-x/2}\, dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{(t-\frac{1}{2})x}\, dx = \frac{1}{2} \int_0^{\infty} e^{-(\frac{1}{2}-t)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(\frac{1}{2}-t)x}}{-\left(\frac{1}{2}-t\right)}\right]_0^{\infty} = -\frac{1}{2}\left[\frac{e^{-(\frac{1}{2}-t)x}}{\frac{1}{2}-t}\right]_0^{\infty}$$

$$= -\frac{1}{2}\left[0 - \frac{1}{\frac{1}{2}-t}\right] = -\frac{1}{2}\left[-\frac{1}{\frac{1-2t}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-2t}\right] = \frac{1}{1-2t}$$

$$E(X) = \text{Mean} = M_X'(0) = \frac{d}{dt}\left[\frac{1}{1-2t}\right]_{t=0}$$

$$= \left[\frac{-1}{(1-2t)^2}(-2)\right]_{t=0} = 2$$

$$E(X^2) = M_X''(0) = \frac{d}{dt}\left[M_X'(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{2}{(1-2t)^2}\right]_{t=0} = \left[\frac{-4}{(1-2t)^3}(-2)\right]_{t=0}$$

$$= \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$\text{Varience} = E(X^2) - (E(X))^2$$

$$= 8 - (2)^2 = 8 - 4 = 4$$

Example 1.3.7

The density function of a random variable x is given by $f(x) = Kx(2-x)$, $0 \le x \le 2$. Find K, mean, variance and rth moment.

[A.U. N/D 2006] [A.U. M/J 2007] [A.U Trichy A/M 2010]

Given : $f(x) = Kx(2-x)$, $0 \le x \le 2$ is a p.d.f.

We know that, if $f(x)$ is a p.d.f then,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad \int_0^2 Kx(2-x)\,dx = 1$$

$$K\int_0^2 (2x - x^2)\,dx = 1 \quad \Rightarrow \quad K\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$K\left[\left(4 - \frac{8}{3}\right) - (0-0)\right] = 1 \quad \Rightarrow \quad K\left[\frac{4}{3}\right] = 1 \Rightarrow K = \frac{3}{4}$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^2 x Kx(2-x)\, dx$$

$$= \int_0^2 \frac{3}{4}(2x^2 - x^3)\, dx \qquad [\because K = \frac{3}{4}]$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\left(\frac{16}{3} - \frac{16}{4}\right) - (0-0)\right]$$

$$= \frac{3}{4}(16)\left[\frac{1}{3} - \frac{1}{4}\right] = 12\left[\frac{1}{12}\right] = 1$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 x^2 Kx(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\, dx \qquad [\because K = \frac{3}{4}]$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left[\left(8 - \frac{32}{5}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{40-32}{5}\right] = \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\mu_r' = E[X^r] = \int x^r f(x)\, dx = \int_0^2 x^r kx(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\, dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\left(\frac{2^{r+2}}{r+2}\right) - \frac{2^{r+3}}{r+3}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{(3)\,(2^{r+3})}{4}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= \frac{(3)\,(2^{r+3})}{4}\left[\frac{r+3-r-2}{(r+2)\,(r+3)}\right]$$

$$= \frac{(3)\;(2^{r+1})}{(r+2)\,(r+3)}$$

Example 1.3.8

A continuous R.V. X has the p.d.f f(x) given by f(x) = $c\,e^{-|x|}$, $-\infty < x < \infty$. Find the value of c and moment generating function of X. **[A.U. M/J 2007]**

Solution : Given : $f(x) = c\,e^{-|x|}$

Given $f(x)$ is a p.d.f.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} c\,e^{-|x|}\,dx = 1$$

$$\Rightarrow \quad 2\int_{0}^{\infty} ce^{-x}\,dx = 1 \quad \Rightarrow \quad 2c\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow \quad -2c\left[e^{-x}\right]_0^{\infty} = 1 \quad \Rightarrow \quad -2c\,[0-1] = 1$$

$$\Rightarrow \quad 2c = 1 \quad \Rightarrow \quad c = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{2}e^{-|x|}$$

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx}\,\frac{1}{2}e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}\,e^{-|x|}\,dx = \frac{1}{2}\,2\int_{0}^{\infty} e^{tx}\,e^{-x}\,dx$$