Chapter 9

Tests of Hypotheses

INTRODUCTION

Every statistical investigation aims at collecting information about some aggregate or collection of individuals or of their attributes, rather than the individuals themselves. In statistical language, such a collection is called *a population* or *universe*. For example, we have the population of products turned out by a machine, of lives of electric bulbs manufactured by a company etc. A population is finite or infinite, according as the number of elements is finite or infinite. In most situations, the population may be considered infinitely large. A finite subset of a population is called *a sample* and the process of selection of such samples is called *sampling*. The basic objective of the theory of sampling is to draw inference about the population using the information of the sample.

Parameters and Statistics

Generally in statistical investigations, our ultimate interest will lie in one or more characteristics possessed by the members of the population. If there is only one characteristic of importance, it can be assumed to be a variable x. If x_i be the value of x for the ith member of the sample, then $(x_1, x_2, ..., x_n)$ are referred to as sample observations. Our primary interest will be to know the values of different statistical measures such as mean and variance of the population distribution of x. Statistical measures, calculated on the basis of population values of x are called *parameters*. Corresponding measures computed on the basis of sample observations are called *statistics*.

Sampling Distribution

If a number of samples, each of size n, (i.e. each containing n elements) are drawn from the same population and if for each sample the value of some statistic, say, mean is calculated, a set of values of the statistic will be obtained.

Note *The values of the statistic will normally vary from one sample to another, as the values of the population members included in different samples, though drawn from the same population, may be different. These differences in the values of a statistic are said to be due to sampling fluctuations.*

If the number of samples is large, the values of the statistic may be classified in the form of a frequency table. The probability distribution of the statistic that would be obtained if the number of samples, each of same size were infinitely large is called the *sampling distribution* of the statistic. If we adopt random sampling technique that is the most popular and frequently used method of sampling [the discussion of which is beyond the scope of this book], the nature of the sampling distribution of a statistic can be obtained theoretically, using the theory of probability, provided the nature of the population distribution is known.

Like any other distribution, a sampling distribution will have its mean, standard deviation and moments of higher order. The standard deviation of the sampling distribution of a statistic is of particular importance in tests of Hypotheses and is called *the standard error* of the statistic.

Estimation and Testing of Hypotheses

In sampling theory, we are primarily concerned with two types of problems which are given below:

(i) Some characteristic or feature of the population in which we are interested may be completely unknown to us and we may like to make a guess about this characteristic entirely on the basis of a random sample drawn from the population. This type of problem is known as the problem of *estimation*.

(ii) Some information regarding the characteristic or feature of the population may be available to us and we may like to know whether the information is tenable (or can be accepted) in the light of the random sample drawn from the population and if it can be accepted, with what degree of confidence it can be accepted. This type of problem is known as the problem of *testing of hypotheses*.

Tests of Hypotheses and Tests of Significance

When we attempt to make decisions about the population on the basis of sample information, we have to make assumptions or guesses about the nature of the population involved or about the value of some parameter of the population. Such assumptions, which may or may not be true, are called *statistical hypotheses*. Very often, we set up a hypothesis which assumes that there is no significant difference between the sample statistic and the corresponding population parameter or between two sample statistics. Such a hypothesis of no difference is called a *null hypothesis* and is denoted by H_0. A hypothesis that is different from (or complementary to) the null hypothesis is called an *alternative hypothesis* and is denoted by H_1. A procedure for deciding whether to accept or to reject a null

hypothesis (and hence to reject or to accept the alternative hypothesis respectively) is called *the test of hypothesis*.

If θ_0 is a parameter of the population and θ is the corresponding sample statistic, usually there will be some difference between θ_0 and θ since θ is based on sample observations and is different for different samples. Such a difference which is caused due to sampling fluctuations is called *insignificant difference*. The difference that arises due to the reason that either the sampling procedure is not purely random or that the sample has not been drawn from the given population is known as *significant difference*. This procedure of testing whether the difference between θ_0 and θ is significant or not is called *the test of significance*.

Critical Region and Level of Significance

If we are prepared to reject a null hypothesis when it is true or if we are prepared to accept that the difference between a sample statistic and the corresponding parameter is significant, when the sample statistic lies in a certain region or interval, then that region is called the *critical region* or *region of rejection*. The region complementary to the critical region is called *the region of acceptance*.

In the case of large samples, the sampling distributions of many statistics tend to become normal distributions. If 't' is a statistic in large samples, then t follows a normal distribution with mean $E(t)$, which is the corresponding population parameter, and S.D. equal to S.E. (t). Hence $Z = \dfrac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}$ is a standard normal variate i.e., Z (called *the test statistic*) follows a normal distribution with mean zero and S.D unity.

It is known from the study of normal distribution, that the area under the standard normal curve between $t = -1.96$ and $t = +1.96$ is 0.95. Equivalently the area under the general normal curve of 't' between $[E(t) - 1.96 \text{ S.E. }(t)]$ and $[E(t) + 1.96 \text{ S.E. }(t)]$ is 0.95. In other words, 95 per cent of the values of t will lie between $[E(t) \mp 1.96 \text{ S.E. }(t)]$ or only 5 per cent of values of t will lie outside this interval.

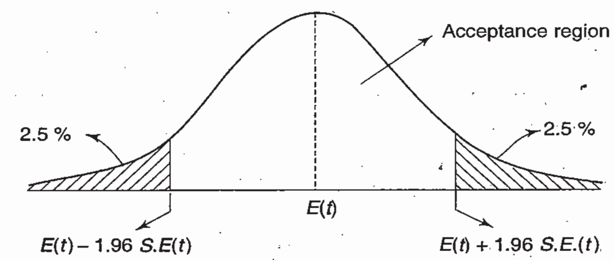

Fig. 9.1

If we are prepared to accept that the difference between t and $E(t)$ is significant when t lies in either of the regions $[-\infty, E(t) - 1.96 \text{ S.E. }(t)]$ and $[E(t) + 1.96 \text{ S.E. }(t), \infty]$ then these two regions constitute the critical region of 't'.

The probability 'α' that a random value of the statistic lies in the critical region is called *the level of significance* and is usually expressed as a percentage.

From the study of normal distributions, it is known that

$$P\{E(t) - 1.96 \text{ S.E. }(t) < t < E(t) + 1.96 \text{ S.E. }(t)\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{S \cdot E \cdot (t)}\right| < 1.96\right\} = 0.95$$

i.e. $$P\{|Z| > 1.96\} = 0.05 \quad \text{or} \quad 5\%$$

Thus when t lies in either of the two regions constituting the critical region given above, the level of significance is 5 per cent.

Note *The level of significance can also be defined as the maximum probability with which we are prepared to reject H_0 when it is true. In other words, the total area of the region of rejection expressed as a percentage is called the level of significance.*

(The specification of critical region and the choice of level of significance will depend upon the nature of the problem and is a matter of judgement for those who carry out the investigation. Usually the levels of significance are taken as 5%, 2% or 1%.)

Errors in Hypotheses Testing

The level of significance is fixed by the investigator and as such it may be fixed at a higher level by his wrong judgement. Due to this, the region of rejection becomes larger and the probability of rejecting a null hypothesis, when it is true, becomes grater. The error committed in rejecting H_0, when it is really true, is called *Type I error*. This is similar to a good product being rejected by the consumer and hence Type I error is also known as *producer's risk*. The error committed in accepting H_0, when it is false, is called *Type II error*. As this error is similar to that of accepting a product of inferior quality, it is also known as *consumer's risk*.

The probabilities of committing Type I and II errors are denoted by α and β respectively. It is to be noted that the probability α of committing Type I error is the level of significance.

One-Tailed and Two-Tailed Tests

If θ_0 is a population parameter and θ is the corresponding sample statistic and if we set up the null hypothesis $H_0 : \theta = \theta_0$, then the alternative hypothesis which is complementary to H_0 can be any one of the following:

(i) $H_1 : \theta \neq \theta_0$, i.e. $\theta > \theta_0$ or $\theta < \theta_0$
(ii) $H_1 : \theta > \theta_0$
(iii) $H_1 : \theta < \theta_0$.

H_1 given in (i) is called a two tailed alternative hypothesis, whereas H_1 given in (ii) is called a right-tailed alternative hypothesis and H_1 given in (iii) is called a left-tailed alternative hypothesis.

When H_0 is tested while H_1 is a one-tailed alternative (right or left), the test of hypothesis is called *a one-tailed test*.

When H_0 is tested while H_1 is two-tailed alternative, the test of hypothesis is called *a two-tailed test*.

The application of one-tailed or two-tailed test depends upon the nature of the alternative hypothesis. The choice of the appropriate alternative hypothesis depends on the situation and the nature of the problem concerned.

Critical Values or Significant Values

The value of the test statistic z for which the critical region and acceptance region are separated is called *the critical value* or *the significant value* of z and denoted by z_α, when α is the level of significance. It is clear that the value of z_α depends not only on α but also on the nature of alternative hypothesis.

When $$z = \frac{t - E(t)}{S \cdot E \cdot (t)},$$ we have seen that

$$P(|z| < 1.96) = 95 \text{ per cent and } \quad P(|z| > 1.96) = 5 \text{ per cent}.$$

Thus $z = \pm 1.96$ separate the critical region and the acceptance region at 5% level of significance for a two-tailed test. That is the critical values of z in this case are ± 1.96.

In general, the critical value z_α for the level of significance α is a given by the equation $P(|z| > z_\alpha) = \alpha$ for a two-tailed test, by the equation $P(z > z_\alpha) = \alpha$ for the right-tailed test and by the equation

$P(z < -z_\alpha) = \alpha$ for the left-tailed test. These equations are solved by using the normal tables.

Note *If z_α is the critical value of z corresponding to the level of significance α in the right-tailed test, then $P(z > z_\alpha) = \alpha$.*

By symmetry of the standard normal distribution followed by z, $P(z < -z_\alpha) = \alpha$.

$$\therefore \quad P(|z| > z_\alpha) = P\{(z > z_\alpha) + (z < -z_\alpha)\}$$
$$= P\{z > z_\alpha) + P(z < -z_\alpha)$$
$$= 2\alpha.$$

That is z_α is the critical value of z corresponding to the LOS (level of significance) 2α.

Thus the critical value of z for a single tailed test (right or left) at LOS 'α' is the same as that for a two-tailed test of LOS '2α'].

The critical values for some standard LOS's are given in the following table both for two-tailed and one-tailed tests

Table 9.1

Nature of test | LOS	1% (.01)	2% (.02)	5% (.05)	10% (.1)
Two-tailed	$\lvert z_\alpha \rvert = 2.58$	$\lvert z_\alpha \rvert = 2.33$	$\lvert z_\alpha \rvert = 1.96$	$\lvert z_\alpha \rvert = 1.645$
Right-tailed	$z_\alpha = 2.33$	$z_\alpha = 2.055$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left-tailed	$z_\alpha = -2.33$	$z_\alpha = -2.055$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

Procedure for Testing of Hypothesis

1. Null hypothesis H_0 is defined.
2. Alternative hypothesis H_1 is also defined after a careful study of the problem and also the nature of the test (whether one-tailed or two tailed) is decided.
3. LOS 'α' is fixed or taken from the problem if specified and z_α is noted.
4. The test-statistic $z = \dfrac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}$ is computed.
5. Comparison is made between $|z|$ and z_α. If $|z| < z_\alpha$, H_0 is accepted or H_1 is rejected, i.e. it is concluded that the difference between t and $E(t)$ is not significant at α % L.O.S.

On the other hand, if $|z| > z_\alpha$, H_0 is rejected or H_1 is accepted, i.e. it is concluded that the difference between t and $E(t)$ is significant at α % L.O.S.

Interval Estimation of Population Parameters

It was pointed out that the objective of the theory of sampling is to estimate population parameters with the help of the corresponding sample statistics. Estimation of a parameter by single value is referred to as *point estimation*, the study of which is beyond the scope of this book. However, an alternative procedure is to give an interval within which the parameter may be supposed to lie. This is called *interval estimation*. The interval within which the parameter is expected to lie is called *the confidence interval* for that parameter. The end points of the confidence interval are called *confidence limits* or *fiducial limits*.

We have already seen that

$$P\{|z| \le 1.96\} = 0.95$$

i.e. $$P\left\{\left|\frac{t - E(t)}{\text{S}\cdot\text{E}\cdot(t)}\right| \le 1.96\right\} = 0.95$$

i.e. $$P\{t - 1.96 \text{ S.E. } (t) \le E(t) \le t + 1.96 \text{ S.E. } (t)\} = 0.95$$

This means that we can assert, with 95% confidence, that the parameter $E(t)$ will lie between $t - 1.96$ S.E. (t) and $t + 1.96$ S.E. (t). Thus $\{t - 1.96$ S.E. (t), $t + 1.96$ S.E. (t) are the 95% confidence limits for $E(t)$.

Similarly $\{t - 2.58$ S.E. (t), $t + 2.58$ S.E. $(t)\}$ is the 99% confidence interval for $E(t)$.

Tests of Significance for Large Samples

It is generally agreed that, if the size of the sample exceeds 30, it should be regarded as a large sample. The tests of significance used for large samples are different from the ones used for small samples for the reason that the following assumptions made for large samples do not hold for small samples

1. The sampling distribution of a statistic is approximately normal, irrespective of whether the distribution of the population is normal or not.
2. Sample statistics are sufficiently close to the corresponding population parameters and hence may be used to calculate the standard error of the sampling distribution.

Test 1

Test of significance of the difference between sample proportion and population proportion.

Let X be the number of successes in n independent Bernoulli trials in which the probability of success for each trial is a constant $= P$ (say). Then it is known that X follows a binomial distribution with mean $E(X) = nP$ and variance $V(X) = nPQ$.

When n is large, X follows $N(nP, \sqrt{nPQ})$, i.e. a normal distribution with mean nP and S.D. $\sqrt{nPQ}$, where $Q = 1 - P$.

$$\therefore \qquad \frac{X}{n} \text{ follows } N\left\{\frac{nP}{n}, \sqrt{\frac{nPQ}{n^2}}\right\}$$

Now $\dfrac{X}{n}$ is the proportion of successes in the sample consisting of n trials, that is denoted by p. Thus the sample proportion p follows $N\left(P, \sqrt{\dfrac{PQ}{n}}\right)$. Therefore test statistic $z = \dfrac{p - P}{\sqrt{\dfrac{PQ}{n}}}$.

If $|z| \le z_\alpha$, the difference between the sample proportion p and the population proportion P is not significant at α % L.O.S.

Note *1. If P is not known, we assume that p is nearly equal to P and hence S.E. (p) is taken as* $\sqrt{\dfrac{pq}{n}}$. *Thus* $z = \dfrac{p - P}{\sqrt{\dfrac{pq}{n}}}$.

2. 95 per cent confidence limits for P are then given by $\dfrac{|P - p|}{\sqrt{\dfrac{Pq}{n}}} \le 1.96$, *i.e. they are* $\left(p - 1.96\sqrt{\dfrac{pq}{n}},\ p + 1.96\sqrt{\dfrac{pq}{n}}\right)$.

Test 2

Test of significance of the difference between two sample proportions.

Let p_1 and p_2 be the proportions of successes in two large samples of size n_1 and n_2 respectively drawn from the same population or from two population with the same proportion P.

Then p_1 follows $N\left(P, \sqrt{\dfrac{PQ}{n_1}}\right)$ and p_2 follows $N\left(P, \sqrt{\dfrac{PQ}{n_2}}\right)$.

Therefore $p_1 - p_2$, which is a linear combination of two normal variables, also follows a normal distribution.

Now $E(p_1 - p_2) = E(p_1) - E(p_2) = P - P = 0$

$$V(p_1 - p_2) = V(p_1) + V(p_2) \qquad (\because \text{ the two samples are independent})$$

$$= PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$\therefore$ $(p_1 - p_2)$ follows $N\left\{0, \sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}\right\}$

$\therefore$ The test statistic $z = \dfrac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$.

If P is not known, an unbiased estimate of P based on both samples, given by

$$\hat{P} = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}, \text{ is used in the place of } P.$$

As before, if $|z| \le z_\alpha$, the difference between the two sample proportions p_1 and p_2 is not significant at α per cent L.O.S.

Note *A sample statistic θ is said to be an unbiased estimate of the parameter θ_0, if $E(\theta) = \theta_0$. In the present case,*

$$E\left\{\frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}\right\} = \frac{1}{n_1 + n_2}\{n_1 E(p_1) + n_2 E(p_2)\}$$

$$= \frac{1}{n_1 + n_2}(n_1 P + n_2 P) = P.$$

$\therefore$ An unbiased estimate of P is $\left(\dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}\right)$.

Test 3

Test of significance of the difference between sample mean and population mean.

Let $X_1, X_2, \ldots, X_n$ be the sample observations in a sample of size n, drawn from a population that is $N(\mu, \sigma)$.

Then each X_i follows $N(\mu, \sigma)$.

It is known that if X_i $(i = 1, 2, \ldots, n)$ are independent normal variates with mean μ_i and variance σ_i^2, then $\Sigma c_i x_i$ is a normal variate with mean $\mu = \Sigma c_i \mu_i$ and variance $\sigma^2 = \Sigma c_i^2 \sigma_i^2$.

Now putting $c_i = \dfrac{1}{n}$, $\mu_i = \mu$ and $\sigma_i = \sigma$, we get

$$\Sigma c_i x_i = \frac{1}{n}\Sigma x_i = \bar{X}, \quad \Sigma c_i \mu_i = \frac{1}{n}\mu + \frac{1}{n}\mu + \ldots + \frac{1}{n}\mu \ (n \text{ terms}) = \mu$$

and $$\Sigma c_i^2 \sigma_i^2 = \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \ldots + \frac{1}{n^2}\sigma^2 \ (n \text{ terms})$$

$$= \frac{\sigma^2}{n}.$$

Thus, if X_i are n independent normal variates with the same mean μ and same variance σ^2, then their mean $\bar{X}$ follows a $N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$. Even if the population, from which the sample is drawn, is non-normal, it is known (from central limit theorem) that the above result holds good, provided n is large.

$\therefore$ The test statistic $z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$.

As usual, if $|z| \le z_\alpha$, the difference between the sample mean $\bar{X}$ and the population mean μ is not significant at α % L.O.S.

Note *1. If σ is not known, the sample S.D. 's' can be used in its place, as s is nearly equal to σ when n is large.*

2. 95% confidence limits for μ are given by $\dfrac{|\mu - \bar{X}|}{\sigma/\sqrt{n}} \le 1.96$, i.e. $\left(\bar{X} - 1.96\dfrac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\dfrac{\sigma}{\sqrt{n}}\right)$, if σ is known. If σ is not known, then the 95% confidence interval is $\left(\bar{X} - \dfrac{1.96\, s}{\sqrt{n}}, \bar{X} + \dfrac{1.96\, s}{\sqrt{n}}\right)$

Test 4

Test of significance of the difference between the means of two samples.

Let $\bar{X}_1$ and $\bar{X}_2$ be the means of two large samples of sizes n_1 and n_2 drawn from two populations (normal or non-normal) with the same mean μ and variances σ_1^2 and σ_2^2 respectively.

Then $\bar{X}_1$ follows a $N\left(\mu, \dfrac{\sigma_1}{\sqrt{n_1}}\right)$ and $\bar{X}_2$ follows a $N\left(\mu, \dfrac{\sigma_2}{\sqrt{n_1}}\right)$ either exactly or approximately.

$\therefore$ $\overline{X}_1 - \overline{X}_2$ also follows a normal distribution.

$$E(\overline{X}_1 - \overline{X}_2) = E(\overline{X}_1) - E(\overline{X}_2) = \mu - \mu = 0.$$

$$V(\overline{X}_1 - \overline{X}_2) = V(\overline{X}_1) + V(\overline{X}_2)$$

($\because$ $\overline{X}_1$ and $\overline{X}_2$ are independent, as the samples are independent)

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Thus $(\overline{X}_1 - \overline{X}_2)$ follows a $N\left\{0, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right\}$

$\therefore$ The test statistic $$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \qquad (1)$$

If $|z| \le z_\alpha$, the difference between $(\overline{X}_1 - \overline{X}_2)$ and 0 or the difference between $\overline{X}_1$ and $\overline{X}_2$ is not significant at α per cent L.O.S.

Note *1. If the samples are drawn from the same population, i.e. if $\sigma_1 = \sigma_2 = \sigma$ then*

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad (2)$$

2. If σ_1 and σ_2 are not known and $\sigma_1 \ne \sigma_2$, σ_1 and σ_2 can be approximated by the sample S.D.'s σ_1 and σ_2. Hence, in such a situation,

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \qquad (3)\ [\text{from } (1)]$$

3. If σ_1 and σ_2 are equal and not known, then $\sigma_1 = \sigma_2 = \sigma$ is approximated by $\sigma^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$. Hence, in such a situation,

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \text{from } (2)$$

i.e. $$z = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}} \qquad (4)$$

4. The difference in the denominators of the values of z given in (3) and (4) may be noted.

Test 5. *Test of significance of the difference between sample S.D. and population S.D.*

Let 's' be the S.D. of a large sample of size n drawn from a normal population with S.D. σ. Then it is known that s follows a $N\left(\sigma, \frac{\sigma}{\sqrt{2n}}\right)$ approximately.

$\therefore$ The test statistic $$z = \frac{s - \sigma}{\sigma/\sqrt{2n}}$$

As before, the significance of the difference between s and σ is tested.

Test 6. *Test of significance of the difference between S.D.'s of two large samples.*

Let s_1 and s_2 be the S.D.'s of two large samples of sizes n_1 and n_2 drawn from a normal population with S.D. σ.

s_1 follows a $N\left(\sigma, \frac{\sigma}{\sqrt{2n_1}}\right)$ and s_2 follows a $N\left(\sigma, \frac{\sigma}{\sqrt{2n_2}}\right)$.

$\therefore$ $(s_1 - s_2)$ follows a $N\left\{0, \sigma\sqrt{\frac{1}{2n_1} + \frac{1}{2n_2}}\right\}$.

$\therefore$ The test statistic $$z = \frac{s_1 - s_2}{\sigma\sqrt{\frac{1}{2n_1} + \frac{1}{2n_2}}}.$$

As usual, the significance of the difference between s_1 and s_2 is tested.

Note *If σ is not known, it is approximated by $\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)}$, when n_1 and n_2 are large. In this situation,*

$$z = \frac{s_1 - s_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\right)\left(\frac{1}{2n_1} + \frac{1}{2n_2}\right)}}$$

i.e. $$z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_2} + \frac{s_2^2}{2n_1}}}$$

Worked Example 9(A)

Example 1

Experience has shown that 20 per cent of a manufactured product is of top quality. In one day's production of 400 articles, only 50 are of top quality. Show that either the production of the day chosen was not a representative sample or the hypothesis of 20 per cent was wrong. Based on the particular day's production, find also the 95 per cent confidence limits for the percentage of top quality product.

$H_0 : P = \frac{1}{5}$, i.e. 20 per cent of the products manufactured is of top quality.

$H_1 : P \neq \frac{1}{5}$.

p = proportion of top quality products in the sample

$= \frac{50}{400} = \frac{1}{8}$.

From the alternative hypothesis H_1, we note that two-tailed test is to be used. Let us assume that LOS (level of significance)

$= 5\%. \quad \therefore z_\alpha = 1.96$

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{\frac{1}{8}-\frac{1}{5}}{\sqrt{\frac{1}{5}\times\frac{4}{5}\times\frac{1}{400}}}, \text{ since the size of the sample} = 400.$$

$$= -\frac{3}{40} \times 50 = -3.75$$

Now $|z| = 3.75 > 1.96$.

The difference between p and P is significant at 5 per cent level.

Also H_0 is rejected. Hence H_0 is wrong or the production of the particular day chosen is not a representative sample.

95 per cent confidence limits for P are given by

$$\frac{|p-P|}{\sqrt{\frac{pq}{n}}} \leq 1.96$$

Note

We have taken $\sqrt{\frac{pq}{n}}$ *in the denominator, because P is assumed to be unknown, for which we are trying to find the confidence limits and P is nearly equal to p.*

i.e. $$p - \sqrt{\frac{pq}{n}} \times 1.96 \leq P \leq p + \sqrt{\frac{pq}{n}} \times 1.96$$

i.e. $$0.125 - \sqrt{\frac{1}{8}\times\frac{7}{8}\times\frac{1}{400}} \times 1.96 \leq P \leq 0.125 + \sqrt{\frac{1}{8}\times\frac{7}{8}\times\frac{1}{400}} \times 1.96$$

i.e. $$0.093 \leq P \leq 0.157$$

$\therefore$ 95 per cent confidence limits for the percentage of top quality product are 9.3 and 15.7.

Example 2

The fatality rate of typhoid patients is believed to be 17.26 per cent. In a certain year 640 patients suffering from typhoid were treated in a metropolitan hospital and only 63 patients died. Can you consider the hospital efficient?

$H_0 : p = P$, i.e. the hospital is not efficient. $H_1 : p < P$.

One-tailed (left-tailed) test is to be used

Let us assume that LOS = 1%. $\therefore z_\alpha = -2.33$

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}, \text{ where } p = \frac{63}{640} = 0.0984 \text{ and}$$

$P = 0.1726$ and hence $Q = 0.8274$.

$$z = \frac{0.0984 - 0.1726}{\sqrt{\frac{0.1726 \times 0.8274}{640}}} = -4.96$$

$\therefore \quad |z| > |z_\alpha|$

$\therefore$ The difference between p and P is significant. i.e., H_0 is rejected and H_1 is accepted.

i.e. The hospital is efficient in bringing down the fatality rate of typhoid patients.

Example 3

A salesman in a departmental store claims that at most 60 percent of the shoppers entering the store leaves without making a purchase. A random sample of 50 shoppers showed that 35 of them left without making a purchase. Are these sample results consistent with the claim of the salesman? Use a level of significance of 0.05.

Let P and p denote the population and sample proportions of shoppers not making a purchase.

$H_0 : p = P$

$H_1 : p > P$, since $p = 0.7$ and $P = 0.6$

One-tailed (right-tailed) test is to be used.

LOS = 5% $\therefore z_\alpha = 1.645$

$$z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.7-0.6}{\sqrt{\frac{0.6\times 0.4}{50}}} = 1.443$$

$\therefore$ $|z| < z_\alpha$

$\therefore$ The difference between p and P is not significant at 5 percent level.
i.e. H_0 is accepted and H_1 is rejected.
i.e. the sample results are consistent with the claim of the salesman.

Example 4

Show that for a random sample of size 100, drawn with replacement, the standard error of sample proportion cannot exceed 0.05.

The items of the sample are drawn one after another with replacement.

$\therefore$ The proportion (probability) of success in the population, i.e. P remains a constant.

We know that the sample proportion p follows a $N\left(P, \sqrt{\frac{PQ}{n}}\right)$

i.e. standard error of $p = \sqrt{\frac{PQ}{n}} = \frac{1}{10}\sqrt{PQ}$ $(\because n = 100)$ (1)

Now $\left(\sqrt{P}-\sqrt{Q}\right)^2 \geq 0$

i.e. $P + Q - 2\sqrt{PQ} \geq 0$

i.e. $1 - 2\sqrt{PQ} \geq 0$ or $\sqrt{PQ} \leq \frac{1}{2}$ (2)

Using (2) in (1), we get,

S.E. of $p \leq \frac{1}{20}$. i.e. S.E. of p cannot exceed 0.05.

Example 5

A cubical die is thrown 9000 times and a throw of three or four is observed 3240 times. Show that the die cannot be regarded as an unbiased one and find the extreme limits between which the probability of a throw of three or four lies.

H_0 : the die is unbiased, i.e. $P = \frac{1}{3}$ (= the probability of getting 3 or 4)

$H_1 : P \neq \frac{1}{3}$

Two tailed test is to be used.

Let LOS = 5% $\therefore z_\alpha = 1.96$

Though we may test the significance of the difference between the sample and population proportions, we shall test the significance of the difference between the number X of successes in the sample and that in the population.

When n is large, X follows a $N\left(nP, \sqrt{nPQ}\right)$ [Refer to Test 1].

$\therefore$
$$z = \frac{X-nP}{\sqrt{nPQ}} = \frac{3240-\left(9000\times\frac{1}{3}\right)}{\sqrt{9000\times\frac{1}{3}\times\frac{2}{3}}} = 5.37$$

$\therefore$ $|z| > z_\alpha$

$\therefore$ The difference between X and nP is significant. i.e. H_0 is rejected.
i.e. The die cannot be regarded as unbiased.

If X follows a $N(\mu, \sigma)$, then the reader can easily varify that $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = .9974$.

The limits $\mu \pm 3\sigma$ are considered as the extreme (confidence) limits within which X lies.

Accordingly, the extreme limits for P are given by

$$\frac{|P-p|}{\sqrt{\frac{pq}{n}}} \leq 3$$ [Refer to Example (1)]

i.e. $p - 3\sqrt{\frac{pq}{n}} \leq P \leq p + 3\sqrt{\frac{pq}{n}}$

i.e. $0.36 - 3\sqrt{\frac{0.36\times 0.64}{9000}} \leq P \leq 0.36 + 3\sqrt{\frac{0.36\times 0.64}{9000}}$

i.e. $0.345 \leq P \leq 0.375.$

Example 6

In a large city A, 20 per cent of a random sample of 900 school boys had a slight physical defect. In another large city B, 18.5 percent of a random sample of 1600 school boys had the same defect. Is the difference between the proportions significant?

$p_1 = 0.2$, $p_2 = 0.185$, $n_1 = 900$ and $n_2 = 1600$

$H_0 : p_1 = p_2$

$H_1 : p_1 \neq p_2$

Two tailed test is to be used.

Let L.O.S. be 5% $\therefore z_\alpha = 1.96$

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \quad (1)$$

Since P, the population proportion, is not given, we estimate it as $\hat{P} = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \dfrac{180 + 296}{900 + 1600} = 0.1904$.

Using in (1), we have

$$z = \frac{0.2 - 0.185}{\sqrt{0.1904 \times 0.8096 \times \left(\frac{1}{900} + \frac{1}{1600}\right)}} = 0.92$$

$|z| \le z_\alpha$. Therefore The difference between p_1 and p_2 is not significant at 5 per cent level.

Example 7

Before an increase in excise duty on tea, 800 people out of a sample of 1000 were consumers of tea. After the increase in duty, 800 people were consumers of tea in a sample of 1200 persons. Find whether there is significant decrease in the consumption of tea after the increase in duty.

Let p_1 and p_2 be the proportions of the consumers before and after the increase in duty respectively.

Then $\quad p_1 = \dfrac{800}{1000} = \dfrac{4}{5}$ and $p_2 = \dfrac{800}{1200} = \dfrac{2}{3}$.

$H_0 : p_1 = p_2$

$H_1 : p_1 > p_2$

One-tailed (right-tailed) test is to be used. Let LOS be 1%. $\therefore z_\alpha = 2.33$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \quad \text{where } P \simeq \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$= \frac{800 + 800}{2200} = 0.7273$$

$$= \frac{0.8 - 0.67}{\sqrt{0.7273 \times 0.2727 \times \left(\frac{1}{1000} + \frac{1}{1200}\right)}}$$

$$= \frac{0.13 \times \sqrt{1000 \times 1200}}{\sqrt{0.7273 \times 0.2727 \times 2200}} = 6.82$$

Now $\quad |z| > z_\alpha$.

$\therefore$ The difference between p_1 and p_2 is significant at 1% level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. there is significant decrease in the consumption of tea after the increase in duty.

Example 8

15.5 per cent of a random sample of 1600 undergraduates were smokers, wherea 20% of a random sample of 900 postgraduates were smokers in a state. Can w conclude that less number of undergraduates are smokers than the postgradu ates?

$p_1 = 0.155$ and $p_2 = 0.2$; $n_1 = 1600$ and $n_2 = 900$

$H_0 : p_1 = p_2$

$H_1 : p_1 < p_2$

One-tailed (left-tailed) test is to be used. Let LOS be 5%. $\therefore z_\alpha = -1.645$.

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}; \quad \text{where } P \simeq \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = 0.1712$$

$$= \frac{0.155 - 0.2}{\sqrt{0.1712 \times 0.8288 \times \left(\frac{1}{1600} + \frac{1}{900}\right)}} = \frac{-0.045 \times 1200}{\sqrt{0.1712 \times 0.8288 \times 2500}}$$

$$= -2.87$$

Now $\quad |z| > |z_\alpha|$

$\therefore$ The difference between p_1 and p_2 is significant.

i.e. H_0 is rejected and H_1 is accepted.

i.e. The habit of smoking is less among the undergraduates than among the postgraduates.

Example 9

A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that, in the population, the mean height is 165 cm, and the S.D. is 10 cm?

$\bar{x} = 160$, $n = 100$, $\mu = 165$ and $\sigma = 10$.

$H_0 : \bar{x} = \mu$ (i.e. the difference between $\bar{x}$ and μ is not significant)

$H_1 : \bar{x} \ne \mu$.

Two-tailed test is to be used.

Let LOS be 1% $\therefore z_\alpha = 2.58$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{160 - 165}{10 / \sqrt{100}} = -5$$

Now $\quad |z| > z_\alpha$.

$\therefore$ The difference between $\bar{x}$ and μ is significant at 1% level.

i.e. H_0 is rejected.

i.e. it is not statistically correct to assume that $\mu = 165$.

Example 10

The mean breaking strength of the cables supplied by a manufacturer is 1800 with a S.D. of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. In order to test this claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1 per cent level of significance?

$$\bar{x} = 1850, \quad n = 50, \quad \mu = 1800 \quad \text{and} \quad \sigma = 100$$

$$H_0 : \bar{x} = \mu$$

$$H_1 : \bar{x} > \mu$$

One-tailed (right-tailed) test is to be used.

$$\text{LOS} = 1\% \quad \therefore z_\alpha = 2.33$$

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{1850 - 1800}{100 / \sqrt{50}} = 3.54$$

Now $\quad |z| > z_\alpha.$

$\therefore$ The difference between $\bar{x}$ and μ is significant at 1 per cent level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. based on the sample data, we may support the claim of increase in breaking strength.

Example 11

The mean value of a random sample of 60 items was found to be 145 with a S.D. of 40. Find the 95% confidence limits for the population mean. What size of the sample is required to estimate the population mean within five of its actual value with 95% or more confidence, using the sample mean?

95% confidence limits for μ are given by

$$\frac{|\mu - \bar{x}|}{\sigma / \sqrt{n}} \le 1.96$$

Since the population S.D. σ too is not given, we can approximate it by the sample S.D.s. therefore 95% confidence limits for μ are given by $\dfrac{|\mu - \bar{x}|}{s / \sqrt{n}} \le 1.96$

i.e. $\quad \bar{x} - 1.96 \dfrac{s}{\sqrt{n}} \le \mu \le \bar{x} + 1.96 \dfrac{s}{\sqrt{n}}.$

i.e. $\quad 145 - \dfrac{1.96 \times 40}{\sqrt{60}} \le \mu \le 145 + \dfrac{1.96 \times 40}{\sqrt{60}}$

i.e. $\quad 134.9 \le \mu \le 155.1$

We have to find the value of n such that

$$P\{\bar{x} - 5 \le \mu \le \bar{x} + 5\} \ge \cdot 95$$

i.e. $\quad P\{-5 \le \mu - \bar{x} \le 5\} \ge \cdot 95$

i.e. $\quad P\{|\mu - \bar{x}| \le 5\} \ge \cdot 95$ or

$$P\{|\bar{x} - \mu| \le 5\} \ge \cdot 95$$

$\therefore \quad P\left\{\dfrac{|\bar{x} - \mu|}{\sigma / \sqrt{n}} \le \dfrac{5}{\sigma / \sqrt{n}}\right\} \ge \cdot 95$

i.e. $\quad P\left\{|z| \le \dfrac{5\sqrt{n}}{\sigma}\right\} \ge \cdot 95$, where z is the standard normal variate (1)

We know that $P\{|z| \le 1.96\} = \cdot 95$

$\therefore$ The least value of $n = n_L$ that will satisfy (1) is given by $\dfrac{5\sqrt{n_L}}{\sigma} = 1.96$

i.e. $\quad \sqrt{n_L} = \dfrac{1.96\, s}{5} \quad (\because \ \sigma \simeq s)$

i.e. $\quad n_L = \left(\dfrac{1.96 \times 40}{5}\right)^2$

i.e. $\quad n_L = 245.86$

$\therefore$ The least size of the sample = 246.

Example 12

A normal population has a mean of 0.1 and S.D. of 2.1. Find the probability that the mean of a sample of size 900 drawn from this population will be negative.

Since $\bar{x}$ follows a $N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$, $z = \dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$ is the standard normal variate.

Now $\quad P(\bar{x} < 0) = P\{\bar{x} - 0.1 < -0.1\}$

$$= P\left\{\frac{\bar{x} - 0.1}{(2.1)/\sqrt{900}} < \frac{-0.1}{(2.1)/\sqrt{900}}\right\}$$

$$= P\{z < -1.43\}$$

$$= P\{z > 1.43\},$$

by symmetry of the standard normal distribution.

$$= 0.5 - P\{0 < z < 1.43\}$$

$$= 0.5 - 0.4236 \text{ (from the normal tables)}$$

$$= 0.0764.$$

Example 13

In a random sample of size 500, the mean is found to be 20. In another independent sample of size 400, the mean is 15. Could the samples have been drawn from the same population with S.D. 4?

$$\bar{x}_1 = 20, \quad n_1 = 500; \quad \bar{x}_2 = 15, \quad n_2 = 400; \quad \sigma = 4$$

$H_0 : \bar{x}_1 = \bar{x}_2$, i.e. the samples have been drawn from the same population.

$H_1 : \bar{x}_1 \neq \bar{x}_2$.

Two-tailed test is to be used.

Let LOS be 1% $\therefore z_\alpha = 2.58$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \qquad \text{(Refer to Note 1 under Test 4)}$$

$$= \frac{20 - 15}{4\sqrt{\dfrac{1}{500} + \dfrac{1}{400}}} = 18.6$$

Now $|z| > z_\alpha$

$\therefore$ The difference between $\bar{x}_1$ and $\bar{x}_2$ is significant at 1% level.

i.e. H_0 is rejected

i.e. the samples could not have been drawn from the same population.

Example 14

A simple sample of heights of 6400 English men has a mean of 170 cm and a S.D. of 6.4 cm, while a simple sample of heights of 1600 Americans has a mean of 172 cm and a S.D. of 6.3 cm. Do the data indicate that Americans are, on the average, taller than the Englishmen?

$$n_1 = 6400, \ \bar{x}_1 = 170 \quad \text{and} \quad s_1 = 6.4$$

$$n_2 = 1600, \ \bar{x}_2 = 172 \quad \text{and} \quad s_2 = 6.3$$

$H_0 : \mu_1 = \mu_2$ or $\bar{x}_1 = \bar{x}_2$,

i.e. the samples have been drawn from two different populations with the same mean.

$H_1 : \bar{x}_1 < \bar{x}_2$ or $\mu_1 < \mu_2$.

Left-tailed test is to be used.

Let LOS be 1%. $\therefore z_\alpha = -2.33$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

[$\because \sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$. Refer to Note 2 under Test 4]

$$= \frac{170 - 172}{\sqrt{\dfrac{(6.4)^2}{6400} + \dfrac{(6.3)^2}{1600}}} = -11.32$$

Now $|z| > |z_\alpha|$

$\therefore$ The difference between $\bar{x}_1$ and $\bar{x}_2$ (or μ_1 and μ_2) is significant at 1% level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. The Americans are, on the average, taller than the Englishmen.

Example 15

Test the significance of the difference between the means of the samples, drawn from two normal populations with the same S.D. from the following data:

Table 9.2

	Size	*Mean*	*S.D.*
Sample 1	100	61	4
Sample 2	200	63	6

$H_0 : \bar{x}_1 = \bar{x}_2$ or $\mu_1 = \mu_2$

$H_1 : \bar{x}_1 \neq \bar{x}_2$ or $\mu_1 \neq \mu_2$

Two-tailed test is to be used.

Let LOS be 5% $\therefore z_\alpha = 1.96$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}}}$$

[Refer to Note 3 under Test 4; The populations have the same S.D.]

$$= \frac{61 - 63}{\sqrt{\dfrac{4^2}{200} + \dfrac{6^2}{100}}} = -3.02$$

Now $|z| > z_\alpha$

$\therefore$ The difference between $\bar{x}_1$ and $\bar{x}_2$ (or μ_1 and μ_2) is significant at 5% level.

i.e. H_0 is rejected and H_1 is accepted.

i.e. The two normal populations, from which the samples are drawn, may not have the same mean, though they may have the same S.D.

Example 16

The average marks scored by 32 boys is 72 with a S.D. of 8, while that for 36 girls is 70 with a S.D. of 6. Test at 1% level of significance whether the boys perform better than girls.

$$H_0 : \bar{x}_1 = \bar{x}_2 \quad (\text{or } \mu_1 = \mu_2)$$
$$H_1 : \bar{x}_1 > \bar{x}_2$$

Right-tailed test is to be used.

$$\text{LOS} = 1\% \quad \therefore z_\alpha = 2.33$$

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

(The two populations are assumed to have S.D.'s $\sigma_1 \simeq s_1$ and $\sigma_2 \simeq s_2$)

$$= \frac{72 - 70}{\sqrt{\frac{8^2}{32} + \frac{6^2}{36}}} = 1.15$$

$|z| < z_\alpha$.

$\therefore$ The difference between $\bar{x}_1$ and $\bar{x}_2$ (μ_1 and μ_2) is not significant at 1% level.
i.e. H_0 is accepted and H_1 is rejected.
i.e. Statistically, we cannot conclude that boys perform better than girls.

Example 17

The heights of men in a city are normally distributed with mean 171 cm and S.D. 7 cm., while the corresponding values for women in the same city are 165 cm and 6 cm respectively. If a man and a woman are chosen at random from this city, find the probability that the woman is taller than the man.

Let $\bar{x}_1$ and $\bar{x}_2$ denote the mean heights of men and women respectively.
Then $\bar{x}_1$ follows a N (171, 7) and $\bar{x}_2$ follows a N (165, 6)

$\therefore$ $\bar{x}_1 - \bar{x}_2$ also a follows a normal distribution.

$$E(\bar{x}_1 - \bar{x}_2) = E(x_1) - E(\bar{x}_2) = 171 - 165 = 6$$

$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2) = 49 + 36 = 85 \text{ (Refer to Test 4)}$$

$\therefore$ S.D. of $(\bar{x}_1 - \bar{x}_2) = \sqrt{85} = 9.22$.

$\therefore$ $\bar{x}_1 - \bar{x}_2$ follows a N (6, 9.22)

Now $P(\bar{x}_2 > \bar{x}_1) = P(\bar{x}_1 - \bar{x}_2 < 0)$

$$= P\left\{\frac{(\bar{x}_1 - \bar{x}_2) - 6}{9.22} < \frac{-6}{9.22}\right\}$$

$= P\{z < -0.65\}$, where z is the standard normal variate.

$= P\{z > 0.65\}$, by symmetry.

$$= 0.5 - P(0 < z < 0.65)$$
$$= 0.5 - 0.2422 = 0.2578.$$

Example 18

Two populations have the same mean, but the S.D. of one is twice that of the other. Show that in samples, each of size 500, drawn under simple random conditions, the difference of the means will, in all probability, not exceed 0.3 σ, where σ is the smaller S.D.

Let $\bar{x}_1$ and $\bar{x}_2$ be the means of the samples of size 500 each. Let their S.D.'s be σ and 2σ respectively.

$$\bar{x}_1 \text{ follows a } N\left(\mu, \frac{\sigma}{\sqrt{500}}\right) \text{ and}$$

$$\bar{x}_2 \text{ follows a } N\left(\mu, \frac{2\sigma}{\sqrt{500}}\right)$$

$\therefore$ $\bar{x}_1 - \bar{x}_2$ also follows a normal distribution

$$E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu - \mu = 0$$
$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2)$$
$$= \frac{\sigma^2}{500} + \frac{4\sigma^2}{500} = \frac{\sigma^2}{100}$$

$\therefore$ S.D. of $(\bar{x}_1 - \bar{x}_2) = \frac{\sigma}{10}$

Thus $(\bar{x}_1 - \bar{x}_2)$ follows a $N\left(0, \frac{\sigma}{10}\right)$.

$\therefore$ $P\{|\bar{x}_1 - \bar{x}_2| \le 0.3\,\sigma\}$

$$= P\left\{\frac{|(\bar{x}_1 - \bar{x}_2) - 0|}{\sigma/10} \le \frac{0.3\,\sigma}{\sigma/10}\right\}$$

$= P\{|z| \le 3\}$, where z is the standard normal variate

$= 0.9974 \simeq 1$.

$\therefore$ $|\bar{x}_1 - \bar{x}_2|$ will not exceed 0.3 σ almost certainly.

Example 19

A manufacturer of electric bulbs, according to a certain process, finds the S.D. of the life of lamps to be 100 hours. He wants to change the process, if the new process results in a smaller variation in the life of lamps. In adopting a new process, a sample of 150 bulbs gave an S.D. of 95 hours. Is the manufacturer justified in changing the process?

$$\sigma = 100, \quad n = 150 \quad \text{and} \quad s = 95$$

$H_0 : s = \sigma$

$H_1 : s < \sigma$

Left-tailed test is to be used.

Let LOS be 5%. $\therefore z_\alpha = -1.645$

$$z = \frac{s - \sigma}{\sigma / \sqrt{2n}} = \frac{95 - 100}{100 / \sqrt{300}} = -0.866$$

Now $|z| < |z_\alpha|$

$\therefore$ The difference between s and σ is not significant at 5% level.

i.e. H_0 is accepted and H_1 is rejected.

i.e. The manufacturer is not justified in changing the process.

Example 20

The S.D. of a random sample of 1000 is found to be 2.6 and the S.D. of another random sample of 500 is 2.7. Assuming the samples to be independent, find whether the two samples could have come from populations with the same S.D.

$n_1 = 1000,\quad s_1 = 2.6\ ;\quad n_2 = 500,\quad s_2 = 2.7$

$H_0 : s_1 = s_2 \quad (\text{or } \sigma_1 = \sigma_2)$

$H_1 : s_1 \neq s_2$

Two tailed test is to be used.

Let LOS be 5%. $\therefore z_\alpha = 1.96$

$$z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_2} + \frac{s_2^2}{2n_1}}}, \text{ since } \sigma \text{ is not known.}$$

$$= \frac{2.6 - 2.7}{\sqrt{\frac{(2.6)^2}{1000} + \frac{(2.7)^2}{2000}}} = -0.98$$

Now $|z| < z_\alpha$

$\therefore$ The difference between s_1 and s_2 (and hence between σ_1 and σ_2) is not significant at 5% level.

i.e. H_0 is accepted.

i.e. the two samples could have come from populations with the same S.D.

Exercise 9(A)

Part A

(Short Answer Questions)

1. What is the difference between population and sample?

2. Distinguish between parameter and statistic.
3. What do you mean by sampling distribution?
4. What is meant by standard error?
5. What do you mean by estimation?
6. What is meant by hypothesis testing?
7. Define null hypothesis and alternative hypothesis.
8. What is meant by test of significance?
9. What do you mean by critical region and acceptance region?
10. Define level of significance.
11. Give the general form of a test statistic.
12. Define type I and type II errors.
13. Define producer's risk and consumer's risk.
14. What is the relation between type I error and level of significance?
15. Define one-tailed and two-tailed tests.
16. Define critical value of a test statistic.
17. What is the relation between the critical value and level of significance?
18. What is the relation between the critical values for a single tailed test and a two-tailed test?
19. Write down the 1% and 5% critical values for right-tailed and two-tailed tests.
20. What do you mean by interval estimation and confidence limits?
21. Write down the general form of 95% confidence limits of a population parameter in terms of the corresponding sample statistic.
22. What is the standard error of the sample proportion, when the population proportion is (i) known; and (ii) not known?
23. What is the standard error of the difference between two sample proportions when the population proportion is (i) known and (ii) not known?
24. What do you mean by unbiased estimate? Give an example.
25. Write down the form of the 98% confidence interval for the population mean in terms of (i) population S.D.; and (ii) Sample S.D.
26. What is the standard error of the difference between the means of two large samples drawn from different populations with (i) known S.D.'s and; (ii) unknown S.D.'s?
27. What is the standard error of the difference between the means of two large samples drawn from the same population with (i) known S.D. and; (ii) unknown S.D.?
28. What is the standard error of the difference between the S.D.'s of two large samples drawn from the same population with (i) known S.D. and; (ii) unknown S.D.?

Part B

29. Out of 200 individuals, 40 per cent show a certain trait and the number expected on a certain theory is 50 per cent . Find whether the number observed differs significantly from expectation.

30. A coin is thrown 400 times and is found to result in 'Head' 245 times. Test whether the coin is a fair one.
31. A manufacturer of light bulbs claims that on the average 2 per cent of the bulbs manufactured by his firm are defective. A random sample of 400 bulbs contained 13 defective bulbs. On the basis of the this sample, can you support the manufacturer's claim at 5% level of significance?
32. 100 people were affected by cholera and out of them only 90 survived. Would you reject the hypothesis that the survival rate, if affected by cholera, is 85 per cent in favour of the hypothesis that it is more at 5 per cent level of significance?
33. A random sample of 400 mangoes was taken from a big consignment and 40 were found to be bad. Prove that the percentage of bad mangoes in the consignment will, in all probability, lie between 5.5 and 14.5.
34. A random sample of 64 articles produced by a machine contained 14 defectives. Is it reasonable to assume that only 10 per cent of the articles produced by the machine are defective? If not, find the 99 per cent confidence limits for the percentage of defective articles produced by the machine.
35. Certain crosses of the pea gave 5321 yellow and 1804 green seeds. The expectation is 25 per cent of green seeds based on a certain theory. Is this divergence significant or due to sampling fluctuations?
36. During a countrywide investigation, the incidence of T.B. was found to be 1 per cent. In a college with 400 students, 5 are reported to be affected whereas in another with 1200 students, 10 are found to be affected. Does this indicate any significant difference?
37. A random sample of 600 men chosen from a certain city contained 400 smokers. In another sample of 900 men chosen from another city, there were 450 smokers. Do the data indicate that (i) the cities are significantly different with respect to smoking habit among men? (ii) the first city contains more smokers than the second?
38. A sample of 300 spare parts produced by a machine contained 48 defectives. Another sample of 100 spare parts produced by another machine contained 24 defectives. Can you conclude that the first machine is better than the second?
39. In two large populations, there are 30 per cent and 25 per cent respectively of fair haired people. Is this difference likely to be hidden in samples of sizes 1200 and 900 respectively drawn from the two populations?

[**Hint** : $H_0 : P_1 - P_2 = 0$, or $P_1 = P_2$ and

$$H_1 : P_1 \neq P_2 \text{ and } z = \frac{p_1 - p_2}{\sqrt{\dfrac{P_1 Q_1}{n_1} + \dfrac{P_2 Q_2}{n_2}}}$$]

40. A machine produces 16 defective bolts in a batch of 500 bolts. After the machine is overhauled, it produces three defective bolts in a batch of 100 bolts. Has the machine improved?
41. There were 956 births in a year in town A, of which 52.5 per cent were males, while in towns A and B combined together this proportion in a total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?
42. A cigarette manufacturing company claims that its brand A cigarettes outsells its brand B by 8 per cent. It is found that 42 out of a sample of 200 smokers prefer brand A and 18 out of another sample of 100 smokers prefer brand B. Test at 5 per cent L.O.S. whether the 8 per cent difference is a valid claim.

[**Hint** : $H_0 : P_1 - P_2 = .08$; $H_1 : P_1 - P_2 \neq .08$ and

$$Z = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}, \quad \text{where } P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$]

43. A sample of 900 items is found to have a mean of 3.47 cm. Can it be reasonably regarded as a simple sample from a population with mean 3.23 cm and S.D. 2.31 cm?
44. A sample of 400 observations has mean 95 and S.D. 12. Could it be a random sample from a population with mean 98? What should be the maximum value of the population mean so that the sample can be regarded as one drawn from it almost certainly?
45. A manufacturer claims that, the mean breaking strength of safety belts for air passengers produced in his factory is 1275 kgs. A sample of 100 belts was tested and the mean breaking strength and S.D. were found to be 1258 kgs and 90 kgs respectively. Test the manufacturer's claim at 5 per cent level of significance.
46. An I.Q. test was given to a large group of boys in the age group of 18 to 20 years, who scored an average of 62.5 marks. The same test was given to a fresh group of 100 boys of the same age group. They scored an average of 64.5 marks with a S.D. 12.5 marks. Can we conclude that the fresh group of boys have better I.Q.?
47. The guaranteed average life of a certain brand of electric bulb is 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90 per cent of the bulbs do not fall short of the guaranteed

average by more than 2.5 per cent . What should be the minimum sample size?

48. A random sample of 100 students gave a mean weight of 58 kg with a S.D. of 4 kg. Find the 95 per cent and 99 per cent confidence limits of the mean of the population.
49. The means of two simple samples of 1000 and 2000 items are 170 cm and 169 cm. Can the samples be regarded as drawn from the same population with S.D. 10, at 5 per cent level of significance?
50. The mean and S.D. of a sample of size 400 are 250 and 40 respectively. Those of another sample of size 400 are 220 and 55. Test at 1% level of significance whether the means of the two populations from which the samples have been drawn are equal.
51. Intelligence tests were given to two groups of boys and girls of the same age group chosen from the same college and the following results were got:

Table 9.3

	Size	*Mean*	*S.D.*
Boys	100	73	10
Girls	60	75	8

Examine if the difference between the means is significant.

52. A sample of 100 bulbs of brand *A* gave a mean lifetime of 1200 hours with a S.D. of 70 hours, while another sample of 120 bulbs of brand *B* gave a mean lifetime of 1150 hours with a S.D. of 85 hours. Can we conclude that brand *A* bulbs are superior to brand *B* bulbs?
53. In a college, 60 junior students are found to have a mean height of 171.5 cm and 50 senior students are found to have a mean height of 173.8 cm. Can we conclude, based on this data, that the juniors are shorter than seniors at (i) 5% level of significance and (ii) 1% level of significance, assuming that the S.D. of students of that college is 6.2 cm?
54. Two samples drawn from two different populations gave the following results:

Table 9.4

	Size	*Mean*	*S.D.*
Sample I	400	124	14
Sample II	250	120	12

Find the 95% confidence limits for the difference of the population means.

$$\left[\textbf{Hint}: (\bar{x}_1 - \bar{x}_2) - 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + 1.96\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right]$$

55. Two samples drawn from two different populations gave the following results:

Table 9.5

	Size	*Mean*	*S.D.*
Sample I	100	582	24
Sample II	100	540	28

Test the hypothesis, at 5% level of significance, that the difference of the means of the populations is 35.

$$\left[\textbf{Hint}: z = \frac{(\bar{x}_1 - \bar{x}_2) - 35}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}\right]$$

56. Two populations have their means equal, but the S.D. of one is twice the other. Show that, in the samples of size 2000 drawn one from each, the difference of the means will in all probability, not exceed 0.15 σ, where σ is the smaller S.D.
57. In a certain random sample of 72 items, the S.D. is found to be 8. Is it reasonable to suppose that it has been drawn from a population with S.D.7?
58. In a random sample of 200 items, drawn from a population with S.D. 0.8, the sample S.D. is 0.7. Can we conclude that the sample S.D. is less than the population S.D. at 1% level of significance?
59. The S.D. of a random sample of 900 members is 4.6 and that of another independent sample of 1600 members is 4.8. Examine if the two samples could have been drawn from a population with S.D. 4.0?
60. Examine whether the two samples for which the data are given in Table 9.6 could have been drawn from populations with the same S.D.:

Table 9.6

	Size	*S.D.*
Sample I	100	5
Sample II	200	7