1. Random variable **[A.U CBT Dec. 2009]**

A real-valued function defined on the outcome of a probability experiment is called a random variable.

2. Probability distribution function of X.

If X is a random variable, then the function $F(x)$, defined by

$F(x) = P\{X \leq x\}$ is called the distribution function of X.

1.1.(a) (i) Discrete random variable

A random variable whose set of possible values is either finite or countably infinite is called discrete random variable.

Example

1. Let X represent the sum of the numbers on the 2 dice, when two dice are thrown. In this case the random variable X takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. So X is a discrete random variable.

2. Number of transmitted bits received in error.

(ii) Probability mass function

If X is a discrete random variable, then the function $P(x) = P[X=x]$ is called the probability mass function of X.

(iii) Probability distribution

The values assumed by the random variable X presented with corresponding probabilities is known as the probability distribution of X.

X	x_1	x_2	x_3	...
$P[X = x]$	p_1	p_2	p_3	...

(iv) Cumulative distribution or Distribution function of X [for discrete R.V]

The cumulative distribution function F(x) of a discrete random variable X with probability distribution P (x) is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} p(t)$$

$$x = -\infty, \ldots -2, -1, 0, 1, 2, \ldots \infty.$$

(v) Properties of distribution function :

1. $F(-\infty) = 0$
2. $F(\infty) = 1$
3. $0 \leq F(x) \leq 1$
4. $F(x_1) \leq F(x_2)$ if $x_1 < x_2$
5. $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$
6. $P(x_1 \leq X \leq x_2) = F(x_2) - F(x_1) + P[X = x_1]$
7. $P(x_1 < X < x_2) = F(x_2) - F(x_1) - P[X = x_2]$
8. $P(x_1 \leq X < x_2) = F(x_2) - F(x_1) - P(X = x_2) + P(X = x_1)$

(vi) Results.

1. $P(X \leq \infty) = 1$
2. $P(X \leq -\infty) = 0$
3. If $x_1 \leq x_2$ then $P(X = x_1) \leq P(X = x_2)$
4. $P(X > x) = 1 - P[X \leq x]$
5. $P(X \leq x) = 1 - P(X > x)$

(vii) Expected value of a Discrete random variable X

Let X be a discrete random variable assuming values $x_1, x_2, \ldots, x_n$ with corresponding probabilities $P_1, P_2, \ldots, P_n$. Then

$$E(X) = \sum_{i} x_i \, p(x_i)$$

is called the Expected value of X.

E(X) is also called commonly the mean or the expectation of X.

A useful identity states that for a function *g*,

$$E[g(x)] = \sum_{x_i} g(x_i) \, p(x_i)$$

(viii) The variance of a random varaible X.

It is defined by $Var(X) = E[X - E(X)]^2$

The variance, which is equal to the expected value of the square of the difference between X and its expected value. It is a measure of the spread of the possible values of X.

A useful identity is that

$$Var(X) = E[X^2] - [E(X)]^2$$

The quantity $\sqrt{Var(X)}$ is called the **standard deviation** of X.

FORMULA

1. $\sum_{i} p(x_i) = 1$

2. $F(x) = P[X \le x]$

   e.g., $P[X \le 4] = F[4]$

   $F[1] = P[0] + P[1]$

   $F[2] = P[0] + P[1] + P[2] = F[1] + P[2]$

   $F[3] = P[0] + P[1] + P[2] + P[3] = F[2] + P[3]$

   ... ...

3. $P[1] = F[1] - F[0]$

   $P[2] = F[2] - F[1]$

   $P[3] = F[3] - F[2]$

4. Mean $= E[X] = \Sigma x_i p(x_i) =$ Expected value

5. $E[X^2] = \Sigma x_i^2 p(x_i)$

6. Variance $= Var[X] = E[X^2] - [E[X]]^2$

7. $E[aX + b] = a\,E[X] + b$

8. $Var[aX \pm b] = a^2\, Var\, X$

9. Probability mass function $p(x) = P[X = x]$

10. Standard deviation $= \sqrt{Var(X)}$

Note :

1. Prove that E(aX + b) = a E(X) + b.

Proof : $E[aX + b] = \sum_{x_i} (ax_i + b)\, p(x_i)$

$$= a \sum_{x_i} x_i\, p(x_i) + b \sum_{x_i} p(x_i)$$

$$= a\, E(X) + b \qquad [\because \sum_{x_i} p(x_i) = 1]$$

2. If X is a random variable then show that $Var(aX + b) = a^2 Var(X)$

Proof : Let $Y = aX + b$

$$E(Y) = E[aX + b]$$

W.K.T $E[aX + b] = a\, E[X] + b$

$$\therefore E(Y) = a\, E(X) + b$$

$$Y - E(Y) = (aX + b) - a\,E(X) - b$$

$$Y - E(Y) = aX - aE(X)$$

$$Y - E(Y) = a\,[X - E(X)]$$

$$[Y - E(Y)]^2 = a^2\,[X - E(X)]^2$$

$$Var(Y) = a^2\, Var(X)$$

i.e., $Var(aX + b) = a^2\, Var(X)$

3. $P[X = x_i] = p(x)$

→ Probability function (or) Probability distribution (or) p.m.f (probability mass function)

4. $f(x)$ → p.d.f (probability density function) (or) density function.

5. $F(x)$ → c.d.f (cumulative distribution function) (or) distribution function.

Problems under the distribution function for discrete random variable

Example 1.1.1 **[Given $P(x_i)$ to find $F(x_i)$ type]**

For the following probability distribution (i) Find the distribution function of X. (ii) What is the smallest value of x for which $P(X \leq x) > 0.5$

$X = x_i$:	0	1	2
$P(X = x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Solution : (i)

x_i	$P(x_i)$	$F(x_i) = P(X \leq x_i)$
0	$P(0) = \frac{1}{4}$	$F(0) = P(0) = \frac{1}{4} = 0.25$
1	$P(1) = \frac{2}{4}$	$F(1) = F(0) + P(1)$ $= \frac{1}{4} + \frac{2}{4} = \frac{3}{4} = 0.75$
2	$P(2) = \frac{1}{4}$	$F(2) = F(1) + P(2)$ $= \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$

(ii) The smallest value of x for which $P(X \leq x_i) > 0.5$ is 1.

Note : The smallest value of x for which $P(X \leq x_i) > 0.1$ is 0 and The smallest value of x for which $P(X \leq x_i) > 0.8$ is 2.

Determine the mean, variance, E[(2X+1)], Var(2X+1) of a discrete random variable X given its distribution as follows :

$X = x_i$	1	2	3	4	5	6
$F(X = x_i)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

Solution :

Mean $= E[X] = \Sigma x_i p(x_i)$; $E[X^2] = \Sigma x_i^2 p(x_i)$

$\text{Var}(X) = E[X^2] - [E(X)]^2$; $E[aX + b] = aE[X] + b$

$\text{Var}[aX \pm b] = a^2 \text{ Var } X$

x_i	$F(x)$	$P(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$
1	$F[1] = \frac{1}{6}$	$P(1) = F(1) = \frac{1}{6}$	$(1)\left(\frac{1}{6}\right) = \frac{1}{6}$	1	$(1)\left(\frac{1}{6}\right) = \frac{1}{6}$
2	$F[2] = \frac{2}{6}$	$P(2) = F(2) - F(1) = \frac{2}{6} - \frac{1}{6} = \frac{1}{6}$	$(2)\left(\frac{1}{6}\right) = \frac{2}{6}$	4	$(4)\left(\frac{1}{6}\right) = \frac{4}{6}$
3	$F[3] = \frac{3}{6}$	$P(3) = F(3) - F(2) = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$	$(3)\left(\frac{1}{6}\right) = \frac{3}{6}$	9	$(9)\left(\frac{1}{6}\right) = \frac{9}{6}$
4	$F[4] = \frac{4}{6}$	$P(4) = F(4) - F(3) = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$	$(4)\left(\frac{1}{6}\right) = \frac{4}{6}$	16	$(16)\left(\frac{1}{6}\right) = \frac{16}{6}$
5	$F[5] = \frac{5}{6}$	$P(5) = F(5) - F(4) = \frac{5}{6} - \frac{4}{6} = \frac{1}{6}$	$(5)\left(\frac{1}{6}\right) = \frac{5}{6}$	25	$(25)\left(\frac{1}{6}\right) = \frac{25}{6}$
6	$F[6] = \frac{6}{6}$	$P(6) = F(6) - F(5) = \frac{6}{6} - \frac{5}{6} = \frac{1}{6}$	$(6)\left(\frac{1}{6}\right) = \frac{6}{6}$	36	$(36)\left(\frac{1}{6}\right) = \frac{36}{6}$
			$E[X] = \frac{21}{6} = \frac{7}{2}$		$E[X^2] = \frac{91}{6}$

$$\text{Var}(X) = E[X^2] - [E(X)]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

$$E[2X+1] = 2E[X] + 1 = 2\left[\frac{7}{2}\right] + 1 = 7 + 1 = 8$$

$$\text{Var}[2X+1] = 2^2 \text{Var}[X] = 4\left(\frac{35}{12}\right) = \frac{35}{3}$$

Example 1.1.5

The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Determine the expected demand for watches. Also compute the variance. **[A.U. N/D 2006]**

Solution : Let X denote the demand

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$	
1	0.08	0.08	1	0.08	$\text{Var}[X]$
2	0.12	0.24	4	0.48	$= E[X^2] - [E(X)]^2$
3	0.19	0.57	9	1.71	$= 19.70 - (4.06)^2$
4	0.24	0.96	16	3.84	$= 19.70 - 16.4836$
5	0.16	0.80	25	4.00	$= 3.2164$
6	0.10	0.60	36	3.60	
7	0.07	0.49	49	3.43	
8	0.04	0.32	64	2.56	
		$E[X] = \Sigma x_i p(x_i) = 4.06$		$E[X^2] = \Sigma x_i^2 p(x_i) = 19.70$	

When a die is thrown, X denotes the number that turns up. Find E (X), E (X^2) and var (X). **[A.U. M/J 2006]**

Solution : X is a discrete random variable taking values, 1, 2, 3, 4, 5, 6 and with probability $\frac{1}{6}$ for each

$E[X] = \Sigma x_i p(x_i)$	$E[X^2] = \Sigma x_i^2 p(x_i)$	$Var(X) = E[X^2] - [E(x)]^2$

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$	Var[X]
1	1/6	1/6	1	1/6	$=E[X^2]-[E(X)]^2$
2	1/6	2/6	4	4/6	$=\frac{91}{6}-\left(\frac{7}{2}\right)^2$
3	1/6	3/6	9	9/6	
4	1/6	4/6	16	16/6	$=\frac{91}{6}-\frac{49}{4}$
5	1/6	5/6	25	25/6	
6	1/6	6/6	36	36/6	$=\frac{35}{12}$
		$E[x]=\frac{21}{6}=\frac{7}{2}$		$E[X^2]=\frac{91}{6}$	

Example 1.1.7

Determine the constant K given the following probability distribution of discrete random variable X. Also find mean and Variance of X.

$X = x_i$	1	2	3	4	5	Total
$P(X = x_i)$	0.1	0.2	K	2K	0.1	1.0

Solution : We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=1}^{5} P(x_i) = 1$

we have, $0.1 + 0.2 + K + 2K + 0.1 = 1$

$$\text{i.e., } 3K = 1 - 0.4 = 0.6$$

$$K = 0.2$$

Mean $= E[X] = \Sigma x_i p(x_i)$	$E[X^2] = \Sigma x_i^2 p(x_i)$

x_i	$p(x_i)$	$x_i\, p(x_i)$	x_i^2	$x_i^2\, p(x_i)$
1	0.1	0.1	1	0.1
2	0.2	0.4	4	0.8
3	0.2	0.6	9	1.8
4	0.4	1.6	16	6.4
5	0.1	0.5	25	2.5
		$E[X] = 3.2$		$E[X^2] = 11.6$

$$\text{Var}[X] = E[X^2] - [E(x)]^2$$

$$= 11.6 - (3.2)^2 = 11.6 - 10.24 = 1.36$$

Example 1.1.8

If X has the distribution function

$$F[x] = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{3} & \text{for } 1 \le x < 4 \\ \frac{1}{2} & \text{for } 4 \le x < 6 \\ \frac{5}{6} & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

Find **(1) The probability distribution of X.**
(2) P (2 < X < 6)
(3) Mean of X
(4) Variance of X. **[A.U. A/M. 2008]**

Solution : (1) The probability distribution of X.

Mean $= E[X] = \Sigma x_i\, p(x_i)$	$E[X^2] = \Sigma x_i^2\, p(x_i)$

$$\boxed{\text{Var}(X) = E[X^2] - [E(x)]^2}$$

$$P[x_1 < X < x_2] = F(x_2) - F(x_1) - P(X = x_2)$$

x_i	$F(x)$	$p(x_i)$	$x_i\,p(x_i)$	x_i^2	$x_i^2\,p(x_i)$
0	$F(0)=0$	$p(0)= F(0)=0$	0	0	0
1	$F(1)=\frac{1}{3}$	$p(1)= F(1)-F(0)$ $=\frac{1}{3}-0=\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{1}{3}$
2	$F(2)=\frac{1}{3}$	$p(2)=F(2)-F(1)$ $=\frac{1}{3}-\frac{1}{3}=0$	0	4	0
3	$F(3)=\frac{1}{3}$	$p(3)=F(3)-F(2)$ $=\frac{1}{3}-\frac{1}{3}=0$	0	9	0
4	$F(4)=\frac{1}{2}$	$p(4)=F(4)-F(3)$ $=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$	$\frac{4}{6}$	16	$\frac{16}{6}$
5	$F(5)=\frac{1}{2}$	$p(5)=F(5)-F(4)$ $=\frac{1}{2}-\frac{1}{2}=0$	0	25	0
6	$F(6)=\frac{5}{6}$	$p(6)=F(6)-F(5)$ $=\frac{5}{6}-\frac{1}{2}=\frac{2}{6}$	$\frac{12}{6}$	36	$\frac{72}{6}$
7	$F(7)=\frac{5}{6}$	$p(7)=F(7)-F(6)$ $=\frac{5}{6}-\frac{5}{6}=0$	0	49	0
8	$F(8)=\frac{5}{6}$	$p(8)=F(8)-F(7)$ $=\frac{5}{6}-\frac{5}{6}=0$	0	64	0
9	$F(9)=\frac{5}{6}$	$p(9)=F(9)-F(8)$ $=\frac{5}{6}-\frac{5}{6}=0$	0	81	0
10	$F(10)=1$	$p(10)=F(10)-F(9)$ $=1-\frac{5}{6}=\frac{1}{6}$	$\frac{10}{6}$	100	$\frac{100}{6}$
			$E(X)=\frac{28}{6}$		$E[X^2]=\frac{190}{6}$

Example 1.1.9

A random variable X has the following probability distribution.

$X = x_i$:	-2	-1	0	1	2	3
$P(X = x_i)$:	0.1	k	0.2	$2k$	0.3	$3k$

Find **[A.U. N/D 2007, M/J 2009, A.U CBT M/J 2010]**

(1) **The value of k,** **[A.U N/D 2011]**

(2) **Evaluate P(X < 2) and P(−2 < X < 2)**

(3) **Find the cumulative distribution of X and**

(4) **Evaluate the mean of X.**

Solution : (1) We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=1}^{3} P(x_i) = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$6k + 0.6 = 1 \Rightarrow 6k = 1 - 0.6 \Rightarrow 6k = 0.4$$

$$\Rightarrow k = \frac{0.4}{6} = \frac{1}{15}$$

(2) (i) $P[X < 2]$

$$= P[X = -2] + P[X = -1] + P[X = 0] + P[X = 1]$$

$$= 0.1 + \frac{1}{15} + 0.2 + \frac{2}{15} = 0.3 + \frac{3}{15} = \frac{1}{2}$$

(ii) P $[-2 < X < 2]$

$$= P[X = -1] + P[X = 0] + P[X = 1]$$

$$= \frac{1}{15} + 0.2 + \frac{2}{15} = 0.2 + \frac{3}{15} = \frac{2}{5}$$

(3) & (4)

x_i	$p(x_i)$	Cumulative distribution $F(x_i)$	E[X] $x_i p(x_i)$
-2	$p(-2) = 0.1$	$F(-2) = p(-2) = 0.1$	-0.2
-1	$p(-1) = \frac{1}{15}$	$F(-1) = F(-2) + P(-1) = 0.1 + \frac{1}{15} = 0.17$	$-\frac{1}{15}$
0	$p(0) = 0.2$	$F(0) = F(-1) + p(0) = 0.17 + 0.2 = 0.37$	0
1	$p(1) = \frac{2}{15}$	$F(1) = F(0) + p(1) = 0.37 + \frac{2}{15} = 0.50$	$\frac{2}{15}$
2	$p(2) = 0.3$	$F(2) = F(1) + p(2) = 0.50 + 0.3 = 0.80$	0.6
3	$p(3) = \frac{3}{15}$	$F(3) = F(2) + p(3) = 0.80 + \frac{3}{15} = 1.00$	$\frac{9}{15}$

(4) Mean $= E[X] = \Sigma x_i p(x_i) = 16/15$

Example 1.1.10

A discrete random variable X has the following probability distribution.

x	0	1	2	3	4	5	6	7	8
P (X = x)	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a. **[A.U CBT N/D 2011, Trichy M/J 2011]**

(ii) Find P(X < 3), P(0 < X < 3), P(X ≥ 3)

(iii) Find the distribution function of X. **[AU Tvli. A/M 2009]**

Solution : (i) We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=1}^{8} P(x_i) = 1$

$$a + 3a + 5a + 7a + 11a + 13a + 15a + 17a = 1$$

(i.e.,) $81a = 1 \therefore \boxed{a = \frac{1}{81}}$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= a + 3a + 5a = 9a = 9 \times \frac{1}{81} = \frac{1}{9} \qquad \text{... (1)}$$

$$P(0<X<3) = P(X = 1) + P(X = 2)$$

$$= 3a + 5a = 8a = 8 \times \frac{1}{81} = \frac{8}{81}$$

$$P[X \geq 3] = 1 - P[X < 3]$$

$$= 1 - \frac{1}{9} = \frac{8}{9} \quad \text{by (1)}$$

Distribution function $F(x)$ of X.

x_i	$p(x_i)$	$F(x_i)$
0	$p(0) = \frac{1}{81}$	$F(0) = p(0) = \frac{1}{81}$
1	$p(1) = \frac{3}{81}$	$F(1) = F(0) + p(1) = \frac{1}{81} + \frac{3}{81} = \frac{4}{81}$
2	$p(2) = \frac{5}{81}$	$F(2) = F(1) + p(2) = \frac{4}{81} + \frac{5}{81} = \frac{9}{81}$
3	$p(3) = \frac{7}{81}$	$F(3) = F(2) + p(3) = \frac{9}{81} + \frac{7}{81} = \frac{16}{81}$
4	$p(4) = \frac{9}{81}$	$F(4) = F(3) + p(4) = \frac{16}{81} + \frac{9}{81} = \frac{25}{81}$
5	$p(5) = \frac{11}{81}$	$F(5) = F(4) + p(5) = \frac{25}{81} + \frac{11}{81} = \frac{36}{81}$
6	$p(6) = \frac{13}{81}$	$F(6) = F(5) + p(6) = \frac{36}{81} + \frac{13}{81} = \frac{49}{81}$
7	$p(7) = \frac{15}{81}$	$F(7) = F(6) + p(7) = \frac{49}{81} + \frac{15}{81} = \frac{64}{81}$
8	$p(8) = \frac{17}{81}$	$F(8) = F(7) + p(8) = \frac{64}{81} + \frac{17}{81} = \frac{81}{81} = 1$

A random variable X has the following probability function :

$X = x_i$:	0	1	2	3	4	5	6	7
$P(X = x_i)$:	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2 + K$

(a) Find K **[A.U N/D 2010, M/J 2012, M//J 2014]**

(b) Evaluate P[X < 6], P[X ≥ 6]

(c) If $P[X \leq C] > \frac{1}{2}$, **then find the minimum value of C.**

(d) Evaluate P [1.5 < X < 4.5/X > 2]

(e) Find P [X < 2], P[X > 3], P[1<X<5]

[A.U M/J 2007, N/D 2008] [A.U. Tvli A/M 2009]

Solution : (a) We know that, $\sum_i P(x_i) = 1$. Here, $\sum_{i=0}^{7} P(x_i) = 1$

i.e., $0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$$\text{i.e.,} \quad 10K^2 + 9K - 1 = 0$$

$$\Rightarrow K = -1 \text{ or } K = 1/10$$

Since, $P(X) \geq 0$ the value $K = -1$ is not permissible

Hence, we have $K = \frac{1}{10}$

∴

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(b) (i) $P[X \geq 6] = P[X = 6] + P[X = 7]$

$$= \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$$

(ii) $P[X < 6] = 1 - P[X \geq 6]$.

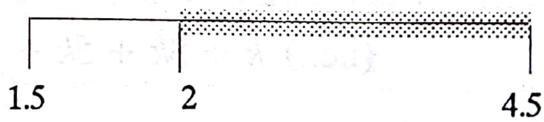
$$= 1 - \frac{19}{100} = \frac{81}{100}$$

(c)

x_i	$p(x_i)$	$F(x_i)$
0	0	$F(0) = p(0) = 0$
1	$\frac{1}{10}$	$F(1) = F(0) + p(1) = 0 + \frac{1}{10} = \frac{1}{10} < \frac{1}{2}$
2	$\frac{2}{10}$	$F(2) = F(1) + p(2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} < \frac{1}{2}$
3	$\frac{2}{10}$	$F(3) = F(2) + p(3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$
4	$\frac{3}{10}$	$F(4) = F(3) + p(4) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} > \frac{1}{2}$
5	$\frac{1}{100}$	$F(5) = F(4) + p(5) = \frac{8}{10} + \frac{1}{100} = \frac{81}{100} > \frac{1}{2}$
6	$\frac{2}{100}$	$F(6) = F(5) + p(6) = \frac{81}{100} + \frac{2}{100} = \frac{83}{100} > \frac{1}{2}$
7	$\frac{7}{100} + \frac{1}{10} = \frac{17}{100}$	$F(7) = F(6) + p(7) = \frac{83}{100} + \frac{17}{100} = \frac{100}{100} = 1 > \frac{1}{2}$

$\therefore$ The minimum value of $c = 4$. [$\because P[X \leq C] > 1/2$]

(d) $P[1.5 < X < 4.5/X > 2]$



$$= \frac{P[(1.5 < X < 4.5) \cap X > 2]}{P(X > 2)}$$

$$P\left(W_1/W_2\right) = \frac{p(w_1 \cap w_2)}{p(w_2)}$$

conditional probability

$$= \frac{P[2 < X < 4.5]}{1 - P[X \leq 2]}$$

$$= \frac{P(3) + P(4)}{1 - [P(0) + P(1) + P(2)]}$$

$$= \frac{\frac{2}{10} + \frac{3}{10}}{1 - \left[0 + \frac{1}{10} + \frac{2}{10}\right]} = \frac{\frac{5}{10}}{1 - \frac{3}{10}} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(e) (i) $P(X < 2) = P[X = 0] + P[X = 1] = 0 + k = k = \frac{1}{10}$

(ii) $P(X > 3) = 1 - P(X \le 3)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$

$= 1 - [0 + k + 2k + 2k]$

$= 1 - 5k = 1 - \frac{5}{10} = 1 - \frac{1}{2} = \frac{1}{2}$

(iii) $P(1 < X < 5) = P(X = 2) + P(X = 3) + P(X = 4)$

$= 2k + 2k + 3k = 7k = \frac{7}{10}$

Example 1.1.12

A random variable 'X' has the following probability function :

$X = x_i$:	0	1	2	3	4
$P(X = x_i)$:	k	$3k$	$5k$	$7k$	$9k$

Find k, P [X ≥ 3] and P (0 < X < 4) **[A.U. Tvli. A/M 2009]**

Solution : (i) We know that, $\sum_i P(x_i) = 1$, Here, $\sum_{i=0}^{4} P(x_i) = 1$

(i.e.,) $k + 3k + 5k + 7k + 9k = 1$

$$25k = 1 \Rightarrow k = \frac{1}{25}$$

(ii) $P(X \ge 3) = P(X = 3) + P(X = 4)$

$= 7k + 9k = 16k = 16\left(\frac{1}{25}\right) = \frac{16}{25}$

(iii) $P(0 < X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 3k + 5k + 7k = 15k$

$= 15\left(\frac{1}{25}\right) = \frac{3}{5}$

Example 1.1.13

Let X be a random variable such that P(X = –2) = P(X = –1) = P(X = 1) = P(X = 2) and P(X < 0) = P(X = 0) = P(X>0). Determine the probability mass function and the distribution of X.

Solution : **[AU N/D 2006]**

Given : $P[X < 0] = P[X = 0] = P[X > 0]$ ... (1)

$P[X = -2] = P[X - 1] = P[X = 1] = P[X = 2]$ ... (2)

Let $P[X = -2] = P[X = -1] = P[X = 1] = P[X = 2] = a$

We know that,

$$P[X < 0] = P[X = -2] + P[X = -1] = a + a = 2a$$

$\therefore$ (1) $\Rightarrow$ $P[X < 0] = P[X = 0] = P[X > 0] = 2a$

$X = x_i$	-2	-1	0	1	2
$P[X = x_i]$	a	a	$2a$	a	a

We know that, $\sum\limits_{i} P(x_i) = 1.$

Here, $\sum\limits_{i=-2}^{2} P(x_i) = 1$

i.e., $a + a + 2a + a + a = 1 \quad \Rightarrow 6a = 1 \quad \Rightarrow a = \frac{1}{6}$

x_i	$p(x_i)$	$F(x_i)$
-2	$p(-2) = \frac{1}{6}$	$F(-2) = p(-2) = \frac{1}{6}$
-1	$p(-1) = \frac{1}{6}$	$F(-1) = F(-2) + p(-1) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$
0	$p(0) = \frac{1}{3} = \frac{2}{6}$	$F(0) = F(-1) + p(0) = \frac{2}{6} + \frac{2}{6} = \frac{4}{6}$
1	$p(1) = \frac{1}{6}$	$F(1) = F(0) + p(1) = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$
2	$p(2) = \frac{1}{6}$	$F(2) = F(1) + p(2) = \frac{5}{6} + \frac{1}{6} = \frac{6}{6} = 1$

Example 1.1.14

If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ find the probability distribution and cumulative distribution function of X.

Solution : **[A.U CBT A/M 2011] [A.U N/D 2012]**

X is a discrete random variable.

Given : $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4]$

Let $2P[X=1] = 3P[X=2] = P[X=3] = 5P[X=4] = k$ ... (1)

(1) ⇒

$X = x_i$	1	2	3	4
$P[X = x_i]$	$\frac{k}{2}$	$\frac{k}{3}$	k	$\frac{k}{5}$

We know that, $\sum_i P[x_i] = 1$, Here, $\sum_{i=1}^{4} P(x_i) = 1$

$$\text{i.e., } \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\frac{61}{30}k = 1 \Rightarrow k = \frac{30}{61}$$

x_i	$p(x_i)$	$F(x_i)$
1	$p(1) = \frac{k}{2} = \frac{15}{61}$	$F(1) = p(1) = \frac{15}{61}$
2	$p(2) = \frac{k}{3} = \frac{10}{61}$	$F(2) = F(1) + p(2) = \frac{15}{61} + \frac{10}{61} = \frac{25}{61}$
3	$p(3) = k = \frac{30}{61}$	$F(3) = F(2) + p(3) = \frac{25}{61} + \frac{30}{61} = \frac{55}{61}$
4	$p(4) = \frac{k}{5} = \frac{6}{61}$	$F(4) = F(3) + p(4) = \frac{55}{61} + \frac{6}{61} = \frac{61}{61} = 1$

Example 1.1.15

The probability function of an infinite discrete distribution is given by $P[X = j] = \frac{1}{2^j}$, $j = 1, 2, \ldots, \infty$. Find the mean and variance of the distribution. Also find P[X is even], $P[X \geq 5]$ and P[X is divisible by 3]. **[A.U. N/D 2006] [A.U N/D 2011]**

Solution : Given : $P[X = j] = \frac{1}{2^j}$

$$\text{Mean} = E[X] = \sum_{j=1}^{\infty} x_j \, p(x_j)$$

$$= (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right)^2 + (3)\left(\frac{1}{2}\right)^3 + \ldots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots\right]$$

$$= \frac{1}{2}[1 - x]^{-2} \qquad \text{Here, } x = \frac{1}{2}$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} = \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = \frac{1}{2}\, 4 = 2$$

$$E[X^2] = \sum_{j=1}^{\infty} x_j^2 \, p(x_j) = \sum_{j=1}^{\infty} (x_j)(x_j + 1)\, p(x_j) - \sum_{j=1}^{\infty} x_j \, p(x_j)$$

$$= \left[(1)(2)\frac{1}{2} + (2)(3)\left(\frac{1}{2}\right)^2 + (3)(4)\left(\frac{1}{2}\right)^3 + \ldots\right] - 2$$

$$= \frac{1}{2}\left[1.2 + 2.3\left(\frac{1}{2}\right) + 3.4\left(\frac{1}{2}\right)^2 + \ldots\right] - 2$$

$$= \frac{1}{2}\left[2[1 - x]^{-3}\right] - 2 \qquad \text{where } x = \frac{1}{2}$$

$$= \frac{1}{2} 2 \left(1 - \frac{1}{2}\right)^{-3} - 2 = \frac{1}{2}\ 2\ \left(\frac{1}{2}\right)^{-3} - 2 = (8) - 2 = 6$$

<u>*FORMULA*</u>	$(1 - x)^{-2} = 1 + 2x + 3x^2 + \dots$
	$(1 - x)^{-3} = \frac{1}{2}[1.2 + 2.3\,x + 3.4\,x^2 + \dots]$

$$\text{Variance of X} = \text{Var(X)} = E[X^2] - \left[E[X]\right]^2$$

$$= 6 - (2)^2 = 6 - 4 = 2$$

(1) P[X is even] $= P[X = 2] + P[X = 4] + \dots$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \dots$$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots$$

$$= (1 - x)^{-1} - 1 \text{ Here, } x = \frac{1}{4} \qquad [\because (1 - x)^{-1} = 1 + x + x^2 + \dots]$$

$$= \left(1 - \frac{1}{4}\right)^{-1} - 1 = \left(\frac{3}{4}\right)^{-1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

(2) $\mathbf{P[X \geq 5] = P[X = 5] + P[X = 6] + \dots}$

$$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \dots$$

$$= \left(\frac{1}{2}\right)^5 \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \dots\right]$$

$$= \left(\frac{1}{2}\right)^5 [1 - x]^{-1} \qquad \text{Here, } x = \frac{1}{2}$$

$$= \left(\frac{1}{2}\right)^5 \left[1 - \frac{1}{2}\right]^{-1} = \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{-1} = \frac{1}{2^4} = \frac{1}{16}$$

(3) P[X is divisible by 3] (or) P[X is multiple of 3]

$$= P[X = 3] + P[X = 6] + \dots$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \dots$$

$$= \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \ldots$$

$$= (1-x)^{-1} - 1. \qquad \text{Here, } x = \frac{1}{8}$$

$$= \left(1 - \frac{1}{8}\right)^{-1} - 1 = \left(\frac{7}{8}\right)^{-1} - 1 = \frac{8}{7} - 1 = \frac{1}{7}$$

Example 1.1.16

Given the probability function of a random variable X as

x :	0	1	2	3	4
P (x) :	0.1	0.2	0.3	0.2	0.2

Find the probability function, means and variance of $Y = 3X + X^2$.

Solution : The random variable $Y = X^2 + 3X$ takes the values 0, 4, 10, 18 and 28 respectively for X = 0, 1, 2, 3 and 4. From the probability values for X, we have $P(Y = 0) = P(X = 0) = 0.1$

$$P(Y = 4) = P(X = 1) = 0.2 ;$$

$$P(Y = 10) = P(X = 2) = 0.3 ;$$

$$P(Y = 18) = P(X = 3) = 0.2 ;$$

$$P(Y = 28) = P(X = 4) = 0.2$$

Hence, the probability function of Y is as follows :

Y	0	4	10	18	28
P(Y)	0.1	0.2	0.3	0.2	0.2

$$E(Y) = \Sigma\,[y_i\, P(Y_i)] = y_1\, P(Y_1) + y_2\, P(Y_2) + \ldots + y_5\, P(y_5)$$

$$= 0 \times 0.1 + 4 \times 0.2 + 10 \times 0.3 + 18 \times 0.2 + 28 \times 0.2 = 13$$

$$E(Y^2) = \Sigma_i\,[y_i^2\, P(Y_i)] = 0 \times 0.1 + 16 \times 0.2 + 100 \times 0.3 + 324 \times 0.2 + 78.4 \times 0.2$$

$$= 254.8$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= 254.8 - 169 = 85.8$$

Example 1.1.17

If Var(X) = 4, then find Var(3X + 8), where X is a random variable.

[A.U, Model] [A.U Tvli M/J 2011]

Solution : We know that, $Var(aX + b) = a^2 \, Var(X)$

$$Var(3X + 8) = 3^2 \, Var(X)$$

$$= 3^2 [4] = 36$$

Example 1.1.18

Let X be a Random Variable with E(X) = 1 and E[X (X – 1)] = 4. Find Var X and Var (2 – 3X), Var $\left[\frac{X}{2}\right]$

[A.U. May, 2000], [A.U. A/M. 2008]

Solution : Given : $E(X) = 1$, $E[X(X-1)] = 4$

$$E[X(X-1)] = E[X^2 - X] = E[X^2] - E[X]$$

$$4 = E(X^2) - 1$$

$$E(X^2) = 4 + 1 = 5$$

$$Var\,X = E(X^2) - [E(X)]^2 = 5 - 1 = 4$$

$$Var(2 - 3X) = (-3)^2 \, Var\,X \qquad [\because Var(aX \pm b) = a^2 \, Var\,X]$$

$$= 9(4) = 36$$

$$Var\left[\frac{X}{2}\right] = \left(\frac{1}{2}\right)^2 Var\,X = \frac{1}{4}[4] = 1$$

Example 1.1.19.

If the probability mass function of a r.v. X is given by $P(X = r) = k r^3$, $r = 1, 2, 3, 4$ find

(i) the value of k, (ii) $P\left(\frac{1}{2} < X < \frac{5}{2} / X > 1\right)$

(iii) the mean and variance of X and

(iv) the distribution function of X

Solution : **[A.U Tvli. M/J 2010]**

We know that, $\sum_i p(x_i) = 1 \Rightarrow 100k = 1 \Rightarrow k = \frac{1}{100}$

Given : $P(X = r) = kr^3$, $r = 1, 2, 3, 4$

x_i	$p(x_i)$	$x_i p(x_i)$	x_i^2	$x_i^2 p(x_i)$	$F[X]$
1	$k = \frac{1}{100}$	$\frac{1}{100}$	1	$\frac{1}{100}$	$F[1] = p(1) = \frac{1}{100}$
2	$8k = \frac{8}{100}$	$\frac{16}{100}$	4	$\frac{32}{100}$	$F[2] = F[1] + p(2) = \frac{1}{100} + \frac{8}{100} = \frac{9}{100}$
3	$27k = \frac{27}{100}$	$\frac{81}{100}$	9	$\frac{243}{100}$	$F[3] = F[2] + p(3) = \frac{9}{100} + \frac{27}{100} = \frac{36}{100}$
4	$64k = \frac{64}{100}$	$\frac{256}{100}$	16	$\frac{1024}{100}$	$F[4] = F[3] + p(4) = \frac{36}{100} + \frac{64}{100} = 1$
	$100k = 1$	$\frac{354}{100}$		$\frac{1300}{100}$	

$$E[X] = \Sigma x_i p(x_i) = \frac{354}{100} = 3.54$$

$$E[X^2] = \Sigma x_i^2 p(x_i) = (1300)\left(\frac{1}{100}\right) = 13$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 13 - \left(\frac{354}{100}\right)^2 = 13 - [3.54]^2 = 13 - 12.5316 = 0.4684$$

$$P\left[\frac{1}{2} < X < \frac{5}{2} / X > 1\right] = \frac{P\left[\left(\frac{1}{2} < X < \frac{5}{2} \cap X > 1\right)\right]}{P[X > 1]}$$

$$= \frac{P[X=2]}{P[X>1]} = \frac{P[X=2]}{P[X=2] + P[X=3] + P[X=4]}$$

$$= \frac{\frac{8}{100}}{\frac{99}{100}} = \frac{8}{99}$$

1.2 Continuous Random Variables

(i) Definition : Continuous Random Variable

A random variable X is said to be continuous if it takes all possible values between certain limits say from real number 'a' to real number 'b'.

Example : The length of time during which a vacuum tube installed in a circuit functions is a continuous random variable.

Note : If X is a continuous random variable for any x_1 and x_2

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2)$$

(ii) Probability density function :

For a continous random variable X, a probability density function is a function such that

(1) $f(x) \ge 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \le X \le b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b

for any a and b.