# BINOMIAL DISTRIBUTION

$$P(X = x) = p(x) = nc_x\, p^x\, q^{n-x}$$

The m.g.f $\quad M_X(t) = E[e^{tx}]$

$$= \sum_{x=0}^{n} e^{tx}\, nc_x\, p^x\, q^{n-x}$$

$$= nc_0 (pe^t)^0 q^n + nc_1 (pe^t) q^{n-1} + nc_2 (pe^t)^2 q^{n-2} + \ldots + nc_n (pe^t)^n q^0$$

$$= q^n + nc_1 (pe^t) q^{n-1} + nc_2 (pe^t)^2 q^{n-2} + \ldots + (pe^t)^n = (pe^t + q)^n$$

**Mean E[X]** $= \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0}$

$$= \left[\frac{d}{dt}(pe^t + q)^n\right]_{t=0} = \left[n(pe^t + q)^{n-1} pe^t\right]_{t=0}$$

$$= np(p+q)^{n-1} = np \quad [\because p + q = 1]$$

$$E[X^2] = \left[\frac{d^2}{dt^2}\left[M_X(t)\right]\right]_{t=0} = \left[\frac{d}{dt}\left[np(pe^t + q)^{n-1} e^t\right]\right]_{t=0}$$

$$= \left[n(n-1)(pe^t + q)^{n-2} p^2 (e^t)^2 + np(pe^t + q)^{n-1} e^t\right]_{t=0}$$

$$= n(n-1)(p+q)^{n-2} p^2 + np(p+q)^{n-1}$$

$$= n(n-1)p^2 + np \qquad [\because p + q = 1]$$

$$= n^2p^2 - np^2 + np$$

**Var [X]** $= E[X^2] - \left[E[X]\right]^2 = n^2p^2 - np^2 + np - (np)^2$

$$= n^2p^2 - np^2 + np - n^2p^2 = np - np^2$$

$$= np(1-p) = npq \qquad [\because p + q = 1]$$

# POISSON DISTRIBUTION

$$P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

**The m.g.f** $\quad M_X(t) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$

$$= \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x} e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \dots\right]$$

$$= e^{-\lambda} e^{\lambda e^{t}} = e^{\lambda[e^{t}-1]}$$

**Mean E[X]** $\quad = \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0}$

$$= \left[\frac{d}{dt}[e^{\lambda(e^{t}-1)}]\right]_{t=0} = \left[\frac{d}{dt}[e^{\lambda e^{t}} e^{-\lambda}]\right]_{t=0}$$

$$= \left[e^{-\lambda} \frac{d}{dt}[e^{\lambda e^{t}}]\right]_{t=0} = \left[e^{-\lambda} e^{\lambda e^{t}} \lambda e^{t}\right]_{t=0}$$

$$= e^{-\lambda} e^{\lambda} \lambda = \lambda$$

$E[X^2] \quad = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}[\lambda e^{-\lambda} e^{\lambda e^{t}} e^{t}]\right]_{t=0}$

$$= \left[\lambda e^{-\lambda} \frac{d}{dt}[e^{\lambda e^{t}} e^{t}]\right]_{t=0}$$

$$= \left[\lambda e^{-\lambda}[e^{\lambda e^{t}} e^{t} + e^{t}\lambda e^{t} e^{\lambda e^{t}}]\right]_{t=0} = \lambda e^{-\lambda}[e^{\lambda} + \lambda e^{\lambda}]$$

$$= \lambda + \lambda^2$$

**Var[X]** $\quad = E[X^2] - \left[E[X]\right]^2 = \lambda + \lambda^2 - [\lambda]^2$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

# GEOMETRIC DISTRIBUTION

$$P(X = x) = q^{x-1} p$$

**The m.g.f**

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \sum_{x=1}^{\infty} p e^t (qe^t)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} = pe^t [1 + qe^t + (qe^t)^2 + \ldots]$$

$$= pe^t [1 - qe^t]^{-1} = \frac{pe^t}{1 - qe^t} = \frac{p}{e^{-t} - q}$$

**Mean E[X]** $= \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{p}{e^{-t} - q}\right)\right]_{t=0}$

$$= \left[\frac{0 - p(-1)e^{-t}}{(e^{-t} - q)^2}\right]_{t=0} = \left[\frac{pe^{-t}}{(e^{-t} - q)^2}\right]_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E[X^2] = \left[\frac{d^2}{dt^2}\left(\frac{p}{e^{-t} - q}\right)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{pe^{-t}}{(e^{-t} - q)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(e^{-t} - q)^2 (-p) e^{-t} - pe^{-t} 2 (e^{-t} - q)(-e^{-t})}{(e^{-t} - q)^4}\right]_{t=0}$$

$$= \frac{(1-q)^2 (-p) - p\, 2 (1-q)(-1)}{(1-q)^4} = \frac{p^2(-p) + 2p(p)}{p^4}$$

$$= \frac{-p^3 + 2p^2}{p^4} = \frac{-p + 2}{p^2} = \frac{-1}{p} + \frac{2}{p^2}$$

$$\text{Var }[X] = E[X^2] - \left[E[X]\right]^2 = \frac{2}{p^2} - \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

# UNIFORM DISTRIBUTION

$$f(x) = \frac{1}{b-a}, \; a < x < b$$

**The m.g.f** $M_X(t) = \int_a^b e^{tx} f(x)\,dx = \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$

$$= \frac{1}{b-a}\left[\frac{e^{bt}-e^{at}}{t}\right] = \frac{e^{bt}-e^{at}}{(b-a)\,t}$$

$$= \frac{\left[1+\frac{bt}{1!}+\frac{(bt)^2}{2!}+\ldots\right] - \left[1+\frac{at}{1!}+\frac{(at)^2}{2!}+\ldots\right]}{(b-a)\,t}$$

$$= \frac{\frac{(b-a)\,t}{1!}+\frac{(b^2-a^2)\,t^2}{2!}+\frac{(b^3-a^3)\,t^3}{3!}+\ldots}{(b-a)\,t}$$

$$= 1+\frac{(b+a)\,t}{2!}+\frac{(b^2+ba+a^2)\,t^2}{3!}+\ldots.$$

**Mean E[X]** $= \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0}$

$$= \left[\frac{d}{dt}\left(1+\frac{(b+a)\,t}{2!}+\frac{(b^2+ba+a^2)\,t^2}{3!}+\ldots\right)\right]_{t=0}$$

$$= \left[0+\frac{b+a}{2}+\frac{b^2+ab+a^2}{6}2t+\ldots\right]_{t=0} = \frac{b+a}{2}$$

$$E[X^2] = \left[\frac{d^2}{dt^2}\left(1+\frac{(b+a)\,t}{2!}+\frac{(b^2+ba+a^2)\,t^2}{3!}+\ldots\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{b+a}{2!}+\frac{b^2+ba+a^2}{3!}2t+\ldots\right)\right]_{t=0}$$

$$= \left[\frac{b^2+ba+a^2}{6}2+\ldots\right]_{t=0} = \frac{1}{3}(b^2+ba+a^2)$$

**Var[X]** $= E[X^2] - \left[E[X]\right]^2 = \frac{1}{3}(b^2+ba+a^2) - \left(\frac{a+b}{2}\right)^2 = \frac{1}{12}[b-a]^2$

# EXPONENTIAL DISTRIBUTION

$$f(x) = \lambda e^{-\lambda x}, \; x \geq 0, \; \lambda > 0$$

The m.g.f $\quad M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\, dx = \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda - t)}\right]_0^{\infty}$$

$$= \frac{-\lambda}{\lambda - t}\left[e^{-(\lambda-t)x}\right]_0^{\infty}$$

$$= \frac{-\lambda}{\lambda - t}[0 - 1] = \frac{\lambda}{\lambda - t}$$

**Mean E[X]** $= \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0}$

$$E[X] = \frac{d}{dt}\left[\frac{\lambda}{\lambda - t}\right]_{t=0} = \left[\frac{\lambda}{(\lambda - t)^2}\right]_{t=0}$$

$$= \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$E[X^2] = \frac{d^2}{dt^2}\left[\frac{\lambda}{(\lambda - t)}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{\lambda}{(\lambda - t)^2}\right)\right]_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

**Var[X]** $= E[X^2] - \left[E[X]\right]^2 = \frac{2}{\lambda^2} - \left[\frac{1}{\lambda}\right]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

# GAMMA DISTRIBUTION

$$f(x) = \frac{e^{-x} x^{\lambda - 1}}{\Gamma_\lambda}, \qquad \lambda > 0,\ 0 < x < \infty$$

**The m.g.f**

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma_\lambda}\, dx$$

$$= \frac{1}{\Gamma_\lambda} \int_0^\infty e^{-(1-t)x} x^{\lambda-1}\, dx$$

$$= \frac{1}{\Gamma_\lambda} \frac{\Gamma_\lambda}{(1-t)^\lambda} = \frac{1}{(1-t)^\lambda}$$

**Mean E[X]** $= \left[\frac{d}{dt}[M_x(t)]\right]_{t=0}$

$$= \left[\frac{d}{dt}\left[\frac{1}{(1-t)^\lambda}\right]\right]_{t=0} = \left[\frac{-\lambda}{(1-t)^{\lambda+1}}(-1)\right]_{t=0}$$

$$= \left[\frac{\lambda}{(1-t)^{\lambda+1}}\right]_{t=0} = \lambda$$

$$E[X^2] = \left[\frac{d^2}{dt^2}[M_x(t)]\right]_{t=0} = \left[\frac{d}{dt}\left[\frac{\lambda}{(1-t)^{\lambda+1}}\right]\right]_{t=0}$$

$$= \left[\frac{-\lambda(\lambda+1)}{(1-t)^{\lambda+2}}(-1)\right]_{t=0} = \left[\frac{\lambda(\lambda+1)}{(1-t)^{\lambda+2}}\right]_{t=0} = \lambda(\lambda+1$$

# NORMAL DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

**The m.g.f** $\quad M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx,$$

Put $z = \frac{x-\mu}{\sigma}$, $\sigma\,dz = dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)}\, e^{\frac{-z^2}{2}}\sigma\,dz = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z^2-2t\sigma z)}{2}}\,dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(z-\sigma t)^2+\frac{\sigma^2 t^2}{2}}\,dz = e^{\mu t+\frac{t^2\sigma^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(z-\sigma t)^2}\,dz$$

Put $u = z - \sigma t$, $du = dz$

$$= e^{\mu t+\frac{t^2\sigma^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}}\,du = e^{\mu t+\frac{t^2\sigma^2}{2}} \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}$$

$$= e^{\mu t+\frac{t^2\sigma^2}{2}} \quad \text{since} \int_{-\infty}^{\infty} e^{\frac{-u^2}{2}}\,du = \sqrt{2\pi}$$

**Mean E[X]** $\quad = \left[\frac{d}{dt}\left[M_X(t)\right]\right]_{t=0} = \left[\frac{d}{dt}\left[e^{\mu t+\frac{t^2\sigma^2}{2}}\right]\right]_{t=0}$

$$= \left[e^{\mu t+\frac{t^2\sigma^2}{2}}\left[\mu + t\sigma^2\right]\right]_{t=0} = \mu$$

$E[X^2] \quad = \left[\frac{d^2}{dt^2}\left[M_X(t)\right]\right]_{t=0} = \left[\frac{d}{dt}\left[e^{\mu t+\frac{t^2\sigma^2}{2}}\left[\mu + t\sigma^2\right]\right]\right]_{t=0}$

$$= \left[(\mu + t\sigma^2)^2\left(e^{\mu t+\frac{t^2\sigma^2}{2}}\right) + \left(e^{\mu t+\frac{t^2\sigma^2}{2}}\right)\sigma^2\right]_{t=0} = \mu^2+\sigma^2$$

**Var[X]** $\quad = E[X^2] - \left[E[X]\right]^2 = \mu^2+\sigma^2-\mu^2 = \sigma^2$