MAT2001
Statistics for Engineers

Module 6
Hypothesis Testing II

Syllabus

Hypothesis Testing II:
Small sample tests- Student's t-test, F-test- chi-square test- goodness of fit - independence of attributes- Design of Experiments - Analysis of variance – one and two way classifications - CRD- RBD- LSD.

Types of Sampling Theory

1. Large Sample:

Size of the Sample (n) is greater than or equal to 30.

2. Small Sample:

Size of the Sample (n) is less than 30.

Small Sample Tests

TESTS OF SIGNIFICANCE FOR SMALL SAMPLES

The tests of significance discussed in the previous section hold good only for large samples, i.e. only when the size of the sample $n \geq 30$. When the sample is small, i.e. $n < 30$, the sampling distributions of many statistics are not normal, even though the parent populations may be normal. Moreover the assumption of near equality of population parameters and the corresponding sample statistics will not be justified for small samples. Consequently we have to develop entirely different tests of significance that are applicable to small samples.

Student's *t*-Distribution $t(\nu)$

A random variable T is said to follow student's t-distribution or simply t-distribution, if its probability density function is given by

$$f(t) = \frac{1}{\sqrt{\nu}\,\beta\left(\frac{\nu}{2}, \frac{1}{2}\right)} \cdot \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \quad -\infty < t < \infty.$$

ν is called the number of degrees of freedom of the t-distribution.

(*Note*: t-distribution was defined by the mathematician *W.S.D. Gosset* whose pen name is Student.)

Properties of *t*-Distribution

1. The probability curve of the t-distribution is similar to the standard normal curve and is symmetric about $t = 0$, bell-shaped and asymptotic to the t-axis
2. For sufficiently large value of n, the t-distribution tends to the standard normal distribution.
3. The mean of the t-distribution is zero.
4. The variance of the t-distribution is $\dfrac{\nu}{\nu - 2}$, if $n > 2$ and is greater than 1, but it tends to 1 as $\nu \to \infty$.

Uses of *t*-Distribution

The *t*-distribution is used to test the significance of the difference between

1. The mean of a small sample and the mean of the population.
2. The means of two small samples

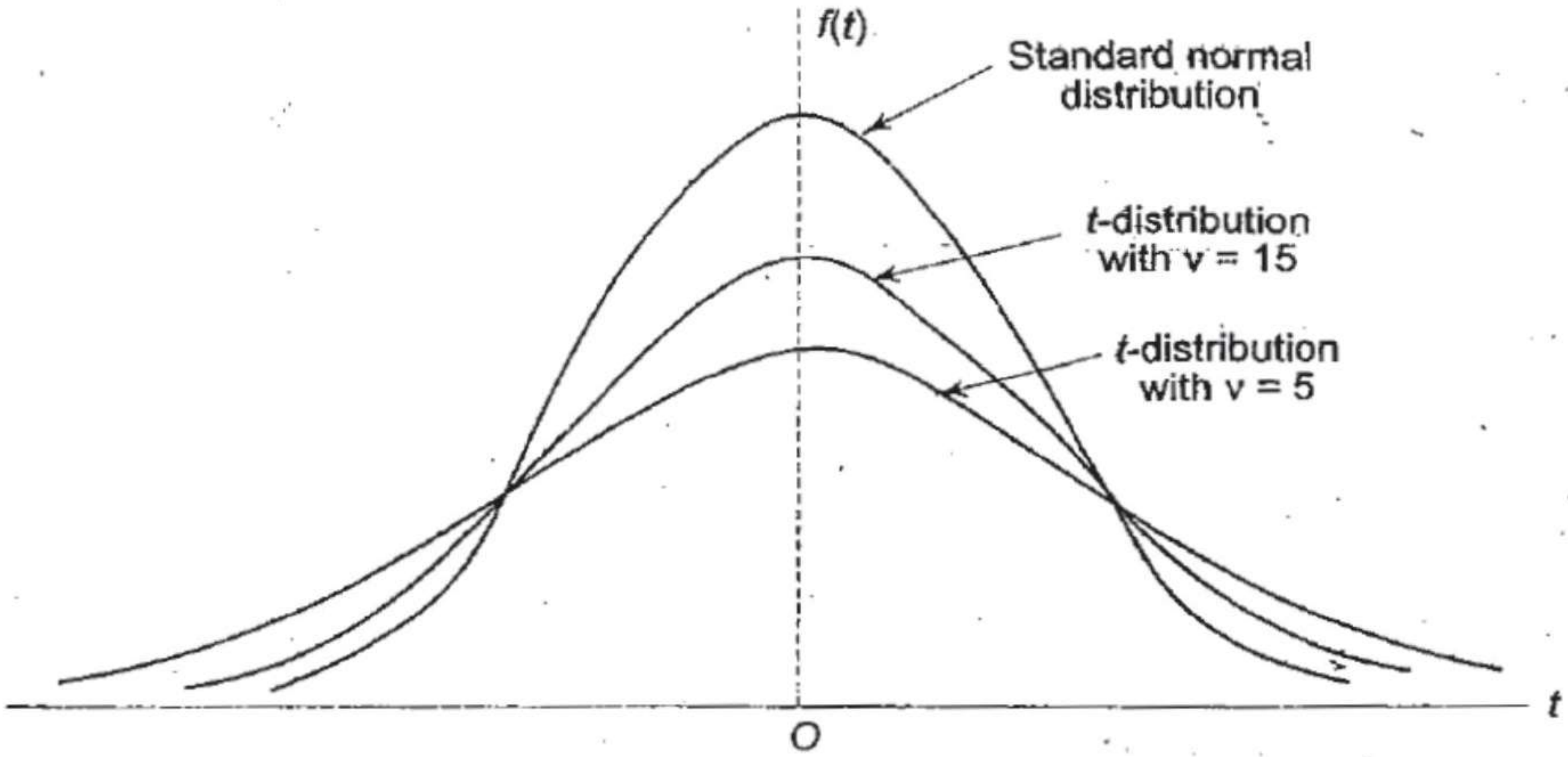

Table for *t*-Test

t* Distribution: Critical Values of *t

		Significance level					
Degrees of freedom	*Two-tailed test:*	10%	5%	2%	1%	0.2%	0.1%
	One-tailed test:	5%	2.5%	1%	0.5%	0.1%	0.05%
1		6.314	12.706	31.821	63.657	318.309	636.619
2		2.920	4.303	6.965	9.925	22.327	31.599
3		2.353	3.182	4.541	5.841	10.215	12.924
4		2.132	2.776	3.747	4.604	7.173	8.610
5		2.015	2.571	3.365	4.032	5.893	6.869
6		1.943	2.447	3.143	3.707	5.208	5.959
7		1.894	2.365	2.998	3.499	4.785	5.408
8		1.860	2.306	2.896	3.355	4.501	5.041
9		1.833	2.262	2.821	3.250	4.297	4.781
10		1.812	2.228	2.764	3.169	4.144	4.587
11		1.796	2.201	2.718	3.106	4.025	4.437
12		1.782	2.179	2.681	3.055	3.930	4.318
13		1.771	2.160	2.650	3.012	3.852	4.221
14		1.761	2.145	2.624	2.977	3.787	4.140
15		1.753	2.131	2.602	2.947	3.733	4.073
16		1.746	2.120	2.583	2.921	3.686	4.015
17		1.740	2.110	2.567	2.898	3.646	3.965
18		1.734	2.101	2.552	2.878	3.610	3.922
19		1.729	2.093	2.539	2.861	3.579	3.883
20		1.725	2.086	2.528	2.845	3.552	3.850
21		1.721	2.080	2.518	2.831	3.527	3.819
22		1.717	2.074	2.508	2.819	3.505	3.792
23		1.714	2.069	2.500	2.807	3.485	3.768
24		1.711	2.064	2.492	2.797	3.467	3.745
25		1.708	2.060	2.485	2.787	3.450	3.725
26		1.706	2.056	2.479	2.779	3.435	3.707
27		1.703	2.052	2.473	2.771	3.421	3.690
28		1.701	2.048	2.467	2.763	3.408	3.674
29		1.699	2.045	2.462	2.756	3.396	3.659
30		1.697	2.042	2.457	2.750	3.385	3.646
32		1.694	2.037	2.449	2.738	3.365	3.622
34		1.691	2.032	2.441	2.728	3.348	3.601
36		1.688	2.028	2.434	2.719	3.333	3.582
38		1.686	2.024	2.429	2.712	3.319	3.566
40		1.684	2.021	2.423	2.704	3.307	3.551
42		1.682	2.018	2.418	2.698	3.296	3.538
44		1.680	2.015	2.414	2.692	3.286	3.526
46		1.679	2.013	2.410	2.687	3.277	3.515
48		1.677	2.011	2.407	2.682	3.269	3.505
50		1.676	2.009	2.403	2.678	3.261	3.496
60		1.671	2.000	2.390	2.660	3.232	3.460
70		1.667	1.994	2.381	2.648	3.211	3.435
80		1.664	1.990	2.374	2.639	3.195	3.416
90		1.662	1.987	2.368	2.632	3.183	3.402
100		1.660	1.984	2.364	2.626	3.174	3.390
120		1.658	1.980	2.358	2.617	3.160	3.373
150		1.655	1.976	2.351	2.609	3.145	3.357
200		1.653	1.972	2.345	2.601	3.131	3.340
300		1.650	1.968	2.339	2.592	3.118	3.323
400		1.649	1.966	2.336	2.588	3.111	3.315
500		1.648	1.965	2.334	2.586	3.107	3.310
600		1.647	1.964	2.333	2.584	3.104	3.307
∞		1.645	1.960	2.326	2.576	3.090	3.291

t-Test for Single Mean

Test I

Test of significance of the difference between sample mean and population mean.

the test-statistic $\boxed{t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}}}$

degrees of freedom $\nu = n - 1$

$\bar{x}$ - Mean (S)

μ - Mean (P)

s - S.D (S)

n - size (S)

t-Test for Difference of Means

Test 2

Test of significance of the difference between means of two small samples drawn from the same normal population.

the test-statistic $$\boxed{t = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}}$$

degrees of freedom $\underline{\nu = (n_1 + n_2 - 2)}$

Note:

If σ is not known, we may assume that $\sigma \simeq \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}}$

with $(n_1 + n_2 - 2)$ degrees of freedom, the test statistic becomes

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

t-Test for Difference of Means

Note 1. If $n_1 = n_2 = n$ and if the samples are independent i.e., the observations in the two samples are not at all related, then the test statistic is given by

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}} \quad \text{with} \quad \nu = 2n - 2 \tag{2}$$

Paired t-Test

2. If $n_1 = n_2 = n$ and if the pairs of values of x_1 and x_2 are associated in some way (or correlated), the formula (2) for t in Note (1) should not be used. In this case, we shall assume that $H_0 : \bar{d}\ (= \bar{x} - \bar{y}) = 0$ and test the significance of the difference between $\bar{d}$ and 0, using the test statistic $\boxed{t = \dfrac{\bar{d}}{s/\sqrt{n-1}}}$ with $\nu = n - 1$, where $d_i = x_i - y_i$ $(i = 1, 2, \ldots\ldots, n)$, $\bar{d} = \bar{x} - \bar{y}$; and $s = \text{S.D. of } d\text{'s} = \frac{1}{n} \sum_{i=1}^{n} \left(d_i - \bar{d}\right)^2$.

Example:

Tests made on the breaking strength of 10 pieces of a metal wire gave the results: 578, 572, 570, 568, 572, 570, 570, 572, 596 and 584 kg. Test if the mean breaking strength of the wire can be assumed as 577 kg.

Solution:

Let us first compute sample mean $\bar{x}$ and sample S.D.'s' and then test if $\bar{x}$ differs significantly from the population mean $\mu = 577$.

We take the assumed mean $A = \frac{568 + 596}{2} = 582$

$$di = x_i - A$$

$$\therefore \quad x_i = d_i + A$$

$$\therefore \quad \bar{x} = \frac{1}{n}\sum x_i = \frac{1}{n}\sum d_i + A$$

$$= \frac{1}{10} \times (-68) + 582 = 575.2 \text{ (see Table 9.7 given below)}$$

Table 9.7

x_i	$d_i = x_i - A$	d_i^2
578	− 4	16
572	− 10	100
570	− 12	144
568	− 14	196
572	− 10	100
570	− 12	144
570	− 12	144
572	− 10	100
596	14	196
584	2	4
Total	− 68	1144

$$s^2 = \frac{1}{n}\sum d_i^2 - \left(\frac{1}{n}\sum d_i\right)^2$$

$$= \frac{1}{10} \times 1144 - \left(\frac{1}{10} \times -68\right)^2 = 68.16$$

$$\therefore \quad s = 8.26$$

Solution (Continued):

Now $$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{575{\cdot}2 - 577}{8{\cdot}26/\sqrt{9}}$$

$$= -0.65$$

and $$\nu = n - 1 = 9.$$

$$H_0 : \bar{x} = \mu \quad \text{and} \quad H_1 : \bar{x} \neq \mu.$$

Let LOS be 5%. Two tailed test is to be used.

From the t-table, for $\nu = 9, t_{5\%} = 2.26$. Since $|t| < t_{5\%}$, the difference between $\bar{x}$ and μ is not significant or H_0 is accepted. $\therefore$ The mean breaking strength of the wire can be assumed as 577 kg at 5% LOS

Exercise:

The mean lifetime of a sample of 25 bulbs is found as 1550 hours with a S.D. of 120 hours. The company manufacturing the bulbs claims that the average life of their bulbs is 1600 hours. Is the claim acceptable at 5% level of significance?

Example:

Two independent samples of sizes 8 and 7 contained the following values:

Sample I :	19,	17,	15,	21,	16,	18,	16,	14
Sample II :	15,	14,	15,	19,	15,	18,	16	

Is the difference between the sample means significant?

Solution:

Sample I			Sample II		
x_1	$d_1 = x_1 - 18$	d_1^2	x_2	$d_2 = x_2 - 16$	d_2^2
19	1	1	15	−1	1
17	−1	1	14	−2	4
15	−3	9	15	−1	1
21	3	9	19	3	9
16	−2	4	15	−1	1
18	0	0	18	2	4
16	−2	4	16	0	0
14	−4	16			
Total	−8	44	Total	0	20

For sample I, $\bar{x}_1 = 18 + \bar{d}_1 = 18 + \frac{1}{8}\sum d_1$

$$= 18 + \frac{1}{8} \times (-8) = 17.$$

$$s_1^2 = \frac{1}{n_1}\sum d_1^2 - \left(\frac{1}{n_1}\sum d_1\right)^2$$

$$= \frac{1}{8} \times 44 - \left(\frac{1}{8} \times -8\right)^2 = 4.5$$

$\therefore$ $s_1 = 2.12.$

For sample II, $\bar{x}_2 = 16 + \bar{d}_2 = 16 + \frac{1}{7}\sum d_2 = 16$

$$s_2^2 = \frac{1}{n_2}\sum d_2^2 - \left(\frac{1}{n_2}\sum d_2\right)^2$$

$$= \frac{1}{7} \times 20 - \left(\frac{1}{7} \times 0\right)^2 = 2.857$$

$\therefore$ $s_2 = 1.69$

Solution (Continued):

Two-tailed test is to be used. Let LOS be 5 %

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{17 - 16}{\sqrt{\left(\frac{8 \times 4\cdot 5 + 7 \times 2\cdot 857}{13}\right)\left(\frac{1}{8} + \frac{1}{7}\right)}}$$

$$= 0.93$$

Also $\nu = n_1 + n_2 - 2 = 13.$

From the t-table, $t_{5\%}$ $(\nu = 13) = 2.16$

Since $|t| < t_{5\%}$, H_0 is accepted and H_1 is rejected.

i.e. the two sample means do not differ significantly at 5% LOS

Exercise:

The mean height and the S.D. height of eight randomly chosen soliders are 166.9 cm. and 8.29 cm. respectively. The corresponding values of six randomly chosen sailors are 170.3 cm and 8.50 cm. respectively. Based on this data, can we conclude that soldiers are, in general, shorter than sailors?

Example:

The following data relate to the marks obtained by 11 students in two tests, one held at the beginning of a year and the other at the end of the year after intensive coaching. Do the data indicate that the students have benefited by coaching?

Test 1:	19,	23,	16,	24,	17,	18,	20,	18,	21,	19,	20
Test 2:	17,	24,	20,	24,	20,	22,	20,	20,	18,	22,	19

Note:

The given data relate to the marks obtained in two tests by the same set of students. Hence the marks in the two tests can be regarded as correlated and so the *t*-test for paired values should be used.

Solution:

Let $\qquad d = x_1 - x_2,$

where x_1, x_2 denote the marks in the two tests.

Thus the values of d are 2, – 1 – 4, 0, – 3, – 4, 0, – 2, 3, – 3, 1.

$$\Sigma d = -11 \quad \text{and} \quad \Sigma d^2 = 69$$

$$\therefore \qquad \bar{d} = \frac{1}{n}\Sigma d = \frac{1}{11} \times -11 = -1$$

$$s^2 = s_d^2 = \frac{1}{n}\Sigma d^2 - (\bar{d})^2 = \frac{1}{11} \times 69 - (-1)^2 = 5.27$$

$$\therefore \qquad s = 2.296$$

$H_0 : \bar{d} = 0$, i.e. the students have not benefited by coaching; $H_1 : \bar{d} < 0$ (i.e. $\bar{x}_1 < \bar{x}_2$).

One-tailed test is to be used. Let LOS be 5%

$$t = \frac{\bar{d}}{s/\sqrt{n-1}} = \frac{-1}{2\cdot 296/\sqrt{10}} = -1.38 \quad \text{and} \quad \nu = 10$$

$t_{5\%}$ ($\nu = 10$) for one-tailed test = $t_{10\%}$ ($\nu = 10$) for two-tailed test = 1.81 (from t-table).

Now $\quad |t| < t_{10\%}\ (\nu = 10)$

$\therefore$ H_0 is accepted and H_1 is rejected.

i.e. there is no significant difference between the two sets of marks.

i.e. the students have not benefitted by coaching.

Snedecor's *F*-Distribution $F(\nu_1, \nu_2)$

A random variable F is said to follow snedecor's F-distribution or simply F-distribution, if its probability density function is given by

$$\boxed{f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \cdot \frac{F^{\frac{\nu_1}{2}-1}}{\left(1+\frac{\nu_1 F}{\nu_2}\right)^{(\nu_1+\nu_2)/2}}, \quad F > 0.}$$

Note *(The mathematical variable corresponding to the random variable F is also taken as F.)*
ν_1 and ν_2 used in $f(F)$ are the degrees of freedom associated with the F-distribution.

$\beta(m,n) \rightarrow$ Beta Function

Properties of *F*-Distribution

1. The probability curve of the F-distribution is roughly sketched in Fig.

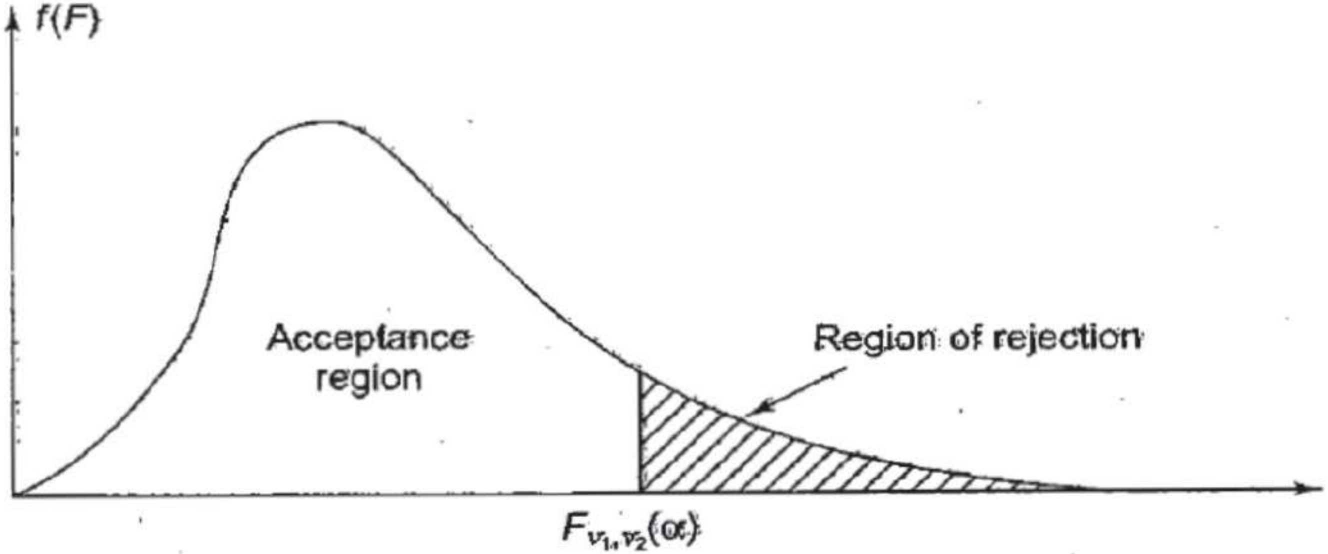

Fig.

2. The square of the t-variate with n degrees of freedom follows a F-distribution with 1 and n degrees of freedom.

3. The mean of the F-distribution is $\dfrac{\nu_2}{\nu_2 - 2}$ $(\nu_2 > 2)$.

4. The variance of the F-distribution is

$$\frac{2\nu_2^2\left(\nu_1+\nu_2-2\right)}{\nu_1\left(\nu_2-2\right)^2\left(\nu_2-4\right)}\ (\nu_2>4).$$

Uses of *F*-Distribution

F-distribution is used to test the equality of the variance of the populations from which two small samples have been drawn.

Table for *F*-Test

***F* Distribution: Critical Values of *F* (5% significance level)**

ν_1 / ν_2	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.36	246.46	247.32	248.01
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.42	19.43	19.44	19.45
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.71	8.69	8.67	8.66
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.87	5.84	5.82	5.80
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.64	4.60	4.58	4.56
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.96	3.92	3.90	3.87
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.53	3.49	3.47	3.44
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.24	3.20	3.17	3.15
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.03	2.99	2.96	2.94
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.86	2.83	2.80	2.77
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.74	2.70	2.67	2.65
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.64	2.60	2.57	2.54
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.55	2.51	2.48	2.46
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.48	2.44	2.41	2.39
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.42	2.38	2.35	2.33
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.37	2.33	2.30	2.28
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.33	2.29	2.26	2.23
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.29	2.25	2.22	2.19
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.26	2.21	2.18	2.16
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.22	2.18	2.15	2.12
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.20	2.16	2.12	2.10
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.17	2.13	2.10	2.07
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.15	2.11	2.08	2.05
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.13	2.09	2.05	2.03
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.11	2.07	2.04	2.01
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.09	2.05	2.02	1.99
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.08	2.04	2.00	1.97
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.06	2.02	1.99	1.96
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.05	2.01	1.97	1.94
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.04	1.99	1.96	1.93
35	4.12	3.27	2.87	2.64	2.49	2.37	2.29	2.22	2.16	2.11	2.04	1.99	1.94	1.91	1.88
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.95	1.90	1.87	1.84
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.89	1.85	1.81	1.78
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.86	1.82	1.78	1.75
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.89	1.84	1.79	1.75	1.72
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.82	1.77	1.73	1.70
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94	1.86	1.80	1.76	1.72	1.69
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.79	1.75	1.71	1.68
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.78	1.73	1.69	1.66
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.82	1.76	1.71	1.67	1.64
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.80	1.74	1.69	1.66	1.62
250	3.88	3.03	2.64	2.41	2.25	2.13	2.05	1.98	1.92	1.87	1.79	1.73	1.68	1.65	1.61
300	3.87	3.03	2.63	2.40	2.24	2.13	2.04	1.97	1.91	1.86	1.78	1.72	1.68	1.64	1.61
400	3.86	3.02	2.63	2.39	2.24	2.12	2.03	1.96	1.90	1.85	1.78	1.72	1.67	1.63	1.60
500	3.86	3.01	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.77	1.71	1.66	1.62	1.59
600	3.86	3.01	2.62	2.39	2.23	2.11	2.02	1.95	1.90	1.85	1.77	1.71	1.66	1.62	1.59
750	3.85	3.01	2.62	2.38	2.23	2.11	2.02	1.95	1.89	1.84	1.77	1.70	1.66	1.62	1.58
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.76	1.70	1.65	1.61	1.58

Table for *F*-Test (Continued)

F Distribution: Critical Values of *F* (5% significance level)

ν_2 \ ν_1	25	30	35	40	50	60	75	100	150	200
1	249.26	250.10	250.69	251.14	251.77	252.20	252.62	253.04	253.46	253.68
2	19.46	19.46	19.47	19.47	19.48	19.48	19.48	19.49	19.49	19.49
3	8.63	8.62	8.60	8.59	8.58	8.57	8.56	8.55	8.54	8.54
4	5.77	5.75	5.73	5.72	5.70	5.69	5.68	5.66	5.65	5.65
5	4.52	4.50	4.48	4.46	4.44	4.43	4.42	4.41	4.39	4.39
6	3.83	3.81	3.79	3.77	3.75	3.74	3.73	3.71	3.70	3.69
7	3.40	3.38	3.36	3.34	3.32	3.30	3.29	3.27	3.26	3.25
8	3.11	3.08	3.06	3.04	3.02	3.01	2.99	2.97	2.96	2.95
9	2.89	2.86	2.84	2.83	2.80	2.79	2.77	2.76	2.74	2.73
10	2.73	2.70	2.68	2.66	2.64	2.62	2.60	2.59	2.57	2.56
11	2.60	2.57	2.55	2.53	2.51	2.49	2.47	2.46	2.44	2.43
12	2.50	2.47	2.44	2.43	2.40	2.38	2.37	2.35	2.33	2.32
13	2.41	2.38	2.36	2.34	2.31	2.30	2.28	2.26	2.24	2.23
14	2.34	2.31	2.28	2.27	2.24	2.22	2.21	2.19	2.17	2.16
15	2.28	2.25	2.22	2.20	2.18	2.16	2.14	2.12	2.10	2.10
16	2.23	2.19	2.17	2.15	2.12	2.11	2.09	2.07	2.05	2.04
17	2.18	2.15	2.12	2.10	2.08	2.06	2.04	2.02	2.00	1.99
18	2.14	2.11	2.08	2.06	2.04	2.02	2.00	1.98	1.96	1.95
19	2.11	2.07	2.05	2.03	2.00	1.98	1.96	1.94	1.92	1.91
20	2.07	2.04	2.01	1.99	1.97	1.95	1.93	1.91	1.89	1.88
21	2.05	2.01	1.98	1.96	1.94	1.92	1.90	1.88	1.86	1.84
22	2.02	1.98	1.96	1.94	1.91	1.89	1.87	1.85	1.83	1.82
23	2.00	1.96	1.93	1.91	1.88	1.86	1.84	1.82	1.80	1.79
24	1.97	1.94	1.91	1.89	1.86	1.84	1.82	1.80	1.78	1.77
25	1.96	1.92	1.89	1.87	1.84	1.82	1.80	1.78	1.76	1.75
26	1.94	1.90	1.87	1.85	1.82	1.80	1.78	1.76	1.74	1.73
27	1.92	1.88	1.86	1.84	1.81	1.79	1.76	1.74	1.72	1.71
28	1.91	1.87	1.84	1.82	1.79	1.77	1.75	1.73	1.70	1.69
29	1.89	1.85	1.83	1.81	1.77	1.75	1.73	1.71	1.69	1.67
30	1.88	1.84	1.81	1.79	1.76	1.74	1.72	1.70	1.67	1.66
35	1.82	1.79	1.76	1.74	1.70	1.68	1.66	1.63	1.61	1.60
40	1.78	1.74	1.72	1.69	1.66	1.64	1.61	1.59	1.56	1.55
50	1.73	1.69	1.66	1.63	1.60	1.58	1.55	1.52	1.50	1.48
60	1.69	1.65	1.62	1.59	1.56	1.53	1.51	1.48	1.45	1.44
70	1.66	1.62	1.59	1.57	1.53	1.50	1.48	1.45	1.42	1.40
80	1.64	1.60	1.57	1.54	1.51	1.48	1.45	1.43	1.39	1.38
90	1.63	1.59	1.55	1.53	1.49	1.46	1.44	1.41	1.38	1.36
100	1.62	1.57	1.54	1.52	1.48	1.45	1.42	1.39	1.36	1.34
120	1.60	1.55	1.52	1.50	1.46	1.43	1.40	1.37	1.33	1.32
150	1.58	1.54	1.50	1.48	1.44	1.41	1.38	1.34	1.31	1.29
200	1.56	1.52	1.48	1.46	1.41	1.39	1.35	1.32	1.28	1.26
250	1.55	1.50	1.47	1.44	1.40	1.37	1.34	1.31	1.27	1.25
300	1.54	1.50	1.46	1.43	1.39	1.36	1.33	1.30	1.26	1.23
400	1.53	1.49	1.45	1.42	1.38	1.35	1.32	1.28	1.24	1.22
500	1.53	1.48	1.45	1.42	1.38	1.35	1.31	1.28	1.23	1.21
600	1.52	1.48	1.44	1.41	1.37	1.34	1.31	1.27	1.23	1.20
750	1.52	1.47	1.44	1.41	1.37	1.34	1.30	1.26	1.22	1.20
1000	1.52	1.47	1.43	1.41	1.36	1.33	1.30	1.26	1.22	1.19

Table for *F*-Test (Continued)

***F* Distribution: Critical Values of *F* (1% significance level)**

ν_2 \ ν_1	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47	6055.85	6106.32	6142.67	6170.10	6191.53	6208.73
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.44	99.44	99.45
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.92	26.83	26.75	26.69
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.25	14.15	14.08	14.02
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.77	9.68	9.61	9.55
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.60	7.52	7.45	7.40
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.36	6.28	6.21	6.16
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.56	5.48	5.41	5.36
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	5.01	4.92	4.86	4.81
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.60	4.52	4.46	4.41
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.29	4.21	4.15	4.10
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.05	3.97	3.91	3.86
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.86	3.78	3.72	3.66
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.70	3.62	3.56	3.51
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.56	3.49	3.42	3.37
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.45	3.37	3.31	3.26
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.35	3.27	3.21	3.16
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.27	3.19	3.13	3.08
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.19	3.12	3.05	3.00
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.13	3.05	2.99	2.94
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.07	2.99	2.93	2.88
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	3.02	2.94	2.88	2.83
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.97	2.89	2.83	2.78
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.93	2.85	2.79	2.74
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.89	2.81	2.75	2.70
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.86	2.78	2.72	2.66
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.82	2.75	2.68	2.63
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.79	2.72	2.65	2.60
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.77	2.69	2.63	2.57
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.74	2.66	2.60	2.55
35	7.42	5.27	4.40	3.91	3.59	3.37	3.20	3.07	2.96	2.88	2.74	2.64	2.56	2.50	2.44
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.56	2.48	2.42	2.37
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.46	2.38	2.32	2.27
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.39	2.31	2.25	2.20
70	7.01	4.92	4.07	3.60	3.29	3.07	2.91	2.78	2.67	2.59	2.45	2.35	2.27	2.20	2.15
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.42	2.31	2.23	2.17	2.12
90	6.93	4.85	4.01	3.53	3.23	3.01	2.84	2.72	2.61	2.52	2.39	2.29	2.21	2.14	2.09
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.37	2.27	2.19	2.12	2.07
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.23	2.15	2.09	2.03
150	6.81	4.75	3.91	3.45	3.14	2.92	2.76	2.63	2.53	2.44	2.31	2.20	2.12	2.06	2.00
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.27	2.17	2.09	2.03	1.97
250	6.74	4.69	3.86	3.40	3.09	2.87	2.71	2.58	2.48	2.39	2.26	2.15	2.07	2.01	1.95
300	6.72	4.68	3.85	3.38	3.08	2.86	2.70	2.57	2.47	2.38	2.24	2.14	2.06	1.99	1.94
400	6.70	4.66	3.83	3.37	3.06	2.85	2.68	2.56	2.45	2.37	2.23	2.13	2.05	1.98	1.92
500	6.69	4.65	3.82	3.36	3.05	2.84	2.68	2.55	2.44	2.36	2.22	2.12	2.04	1.97	1.92
600	6.68	4.64	3.81	3.35	3.05	2.83	2.67	2.54	2.44	2.35	2.21	2.11	2.03	1.96	1.91
750	6.67	4.63	3.81	3.34	3.04	2.83	2.66	2.53	2.43	2.34	2.21	2.11	2.02	1.96	1.90
1000	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43	2.34	2.20	2.10	2.02	1.95	1.90

Table for *F*-Test (Continued)

F Distribution: Critical Values of *F* (1% significance level)

v_2 \ v_1	25	30	35	40	50	60	75	100	150	200
1	6239.83	6260.65	6275.57	6286.78	6302.52	6313.03	6323.56	6334.11	6344.68	6349.97
2	99.46	99.47	99.47	99.47	99.48	99.48	99.49	99.49	99.49	99.49
3	26.58	26.50	26.45	26.41	26.35	26.32	26.28	26.24	26.20	26.18
4	13.91	13.84	13.79	13.75	13.69	13.65	13.61	13.58	13.54	13.52
5	9.45	9.38	9.33	9.29	9.24	9.20	9.17	9.13	9.09	9.08
6	7.30	7.23	7.18	7.14	7.09	7.06	7.02	6.99	6.95	6.93
7	6.06	5.99	5.94	5.91	5.86	5.82	5.79	5.75	5.72	5.70
8	5.26	5.20	5.15	5.12	5.07	5.03	5.00	4.96	4.93	4.91
9	4.71	4.65	4.60	4.57	4.52	4.48	4.45	4.41	4.38	4.36
10	4.31	4.25	4.20	4.17	4.12	4.08	4.05	4.01	3.98	3.96
11	4.01	3.94	3.89	3.86	3.81	3.78	3.74	3.71	3.67	3.66
12	3.76	3.70	3.65	3.62	3.57	3.54	3.50	3.47	3.43	3.41
13	3.57	3.51	3.46	3.43	3.38	3.34	3.31	3.27	3.24	3.22
14	3.41	3.35	3.30	3.27	3.22	3.18	3.15	3.11	3.08	3.06
15	3.28	3.21	3.17	3.13	3.08	3.05	3.01	2.98	2.94	2.92
16	3.16	3.10	3.05	3.02	2.97	2.93	2.90	2.86	2.83	2.81
17	3.07	3.00	2.96	2.92	2.87	2.83	2.80	2.76	2.73	2.71
18	2.98	2.92	2.87	2.84	2.78	2.75	2.71	2.68	2.64	2.62
19	2.91	2.84	2.80	2.76	2.71	2.67	2.64	2.60	2.57	2.55
20	2.84	2.78	2.73	2.69	2.64	2.61	2.57	2.54	2.50	2.48
21	2.79	2.72	2.67	2.64	2.58	2.55	2.51	2.48	2.44	2.42
22	2.73	2.67	2.62	2.58	2.53	2.50	2.46	2.42	2.38	2.36
23	2.69	2.62	2.57	2.54	2.48	2.45	2.41	2.37	2.34	2.32
24	2.64	2.58	2.53	2.49	2.44	2.40	2.37	2.33	2.29	2.27
25	2.60	2.54	2.49	2.45	2.40	2.36	2.33	2.29	2.25	2.23
26	2.57	2.50	2.45	2.42	2.36	2.33	2.29	2.25	2.21	2.19
27	2.54	2.47	2.42	2.38	2.33	2.29	2.26	2.22	2.18	2.16
28	2.51	2.44	2.39	2.35	2.30	2.26	2.23	2.19	2.15	2.13
29	2.48	2.41	2.36	2.33	2.27	2.23	2.20	2.16	2.12	2.10
30	2.45	2.39	2.34	2.30	2.25	2.21	2.17	2.13	2.09	2.07
35	2.35	2.28	2.23	2.19	2.14	2.10	2.06	2.02	1.98	1.96
40	2.27	2.20	2.15	2.11	2.06	2.02	1.98	1.94	1.90	1.87
50	2.17	2.10	2.05	2.01	1.95	1.91	1.87	1.82	1.78	1.76
60	2.10	2.03	1.98	1.94	1.88	1.84	1.79	1.75	1.70	1.68
70	2.05	1.98	1.93	1.89	1.83	1.78	1.74	1.70	1.65	1.62
80	2.01	1.94	1.89	1.85	1.79	1.75	1.70	1.65	1.61	1.58
90	1.99	1.92	1.86	1.82	1.76	1.72	1.67	1.62	1.57	1.55
100	1.97	1.89	1.84	1.80	1.74	1.69	1.65	1.60	1.55	1.52
120	1.93	1.86	1.81	1.76	1.70	1.66	1.61	1.56	1.51	1.48
150	1.90	1.83	1.77	1.73	1.66	1.62	1.57	1.52	1.46	1.43
200	1.87	1.79	1.74	1.69	1.63	1.58	1.53	1.48	1.42	1.39
250	1.85	1.77	1.72	1.67	1.61	1.56	1.51	1.46	1.40	1.36
300	1.84	1.76	1.70	1.66	1.59	1.55	1.50	1.44	1.38	1.35
400	1.82	1.75	1.69	1.64	1.58	1.53	1.48	1.42	1.36	1.32
500	1.81	1.74	1.68	1.63	1.57	1.52	1.47	1.41	1.34	1.31
600	1.80	1.73	1.67	1.63	1.56	1.51	1.46	1.40	1.34	1.30
750	1.80	1.72	1.66	1.62	1.55	1.50	1.45	1.39	1.33	1.29
1000	1.79	1.72	1.66	1.61	1.54	1.50	1.44	1.38	1.32	1.28

F-Test for Differences of Population Variances

F-test of significance of the difference between population variances and F-table.

To test the significance of the difference between population variances, we shall first find their estimates, $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ based on the sample variances s_1^2 and s_2^2 and then test their equality. It is known that $\boxed{\hat{\sigma}_1^2 = \dfrac{n_1 s_1^2}{n_1 - 1}}$ with the number of degree of freedom $v_1 = n_1 - 1$ and $\boxed{\hat{\sigma}_2^2 = \dfrac{n_2 s_2^2}{n_2 - 1}}$ with the number of degrees of freedom $v_2 = n_2 - 1$.

It is also known that $\boxed{F = \dfrac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}}$ follows a F-distribution with v_1 and v_2 degrees of freedom. If $\hat{\sigma}_1^2 = \hat{\sigma}_2^2$, then $F = 1$. Hence our aim is to find how far any observed value of F can differ from unity due to fluctuations of sampling.

Note: $F > 1$

Example:

A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 gave an estimate of 2.5. Could both samples be from populations with the same variance?

Soln/.

$n_1 = 13$ | $\sigma_1^2 = 3.0$ | $\nu_1 = n_1 - 1$

$n_2 = 15$ | $\sigma_2^2 = 2.5$ | $\nu_2 = n_2 - 1$

① $H_0: \sigma_1^2 = \sigma_2^2$

② $H_1: \sigma_1^2 \neq \sigma_2^2$

③. $\alpha = 5\%$

$F_{Tab} = F_{5\%}(12, 14) = \underline{\quad ?}$

④. $F_{cal} = \dfrac{\sigma_1^2}{\sigma_2^2} = \underline{\quad ?}$

⑤. Comparison & Conclusion

$|F_{cal}| \lessgtr |F_{Tab}|$?

Solution:

$$n_1 = 13, \quad \hat{\sigma}_1^2 = 3.0 \quad \text{and} \quad \nu_1 = 12$$

$$n_2 = 15, \quad \hat{\sigma}_2^2 = 2.5 \quad \text{and} \quad \nu_2 = 14.$$

$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$, i.e. The two samples have been drawn from populations with the same variance.

$H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$. Let L.O.S. be 5%

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{3\cdot 0}{2\cdot 5} = 1.2$$

$$\nu_1 = 12 \quad \text{and} \quad \nu_2 = 14.$$

$F_{5\%}\ (\nu_1 = 12, \quad \nu_2 = 14) = 2.53$, from the F-table.

$F < F_{5\%}$. $\therefore$ H_0 is accepted

i.e. the two samples could have come from two normal populations with the same variance.

Example:

Two independent samples of eight and seven items respectively had the following values of the variable.

Sample 1 :	9,	11,	13,	11,	15,	9,	12,	14
Sample 2 :	10,	12,	10,	14,	9,	8,	10	

Do the two estimates of population variance differ significantly at 5% level of significance?

Soln/.

$n_1 = 8$ | $\bar{x}_1 = ?$ | $s_1 = ?$

$n_2 = 7$ | $\bar{x}_2 = ?$ | $s_2 = ?$

$\therefore \sigma_1^2 = \frac{n_1 \cdot s_1^2}{(n_1 - 1)}$ & $\sigma_2^2 = \frac{n_2 \cdot s_2^2}{(n_2 - 1)}$

① $H_0 : \sigma_1^2 = \sigma_2^2$

②. $H_1 : \sigma_1^2 \neq \sigma_2^2$

③. LOS $= \alpha = 5\%$

$F_{Tab} = F_{\alpha = 5\%}(\nu_1 = 7, \nu_2 = 6) = \underline{?}$

④. Test statistic

$F_{cal} = \frac{\sigma_1^2}{\sigma_2^2}$ or $\frac{\sigma_2^2}{\sigma_1^2}$

⑤ Comparison and Conclusion

$|F_{cal}| \lessgtr |F_{Tab}|$

Solution:

For the first sample, $\Sigma x_1 = 94$ and $\Sigma x_1^2 = 1138$

$$\therefore \quad s_1^2 = \frac{1}{n_1}\sum x_1^2 - \left(\frac{1}{n_1}\sum x_1\right)^2$$

$$= \frac{1}{8} \times 1138 - \left(\frac{1}{8} \times 94\right)^2 = 4.19$$

For the second sample, $\Sigma x_2 = 73$ and $\Sigma x_2^2 = 785$

$$\therefore \quad s_2^2 = \frac{1}{n_2}\sum x_2^2 - \left(\frac{1}{n_2}\sum x_2\right)^2$$

$$= \frac{1}{7} \times 785 - \left(\frac{1}{7} \times 73\right)^2 = 3.39$$

$$\hat{\sigma}_1^2 = \frac{n_1}{n_1 - 1} s_1^2 = 4.79 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{n_2}{n_2 - 1} s_2^2 = 3.96$$

since $\quad \hat{\sigma}_1^2 > \hat{\sigma}_2^2, \ \nu_1 = 7 \quad \text{and} \quad \nu_2 = 6$

$$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2 \quad \text{and} \quad H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$$

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{4{\cdot}79}{3{\cdot}96} = 1.21$$

$F_{5\%}(\nu_1 = 7, \nu_2 = 6) = 4.21$, from the F-table. Since $F < F_{5\%}$, H_0 is accepted.

i.e. $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ do not differ significantly at 5% level of significance.

Example:

Two samples of sizes nine and eight gave the sums of squares of deviations from their respective means equal to 160 and 91 respectively. Can they be regarded as drawn from the same normal population?

Soln/.

$n_1 = 9$

$n_2 = 8$

$$\frac{1}{n_1}\sum_{i=1}^{n_1}(x_1 - \bar{x}_1)^2 = \frac{160}{n_1}$$

$$\Rightarrow n_1 \cdot s_1^2 = 160 \Rightarrow \sigma_1^2 = \frac{160}{(n_1-1)}$$

$$\frac{1}{n_2}\sum_{i=1}^{n_2}(x_2 - \bar{x}_2)^2 = \frac{91}{n_2}$$

$$\Rightarrow n_2 \cdot s_2^2 = 91 \Rightarrow \sigma_2^2 = \frac{91}{(n_2-1)}$$

1. $H_0 := \sigma_1^2 = \sigma_2^2$
2. $H_2 := \sigma_1^2 \neq \sigma_2^2$
3. α, F_{Tab}
4. $F_{cal} = \sigma_1^2/\sigma_2^2$ (or) σ_2^2/σ_1^2
5. Comparison and Conclusion

Solution:

$$n_1 = 9, \quad \Sigma(x_i - \bar{x})^2 = 160, \quad \text{i.e. } n_1 s_1^2 = 160$$

$$n_2 = 8, \quad \Sigma(y_i - \bar{y})^2 = 91, \quad \text{i.e. } n_2 s_2^2 = 91$$

$$\hat{\sigma}_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{1}{8} \times 160 = 20\,; \quad \hat{\sigma}_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{1}{7} \times 91 = 13$$

Since $\hat{\sigma}_1^2 > \hat{\sigma}_2^2$, $v_1 = n_1 - 1 = 8$ and $v_2 = n_2 - 1 = 7$

$H_0 : \hat{\sigma}_1^2 = \hat{\sigma}_2^2$ and $H_1 : \hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$.

Let the LOS be 5%

$$F = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{20}{13} = 1.54$$

$F_{5\%}(v_1 = 8, v_2 = 7) = 3.73$, from the F-table.

Since $F < F_{5\%}$, H_0 is accepted.

i.e. the two samples could have come from two normal populations with the same variance.

Exercise:

Two random samples gave the following data:

	Size	Mean	Variance
Sample I	8	9.6	1.2
Sample II	11	16.5	2.5

Can we conclude that the two samples have been drawn from the same normal population?

Exercise:

The nicotine contents in two random samples of tobacco are given below.

Sample I :	21	24	25	26	27	
Sample II :	22	27	28	30	31	36.

Can you say that the two samples came from the same population?

Chi-square Distribution $\chi^2(\nu)$

Definition:

The pdf is,

$$f(\chi^2) = \frac{1}{2^{\nu/2}\sqrt{\left(\frac{\nu}{2}\right)}} \cdot (\chi^2)^{\nu/2-1} \; e^{-\chi^2/2}$$

$0 < \chi^2 < \infty$, where ν is the number of degrees of freedom.

Properties of χ^2-Distribution

1. A rough sketch of the probability curve of the χ^2-distribution for $\nu = 3$ and $\nu = 6$ is given in Fig.
2. As ν becomes smaller and smaller, the curve is skewed more and more to the right. As ν increases, the curve becomes more and more symmetrical.
3. The mean and variance of the χ^2-distribution are ν and 2ν respectively.

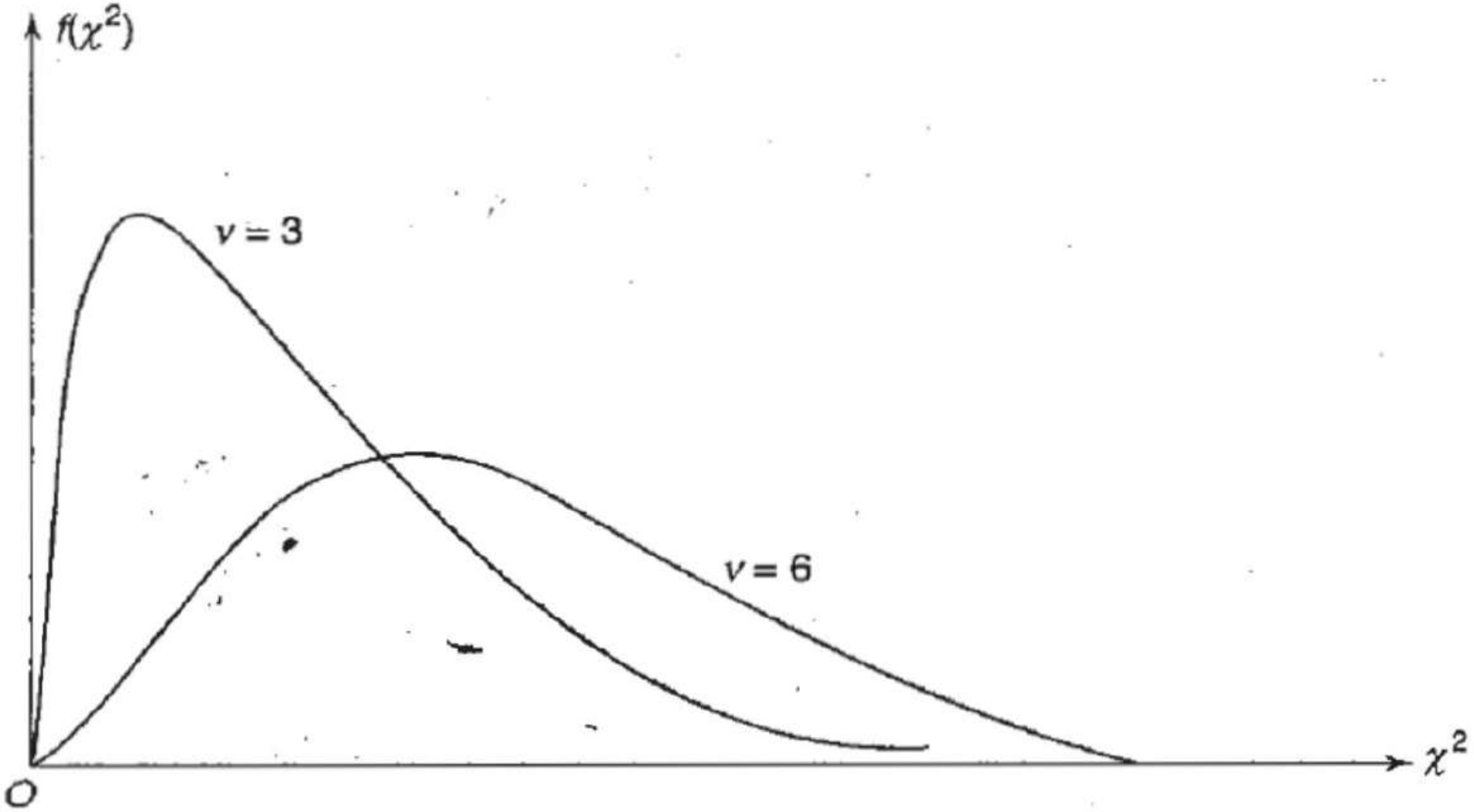

Fig.

4. As n tends to ∞, the χ^2-distribution becomes a normal distribution.

Uses of χ^2-Distribution

1. χ^2-distribution is used to test the goodness of fit. i.e., it is used to judge whether a given sample may be reasonably regarded as *a* simple sample from a certain hypothetical population.
2. It is used to test the independence of attributes. i.e. If a population is known to have two attributes (or traits), then χ^2-distribution is used to test whether the two attributes are associated or independent, based on a sample drawn from the population.

Table for Chi-square Test

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

χ^2-Test of Goodness of Fit

On the basis of the hypothesis assumed about the population, we find the expected frequencies $E_i (i = 1, 2, ..., n)$, corresponding to the observed frequencies $O_i (i = 1, 2, ..., n)$ such that $\boxed{\sum E_i = \sum O_i}$. It is known that $\boxed{\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}}$ follows approximately a χ^2-distribution with degrees of freedom equal to the number of independent frequencies. In order to test the goodness of fit, we have to determine how far the differences between O_i and E_i can be attributed to fluctuations of sampling and when we can assert that the differences are large enough to conclude that the sample is not a simple sample from the hypothetical population. In other words, we have to determine how large a value of χ^2 we can get so as to assume that the sample is a simple sample from the hypothetical population.

<u>Note</u>: $\nu = n - 1$

If the calculated $\chi^2 < \chi^2_\nu(\alpha)$, we will accept the null hypothesis H_0 which assumes that the given sample is one drawn from the hypothetical population, i.e. we will conclude that the difference between the observed and expected frequencies is not significant at α % LOS If $\chi^2 > \chi^2_\nu(\alpha)$, we will reject H_0 and conclude that the difference is significant.

Conditions for the Validity of χ^2-Test

1. The number of observations N in the sample must be reasonably large, say ≥ 50.
2. Individual frequencies must not be too small, i.e. $O_i \geq 10$. In case $O_i < 10$, it is combined with the neighbouring frequencies, so that the combined frequency is ≥ 10.
3. The number of classes n must be neither too small nor too large i.e., $4 \leq n \leq 16$.

χ^2-Test of Independence of Attributes

If the population is known to have two major attributes A and B, then A can be divided into m categories $A_1, A_2, ..., A_m$ and B can be divided into n categories $B_1, B_2, ..., B_n$. Accordingly the members of the population and hence those of the sample can be divided into mn classes. In this case, the sample data may be presented in the form of a matrix containing m rows and n columns and hence mn cells and showing the observed frequencies O_{ij}, in the various cells, where $i = 1, 2, ..., m$ and $j = 1, 2, ... n$. O_{ij} means the number of observed frequencies possessing the attributes A_i and B_j. The matrix or tabular form of the sample data, called an $(m \times n)$ *contingency table* is given below:

A\B	B_1	B_2	-	B_j	-	B_n	Row Total
A_1	O_{11}	O_{12}	-	O_{1j}	-	O_{1n}	O_{1*}
A_2	O_{21}	O_{22}	-	O_{2j}	-	O_{2n}	O_{2*}
:	-	-	-	-	-	-	-
A_i	O_{i1}	O_{i2}	-	O_{ij}	-	O_{in}	O_{i*}
:	-	-	-	-	-	-	-
A_m	O_{m1}	O_{m2}		O_{mj}	-	O_{mn}	O_{m*}
Column Total	O_{*1}	O_{*2}	-	O_{*j}	-	O_{*n}	N

Now, based on the null hypothesis H_0 i.e. the assumption that the two attributes A and B are independent, we compute the expected frequencies E_{ij} for various cells, using the following formula $\boxed{E_{ij} = \frac{O_{i*} \cdot O_{*j}}{N}}$, $i = 1, 2, \ldots m$; and $j = 1, 2, \ldots n$

i.e. $$E_{ij} = \left\{ \frac{(\text{Total of observed frequencies in the } i^{\text{th}} \text{ row}) \times (\text{total of observed frequencies in the } j^{\text{th}} \text{ column})}{\text{Total of all cell frequecies}} \right\}$$

Then we compute $$\boxed{\chi^2 = \sum_{i=1}^{m} \sum_{J=1}^{n} \left\{ \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \right\}}$$

The number of degrees of freedom for this χ^2 computed from the $(m \times n)$ contingency table is $\boxed{\nu = (m-1)(n-1)}$.

If $\chi^2 < \chi^2_\nu(\alpha)$, H_0 is accepted at α % LOS i.e. the attributes A and B are independent.

If $\chi^2 > \chi^2_\nu(\alpha)$, H_0 is rejected at α % LOS i.e. A and B are not independent.

Example:

Table gives the number of air-craft accidents that occurred during the various days of a week. Test whether the accidents are uniformly distributed over the week.

Table

Day:	Mon	Tues	Wed	Thu	Fri	Sat	Total
No. of accidents:	15	19	13	12	16	15	90

Soln/. $N = 90$ & $n = 6$.

①. H_0: The accidents are distributed uniformly.

②. H_1: Not uniformly distributed.

③. LOS $= \alpha = 5\% = 0.05$

$$\chi^2_{Tab} = \chi^2_{5\%}(\nu = 5) = ?$$

④. Test Statistic:

$$\chi^2_{cal} = \sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i}$$

$$E_i = \frac{90}{6} = 15 \; ; \quad i = 1, 2, \ldots, 6$$

[Since uniform distribution]

⑤. Comparison and Conclusion

Solution:

H_0 : Accidents occur uniformly over the week.

Total number of accidents = 90

Based on H_0, the expected number of accidents on any day $= \frac{90}{6} = 15$.

O_i	:	15	19	13	12	16	15
E_i	:	15	15	15	15	15	15

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{1}{15}(0 + 16 + 4 + 9 + 1 + 0) = 2.$$

Since $\sum E_i = \sum O_i, \quad \nu = 6 - 1 = 5$

From the χ^2-table, $\chi^2_{5\%}(\nu = 5) = 11.07$.

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. accidents may be regarded to occur uniformly over the week.

Example:

Theory predicts that the proportion of beans in four groups A, B, C, D should be $9:3:3:1$. In an experiment among 1600 beans, the numbers in the four groups were 882, 313, 287 and 118. Does the experiment support the theory?

Solution:

H_0 : The experiment supports the theory, i.e. the numbers of beans in the four groups are in the ratio 9 : 3 : 3 : 1

Based on H_0, the expected numbers of beans in the four groups are as follows

$$E_i : \quad \frac{9}{16} \times 1600, \quad \frac{3}{16} \times 1600, \quad \frac{3}{16} \times 1600, \quad \frac{1}{16} \times 1600$$

i.e.

$$E_i : \quad 900, \quad 300, \quad 300, \quad 100$$

$$O_i : \quad 882, \quad 313, \quad 287, \quad 118$$

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{18^2}{900} + \frac{13^2}{300} + \frac{13^2}{300} + \frac{18^2}{100} = 4.73$$

Since $\sum E_i = \sum O_i$, $\nu = 4 - 1 = 3$

From the χ^2-table, $\chi^2_{5\%}(\nu = 3) = 7.82$

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

i.e. the experimental data support the theory.

Example:

Fit a binomial distribution for the following data and also test the goodness of fit.

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80

Solution:

To find the binomial distribution $N(q+p)^n$, which fits the given data, we require p.

We know that the mean of the binomial distribution is np, from which we can find p. Now the mean of the given distribution is found out and is equated to np.

x:	0	1	2	3	4	5	6	Total
f:	5	18	28	12	7	6	4	80
fx:	0	18	56	36	28	30	24	192

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{192}{80} = 2.4$$

i.e. $np = 2.4$ or $6p = 2.4$, since the maximum value taken by x is n.

$\therefore$ $p = 0.4$ and hence $q = 0.6$

$\therefore$ The expected frequencies are given by the successive terms in the expansion of $80(0.6+0.4)^6$.

Thus E_i: 3.73, 14.93, 24.88, 22.12, 11.06, 2.95, 0.33

Converting the E_i's into whole number such that $\sum E_i = \sum O_i = 80$, we get

E_i: 4 15 25 22 11 3 0

Let us now proceed to test the goodness of binomial fit.

O_i: 5 18 28 12 7 6 4

Solution (Continued):

The first class is combined with the second and the last two classes are combined with the last but second class in order to make the expected frequency in each class greater than or equal to 10. Thus, after regrouping, we have,

E_i :	19	25	22	14
O_i :	23	28	12	17

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{4^2}{19} + \frac{3^2}{25} + \frac{10^2}{22} + \frac{3^2}{14} = 6.39$$

We have used the given sample to find

$\sum E_i \; (= \sum O_i)$ and p through its mean.

Hence $\quad \nu = n - k$

$$= 4 - 2 = 2$$

$\chi^2_{5\%} \; (\nu = 2) = 5.99$, from the χ^2-table.

Since $\chi^2 > \chi^2_{5\%}$, H_0, which assumes that the given distribution is approximately a binomial distribution, is rejected. i.e. the binomial fit for the given distribution is not satisfactory.

Exercise:

The following table shows the distribution of digits in the numbers chosen at random from a telephone directory:

Table

Digit:	0	1	2	3	4	5	6	7	8	9	Total
Frequency:	1026,	1107,	997,	966,	1075	933,	1107,	972,	964,	853	10,000

Test whether the digits may be taken to occur equally frequently in the directory.

Exercise:

According to genetic theory, children having one parent of blood type *M* and the other of blood type *N* will always be one of the three types-*M*, *MN* and *N* and the average proportions of these types will be 1 : 2 : 1. Out of 300 children, having one *M* parent and one N parent, 30 per cent were found to be of type *M*, 45 per cent of type *MN* and the remaining of type *N*. Test the genetic theory by χ^2-test.

Exercise:

Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x:	0	1	2	3	4	5	Total
f:	142	156	69	27	5	1	400

Example:

The following data are collected on two characters

	Smokers	*Non-smokers*
Literates :	83	57
Illiterates :	45	68

Based on this, can you say that there is no relation between smoking and literacy?

Solution:

H_0 : Literacy and smoking habit are independent

	Smokers	*Non-smokers*	*Total*
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

O	E	E (rounded)	$(O-E)^2/E$
83	$\dfrac{128 \times 140}{253} = 70.83$	71	$122/71 = 2.03$
57	$\dfrac{125 \times 140}{253} = 69.17$	69	$122/69 = 2.09$
45	$\dfrac{128 \times 113}{253} = 57.17$	57	$122/57 = 2.53$
68	$\dfrac{125 \times 113}{253} = 55.83$	56	$122/56 = 2.57$
			$\chi 2 = 9.22$

$$\nu = (m-1)(n-1)$$
$$= (2-1(2-1) = 1.$$

From the χ^2-table, $\chi^2_{5\%}(\nu = 1) = 3.84$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

i.e. there is some association between literacy and smoking.

Example:

A total number of 3759 individuals were interviewed in a public opinion survey on a political proposal. Of them, 1872 were men and the rest women. 2257 individuals were in favour of the proposal and 917 were opposed to it. 243 men were undecided and 442 women were opposed to the proposal. Do you justify or contradict the hypothesis that there is no association between sex and attitude?

Table:

	Favoured	*Opposed*	*Undecided*	*Total*
Men	1154	475	243	1872
Women	1103	442	342	1887
Total	2257	917	585	3759

Solution:

H_0 : Sex and attitude are independent, i.e. there is no association between sex and attitude.

O	E (rounded E)	$(O-E)^2/E$
1154	$\frac{1872\times 2257}{3759} \simeq 1124$	$30^2/1124 = 0.80$
475	$\frac{1872\times 917}{3759} \simeq 457$	$18^2/457 = 0.71$
243	$\frac{1872\times 585}{3759} \simeq 291$	$48^2/291 = 7.92$
1103	$\frac{1887\times 2257}{3759} \simeq 1133$	$30^2/1133 = 0.79$
442	$\frac{1887\times 917}{3759} \simeq 460$	$18^2/460 = 0.70$
342	$\frac{1887\times 585}{3759} \simeq 294$	$48^2/294 = 7.84$
$\nu = (3-1)(2-1) = 2$		$\chi^2 = 18.76$

From the χ^2-table, $\chi^2_{5\%}(\nu = 2) = 5.99$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

That is, sex and attitude are not independent i.e. there is some association between sex and attitude.

Exercise:

A survey of radio listeners' preference for two types of music under various age groups gave the following information.

Table

Type of music	*Age group*		
	19-25	26-35	Above 36
Carnatic music :	80	60	90
Film music :	210	325	44
Indifferent :	16	45	132

Is preference for type of music influenced by age?

Design of Experiments

By 'experiment', we mean collection of data (which usually consist of a series of measurement of some feature of an object) for a scientific investigation, according to certain specified sampling procedures. Statistics provides not only the principles and the basis for the proper planning of the experiments but also the methods for proper interpretation of the results of the experiment.

In the beginning, the study of the design of experiments was associated only with agricultural experimentation. The need to save time and money has led to a study of ways to obtain maximum information with the minimum cost and labour. Such motivations resulted in the subsequent acceptance and wide use of the design of experiments and the related analysis of variance techniques in all fields of scientific experimentation. In this chapter we consider some aspects of experimental design briefly and analysis of data from such experiments using analysis of variance techniques.

Aim of the Design of Experiments

A statistical experiment in any field is performed to verify a particular hypothesis. For example, an agricultural experiment may be performed to verify the claim that a particular manure has got the effect of increasing the yield of paddy. Here the quantity of the manure used and the amount of yield are the two variables involved directly. They are called *experimental variables*. Apart from these two, there are other variables such as the fertility of the soil, the quality of the seed used and the amount of rainfall, which also affect the yield of paddy. Such variables are called *extraneous variables*. The main aim of the design of experiments is to control the extraneous variables and hence to minimise the experimental error so that the results of the experiments could be attributed only to the experimental variables.

Basic Principles of the Experimental Design

In order to achieve the objective mentioned above, the following three principles are adopted while designing the experiments– (1) randomisation, (2) replication and (3) local control.

1. Randomisation

As it is not possible to eliminate completely the contribution of extraneous variables to the value of the response variable (the amount of yield of paddy), we try to control it by randomisation. The group of experimental units (plots of the same size) in which the manure is used is called the *experimental group* and the other group of plots in which the manure is not used and which will provide a basis for comparison is called the *control group*. If any information regarding the extraneous variables and the nature and magnitude of their effect on the response variable in question is not available, we resort to randomisation. That is, we select the plots for *the* experimental and control groups in a random manner, which provides the most effective way of eliminating any unknown bias in the experiment.

2. Replication

In a comparative experiment, in which the effects of different manures on the yield are studied, each manure is used in more than one plot. In other words, we resort to replication which means repetition. It is essential to carry out more than one test on each manure in order to estimate the amount of the experimental error and hence to get some idea of the precision of the estimates of the manure effects.

3. Local Control

To provide adequate control of extraneous variables, another essential principle used in the experimental design is the local control. This includes techniques such as grouping, blocking and balancing of the experimental units used in the experimental design. By *grouping*, we mean combining sets of homogeneous plots into groups, so that different manures may be used in different groups. The number of plots in different groups need not necessarily be the same. By *blocking*, we mean assigning the same number of plots in different blocks. The plots in the same block may be assumed to be relatively homogeneous. We use as many manures as the number of plots in a block in a random manner. By *balancing*, we mean adjusting the procedures of grouping, blocking and assigning the manures in such a manner that a balanced configuration is obtained.

Basic Designs of Experiment

1. Completely Randomised Design (CRD)

Analysis of Variance (ANOVA) for One Way Classification

2. Randomised Block Design (RBD)

Analysis of Variance (ANOVA) for Two Way Classification

3. Latin Square Design (LSD)

Analysis of Variance (ANOVA) for Three Way Classification

1. Completely Randomised Design (CRD)

Let us suppose that we wish to compare 'h' treatments (use of 'h' different manures) and there are n plots available for the experiment.

Let the ith treatment be replicated (repeated) n_i times, so that $n_1 + n_2 + \ldots + n_h = n$.

The plots to which the different treatments are to be given are found by the following randomisation principle. The plots are numbered from 1 to n serially. n identical cards are taken, numbered from 1 to n and shuffled thoroughly. The numbers on the first n_1 cards drawn randomly give the numbers of the plots to which the first treatment is to be given. The numbers on the next n_2 cards drawn at random give the numbers of the plots to which the second treatment is to be given and so on. This design is called a completely randomised design, which is used when the plots are homogeneous or the pattern of heterogeneity of the plots is unknown.

2. Randomised Block Design (RBD)

Let us consider an agricultural experiment using which we wish to test the effect of 'k' fertilizing treatments on the yield of a crop. We assume that we know some information about the soil fertility of the plots. Then we divide the plots into 'h' blocks, according to the soil fertility, each block containing 'k' plots. Thus the plots in each block will be of homogeneous fertility as far as possible.

Within each block, the 'k' treatments are given to the 'k' plots in a perfectly random manner, such that each treatment occurs only once in any block. But the same 'k' treatments are repeated from block to block. This design is called Randomised Block Design.

3. Latin Square Design (LSD)

We consider an agricultural experiment, in which n^2 plots are taken and arranged in the form of an $n \times n$ square, such that the plots in each row will be homogeneous as far as possible with respect to one factor of classification, say, soil fertility and plots in each column will be homogeneous as far as possible with respect to another factor of classification, say, seed quality.

Then n treatments are given to these plots such that each treatment occurs only once in each row and only once in each column. The various possible arrangements obtained in this manner are known as Latin squares of order n. This design of experiment is called the Latin Square Design.

Comparison of RBD and LSD

1. The number of replications of each treatment is equal to the number of treatments in LSD, whereas there is no such restrictions on treatments and replication in RBD.
2. LSD can be performed on a square field, while RBD can be performed either on a square field or a rectangular field.
3. LSD is known to be suitable for the case when the number of treatments is between 5 and 12, whereas RBD can be used for any number of treatments.
4. The main advantage of LSD is that it controls the effect of two extraneous variables, whereas RBD controls the effect of only one extraneous variable. Hence the experimental error is reduced to a larger extent in LSD than in RBD.

Analysis of Variance (ANOVA)

The analysis of variance is a widely used technique developed by R.A. Fisher. It enables us to divide the total variation (represented by variance) in a group into parts which are ascribable to different factors and a residual random variation which could not be accounted for by any of these factors. The variation due to any specific factor is compared with the residual variation for significance by applying the F-test, with which the reader is familiar. The details of the procedure will be explained in the sequel.

Analysis of Variance (ANOVA) for One Way Classification

ANOVA table for one factor of classification

Source of variation (S.V.)	*Sum of squares (S.S.)*	*Degree of freedom (d.f.)*	*Mean square (M.S.)*	*Variance ratio (F)*
Between classes	Q_1	$h-1$	$Q_1/(h-1)$	$\frac{Q_1/(h-1)}{Q_2/(N-h)}$ (OR)
Within classes classes	Q_2	$N-h$	$Q_2/(N-h)$	$\frac{Q_2/(N-h)}{Q_1/(h-1)}$
Total	Q	$N-1$	–	–

Note *For calculating Q, Q_1, Q_2, the following computational formulas may be used:*

$$Q = N\left\{\frac{1}{N}\sum\sum x_{ij}^2 - \bar{x}^2\right\}$$

$$= N\left\{\frac{1}{N}\sum\sum x_{ij}^2 - \left(\frac{1}{N}\sum\sum x_{ij}\right)^2\right\}$$

$$= \sum\sum x_{ij}^2 - \frac{T^2}{N}, \text{ where } T = \sum\sum x_{ij}$$

Similarly, for the ith class,

$$\sum_j (x_{ij} - \bar{x}_i)^2 = \sum_j x_{ij}^2 - \frac{T_i^2}{n_i}, \text{ where } T_i = \sum_j x_{ij}.$$

$$\therefore \quad Q_2 = \sum_i\sum_j (x_{ij} - \bar{x}_i)^2 = \sum_i\sum_j x_{ij}^2 - \sum_i \frac{T_i^2}{n_i}$$

Hence

$$Q_1 = Q - Q_2$$

$$= \sum_i \frac{T_i^2}{n_i} - \frac{T^2}{N}$$

Analysis of Variance (ANOVA) for Two Way Classification

The ANOVA table for the two factors of classifications

S.V.	*S.S.*	*d.f.*	*M.S.*	*F*
Between rows	Q_1	$h-1$	$Q_1/(h-1)$	$\left[\dfrac{Q_1/(h-1)}{Q_3/(h-1)(k-1)}\right]^{\pm 1}$
Between columns	Q_2	$k-1$	$Q_2/(k-1)$	$\left[\dfrac{Q_2/(k-1)}{Q_3/(h-1)(k-1)}\right]^{\pm 1}$
Residual	Q_3	$(h-1)(k-1)$	$Q_3/(h-1)(k-1)$	–
Total	Q	$hk-1$	–	–

Note *The following working formulas that can be easily derived may be used to compute Q, Q_1, Q_2 and Q_3:*

1. $Q = \sum\sum x_{ij}^2 - \dfrac{T^2}{N}$, *where* $T = \sum\sum x_{ij}$

2. $Q_1 = \dfrac{1}{k}\sum T_i^2 - \dfrac{T^2}{N}$, *where* $T_i = \sum_{j=1}^{k} x_{ij}$

3. $Q_2 = \dfrac{1}{h}\sum T_j^2 - \dfrac{T^2}{N}$, *where* $T_j = \sum_{i=1}^{h} x_{ij}$

4. $Q_3 = Q - Q_1 - Q_2$

It may be verified that $\sum_i T_i = \sum_j T_j = T$.

Analysis of Variance (ANOVA) for Three Way Classification

The ANOVA table for three factors of **classification**

S.V.	*S.S.*	*d.f.*	*M.S.*	*F*
Between rows	Q_1	$n-1$	$Q_1/(n-1) = M_1$	$\left(\frac{M_1}{M_4}\right)^{\pm 1}$
Between columns	Q_2	$n-1$	$Q_2/(n-1) = M_2$	$\left(\frac{M_2}{M_4}\right)^{\pm 1}$
Between letters	Q_3	$n-1$	$Q_3/(n-1) = M_3$	$\left(\frac{M_3}{M_4}\right)^{\pm 1}$
Residual	Q_4	$(n-1)(n-2)$	$Q_4/(n-1)(n-2) = M_4$	–
Total	Q	n^2-1	–	–

Note *The following working formulas may be used to compute the Q's:*

1. $Q = \sum\sum x_{ij}^2 - \frac{T^2}{n^2}$, *where* $T = \sum\sum x_{ij}$

2. $Q_1 = \frac{1}{n}\sum T_i^2 - \frac{T^2}{n^2}$, *where* $T_i = \sum_{j=1}^{n} x_{ij}$

3. $Q_2 = \frac{1}{n}\sum T_j^2 - \frac{T^2}{n^2}$, *where* $T_j = \sum_{i=1}^{n} x_{ij}$

4. $Q_3 = \frac{1}{n}\sum T_k^2 - \frac{T^2}{n^2}$, *where* T_k *is the sum of all* x_{ij}*s receiving the* k^{th} *treatment.*

5. $Q_4 = Q - Q_1 - Q_2 - Q_3$.

Also $T = \sum_i T_i = \sum_j T_j = \sum_k T_k$

Example:

A completely randomised design experiment with 10 plots and 3 treatments gave the following results:

Plot No. :	1	2	3	4	5	6	7	8	9	10
Treatment :	A	B	C	A	C	C	A	B	A	B
Yield :	5	4	3	7	5	1	3	4	1	7

Analyse the results for treatment effects.

Solution:

Rearranging the data according to the treatments, we have the following table:

Treatment	*Yield from plots* (x_{ij})				T_i	T_i^2	n_i	$\frac{T_i^2}{n_i}$
A	5	7	3	1	16	256	4	64
B	4	4	7	–	15	225	3	75
C	3	5	1	–	9	81	3	27
				Total	$T = 40$	–	$N = 10$	166

$$\sum\sum x_{ij}^2 = (25 + 49 + 9 + 1) + (16 + 16 + 49) + (9 + 25 + 1)$$
$$= 84 + 81 + 35 = 200$$

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 200 - \frac{40^2}{10} = 200 - 160 = 40$$

$$Q_1 = \sum\frac{T_i^2}{n_i} - \frac{T^2}{N} = 166 - 160 = 6$$

$$Q_2 = Q - Q_1 = 40 - 6 = 34$$

Solution (Continued):

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between classes (treatments)	$Q_1 = 6$	$h - 1 = 2$	3.0	$\frac{4.86}{3.0}$
Within classes	$Q_2 = 34$	$N - h = 7$	4.86	$= 1.62$
Total	$Q = 40$	$N - 1 = 9$	–	–

From the F-table, $F_{5\%}$ $(\nu_1 = 7, \nu_2 = 2) = 19.35$

We note that $F_0 < F_{5\%}$

Let H_0 : The treatments do not differ significantly.

$\therefore$ The null hypothesis is accepted.

i.e., the treatments are not significantly different.

Example:

Three varieties of a crop are tested in a randomised block design with four replications, the layout being as given below: The yields are given in kilograms. Analyse for significance

C48	A51	B52	A49
A47	B49	C52	C51
B49	C53	A49	B50

Solution:

Rewriting the data such that the rows represent the blocks and the columns represent the varieties of the crop (as assumed in the discussion of analysis of variance for two factors of classification), we have the following table:

Crops

Blocks	*A*	*B*	*C*
1	47	49	48
2	51	49	53
3	49	52	52
4	49	50	51

We shift the origin to 50 and work out with the new values of x_{ij}.

Solution (Continued):

Crops

Blocks	A	B	C	T_i	T_i^2/k	$\sum_j x_{ij}^2$
1	-3	-1	-2	-6	$36/3 = 12$	14
2	1	-1	3	3	$9/3 = 3$	11
3	-1	2	2	3	$9/3 = 3$	9
4	-1	0	1	0	$0/3 = 0$	2
T_j	-4	0	4	$T = 0$	$\sum \frac{T_i^2}{k} = 18$	36
T_j^2/h	$\frac{16}{4} = 4$	$\frac{0}{4} = 0$	$\frac{16}{4} = 4$	$\sum \frac{T_j^2}{h} = 8$		
$\sum_i x_{ij}^2$	12	6	18	36		

$$Q = \sum\sum x_{ij}^2 - \frac{T^2}{N} = 36 - \frac{0^2}{12} = 36$$

$$Q_1 = \frac{1}{k}\sum T_i^2 - \frac{T^2}{N} = 18 - 0 = 18$$

$$Q_2 = \frac{1}{h}\sum T_j^2 - \frac{T^2}{N} = 8 - 0 = 8$$

$$Q_3 = Q - Q_1 - Q_2 = 36 - 18 - 8 = 10$$

Solution (Continued):

ANOVA table

S.V.	*S.S.*	*d.f.*	*M.S.*	F_0
Between rows (blocks)	$Q_1 = 18$	$h-1=3$	6	$\frac{6}{1.67} = 3.6$
Between columns (crops)	$Q_2 = 8$	$k-1=2$	4	$\frac{4}{1.67} = 2.4$
Residual	$Q_3 = 10$	$(h-1)(k-1) = 6$	1.67	–
Total	$Q = 36$	$hk-1=11$	–	–

From F-tables, $F_{5\%}$ $(v_1 = 3, v_2 = 6) = 4.76$ and $F_{5\%}$ $(v_1 = 2, v_2 = 6) = 5.14$ Considering the difference between rows, we see that F_0 $(= 3.6) < F_{5\%}$ $(= 4.76)$ Hence the difference between the rows is not significant. (H_0 is accepted) viz., the blocks do not differ significantly with respect to the yield.

Considering the difference between columns, we see that F_0 $(= 2.4) < F_{5\%}$ $(= 5.14)$

Hence the difference between the columns is not significant. (H_0 is accepted) viz., the varieties of crop do not differ significantly with respect to the yield.

Exercise:

The following table shows the lives in hours of four brands of electric lamps:

Brand

A : 1610, 1610, 1650, 1680, 1700, 1720, 1800

B : 1580, 1640, 1640, 1700, 1750

C : 1460, 1550, 1600, 1620, 1640, 1660, 1740, 1820

D : 1510, 1520, 1530, 1570, 1600, 1680

Perform an analysis of variance and test the homogeneity of the mean lives of the four brands of lamps.

Exercise:

In order to determine whether there is significant difference in the durability of : makes of computers, samples of size 5 are selected from each make and the fre quency of repair during the first year of purchase is observed. The results are a follows :

Makes

A	B	C
5	8	7
6	10	3
8	11	5
9	12	4
7	4	1

In view of the above data, what conclusion can you draw?

Exercise:

The following data represent the number of units of production per day turned out by 5 different workers using 4 different types of machines:

		Machine Type			
		A	*B*	*C*	*D*
	1	44	38	47	36
	2	46	40	52	43
Workers:	3	34	36	44	32
	4	43	38	46	33
	5	38	42	49	39

(a) Test whether the five men differ with respect to mean productivity.

(b) Test whether the mean productivity is the same for the four different machine types.

Exercise:

Four doctors each test four treatments for a certain disease and observe the number of days each patient takes to recover. The results are as follows (recovery time in days)

	Treatment			
Doctor	*1*	*2*	*3*	*4*
A	10	14	19	20
B	11	15	17	21
C	9	12	16	19
D	8	13	17	20

Discuss the difference between (a) doctors and (b) treatments.