1. Consider the scenario that there are three outlaws $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ placed in three distinct prison cells. Further, one of them will be executed in the subsequent sunrise. Suppose outlaw $\boldsymbol{A}$ is extremely anxious, as he has $\mathbf{1/3}$ chance to be the one. He attempts to acquire some information from the janitor: "I know you cannot tell me whether I will be executed in the next morning, but can you tell me which of my inmates $\boldsymbol{B}$ and $\boldsymbol{C}$ will not be executed? Because one of them will not be executed anyhow, by pointing out who will not be executed, you are not telling me any information." This sounds quite logical. So the Janitor tells $\boldsymbol{A}$ that $\boldsymbol{C}$ will not be executed. At second thought, $\boldsymbol{A}$ becomes much more bothered. Before he asked the janitor, he thought he had a $\mathbf{1/3}$ chance, however, with $\boldsymbol{C}$ being excluded, he seems to have a $\mathbf{1/2}$ chance. $\boldsymbol{A}$ says to himself: "What did I do wrong? Why did I ask the janitor?" Use Bayes rule to determine the solutions for the following queries.

   (i) Give details about the random variables and the input data and also give the significance of the prior and posterior probabilities for this scenario.
   <u>Answer:</u>

   Since who's going to be executed tomorrow has already been decided before A asked the janitor - otherwise the janitor won't be able to tell A which one of A's inmates will live - we have:
   The random variable is all the possible answers from the janitor; the input data (observation) is the janitor's answer; the prior probability is the chance of A being executed before observing the janitor's answer; the posterior probability is the chance of A being executed after observing the janitor's answer.

   (ii) Determine the probability values for the prior.
   <u>Answer:</u>

   Let E_X, where X = {A, B, C}, denote the event that X is going to be executed. The prior probability of A being executed tomorrow is $P(E_A) = P(E_B) = P(E_C) = \frac{1}{3}$ (suppose they kill at random).

   (iii) Determine the probability values for the likelihood.
   <u>Answer:</u>

   Let L_Y, where Y = {B, C}, denote the event that: knowing that X will be executed tomorrow, the likelihood of the janitor telling A that Y will live. The likelihoods are: $P(L_B|E_A) = P(L_C|E_A) = \frac{1}{2}$; $P(L_B|E_B) = 0, P(L_C|E_B) = 1$; $P(L_B|E_C) = 1, P(L_C|E_C) = 0$.

(iv) Derive the probability values and determine the posterior probability.
Answer:

The posterior probability of A being executed is

$$\begin{aligned}
P(E_A|L_C) &= \frac{P(L_C|E_A) \cdot P(E_A)}{P(L_C)} \\
&= \frac{P(L_C|E_A) \cdot P(E_A)}{P(L_C|E_A) \cdot P(E_A) + P(L_C|E_B) \cdot P(E_B) + P(L_C|E_C) \cdot P(E_C)} \\
&= \frac{1/2 \cdot 1/3}{1/2 \cdot 1/3 + 1 \cdot 1/3 + 0 \cdot 1/3} \\
&= \frac{1}{3}.
\end{aligned}$$

(v) What is the probability of ***A*** being executed after he knows that ***C*** is excluded?
Answer:

As shown in the posterior probability, it's still $\frac{1}{3}$.

(vi) Did the janitor express any information about ***A's*** destiny?
Answer:

No

(vii) Elucidate how the Bayes rule is useful to you in this scenario.
Answer:

Helps decompose complicated problems into tractable parts, especially to determine the probability of an event A after observing the happening of event B.