1 Bayes Theorem

1.1 Concepts

1. We use **Bayes theorem** when we want to find the probability of A given B but we are told the opposite probability, the probability of B given A. There are several forms of Bayes Theorem as follows:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})} = \frac{1}{1 + \frac{P(B|\bar{A})P(\bar{A})}{P(B|A)P(A)}}.$$

   In order to discern which form to use, look at the information you are given. If you are told $P(B|A)$ as well as $P(B|\bar{A})$, use the latter two methods but if you are only told $P(B)$, then use the first form.

   We say that two events A, B are **independent** if $P(A \cap B) = P(A)P(B)$.

1.2 Examples

2. There are 10 red and 10 blue balls in a bag. Someone randomly picks out a ball and then places it back and puts 10 more balls of that color into the bag. Then you draw a ball. What is the probability that the 10 balls added were red, given that you drew out a red ball?

Solution: We use Bayes Theorem to get

$$P(AddRed|DrawRed) = \frac{1}{1 + \frac{P(DrawRed|AddBlue)P(AddBlue)}{P(DrawRed|AddRed)P(AddRed)}}$$

$$= \frac{1}{1 + \frac{10/30 \cdot 1/2}{20/30 \cdot 1/2}} = \frac{1}{1 + 1/2} = \frac{2}{3}.$$

3. Out of those brought to court, there are 60% which are actually guilty. Of those that are guilty, 95% of them are convicted. But there are 1% of innocent people who get falsely convicted. What is the probability that you are actually innocent given that you are convicted?

Solution: We use Bayes rule which tells us that $P(Innocent|Convicted)$

$$= \frac{P(Convicted|Innocent)P(Innocent)}{P(Convicted|Innocent)P(Innocent) + P(Convicted|Guilty)P(Guilty)}$$

$$= \frac{0.01 \cdot 0.4}{0.01 \cdot 0.4 + 0.95 \cdot 0.6} \approx 0.7\%$$

1.3 Problems

4. True **FALSE** We can always use the formula $P(A|B) = \frac{1}{1+\frac{P(B|\bar{A})P(\bar{A})}{P(B|A)P(A)}}$.

Solution: This is undefined if $P(B) = 0$.

5. I have two boxes of apples and oranges. In box 1, there are 5 oranges and 6 apples, in box 2 there are 6 oranges and 5 apples. I randomly pick a box and then in this box randomly pick a fruit. What is the probability that I picked box 1 given that I picked an orange?

Solution: We have that

$$P(Box1|Orange) = \frac{P(Box1 \cap Orange)}{P(Orange)}$$

$$= \frac{P(Orange|Box1)P(Box1)}{P(Orange|Box1)P(Box1) + P(Orange|Box2)P(Box2)} = \frac{5/11 \cdot 1/2}{5/11 \cdot 1/2 + 6/11 \cdot 1/2} = \frac{5}{11}.$$

6. An exam has a 99% chance of testing positive if you have the disease and 1% chance of testing positive if you do not have the disease. Give that 0.5% of people have this disease, what is the probability that you have the disease given that you tested positive?

Solution:

$$P(Sick|Positive) = \frac{P(Positive|Sick)P(Sick)}{P(Positive|Sick)P(Sick) + P(Positive|Notsick)P(Notsick)}$$

$$= \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995} \approx 0.33.$$

7. About 2/3 of drivers use their cell phone while driving. Suppose that you are 5 times more likely to get into an accident if you text and drive, and if you don't use your cell phone, you have a 1% chance of getting into an accident. What is the probability that someone was texting given that they got into an accident?

Solution: We have that

$$P(Text|Accident) = \frac{P(Accident|Text)P(Text)}{P(Accident|Text)P(Text) + P(Accident|NoText)P(NoText)}$$

$$= \frac{0.05 \cdot 2/3}{0.05 \cdot 2/3 + 0.01 \cdot 1/3} = \frac{10}{11}.$$