AI1103 : Assignment 2

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Download all python codes from

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https://github.com/prashanthsriram-s/AI1103/tree/
    main/Assignment2/codes/
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and latex codes from

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https://github.com/prashanthsriram-s/AI1103/tree/
    main/Assignment2/Assignment2.tex
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Problem statement(GATE 67)

Let X and Y be random variables having the joining probability density function

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & -\infty < x < \infty, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

The Variance of the random variable X is

a) $\frac{1}{12}$

b) $\frac{1}{4}$

c) $\frac{7}{12}$

d) $\frac{5}{12}$

Solution(GATE 67)

Variance of the random variable X is

$$V(X) = E\left(X^2\right) - (E(X))^2 \tag{0.0.2}$$

Lemma 0.1.

$$\int_{-\infty}^{\infty} xe^{-\frac{1}{2y}(x-y)^2}\, dx = \sqrt{2\pi}y^{\frac{3}{2}} \tag{0.0.3}$$

Proof.

$$\int_{-\infty}^{\infty} xe^{-\frac{1}{2y}(x-y)^2}\, dx \tag{0.0.4}$$

$$= \int_{-\infty}^{\infty} (x-y)\, e^{-\frac{1}{2y}(x-y)^2}\, dx + y\int_{-\infty}^{\infty} e^{-\frac{1}{2y}(x-y)^2}\, dx \tag{0.0.5}$$

$$=0 + \sqrt{2\pi}y^{\frac{3}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{\frac{-1}{2(\sqrt{y})^2}(x-y)^2}\, dx \tag{0.0.6}$$

$$= \sqrt{2\pi}y^{\frac{3}{2}} \lim_{x_0\to-\infty} Q\left(\frac{x_0 - y}{\sqrt{y}}\right) \tag{0.0.7}$$

$$= \sqrt{2\pi}y^{\frac{3}{2}} \tag{0.0.8}$$

□

Lemma 0.2.

$$E(X) = \frac{1}{2} \tag{0.0.9}$$

Proof.

$$E(X) = \int_0^1 \int_{-\infty}^{\infty} x f_{XY}(x,y)\, dx\, dy \tag{0.0.10}$$

$$= \int_0^1 \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2}\, dx\, dy \tag{0.0.11}$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} xe^{-\frac{1}{2y}(x-y)^2}\, dx\right) dy \tag{0.0.12}$$

From (0.0.12) and (0.0.8),

$$E(X) = \int_0^1 y\, dy \tag{0.0.13}$$

$$E(X) = \frac{1}{2} \tag{0.0.14}$$

□

Lemma 0.3.

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2}\, dx = \sqrt{2\pi}y^{\frac{3}{2}}(y+1) \tag{0.0.15}$$

Proof.

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2}\, dx \tag{0.0.16}$$

$$= \left(\sqrt{\frac{\pi}{2}} y^{\frac{3}{2}} (y+1) \left(1 - 2Q\left(\frac{x-y}{\sqrt{y}}\right)\right) - y e^{\frac{-(x-y)^2}{2y}} (x+y)\right) \Big]_{-\infty}^{\infty} \tag{0.0.17}$$

$$= 0 - \sqrt{2\pi} y^{\frac{3}{2}} (y+1)\, Q\left(\frac{x-y}{\sqrt{y}}\right) \Big]_{-\infty}^{\infty} - 0 \tag{0.0.18}$$

$$= \sqrt{2\pi} y^{\frac{3}{2}} (y+1) \tag{0.0.19}$$

□

Lemma 0.4.

$$E\left(X^2\right) = \frac{5}{6} \tag{0.0.20}$$

Proof.

$$E\left(X^2\right) = \int_0^1 \int_{-\infty}^{\infty} x^2 f_{XY}(x, y)\; dx\, dy \tag{0.0.21}$$

$$= \int_0^1 \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2}\, dx\, dy \tag{0.0.22}$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2}\, dx \right) dy \tag{0.0.23}$$

From (0.0.23) and (0.0.19), we get

$$E\left(X^2\right) = \int_0^1 y(y+1)\, dy \tag{0.0.24}$$

$$= \left(\frac{y^3}{3} + \frac{y^2}{2}\right) \Big]_0^1 \tag{0.0.25}$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \tag{0.0.26}$$

□

From (0.0.14) and (0.0.26), we get

$$V(X) = E\left(X^2\right) - (E(X))^2 \tag{0.0.27}$$

$$= \frac{5}{6} - \frac{1}{4} \tag{0.0.28}$$

$$= \frac{7}{12} \tag{0.0.29}$$

Therefore, the answer is (C)$\frac{7}{12}$