AI1103 : Assignment 2

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Download all python codes from

https://github.com/prashanthsriram−s/AI1103/tree/main/Assignment2/codes/

and latex codes from

https://github.com/prashanthsriram−s/AI1103/tree/main/Assignment2/Assignment2.tex

Problem statement(GATE 67)

Let X and Y be random variables having the joining probability density function

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2} & -\infty < x < \infty, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{0.0.1}$$

The Variance of the random variable X is

(A) $\frac{1}{12}$ (C) $\frac{7}{12}$

(B) $\frac{1}{4}$ (D) $\frac{5}{12}$

Solution(GATE 67)

Variance of the random variable X is $E\left(X^2\right) - \left(E\left(X\right)\right)^2$

Lemma 0.1. $E(X) = \frac{1}{2}$

Proof.

$$E\left(X\right) = \int_0^1 \int_{-\infty}^{\infty} x f_{XY}(x,y)\,dx\,dy \tag{0.0.2}$$

$$= \int_0^1 \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2}\,dx\,dy \tag{0.0.3}$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx\right) dy \tag{0.0.4}$$

Lemma 0.2. $\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx = \sqrt{2\pi} y^{\frac{3}{2}}$

Proof.

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2}\,dx \tag{0.0.5}$$

$$= \int_{-\infty}^{\infty} (x-y) e^{-\frac{1}{2y}(x-y)^2}\,dx + y \int_{-\infty}^{\infty} e^{-\frac{1}{2y}(x-y)^2}\,dx \tag{0.0.6}$$

$$= 0 + \sqrt{2\pi} y^{\frac{3}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{y}\sqrt{2\pi}} e^{\frac{-1}{2(\sqrt{y})^2}(x-y)^2}\,dx \tag{0.0.7}$$

$$= \sqrt{2\pi} y^{\frac{3}{2}} \lim_{x_0 \to -\infty} Q\left(\frac{x_0 - y}{\sqrt{y}}\right) \tag{0.0.8}$$

$$= \sqrt{2\pi} y^{\frac{3}{2}} \tag{0.0.9}$$

□

From (0.0.4) and (0.0.9),

$$E\left(X\right) = \int_0^1 y\,dy \tag{0.0.10}$$

$$E\left(X\right) = \frac{1}{2} \tag{0.0.11}$$

□

Lemma 0.3. $E(X^2) = \frac{5}{6}$

Proof.

$$E\left(X^2\right) = \int_0^1 \int_{-\infty}^{\infty} x^2 f_{XY}(x,y)\,dx\,dy \tag{0.0.12}$$

$$= \int_0^1 \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi y}} e^{\frac{-1}{2y}(x-y)^2}\,dx\,dy \tag{0.0.13}$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2}\,dx\right) dy \tag{0.0.14}$$

Lemma 0.4. $\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2}\,dx = \sqrt{2\pi} y^{\frac{3}{2}}(y+1)$

Proof.

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2}\, dx \tag{0.0.15}$$

$$= (\sqrt{\frac{\pi}{2}} y^{\frac{3}{2}}(y+1)(1-2Q(\frac{x-y}{\sqrt{y}})) - ye^{\frac{-(x-y)^2}{2y}}(x+y))]_{-\infty}^{\infty} \tag{0.0.16}$$

$$= 0 - \sqrt{2\pi} y^{\frac{3}{2}}(y+1)Q(\frac{x-y}{\sqrt{y}})]_{-\infty}^{\infty} - 0 \tag{0.0.17}$$

$$= \sqrt{2\pi} y^{\frac{3}{2}}(y+1) \tag{0.0.18}$$

□

From (0.0.14) and (0.0.18), we get

$$E(X^2) = \int_0^1 y(y+1)dy \tag{0.0.19}$$

$$= (\frac{y^3}{3} + \frac{y^2}{2})]_0^1 \tag{0.0.20}$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \tag{0.0.21}$$

□

From (0.0.11) and (0.0.21), we get

$$V(X) = E(X^2) - (E(X))^2 \tag{0.0.22}$$

$$= \frac{5}{6} - \frac{1}{4} \tag{0.0.23}$$

$$= \frac{7}{12} \tag{0.0.24}$$

Therefore, the answer is (C)$\frac{7}{12}$