

AI1103 : Assignment 2

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Download all python codes from

<https://github.com/prashanthssriram-s/AI1103/tree/main/Assignment2/codes/>

and latex codes from

<https://github.com/prashanthssriram-s/AI1103/tree/main/Assignment2/Assignment2.tex>

Proof.

$$E(X) = \int_0^1 \int_{-\infty}^{\infty} x f_{XY}(x, y) dx dy \quad (0.0.4)$$

$$= \int_0^1 \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2y}(x-y)^2} dx dy \quad (0.0.5)$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2} dx \right) dy \quad (0.0.6)$$

PROBLEM STATEMENT(GATE 67)

Let X and Y be random variables having the joining probability density function

$$f(x, y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2y}(x-y)^2} & -\infty < x < \infty, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \quad (0.0.1)$$

The Variance of the random variable X is

- a) $\frac{1}{12}$
- b) $\frac{1}{4}$
- c) $\frac{7}{12}$
- d) $\frac{5}{12}$

SOLUTION(GATE 67)

Variance of the random variable X is

$$V(X) = E(X^2) - (E(X))^2 \quad (0.0.2)$$

Lemma 0.1.

$$E(X) = \frac{1}{2} \quad (0.0.3)$$

Lemma 0.2.

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2} dx = \sqrt{2\pi y}^{\frac{3}{2}} \quad (0.0.7)$$

Proof.

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2y}(x-y)^2} dx \quad (0.0.8)$$

$$= \int_{-\infty}^{\infty} (x - y) e^{-\frac{1}{2y}(x-y)^2} dx + y \int_{-\infty}^{\infty} e^{-\frac{1}{2y}(x-y)^2} dx \quad (0.0.9)$$

$$= 0 + \sqrt{2\pi y}^{\frac{3}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{y} \sqrt{2\pi}} e^{-\frac{1}{2(\sqrt{y})^2}(x-y)^2} dx \quad (0.0.10)$$

$$= \sqrt{2\pi y}^{\frac{3}{2}} \lim_{x_0 \rightarrow -\infty} Q\left(\frac{x_0 - y}{\sqrt{y}}\right) \quad (0.0.11)$$

$$= \sqrt{2\pi y}^{\frac{3}{2}} \quad (0.0.12)$$

□

From (0.0.6) and (0.0.12),

$$E(X) = \int_0^1 y dy \quad (0.0.13)$$

$$E(X) = \frac{1}{2} \quad (0.0.14)$$

□

Lemma 0.3.

$$E(X^2) = \frac{5}{6} \quad (0.0.15)$$

Proof.

$$E(X^2) = \int_0^1 \int_{-\infty}^{\infty} x^2 f_{XY}(x, y) dx dy \quad (0.0.16)$$

$$= \int_0^1 \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2y}(x-y)^2} dx dy \quad (0.0.17)$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi y}} \left(\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2} dx \right) dy \quad (0.0.18)$$

Lemma 0.4.

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2} dx = \sqrt{2\pi y^{\frac{3}{2}}} (y+1) \quad (0.0.19)$$

Proof.

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2y}(x-y)^2} dx \quad (0.0.20)$$

$$= \left(\sqrt{\frac{\pi}{2}} y^{\frac{3}{2}} (y+1) \left(1 - 2Q\left(\frac{x-y}{\sqrt{y}}\right) \right) - y e^{-\frac{(x-y)^2}{2y}} (x+y) \right) \Big|_{-\infty}^{\infty} \quad (0.0.21)$$

$$= 0 - \sqrt{2\pi y^{\frac{3}{2}}} (y+1) Q\left(\frac{x-y}{\sqrt{y}}\right) \Big|_{-\infty}^{\infty} - 0 \quad (0.0.22)$$

$$= \sqrt{2\pi y^{\frac{3}{2}}} (y+1) \quad (0.0.23)$$

□

From (0.0.18) and (0.0.23), we get

$$E(X^2) = \int_0^1 y(y+1) dy \quad (0.0.24)$$

$$= \left(\frac{y^3}{3} + \frac{y^2}{2} \right) \Big|_0^1 \quad (0.0.25)$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \quad (0.0.26)$$

□

From (0.0.14) and (0.0.26), we get

$$V(X) = E(X^2) - (E(X))^2 \quad (0.0.27)$$

$$= \frac{5}{6} - \frac{1}{4} \quad (0.0.28)$$

$$= \frac{7}{12} \quad (0.0.29)$$

Therefore, the answer is (C) $\frac{7}{12}$