Assignment 3 - (AS3)=UGC / MATH (mathA June 2017), Q.118

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Question

Suppose the random variable X has the following probability density function

$$f(x) = \begin{cases} \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^{\alpha}} & x > \mu \\ 0 & x \le \mu, \end{cases} \tag{1}$$

Where $\alpha > 0, -\infty < \mu < \infty$. Which of the following statements are correct? The Hazard function of X is

1. A) an increasing function for all $\alpha > 0$
2. B) a decreasing function for all $\alpha > 0$
3. C) an increasing function for some $\alpha > 0$
4. D) a decreasing function for some $\alpha > 0$

Required Concepts

Let T be a non-negative random variable representing the waiting time until the occurrence of an event. For simplicity, we will take the event as death and the waiting time as Survival time.

Survival Function

Let us assume that T is a continuous random variable with p.d.f $f(t)$ and c.d.f $F(t)$

$$S(t) = \Pr\{T \geq t\} = 1 - F(t) = \int_{x}^{\infty} f(x)dx \tag{2}$$

Required Concepts

Hazard Function

Hazard function gives the instantaneous rate of occurrence of the event

$$\lambda(t) = \lim_{dt \to 0} \frac{\Pr\{t \leq T \leq t + dt | T \geq t\}}{dt} \tag{3}$$

$$= \lim_{dt \to 0} \frac{\Pr\{t \leq T \leq t + dt\}}{S(t)dt} \tag{4}$$

$$= \frac{f(t)}{S(t)} \tag{5}$$

Required Concepts

Hazard Function

From (2),

$$\frac{d}{dt}S(t) = -f(t) \tag{6}$$

From (5) and (6)

$$\lambda(t) = -\frac{d}{dt}log(S(t)) \tag{7}$$

Integrating on both sides from 0 to t and using $S(0) = 1$,

$$-\int_0^t \lambda(x)dx = log(S(t)) - 0 \tag{8}$$

Required Concepts

Hazard Function

$$S(t) = e^{-\int_0^t \lambda(x)dx} \tag{9}$$

From (20) and (5), we can find $\lambda(t)$

Solving the question

Finding the Survival Function

$$f(x) = \begin{cases} \alpha(x-\mu)^{\alpha-1}e^{-(x-\mu)^{\alpha}} & x > \mu \\ 0 & x \leq \mu, \end{cases} \tag{10}$$

Lemma

$$S(x) = \begin{cases} e^{-(x-\mu)^{\alpha}} & x > \mu \\ 1 & x \leq \mu \end{cases} \tag{11}$$

Finding Survival function

Proof.

$$\int f(t)dt = \int \alpha(t-\mu)^{\alpha-1} e^{-(t-\mu)^{\alpha}} dt \tag{12}$$

$$= -e^{-(t-\mu)^{\alpha}} + C \tag{13}$$

If $x > \mu$,

$$S(x) = \int_x^{\infty} \alpha(t-\mu)^{\alpha-1} e^{-(t-\mu)^{\alpha}} dt \tag{14}$$

$$= -e^{-(t-\mu)^{\alpha}}]_x^{\infty} \tag{15}$$

$$= e^{-(x-\mu)^{\alpha}} \tag{16}$$

□

Finding Survival function

Proof.

If $x \leq \mu$,

$$S(x) = \int_{x}^{\mu} f(t)dt + \int_{\mu}^{\infty} f(t)dt \tag{17}$$
$$= 0 + e^{-(\mu-\mu)^{\alpha}} \tag{18}$$
$$= 1 \tag{19}$$

□

From (20) and (16), the lemma is proved

Finding Hazard function

Finding $\lambda(x)$

Using (16) and (10), we get

$$\lambda(x) = \begin{cases} \alpha(x-\mu)^{\alpha-1} & x > \mu \\ 0 & x \leq \mu \end{cases} \tag{20}$$

Analysing this function for different values of α

So,
if $\alpha > 1$, $\lambda(x)$ is an increasing function

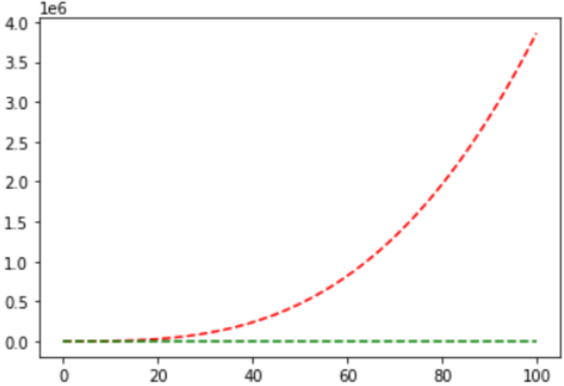

Figure: $\alpha = 2$ for red. $\alpha = 1$ for green, $\mu = 1$ for both

Analysing this function for different values of α

if $0 < \alpha < 1$, $\lambda(x)$ is a decreasing function

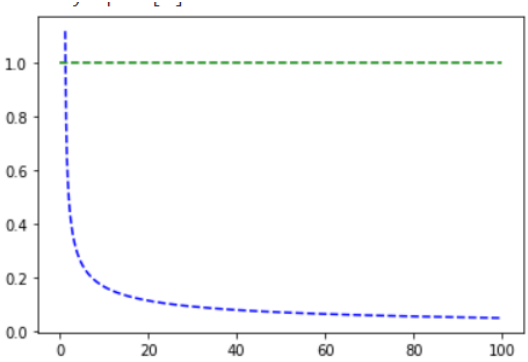

Figure: $\alpha = 0.5$ for blue. $\alpha = 1$ for green, $\mu = 1$ for both

For $\alpha = 1$, $\lambda(x) = 1$, a constant function.

Conclusion

So, for some values of α, it is an increasing, for some it is a decreasing function
Hence, the answer is (C) and (D)