Budget and Cost Constraints

Suppose that an individual buys x_1 units of a good G_1 and x_2 units a good G_2. If the prices of the goods are p_1 and p_2 respectively, and the individual has a fixed budget B, then the equation $p_1x_1 + p_2x_2 = B$ is called the budget constraint. It shows the combinations of two goods that it is possible to buy with a given amount of money and a given set of prices.Graphically, the equation represents a straight line called the budget line. The slope of the budget constraint is $m = -\frac{p_1}{p_2}$ from which we can conclude that

- if the price ratio changes the slope of the budget line changes,
- if the budget alters but the prices remain unchanged, the slope of the budget linedoes not alter.

In the similar fashion, let us assume that a firm wants to maximize output and that the production function is of the form f(K, L) where K is the number of units of capital and L is the number of units of labor. If p_K denotes the cost of each unit of capital and p_K denotes the cost of each unit of labor, the cost to the firm of using as input K units of capital and L units of labor is $p_KK + p_LL$. If the firm has fixed amount C to spend on these inputs, then $Lp_L + Kp_k = C$ is called the cost constraint (or budget constraint) for the firm.

EXAMPLE 2

The total cost function of a firm is $TC = 3x^2 + 2xy + 7y^2$ where x and y denote the number of items of goods G_x and G_y, respectively, that are produced. Find the values of x and y which minimize costs if a total of 40 goods must be produced.

Solution:

Here, $TC = 3x^2 + 2xy + 7y^2$

Since, a total of 40 goods must be produced,

$x + y = 40 \qquad y = 40 - x$

Thus $TC = 3x^2 + 2x(40 - x) + 7(40 - x)^2 = 8x^2 - 480x + 11200$

Differentiating

$TC' = 16x - 480 TC'' = 16$

Since $TC'' > 0$, for the maximum value of TC,

$TC' = 0$

$16x - 480 = 0$

$x = 3$

$y = 40 - 3 = 37$

Hence, the required values are: $x = 3$, $y = 37$.

Exercise 23.5

1. A consumer has an income of Rs. 2000 to spend on the two goods G_1 and G_2 with prices are Rs 200 and Rs 50 each, respectively.
   a. Formulate and graph the consumer's budget constraint.
   b. What is the slope of the constraint?
   c. What happens to the slope if the price of G_2riss to Rs.100?
   d. What happens if the income then falls to Rs.1500?
2. A firm has a budget of Rs.80,000 per week to spend on the two inputs K and L. One week it is observed to buy 120 units of L and 25 of K. Another week it is cbserved to buy 80 units of L and 50 of K. Find out the prices of K and L, which are assumed to be unchanged from one week to the next.
3. If a consumer's income doubles and the prices of the two goods that the consumerspends the entire income on also double, what happens to the budgetconstraint?
4. Find the minimum value of $3x^2 + 2xy + y^2$ subject to $x + y = 40$.
5. A firm's unit capital and labor costs are Rs.100 and Rs.200, respectively. If the production function is given by $Q = 4LK + L^2$. Find the maximum output and levels of K and L at which it is achieved when total input costs are fixed at Rs. 21000.
6. An individual's utility function is given by $U = x_1x_2$, where x_1 and x_2 denote the number of items of goods G_1 and G_2. The prices of the goods are Rs. 2 and Rs.10 respectively. Assuming that the individual has Rs. 400 available to spend on these good, find the utility-maximizing values of x_1 and x_2.
7. A firm's production function is given by $Q = 2K^{1/2}L^{1/2}$. Unit capital and labor costs are \$4 and \$3 respectively. Find the values of K and L which minimizes the total input costs if the firm is contracted to provide 160 units of output.

☑ Answers:

1. a. $4x_1 + x_2 = 40$, b. -4 c. slope increases to -2
   d. slope remains the same, budget line moves towards the origin.
2. Rs. 800, Rs. 500
3. The budget constraint does not change
4. 1600
5. Rs. 25,200, K = 90, L = 60
6. $x_1 = 100$, $x_2 = 20$
7. $K = 40\sqrt{3}$, $L = 160/\sqrt{3}$

Exercise 23.5

1. A consumer has an income of Rs. 2000 to spend on the two goods G_1 and G_2 with prices are Rs 200 and Rs 50 each, respectively.
   a. Formulate and graph the consumer's budget constraint.
   b. What is the slope of the constraint?
   c. What happens to the slope if the price of G_2 rises to Rs 100?
   d. What happens if the income then falls to Rs1500?

Solution:

a. Given, G_1 and G_2 be two types of good with prices is 200 and Rs.50 respectively. Also, a consumers has an income = Rs.2000.

Let x_1 and x_2 be the quantity of goods of type G_1 and G_2 respectively. Then total price will be $200x_1 + 50x_2$.

Since, budget is limited to Rs.2000, the required budget constraint is given by

$200x_1 + 50x_2 = 2000$

i.e. $4x_1 + x_2 = 40$

b. We have budget equation

$$200x_1 + 50x_2 = 2000$$

$$\therefore \text{Slope (m)} = -\frac{\text{Coefficient of } x_1}{\text{Coefficient of } x_2} = -\frac{200}{50} = -4$$

c. When the price of good G_2 rises to Rs.100 then budget equation will be

$$200x_1 + 100x_2 = 2000$$

$$\therefore \text{Slope} = -\frac{200}{100} = -2$$

$\therefore$ Slope increases to –2.

d. When income falls to Rs.1500, then budget constraint becomes

$$200x_1 + 50x_2 = 1500$$

$$\therefore \text{Slope} = -\frac{200}{50} = -4$$

$\therefore$ Slope remains same.

2. A firm has a budget of Rs80,000 per week to spend on the two inputs K and L. One week it is observed to buy 120 units of L and 25 of K. Another week it is observed to buy 80 units of L and 50 of K. Find out the prices of K and L, which are assumed to be unchanged from one week to the next.

Then, budget equation is given by

120L + 25K = 80000 ... (i)

Also, on another week it is observed to buy 80 units of L and 50 units of K, then

80L + 50K = 80000 ... (ii)

Solving (i) and (ii), we get

$$
\begin{array}{rcrcl}
240L & + & 50K & = & 160000 \\
80L & + & 50k & = & 80000 \\
- & & - & & - \\
\hline
& & 160L & = & 80000
\end{array}
$$

$$L = \frac{80000}{160} = 500$$

Substituing the value of L in equation (i) we get,

K = 800

Therefore, the prices of K and L are Rs. 800 and Rs.500 respectively.

3. If a consumer's income doubles and the prices of the two goods that the consumer spends the entire income on also double, what happens to the budget constraint?

Solution:

Let Rs. y be the income of a certain consumer, which need to be consumed on two goods costing Rs. x_1 and Rs. x_2. Then budget constraint is written as

$ax_1 + bx_2 = y$ ... (i)

Where a and b are respective quantities of two goods.

According to question, when consumeers income doubles and the price of two goods also double, then budget constrain becomes

$a(2x_1) + b(2x_2) = 2y$

or, $ax_1 + bx_2 = y$ ... (ii)

∴ Equation (i) and (ii) shows budget constriant doesn't change.

4. Find the minimum value of $3x^2 + 2xy + y^2$ subject to $x + y = 40$.

Solution:

Let $f(x, y) = 3x^2 + 2xy + y^2$ with constraint $x + y = 40$

$\therefore f(x) = 3x^2 + 2x(40 - x) + (40 - x)^2$ [$\because y = 40 - x$]

Differentating bothd sides w.r.to x

$f'(x) = 6x + 80 - 4x - 2(40 - x) = 2x + 80 - 80 + 2x = 4x$

Again, Differentating w.r.to x

$f''(x) = 4 > 0$ (case of minima)

For minima, $f'(x) = 0$

i.e. $4x = 0$

$x = 0$

Putting the value of x in $x + y = 40$, we get $y = 40$

$\therefore f(x, y) = 3x^2 + 2xy + y^2$ gives minimum value at $x = 0$ and $y = 40$

$\therefore$ The minimum value is $3 \times 10^2 + 2 \times 0 \times 40 + 40^2 = 1600$

5. A firm's unit capital and labor costs are Rs100 and Rs200, respectively. If the production function is given by $Q = 4LK + L^2$. Find the maximum output and levels of K and L at which it is achieved when total input costs are fixed at Rs. 21000.

Solution:

Given, production function

$Q = 4LK + L^2$

Also, it is given that unit labor and capital costs are Rs.200 and Rs.100 respeectively.

Sincce total input cost is Rs.21000, the constraint is given by

$200L + 100K = 21000$

i.e. $2L + K = 210$

$K = 210 - 2L$

Then, $Q = 4L(210 - 2L) + L^2 = 840L - 8L^2 + L^2$

$Q = 840L - 7L^2$

Differentating both sides w.r.to L

$Q' = 840 - 14L$

Again Differentating w.r.to L

$Q'' = -14$

i.e. $840 - 14L = 0$

$L = 60$

When $L = 60$ Then $K = 210 - 2L = 210 - 120 = 90$

Since $Q'' < 0$, the production function is maximum when $L = 60$ and $K = 90$.

$\therefore$ Maximum $Q = 4LK + L^2 = 4 \times 60 \times 90 + 60^2 = 21600 + 3600 = 25,200$

6. An individual's utility function is given by $U = x_1x_2$, where x_1 and x_2 denote the number of items of goods G_1 and G_2. The prices of the goods are Rs. 2 and Rs.10 respectively. Assuming that the individual has Rs. 400 available to spend on these good, find the utility-maximizing values of x_1 and x_2.

Solution:

Given utility function $U = x_1 x_2$ ... (i)

Also, $2x_1 + 10x_2 = 400$

$x_1 + 5x_2 = 200$ $\qquad$ $x_1 = 200 - 5x_2$ ... (ii)

Then (i) becomes

$U = (200 - 5x_2)\, x_2 = 200x_2 - 5x_2^2$

Differentating w.r. to x_2

$U'' = -10$

For maximum or minimum $U' = 0$

i.e. $200 - 10x_2 = 0$

$\therefore\ x_2 = 20$

From (ii) $x_1 = 200 - 5x_2 = 200 - 5 \times 20 = 100$

$\therefore\ x_1 = 100$ and $x_2 = 20$

From (ii) $x_1 = 200 - 5x_2 = 200 - 5 \times 20 = 100$

$\therefore\ x_1 = 100$ and $x_2 = 20$

Since, $U'' = -10 < 0$, utility function has maximum value when $x_1 = 100$ and $x_2 = 20$.

7. A firm's production function is given by $Q = 2K^{1/2}L^{1/2}$. Unit capital and labor costs are \$4 and \$3 respectively. Find the values of K and L which minimizes the total input costs if the firm is contracted to provide 160 units of output.

Solution:

Given production function

$Q = 2K^{\frac{1}{2}} L^{\frac{1}{2}}$, $Q = 2\sqrt{KL}$

Let C can be the total input cost.

Given that \$3 and \$4 are the costs of unit labor and unit capital respectively.

Then, $C = 3L + 4K$ ... (i)

But total output (Q) is 160 units

i.e. $2\sqrt{KL} = 160$

$\sqrt{KL} = 80$

$KL = 6400 = \frac{6400}{L}$ ... (ii)

From (i)

$C = 3L + 4 \times \frac{6400}{L} = 3L + \frac{25600}{L}$

Differentating w.r.to x L

$C' = 3 - \frac{25600}{L^2}$

For maximum or minimum, $C' = 0$

i.e. $3L^2 = 25600$

$L^2 = \frac{25600}{3} = \frac{160}{\sqrt{3}}$

From (ii), $K = 40\sqrt{3}$

Since, $C'' = \frac{51200}{L^3} > 0$ [$\because L = \frac{160}{\sqrt{3}}$]

$\therefore$ Cost is minimum

Hence cost is minimum at $L = \frac{160}{\sqrt{3}}$ and $K = 40\sqrt{3}$