TRIGONOMETRY

Exercise 8.1

1. In any Δ ABC, prove that

a. $2a\sin\frac{B}{2}.\operatorname{Sin}\frac{C}{2} = (b + c - a)\sin\frac{A}{2}$

b. $ac\cos B - bc\cos A = a^2 - b^2$

c. $a\cos^2\frac{B}{2} + b\cos^2\frac{A}{2} = s$

d. $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a+b+c$

e. $2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = c + a - b$

Solution:

a. R.H.S. $= (b + c - a)\sin\frac{A}{2} = (2R\sin B + 2R\sin C - 2R\sin A)\sin\frac{A}{2}$

$= 2R\sin\frac{A}{2}[\sin B + \sin C - \sin A] = 2R\sin\frac{A}{2}\left[2\sin\left(\frac{B+C}{2}\right).\cos\left(\frac{B-C}{2}\right) - 2\sin\frac{A}{2}.\cos\frac{A}{2}\right]$

$= 2R\sin\frac{A}{2}\left[2\cos\frac{A}{2}.\cos\left(\frac{B-C}{2}\right) - 2\sin\frac{A}{2}.\cos\frac{A}{2}\right] = 4R\sin\frac{A}{2}.\cos\frac{A}{2}\left[\cos\left(\frac{B-C}{2}\right) - \sin\frac{A}{2}\right]$

$= 2R\times\sin A\left[\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right)\right] = a\left[2\sin\frac{B}{2}.\sin\frac{C}{2}\right] = 2a\sin\frac{B}{2}.\sin\frac{C}{2}$

b. $$\text{L.H.S.} = ac\cos B - bc\cos A = ac\frac{a^2+c^2-b^2}{2ac} - bc\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2+c^2-b^2}{2} - \frac{b^2+c^2-a^2}{2} = \frac{a^2+c^2-b^2-b^2-c^2+a^2}{2} = \frac{2(a^2-b^2)}{2} = a^2-b^2$$

c. $$a\cos^2\frac{B}{2} + b\cos^2\frac{A}{2} = s$$

$$= a.\frac{s(s-b)}{ca} + b.\frac{s(s-a)}{bc} = \frac{s(s-b)}{c} + \frac{s(s-a)}{c} = \frac{s^2-sb+s^2-sa}{c} = \frac{2s^2-s(a+b)}{c}$$

$$= \frac{2s^2-s(2s-c)}{c} \qquad \left[s = \frac{a+b+c}{2} \Rightarrow 2s-c = a+b \right]$$

$$= \frac{2s^2-2s^2+sc}{c} = \frac{sc}{c} = s$$

d. $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$

L.H.S. $= (b + c)\cos A + (c + a)\cos B + (a + b)\cos C$

$= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C = a + b + c$ [using projection law]

e. $2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = c + a - b$

L.H.S. $= 2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = 2\left[a.\frac{(s-a)(s-b)}{ab} + c.\frac{(s-b)(s-c)}{bc}\right]$

$= 2\left[\frac{(s-a)(s-b)}{b} + \frac{(s-b)(s-c)}{b}\right] = \frac{2}{b}(s-b)[s-a+s-c]$

$= \frac{2(s-b)}{b}[2s-a-c] = \frac{2(s-b)}{b} \times [a+b+c-a-c] = \frac{2(s-b)}{b} \times b$

$= 2s - 2b = a + b + c - 2b = a - b + c$

2. Prove the followings.

a. $\dfrac{b-c}{a}\cos\dfrac{A}{2} = \sin\dfrac{B-C}{2}$

b. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$

c. $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{c\sin(A-B)}{a^2-b^2}$

Solution:

a. $\dfrac{b-c}{a}.\cos\dfrac{A}{2} = \sin\left(\dfrac{B-C}{2}\right)$

$$\text{L.H.S.} = \frac{b-c}{a}.\cos\frac{A}{2} = \frac{2R\sin B - 2R\sin C}{2R\sin A}.\cos\frac{A}{2} = \frac{\sin B - \sin C}{\sin A}.\cos\frac{A}{2}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}.\cos\frac{A}{2}} \times \cos\frac{A}{2} = \frac{\sin\frac{A}{2}.\sin\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}} = \sin\left(\frac{B-C}{2}\right)$$

b. $\dfrac{c - b\cos A}{b - c\cos A} = \dfrac{\cos B}{\cos C}$

$$\text{L.H.S.} = \frac{c - b\cos A}{b - c\cos A} = \frac{a\cos B + b\cos A - b\cos A}{a\cos C + c\cos A - c\cos A} = \frac{a\cos B}{a\cos C} = \frac{\cos B}{\cos C}$$

c. $\dfrac{a\sin(B - C)}{b^2 - c^2} = \dfrac{b\sin(C - A)}{c^2 - a^2} = \dfrac{c\sin(A - B)}{a^2 - b^2}$

$$\text{L.H.S.} = \frac{a\sin(B - C)}{b^2 - c^2} = \frac{2R\sin A.\sin(B - C)}{b^2 - c^2} = \frac{2R\sin(B + C).\sin(B - C)}{b^2 - c^2} = \frac{2R(\sin^2 B - \sin^2 C)}{b^2 - c^2}$$

$$= \frac{2R\sin^2 B - 2R\sin^2 C}{b^2 - c^2} = \frac{b\sin B - c\sin C}{b^2 - c^2} = \frac{b.\frac{b}{2R} - c.\frac{c}{2R}}{b^2 - c^2} = \frac{1}{2R} \times \frac{b^2 - c^2}{b^2 - c^2} = \frac{1}{2R}$$

$$\text{Now, M.H.S.} = \frac{b\sin(C - A)}{c^2 - a^2} = \frac{2R\sin B.\sin(C - A)}{c^2 - a^2} = \frac{2R\sin(C + A).\sin(C - A)}{c^2 - a^2}$$

$$= \frac{2R(\sin^2 C - \sin^2 A)}{c^2 - a^2} = \frac{2R\sin^2 C - 2R\sin^2 A}{c^2 - a^2} = \frac{c\sin C - a\sin A}{c^2 - a^2} = \frac{c.\frac{c}{2R} - a.\frac{a}{2R}}{c^2 - a^2} = \frac{1}{2R}$$

$$\text{Again, R.H.S.} = \frac{c\sin(A - B)}{a^2 - b^2} = \frac{2R\sin C.\sin(A - B)}{a^2 - b^2} = \frac{2R\sin(A + B).\sin(A - B)}{a^2 - b^2}$$

$$= \frac{2R(\sin^2 A - \sin^2 B)}{a^2 - b^2} = \frac{2R\sin^2 A - 2R\sin^2 B}{a^2 - b^2} = \frac{a\sin A - b\sin B}{a^2 - b^2} = \frac{a.\frac{a}{2R} - b.\frac{b}{2R}}{a^2 - b^2} = \frac{1}{2R}$$

Hence, L.H.S. = M.H.S. = R.H.S.

3. a. $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = \frac{1}{2}(a+b+c)$ b. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

c. $\sin(A+B) : \sin(A-B) = c^2 : (a^2 - b^2)$ d. $1 - \tan\frac{A}{2} \cdot \tan\frac{B}{2} = \frac{2c}{a+b+c}$

e. $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2 - c^2}{a^2}$

Solution:

a. $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = \frac{1}{2}(a+b+c)$

L.H.S. $= b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = b.\frac{s(s-c)}{ab} + c.\frac{s(s-b)}{ca} = \frac{s(s-c)}{a} + \frac{s(s-b)}{a} = \frac{s^2 - sc + s^2 - sb}{a}$

$= \frac{2s^2 - s(b+c)}{a} = \frac{2s^2 - s(2s-a)}{a} = \frac{2s^2 - 2s^2 + sa}{a} = \frac{sa}{a} = s = \frac{1}{2}(a+b+c)$

b. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

L.H.S.

$= a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= 2R\sin A.\sin(B-C) + 2R\sin B.\ \text{Sin}(C-A) + 2R\sin C.\sin(A-B)$

$= 2R[\sin(B+C).\sin(B-C) + \sin(C+A).\sin(C-A) + \sin(A+B).\sin(A-B)]$

$= 2R[\sin^2B - \sin^2C + \sin^2C - \sin^2A + \sin^2A - \sin^2B] = 2R \times 0 = 0$

c. $\sin(A+B) : \sin(A-B) = c^2 : (a^2 - b^2)$

L.H.S. $= \sin(A+B) : \sin(A-B) = \frac{\sin(A+B)}{\sin(A-B)}$

$= \frac{\sin C}{\sin(A-B)} \times \frac{\sin(A+B)}{\sin(A+B)} = \frac{\sin C \times \sin C}{\sin^2A - \sin^2B}$

$= \frac{\sin^2C}{\sin^2A - \sin^2B} = \frac{\frac{c^2}{4R^2}}{\frac{a^2}{4R^2} - \frac{b^2}{4R^2}} = \frac{c^2}{a^2 - b^2} = c^2 : (a^2 - b^2)$

d. $1 - \tan\frac{A}{2}.\tan\frac{B}{2} = \frac{2c}{a+b+c}$

L.H.S. $= 1 - \tan\frac{A}{2}.\tan\frac{B}{2} = 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = 1 - \sqrt{\frac{(s-a)(s-b)(s-c)^2}{s^2(s-a)(s-b)}}$

$= 1 - \frac{s-c}{s} = \frac{s-s+c}{s} = \frac{c}{\frac{a+b+c}{2}} = \frac{2c}{a+b+c}$

e. $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2-c^2}{a^2}$

L.H.S. $= \frac{\sin(B-C)}{\sin(B+C)} = \frac{\sin(B-C)}{\sin(B+C)} \times \frac{\sin(B+C)}{\sin(B+C)} = \frac{\sin^2B - \sin^2C}{\sin^2A} = \frac{\frac{b^2}{4R^2} - \frac{c^2}{4R^2}}{\frac{a^2}{4R^2}} = \frac{b^2-c^2}{a^2}$

4. Prove the following:

a. $\frac{a+b-c}{a+b+c} = \tan\frac{A}{2}\,.\,\tan\frac{B}{2}$

b. $\frac{b^2+c^2-a^2}{4\cot A} = \Delta$

c. $\frac{\cos A}{a} + \frac{a}{bc} = \frac{\cos B}{b} + \frac{b}{ca} = \frac{\cos C}{c} + \frac{c}{ab}$

d. $\frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(A-B)}{\sin C} = 0$

Solution:

a. $\frac{a+b-c}{a+b+c} = \tan\frac{A}{2}.\tan\frac{B}{2}$

R.H.S. $= \tan\frac{A}{2}.\tan\frac{B}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{(s-a)(s-b)(s-c)^2}{s^2(s-a)(s-b)}} = \frac{s-c}{s}$

$= \frac{\frac{a+b+c}{2} - c}{\frac{a+b+c}{2}} = \frac{a+b-c}{a+b+c}$

b. $\frac{b^2+c^2-a^2}{4\cot A} = \Delta$

L.H.S. $= \frac{b^2+c^2-a^2}{4\cot A} = \frac{2bc\cos A}{4\cos A} \times \sin A$ $\left[\cos A = \frac{b^2+c^2-a^2}{2bc} \Rightarrow 2bc\cos A = b^2+c^2-a^2\right]$

$= \frac{bc\sin A}{2} = \Delta$

c. $$\frac{\cos A}{a}+\frac{a}{bc}=\frac{\cos B}{b}+\frac{b}{ca}=\frac{\cos C}{c}+\frac{c}{ab}$$

$$\text{L.H.S.}=\frac{\cos A}{a}+\frac{a}{bc}=\frac{b^2+c^2-a^2}{2abc}+\frac{a}{bc}=\frac{b^2+c^2-a^2+2a^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$$

$$\text{Now, M.H.S.}=\frac{\cos B}{b}+\frac{b}{ca}=\frac{a^2+c^2-b^2}{2abc}+\frac{b}{ca}=\frac{a^2+c^2-b^2+2b^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$$

$$\text{Again, R.H.S.}=\frac{\cos C}{c}+\frac{c}{ab}=\frac{a^2+b^2-c^2}{2abc}+\frac{c}{ab}=\frac{a^2+b^2-c^2+2c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$$

Hence, L.H.S. = M.H.S. = R.H.S.

d. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0$

$$\text{L.H.S.} = \frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}$$

$$= 2R.a.\sin(B-C) + 2R.b.\sin(C-A) + 2R.c.\sin(A-B)$$

$$= 2R[2R\sin A.\sin(B-C) + 2R\sin B.\sin(C-A) + 2R\sin C.\sin(A-B)]$$

$$= 4R^2[\sin(B+C).\sin(B-C) + \sin(C+A).\sin(C-A) + \sin(A+B).\sin(A-B)]$$

$$= 4R^2[\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B] = 4R^2 \times 0 = 0$$

5. Prove the following:

a. $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{4R}{\Delta}$

b. $\tan^2\frac{A}{2}\cdot\tan^2\frac{B}{2}\cdot\tan^2\frac{C}{2}=\left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{s}\right)$

c. $\frac{b^2-c^2}{a^2}\text{Sin}2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

d. $a\cos A + b\cos B + c\cos C = 4R\sin A.\ \text{Sin}B.\ \text{Sin}C$

e. $(b+c-a)\left[\cot\frac{B}{2}+\cot\frac{C}{2}\right]=2a\cot\frac{A}{2}$

Solution:

a. $$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{4R}{\Delta}$$

$$\text{L.H.S.} =\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{1}{s-a}-\frac{1}{s}+\frac{1}{s-b}+\frac{1}{s-c}=\frac{s-s+a}{s(s-a)}+\frac{s-c+s-b}{(s-b)(s-c)}$$

$$=\frac{a}{s(s-a)}+\frac{a+b+c-c-b}{(s-b)(s-c)}=\frac{a}{\Delta\cot\frac{A}{2}}+\frac{a}{\Delta\tan\frac{A}{2}}$$

$$\text{Since, }\cot\frac{A}{2}=\frac{s(s-a)}{\Delta}$$

$$\tan\frac{A}{2}=\frac{(s-b)(s-c)}{\Delta}=\frac{a}{\Delta}\left[\tan\frac{A}{2}+\frac{1}{\tan\frac{A}{2}}\right]=\frac{a}{\Delta}\left[\frac{1+\tan^2\frac{A}{2}}{\tan\frac{A}{2}}\right]=\frac{a}{\Delta}\left[\frac{\sec^2\frac{A}{2}}{\tan\frac{A}{2}}\right]$$

$$=\frac{a}{\Delta}\frac{1}{\sin\frac{A}{2}.\sin\frac{A}{2}}=\frac{2a}{\sin A.\Delta}=\frac{2.2R}{\Delta}=\frac{4R}{\Delta}$$

b. $\text{Tan}^2\frac{A}{2}.\tan^2\frac{B}{2}.\tan^2\frac{C}{2} = \left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{s}\right)$

L.H.S. = $\text{Tan}^2\frac{A}{2}.\tan^2\frac{B}{2}.\tan^2\frac{C}{2}$

Now, $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Then, $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

Again, $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

So, $\frac{(s-b)(s-c)}{s(s-a)} \times \frac{(s-c)(s-a)}{s(s-b)} \times \frac{(s-a)(s-b)}{s(s-c)} = \frac{(s-a)(s-b)(s-c)}{s^3} = \left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{c}\right)$

c. $\frac{b^2-c^2}{a^2}.\sin 2A + \frac{c^2-a^2}{b^2}.\sin 2B + \frac{a^2-b^2}{c^2}.\sin 2C = 0$

$$\text{L.H.S.} = \frac{b^2-c^2}{a^2}.\sin 2A + \frac{c^2-a^2}{b^2}.\sin 2B + \frac{a^2-b^2}{c^2}.\sin 2C$$

$$= \frac{b^2-c^2}{a^2}.2\sin A.\cos A + \frac{c^2-a^2}{b^2}.2\sin B.\cos B + \frac{a^2-b^2}{c^2}.2\sin C.\cos C$$

$$= 2.\frac{b^2-c^2}{a^2}.\frac{a}{2R}.\frac{b^2+c^2-a^2}{2bc} + \frac{c^2-a^2}{b^2}.2.\frac{b}{2R}.\frac{a^2+c^2-b^2}{2ac} + \frac{a^2-b^2}{c^2}.2.\frac{c}{2R}.\frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{(b^2-c^2)(b^2+c^2-a^2)}{2abcR} + \frac{(c^2-a^2)(a^2+c^2-b^2)}{2abcR} + \frac{(a^2-b^2)(a^2+b^2-c^2)}{2abcR}$$

$$= \frac{(b^2-c^2)(b^2+c^2-a^2) + (c^2-a^2)(a^2+c^2-b^2) + (a^2-b^2)(a^2+b^2-c^2)}{2abcR} = \frac{0}{2abcR} = 0$$

d. $a\cos A + b\cos B + c\cos C = 4R\sin A.\sin B.\sin C$

L.H.S. $= a\cos A + b\cos B + c\cos C = 2R\sin A.\cos A + 2R\sin B.\cos B + 2R\sin C.\cos C$

$= R[\sin 2A + \sin 2B + \sin 2C]$

$= R[2\sin(A+B).\cos(A-B) + 2\sin C.\cos C] = R[2\sin C.\cos(A-B) + 2\sin C.\cos C]$

$= 2\sin C.R[\cos(A-B) - \cos(A+B)] = 4R\sin A.\sin B.\sin C$

e. $(b+c-a)\left[\cot\frac{B}{2}+\cot\frac{C}{2}\right]=2a\cot\frac{A}{2}$

$$\text{L.H.S.} = (b+c-a)\left[\cot\frac{B}{2}+\cot\frac{C}{2}\right]$$

$$= (a+b+c-2a)\left[\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}\right]$$

$$= (2s-2a)\left[\frac{(s-b)\sqrt{s}+(s-c)\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}\right] = 2(s-a)\sqrt{\frac{s}{(s-a)(s-b)(s-c)}}\,.\,[(s-b)+(s-c)]$$

$$= 2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\,.(a+b+c-b-c) = 2\cot\frac{A}{2}.a = 2a\cot\frac{A}{2}$$

6. In ΔABC, if $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ prove that $C = 60°$

Solution:

Given, $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

or, $\frac{b+c+a+c}{(a+c)(b+c)} = \frac{3}{a+b+c}$

or, $(a+b+2c)(a+b+c) = 3(a+c)(b+c)$

or, $a^2 + ab + ac + ab + b^2 + bc + 2ac + 2bc + 2c^2 = 3[ab + ac + bc + c^2]$

or, $a^2 + b^2 + 2c^2 + 2ab + 3ac + 3bc = 3ab + 3ac + 3bc + 3c^2$

or, $a^2 + b^2 - c^2 - ab = 0$

or, $a^2 + b^2 - c^2 = ab$

or, $\frac{a^2 + b^2 - c^2}{2ab} \times 2 = 1$

or, $\cos C = \frac{1}{2} = 60°$

7. In ΔABC, if $(a + b + c)(a - b - c) + 3bc = 0$, find A.

Solution:

Given, $(a + b + c)[a - (b + c)] + 3bc = 0$

or, $a^2 - (b + c)^2 + 3bc = 0$

or, $a^2 - b^2 - 2bc - c^2 + 3bc = 0$

or, $3bc - 2bc = b^2 + c^2 - a^2$

or, $bc = b^2 + c^2 - a^2$

or, $\frac{2bc}{2} = b^2 + c^2 - a^2$

or, $\frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}$

or, $\cos A = \frac{1}{2}$

$\therefore$ $A = 60°$

8. If $A = 2B$, then prove that either $b = c$ or $a^2 = b(c + b)$.

Solution:

We know, $\frac{a}{\sin A} = \frac{b}{\sin B}$

or, $\frac{a}{\sin 2B} = \frac{b}{\sin B}$

or, $\frac{a}{2\sin B.\cos B} = \frac{b}{\sin B}$

or, $\frac{a}{2b} = \cos B$ ....(i)

Again, $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ ....(ii)

From (i) and (ii), we get

$\frac{a}{2b} = \frac{a^2 + c^2 - b^2}{2ac}$

or, $a^2c = a^2b + bc^2 - b^3$

or, $a^2(c - b) - b(c^2 - b^2) = 0$

or, $(c - b)[a^2 - b(c + b)] = 0$

Either, $c - b = 0$

$\Rightarrow c = b$

or, $a^2 - b(c + b) = 0$

$\therefore \quad a^2 = b(c + b)$

9. If $a^4 + b^4 + c^4 - 2c^2(a^2 + b^2) + a^2.b^2 = 0$ show that $\angle C = 60°$ or $120°$

Solution:

Given, $a^4+b^4+c^4 - 2c^2(a^2 + b^2) + a^2.b^2 = 0$

or, $a^2b^2 = 2c^2a^2 + 2b^2c^2 - a^4 - b^4 - c^4$

or, $a^2b^2 + 2a^2b^2 = 2a^2b^2 + 2c^2a^2 + 2b^2c^2 - a^4 - b^4 - c^4$

or, $3a^2b^2 = 16\Delta^2$

$[\ \Delta = \frac{1}{4}\sqrt{2a^2b^2 + 2c^2a^2 + 2b^2c^2 - a^4 - b^4 - c^4}\ ]$

or, $3a^2b^2 = 16.\left(\frac{1}{2}ab\sin C\right)^2$

or, $3a^2b^2 = 16 \times \frac{1}{4} \times a^2b^2\sin^2C$

or, $\frac{3}{4} = \sin^2C$

or, $\sin C = \pm\frac{\sqrt{3}}{2}$

$\Longrightarrow$ $\sin C = \sin 60°, \sin 120°$

$\therefore$ $C = 60°$ or $120°$

10. In ΔABC if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in AP.

Solution:

Given, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$

or, $\sin A.\sin(B-C) = \sin C.\sin(A-B)$

or, $\sin(B+C).\sin(B-C) = \sin(A+B).\sin(A-B)$

or, $\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

or, $2\sin^2 B = \sin^2 A + \sin^2 C$

or, $2.\frac{b^2}{4R^2} = \frac{a^2}{4R^2} + \frac{c^2}{4R^2}$

or, $2b^2 = a^2 + c^2$

or, $b^2 = \frac{a^2 + c^2}{2}$

Hence, a^2, b^2 and c^2 are in AP.

11. If $\frac{\sin(A - B)}{\sin(A + B)} = \frac{a^2 - b^2}{a^2 + b^2}$ Prove that the ΔABC is either isosceles or right angled triangle.

Solution:

Given, $\frac{\sin(A - B)}{\sin(A + B)} = \frac{a^2 - b^2}{a^2 + b^2}$

or, $\frac{\sin(A - B)}{\sin(A + B)} \times \frac{\sin(A + B)}{\sin(A + B)} = \frac{a^2 - b^2}{a^2 + b^2}$

or, $\frac{\sin^2 A - \sin^2 B}{\sin^2 C} = \frac{a^2 - b^2}{a^2 + b^2}$

or, $\frac{\frac{a^2}{4R^2} - \frac{b^2}{4R^2}}{\frac{c^2}{4R^2}} = \frac{a^2 - b^2}{a^2 + b^2}$

or, $\frac{a^2 - b^2}{c^2} = \frac{a^2 - b^2}{a^2 + b^2}$

or, $(a^2 - b^2)(a^2 + b^2) - (a^2 - b^2)c^2 = 0$

or, $(a^2 - b^2)(a^2 + b^2 - c^2) = 0$

Either, $a^2 = b^2 \Rightarrow a = b$

This implies the triangle is an isosceles triangle.

Or, $a^2 + b^2 - c^2 = 0 \Rightarrow a^2 + b^2 = c^2$

Which is the Pythagoras theorem with right angled at C.

Which implies the triangle is right angled triangle.

12. In ΔABC, if $a \cos A = b \cos B$ show that the triangle is either isosceles or right angled.

Solution:

or, $a \cos A = b \cos B$

or, $a.\dfrac{b^2 + c^2 - a^2}{2bc} = b.\dfrac{a^2 + c^2 - b^2}{2ac}$

or, $a^2(b^2 + c^2 - a^2) = b^2(a^2 + c^2 - b^2)$

or, $a^2b^2 + a^2c^2 - a^4 = a^2b^2 + b^2c^2 - b^4$

or, $a^2c^2 - b^2c^2 - a^4 + b^4 = 0$

or, $c^2(a^2 - b^2) - [(a^2)^2 - (b^2)^2] = 0$

or, $c^2(a^2 - b^2) - [(a^2 + b^2)(a^2 - b^2)] = 0$

or, $(a^2 - b^2)[c^2 - (a^2 + b^2)] = 0$

Either, $a^2 - b^2 = 0 \Rightarrow a = b$.

This implies that the triangle is an isosceles triangle.

or, $c^2 - (a^2 + b^2) = 0 \Rightarrow c^2 = a^2 + b^2$

This is Pythagoras theorem with right angled at C. So, this implies that the triangle is a right angled triangle.

13. If the cosine of two angles of a triangle are proportional to the opposite sides, prove that the triangle is isosceles.

Solution:

Given, $\dfrac{\cos A}{a} = \dfrac{\cos B}{b}$

or, $b \cos A = a \cos B$

or, $2R \sin B.\cos A = 2R \sin A.\cos B$

or, $2R[\sin A.\cos B - \cos A.\sin B] = 0$

or, $\sin(A - B) = 0$

or, $A - B = 0$

$\therefore$ $A = B$

It signifies that the triangle is an isosceles triangle.

14. If $b - a = mc$, prove that $\cot\left(\frac{B-A}{2}\right) = \frac{1 + m\cos B}{m\sin B}$

Solution:

Given, $b - a = mc \Rightarrow m = \frac{b-a}{c}$

R.H.S.

$$= \frac{1 + m\cos B}{m\sin B} = \frac{1 + \frac{b-a}{c}.\cos B}{\frac{b-a}{c}.\sin B}$$

$$= \frac{c + b\cos B - a\cos B}{(b-a)\sin B}$$

$$= \frac{a\cos B + b\cos A + b\cos B - a\cos B}{(b-a).\sin B}$$

$$= \frac{b[\cos A + \cos B]}{(b-a).\sin B} = \frac{2R[\cos A + \cos B]}{2R(\sin B - \sin A)}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{B+A}{2}\right).\sin\left(\frac{B-A}{2}\right)} = \frac{\cos\left(\frac{B-A}{2}\right)}{\sin\left(\frac{B-A}{2}\right)} = \cot\left(\frac{B-A}{2}\right)$$

15. In ΔABC, if $a = 3$, $b = 4$ and $c = 5$ prove that $\sin 2A = \frac{24}{25}$.

Solution:

We know, $\text{Sin}2A = 2\sin A.\cos A$

or, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

or, $\cos A = \dfrac{16 + 25 - 9}{2 \times 4 \times 5}$

or, $\cos A = \dfrac{32}{40}$

or, $\cos A = \dfrac{4}{5}$

$\therefore \quad \sin A = \sqrt{1 - \cos^2 A} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \dfrac{3}{5}$

$\therefore \quad \sin 2A = 2\sin A.\cos A = 2 \times \dfrac{3}{5} \times \dfrac{4}{5} = \dfrac{24}{25}$

16. In any ΔABC if $\frac{\sin A + \cos A}{\cos B} = \sqrt{2}$, show that $\angle C = 135°$

Solution:

Given, $\frac{\sin A + \cos A}{\cos B} = \sqrt{2}$

or, $\sin A + \cos A = \sqrt{2}\cos B$

squaring both sides, we get

or, $(\sin A + \cos A)^2 = (\sqrt{2}\cos B)^2$

or, $1 + 2\sin A.\cos A = 2\cos^2 B$

or, $\sin 2A = 2\cos^2 B - 1$

or, $\sin 2A = \cos 2B$

or, $\sin 2A = \sin(90 - 2B)$

or, $2A + 2B = 90°$

$\therefore$ $A + B = 45°$

In any triangle,

$A + B + C = 180°$

or, $C = 180° - (A + B)$

or, $C = 180° - 45°$

$\therefore$ $C = 135°$

17. In any triangle, if a = 13, b = 14 and c = 15 then find Δ, s, $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, $\tan\frac{A}{2}$.

Solution:

or, $s = \frac{a + b + c}{2}$

or, $s = \frac{13 + 14 + 15}{2}$

$\therefore$ $s = 21$

or, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

or, $\Delta = \sqrt{21 \times (21-13)(21-14)(21-15)}$

or, $\Delta = \sqrt{21 \times 8 \times 7 \times 6}$

or, $\Delta = 84$

Also,

$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{(21-14)(21-15)}{14 \times 15}} = \frac{1}{\sqrt{5}}$

Also, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{21 \times (21-13)}{14 \times 15}} = \frac{2}{\sqrt{5}}$

And, $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(21-14)(21-15)}{21 \times (21-13)}} = \frac{1}{2}$

Exercise 8.2

1. Solve the triangle: $A = 60°$, $B = 45°$, $c = 6\sqrt{2}$

Solution:

$C = 180 - (60 + 45)° = 75°$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{a}{\sin 60°} = \frac{b}{\sin 45°} = \frac{6\sqrt{2}}{\sin 75°}$

from (i) and (iii) ratios, we get

or, $a = \frac{6\sqrt{2} \times \frac{\sqrt{3}}{2} \times 4}{\sqrt{6} + \sqrt{2}}$

or, $a = \frac{12\sqrt{6}}{\sqrt{6} + \sqrt{2}}$

or, $a = \frac{12\sqrt{6}}{\sqrt{2}(\sqrt{3} + 1)}$

or, $a = \frac{12\sqrt{3}}{\sqrt{3} + 1}$

or, $a = \frac{12\sqrt{3}}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$

or, $a = \frac{12\sqrt{3}(\sqrt{3} - 1)}{3 - 1}$

$\therefore a = 6\sqrt{3}(\sqrt{3} - 1)$

Again, from (ii) and (iii) ratios, we get

or, $\frac{b}{\sin 45°} = \frac{6\sqrt{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}}$

or, $b = \frac{6 \times 4}{\sqrt{6} + \sqrt{2}}$

or, $b = \frac{24}{\sqrt{2}(\sqrt{3} + 1)} = \frac{24(\sqrt{3} - 1)}{\sqrt{2}(3 - 1)}$

or, $b = \frac{24}{2\sqrt{2}} \times (\sqrt{3} - 1) = 6\sqrt{2}(\sqrt{3} - 1)$

2. Solve $\triangle ABC$, if $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, $C = 60°$

Solution:

We know, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

or, $\cos 60° = \dfrac{(\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 - c^2}{2(\sqrt{3}+1)(\sqrt{3} - 1)}$

or, $\dfrac{1}{2} = \dfrac{3 + 2\sqrt{3} + 1 + 3 - 2\sqrt{3} + 1 - c^2}{2 \times 2}$

or, $2 = 6 + 2 - c^2$

or, $c^2 = 6$

$\therefore$ $c = \sqrt{6}$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or, $b = \dfrac{c \sin B}{\sin C}$

or, $\sqrt{3} - 1 = \dfrac{\sqrt{6} \times \sin B}{\sqrt{3}/2}$

or, $\dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \sin B$

$\therefore$ $B = 15°$

or, $\sin A = \dfrac{a \sin C}{c}$

or, $\sin A = \dfrac{(\sqrt{3} + 1) \times \frac{\sqrt{3}}{2}}{\sqrt{6}}$

or, $\sin A = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

$\therefore$ $A = 105°$

3. In $\triangle ABC$, if $a = 3$, $b = 5$ and $c = 6$. Find (i) Δ (ii) R (iii) $\tan\frac{A}{2}$

Solution:

a. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

or, $s = \dfrac{3+5+6}{2} = 7$

or, $\Delta = \sqrt{7 \times 4 \times 2 \times 1}$

or, $\Delta = \sqrt{56}$

$\therefore$ $\Delta = 2\sqrt{14}$

b. We know, $\Delta = \dfrac{abc}{4R}$

or, $R = \dfrac{abc}{4\Delta}$

or, $R = \dfrac{3 \times 5 \times 6}{4 \times 2\sqrt{14}}$

$\therefore$ $R = \dfrac{45}{4\sqrt{14}}$

c. $\tan\frac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\dfrac{(7-5)(7-6)}{7 \times 4}} = \sqrt{\dfrac{2 \times 1}{28}} = \dfrac{1}{\sqrt{14}}$

4. If $a = 2$, $b = \sqrt{6}$ and $c = \sqrt{3} - 1$. Solve the triangle.

Solution:

or, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

or, $\cos A = \dfrac{6 + (\sqrt{3} - 1)^2 - 4}{2 \times \sqrt{6}(\sqrt{3} - 1)}$

or, $\cos A = \dfrac{2 + 3 - 2\sqrt{3} + 1}{2\sqrt{6}(\sqrt{3} - 1)}$

or, $\cos A = \dfrac{6 - 2\sqrt{3}}{2\sqrt{6}(\sqrt{3} - 1)}$

or, $\cos A = \dfrac{2\sqrt{3}(\sqrt{3} - 1)}{2\sqrt{6}(\sqrt{3} - 1)}$

or, $\cos A = \dfrac{1}{\sqrt{2}}$

$\therefore$ $A = 45°$

or, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

or, $\cos B = \dfrac{4 + 3 - 2\sqrt{3} + 1 - 6}{2.2(\sqrt{3} - 1)}$

or, $\cos B = \dfrac{2 - 2\sqrt{3}}{4(\sqrt{3} - 1)}$

or, $\cos B = \dfrac{-2(\sqrt{3} - 1)}{4(\sqrt{3} - 1)}$

or, $\cos B = \dfrac{-1}{2}$

or, $\cos B = \cos 120°$

$\therefore$ $B = 120°$

Now, $A + B + C = 180°$ $\therefore$ $C = 15°$

5. If three angles of a triangle are in the ratio of 2:3:7 find a:b:c

Solution:

Given, $A:B:C = 2:3:7$

So, $\dfrac{A}{2} = \dfrac{B}{3} = \dfrac{C}{7} = k$ (suppose)

$A = 2k$, $B = 3k$, $C = 7k$

We know, $A + B + C = 180°$

or, $2k + 3k + 7k = 180°$

or, $12k = 180°$

$\therefore$ $k = 15°$

Hence, $A = 30°$, $B = 45°$ and $C = 105°$

We know, from sine law

or, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or, $\dfrac{a}{\sin 30°} = \dfrac{b}{\sin 45°} = \dfrac{c}{\sin 105°}$

or, $\dfrac{a}{1/2} = \dfrac{b}{1/\sqrt{2}} = \dfrac{c}{\sqrt{3} + 1/2\sqrt{2}}$

or, $\dfrac{a}{\sqrt{2}} = \dfrac{b}{2} = \dfrac{c}{\sqrt{3} + 1}$

$\therefore$ $a:b:c = \sqrt{2}:2:\sqrt{3} + 1$

6. In ΔABC, $B = 60°$, $b : c = \sqrt{3}:\sqrt{2}$. Show that $A = 75°$

Solution:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or, $\frac{b}{c} = \frac{\sin B}{\sin C}$

or, $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60°}{\sin C}$

or, $\sin C = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{\sqrt{3}}$

or, $\sin C = \frac{1}{\sqrt{2}} = 45°$

We know,

$A + B + C = 180°$

or, $A + 60° + 45° = 180° = 75°$

7. a:b:c = 4:5:6 in ΔABC, prove that C = 2A.

Solution:

Given, a:b:c = 4:5:6

or, $\frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ (suppose)

or, $a = 4k,\ b = 5k\ ,\ c = 6k$

or, $\cos A = \frac{(5k)^2 + (6k)^2 - (4k)^2}{2.5k.6k}$

or, $\cos A = \frac{45k^2}{60k^2}$

or, $\cos A = \frac{3}{4}$

So, $\cos 2A = 2\cos^2 A - 1$

$= 2 \times \frac{9}{16} - 1 = \frac{2}{16} = \frac{1}{8}$ .....(i)

Again, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

or, $\cos C = \frac{16k^2 + 25k^2 - 36k^2}{2 \times 5k \times 4k}$

or, $\cos C = \frac{5k^2}{40k^2}$

or, $\cos C = \frac{1}{8}$ ....(ii)

from (i) and (ii), we get

or, cos2A = cosC

i.e. C = 2A

8. If $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$ find a:b:c.

Solution:

Given, $\cos A = \frac{4}{5}$

$\Rightarrow \sin A = \sqrt{1 - \cos^2 A}$

$= \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$ $\cos B = \frac{3}{5}$

$\Rightarrow \sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

Now, $\cos(A + B) = \cos A.\cos B - \sin A.\sin B$

or, $\cos(A + B) = \frac{4}{5} \cdot \frac{3}{5} - \frac{3}{5} \times \frac{4}{5}$

or, $\cos(A + B) = 0$

or, $\cos(A + B) = \cos 90^\circ$

$\therefore A + B = 90^\circ$

So, $C = 180 - (A + B)$

or, $C = 180 - 90$

$\therefore C = 90^\circ$

From sine law,

or, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{a}{3/5} = \frac{b}{4/5} = \frac{c}{1}$

or, $\frac{a}{3} = \frac{b}{4} = \frac{c}{5}$

$\therefore$ a:b:c = 3:4:5

9. Solve the triangle:

a. $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3} + 1$

b. $A = 75°$, $B = 60°$, $C = 45°$

c. $a = 1$, $b = \sqrt{3}$ $C = 30°$

d. $a = \sqrt{57}$, $A = 60°$, $\Delta = 2\sqrt{3}$

Solution:

a. $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3} + 1$

or, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

or, $\cos A = \dfrac{2 + (\sqrt{3} + 1)^2 - 4}{2 \times \sqrt{2}(\sqrt{3} + 1)}$

or, $\cos A = \dfrac{3 + 2\sqrt{3} + 1 - 2}{2\sqrt{2}(\sqrt{3} + 1)}$

or, $\cos A = \dfrac{2(\sqrt{3} + 1)}{2\sqrt{2}(\sqrt{3} + 1)}$

or, $\cos A = \dfrac{1}{\sqrt{2}}$

$\therefore$ $A = 45°$

Again, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

or, $\cos B = \dfrac{4 + 3 + 2\sqrt{3} + 1 - 2}{2.2.(\sqrt{3} + 1)}$

or, $\cos B = \dfrac{6 + 2\sqrt{3}}{4(\sqrt{3} + 1)}$

or, $\cos B = \dfrac{2\sqrt{3}(\sqrt{3} + 1)}{4(\sqrt{3} + 1)}$

or, $\cos B = \dfrac{\sqrt{3}}{2}$

$\therefore$ $B = 30°$

We know, $A + B + C = 180°$

or, $45° + 30° + C = 180°$

$\therefore$ $C = 105°$

b. $A = 75°$, $B = 60°$, $C = 45°$

We know that when three angles of a triangle are given then no unique solution is possible. Only ratio of sides can be found not the actual length of sides.

or, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or, $\dfrac{a}{\sin 75°} = \dfrac{b}{\sin 60°} = \dfrac{c}{\sin 45°}$

or, $\dfrac{a}{\frac{\sqrt{3} + 1}{2\sqrt{2}}} = \dfrac{b}{\frac{\sqrt{3}}{2}} = \dfrac{c}{\frac{1}{\sqrt{2}}}$

or, $\dfrac{a}{\sqrt{3} + 1} = \dfrac{b}{\sqrt{6}} = \dfrac{c}{2}$

$\therefore$ $a:b:c = (\sqrt{3} + 1):\sqrt{6}:2$

c. $1a = 1,\ b = \sqrt{3},\ C = 30°$

or, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

or, $\cos 30° = \dfrac{1 + 3 - c^2}{2 \times 1 \times \sqrt{3}}$

or, $\dfrac{\sqrt{3}}{2} = \dfrac{4 - c^2}{2\sqrt{3}}$

or, $3 = 4 - c^2$

$\therefore$ $c = 1$

or, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

or, $\cos B = \dfrac{1 + 1 - 3}{2}$

or, $\cos B = \dfrac{-1}{2}$

$\therefore$ $B = 120°$

We know, $A + B + C = 180°$

or, $A = 180° - 120° - 30°$

$\therefore$ $A = 30°$

d. $a = \sqrt{57},\ A = 60°,\ \Delta = 2\sqrt{3}$

or, $\Delta = \dfrac{1}{2} bc \sin A$

or, $2\sqrt{3} = \dfrac{1}{2} \times bc \times \dfrac{\sqrt{3}}{2}$

or, $bc = 8$ ....(i)

or, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

or, $\cos 60° = \dfrac{b^2 + c^2 - 57}{2 \times 8}$

or, $8 = b^2 + c^2 - 57$

or, $b^2 + c^2 = 65$ ....(ii)

putting the value of $c = \dfrac{8}{b}$ in (ii), we get

or, $b^2 + \dfrac{(8)^2}{b^2} = 65$

or, $b^4 + 64 = 65b^2$

or, $b^4 - 65b^2 + 64 = 0$

Taking quadrant in b^2, we get

or, $b^2 = \dfrac{65 \pm \sqrt{(65)^2 - 4.1.64}}{2 \times 1} = \dfrac{65 \pm 63}{2}$

Taking +ve sign, we get

or, $b^2 = \dfrac{65 + 63}{2}$

or, $b^2 = 64 \ \therefore b = 8$

Again, taking –ve sign, we get

or, $b^2 = 1 \ \therefore b = 1$

Putting the values of b in (i) , we get

$c = 1$ and 8

Hence, required sides are 1 and 8.

10. If a = 2, b = $\sqrt{6}$ and c = $\sqrt{3}$ + 1. Find the greatest and least angle of triangle ABC.

Solution:

Greater angle corresponds to greatest side and least angle corresponds smallest sides.
Since, c is the largest side, so, 'C' will be larger angle and 'A' will be smallest angle.

$$\text{or,}\quad \cos C = \frac{4 + 6 - (\sqrt{3} + 1)^2}{2 \times 2 \times \sqrt{6}}$$

$$\text{or,}\quad \cos C = \frac{10 - 3 - 2\sqrt{3} - 1}{4\sqrt{6}}$$

$$\text{or,}\quad \cos C = \frac{6 - 2\sqrt{3}}{4\sqrt{6}}$$

$$\text{or,}\quad \cos C = \frac{2\sqrt{3}(\sqrt{3} - 1)}{2.2\sqrt{6}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\text{or,}\quad \cos C = \cos 75^\circ \quad \therefore\ C = 75^\circ$$

$$\text{or,}\quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{or,}\quad \cos A = \frac{6 + (\sqrt{3} + 1)^2 - 4}{2 \times \sqrt{6} \times (\sqrt{3} + 1)}$$

$$\text{or,}\quad \cos A = \frac{2 + 3 + 2\sqrt{3} + 1}{2\sqrt{6}(\sqrt{3} + 1)}$$

$$\text{or,}\quad \cos A = \frac{2\sqrt{3}(\sqrt{3} + 1)}{2\sqrt{6}(\sqrt{3} + 1)} = \frac{1}{\sqrt{2}}$$

$$\therefore\quad A = 45^\circ$$

11. If a = 2, b = $\sqrt{3}$ + 1, C = 60°, Solve the triangle.

Solution:

or, $\cos C = \dfrac{b^2 + a^2 - c^2}{2ab}$

or, $\cos 60° = \dfrac{(\sqrt{3} + 1)^2 + 4 \ - c^2}{2 \times 2(\sqrt{3} + 1)}$

or, $\dfrac{1}{2} = \dfrac{3 + 2\sqrt{3} + 1 + 4 - c^2}{4(\sqrt{3} + 1)}$

or, $2(\sqrt{3} + 1) = 8 + 2\sqrt{3} - c^2$

or, $2\sqrt{3} + 2 = 8 + 2\sqrt{3} - c^2 \ \therefore \ c = \sqrt{6}$

or, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or, $\dfrac{2}{\sin A} = \dfrac{\sqrt{3} + 1}{\sin B} = \dfrac{\sqrt{6}}{\sin 60°}$

or, $\sin A = \dfrac{2 \sin 60°}{\sqrt{6}}$

or, $\sin A = \dfrac{2}{\sqrt{6}} \times \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{2}} \ \therefore \ A = 45°$

We know, A + B + C = 180°

or, $45° + B + 60° = 180° \ \therefore \ B = 75°$

12 a. $a = 2, b = \sqrt{3} + 1, A = 45°$ b. $a = 3, b = 3\sqrt{3}, A = 30°$.

Solution:

a. $a = 2, b = \sqrt{3} + 1, A = 45°$.

We know, by sine law

or, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{2}{\sin 45°} = \frac{\sqrt{3} + 1}{\sin B}$

or, $\sin B = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ $\therefore$ B = 75° or 105°

When B = 75°, C = 60°

When B = 105°, C = 30°

When B = 75°, c is given by

or, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2}{\sin 45°} = \frac{c}{\sin 60°}$

or, $c = 2\sqrt{2} \times \frac{\sqrt{3}}{2}$

$\therefore$ $c = \sqrt{6}$

When B = 105°, value of c is given by

or, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{2}{1/\sqrt{2}} = \frac{c}{1/2}$

or, $c = \sqrt{2}$

the solution are

$B = 75°, C = 60°, c = \sqrt{6}$

$B = 105°, C = 30°, c = \sqrt{2}$

b. $a = 3, b = 3\sqrt{3}, A = 30°$

We know, by sine law

or, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

or, $\frac{3}{\sin 30°} = \frac{3\sqrt{3}}{\sin B}$

or, $\sin B = \frac{3\sqrt{3} \times \frac{1}{2}}{3}$

or, $\sin B = \frac{\sqrt{3}}{2}$ $\therefore$ B = 60°, 120°

When B = 60°, C = 90°

When B = 120°, C = 30°

When B = 60°

or, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{3}{\sin 30°} = \frac{c}{\sin 90°}$

$\therefore$ c = 6

When B = 120°

or, $\frac{a}{\sin A} = \frac{c}{\sin C}$

or, $\frac{3}{\sin 30°} = \frac{c}{\sin 30°}$ $\therefore$ c = 3

So, required solution are

B = 60°, C = 90°, c = 6

B = 120°, C = 30°, c = 3