Chapter 2:
Articulated Systems

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Forward Kinematics

Configuration Space

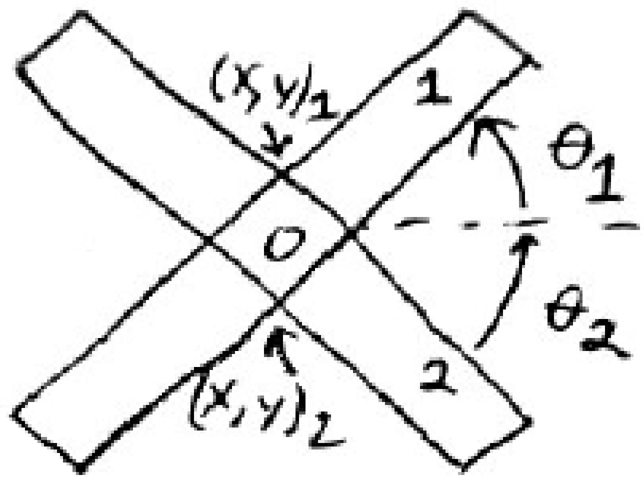
Work Space, position/orientation

$$f : Q \rightarrow W$$

R^2 or *SE(2)*

Accessible Manifold

(x; y)$_1$ = (x; y)$_2$



$$SE(2) \times SE(2) \to SE(2) \times S^1$$

6 dimensions four dimensions

Holonomic Constraints

- a (possibly time-varying) constraint function f on the system's configuration space Q.
  - remove degrees of freedom from a system, reducing the dimensionality of its configuration space
  - In general, the zero set of a real-valued function forms the accessible manifold of the constrained system

Example of Holonomic Constraint

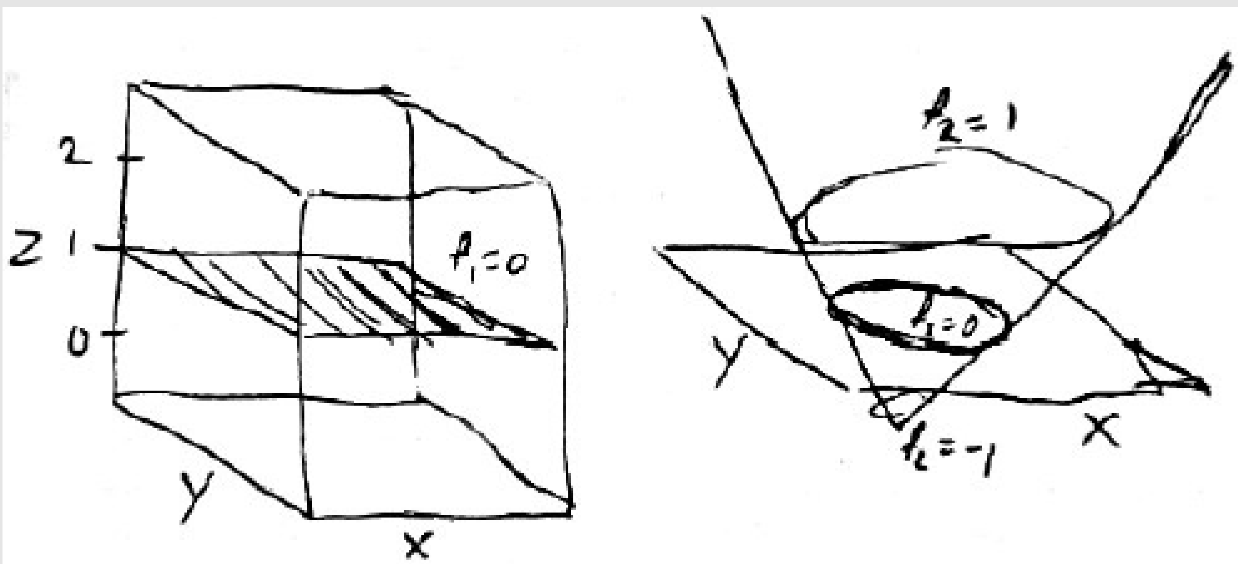
As an example, consider the problem of restricting a point $p = (x, y, z) \in \mathbb{R}^3$ to move only within a unit circle on the plane $z = 1$. The planar condition corresponds to the holonomic constraint function

$$f_1(p, t) = z - 1, \tag{2.i}$$

which has a zero set (and accessible manifold) at the $z = 1$ plane. Once this constraint is made, the configuration space of the system is reduced by one dimension and effectively becomes $p_1 = (x, y) \in \mathbb{R}^2$. Restricting the point to the unit circle is accomplished by the constraint function

$$f_2(p_1, t) = \sqrt{(x^2 + y^2)} - 1. \tag{2.ii}$$

with the final accessible manifold formed by the intersection of a cone representing f_2 with the xy plane. Note that points on the f_2 cone are *not* elements of the original $\mathbb{R}^3$ space.



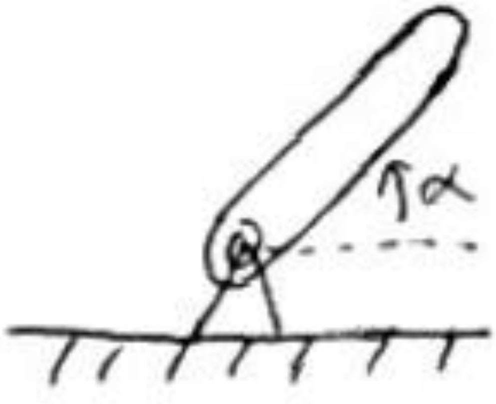
However,

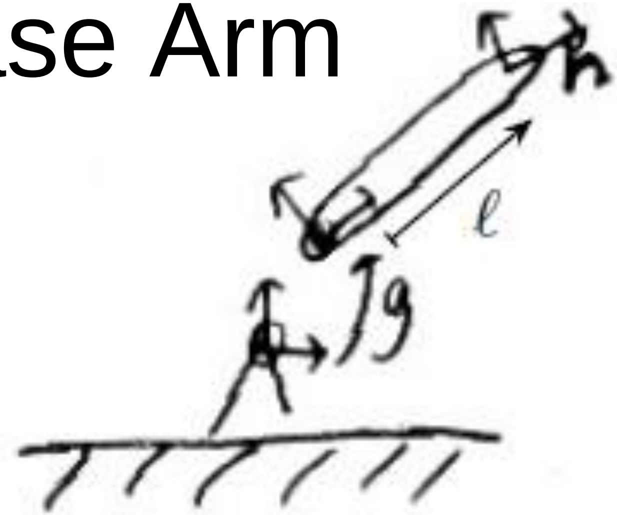
- One thing holonomic constraint do not do is to remove the dependence of the system dynamics on the actual physical positions of the component bodies

- Just because you can reduce the configuration space to a lower-dimensional one, it does not mean that we can ignore the inertial or collision effects in the work space, i.e, if we can reduce the cpsace to a point in the plane, we still need to generate the forces in this plane by evaluating the forces acting on the physical bodies in their own configuration spaces, then projecting them into the constraint manifold that forms the reduced configuration space.

- Therefore, we use forward kinematics to relate the positions of the component bodies to their configuration variables
  - Ultimately, we want to know configuration space forces
  - Configuration space forces are derived from the forces acting on each point on the rigid bodies on the robot (ie forces in the ambient space)
  - Physics can gets us from ambient positions and velocities to ambient forces, ie, your position and velocity could be a result of a force like gravity or contact on an object, and likewise you can have forces that are functions of positions and velocities like drag forces
  - We want a relationship between our Cspace positions and forces and ambient positions and forces
  - This relationship comes from the kinematics map

Apply Constraints on Cpsace: Fixed-base Arm





$$\alpha \in S^1 = Q$$

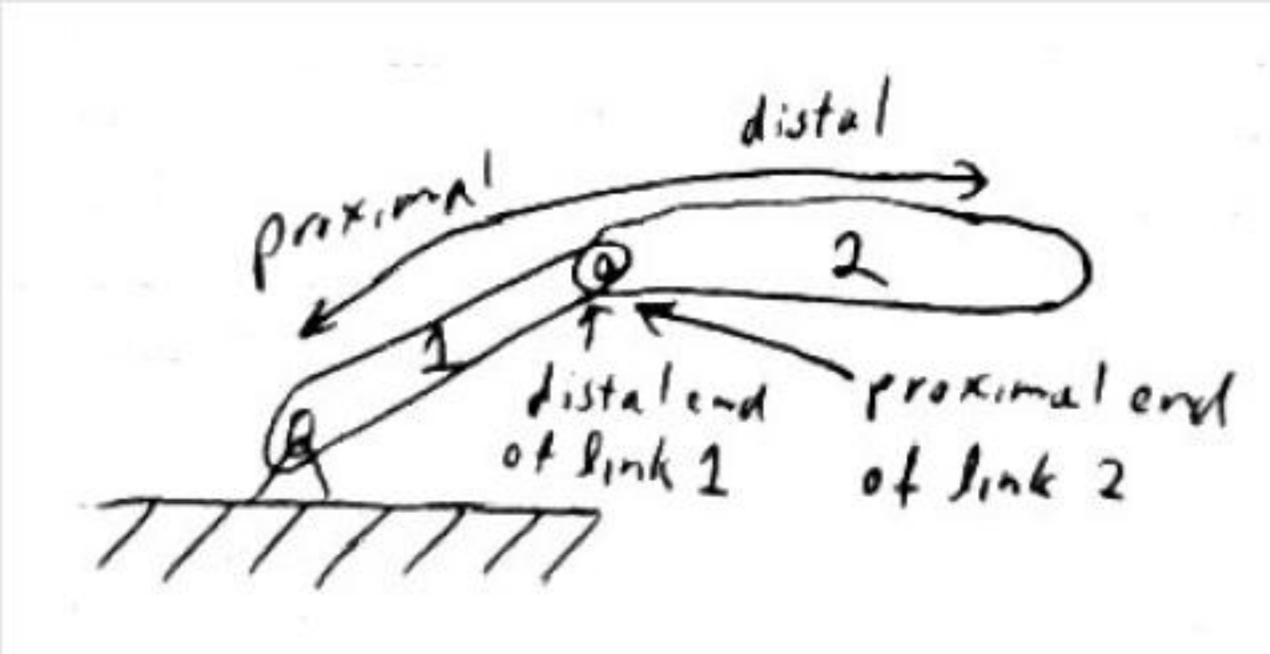
$$\text{Rigid body} : g \in SE(2)$$

$$\text{Holonomic Constraints} : x = 0, y = 0$$

$$g = \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ x,y=0} \equiv \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}^{SO(2)} \equiv \overbrace{\alpha}^{\mathrm{s}^1}$$

Isomorphic

More Notation

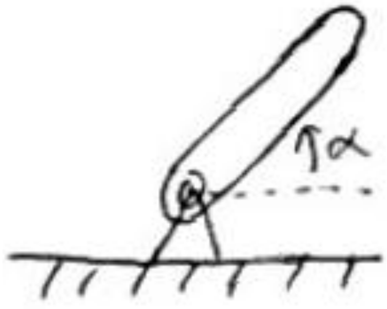


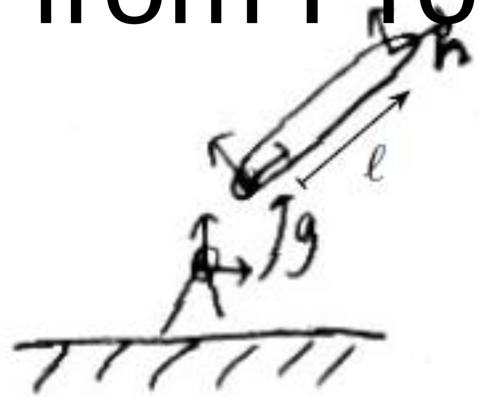
For individual links, these terms
may describe a relationship between two
links ("link 1 is proximal to link 2")

an absolute position in the
chain ("link 2 is the distal link").

"Proximal": Near, like in "proximity"
"Distal": Far, like in "distant"

Calculate Distal from Proximal





$h_g = (\ell, 0, 0)$

$$g = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad h = gh_g = \begin{bmatrix} \cos\alpha & -\sin\alpha & \ell\cos\alpha \\ \sin\alpha & \cos\alpha & \ell\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

Two ways to think of find distal portion of link

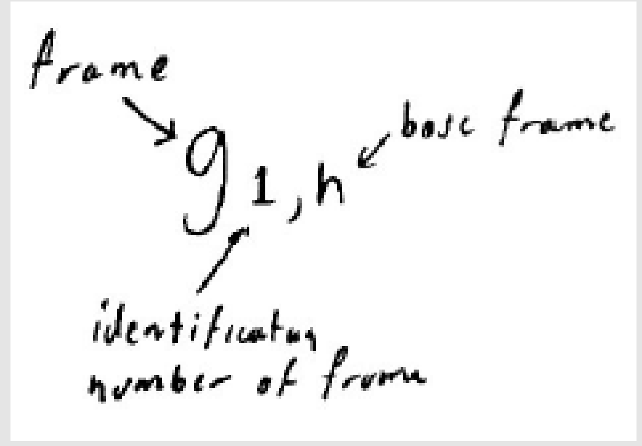
$L_g h_g$ transforming the frame position h_g by g. wrt to global frame

or

$R_{h_g} g$ placing h_g into g Which is same as h_g wrt to g

Think of this as “pretending” the origin is at g
Move from origin by h_g (or really from g by h_g)
Return the origin to where it was before

Notation



$g_{1,h}$ denotes the position of frame g_1 with respect to frame h,

$$g_{1,h} = h^{-1}g.$$

Frames on the left cancel with subscripted frames on the right

$$gh_g = gg^{-1}h = h$$

During this cancellation, base-frame subscripts on the left are transferred to the right

$$g_{1,g_0}h_{g_1} = g_0^{-1}(g_1g_1^{-1})h = g_0^{-1}h = h_{g_0}$$

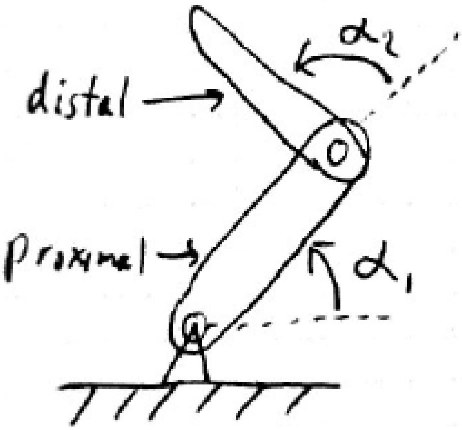
Add a link with a rotary joint

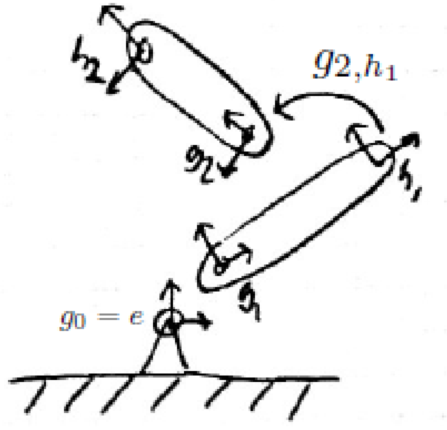
Second body's base position w.r.t. end of first link

$$g_{2,h_1} = (x_{2,h_1}, y_{2,h_1}, \theta_{2,h_1})$$

Apply constraint: *x=0, y=0*

$$g_{2,h_1} = (0, 0, \alpha_2)$$





$$g_2 = \overbrace{(g_1 h_{1,g_1})}^{h_1} g_{2,h_1} = \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1 \cos\alpha_1 \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1 \sin\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} h_2 &= g_2 h_{2,g_2} \\ &= \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1+\alpha_2) \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Add a link with a prismatic joint

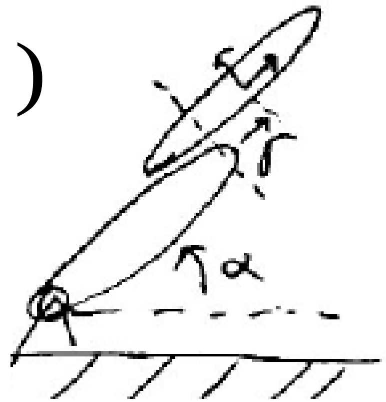
Second body's base position w.r.t. end of first link

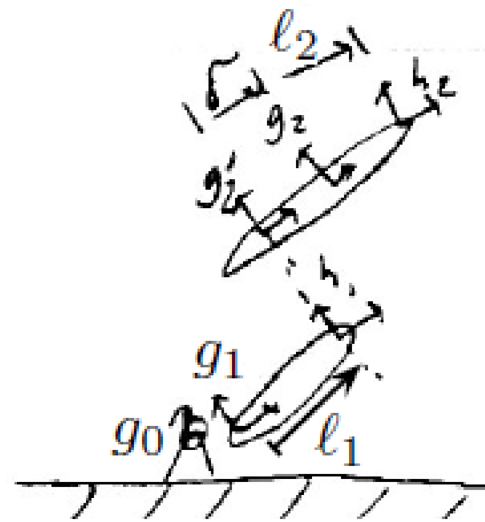
$$g_{2,h_1} = (x_{2,h_1}, y_{2,h_1}, \theta_{2,h_1})$$

Apply constraint: *y=0, δ=0*

$$g_{2,h_1} = (\delta, 0, 0)$$

$$g = \overbrace{\begin{bmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ y,\theta=0} \equiv \overbrace{\begin{bmatrix} \delta \\ 0 \end{bmatrix}}^{\mathbb{R}^2,\ y=0} \equiv \overbrace{\delta}^{\mathbb{R}^1}$$





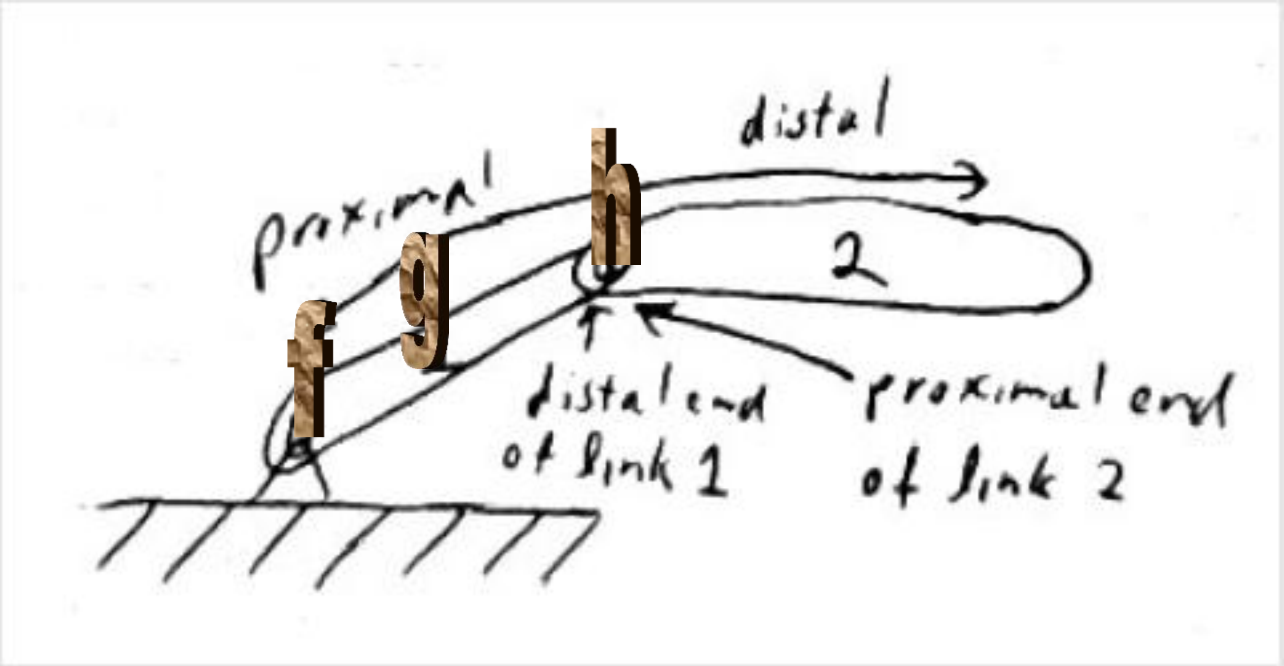
Does this offset bother anyone?

$$g_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

$$h_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta+\ell_2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta+\ell_2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

And so on…

Slight Change Notation



For the next examples, we will use g for the medial frame on the link, and f as the proximal frame.

This is useful for talking about the center of mass of a link as its position, and we like g as the "primary" frame for describing a link

Mobile Articulated Systems

- Position: location and orientation of its *body frame,*

$$g \in G$$

- Shape:
  - placement of the component rigid bodies relative to the system body frame
  - correspond to the configuration variables of fixed-base systems

$$r \in M$$

- Configuration Space

$$Q = G \times M$$

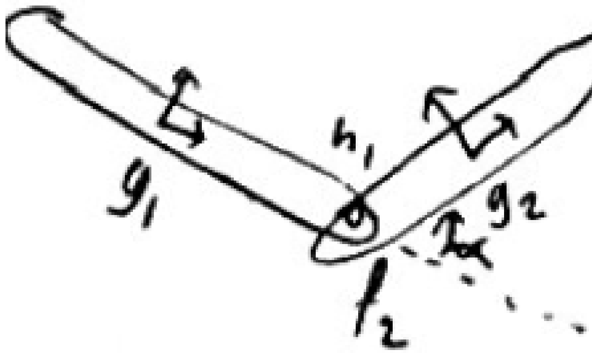
$$q = (g, r) \in Q$$

Body Frame

Body frame: one in which the forward kinematics to every frame on the body is a function of the shape variable

Choice of base link as body frame is a natural choice because all rigid bodies are “jointed” together and hence the forward kinematics are functions of the shape variables

Choice of Body Frame (Position)

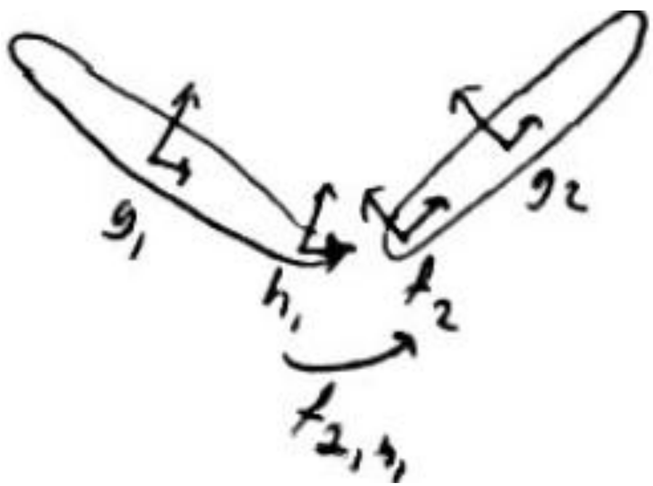


$$Q = SE(2) \times S^1$$

$$g = g_1$$

$$r = \alpha$$

"base link" and shape is configurations of remaining bodies relative to previous ones.



$$g_2 = \overbrace{g_1 \circ h_{1,g_1}}^{\text{link 1}} \circ \overbrace{f_{2,h_1} \circ g_{2,f_2}}^{\text{link 2}}$$

$$= g_1 \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1/2) + (\ell_2/2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1/2) + (\ell_2/2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

Subsequent links can then be added either to the distal end of link 2 or to the proximal end of link 1.

Other choices exist

- select any frame whose position with respect to the base link is a function of the shape variables

- How? *a frame in which the position of every component body (and, by extension, any point on those bodies) is a function of the shape r*

- Two choices
  - On another rigid body: it is like choosing another link as “base link” and relying on SE(2) being invertible

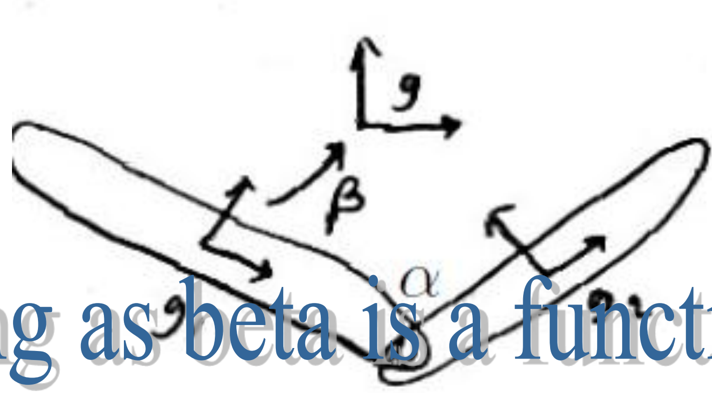
  - Not on another rigid body

Not on a rigid body, cont

Choose g at $\beta \in SE(2)$ w.r.t. base link

wrt body

$g_{1,g} = \beta^{-1}$



$g_{2,g} = \beta^{-1} g_{2,g_1}(\alpha)$

cinctly as: As long as beta is a function of alpha, g is

If g is a *valid body frame*, then $g_{1,g}$ and $g_{2,g}$ must both be functions of alpha because any transformation from g to a link-attached frame must be a function of α

Therefore, given that $g_{1,g} = \beta^{-1}$, then g is valid body frame if and only if β is a function of α

In other words: If (and only if) we can express β as a function of α, then $g_{1,g}$ and $g_{2;g}$ are both functions of the system shape, meeting the necessary and sufficient conditions for g to serve as the system body frame

We will see this is good later on when we want to chose an optimal frame location

Velocity and the Jacobian

Forward kinematic map from configuration to any point/frame on the robot

Notation alert: here g is not the position variable

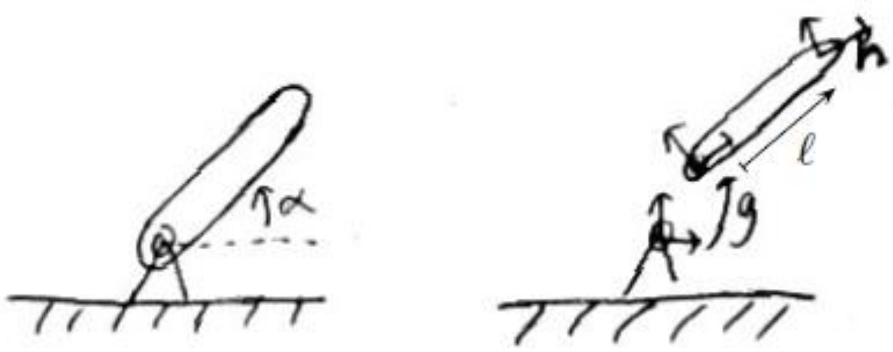
$$\dot{g}(q, \dot{q}) = J_g \dot{q} = \frac{\partial g}{\partial q} \dot{q}.$$

Derivative of forward kinematic map
Generalization of scalar derivative
Mapping from configuration map to velocity of point/frame on robot

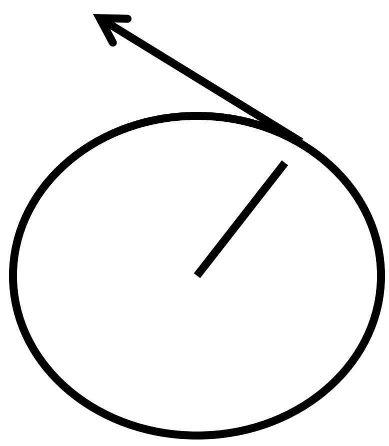
Jacobians from Differentiation

$$h = gh_g = \begin{bmatrix} \cos\alpha & -\sin\alpha & \ell\cos\alpha \\ \sin\alpha & \cos\alpha & \ell\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

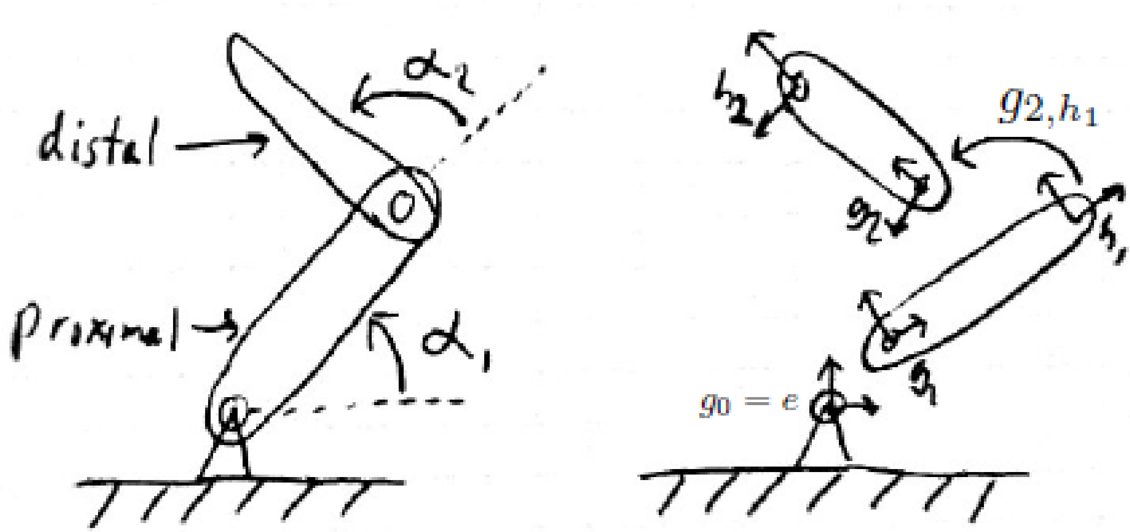
$$h = (\ell\cos\alpha, \ell\sin\alpha, \alpha)$$

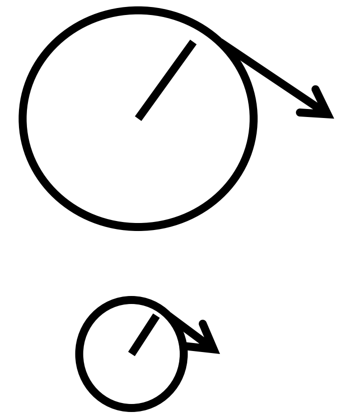


$$\dot{h} = J_h\dot{\alpha} = \frac{\partial h}{\partial \alpha}\dot{\alpha} = \begin{bmatrix} -\ell\sin\alpha \\ \ell\cos\alpha \\ 1 \end{bmatrix}\dot{\alpha}.$$



Two-link Jacobian





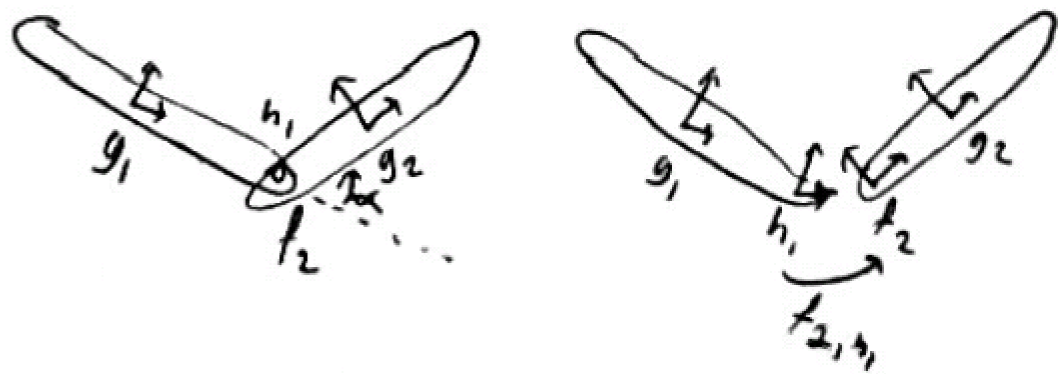
$$h_2(q) \quad = \quad \Big((\ell_1 \cos\alpha_1 \;+\; \ell_2 \cos(\alpha_1 + \alpha_2)), (\ell_1 \sin\alpha_1 \;+\; \ell_2 \sin(\alpha_1 + \alpha_2)), (\alpha_1 \;+\; \alpha_2)\Big)$$

has a two-column Jacobian,

$$\dot{h}_2 = \begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) & -\ell_2 \sin(\alpha_1 + \alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) & \ell_2 \cos(\alpha_1 + \alpha_2) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha_1} \\ \dot{\alpha_2} \end{bmatrix},$$

Two columns because 2 DOF; Three rows because g has position and orientation

Mobile Two-Link



4 columns

3   1

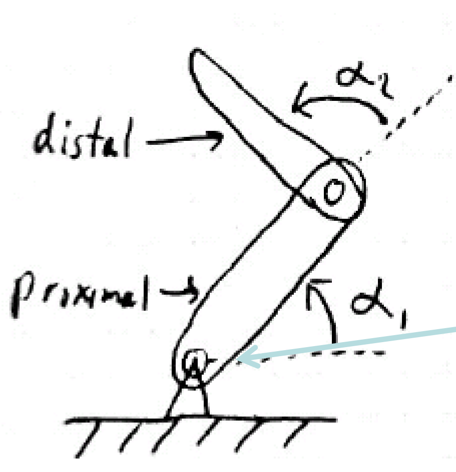
$$\dot{g}_2 = \begin{bmatrix} \frac{\partial g_2}{\partial g} & \frac{\partial g_2}{\partial \alpha} \end{bmatrix} (\dot{g}, \dot{r}).$$

3 rows

What are these?

Iterative Jacobian Assembly

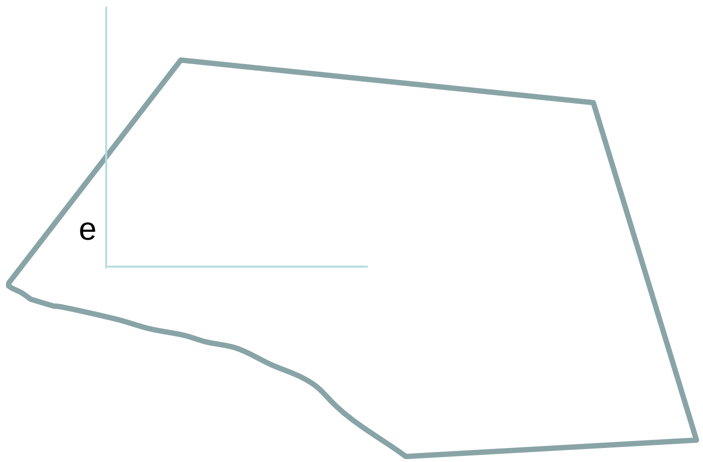
- Pre-differentiation
  - Frames on a rigid body are related by a right action
  - Motion at a joint between two rigid bodies is the same, modulo the relative motion



$$\dot{g}_1 = \dot{g}_0 + (0, 0, \dot{\alpha}_1) = (0, 0, \dot{\alpha}_1)$$

$$\dot{g}_0 = (0, 0, 0)$$

Spatial Velocity Reminder



$$\xi^s = T_g R_{g^{-1}} \dot{g}.$$

$$\dot{g} = (T_g R_{g^{-1}})^{-1} \xi^s = T_e R_g \xi^s.$$

frames

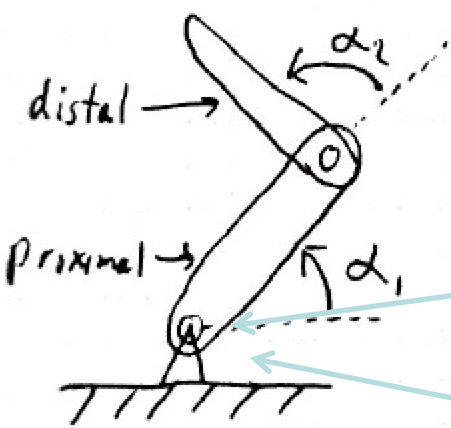
Two ~~points~~ on the same rigid body have the same spatial velocity

Use Spatial Velocity

$$T_{h_1} R_{h_1^{-1}} \dot{h}_1 = T_{g_1} R_{g_1^{-1}} \dot{g}_1 \quad + \quad \dot{g} = (T_g R_{g^{-1}})^{-1} \xi^s = T_e R_g \xi^s$$

$$\dot{h}_1 = \overbrace{(T_e R_{h_1})(T_{g_1} R_{g_1^{-1}})}^{T_{g_1} R_{h_1, g_1}} \dot{g}_1$$

$$\dot{h}_1 = \overbrace{\begin{bmatrix} 1 & 0 & -\ell_1 \sin\alpha_1 \\ 0 & 1 & \ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_e R_{h_1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_1} R_{g_1^{-1}}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_1 \end{bmatrix}}^{\dot{g}_1} = \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 \\ \ell_1 \cos\alpha_1 \\ 1 \end{bmatrix}}^{J_{h_1}} \dot{\alpha}_1$$



$$\dot{g}_1 = \dot{g}_0 + (0, 0, \dot{\alpha}_1) = (0, 0, \dot{\alpha}_1)$$

$$\dot{g}_0 = (0, 0, 0)$$

The Second Link

Proximal

$$\begin{aligned}\dot{g}_2 &= \dot{h}_1 + (0, 0, \dot{\alpha}_2) \\ &= ((-\ell_1 \sin\alpha_1)\dot{\alpha}_1, (\ell_1 \cos\alpha_1)\dot{\alpha}_1, (\dot{\alpha}_1 + \dot{\alpha}_2))\end{aligned}$$

$$\dot{g}_2 = \begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

Distal

$$\dot{h}_2 = \overbrace{\begin{bmatrix} 1 & 0 & -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) \\ 0 & 1 & (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) \\ 0 & 0 & 1 \end{bmatrix}}^{T_e R_{h_2}} \overbrace{\begin{bmatrix} 1 & 0 & \ell_1 \sin\alpha_1 \\ 0 & 1 & -\ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_2} R_{g_2^{-1}}} \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}}^{\dot{g}_2};$$

$$\dot{h}_2 = \overbrace{\begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) & -\ell_2 \sin(\alpha_1 + \alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) & \ell_2 \cos(\alpha_1 + \alpha_2) \\ 1 & 1 \end{bmatrix}}^{J_{h_2}} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

A Pattern is Beginning to Emerge

$$\dot{h}_i = (T_e R_{h_i})(T_{g_i} R_{g_i^{-1}}) \overbrace{(\dot{h}_{i-1} + v_i)}^{\dot{g}_i},$$

where v_i is the velocity of body i with respect to body $i-1$ at joint i.

Body Velocity Formulation

Often it is useful to work with body velocities instead of world velocities

$$\dot{h}_i = (T_e R_{h_i})(T_{g_i} R_{g_i^{-1}}) \overbrace{(\dot{h}_{i-1} + v_i)}^{\dot{g}_i},$$

$$\underbrace{\overbrace{(T_{h_i} L_{h_i^{-1}})}^{\text{new}} \dot{h}_i}_{\xi_{h_i}} = \underbrace{\overbrace{(T_{h_i} L_{h_i^{-1}})}^{\text{new}} (T_e R_{h_i})}_{Ad_{h_i}^{-1}} \underbrace{(T_{g_i} R_{g_i^{-1}}) (T_e L_{g_i})}_{Ad_{g_i}} \overbrace{\phantom{(T_e L_{g_i})(T_{g_i} L_{g_i^{-1}})}}^{\text{Identity}} \underbrace{(T_{g_i} L_{g_i^{-1}}) \dot{g}_i}_{\xi_{g_i}}.$$

$\xi = T_g L_{g^{-1}} \dot{g}$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g} \xi.$$

$\xi = T_g L_{g^{-1}} \dot{g}$

Adjoint Operator

- Definition: product of left lifted action from origin to g followed by right lifted action from g back to origin

$$Ad_g = (T_g R_{g^{-1}})(T_e L_g)$$

- Meaning: A measure of noncommunitivity of the Lie group, ie., measure of how left and right actions produce different results

- Converts from body to spatial velocity

Adjoint Operator in SE(2)

Recall spatial velocity

$$\xi^s = T_g R_{g^{-1}} \dot{g}.$$

$$\xi^s = \overbrace{\begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}}^{T_g R_{g^{-1}}} \underbrace{\overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_e L_g} \xi}_{\dot{g}}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g} \xi.$$

The Inverse

$$Ad_g^{-1} = (T_g L_{g^{-1}})(T_e R_g) = Ad_{g^{-1}}$$

$$\begin{aligned}
\xi &= \overbrace{\begin{bmatrix}1 & 0 & y\\ 0 & 1 & -x\\ 0 & 0 & 1\end{bmatrix}}^{T_g L_{g^{-1}}} \underbrace{\overbrace{\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}}^{T_e R_g} \xi^s}_{\dot{g}} \\
&= \underbrace{\begin{bmatrix}\cos\theta & \sin\theta & x\sin\theta - y\cos\theta\\ -\sin\theta & \cos\theta & x\cos\theta + y\sin\theta\\ 0 & 0 & 1\end{bmatrix}}_{Ad_g^{-1}} \xi^s
\end{aligned}$$

Back to the Jacobian

$$\underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}\dot{h}_i}_{\xi_{h_i}} = \underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}(T_eR_{h_i})}_{Ad_{h_i}^{-1}}\underbrace{(T_{g_i}R_{g_i^{-1}})\overbrace{(T_eL_{g_i})}^{}}_{Ad_{g_i}}\underbrace{\overbrace{(T_{g_i}L_{g_i^{-1}})}^{}\dot{g}_i}_{\xi_{g_i}}.$$

Identity: $(T_eL_{g_i})(T_{g_i}L_{g_i^{-1}})$

Consider three frames.

h_{i-1}, the distal frame on link $i-1$

g_i, the proximal frame on link i

g'_i, the frame on link i that is instantaneously aligned with h_{i-1}

We want g_i fixed to the link and not moving around on link

We want coincident tan spaces between h_{i-1} and g'_i because if h_{i-1} sees the g'_i moving with a certain velocity, that velocity is the same as the body velocity of g'_i because at the origin spatial and body velocities are the same

Note the actuator on hi-1 specifies the spatial velocity of link i (ie all frames on link i) taking h_{i-1} as the origin

$$\xi_{g'_i} = \xi_{h_{i-1}} + v_i \qquad v_i = (\dot{\delta}_{x,i}, \dot{\delta}_{y,i}, \dot{\alpha}_i)$$

The relative motion defined wrt proximal link

How is v_i different from before defined?

Keep Jacobian'

$\xi_{g_i} = Ad_{g_i}^{-1} Ad_{g'_i} \xi_{g'_i}$ Same rigid body

There are missing lines here which should be inser

$$\xi_{h_i} = (Ad_{h_i}^{-1})(Ad_{g'_i})(\xi_{h_{i-1}} + v_i)$$

Look terms with g have been dropped; why is this good?

If x and y components of g'_i and g_i are equal, this conversion reduces to rotation by $-\alpha_i$,

Why is this negative?

Take perspective of body frame

Use relative positions

i

If the relative positions of frames on a link are more convenient to use their absolute positions, a further reduction is possible, by evaluating (2.36) with the origin temporarily placed at g_i'. This change of coordinates transforms the position of each link frame by $(g_i')^{-1}$, so that g_i' and h_1 respectively become e and $h_{1,g_i'}$. As the adjoint action at the origin is an identity matrix, the body velocity of the distal frame simplifies to the inverse adjoint action of that frame relative to g_i'.

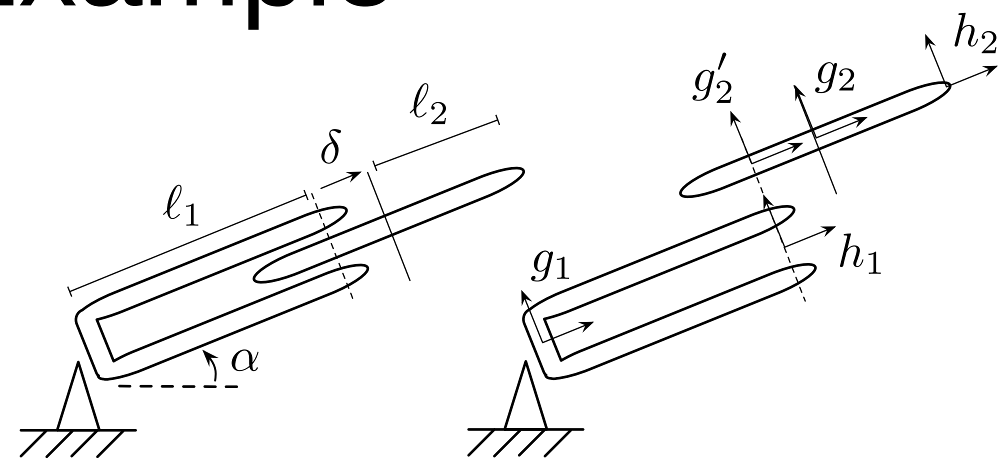
$$\xi_{h_i} = (Ad^{-1}_{h_i,g_i'})(\xi_{h_{i-1}} + v_i). \tag{2.37}$$

How do you get to "temporarily" put the origin anywhere
Note: new chap 1 will have symmetry and discuss what coord. Independence is about

Body and spatial velocities are same at the origin

$$\xi_{h_i} = (Ad^{-1}_{h_i})(Ad_{g_i'})(\xi_{h_{i-1}} + v_i)$$

RP Example



$$\xi_{g_0} = (0,0,0)$$

$$\xi_{g_1'} = \xi_{g_0} + \overbrace{(0,0,\dot\alpha)}^{v_1} = (0,0,\dot\alpha).$$

$$\xi_{g_1} = Ad^{-1}_{g_1,g_1'}\xi_{g_1'} = \overbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_1,g_1'=(0,0,\alpha)}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot\alpha \end{bmatrix}}^{\xi_{g_1'}} = \begin{bmatrix} 0 \\ 0 \\ \dot\alpha \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{J^b_{g_1}} \dot\alpha,$$

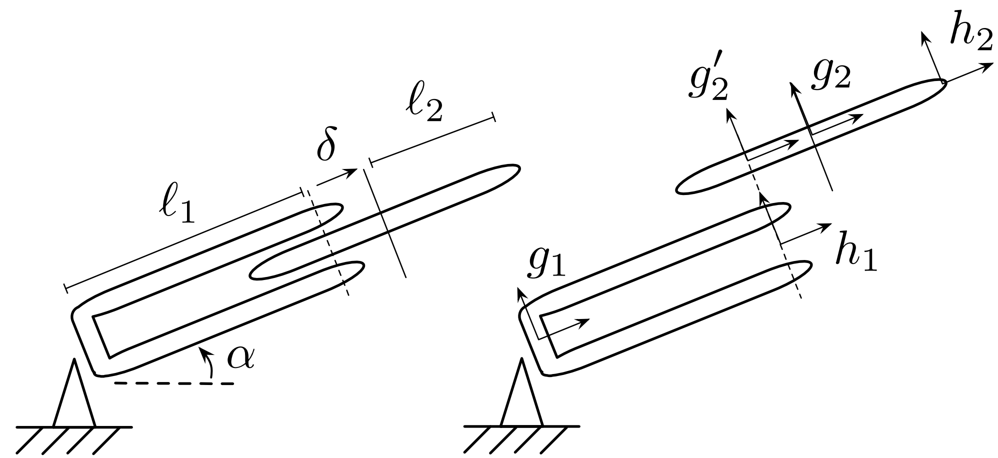
Jacobian for proximal end

$$\xi_{h_1} = Ad^{-1}_{h_1,g_1}\xi_{g_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_1 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_1,g_1=(\ell_1,0,0)}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot\alpha \end{bmatrix}}^{\xi_{g_1}} = \begin{bmatrix} 0 \\ \ell_1\dot\alpha \\ \dot\alpha \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ \ell_1 \\ 1 \end{bmatrix}}^{J^b_{h_1}} \dot\alpha,$$

Jacobian for distal end, look it moves lateral

How does this compare to $\dot{h}$

Next DOF



- Look, no forward kinematics!

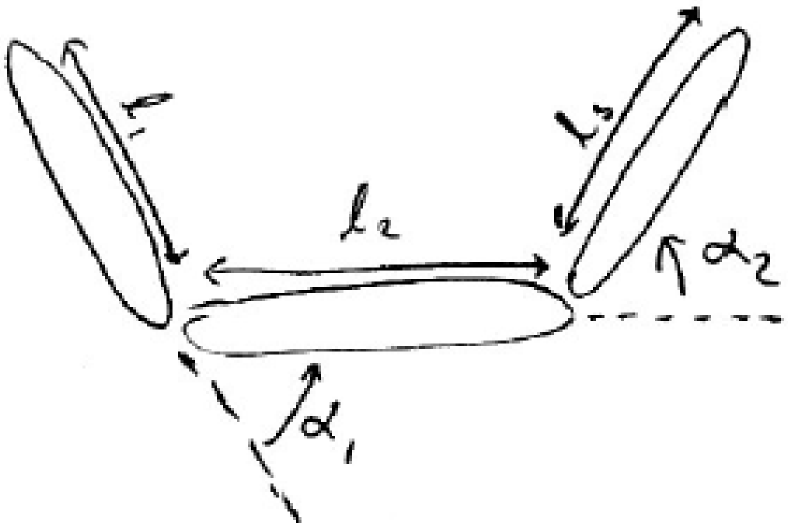
$$v_2 = (\dot\delta, 0, 0) \qquad \xi_{g_2'} = \xi_{h_1} + v_2 = (\dot\delta, \ell_1\dot\alpha, \dot\alpha).$$

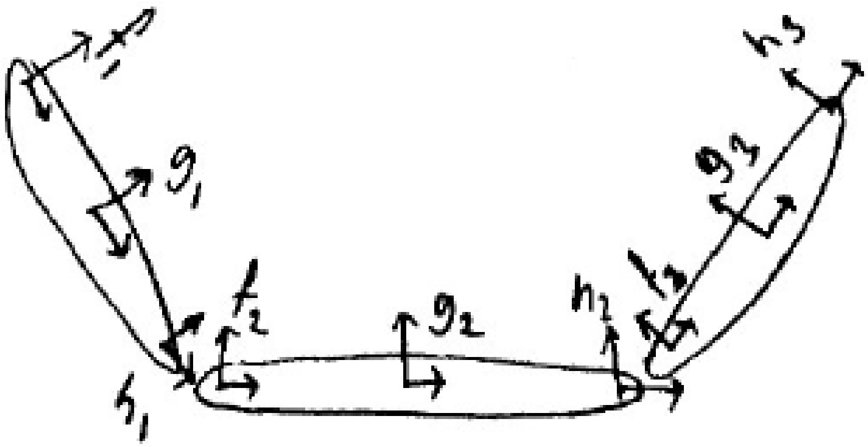
$$\xi_{g_2} = Ad^{-1}_{g_2,g_2'}\xi_{g_2'} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \delta\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{g_2,g_2'=(\delta,0,0)}} \overbrace{\begin{bmatrix}\dot\delta\\ \ell_1\dot\alpha\\ \dot\alpha\end{bmatrix}}^{\xi_{g_2'}} = \begin{bmatrix}\dot\delta\\ (\ell_1+\delta)\dot\alpha\\ \dot\alpha\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1\\ \ell_1+\delta & 0\\ 1 & 0\end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix}\dot\alpha\\ \dot\delta\end{bmatrix}.$$

Above is extraneous

$$\xi_{h_2} = Ad^{-1}_{h_2,g_2'}\xi_{g_2'} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \delta+\ell_2\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{h_2,g_2'=(\delta+\ell_2,0,0)}} \overbrace{\begin{bmatrix}\dot\delta\\ \ell_1\dot\alpha\\ \dot\alpha\end{bmatrix}}^{\xi_{g_2'}} = \begin{bmatrix}\dot\delta\\ (\ell_1+\delta+\ell_2)\dot\alpha\\ \dot\alpha\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1\\ \ell_1+\delta+\ell_2 & 0\\ 1 & 0\end{bmatrix}}^{J^b_{h_2}} \begin{bmatrix}\dot\alpha\\ \dot\delta\end{bmatrix}$$

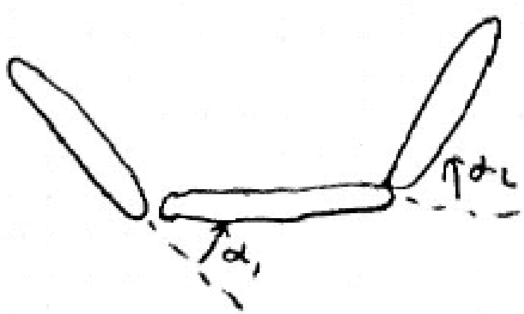
Three-Link System

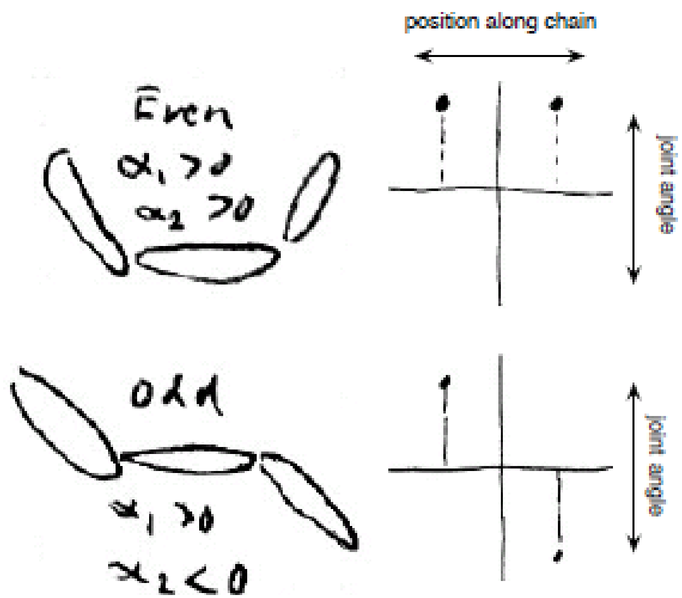




Which way is up (positive)

- For serial links, be consistent from proximal to distal
- “even” and “odd” sets of joint angles (respectively sign-matched and sign-opposite) correspond to physical configurations with even (bilateral) and odd (rotational) symmetries, as shown at the right





The Middle Link

find the body velocity of each link
systems overall body velocity ξ and its shape velocity $\dot{r}$;

Let the body frame be equal to g_2

$$\xi_{g_2} = \xi = \begin{bmatrix} I^{3\times 3} & 0^{3\times 2} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

$$\xi_{f_2} = Ad^{-1}_{f_{2,g_2}} \xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(-\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

$$\xi_{h_2} = Ad^{-1}_{h_{2,g_2}} \xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

$$\xi_{f_2} = \begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

Link 1

Body velocity of two frames are related by the adjoint inverse of the relative transformation

$$\xi_{h_1} = Ad^{-1}_{h_1,h_1'} \xi_{h_1'} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_1,h_1'}=(0,0,-\alpha_1)} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{f_2}} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\alpha}_1 \end{bmatrix} \right)}^{\xi_{h_1'}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}.$$

Link 1 has a rotational velocity of $-\alpha_1$ wrt f_2

$$\xi_{g_1} = Ad^{-1}_{g_1,h_1} \xi_{h_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_1/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_1,h_1}=(0,-\ell_1/2,0)} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}}^{\xi_{h_1}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}$$

$$\xi_{h_2} = \begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

Link 3

$$\xi_{f_3} = Ad^{-1}_{f_3,f_3'}\,\xi_{f_3'} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_3,f_3'=(0,0,\alpha_2)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{h_2}} + \begin{bmatrix} 0 \\ 0 \\ \dot\alpha_2 \end{bmatrix} \right)}^{\xi_{f_3'}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2) \sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2) \cos\alpha_2 \\ \xi^\theta + \dot\alpha_2 \end{bmatrix},$$

Link 1 has a rotational velocity of α_2 wrt h_2

$$\xi_{g_3} = Ad^{-1}_{g_3,f_3}\,\xi_{f_3} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_3/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_3,f_3=(0,\ell_3/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2) \sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2) \cos\alpha_2 \\ \xi^\theta + \dot\alpha_2 \end{bmatrix}}^{\xi_{f_3}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2) \sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2) \cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \\ \xi^\theta + \dot\alpha_2 \end{bmatrix}$$

The Jacobians

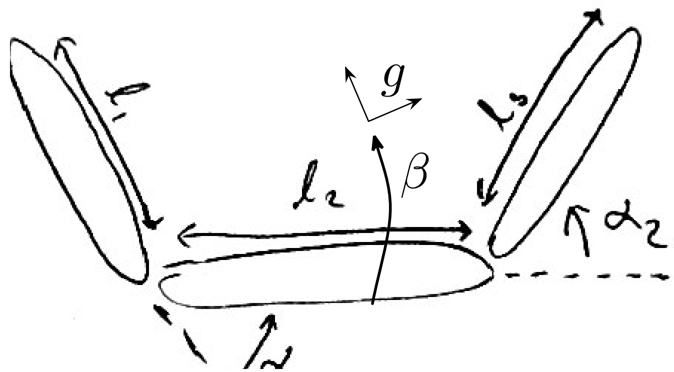
$$\xi_{g_1} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & -(\ell_2/2)\sin\alpha_1 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & (\ell_2/2)\cos\alpha_1 - (\ell_1/2) & \ell_1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}}^{J^b_{g_1}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix},$$

$$\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}$$

$$\xi_{g_3} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & (\ell_2/2)\sin\alpha_2 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2/2)\cos\alpha_2 + (\ell_3/2) & 0 & \ell_3/2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}^{J^b_{g_3}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}.$$

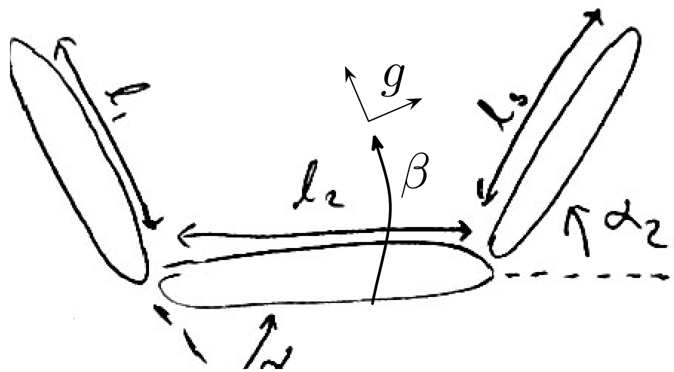
Note how it all depends on *g* and g_i' s

Three-Link Alternate Body Frame Jacobians



1. If we keep the middle link as the system body frame, what is the body velocity of frame g in Figure 2.9?
2. If we take frame g as the system body frame, what are the body velocities of the links?

Three-Link Alternate Body Frame Jacobians



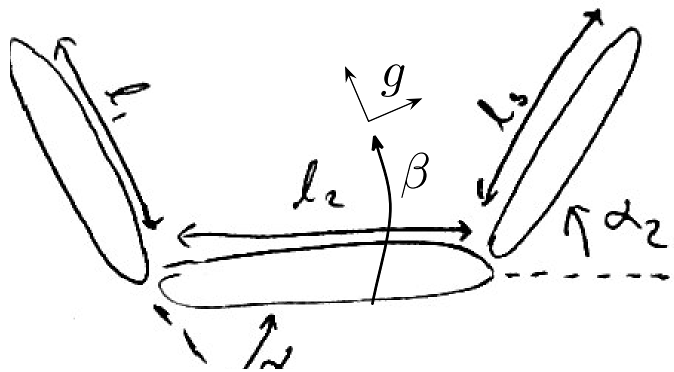
$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + v_\beta)$$

This leaves the question, however, of calculating v_β. We don't just want $\dot{\beta} = (\partial\beta/\partial\alpha)\,\dot{\alpha}$, we want *the velocity with respect to* g_2 *of the frame rigidly attached to* g *and coincident with* g_2. Taking advantage of properties of the spatial velocity, we can use a right action to find this velocity,

$$v_\beta = T_\beta R_{\beta^{-1}}\dot{\beta} = T_\beta R_{\beta^{-1}}\frac{\partial\beta}{\partial\alpha}\dot{\alpha}$$

$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + T_\beta R_{\beta^{-1}}\frac{\partial\beta}{\partial\alpha}\dot{\alpha})$$

Three-Link Alternate Body Frame Jacobians

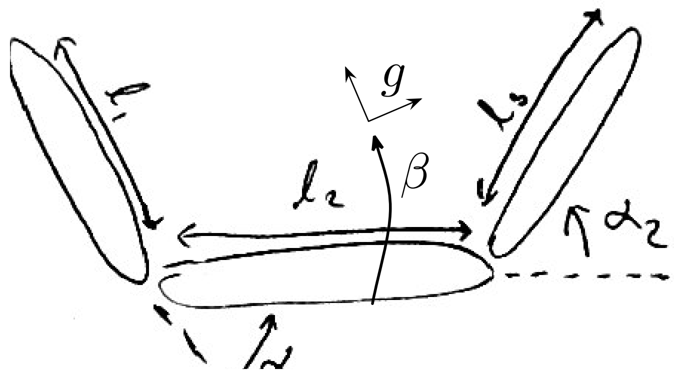


$$\xi_g = Ad_\beta^{-1}(\xi_{g_2} + T_\beta R_{\beta^{-1}} \frac{\partial \beta}{\partial \alpha}\dot{\alpha})$$

$$\xi_g = \begin{bmatrix} \cos\beta^\theta & \sin\beta^\theta & \beta^x \sin\beta^\theta - \beta^y \cos\beta^\theta \\ -\sin\beta^\theta & \cos\beta^\theta & \beta^x \cos\beta^\theta + \beta^y \sin\beta^\theta \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 & \beta^y \\ 0 & 1 & -\beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \beta^x}{\partial \alpha_1} & \frac{\partial \beta^x}{\partial \alpha_2} \\ \frac{\partial \beta^y}{\partial \alpha_1} & \frac{\partial \beta^y}{\partial \alpha_2} \\ \frac{\partial \beta^\theta}{\partial \alpha_1} & \frac{\partial \beta^\theta}{\partial \alpha_2} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \right) \tag{2.63}$$

$$= \begin{bmatrix} \cos\beta^\theta & \sin\beta^\theta & \beta^x \sin\beta^\theta - \beta^y \cos\beta^\theta \\ -\sin\beta^\theta & \cos\beta^\theta & \beta^x \cos\beta^\theta + \beta^y \sin\beta^\theta \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^\theta \end{bmatrix} + \begin{bmatrix} \frac{\partial \beta^x}{\partial \alpha_1} + \frac{\partial \beta^\theta}{\partial \alpha_1}\beta^y & \frac{\partial \beta^x}{\partial \alpha_2} + \frac{\partial \beta^\theta}{\partial \alpha_2}\beta^y \\ \frac{\partial \beta^y}{\partial \alpha_1} - \frac{\partial \beta^\theta}{\partial \alpha_1}\beta^x & \frac{\partial \beta^y}{\partial \alpha_2} - \frac{\partial \beta^\theta}{\partial \alpha_2}\beta^x \\ \frac{\partial \beta^\theta}{\partial \alpha_1} & \frac{\partial \beta^\theta}{\partial \alpha_2} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \right) \tag{2.64}$$

Three-Link Alternate Body Frame Jacobians



For individual link Jacobians, start by getting middle link velocity as function of new frame velocity

$$\xi_{g_2} = Ad_{\beta}\xi_g - v_{\beta}$$

Similar Jacobian extraction process as just covered, and iterative Jacobian evaluation for the outer links can proceed from this expression.