Chapter 1: Configurations and Velocities

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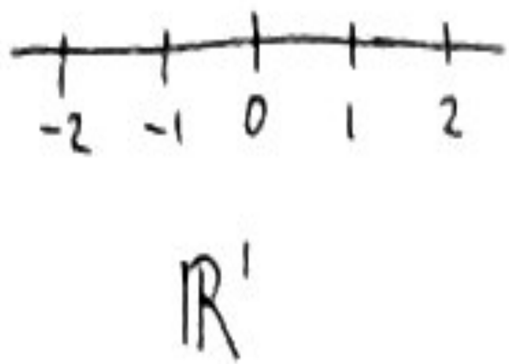
Degrees of Freedom

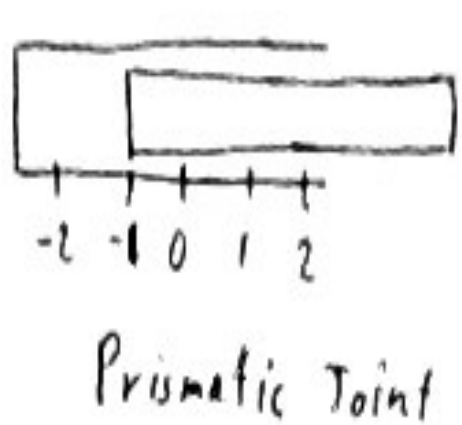
- I used to like relative motion between rigid bodies
- I used to like relative motion from a frame
- Lets go with number of independent ways a system can move
  - Translation
  - Rotation
  - Bending at joint

Configuration Space

- configuration, denoted q, is an arrangement of degrees of freedom that uniquely defines the location in the world of each point on the system
- Configuration space Q is the set of al q
- Dim(Q) = #DOF

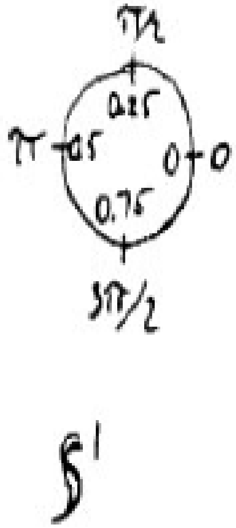
Non-rotational 1 DOF Qs

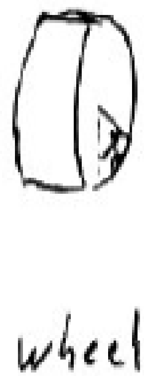


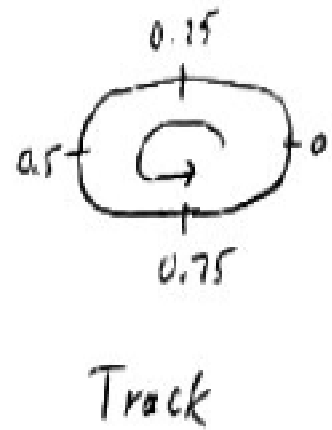


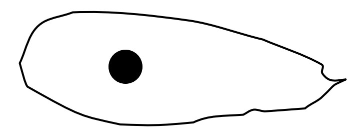
Not linear but related

Rotary 1-DOF Qs









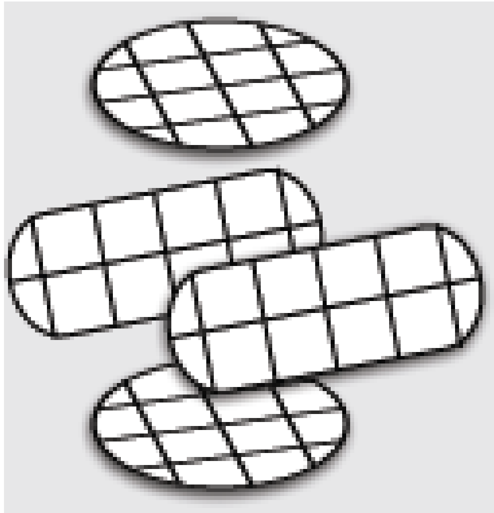
Planar rigid body
“nailed” in place

Not necessarily a circle

What about joint limits

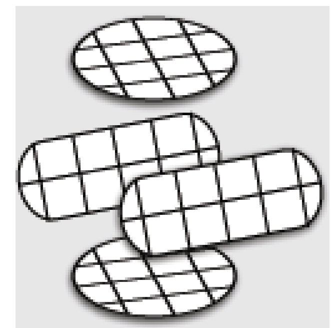
Manifold

- manifold is a space that is locally like a Euclidean space, but may have a more complicated global structure
- described by the an *atlas* of *charts* corresponding to the manifold
  - *chart* is a region of Euclidean space that maps to a region of the manifold
  - *atlas* is a set of overlapping charts that collectively describe the entirety of the manifold

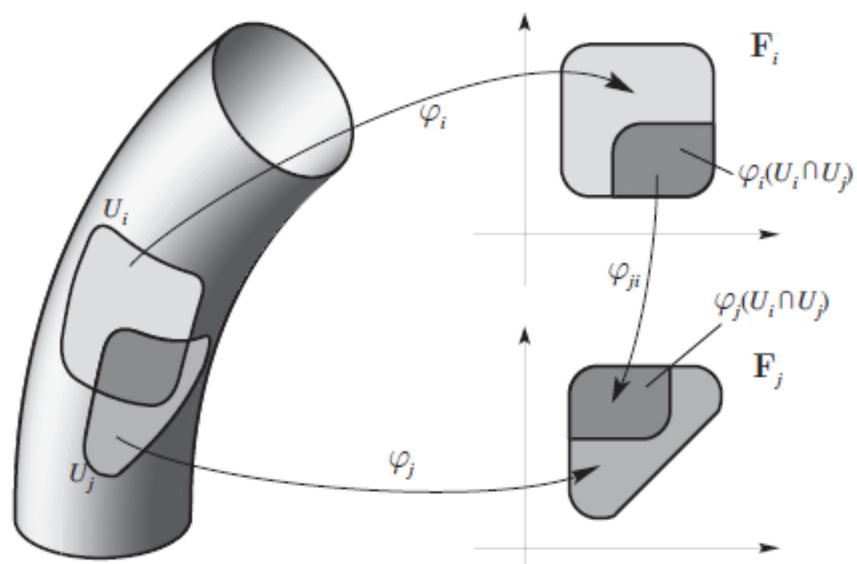


Manifold: Charts

- charts inherently parameterize a space by assigning points in Euclidean space
- multi-chart atlases is to “paper over” singularities: latitude-longitude map is singular at the poles, and so must be combined with additional maps



- Overlap maps: At the overlap, between charts, there is naturally a mapping from each chart into the manifold, then back out into the other chart. These composite functions are the transition maps between charts, describing how to translate coordinates on one chart to coordinates on a second chart



overlap map: $\varphi_{ji} = \varphi_j \circ \varphi_i^{-1}|\varphi_i(U_i \cap U_j)$.

C^k-differentiable Manifolds

1. The mappings from the charts to the manifolds must each be k-times differentiable, i.e., they must be C^k-diffeomorphisms.

2. All overlap maps for charts in the atlas must be C^k-diffeomorphisms

When $k=\infty$, then the manifold is a smooth or differential manifold

Homeo- and Diffeomorphisms

- Recall mappings:
  - $\phi: S \rightarrow T$
  - If each elements of ϕ goes to a unique T, ϕ is *injective* (or 1-1)
  - If each element of T has a corresponding preimage in S, then ϕ is *surjective* (or onto).
  - If ϕ is surjective and injective, then it is bijective (in which case an inverse, ϕ^{-1} exists).
  - ϕ is *smooth* if derivatives of all orders exist (we say ϕ is C^{∞})

- If $\phi: S \rightarrow T$ is a bijection, and both ϕ and ϕ^{-1} are continuous, ϕ is a *homeomorphism;* if such a ϕ exists, S and T are *homeomorphic.*

- If homeomorphism where both ϕ and ϕ^{-1} are smooth is a *diffeomorphism.*

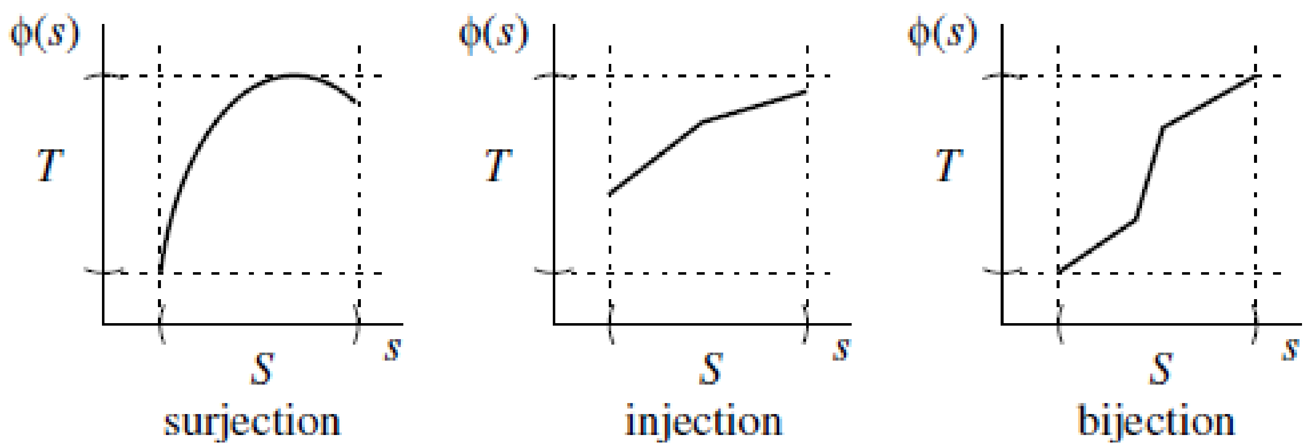
Homeo/Diffeo-morphisms

	Homeomorphism	Diffeomorphism	Ck-Diffeomorphism
f	Continuous, C^0	Smooth, C^∞	C^k differentiable
f^{-1}	Continuous, C^0	Smooth, C^∞	C^k differentiable
f	bijective	bijective	bijective

Surjective: Every point in the range is a function of at least one point in the domain
many to one mapping, e.g., sine is surjective onto [-1,1]

Injective: One to one, e.g., every point in the domain maps to a unique point in the rang

Bijective: BOTH

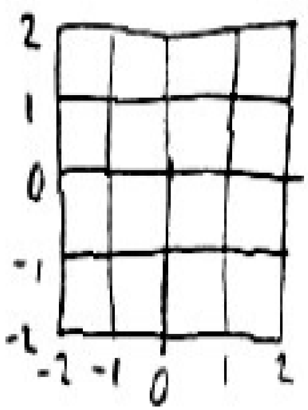


Form Manifolds with Direct Products

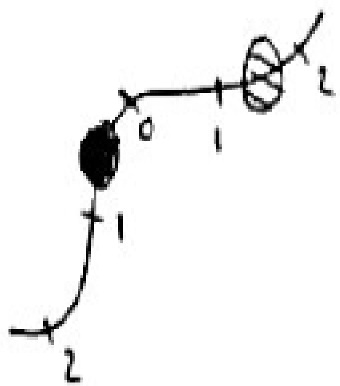
Direct Product

The *direct product*, or *Cartesian product* of two sets or spaces combines them without mixing the elements together. For instance, if we have two systems with configurations $a \in \mathbb{R}^1_a$ and $b \in \mathbb{R}^1_b$, the direct product of the configuration spaces, $\mathbb{R}^2 = \mathbb{R}^1_a \times \mathbb{R}^1_b$, is structured to preserve the independence of the component subspaces,

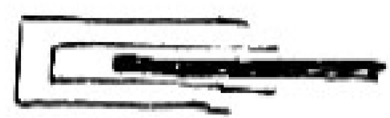
$$(a, b) \in \mathbb{R}^2 \equiv (a \in \mathbb{R}^1_a, b \in \mathbb{R}^1_b). \tag{1.i}$$



$\mathbb{R}^2$

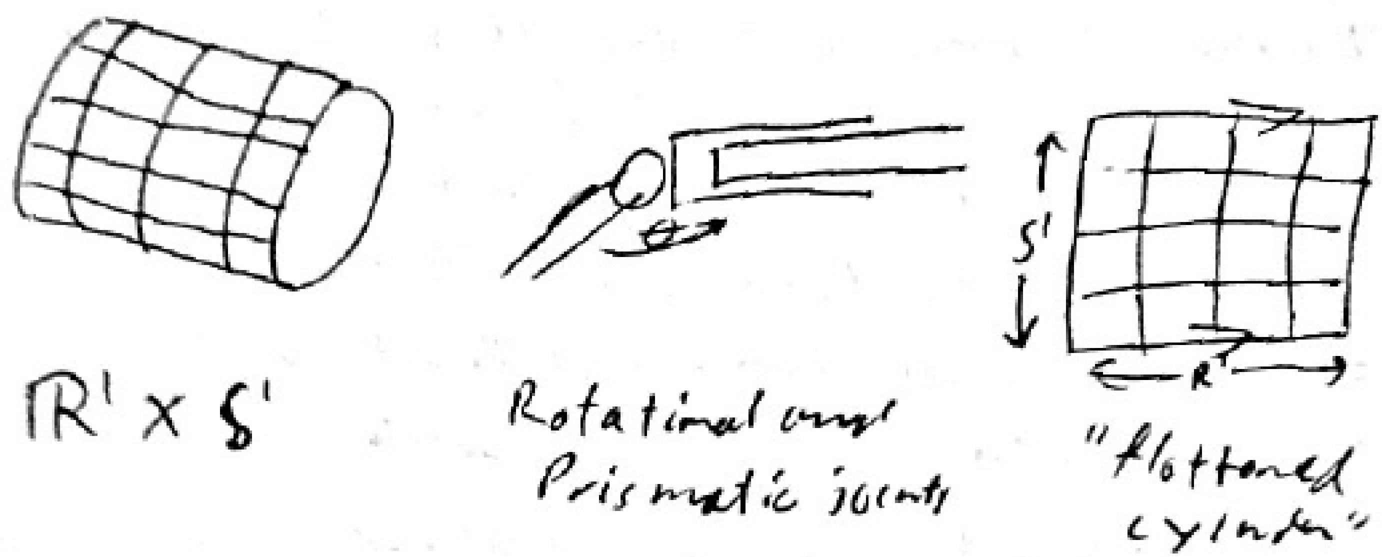


Two beads on a wire

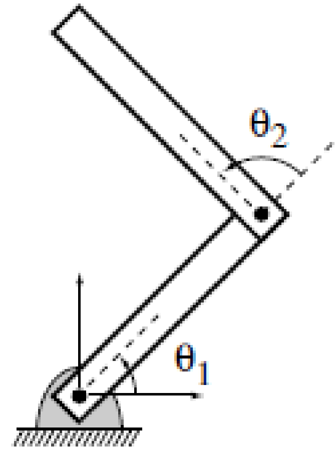


Double prismatic joint

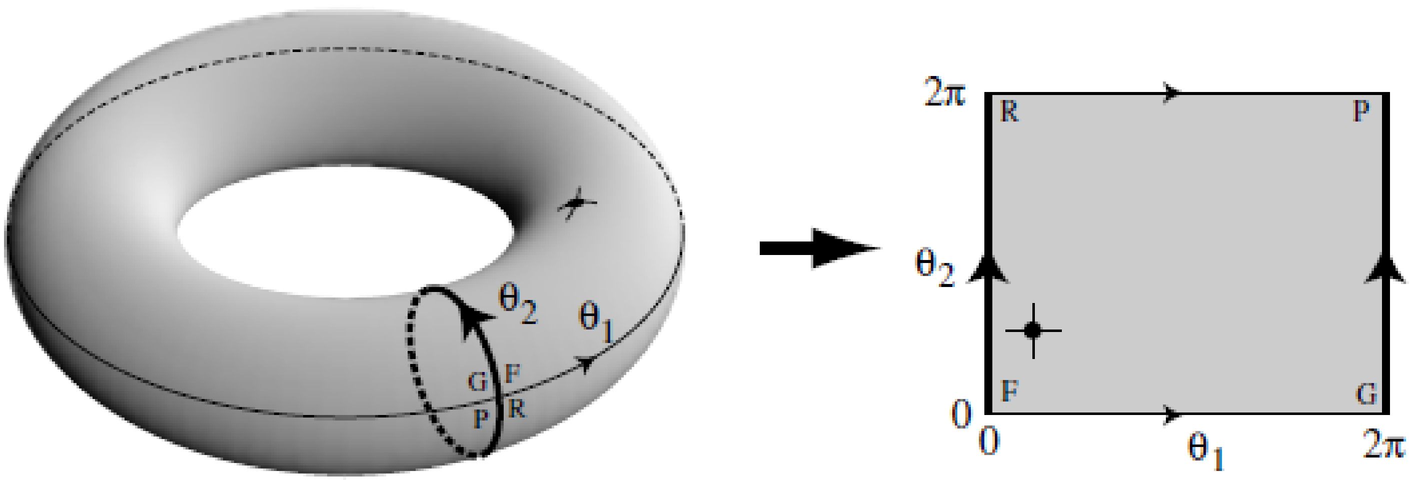
Form Manifolds with Direct Products



Form Manifolds with Direct Products

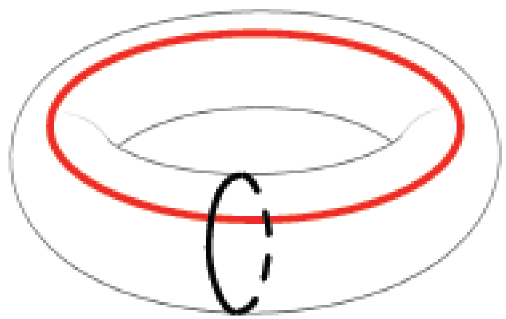


(a)



(b) (c)

The Sphere



Similarities

Differences

Lie Groups: Manifold + Group

- Motivation
  - Perform algebraic operations on configurations, say add or subtract
  - Rigid body motion is nicely described by Lie Groups
  - Another?
- What’s a group?
- Examples of groups
- Same manifold, different group

Groups

$g_1, g_2 \in G$ then $g_1 \circ g_2 \in G$

A group $(G, \circ)$ is the combination of a set G and an operation $\circ$ that satisfies the following properties:

1. **Closure**: The product of any element of G acting on another by the group operation must also be an element of G. More formally, for $g_1, g_2 \in G$,
$$\begin{aligned} g_1 &: G \to G \\ g_2 &\mapsto g_1 \circ g_2. \end{aligned} \tag{1.i}$$
2. **Associativity**: The order in which group operations are evaluated must not affect the product: for all $g_1, g_2, g_3 \in G$,
$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3. \tag{1.ii}$$
3. **Identity element**: The set must contain an identity element e that leaves other elements unchanged when it interacts with them: for $g \in G$, there exists $einG$ such that
$$e \circ g = g = g \circ e \tag{1.iii}$$
4. **Inverse**: The inverse (with respect to the group operation) of each group element must be an element of the group and produce the identity element when operating on or operated on by its respective element: for $g \in G$, there must exist $g^{-1} \in G$ such that
$$g^{-1} \circ g = e = g \circ g^{-1} \tag{1.iv}$$

Left and Right Actions

- Left Action $L_h g = h \circ g$
- Right Action $R_h g = g \circ h$

- Abelian groups (additive)

$$L_h g = h \circ g = h + g = g + h = g \circ h = R_h g$$

- Most groups are not abelian, ie matrix multiplication does not commute

Examples of Groups

$$(R^1, +)$$

$$(R^+, x)$$

$$\{A \in R^{n \times n} \mid \det(A) \neq 0\}$$

$$O(n) = \{A \in R^{n \times n} \mid \det(A) = \pm 1\}$$

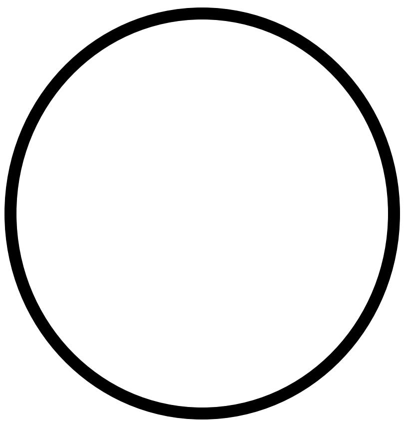
$$SO(n) = \{A \in O(n) \mid \det(A) = 1\}$$

$$NSSO(n) = O(n) \setminus SO(n)?$$

Combinations of lines and circles created via the direct product naturally inherit the group structures of their component spaces, with (modular) addition acting independently along each degree of freedom.

One Manifold, Many Groups

Consider $(SO(2), \times)$ θ is magnitude of rotation



$$R \in SO(2) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

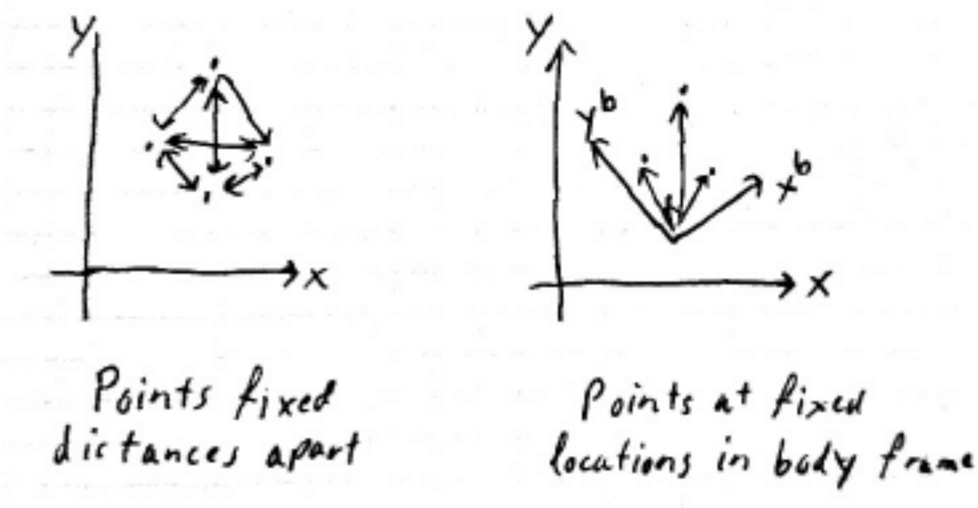
matrices are smooth, cyclic, and unique with respect to theta, so homeomorphic to a circle

Consider $(S^1, + \bmod k)$ modular arithmetic

Mod k for k = 2π, it is the common mod +
What meaning does SO(2) have?
How do they generalize? Not the same way

Rigid Body Configurations

- An object that does not deform in response to external forces
- Set of points with fixed interpoint distances and relative orientation
- Movable reference point and all points are fixed with respect to this frame (infinitely large bodies)



Choices of body frame?
Origin: Center of mass

What about orientation?

Which Lie Group?

Not really $SO(2) \circledS \mathbb{R}^2$.

Semi-direct product

- $(R^2 \times S^1; +)$
- $SE(2) = (R^2,+) \times (SO(2),\times)$

What is special about the special Euclidean group?

$$g \in SE(2) \qquad g = (x, y, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Direct/Semi- Product with Groups

Direct and Semi-direct Products of Groups

When we combine groups together to form larger groups, we must consider not only how their underlying sets combine, but also what the overall group action becomes. Often, the combination of two groups A and B into a new group C is taken to mean the creation of a *direct product group* $C = A \times B$, in which components that started out in A or B only affect other components that started out as elements of the same group, *i.e.*, $c_1c_2 = (a_1a_2, b_1b_2)$. Direct products preserve properties such as being abelian (commutative) – if A or B has this property, then so does the corresponding section of C.

In a semi-direct product group, $D = A \circledS B$, elements of A act not only on each other, but also on elements of B. One such structure that appears in our kinematics discussion takes the form

$$d_1d_2 = (a_1a_2, b_1(a_1b_2)). \tag{1.vi}$$

A key aspect of such groups is that even though they do not possess the full orthogonality of a direct product group, the A components do preserve their original properties, and thus results that depend on these properties can be applied to the corresponding elements of D.

Interpretations of SE(2)

- 1. the position and orientation of rigid bodies
- 2. the position and orientation of coordinate frames
- 3. actions that move a rigid body or coordinate frame with respect to a fixed coordinate frame
- 4. actions that take a point in one coordinate frame, and find the equivalent point in a second coordinate frame.

the position and orientation of a rigid object inherently identifies a body coordinate frame aligned with the object's longitudinal and lateral axes, and vice versa.

Interpretations of SE(2)

- 1. the position and orientation of rigid bodies
- 2. the position and orientation of coordinate frames
- 3. actions that move a rigid body or coordinate frame with respect to a fixed coordinate frame
- 4. actions that take a point in one coordinate frame, and find the equivalent point in a second coordinate frame.

Actions make these groups, and in particular Lie groups

Identity Element in SE(2)

$$I = (0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

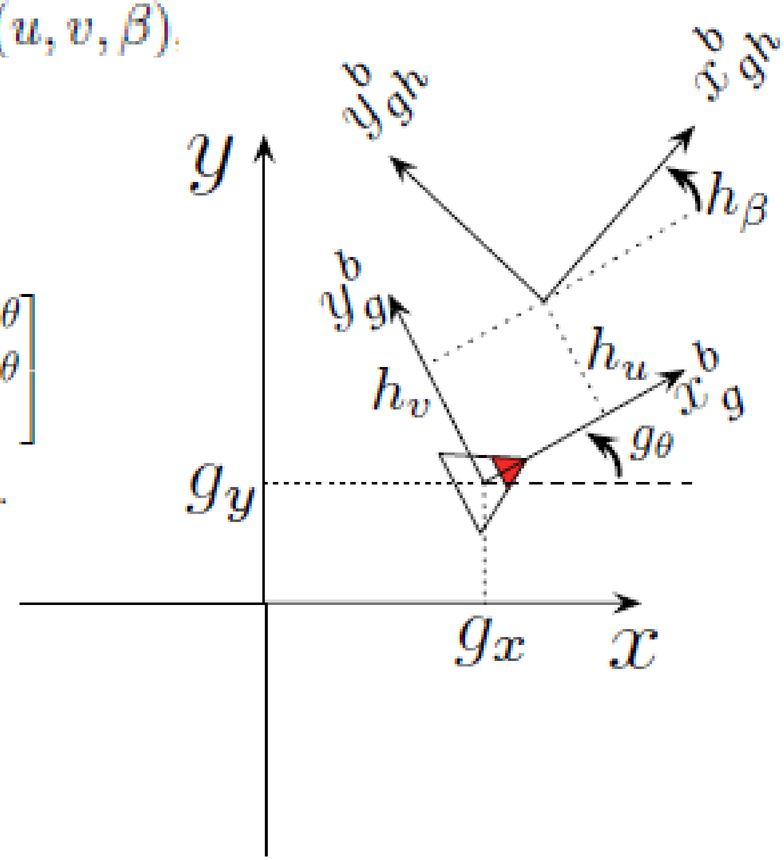
When $h = g^{-1}$ both left and right actions return the identity element

$$g^{-1}g = gg^{-1} = I = (0,0,0).$$

Right Action in SE(2)

$g, h \in SE(2)$, with $g = (x, y, \theta)$ and $h = (u, v, \beta)$.

$$
\begin{aligned}
gh &= \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & u \\ \sin\beta & \cos\beta & v \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) & x + u\cos\theta - v\sin\theta \\ \sin(\theta+\beta) & \cos(\theta+\beta) & y + u\sin\theta + v\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \\
&= (x + u\cos\theta - v\sin\theta, y + u\sin\theta + v\cos\theta, \theta + \beta).
\end{aligned}
$$



starting at position g and moving by h relative to this starting position

or

finding the global position of the point at h relative to g,

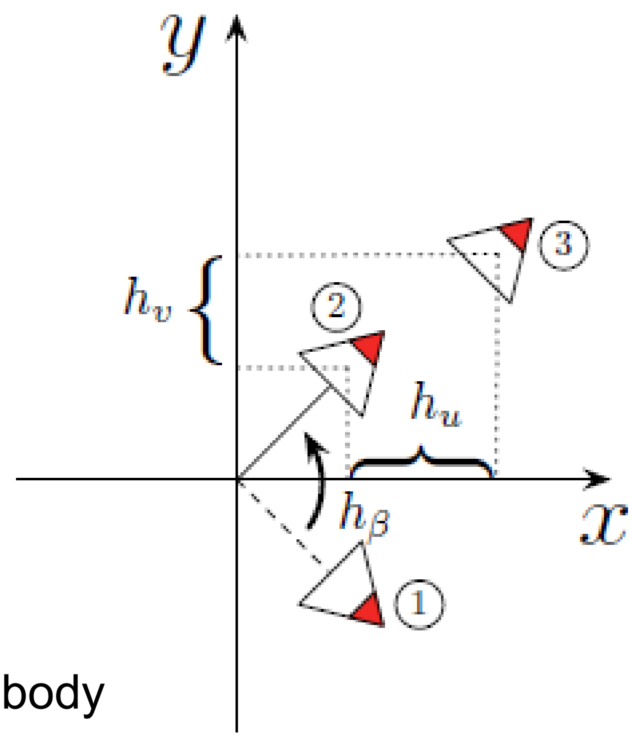
Left Action in SE(2)

$g, h \in SE(2)$, with $g = (x, y, \theta)$ and $h = (u, v, \beta)$.

$$hg = \begin{bmatrix} \cos\beta & -\sin\beta & u \\ \sin\beta & \cos\beta & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) & x\cos\beta - y\sin\beta + u \\ \sin(\theta+\beta) & \cos(\theta+\beta) & x\sin\beta + y\cos\beta + v \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (x\cos\beta - y\sin\beta + u, x\sin\beta + y\cos\beta + v, \theta + \beta).$$

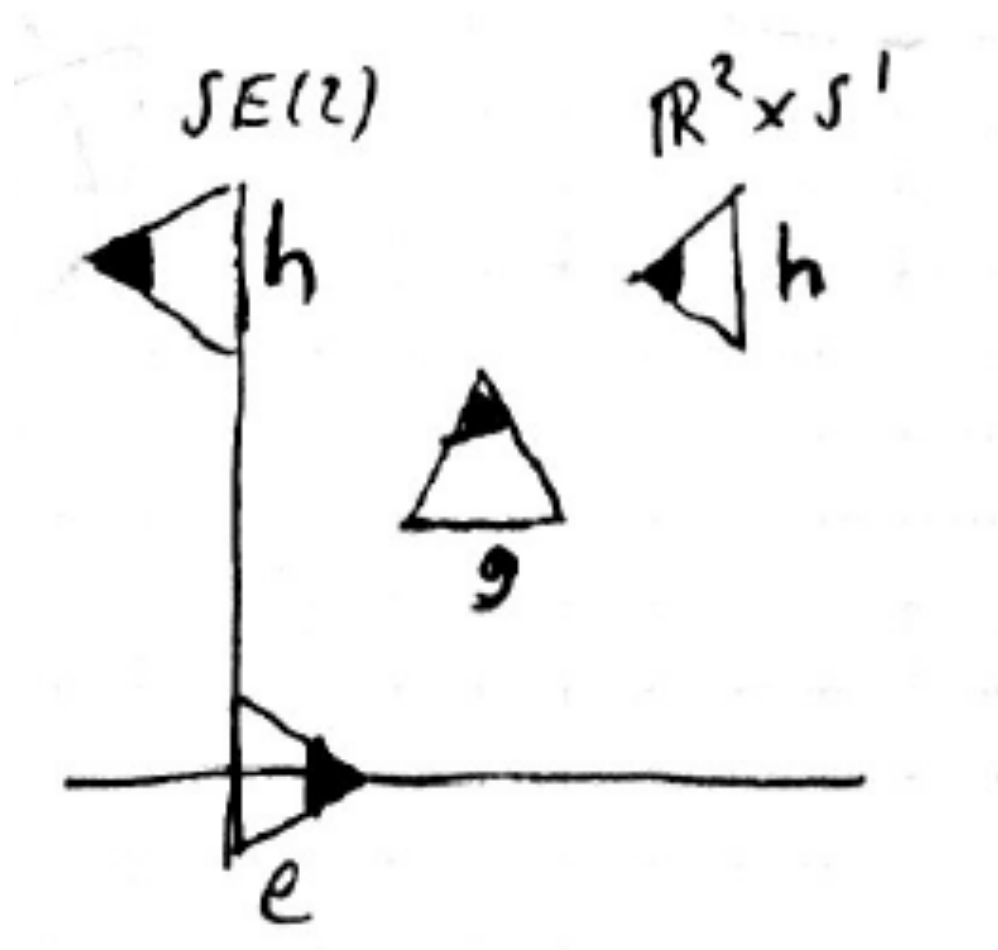


1. g as the (x; y, θ) coordinates of a rigid body, and
2. h as an action that transforms g by first rotating the body around the origin by β,
3. then translating it by (u; v),

OR - (absolute) location of the system as if g were defined with respect to h rather than the origin.

SE(2) vs. (R^2 x S^1,+) (right action)

Symmetries,
Coming soon

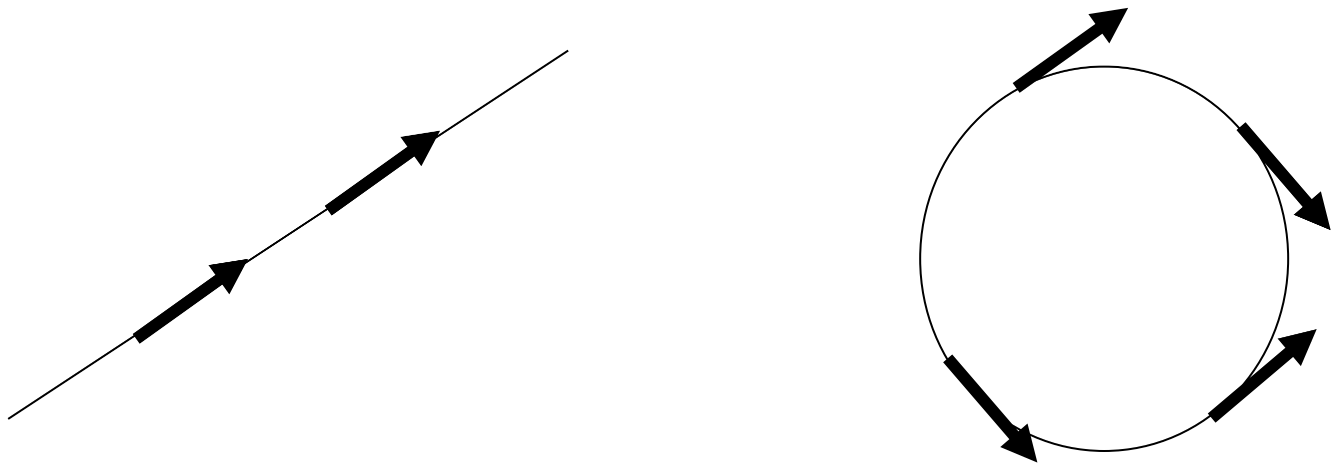


Unify Frames and Points

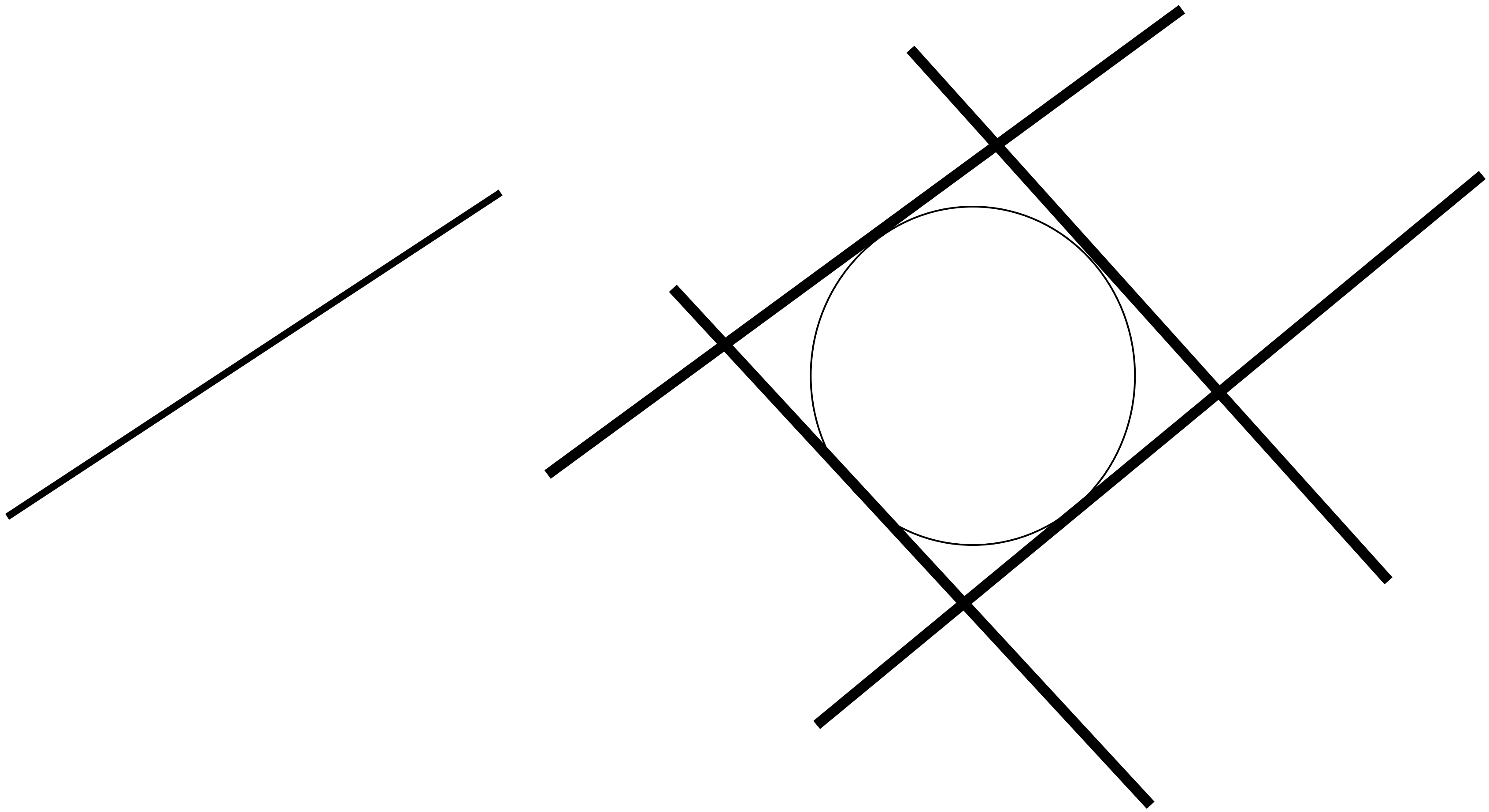
$$\begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\varnothing$}} & p_x \\ & & p_y \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}}^{g} \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\varnothing$}} & p_x^b \\ & & p_y^b \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x^b \\ p_y^b \\ 1 \end{bmatrix},$$

Tangent Spaces



Tangent Spaces



Tangent spaces are “flat” spaces whose elements are vectors based at point

Two Dimensional Tan Space

Notation and Velocity

configuration space

tangent space

For a $q \in Q$, we can have $\dot{q} \in T_q Q$

configuration

velocity

State and Bundles

$$x = \left(q, \dot{q}\right)$$

State

$$TQ = \bigcup_{q \in Q} (q, T_q Q)$$

Tangent Bundle

Vector Fields

Vector field assigns a vector to each point on a manifold

$$X : Q \to TQ \qquad X : Q \to T_q Q$$

$$X : q \mapsto (q, v) \qquad X : q \mapsto v$$

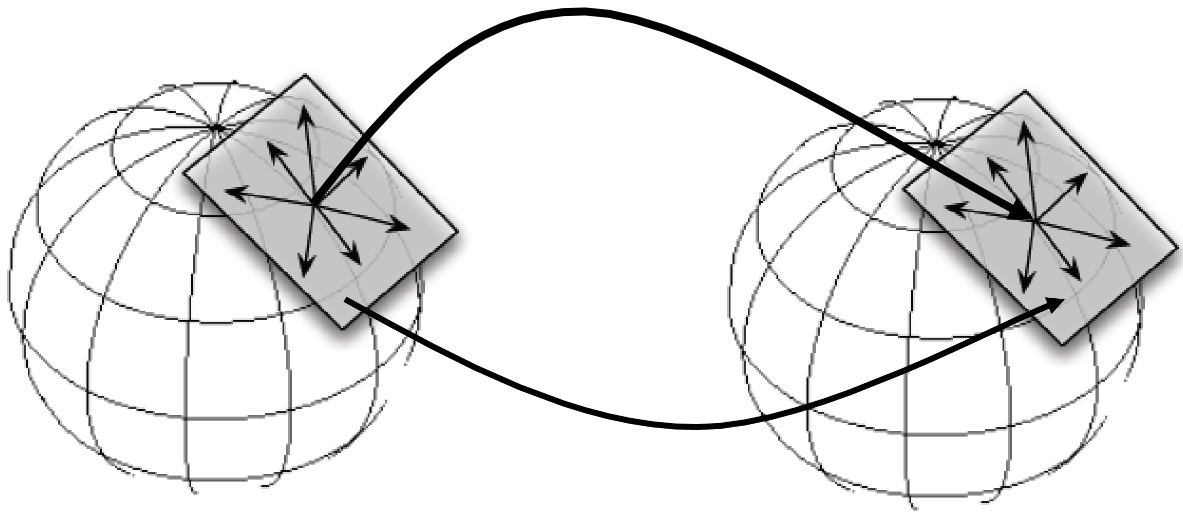
Examples: gradient vector field

potential vector field

constant vector field

Lifted Maps

$f : Q \to Q$



$T_q f : T_q Q \to T_{f(q)} Q$

Equivalence

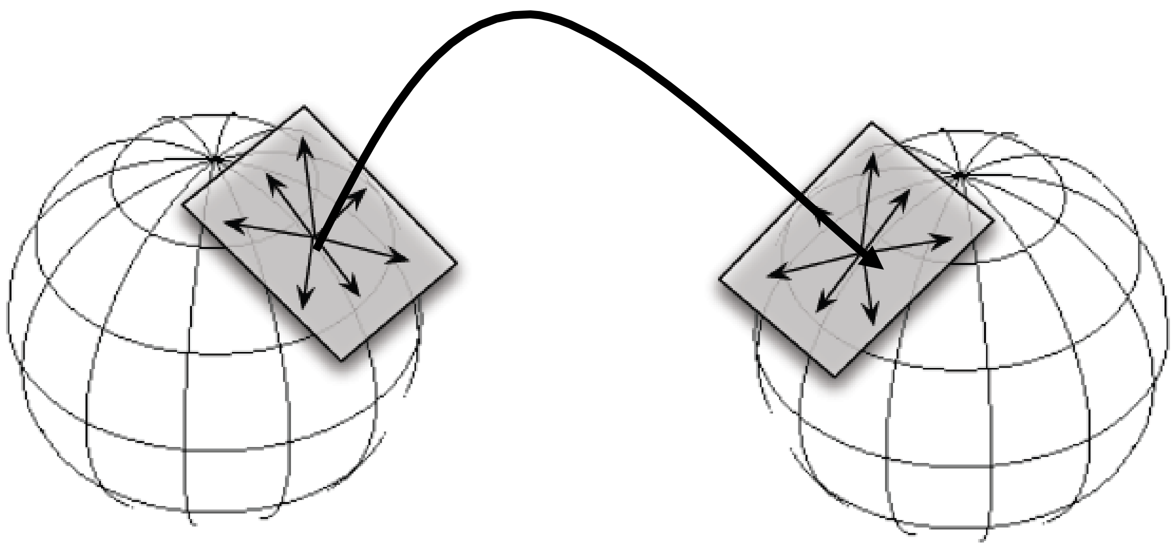
- Sometimes there is a structure that allows us to consider pairs of vectors in different tangent spaces as equivalent

- When this structure is present, the lifted map between these two spaces identifies these two vectors

- For Lie groups, the group provides this structure, e.g., left lifted map, right lifted map

Left Lifted Maps (on Lie Groups)

$$L_h : G \to G \qquad\qquad T_g L_h : T_g G \to T_{hg} G$$
$$g \mapsto hg \qquad\qquad \dot{g} \mapsto T_g L_h \dot{g}$$



Bonus question: what is wrong with this figure?

Left Lifted Maps (on Additive Groups)

$$L_h g = h \circ g = h + g$$

$$T_g L_h = \frac{\partial (L_h g)}{\partial g}$$

$$\frac{\partial([h_1, h_2] + [g_1, g_2])}{\partial [g_1, g_2]}$$

$$\begin{bmatrix} \frac{\partial(h_1 + g_1)}{\partial g_1} & \frac{\partial(h_1 + g_1)}{\partial g_2} \\ \frac{\partial(h_2 + g_2)}{\partial g_1} & \frac{\partial(h_2 + g_2)}{\partial g_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On additive groups, equivalent vectors all have the same components,

Which means we care carefeely and carelessly add vectors wherever we want.

Left Lifted Maps (on Lie Groups)

$$(\mathbb{R}^+, \times)$$

$$L_h g = hg,$$

$$T_g L_h = \frac{\partial(hg)}{\partial g} = h,$$

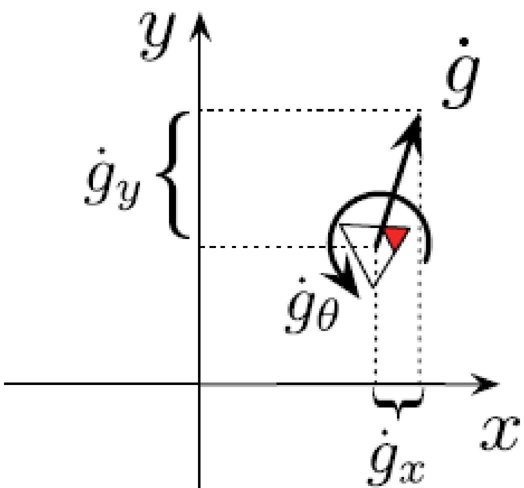
Equivalent vectors are scaled in proportion to the group position

This calculation illustrates that equivalent vectors on multiplicative groups are based on the proportional rate of change

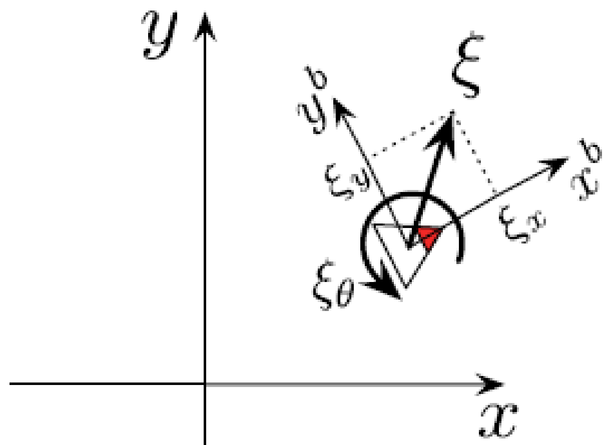
Review of Notation

Notation	Meaning
TQ	Tangent bundle to Q
T_qQ	Tangent space to Q at q
T_gL_h	Left lifted action mapping velocities from T_gG to $T_{hg}G$
T_gR_h	Right lifted action mapping velocities from T_gG to $T_{gh}G$

More Complex Use of Lifted Actions: Body Velocity



World Velocity



Body Velocity

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{g}_x \\ \dot{g}_y \\ \dot{g}_\theta \end{bmatrix}$$

$$\dot{g} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \xi$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi.$$

rotating the translational component by -θ (equivalent to rotating the reference frame by θ) and leaving the rotational component unchanged,

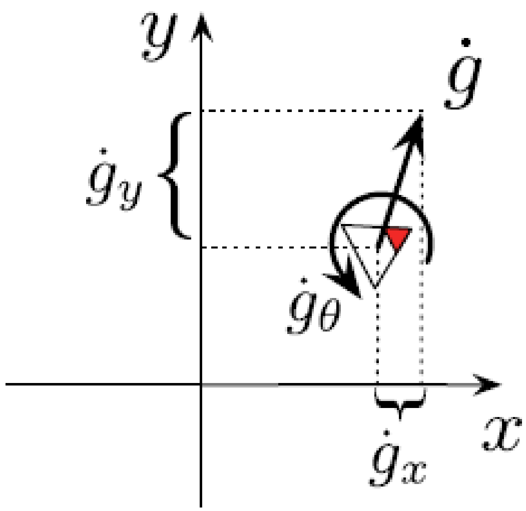
Left Lifted Action on *SE(2)*

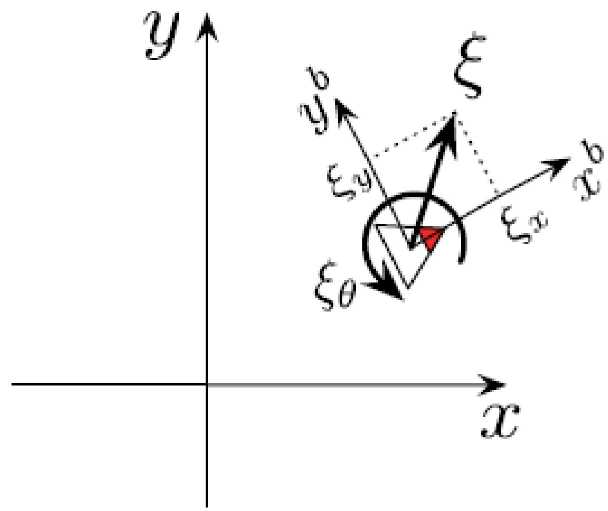
- Gain greater insight into rigid body motion and apply some powerful mathematical tools to these systems
- Let g = (x,y,θ), h = (u,v,β)

$$
\begin{aligned}
T_g L_h &= \frac{\partial(hg)}{\partial g} \\
&= \begin{bmatrix} \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial x} & \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial y} & \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial \theta} \\ \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial x} & \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial y} & \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial \theta} \\ \frac{\partial(\theta+\beta)}{\partial x} & \frac{\partial(\theta+\beta)}{\partial y} & \frac{\partial(\theta+\beta)}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

-Looks similar to matrix on previous slide
-Preserves body velocity (magnitude of translation remains fixed)
-For any L_h that rotates by β, the lifted action rotates the translational component of the velocity vector by β

h = g^{-1}, lifted action gives body velocity

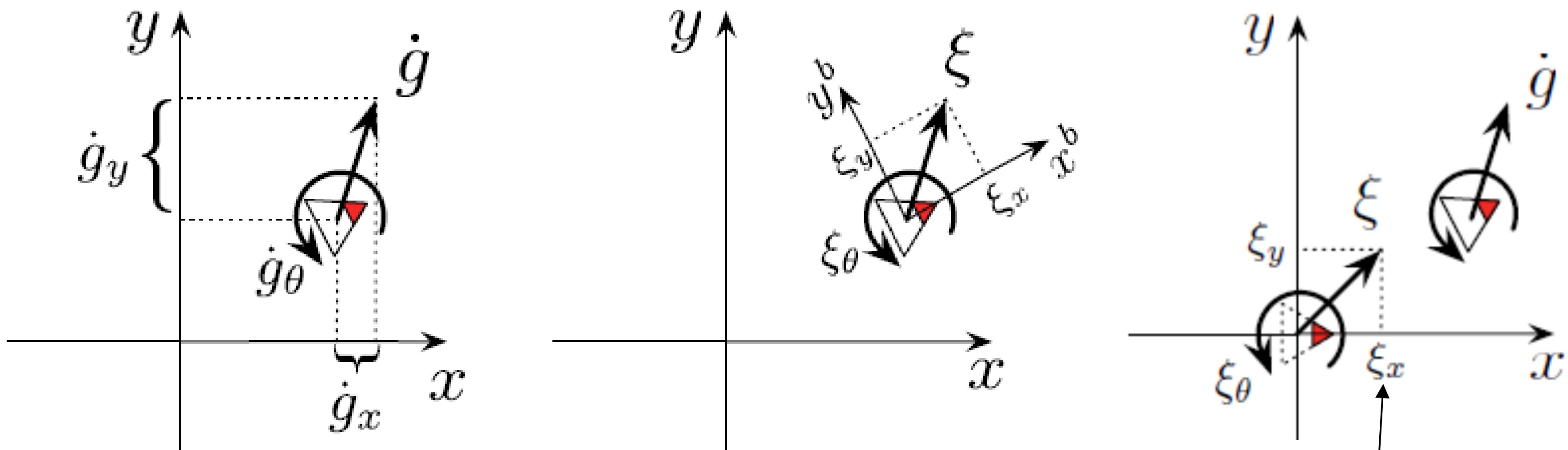




$$\xi = T_g L_{g^{-1}} \dot{g}$$

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = T_g L_{g^{-1}}$$

From $h = g^{-1}$ to Lie Algebra



$L_{g^{-1}}$ and $T_g L_{g^{-1}}$ take a rigid body from g and place it at the origin with equivalent body velocity.

Physically, body velocity is velocity vector in body frame

$$\xi = T_g L_{g^{-1}} \dot{g} \in T_e G$$

Algebraically, body velocity is left-equivalent velocity at the origin. Because moving to the origin is the same as moving the origin to you, this is OK.

$$\dot{g} = (T_g L_{g^{-1}})^{-1} \xi = T_e L_g \xi$$

$T_e G$ is the Lie algebra. One can do a lot in a Lie algebra

Right Lifted Action on *SE(2)*

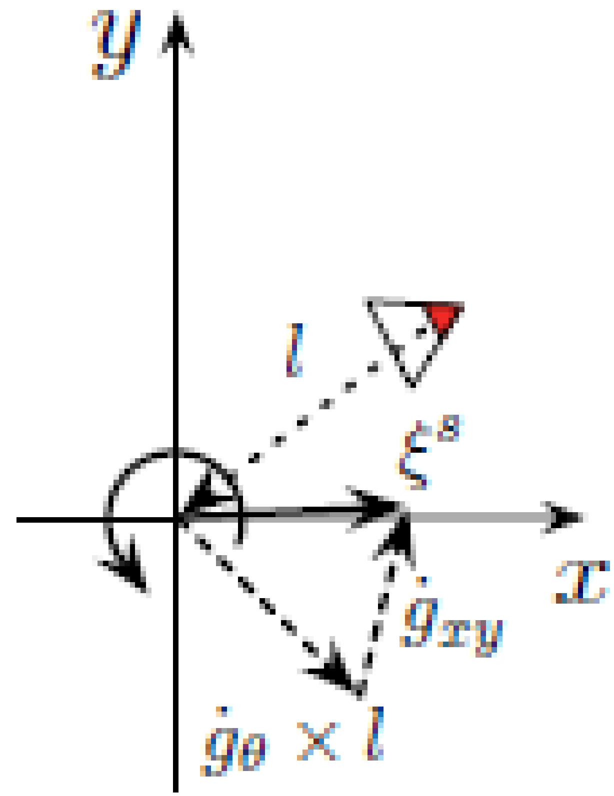
(was a cross product)

The velocity of two rigid frames on the same rigid body given we know the velocity of one of them

$$\dot{(gh)} = \begin{bmatrix} 1 & 0 & -(u\sin\theta + v\cos\theta) \\ 0 & 1 & u\cos\theta - v\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \dot{g}$$

Figure on the right considers a frame attached to the body and over the origin

l = (u, v)



Right Lifted Action on *SE(2)*

- What does $R_g h$ in SE(2) do?
- finds the frame at position and orientation *h* with respect to *g*.

$$T_g R_h = \frac{\partial(gh)}{\partial g}$$

$$= \begin{bmatrix} \frac{\partial(x+u\cos\theta-v\sin\theta)}{\partial x} & \frac{\partial(x+u\cos\theta-v\sin\theta)}{\partial y} & \frac{\partial(x+u\cos\theta-v\sin\theta)}{\partial\theta} \\ \frac{\partial(y+u\sin\theta+v\cos\theta)}{\partial x} & \frac{\partial(y+u\sin\theta+v\cos\theta)}{\partial y} & \frac{\partial(y+u\sin\theta+v\cos\theta)}{\partial\theta} \\ \frac{\partial(\theta+\beta)}{\partial x} & \frac{\partial(\theta+\beta)}{\partial y} & \frac{\partial(\theta+\beta)}{\partial\theta} \end{bmatrix}$$

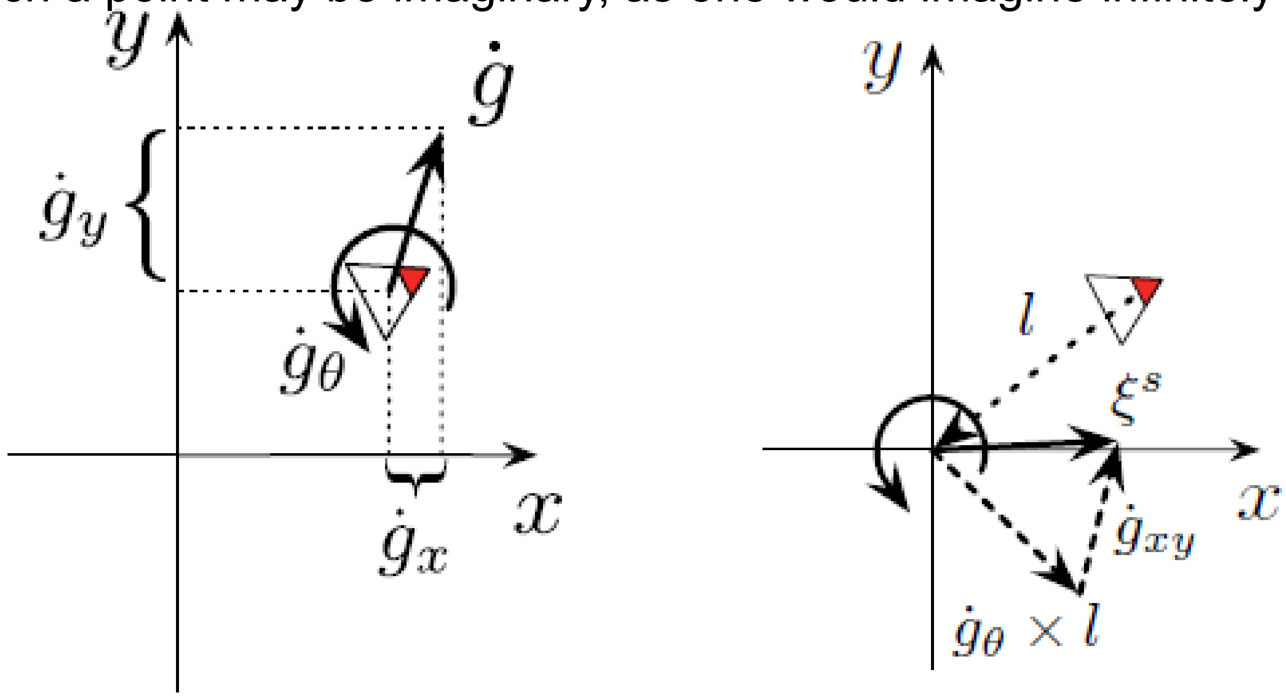
$$= \begin{bmatrix} 1 & 0 & -(u\sin\theta+v\cos\theta) \\ 0 & 1 & u\cos\theta-v\sin\theta \\ 0 & 0 & 1 \end{bmatrix},$$

Same relationship as before!

Spatial Velocity: $h = g^{-1}$

$$\xi^s = T_g R_{g^{-1}} \dot{g}. \qquad \dot{g} = (T_g R_{g^{-1}})^{-1} \xi^s = T_e R_g \xi^s$$

The velocity a point on the rigid body that passes through the origin
Note: such a point may be imaginary, as one would imagine infinitely large rigid bodies



Adjoint

The adjoint action Ad_g on $SE(2)$ maps the body velocity of a rigid body to its spacial velocity,

$$\xi^s = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\dot{g}} \xi$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g} \xi.$$

Similarly, the adjoint inverse, $Ad_g^{-1} = Ad_{g^{-1}}$ maps the spatial velocity of the system to the corresponding body velocity,

$$\xi = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & x\sin\theta - y\cos\theta \\ -\sin\theta & \cos\theta & x\cos\theta + y\sin\theta \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g^{-1}} \xi^s$$

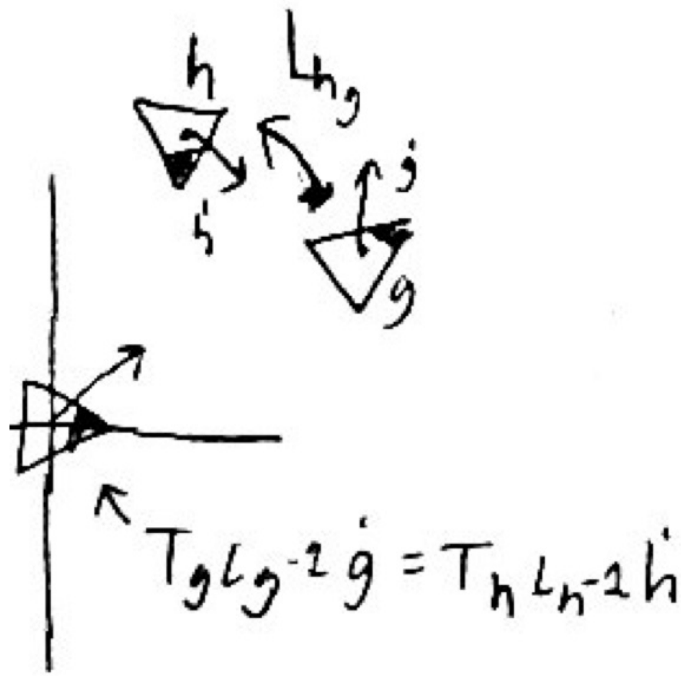
Review of *SE(2)*

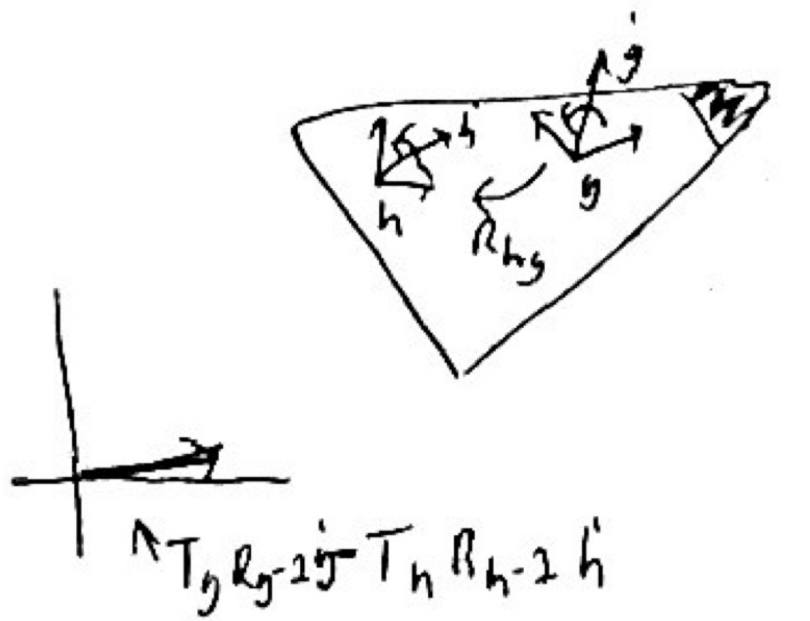
Left actions *move* elements

Right actions concern *relative* positions of elements

Left lifted actions preserve body velocity
- Two frames with left-equivalent velocity are each moving the same *with respect to themselves*

Right lifted actions preserve spatial velocity
-Two frames with right-equivalent velocity are moving as if rigidly attached (i.e., they remain a constant transformation away from each other





Where does this equivalence come from?

Velocity means different things when talking about addition and multiplication.

We can understand this difference better if we think about **Multiplicative Calculus**

	Continuous	Discrete
Addition	$\int$ Integral	$\sum$ Sum
Multiplication	$\prod$ Product-integral	$\prod$ Product

Multiplicative Calculus

In familiar Newton-Leibniz calculus, an integral is difference between the value of a function at two points, and is calculated as the cumulative sum of many small changes to the function:

$$F(b) - F(a) = \int_a^b u(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} u(x_k)\ \Delta x$$

This makes sense on additive groups, where sums sums and differences are well-defined.

On multiplicative groups, however, we don't have these operations. Instead, we have products and quotients. If generalize the notion of an integral to be the cumulative composition of small group elements, then we should be using Volterra's multiplicative calculus, in which integrals are quotients built of small multiplications:

$$\frac{\mathcal{F}(b)}{\mathcal{F}(a)} = \prod_a^b (I + v(x)\ dx) = \lim_{n\to\infty} \prod_{k=1}^{n} (I + v(x_k)\ \Delta x)$$

Multiplicative Calculus: Derivatives

In additive calculus, the derivative of a function is based on the difference between function values at consecutive points:

$$u(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

In multiplicative calculus, the same idea holds, except that we take the quotient of function values, not their difference:

$$v(x) = \lim_{\Delta x \to 0} \frac{\big(F(x + \Delta x)/F(x)\big) - I}{\Delta x}$$

Note that we subtract out the identity term, so that *v* is a small deviation from the identity element.

Derivatives on Multiplicative Lie Groups

Two velocities to consider:

- Velocity through the manifold (i.e. rate of change of the parameters) – Additive
- Group velocity – Multiplicative

When we talk about equivalent velocities, we are typically asking “For a given group velocity, what is the parameter velocity at different configurations?

First step to find this: Transform multiplicative velocity into additive time derivative

$$v(t) = \lim_{\Delta t \to 0} \frac{\big(g(t+\Delta t)/g(t)\big) - I}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{g^{-1}(t)\big(g(t+\Delta t) - g(t)\big)}{\Delta t}$$

$$= g^{-1}(t) \lim_{\Delta t \to 0} \frac{\big(g(t+\Delta t) - g(t)\big)}{\Delta t}$$

$$= g^{-1}(t)\frac{dg}{dt}(t)$$

←if this quantity is equal for two configuration/velocity pairs, the system velocity is equivalent under the group action

Derivatives on Multiplicative Lie Groups

continued

How does this relate to the lifted action?

So far, we have a means of converting between group velocities and the time rate of change of the group parameter:

$$v(t) = g^{-1}(t)\frac{dg}{dt}(t)$$

Earlier, we said that two velocities were equivalent according to the group if they were related by the lifted form of the group action:

$$\dot{g} \in T_gG \equiv (T_gL_h\dot{g}) \in T_{hg}G$$

Are these notions the same?

Derivatives on Multiplicative Lie Groups

continued

Yes!

Two vectors are equivalent (according to the lifted action) if and only if they share an equivalent vector at the group identity:

$$\dot{g} \in T_g G \equiv \dot{h} \in T_h G$$

iff

$$T_g L_{g^{-1}} \dot{g} = T_h L_{h^{-1}} \dot{h} = \xi$$

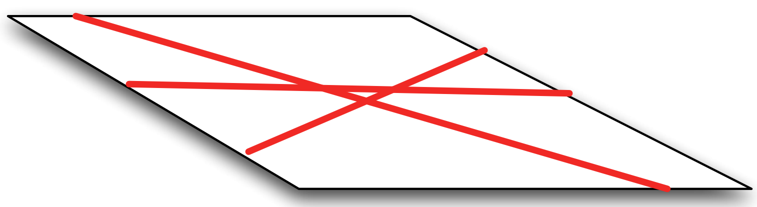
This looks very similar to our test for equivalence of multiplicative velocity:

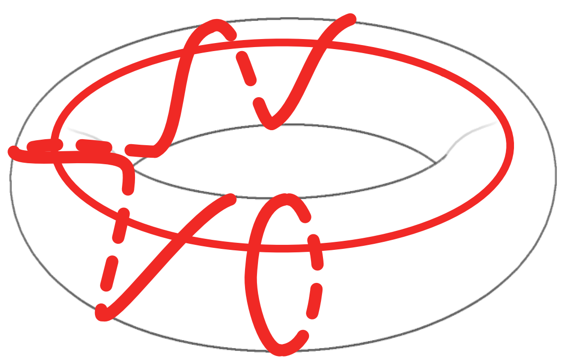
$$\frac{dg}{dt} \equiv \frac{dh}{dt} \qquad \text{iff} \qquad g^{-1} \frac{dg}{dt} = h^{-1} \frac{dh}{dt} = v$$

And is in fact the same test for equivalence, modulo operations to convert between *n*-tuple and matrix representations of groups

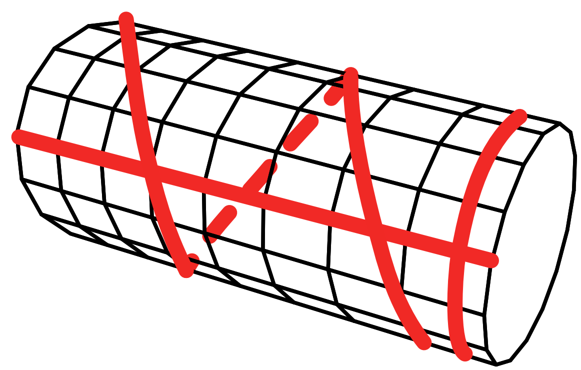
Geodesics

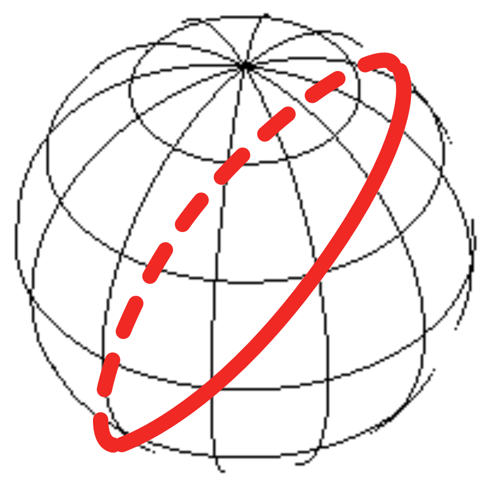
"Straight lines" through a space



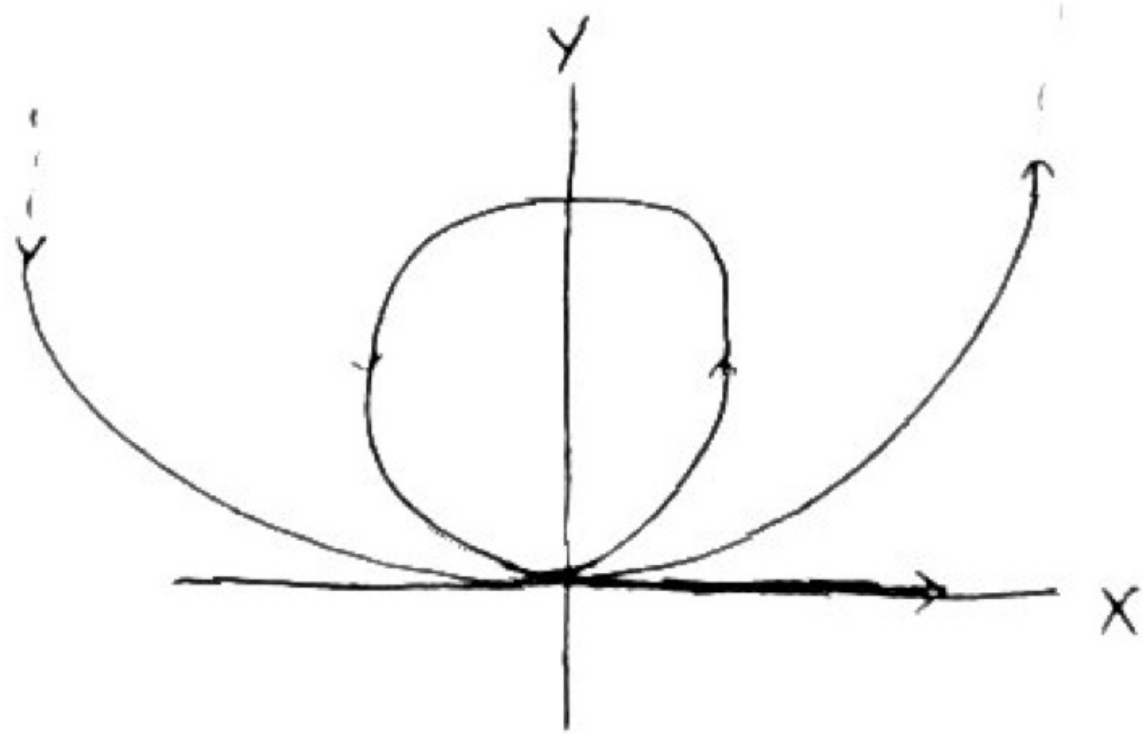


Often also the locally-shortest paths





Geodesics on *SE(2)*

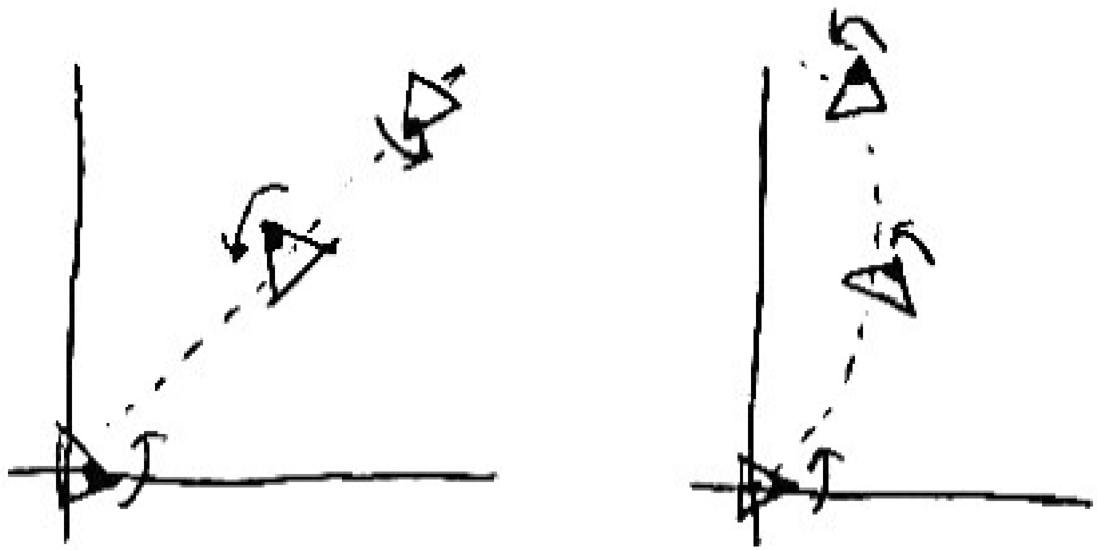


Geodesics on *SE(2)* are trajectories with constant body or spatial velocity

Straight lines if no rotaton, helices (that project to circles on *xy*) if rotation present

Note that even though the *xy* magnitude of right-equivalent velocities is not constant over the whole space, it *is* constant along a geodesic, and that the geodesics are the same for left and right actions – intuitively, if all the factors are the same, it doesn't matter if you multiply from left or right

SE(2) vs. (R^2 x S^1,+), revisited



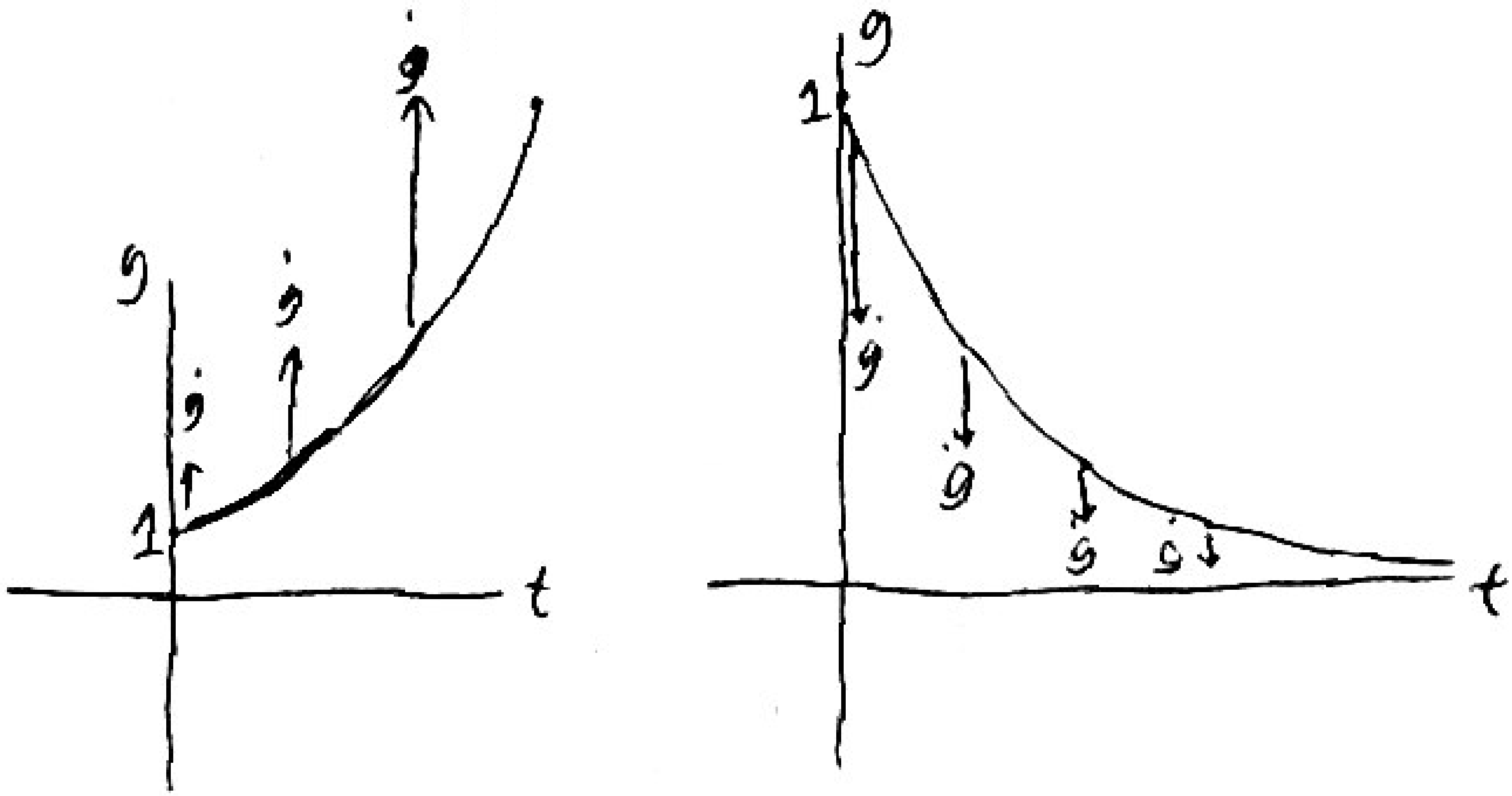
Exponential Map

- Unit-time paths along geodesics
- Equivalently, either flow along a left- or right-invariant field for 1 unit of time, or (on a multiplicative group) take a product-integral for one unit of time with constant group velocity.

Exponential maps and e^x

- How does this relate to standard notion of exponential as exp(x) = ex? (note that e here is not the same as group-identity e)
  - – exp(x) on multiplicative group is unit-time path along always-accelerating (or decelerating) trajectory.

Exponential maps and e^x continued



Exponential maps and e^x continued

- – over any interval, ratio of start and end values will always be the same – this is the same basic definition
- for a function kx
- – *e* is by definition the value of exp(1) on the multiplicative group (unit flow along the exponentia ltrajectory starting with a slope of 1)

Exponential maps and e^x continued

– – once we have $e^1 = \exp(1)$, everything else follows. For example, increasing the magnitude of the argument just pushes the result further along the curve, and we then have identities like

– $\exp(2)/\exp(1) = \exp(1)/\exp(0) \rightarrow \exp(2) = e^2$

Exponential maps and power series

Probably, someone at some point has told you that exp(A) can be found by a power series (or Taylor expansion),

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

How does this relate to our definition of an exponential?

Exponential maps and power series continued

- express exponential as a product integral, and then in limit form:

$$\exp(A) = \prod_0^1 (I + A\,dx) = \lim_{n\to\infty}\left(I + \frac{A}{n}\right)^n \tag{1.49}$$

- coefficients of a binomial raised to a power have a specific form:

$$(x+y)^n = \sum_{k=0}^{n}\left(\frac{n^k}{k!}\right) x^{n-k} y^k \tag{1.50}$$

- apply this form , cancel terms, and power series appears

$$\lim_{n\to\infty}\left(I + \frac{A}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{n^k}{k!}\right) I^{n-k}\left(\frac{A}{n}\right)^k = \sum_{k=0}^{\infty}\frac{A^k}{k!} \tag{1.51}$$

- Note that the power series is as applicable to matrix multiplication as it is to scalar multiplication. In general, the binomial expansion does not apply to matrices or other non-commutative binomials, but as multiplication with the identity commutes ($IA = AI$), it does apply here.

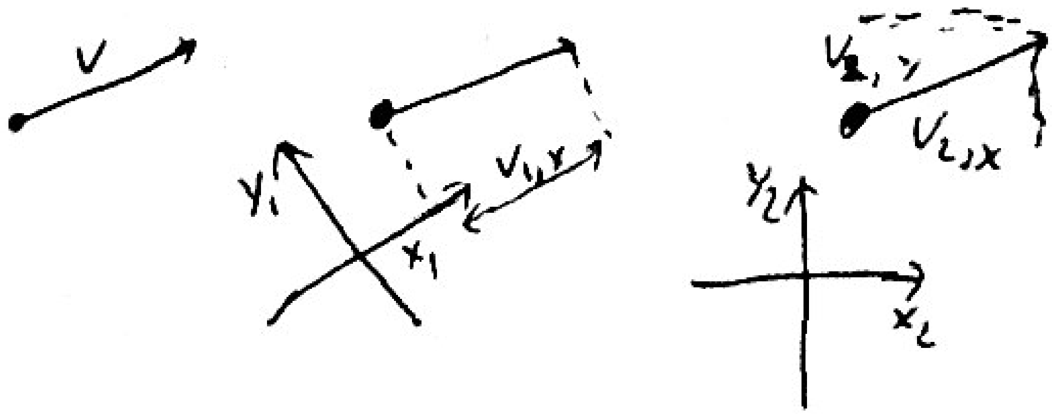
Exponential maps and power series continued

Invariant-field exponentiation is equivalent to group velocity exponentiation

$$\exp\left(\begin{bmatrix} 0 & -\xi_\theta & \xi_x \\ \xi_\theta & 0 & \xi_y \\ 0 & 0 & 0 \end{bmatrix}\right) \equiv \exp(\xi)$$

SE(2) multiplicative velocity as a function of the body velocity parameters

Reference and Coordinate Frames



- Reference frames provide an observer to get relative motion for velocity
- Coordinate frames parameterize motion
- Multiple coordinate frames may be assigned to a reference frame
  - In figure above, velocity vector (shown in triplicate) is motion relative to the page. Two coordinate frames are shown parameterizing the vector, but it exists independently of them, in the reference frame of the page.
- Identical coordinate frames may be defined in different reference frames
- E.g., body velocity is motion of body reference frame, relative to world reference frame, measured in coordinate frame attached to world reference frame but instantaneously identical to body coordinate frame

Frame Labeling

frame

$g_{1,h}$ base frame

identification number of frame

$$g_{1,h} = h^{-1} g_1$$

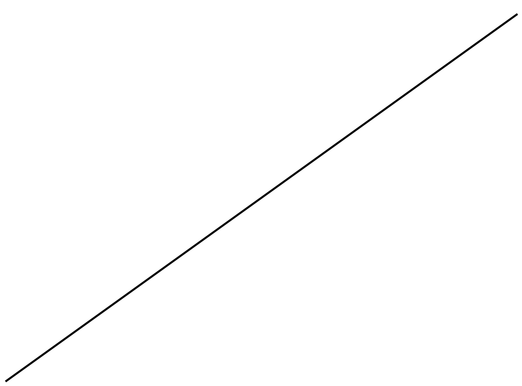
$$g h_g = g g^{-1} h = h$$

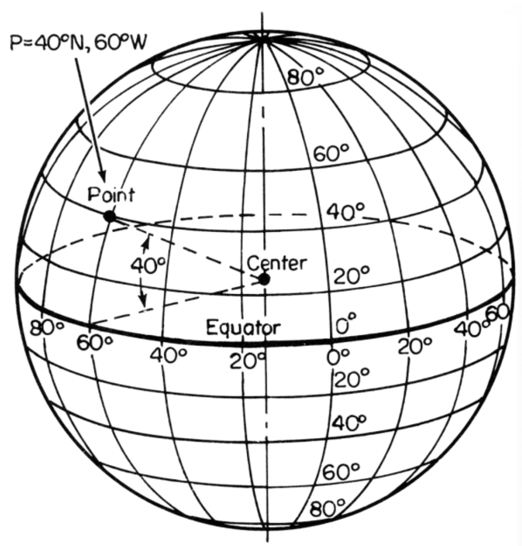
$$g_{1,g_0} h_{g_1} = g_0^{-1} (g_1 g_1^{-1}) h = g_0^{-1} h = h_{g_0}$$

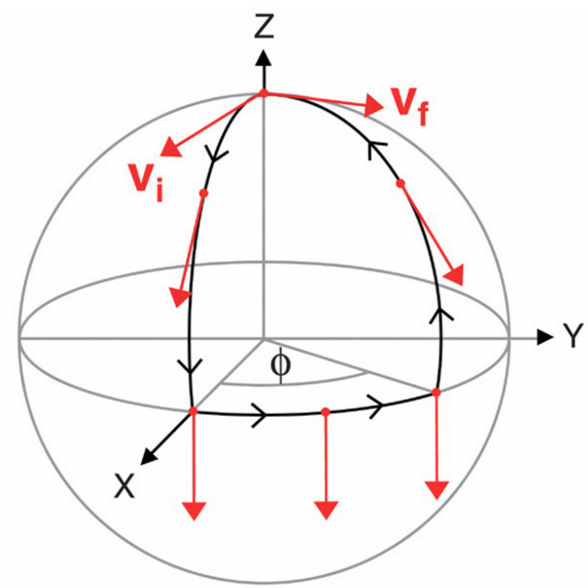
END HERE

Geodesic

- Usually thought of as the shortest path
- Trajectory with constant velocity
  - I like a trajectory with no wiggling and speed ups and downs







Geodesic

- Usually thought of as the shortest path
- Trajectory with constant velocity
- I like a trajectory with no wiggling
- Solutions to differential equations
  - On Lie Groups, left and right invariant vector fields

$$\mathbf{X}_L(g) = T_e L_g v$$

For $v \in T_e G$

$$\mathbf{X}_R(g) = T_e R_g v,$$

Preservation of Velocities

- Left lifted map preserves body velocity
- Right lifted map preserves spatial velocity

- What are body and spatial velocities?