Chapter 3: Kinematics Locomotion

Ross Hatton and Howie Choset

(Fully/Under)Actuated

- Fully Actuated
  - Control all of the DOFs of the system
  - Controlling the joint angles completely specifies the configuration
- Under Actuated
  - Cannot control all of the DOFs of the system
  - Controlling joint angles specifies shape, but not position

Locomotion

- Shape changes can exploit physical constraints on their motions
- Interactions produce reaction forces
  - restrictions on the velocities of points on the body
  - momentum conservation laws
  - Fluid or frictional drag
- Def: Process of using internal reaction forces caused by shape changes to provide position change.

Shape and Position

shape trajectory ψ

$$\psi : [0,T] \to M$$
$$t \mapsto r,$$

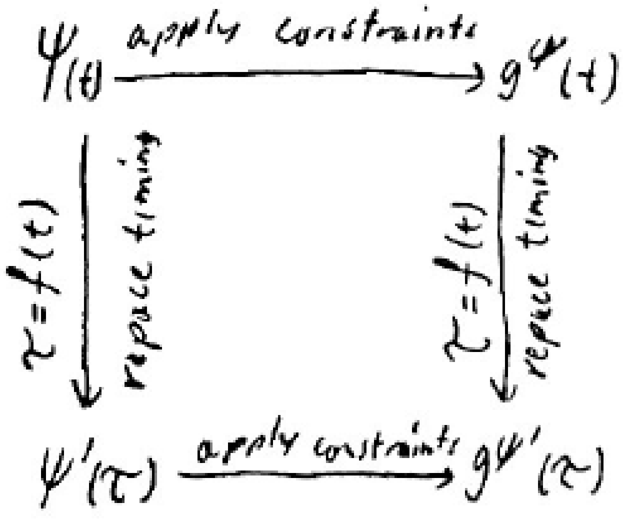
a position trajectory g^{ψ}

$$g^{\psi} : [0,T] \to G$$
$$t \mapsto g,$$

Kinematic Locomotion

- Changing the “pacing” or the time parameterization does not change the path traced by the position trajectory
- Time reparameterization $\psi(t) = \psi'(\tau(t))$

$$g^{\psi}(t) = g^{\psi'}(\tau(t))$$



Ψ’ is a smooth reparameterization of Ψ with respect to time, the induced position trajectories of a kinematic system have the same relationship

displacement induced by a given shape trajectory is entirely a function of the geometric path it follows, and not the rate with it is executed.

Symmetric Linear-Kinematic Locomotors

- Directionally Linear

$$\dot{g} = f(q, \hat{\dot{r}})\|\dot{r}\|$$

- Linear Kinematic

$$\dot{g} = f(q)\dot{r}$$

- Symmetric Linear-Kinematic

$$\xi = f(r)\dot{r}$$

Directional Linearity and Linearity

$$f : A \to B$$
$$a \mapsto b,$$

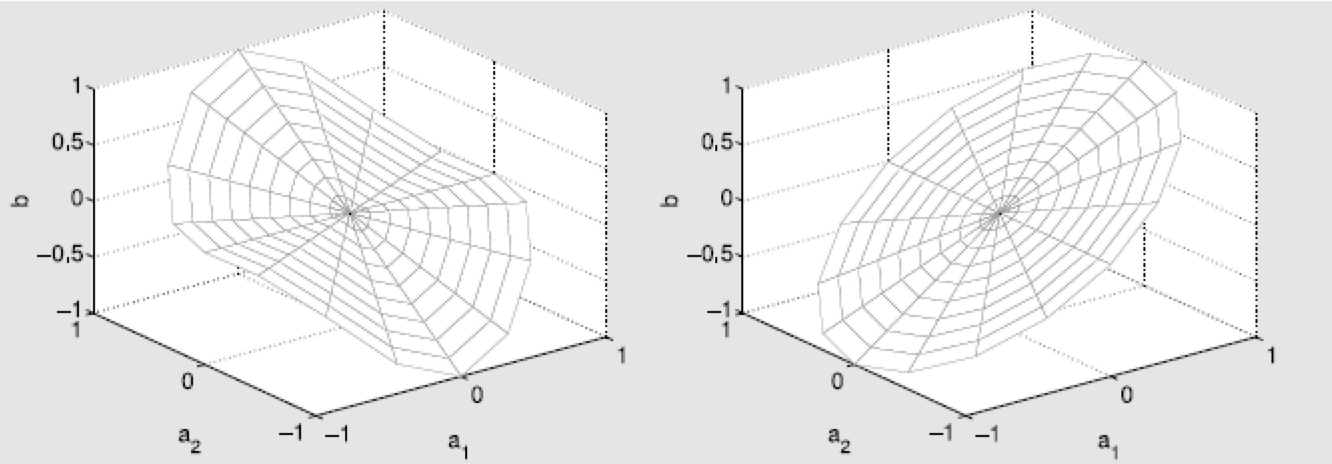
Directional Linearity

each element of b is a linear function of a along each line passing through the origin, *i.e.* it can be written in the form

$$b_i = f_i(\hat{a})\|a\| \text{ with } b(-a) = -b(a)$$

Linearity

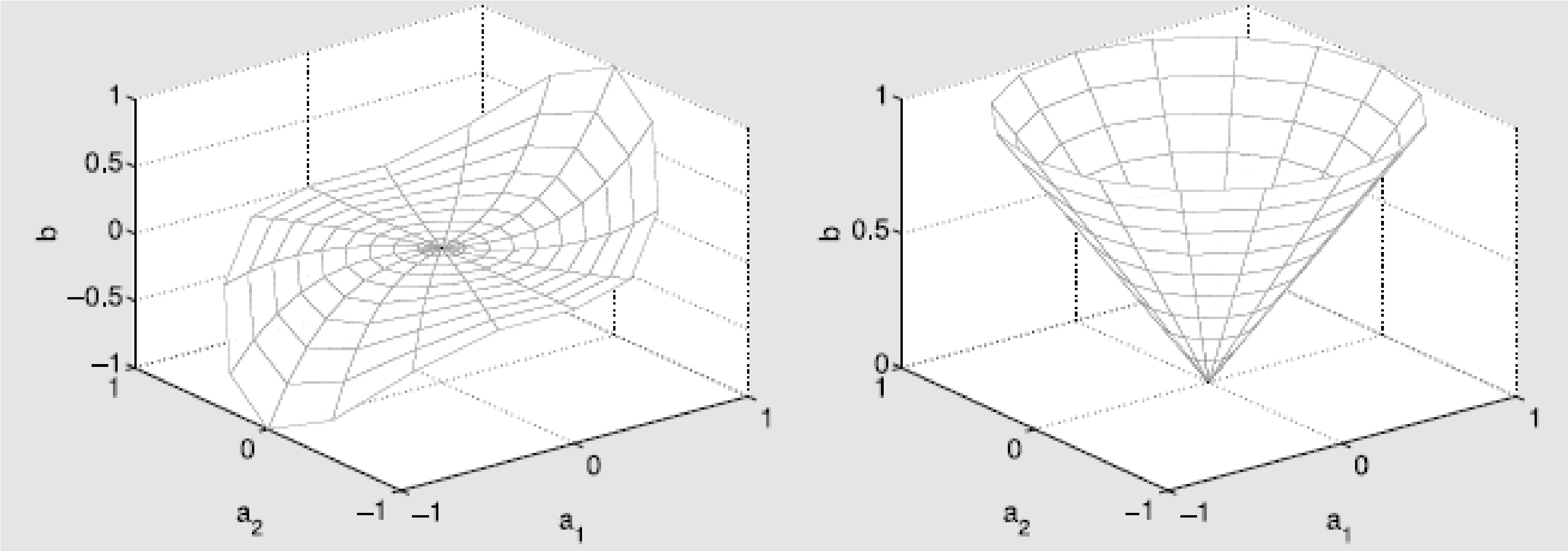
$$b = \overbrace{\mathcal{M}\hat{a}}^{f(\hat{a})}\|a\| = \mathcal{M}a$$



(a) Directionally linear function.
linear along each azimuth

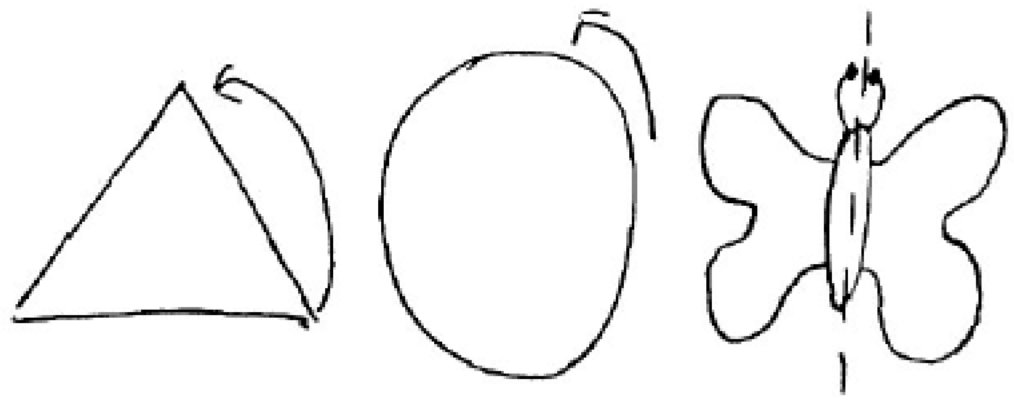
(b) Linear function.
slopes along the azimuths are linearly related.

What linearity are these?



Symmetry

An object is *symmetric* with respect to an action if when the action is applied to the object, the action preserves a property of the object



Examples of geometric symmetries

Notion of indistinguishable

Symmetry Groups

Symmetry is closely linked to mathematical groups structure—the set of actions under which an object is symmetric form a group

- **Closure**: Two symmetry-preserving actions can be concatenated into a third action that is also symmetry preserving
- **Associativity**: If a series of transformations is conducted, the order of operations does not matter (though left-right order may still be important).
- **Identity**: Objects are trivially symmetric under null actions
- I**nverse**: Any symmetry-preserving transformation may be “undone” or reversed, and concatenating this reversal with the original action produces a null action

Symmetry: R^1 vs S^1

- Rotary joint
  - Revolute joint: configuration of the attached link is symmetric with respect to complete revolutions
  - Wheel: number of revolutions is important for other reasons

Symmetry in *SE(2)*

- under any SE(2) transformation, of the system the body-frame positions of particles in the system are unchanged
- characterize the system by its shape in the body frame

Why is symmetry good?

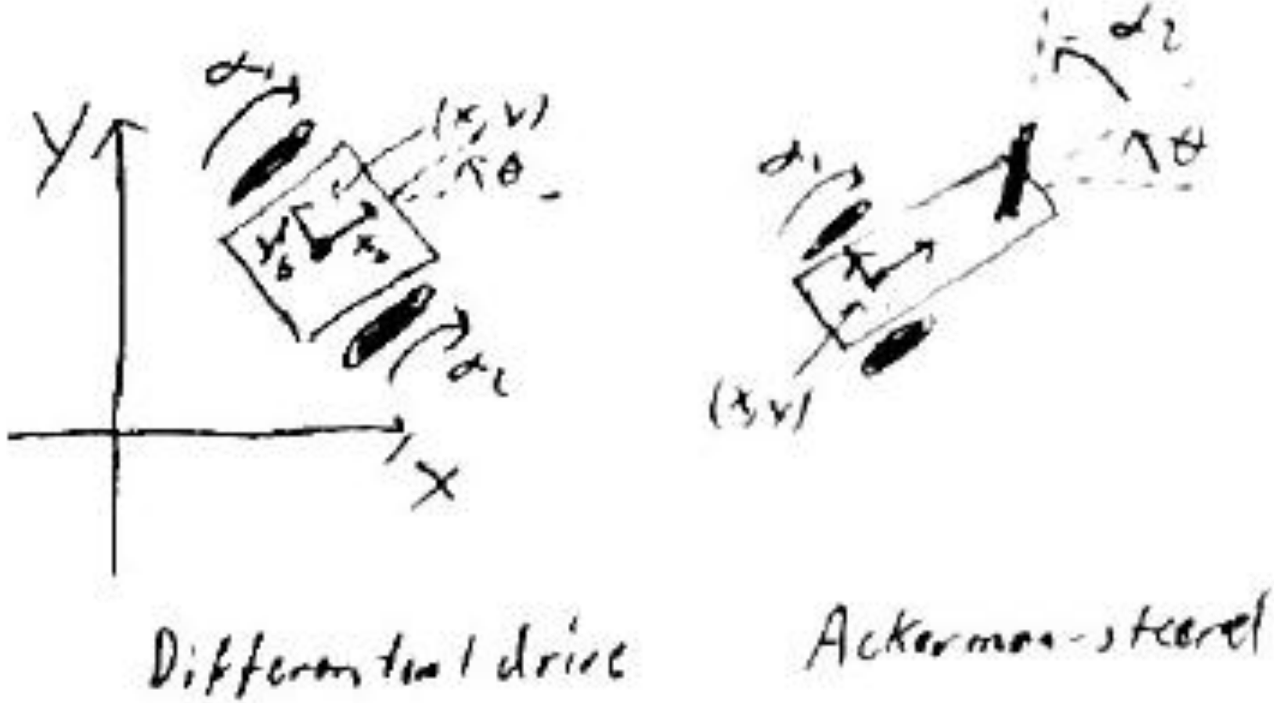
$$\xi = f(r)\dot{r}$$

- This can allow us to reduce the number of degrees of freedom and simplify the system
- By removing position, think in terms of forward and sideways (lateral) velocities
- Use math like operators in T_eG such as the Lie Bracket

Examples: Diff Drive, Ackerman

$g \in SE(2)$ defining body frames

body velocity, ξ^x, ξ^y, and ξ^θ, correspond respectively to the vehicle's longitudinal, lateral, and rotational velocities.

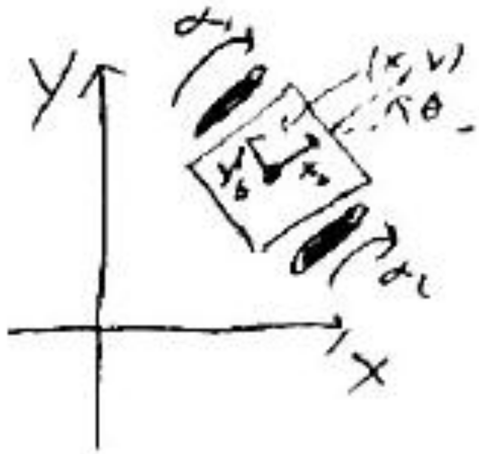


shape

rotation of each wheel around its axle

drive axle and the steering angle.

Examples: Diff Drive, Ackerman



Differential drive

Ackerman-steered

forward when its wheels are turned together, rotates when they are turned oppositely

$$\begin{bmatrix}\xi^x\\ \xi^y\\ \xi^\theta\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}\dot\alpha_1\\ \dot\alpha_2\end{bmatrix}$$

forward when the drive axle is rotated simultaneously rotates at a rate dictated by the steering angle

$$\begin{bmatrix}\xi^x\\ \xi^y\\ \xi^\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\\ \tan\alpha_2 & 0\end{bmatrix}\begin{bmatrix}\dot\alpha_1\\ \dot\alpha_2\end{bmatrix}$$

Kinematic Reconstruction Equation

symmetric linear-kinematic systems $\Longrightarrow$ kinematic locomotors

$$\xi = -\mathbf{A}(r)\dot{r}$$

Local Connection

corresponds to, and can be derived from

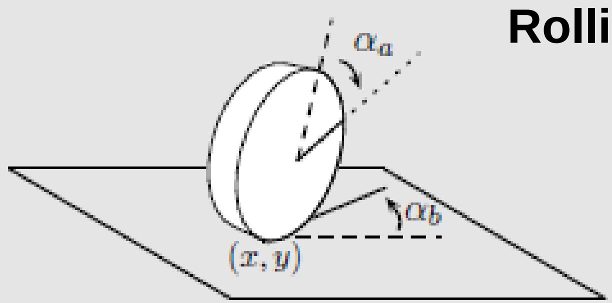
Pfaffian constraint on its configuration velocities,

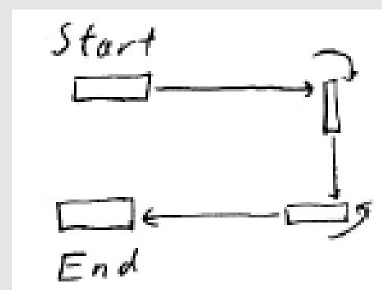
$$0 = \omega(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

Non-holonomic Constraints

- Non-integrable constraints (does not help)
- restrict the velocity with which a system can move, but without restricting the accessible configurations.

Rolling disk





roll forward and backward or turn,
but is **unable to slide sideways**

Can displace sideways

More formally….

- a nonholonomic constraint is defined by a (possibly time-varying) function *c* on the system's configuration tangent bundle, *TQ*

- The zero set of this function defines the system's allowable velocities at each configuration that satisfy the constraint

$\dot{Q}_0 \in T_q Q$ at each point q in the configuration space such that $c(q, \dot{q}, t) = 0$

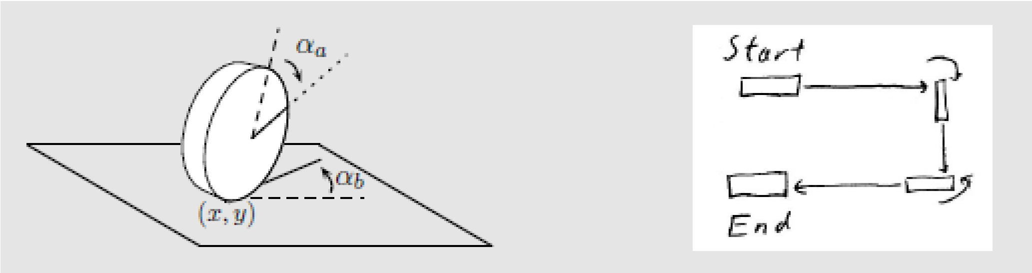
Non-holonomic Constraint for Disk

- No Slide Condition
- World

$$c_{\text{world}} = \dot{y}\cos\theta - \dot{x}\sin\theta.$$

Body

$$c_{\text{body}} = \xi^{y}$$



This constraint prevents lateral motion of the disk, but does not prevent it from achieving arbitrary net lateral displacement. By combining rolling and turning actions into a “parallel parking” motion, the disk can access any point in the SE(2) space despite the constraint on its velocity

Pfaffian Constraints

- A linear nonholonomic constraint

$$c(q,\dot{q}) = \omega(q)\dot{q}$$

- m = number of independent constraints
- n = dim(Q)

$\omega(q) \in R^{m \times n}$

- Null space $\omega(q)$ is allowable velocities
- Collection of these null spaces forms a distribution

What's a null space?

- Unconstrained systems have n-dimensional distributions

Nonintegrability

- To qualify as a nonholonomic constraint function, *c* must not be integrable into a holonomic constraint *f(q, t) = 0*

$$c = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{x}x + \dot{y}y$$

+ Initial conditions

$$f = \sqrt{x^2 + y^2} - R$$

Pfaffian Constraints on *SE(2)*

$$0 = \omega(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix} \Longrightarrow 0^{m \times 1} = \omega^{m \times (3+n)} \begin{bmatrix} \xi^{3 \times 1} \\ \dot{r}^{n \times 1} \end{bmatrix}$$

m individual Pfaffian constraints
n shape variables,

each row of which represents a restriction on the motion of the system by mapping the configuration velocity (the body and shape velocities) to zero.

Kinematic locomoting systems have Pfaffians composed of at least
three independent constraints, one per degree of freedom in the position space

fewer constraints gains the ability to *drift* through the position space without changing shape, and thus is not fully kinematic (pushing analogy bad, coasting analogy good)

more constraints present than positional degrees of freedom, the system is overconstrained from a controls perspective, and only certain shape trajectories can be executed

M = 3 Pfaffians

- As many constraints as DOFs in SE(2), systems shape velocity “uses up” all of the local DOFs
- Body velocity becomes a linear function of shape velocity
- And viola, kinematic system!
- If there are more constraints (m>3), only certain shape trajectories are allowed

three-constraint Pfaffian (m = 3)

- Kinematic without being over constrained
- Unconstrained choices of shape trajectories

three-constraint Pfaffian (m = 3)

- We can compute the connection!

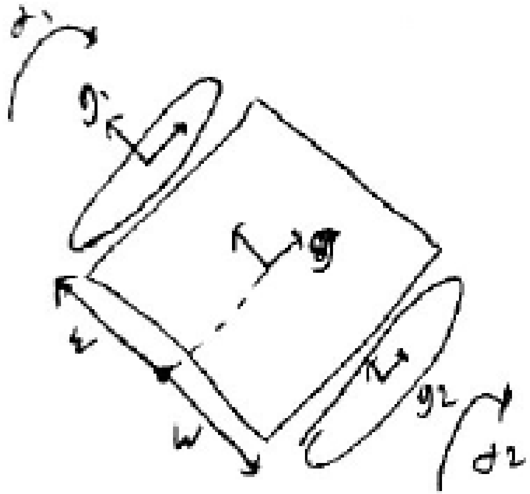
$$\mathbf{0}^{m\times 1} = \omega^{m\times(3+n)} \begin{bmatrix} \xi^{3\times 1} \\ \dot{r}^{n\times 1} \end{bmatrix}$$

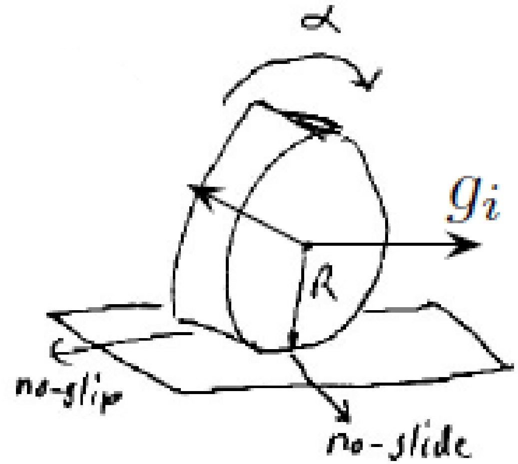
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_{\xi}^{3\times 3} & \omega_{\dot{r}}^{3\times n} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \omega_{\xi}\xi + \omega_{\dot{r}}\dot{r} \qquad \omega_{\xi}\xi = -\omega_{\dot{r}}\dot{r}$$

$$\xi = -\omega_{\xi}^{-1}\omega_{\dot{r}}\dot{r} \qquad \mathbf{A} = \omega_{\xi}^{-1}\omega_{\dot{r}}$$

Differential Drive Example





$$\xi^x_{g_i} - R\dot{\alpha}_i = 0 \qquad \text{(no-slip)}$$

$$\xi^\theta_{g_i} = 0 \qquad \text{(no-slide)}$$

$g_{1,g} = (0, w, 0)$ and $g_{2,g} = (0, -w, 0)$.

more

$$\xi_{g_1} = Ad^{-1}_{g_{1,g}} \xi = \begin{bmatrix} \xi^x - w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix} \qquad \xi_{g_2} = Ad^{-1}_{g_{2,g}} \xi = \begin{bmatrix} \xi^x + w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix}$$

$$\begin{aligned} \xi^x - w\xi^\theta - R\dot{\alpha}_1 &= 0 && \text{(no-slip wheel 1)} \\ \xi^x + w\xi^\theta - R\dot{\alpha}_1 &= 0 && \text{(no-slip wheel 2)} \\ \xi^y &= 0 && \text{(no-slide wheels 1 \& 2).} \end{aligned}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \left[\overbrace{\begin{matrix} 1 & 0 & -w \\ 1 & 0 & w \\ 0 & 1 & 0 \end{matrix}}^{\omega_\xi} \quad \overbrace{\begin{matrix} -R & 0 \\ 0 & -R \\ 0 & 0 \end{matrix}}^{\omega_{\dot r}} \right] \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \xi \\ \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \dot{r} \end{matrix} .$$

more

$$\xi = -\overbrace{\begin{bmatrix}1 & 0 & -w\\ 1 & 0 & w\\ 0 & 1 & 0\end{bmatrix}}^{\omega_\xi}{}^{-1}\overbrace{\begin{bmatrix}-R & 0\\ 0 & -R\\ 0 & 0\end{bmatrix}}^{\omega_{\dot r}}\dot r$$

$$= -\begin{bmatrix}1/2 & 1/2 & 0\\ 0 & 0 & 1\\ -1/(2w) & 1/(2w) & 0\end{bmatrix}\begin{bmatrix}-R & 0\\ 0 & -R\\ 0 & 0\end{bmatrix}\dot r$$

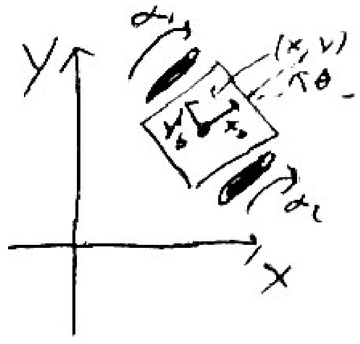
$$= -\underbrace{\begin{bmatrix}-R/2 & -R/2\\ 0 & 0\\ R/(2w) & -R/(2w)\end{bmatrix}}_{\mathbf{A}}\dot r,$$

for $R = 2$ and $w = 1$, exactly matches

Connection Vector Fields

So, we have a local connection. How can we visualize it?

$$\xi = -\mathbf{A}(r)\dot{r}$$



$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = - \underbrace{\begin{bmatrix} -R/2 & -R/2 \\ 0 & 0 \\ R/(2w) & -R/(2w) \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

First, what do the rows and columns of the matrix mean?

Columns: body velocities from moving each shape variable independently

Rows: dependence of each component of the body velocity on the shape velocity

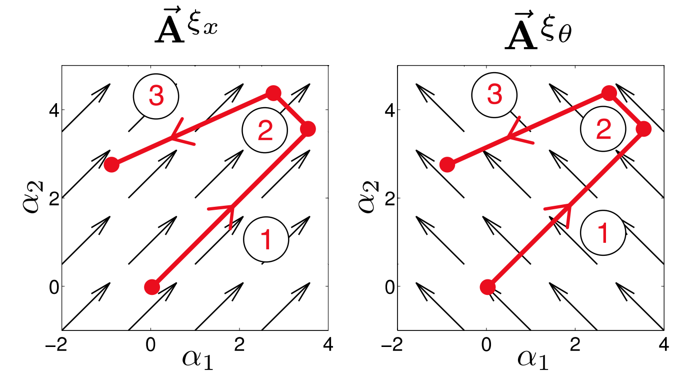
Connection Vector Fields

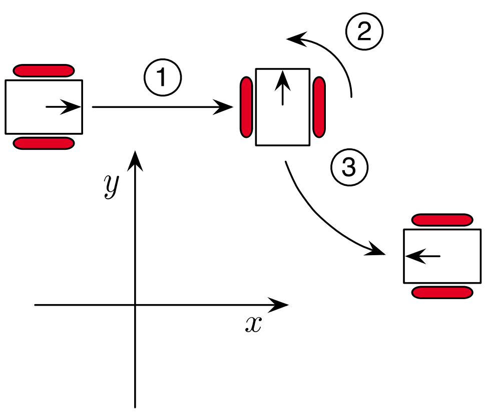
In locomotion we care about how the system's position changes for different shape inputs, so lets look at the rows

$$\begin{bmatrix}\xi^x\\ \xi^y\\ \xi^\theta\end{bmatrix} = -\underbrace{\begin{bmatrix} -R/2 & -R/2 \\ 0 & 0 \\ R/(2w) & -R/(2w)\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}\dot\alpha_1\\ \dot\alpha_2\end{bmatrix}$$

Computationally, each component of the body velocity is the dot product of its corresponding row in **A** and the shape velocity.

We can visually evaluate this dot product by looking at the alignment of the shape velocity with the vector fields formed by the rows of **A**. At right, a simple example using the differential drive car:





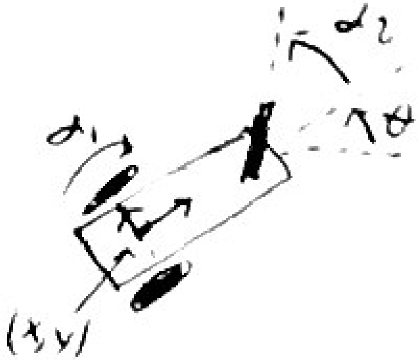
When the shape change is following the x field, the car moves forward, and when it is following the y field, the car rotates CCW. When the trajectories are orthogonal to a field, the car does not move in the corresponding body direction

Alternately, we can say that the rows encode the *local gradient* of position with respect to shape

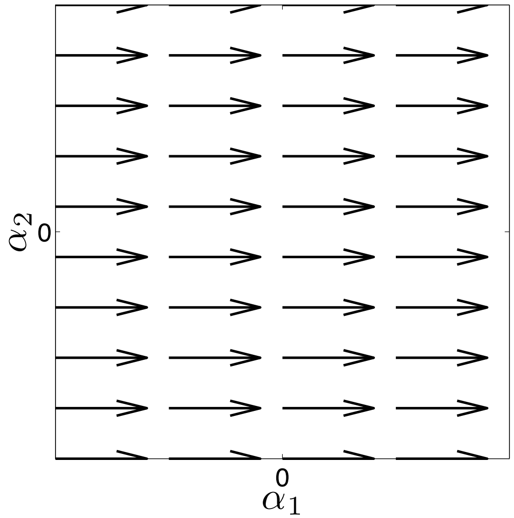
Connection Vector Fields

A more interesting example: the local connection of the Ackerman car varies over the shape space

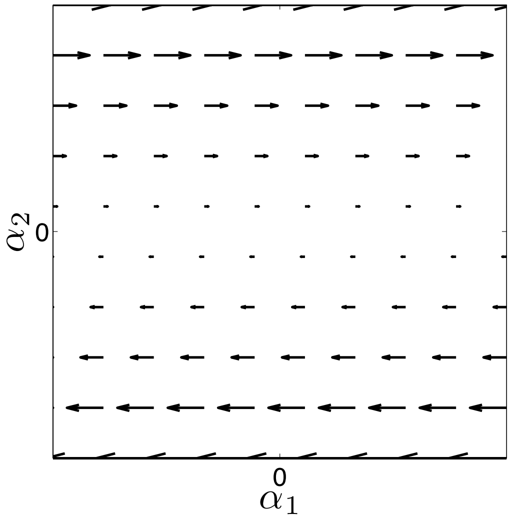
$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \tan\alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$



$\vec{\mathbf{A}}^{\xi_x}$



$\vec{\mathbf{A}}^{\xi_\theta}$



Vectors and Covectors

Vectors are generally represented as columns, and covectors as rows

covector vector

$$\omega v = \langle \omega, v \rangle = \sum_i \omega_i v^i = \begin{bmatrix} \omega_1 & \dots & \omega_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \omega^\sharp \cdot v$$

Many operations we learn as “dot products” are secretly vector-covector pairings.
If we want to talk about them as dot products and still be formal, we can use *musical notation* (sharps and flats) to change vectors into covectors

Co-vectors

meaning that we can consider a covector $\omega \in T_q^*Q$ to be a mapping from vectors to real numbers,

$$\begin{aligned} \omega : T_qQ &\to \mathbb{R} \\ v &\mapsto \langle \omega, v \rangle. \end{aligned} \tag{3.xiv}$$

If ω is a function of the configuration (*i.e.* is a covector field, rather than an isolated covector), it becomes a map from configuration, velocity pairs to real numbers,

$$\begin{aligned} \omega : TQ &\to \mathbb{R} \\ (q, v) &\mapsto \langle \omega(q), v \rangle, \end{aligned} \tag{3.xv}$$

that is linear with respect to v, but may vary nonlinearly with q. Functions of this type are also known as *(differential) one-forms*. Differential forms play an important role in geometric mechanics, and we will examine them in greater depth in Chapter ??.

Vector-valued One Forms

- each element of the output is the product of one of the component covectors with the input vector one-form ω with m components, $(\omega^1, \ldots, \omega^m)$

$$\omega(q)v = \begin{bmatrix} \omega_1^1(q) & \ldots & \omega_n^1(q) \\ \vdots & \ddots & \vdots \\ \omega_1^m(q) & \ldots & \omega_n^m(q) \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

- Jacobians and Pfaffians

Directional Derivative

- rates of change of functions with respect to the underlying space

$$\omega = \mathbf{d}f = \left(\frac{\partial f}{\partial q^1}, \cdots, \frac{\partial f}{\partial q^n} \right)$$

the "derivative" of an *implied* function f with respect to the configuration space

$$D_v f = \langle \mathbf{d}f, v \rangle = \frac{\partial f}{\partial q} v,$$

Gradient Vector field

$$\nabla f = \omega^{\sharp}$$

pointing in the direction in which *f* increases the most quickly

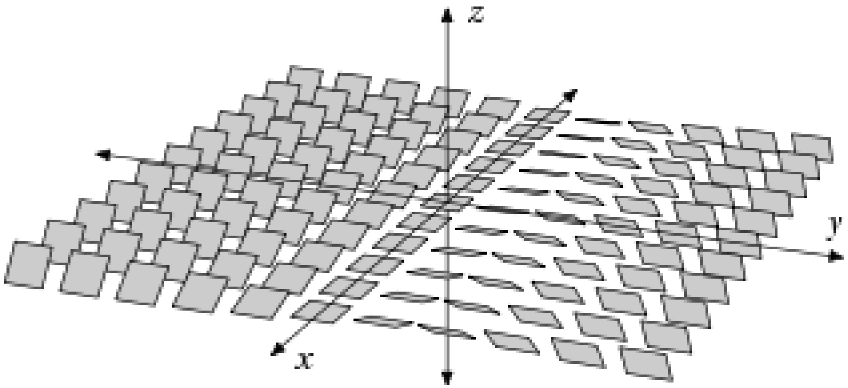
Rate of change of a function over a space is a one-form, not a velocity (recall, a vector is the velocity of something moving through the space, co-vector is the rate of change of a function over the space)

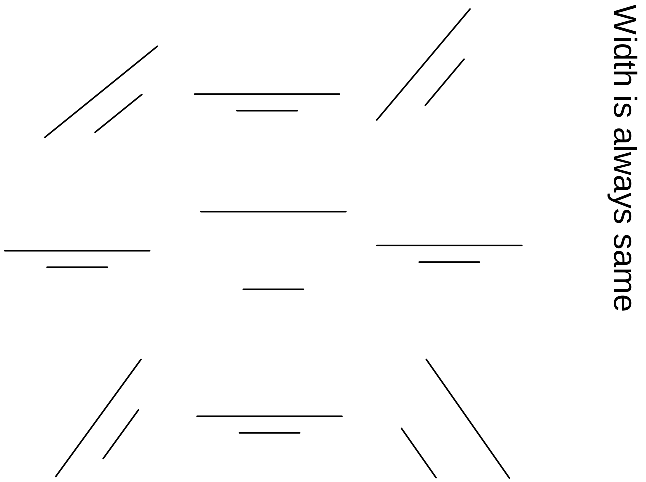
Gradient vector fields are really the natural dual of the rate of change of function

f may be only locally definable – the local connection is the derivative of the system's position with respect to its shape,

Vector and Covector Visualization

As part of the vector/covector distinction, some sources reserve arrows for visualizing vectors, and use planes or *local contour lines* to visualize covectors and covector fields.



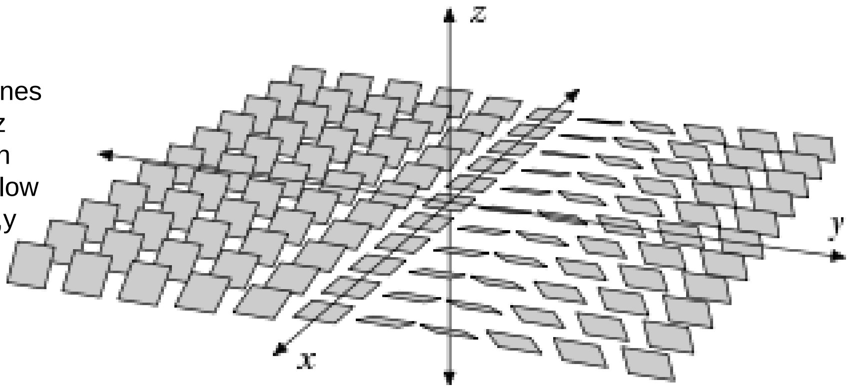


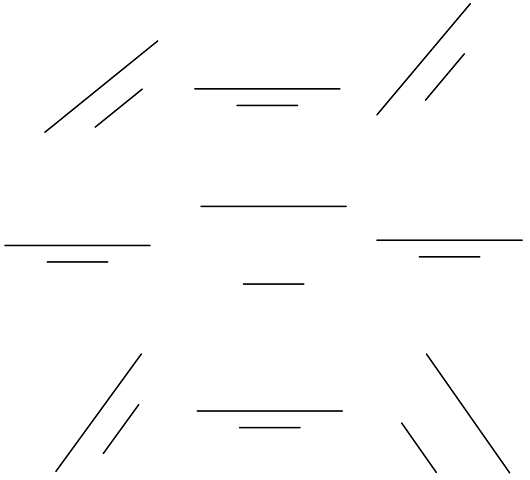
Plane determines flow in z direction during flow along x,y	Closeness of contours represent steepness, and orientation represents directions (closer space is more steep)
Connection field elements are gradient vectors of each local plane	Connection field elements are dual to the local contours

Vector and Covector Visualization

As part of the vector/covector distinction, some sources reserve arrows for visualizing vectors, and use planes or local contour lines to visualize covectors and covector fields.

Plane determines flow in z direction during flow along x,y





Connection rows are really *co*vector fields. Why not use one of these representations?

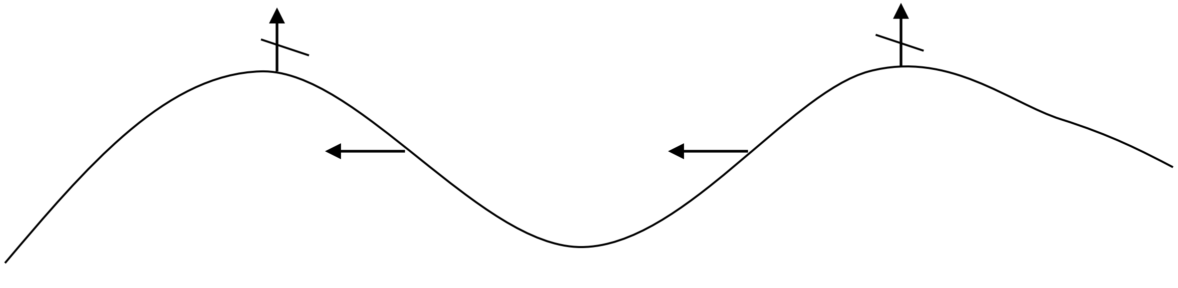
Reasons we don't:

1. <u>Plane fields are unnecessarily 3d</u> – all the information they contain can be presented in the arrow representation of their slope direction and magnitude
2. <u>Local contour plots are space-inefficient</u>, hard to read, and require infinitely-spaced lines for zero-magnitude covectors
3. Objections to using arrows for covectors seem to stem from naming overlap between vectors (velocities) and vectors (arrows representing directional derivatives). If we separate these concepts, then there is little trouble in plotting rows as "vector fields"

Full Body Locomotion

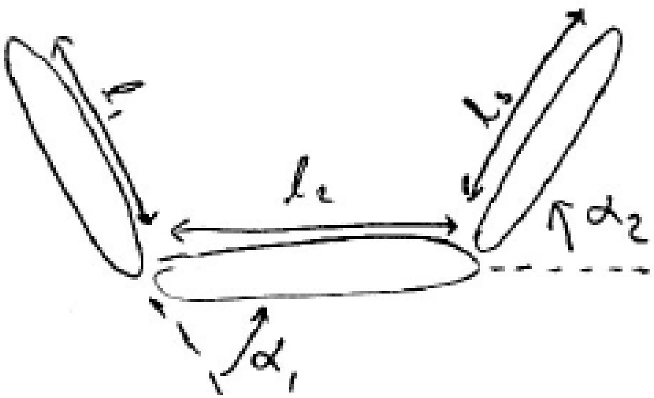
- Instead of a drive element and control surface

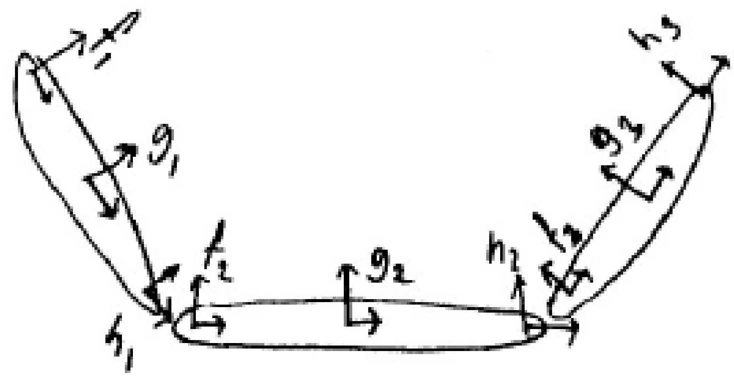
- Segments of the body trade-off and simultaneously act as drive surfaces and control surfaces



Three-link Locomotors

- Review of body velocities

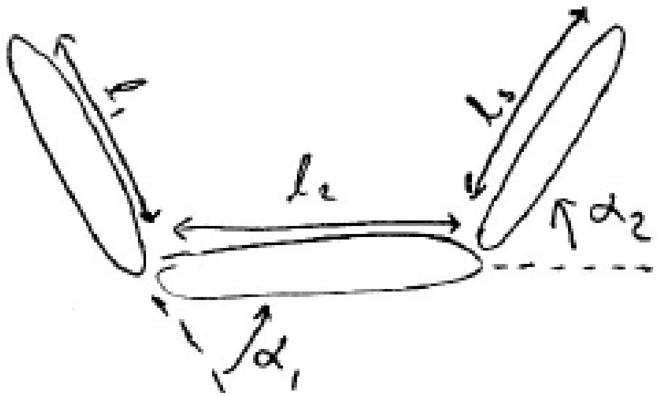


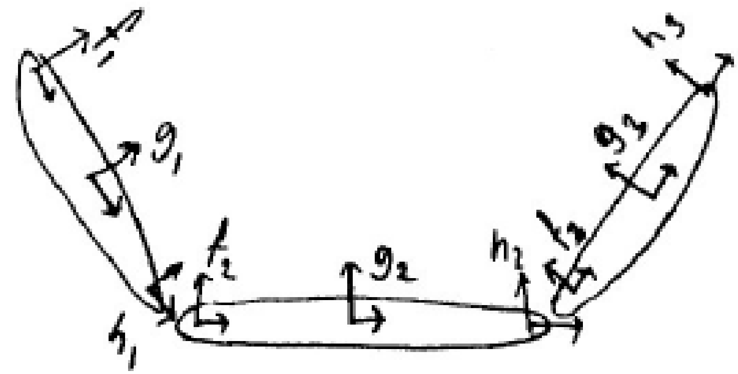


$$\xi_{g2} = \xi = \begin{bmatrix} I^{3\times 3} & \mathbf{0}^{3\times 2} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

$$\xi_{f_2} = Ad^{-1}_{f_{2,g2}} \xi_{g2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g2}=(-\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g2}} = \begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

Link 1





distal end of link 1, which has rotational velocity of $-\dot{\alpha}_1$ with respect to f_2,

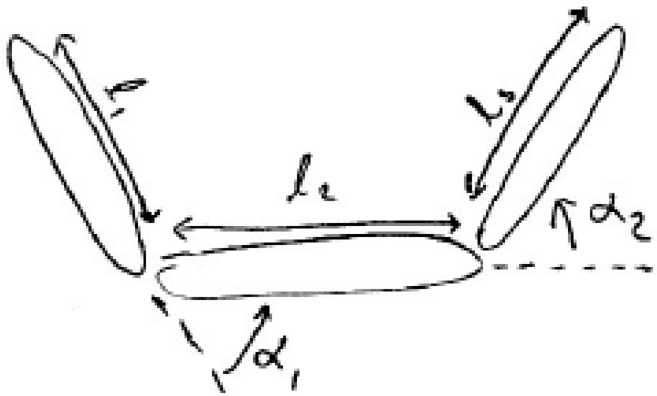
$$\xi_{h_1} = Ad^{-1}_{h_1,h_1'}\xi_{h_1'} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_1,h_1'=(0,0,-\alpha_1)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{f_2}} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\alpha}_1 \end{bmatrix} \right.}^{\xi_{h_1'}}$$

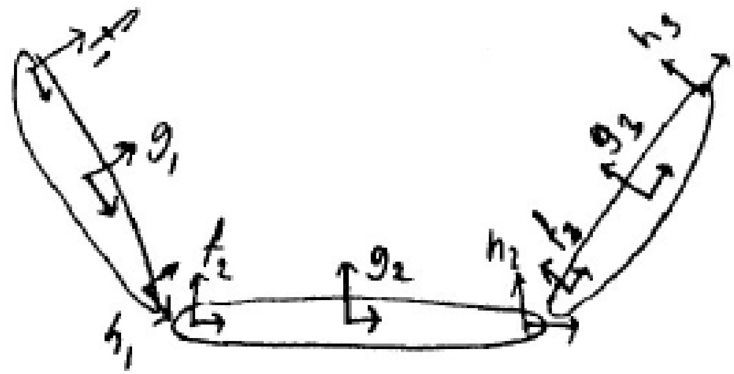
$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}.$$

$$\xi_{g_1} = Ad^{-1}_{g_1,h_1}\xi_{h_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_1/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_1,h_1=(0,-\ell_1/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}}^{\xi_{h_1}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}$$

Link 3





$$\xi_{h_2} = Ad^{-1}_{h_2,g_2}\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_2,g_2=(\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}$$

$$\xi_{f_3} = Ad^{-1}_{f_3,f_3'}\xi_{f_3'} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_3,f_3'=(0,0,\alpha_2)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{h_2}} + \begin{bmatrix} 0 \\ 0 \\ \dot\alpha_2 \end{bmatrix} \right)}^{\xi_{f_3'}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot\alpha_2 \end{bmatrix},$$

$$\xi_{g_3} = Ad^{-1}_{g_3,f_3}\xi_{f_3} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_3/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_3,f_3=(0,\ell_3/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot\alpha_2 \end{bmatrix}}^{\xi_{f_3}}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \\ \xi^\theta + \dot\alpha_2 \end{bmatrix}$$

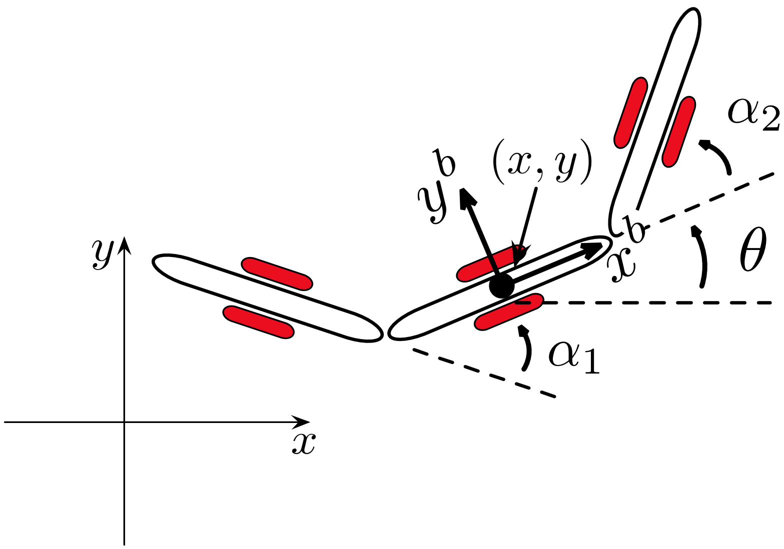
The Velocities

$$\xi_{g_1} = \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}$$

$$\xi_{g_2} = \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}$$

$$\xi_{g_3} = \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot{\alpha}_2) \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}$$

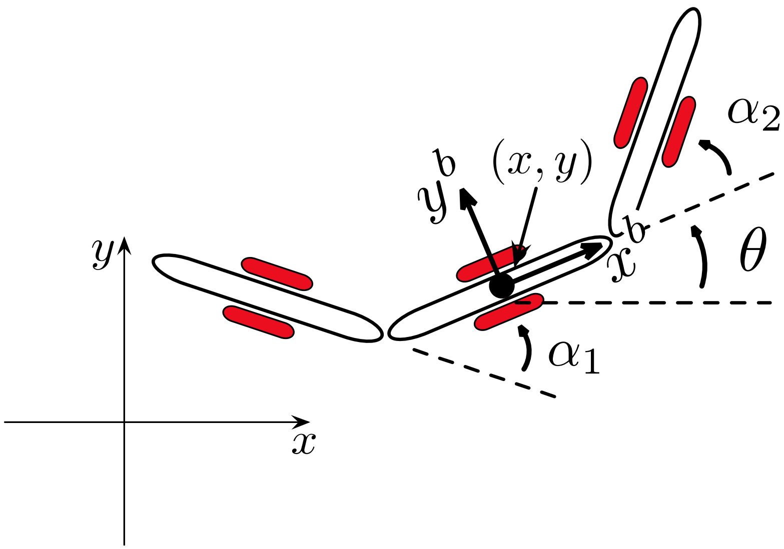
Kinematic Snake



No-slide constraints on the wheels

$$\begin{bmatrix} \xi^y_{g_1} \\ \xi^y_{g_2} \\ \xi^y_{g_3} \end{bmatrix} = \begin{bmatrix} \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^y \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot{\alpha}_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

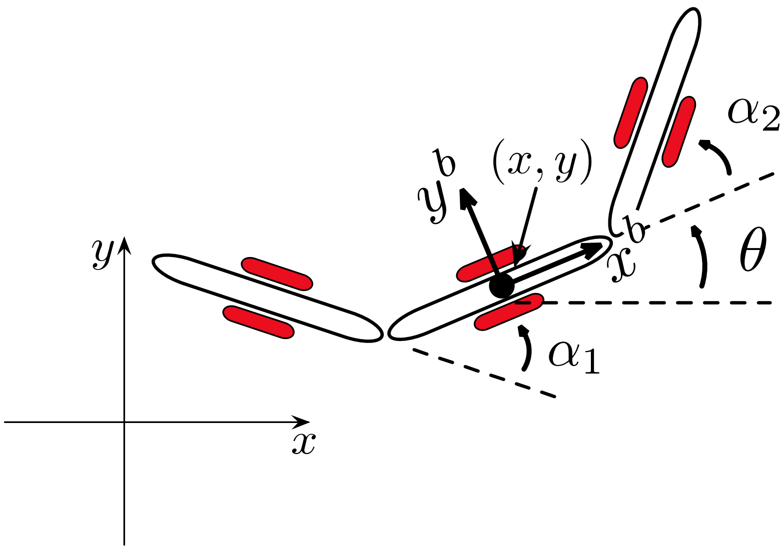
Kinematic Snake



Extract terms to get Pfaffian constraints

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 & -\ell_1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \end{bmatrix} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}$$

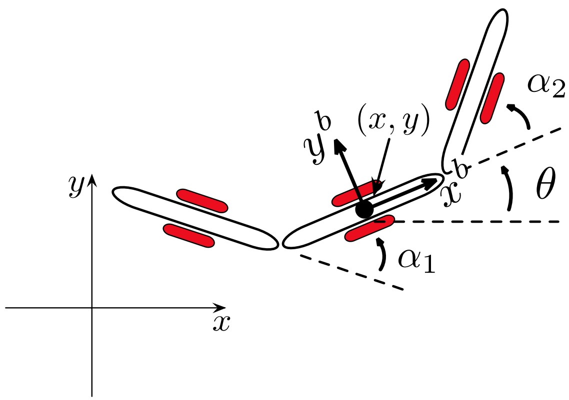
Kinematic Snake



Rearrange terms to solve for local connection

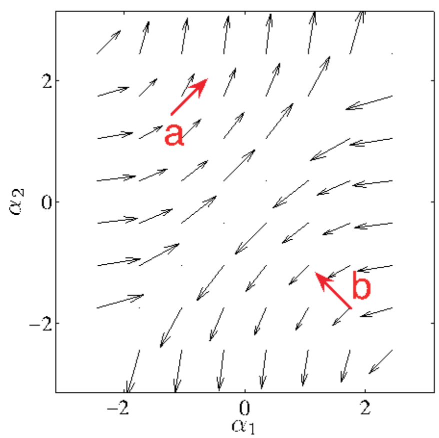
$$\xi = -\underbrace{\begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 \end{bmatrix}^{-1}}_{\omega_\xi} \underbrace{\begin{bmatrix} -\ell_1/2 & 0 \\ 0 & 0 \\ 0 & \ell_3/2 \end{bmatrix}}_{\omega_{\dot r}} \dot\alpha$$

Kinematic Snake

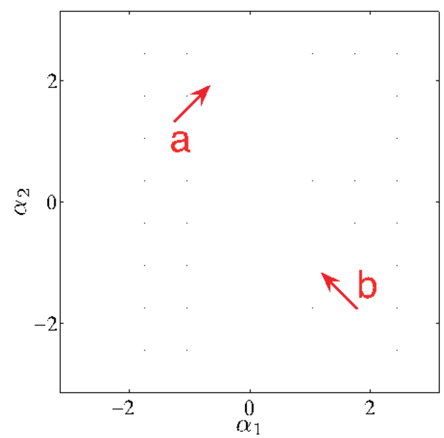


$$\xi = -\frac{1}{D}\overbrace{\begin{bmatrix} -\ell_1(\ell_3 + \ell_2\cos\alpha_2)/2 & \ell_3(\ell_1 - \ell_2\cos\alpha_1)/2 \\ 0 & 0 \\ -\ell_1\sin\alpha_2 & \ell_3\sin\alpha_1 \end{bmatrix}}^{\mathbf{A}}\dot{\alpha}$$

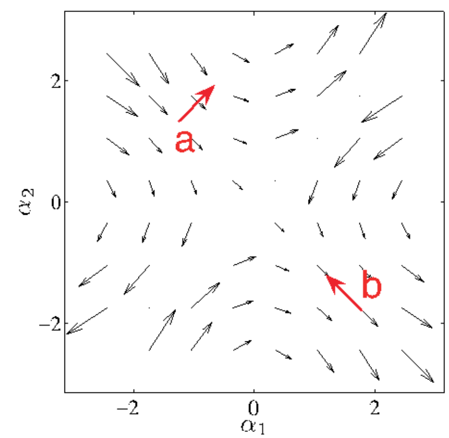
$$D = \ell_2\sin(\alpha_1 + \alpha_2) - \ell_1\sin\alpha_2 + \ell_3\sin\alpha_3$$



$\vec{\mathbf{A}}^{\xi_x}$

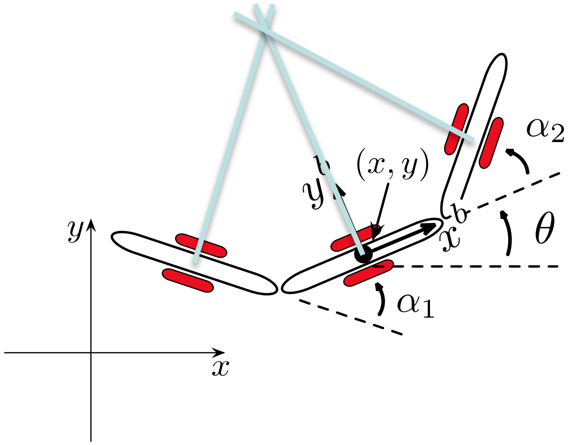


$\vec{\mathbf{A}}^{\xi_y}$



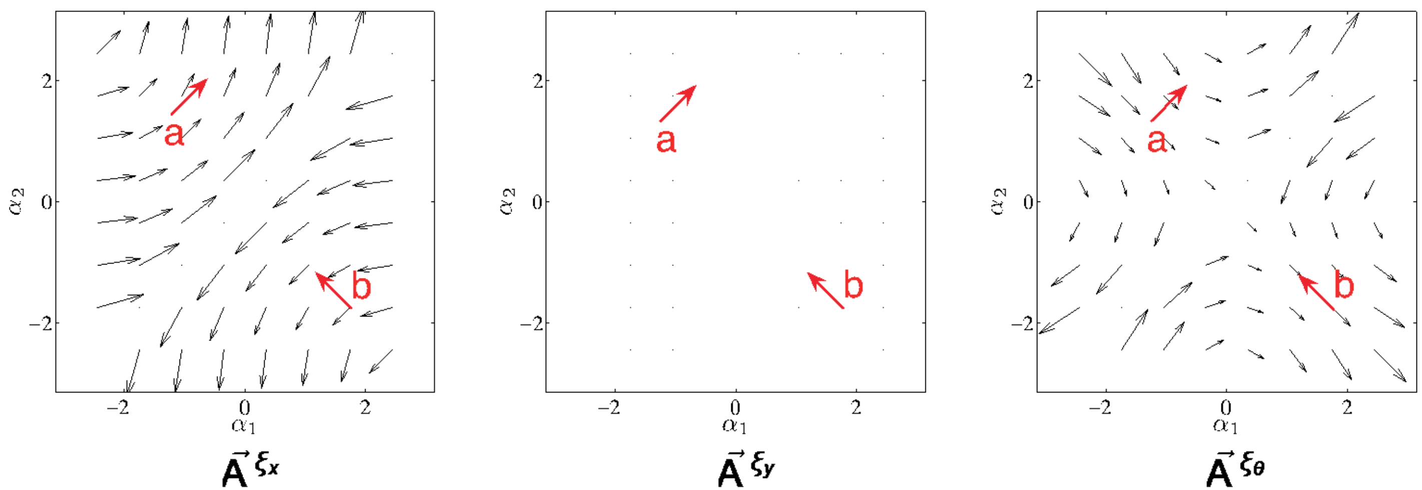
$\vec{\mathbf{A}}^{\xi_\theta}$

Kinematic Snake Singularities



Note singularities along alpha_1 = alpha_2

Constraints all pass through a single point:
1. System can now “drift” or “coast,” rotating around meeting point
2. Joints motions are highly restricted – large opposing constraint forces in response to joint torques



The Velocities

$$\xi_{g_1} = \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot\alpha_1) \\ \xi^\theta - \dot\alpha_1 \end{bmatrix}$$

$$\xi_{g_2} = \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}$$

$$\xi_{g_3} = \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \\ \xi^\theta + \dot\alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \xi^y_{g_1} \\ \xi^y_{g_2} \\ \xi^y_{g_3} \end{bmatrix} = \begin{bmatrix} \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot\alpha_1) \\ \xi^y \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

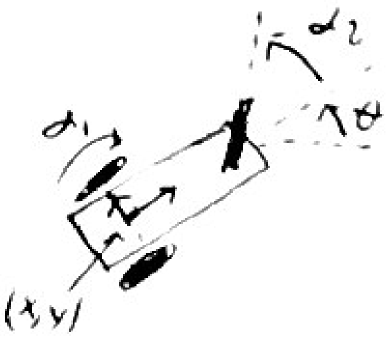
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 & -\ell_1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \end{bmatrix} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}$$

Vectors and Co-vectors

- Tangent vectors (vectors): velocity-like terms that describe motion through the underlying space [Columns]

- Cotangent vectors (covectors) gradient-like terms describing how a quantity varies across the space [Rows]

Ackerman Car Singularities

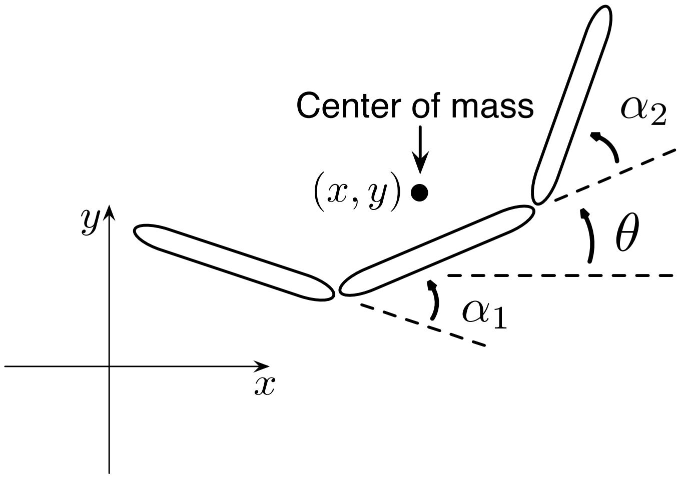


Similar situation with Ackerman car for alpha_2 = p

Drive wheel "forward motion" constraint lines up with steering wheel "lateral motion" constraint

Note that no such problem happens with front wheel drive – front wheel is never *inline* with rear axle

Floating Snake



Inertially-constrained system – think of it as floating in space, or on an air table

Pfaffian constraint is on momentum: If the system starts with zero momentum, it must maintain zero momentum for all time

Could calculate momentum for each link, and use adjoints to collect momentum terms into single body frame.

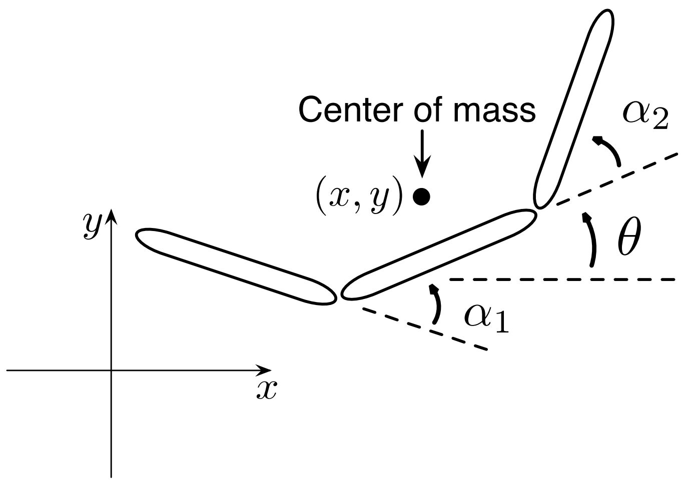
$$KE = \frac{1}{2} \begin{bmatrix} \xi & \dot{r} \end{bmatrix} \mathbb{M}(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$

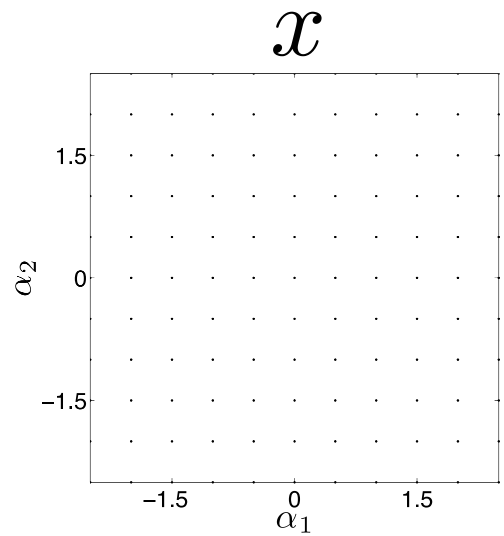
Easier method: If we find the kinetic energy matrix for the system, then the Pfaffian appears as the top two blocks of the matrix

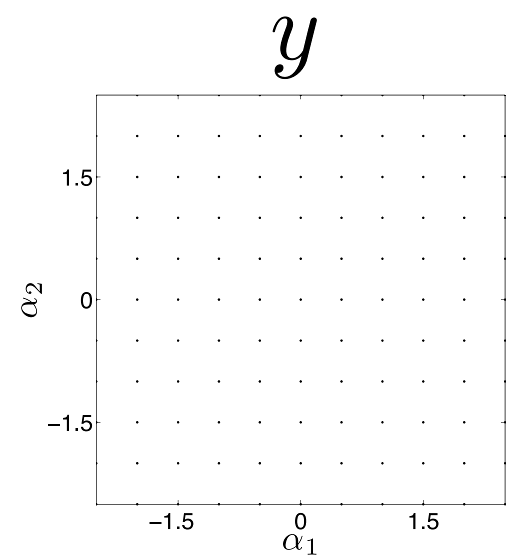
$$\mathbb{M} = \begin{bmatrix} \mathbb{I}(r) & \mathbb{I}(r)\mathbf{A}(r) \\ (\mathbb{I}(r)\mathbf{A}(r))^T & m(r) \end{bmatrix}$$

I(r) is the *locked inertia tensor* – the inertia tensor if the joints are held constant

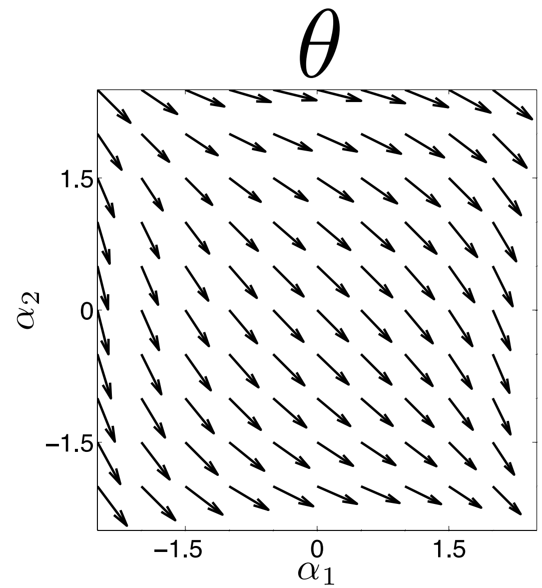
Floating Snake







$$KE = \frac{1}{2} \begin{bmatrix} \xi & \dot{r} \end{bmatrix} \mathbb{M}(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}$$



$$\mathbb{M} = \begin{bmatrix} \mathbb{I}(r) & \mathbb{I}(r)\mathbf{A}(r) \\ (\mathbb{I}(r)\mathbf{A}(r))^T & m(r) \end{bmatrix}$$

Tangent and Co-Tangent Spaces

$$v = (v^1, v^2, ..., v^n) \in T_q Q \qquad w = (w_1, w_2, ..., w_n) \in T_q^* Q$$

Vector fields and co-vector fields

Tangent bundles and co-tangent bundles

When equivalent bases are used $w_i = v^i$ *then they are natural duals with*

$$v^b = w$$

$$w^\# = v$$

Three-Link Swimmers

- Reynolds Number
- Low Reynolds
- High Reynolds