An Introduction to Geometric Mechanics and Differential Geometry

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Configurations and Velocities

The first step in modeling any system is to identify an appropriate mathematical structure to describe it. For a mechanical system, this structure is the *configuration space*. It captures the core aspects of the system, such as the number and character of its degrees of freedom, while abstracting away physical details that affect the system's dynamics but not its fundamental nature.

In this chapter, we examine configuration spaces from first principles. In particular, we review how complex configuration spaces may be built up from basic building blocks, and consider how the internal structure of the spaces affects notions of velocity and relative position within the space. An important aspect of this analysis is a rigorous and principled selection of configuration space structure for *rigid bodies*.

Rigid bodies play a key role in the study and application of geometric mechanics. From a theoretical standpoint, they provide intuitive examples of range of differential geometric concepts such as Lie groups, lifted actions, and exponential maps. On the applications side, mathematical rigid bodies correspond directly to to physical rigid bodies, such as links of a robot or other mechanical system. This correspondence provides an avenue for applying deep mathematical results to practical systems. Further, the basic principles of rigid body motion provide the *body frame* paradigm for mobile articulated systems that we explore in subsequent chapters.

1.1 Configuration Spaces

A mechanical system's *degrees of freedom* are the independent ways in which it can move, such as by translating, rotating, or bending at a joint. Its *configuration*, denoted q, is an arrangement of these degrees of freedom that uniquely defines the location in the world of each point on the system. The system's *configuration space*, Q, is the set of configurations the system can assume, and has as many dimensions as the system has degrees of freedom. Configurations are often expressed in terms of *generalized coordinates*, which correspond to the choice of parameterization used for the configuration space.

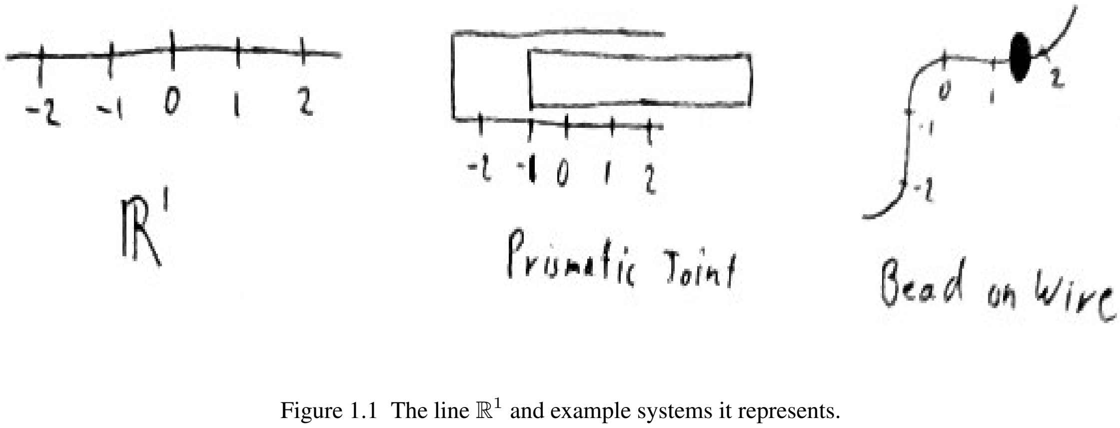


Figure 1.1 The line $\mathbb{R}^1$ and example systems it represents.

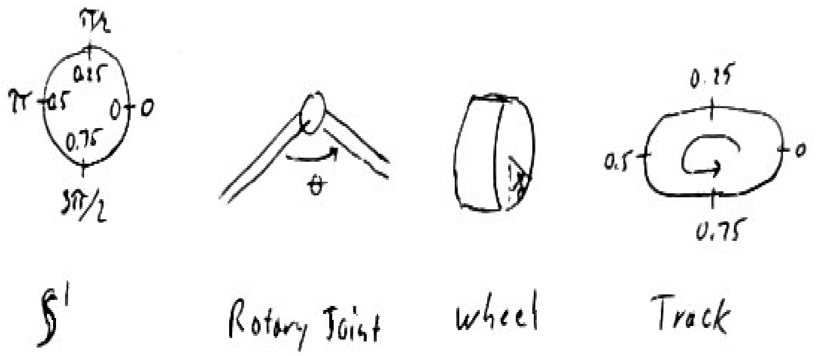


Figure 1.2 The circle $\mathbb{S}^1$ and example systems it represents. The two sets of numbers represent two common parameterizations of the circle: from 0 to 2π radians, and from 0 to 1 cycles.

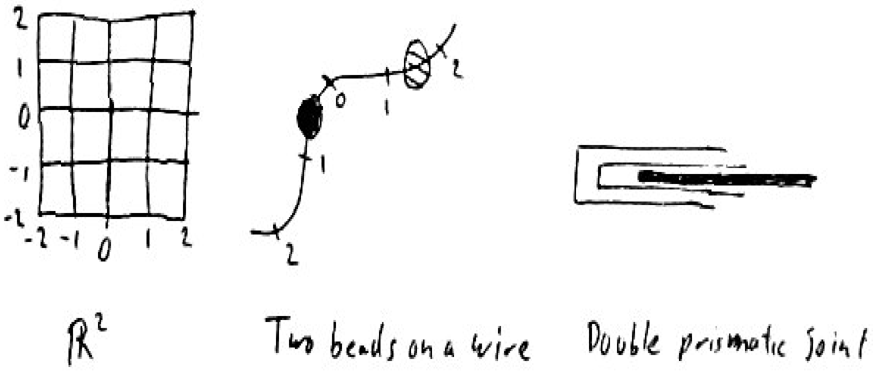


Figure 1.3 The plane $\mathbb{R}^2$ and example systems it represents.

The configuration spaces of many mechanical systems are *manifolds*[1]—differentiable spaces—that can be constructed from three basic building blocks: the line, the circle, and the sphere. On its own, the line serves as the configuration space of simple one-dimensional systems, some examples of which appear in Figure 1.1. These systems may be physically linear, such as a prismatic joint, or not, as in a bead on a curved wire; the important aspect is that a single (real) number (*e.g.* the extension of the joint or the distance along the wire of the bead from a starting point) completely defines the system configuration. The set of ordered real numbers forms a line, and is designated $\mathbb{R}$, or sometimes $\mathbb{R}^1$ to emphasize its one-dimensional nature.

The second configuration building block is the circle $\mathbb{S}^1$ in Figure 1.2, which represents cyclic motion and whose notation reflects that the circle is a sphere with a one-dimensional surface. Cyclic configurations are most commonly encountered in the context of rotating systems where orientations that differ by 2π are equivalent. For example, wheels or other unrestricted rotary joints have circular configuration spaces. Note that it is the cyclic nature, and not the rotational aspect of the motion, that induces the circular configuration space—if a joint cannot make a full rotation, then its configuration space is $\mathbb{R}^1$ and not $\mathbb{S}^1$; conversely, systems moving around a closed track have $\mathbb{S}^1$ configuration spaces, even if the track is not circular.

Combinations of lines and circles provide configuration spaces for more complex systems. The simplest of these, the plane $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$, is formed as the *direct product* of two lines. As illustrated in Figure 1.3, it applies to systems whose configurations are described by two real numbers, such as a mobile robot moving on a plane or other two-dimensional surface, two points on a line, or a series of two prismatic joints.

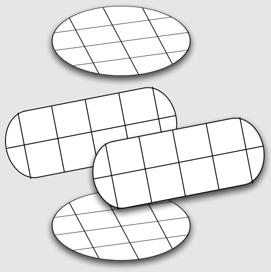
Replacing one of the $\mathbb{R}^1$ components of the plane with a circle generates the cylinder $\mathbb{R}^1 \times \mathbb{S}^1$. Following the same pattern as with other configuration spaces, this space may represent a point moving over a literal cylinder, or a mechanical system such as a prismatic joint in combination with a rotational joint. For ease of

[1] See the box on the facing page.

Manifolds

A *manifold* is a space that is locally like a Euclidean space, but may have a more complicated global structure such as being curved like a wavy sheet, or self-connected like the surface of a cylinder or sphere. This structure is described by the an *atlas* of *charts* corresponding to the manifold. Each chart is a region of Euclidean space that maps to a region of the manifold, generalizing the notion of a "chart" as a literal "map" from a flat sheet of paper onto the surface of the globe. An atlas is a set of overlapping charts that collectively describe the entirety of the manifold, in the same way that an atlas of the globe contains maps that together show the whole surface of the earth.

By assigning points in Euclidean space (n-tuples of real numbers) to points in the manifold, charts inherently parameterize a space and so are often referred to as *coordinate charts*. A key role of multi-chart atlases is to "paper over" singularities that appear when using a single coordinate chart to describe a manifold. Such singularities are well-known on maps of the globe; for instance, a latitude-longitude map is singular at the poles, and so must be combined with additional maps (such as polar projections) to produce an atlas in which each point in the manifold appears at least once.



At the overlap between charts, there is naturally a mapping from each chart into the manifold, then back out into the other chart. These composite functions are the *overlap maps* (referred to in some sources as *transition maps*) between charts, describing how to translate coordinates on one chart to coordinates on a second chart.

continued...

Manifolds *continued*

Formally, a manifold must meet two chief requirements:

1. The *neighborhood*[a] around each point in the manifold must be *homeomorphic*[b] to an open subset of $\mathbb{R}^n$, where n is the *dimensionality* of the manifold.
2. Each point on the manifold must correspond to a point on at least one chart in the atlas.

For many applications, the neighborhoods must share more differentiable structure with $\mathbb{R}^n$ than simple homeomorphism guarantees. Manifolds with this structure are designated as C^k*-(differentiable) manifolds*, and meet two additional requirements:

1. The mappings from the charts to the manifolds must each be k-times differentiable, *i.e.*, they must be C^k-diffeomorphisms.
2. All overlap maps for charts in the atlas must be C^k-diffeomorphisms.

Together, these properties permit the definition of C^k functions on the manifold that maintain their differentiability across regions mapped by different charts. C^∞-manifolds, on which functions can be differentiated arbitrarily many times, are also known as *smooth* or *differential* manifolds.

[a] An infinitesimal region around a point

[b] See the box on the next page.

Homeomorphisms and Diffeomorphisms

Homeomorphism. A *homeomorphism* is a function f mapping between two spaces that is:

1. **Bijective**, *i.e.*, invertible. Bijective functions are both
   a. **Surjective**, or *onto*, meaning that every point in the range is the function of *at least* one point in the domain, and
   b. **Injective**, or *one-to-one* meaning that each point in the range is the function of *no more than* one point in the domain.

   Combining these properties means that there is a full one-to-one correspondence in both directions between points in the domain and points in the range, and thus that
   a. f^{-1} is a true (single-valued) function whose domain is the entire range of f, and
   b. $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identity mappings.
2. **Continuous**, and has a continuous inverse.

Diffeomorphism. A *diffeomorphism* is a homeomorphism for which both f and f^{-1} are additionally differentiable, *i.e.*, they are not only continuous, but all their derivatives are also continuous. A C^k-diffeomorphism is k-times differentiable, meaning that its first k derivatives are continuous.[a] C^∞-diffeomorphisms, for which all of the derivatives are continuous, are referred to as *smooth* diffeomorphisms. Technically, a homeomorphism that is continuous but not differentiable is a C^0-diffeomorphism, but the adjective “diffeomorphic” is generally reserved for mappings that are at least once-differentiable.

[a] C^k is more generally the set of all k-times differentiable functions.

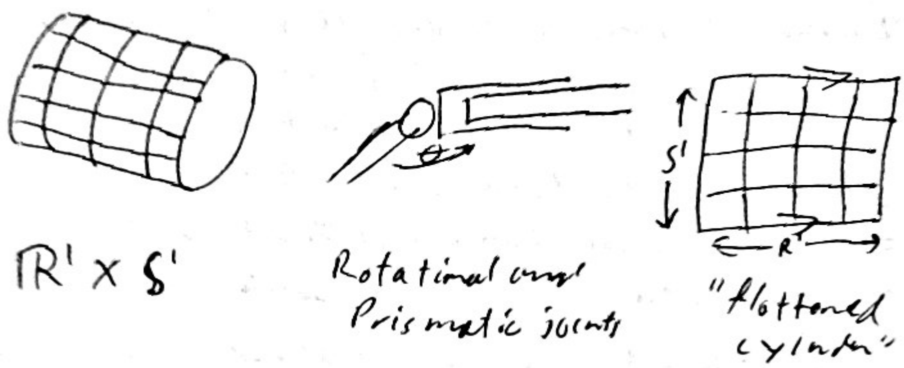


Figure 1.4 The cylinder $\mathbb{R}^1 \times \mathbb{S}^1$, an example system it represents, and its flattened representation.

representation, the cylinder is often depicted in a flattened view, such as that in Figure 1.4, where the $>$ symbols indicate edges identified with each other.

Two $\mathbb{S}^1$ elements together form the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The torus represents systems with two independent cyclic parameters, such as a pair of rotational joints (without joint limits). Similarly to the cylinder, it is often convenient to work with the torus opened out into a planar section with opposite edges identified, as in Figure 1.5.

Considering multi-dimensional configurations brings us to a third fundamental space, the (two-dimensional) sphere $\mathbb{S}^2$. Like the torus, the sphere extends the circle to higher dimensions, but instead of being a set of circles around a circle, the sphere is a set of points equidistant from a common center. This distinction leads to several fundamental differences between the spaces, most notably that any loop on a sphere can be “smoothly deformed” into any other loop or contracted into a point, whereas on a torus there are two distinct loops that can be neither deformed into each other nor contracted to a point, as illustrated in Figure 1.6. Additionally, tori

Direct Product

The *direct product*, or *Cartesian product* of two sets or spaces combines them without mixing the elements together. For instance, if we have two systems with configurations $a \in \mathbb{R}^1_a$ and $b \in \mathbb{R}^1_b$, the direct product of the configuration spaces, $\mathbb{R}^2 = \mathbb{R}^1_a \times \mathbb{R}^1_b$, is structured to preserve the independence of the component subspaces,

$$(a, b) \in \mathbb{R}^2 \equiv (a \in \mathbb{R}^1_a, b \in \mathbb{R}^1_b). \tag{1.i}$$

The direct product stands in contrast to the *semi-direct product* discussed on page 10.

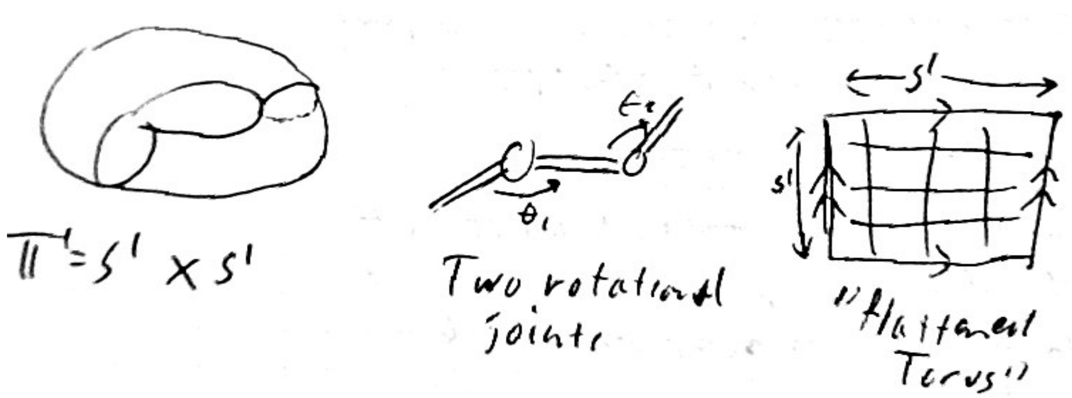


Figure 1.5 The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, an example system it represents, and its flattened representation.

can generally be smoothly parameterized but spherical parameterizations have singularities, such as those at the poles in Figure 1.6, where a single point on the sphere corresponds to multiple latitude, longitude pairings.

Configuration spaces for systems with more than two degrees of freedom can be constructed in the same general same pattern of compounding the basic building blocks.[2] In §1.3, we will examine combining lines and circles to build configuration spaces for rigid bodies moving in the plane, and in Chapter 2, we will extend this approach to mobile articulated systems.

[2] Occasionally new structures (such as hyperspheres) that can only exist in higher-dimensional spaces must also be considered.

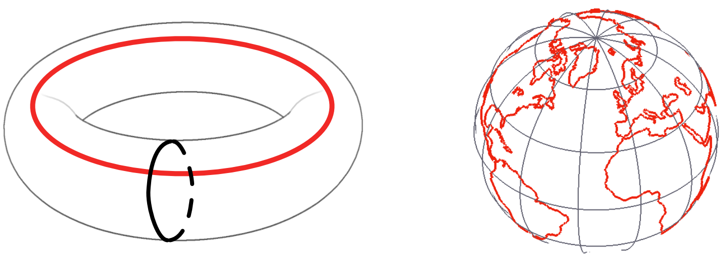


Figure 1.6 (left) The torus has two independent, noncontractable loops. (right) Parameterizations of the spheres have singularities at the poles.

1.2 Lie Groups

In many cases, it is useful to perform algebraic operations such as addition or subtraction on configurations. We may, for instance, want to know the absolute configuration of an object whose position is specified relative to another (combining the configurations), or to find the relative configurations of two objects whose position is known absolutely (taking their difference). Executing these additions and subtractions requires interpreting configurations not only as points in the configuration space, but also as transformations that can be applied to other configurations. This dual interpretation corresponds to the mathematical concept of a *Lie*[3] *group*.

A *group* is the combination of a set and an operation, such that the products of members acting on each other via the operation are themselves within the set.[4] Group structure is often implicit and underlies many basic mathematical concepts, such as the addition of real numbers (the group $(\mathbb{R}, +)$), the multiplication of positive reals (the group $(\mathbb{R}^+, \times)$), or the addition of integers (the group $(\mathbb{Z}, +)$). In a Lie group, the set forms a smooth manifold (as is the case for groups based on real numbers $\mathbb{R}$, but not for discrete groups such those based on integers $\mathbb{Z}$)

For the building-block spaces, the most commonly encountered group operations are addition on the line and modular addition on the circle, respectively producing the Lie groups $(\mathbb{R}^1, +)$ and $(\mathbb{S}^1, + \bmod k)$, where k is the cycle-length in the parameterization of the circle. The group properties match an intuitive understanding of how addition works on these spaces: configurations represent translational offsets along the space, so the natural combination of two configurations is the sum of their offsets. Combinations of lines and circles created via the direct product naturally inherit the group structures of their component spaces, with (modular) addition acting independently along each degree of freedom.

In some cases, more than one choice of group structure is available. For instance, the circle is closely associated with a second Lie group, $(SO(2), \times)$.[5] The *special orthogonal group* $SO(n)$ is the group of rotations in n-dimensional space. In two dimensions, these rotations are 2×2 matrices of the form

$$R \in SO(2) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{1.1}$$

where θ is the magnitude of rotation. These matrices are smooth, cyclic, and unique with respect to theta, and thus inherently correspond to points on $\mathbb{S}^1$ (*i.e.* their spaces are diffeomorphic). Here, the actions of the two Lie groups are also compatible, with the matrix product of two $SO(2)$ elements equivalent to the modular sum of the corresponding $\mathbb{S}^1$ elements. This compatibility means that S^1 and $SO(2)$ are fully *isomorphic*[6] to each other, but this is not a general requirement for groups sharing a manifold structure. In the next section, we consider two non-isomorphic group structures for the configuration spaces of rigid bodies.

Much in the same way that the circle generalizes to higher dimensions either via the direct product as a torus or as a higher-dimensional sphere, direct products of multiple $(\mathbb{S}^1, + \bmod k)$ groups produce toroidal Lie groups and extensions of $SO(n)$ produce rotations through higher-dimensional spaces. There is not, however, a direct correspondence between $\mathbb{S}^{n-1}$ and $SO(n)$. For example, $\mathbb{S}^2$ does not admit any Lie group structure, and $SO(3)$ corresponds to the set of *oriented* positions on the sphere.

1.3 Rigid Body Configurations

With basic definitions of configuration spaces and Lie groups in place, we can now turn our attention to the main topic of this chapter: rigid bodies. First, what is a rigid body? Physically, it is an object that does not deform in response to external forces. Mathematically, we can describe it as a (possibly continuous) set of points with fixed interpoint distances and relative orientation. A more useful mathematical description, illustrated in Figure 1.7, is that a rigid body is composed of a movable reference frame (the *body frame*) and a set of points whose positions are fixed with respect to the body frame. Under this last definition, it becomes clear that the

[3] Pronounced “Lee.”
[4] Group structure also requires several other properties, as described in the box on the following page.
[5] As $(SO(2), \times)$ is a well-known group, its operation (matrix multiplication) is implicit, and we can refer to it simply as $SO(2)$.
[6] See the box on page 9.

Groups

A *group* $(G, \circ)$ is the combination of a set G and an operation $\circ$ that satisfies the following properties:

1. **Closure**: The product of any element of G acting on another by the group operation must also be an element of G. More formally, for $g_1, g_2 \in G$,
$$\begin{aligned}(g_1 \circ\ ) : G &\to G \\ g_2 &\mapsto g_1 \circ g_2.\end{aligned} \tag{1.ii}$$
2. **Associativity**: The order in which group operations are evaluated must not affect the product: for all $g_1, g_2, g_3 \in G$,
$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3. \tag{1.iii}$$
3. **Identity element**: The set must contain an identity element e that leaves other elements unchanged when it interacts with them: for $g \in G$, there exists $e \in G$ such that
$$e \circ g = g = g \circ e \tag{1.iv}$$
4. **Inverse**: The inverse (with respect to the group operation) of each group element must be an element of the group and produce the identity element when operating on or operated on by its respective element: for $g \in G$, there must exist $g^{-1} \in G$ such that
$$g^{-1} \circ g = e = g \circ g^{-1} \tag{1.v}$$

Commonly encountered group actions include addition (for which the identity is a zero element) and multiplication (for which the identity has unit value). If the order of operation is important (such as most matrix multiplication), any group action may be conducted as a *left action*
$$L_g = g\ \circ \tag{1.vi}$$
or a *right action*
$$R_g = \ \circ\, g \tag{1.vii}$$
with the acting element placed correspondingly at the beginning or end of the execution sequence. As a general principle, the left transformation $L_{g_2} g_1$ may be interpreted as "moving the group element g_1 by g_2," while the right action $R_{g_2} g_1$ is used to find "the group element at g_2 relative to g_1.

When the group action is unambiguous the group may be referred to by its underlying set, as is the case for the $SO(n)$ and $SE(n)$ groups discussed below. As additional shorthand notation, the $\circ$ symbol is often dropped, with $g_1 g_2$ being read as equivalent to $g_1 \circ g_2$.

configuration (position and orientation) of the body frame completely defines the location of all points in the body, and therefore that we can treat the configuration spaces of the body and the frame equivalently.

When modeling a rigid body, we have some freedom in our selection of the body frame, corresponding to our freedom to select different parameterizations for the system. For instance, we may place the body frame at the center of mass of an object, at one end, or even at a point that is not actually included within the physical bounds of the body, so long as the positions of all points in the body are constant with respect to the chosen frame.

Once a body frame has been selected, the question arises of what configuration space and Lie group structure best describes its configuration. In the case of bodies moving in the plane, for which the frame is defined by its two-dimensional position and a single orientation component, it might be natural to take these components independently and use an $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ configuration group. There are several advantages, however, to instead describing the frame's configuration via the *special Euclidian group* on 2-dimensional space, $(SE(2), \times)$. Most relevantly to our current discussion, $SE(2)$ better captures the *relative* motion of rigid bodies than does $(\mathbb{R}^2 \times$

Isomorphism

An *isomorphism* is a structure-preserving relationship between two mathematical objects. Two groups A and B are considered isomorphic if there exists a bijective function

$$f : A \to B \tag{1.viii}$$
$$a \mapsto b \tag{1.ix}$$

such that

$$f(a_1 \circ a_2) = f(a_1) \circ f(a_2), \tag{1.x}$$

and (because of the bijectivity requirement),

$$f^{-1}(b_1 \circ a_2) = f^{-1}(b_1) \circ f^{-1}(b_2). \tag{1.xi}$$

Two simple examples of group isomorphisms are

1. The relationship between the multiplicative group of positive real numbers, $(\mathbb{R}^+, \times)$, and the additive group of real numbers, $(\mathbb{R}, +)$. The natural logarithm and exponential functions isomorphically map between these two groups: for $x, y \in (\mathbb{R}^+, \times)$,

$$\log(xy) = \log x + \log y \tag{1.xii}$$

   and for $x, y \in (\mathbb{R}^+, +)$

$$\exp(x + y) = (\exp x)(\exp y) \tag{1.xiii}$$

2. The relationship between the modular additive group of numbers on the circle, $(\mathbb{S}^1, + \bmod 2\pi)$, and the group of two-dimensional rotations, $SO(2)$. Under the matrix construction function for elements of $SO(2)$,

$$R : \mathbb{S}^1 \to SO(2) \tag{1.xiv}$$
$$\theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1.xv}$$

   addition of $\mathbb{S}^1$ elements is equivalent to multiplications of $SO(2)$ elements,

$$R(\theta_1 + \theta_2) = R(\theta_1)R(\theta_2). \tag{1.xvi}$$

$\mathbb{S}^1, +)$, giving rise to *symmetries* that we exploit in future chapters.[7] Before discussing this point further, it is advantageous to examine the details of working on $SE(2)$.

1.3.1 The Special Euclidean Group $SE(2)$

The special Euclidean group in two dimensions, $SE(2)$ is the set of *proper rigid* (non-distorting and non-reflecting) transformations in the plane. It includes translations and rotations, while excluding shearing, scaling, and mirroring motions. An element $g \in SE(2)$ has components (x, y, θ), and is typically represented as a *homogeneous matrix*,

$$g = (x, y, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.2}$$

[7] Additionally, although we will not consider this for now, $SE(n)$ generalizes much better to three dimensional motion than does $(\mathbb{R}^2 \times \mathbb{S}^1, +)$.

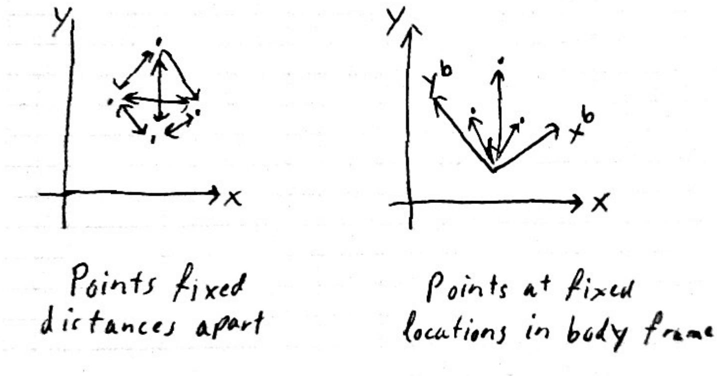


Figure 1.7 A rigid body can be defined either as a set of points at fixed distances from each other (left) or more usefully as a set of points at fixed locations in a movable body frame (right).

Direct and Semi-direct Products of Groups

When we combine groups together to form larger groups, we must consider not only how their underlying sets combine, but also what the overall group action becomes. Often, the combination of two groups A and B into a new group C is taken to mean the creation of a *direct product group*

$$C = A \times B, \tag{1.xvii}$$

in which components that started out in A or B only affect other components that started out as elements of the same group, *i.e.*, $c_1c_2 = (a_1a_2, b_1b_2)$. Direct products preserve properties such as being abelian (commutative) – if A or B has this property, then so does the corresponding section of C.

In a *semi-direct product group*,

$$D = A \circledS B, \tag{1.xviii}$$

elements of A act not only on each other, but also on elements of B. One such structure that appears in our kinematics discussion takes the form

$$d_1d_2 = (a_1a_2, b_1(a_1b_2)). \tag{1.xix}$$

A key aspect of such groups is that even though they do not possess the full orthogonality of a direct product group, the A components do preserve their original properties, and thus results that depend on these properties can be applied to the corresponding elements of D.

This structure, containing an $SO(2)$ element for orientation and an $\mathbb{R}^2$ element for position, represents the *semi-direct product*[8] $SO(2) \circledS \mathbb{R}^2$, with rotations acting on each other and on the translational components, but the translational components only acting on each other.

Elements of $SE(2)$ have four common (related) interpretations for planar systems:

1. the position and orientation of rigid bodies
2. the position and orientation of coordinate frames
3. actions that move a rigid body or coordinate frame with respect to a fixed coordinate frame
4. actions that take a point in one coordinate frame, and find the equivalent point in a second coordinate frame.

As discussed above, the first and second interpretations are closely linked—the position and orientation of a rigid object inherently identifies a body coordinate frame aligned with the object's longitudinal and lateral axes, and vice versa.

[8] See the box on this page.

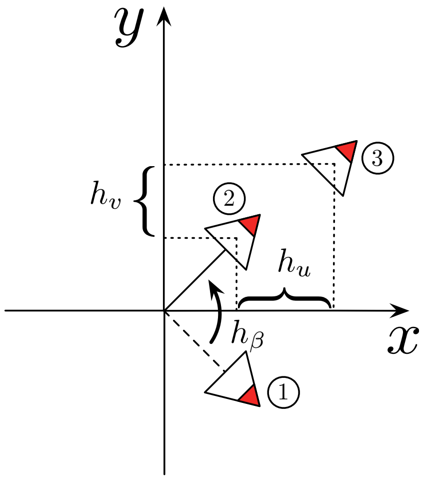


Figure 1.8 The left action of $SE(2)$ interpreted as a change in position. $h = (u, v, \beta)$ first rotates the object around the origin by β, then translates it by (u, v).

Interpretations three and four relate to $SE(2)$ being not just a space but a Lie group, making every configuration also an action. The presence of two action interpretations highlights a feature of groups that did not appear in §1.2. Strictly speaking, a group has both a *left action* and a *right action*, respectively denoted $L_h g = h \circ g$ and $R_h g = g \circ h$ for an element h acting on an element g. Additive groups, such as we considered in §1.2, are abelian (commutative), making the left/right distinction disappear:

$$L_h g = h \circ g = h + g = g + h = g \circ h = R_h g. \tag{1.3}$$

Matrix multiplication, however, does not commute outside of a few special cases such as $SO(2)$, and the left and right actions on $SE(2)$ are not equivalent.

If we take $g, h \in SE(2)$, with $g = (x, y, \theta)$ and $h = (u, v, \beta)$, the product of the left action of h on g is

$$hg = \begin{bmatrix} \cos\beta & -\sin\beta & u \\ \sin\beta & \cos\beta & v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \tag{1.4}$$

$$= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) & x\cos\beta - y\sin\beta + u \\ \sin(\theta+\beta) & \cos(\theta+\beta) & x\sin\beta + y\cos\beta + v \\ 0 & 0 & 1 \end{bmatrix} \tag{1.5}$$

$$= (x\cos\beta - y\sin\beta + u, x\sin\beta + y\cos\beta + v, \theta + \beta). \tag{1.6}$$

Physically, this action corresponds to taking g as the (x, y, θ) coordinates of a rigid body, and h as an action that transforms g by first rotating the body around the origin by β, then translating it by (u, v), as illustrated in Figure 1.8; it also finds the (absolute) location of the system as if g were defined with respect to h rather than the origin.

The product of the right action of h on g is

$$gh = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & u \\ \sin\beta & \cos\beta & v \\ 0 & 0 & 1 \end{bmatrix} \tag{1.7}$$

$$= \begin{bmatrix} \cos(\theta+\beta) & -\sin(\theta+\beta) & x + u\cos\theta - v\sin\theta \\ \sin(\theta+\beta) & \cos(\theta+\beta) & y + u\sin\theta + v\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \tag{1.8}$$

$$= (x + u\cos\theta - v\sin\theta, y + u\sin\theta + v\cos\theta, \theta + \beta). \tag{1.9}$$

This action corresponds to starting at position g and moving by h relative to this starting position, or equivalently, to finding the global position of the point at h relative to g, as illustrated in Fig. 1.9.

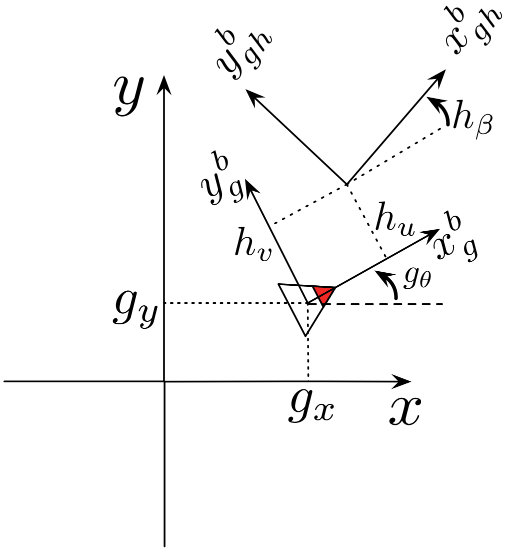


Figure 1.9 The right action of $SE(2)$ finds the coordinates of the frame identified by gh, which is the frame at h relative to g.

Note that when $h = g^{-1}$, both the left and right action return the identity element of $SE(2)$, *i.e.* the origin of the space, as

$$g^{-1}g = gg^{-1} = I = (0, 0, 0). \tag{1.10}$$

1.3.2 $SE(2)$ and Points

Each point p in a rigid body is characterized by its coordinates p^b_x and p^b_y. For a body configuration g, the world positions (p_x, p_y), of these points are calculated as a special case of the general $SE(2)$ action,

$$\begin{bmatrix} \varnothing & & p_x \\ & & p_y \\ 0 & 0 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}}^{g} \begin{bmatrix} \varnothing & & p^b_x \\ & & p^b_y \\ 0 & 0 & 1 \end{bmatrix} \tag{1.11}$$

$$\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p^b_x \\ p^b_y \\ 1 \end{bmatrix}, \tag{1.12}$$

in which, as points have only positions, the orientation component of p is undefined. The process of mapping from g to the world position of p is an aspect of the *forward kinematics* process we discuss in more depth in §2.1.

1.3.3 $SE(2)$ vs. $(\mathbb{R}^2 \times \mathbb{S}^1, +)$

In the choice of group structures to use for rigid body configurations, $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ initially appears to have an advantage from a complexity standpoint—simply adding translations and changes in orientation together is easier than working with matrix multiplications that mix translation and rotation. The advantage of $SE(2)$ lies in the way the group action corresponds to relative positioning. For example, Figure 1.10 shows a rigid body at positions $e = (0, 0, 0)$ (the group identity element, and the origin of the space), $g = (1, 1, \pi/2)$, and at $h = g \circ g$ as calculated with each group action.

In $\mathbb{R}^2 \times \mathbb{S}^1$, the move from e to g is the same as that from g to h in *world* terms: the system experiences the same displacements in x, y, and θ during each transformation. From a *local* perspective, however, the

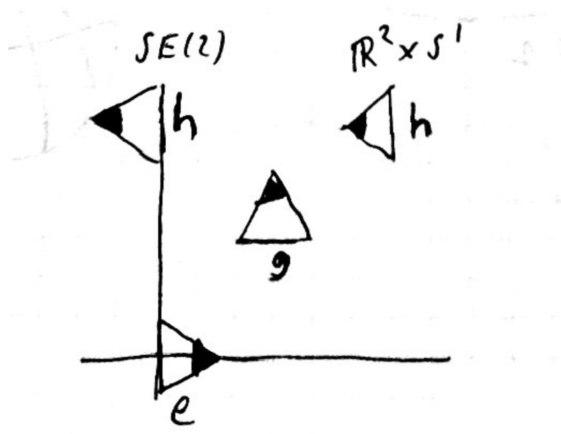


Figure 1.10 In $(\mathbb{R}^2 \times \mathbb{S}^1, +)$, displacements are specified with respect to the world, while in $SE(2)$ they are with respect to the starting frame of the system.

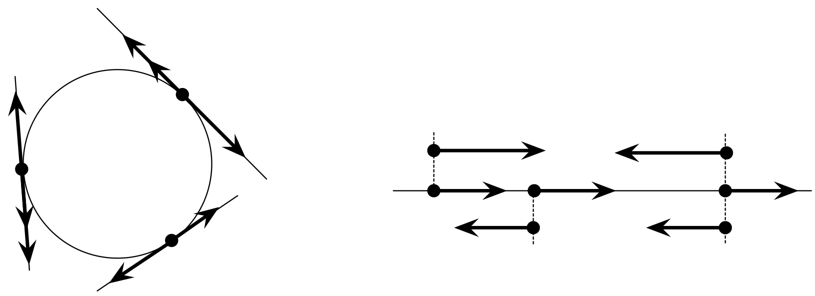


Figure 1.11 Vectors in the tangent spaces to the circle and the line. On the line, at right, some of the vectors have been illustrated parallel to the line for visual clarity.

displacements appear very different. In the body frame, looking down the x axis, the translation from e to g is a motion "forward and to the left," but the translation from g to h is "forward and to the right." With $SE(2)$, the group action takes into account the orientation of the frame at g, ensuring that the second transformation is, like the first, a motion "forward and to the left."

This automatic inclusion of locality is the strength of $SE(2)$. It provides a solid framework for describing relative motion that supports describing the motions of systems in terms of body velocities (§1.5.1) and intuitive tools for working with kinematic chains of linked bodies. Ultimately, the locality reveals the *symmetries* in system dynamics that enable the locomotion analysis starting in Chapter 3.

1.4 Velocities

When analyzing the kinematics and dynamics of a system, it is generally important to consider not only its configuration q, but also the velocity $\dot{q}$ with which the configuration is changing; together these quantities form the system's *state*. Velocity vectors are represented as elements of *tangent spaces* T_qQ associated with the system's configuration space Q. A tangent space to a manifold can be thought of as a "linearization" of the manifold, or the vector space that most closely approximates the manifold at a given point, and can only exist if the manifold is at least C^1 differentiable at that point. Intuition for the definition of a tangent space is shown in Figure 1.11, where the velocity of a point moving along the circle can always be represented as a vector lying in the line tangent to the circle at its current location, with magnitude (distance along the line) corresponding to the point's speed.

This notion of tangency extends naturally to points moving on a line (for which the tangent spaces are overlaid with the line itself, as in Figure 1.11), and to higher-dimension surfaces, such as the sphere in Figure 1.12, with

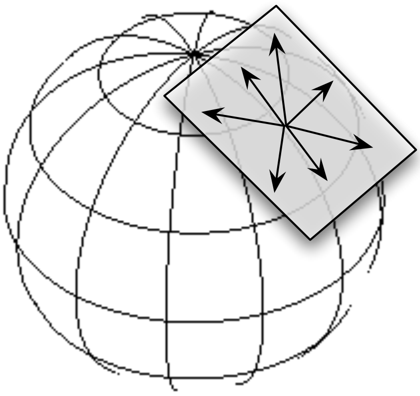


Figure 1.12 Tangent space to the sphere.

the chosen tangent space illustrated as a two-dimensional plane touching the sphere. In general, the tangent spaces to an n-dimensional manifold are each $\mathbb{R}^n$ vector spaces. The collection of all the tangent spaces to a manifold form the manifold's *tangent bundle*, TQ.

1.4.1 Vector Fields

A *vector field* is a (possibly time-varying) assignment of a unique vector to each point in a subset of a manifold. Formally, a vector field $\mathbf{X}$ on $Q_0 \subset Q$ is a mapping

$$\mathbf{X} : Q_0 \times \mathbb{R} \to T_q Q \tag{1.13}$$

$$(q, t) \mapsto v, \tag{1.14}$$

where v is a vector in $T_q Q$. When the manifold Q is a configuration space, vector fields on Q represent *velocity fields* describing how the system's configuration evolves under the influence of a first-order differential equation

$$\dot{q} = \mathbf{X}(q, t). \tag{1.15}$$

Solutions to this differential equation (given an initial configuration $q_0 = q(0)$) are the *integral curves*, or *flows*, of $\mathbf{X}$. These solutions take the forms of trajectories γ through the configuration space,

$$\gamma : [0, T] \to Q \tag{1.16}$$

$$t \mapsto q, \tag{1.17}$$

whose tangent vector (*i.e.* time derivative) at each point is equal to the vector at that point in the underlying vector field,

$$\dot{\gamma}(t) = \mathbf{X}(\gamma(t)). \tag{1.18}$$

Several examples of integral curves on a vector field are shown in Figure 1.13. Note that on a static (time-invariant) field, the theorem of uniqueness of the solutions of differential equations guarantees certain properties of the integral curves. First no two integral curves may cross each other, and no integral curve may cross itself; either occurrence would require that $\mathbf{X}$ contain two different vectors at the intersection point, as illustrated in Figure 1.14. Similarly, two integral curves will never merge with each other, and single integral curve will never merge back into itself (though they may become asymptotically close), as a complete merger would require that the corresponding flow on $-\mathbf{X}$ split into two solutions, and thus that $\mathbf{X}$ have two vectors at the split/merge point. The one exception to this principle is for integral curves that form closed loops, in which the self-connections do not create split points, and thus are compatible with having a unique tangent vector at each point in the space.

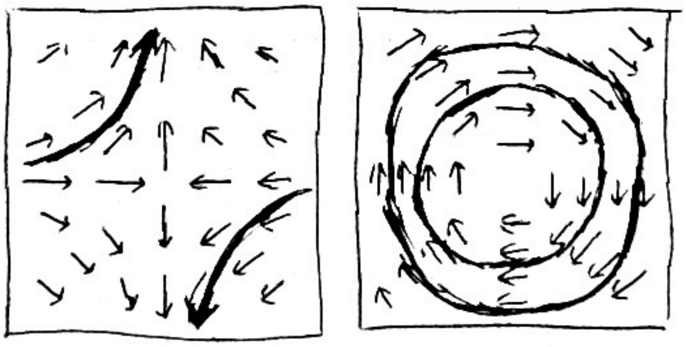


Figure 1.13 Sample integral curves on vector fields.

Figure 1.14 Integral curves on a vector field cannot cross, split, or merge, as this would require there to be two tangent vectors at the same point in the underlying space .

1.4.2 Lifted Actions

Individual tangent spaces in a tangent bundle are independent from one another, so in the general case there is no well-defined method for comparing or adding vectors in different spaces. When transformations are defined between configurations, as is the case for Lie groups, these transformations may have associated *lifted actions* that map between "equivalent" vectors in separate tangent spaces and allow them to be compared. Specifically, on Lie groups a left action L_h applied to an initial configuration g is accompanied by a left lifted action T_gL_h that takes vectors from the starting tangent space T_gG and finds their "equivalent" vectors in the ending tangent space $T_{hg}G$,

$$\begin{aligned} T_gL_h : T_gG &\to T_{hg}G \\ \dot{g} &\mapsto (\dot{hg}) \end{aligned} \tag{1.19}$$

with the *notion of equivalency based on the group action* as discussed below. For groups where the left and right actions differ, a right lifted action, T_gR_h, is similarly defined.[9]

As a general principle, the the left lifted action preserves the *local velocity* of a system—if its velocities at two configurations are related by the left lifted action, then it is moving equivalently *from its own perspective* at both configurations. In contrast, the right lifted action is a *relationship-preserving* velocity transformation—two points moving with right-equivalent velocities maintain themselves a constant group action apart from each other. These equivalencies closely mirror the inherent natures of left and right actions: as we noted in the discussion of groups on page 8, the left action *moves* group elements, while the right action deals with group elements whose positions are defined *relative to each other*.

[9] The notation for tangent spaces and lifted actions is similar. As an aid to the reader, we have collected a recap of these definitions in Table 1.1 on the next page.

Table 1.1 *Tangent space terminology*

Notation	Meaning
TQ	Tangent bundle to Q
T_qQ	Tangent space to Q at q
T_gL_h	Left lifted action mapping velocities from T_gG to $T_{hg}G$
T_gR_h	Right lifted action mapping velocities from T_gG to $T_{gh}G$

Lifted actions are calculated as the differential of the associated action,

$$T_gL_h = \frac{\partial(L_h g)}{\partial g} \tag{1.20}$$

and

$$T_gR_h = \frac{\partial(R_h g)}{\partial g}. \tag{1.21}$$

On additive Lie groups, the linearity of the group action makes the lifted actions trivial identity maps. For example, on $\mathbb{R}^2$, the lifted actions are

$$T_gL_h = T_gR_h = \frac{\partial([h_1,h_2]+[g_1,g_2])}{\partial[g_1,g_2]} = \begin{bmatrix} \dfrac{\partial(h_1+g_1)}{\partial g_1} & \dfrac{\partial(h_1+g_1)}{\partial g_2} \\ \dfrac{\partial(h_2+g_2)}{\partial g_1} & \dfrac{\partial(h_2+g_2)}{\partial g_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.22}$$

As a left lifted action, this property matches an intuitive notion of treating velocity as independent from position—no matter where in $\mathbb{R}^2$ the system is, moving with unit velocity in a given direction has the same meaning. As a right lifted action, it tells us that two points moving in the plane with the same velocity have zero velocity relative to each other, once more matching intuitive expectations.

A simple example of a nontrivial lifted action appears in the multiplication of positive real numbers, $(\mathbb{R}^+, \times)$. Over a left group action of

$$L_h g = hg, \tag{1.23}$$

the lifted action for this group,

$$T_gL_h = \frac{\partial(hg)}{\partial g} = h, \tag{1.24}$$

scales velocity vectors by the same factor of h as is applied to the configuration. Here, left-equivalent velocities are interpreted as being *in proportion to the current value*, much in the same way that interest rates on bank accounts are specified as percentages of money in the account, rather than a flat payout each time period.

This proportional property highlights an interesting tension between the group and manifold interpretations of multiplicative Lie groups. On multiplicative groups, the composition and difference operations are respectively products and quotients, rather than the addition and subtraction seen on additive groups. As such, velocities and integrals on these groups are much better described by *multiplicative calculus*[10] than by standard additive calculus formulations. In particular, velocities corresponding to the group action are given by the *quotient-derivative* with respect to time, *i.e.*, the *multiplicative velocity*, which describes the *proportional* change over time of the group element's value.

The manifold aspect of a Lie group, however, is inherently an additive space, meaning that velocities through the manifold are *difference-derivatives* with respect to time, or *additive velocities*, describing the *linear* change over time of a configuration's position in the manifold. As a consequence of this duality, two configurations with the same group velocity are unlikely to have the same manifold velocity, and vice versa. The left lifted action bridges this discrepancy, by identifying pairs of (additive) manifold velocities in the tangent spaces of different configurations that share the same (multiplicative) group velocities and are thus equivalent with respect to the

[10] See the box on page 18.

group action; integrating these manifold velocities in the standard additive fashion thus gives the same results as evaluating the product-integral of the group velocities.

To close this stage of our discussion on lifted actions on the multiplicative group, we note that the right lifted action,

$$T_g R_h = \frac{\partial(gh)}{\partial g} = h, \tag{1.25}$$

also scales velocities proportionally to the magnitude of the acting element h. Here this indicates that if two points should remain at a given ratio to each other, they should each grow with the same multiplicative velocity through the group, and thus with a proportionally-scaled linear velocity through the manifold.

1.5 Rigid Body Velocities

In §1.3.3, we discussed how assigning an $SE(2)$ structure to planar rigid bodies imbues the configuration space with a sense of locality, in that concatenated group actions are defined with respect to the body frame at the beginning of each action, rather than with respect to a global coordinate frame. This same sense of locality appears in the tangent spaces of $SE(2)$, with the left and right lifted actions each encoding an equivalence between velocity vectors that preserves an aspect of "local" velocity for the rigid body.

The left lifted action preserves a system's *body velocity*: a rigid body whose velocity at two different configurations is related by $T_g L_h$ is moving with the same longitudinal, lateral, and rotational velocity in each configuration. The right lifted action $T_g R_h$ preserves *spatial velocity*, facilitating calculation of the velocities of different points fixed to a common rigid body.

1.5.1 Body Velocity

A system's *body velocity* ξ is its velocity expressed in the instantaneous local coordinate frame, as depicted in Figure 1.15(b). For a planar system, the components of ξ can be characterized as the forward, lateral, and rotational velocities ξ^x, ξ^y, and ξ^θ. The body velocity plays an important role in kinematic and dynamic analysis, where forces and constraints are often defined in terms of body coordinates.

At a strictly mechanical level, the body velocity of a rigid body with orientation θ can be calculated from the world velocity by rotating the translational component by $-\theta$ (equivalent to rotating the reference frame by θ) and leaving the rotational component unchanged,

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{g}_x \\ \dot{g}_y \\ \dot{g}_\theta \end{bmatrix}. \tag{1.26}$$

Similarly, the world velocity can be calculated from the body velocity by inverting this relationship,

$$\dot{g} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \xi \tag{1.27}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi. \tag{1.28}$$

For many purposes, these relationships are all that is needed to make effective use of the body velocity. However, by examining their roots in the structure of $SE(2)$, specifically the left lifted action, we can gain greater insight into rigid body motion and apply some powerful mathematical tools to these systems. As on other spaces, the left lifted action on $SE(2)$ is the differential of the left action. Using the convention of $g = (x, y, \theta)$

Multiplicative Calculus

Standard calculus in the Newton-Leibniz tradition is fundamentally based on the addition and subtraction of infinitesimal quantities. This property is well-suited to working on additive groups, but does not adapt naturally to problems on multiplicative groups, in which products and quotients replace sums and differences. On these groups, it is much more natural to use the *multiplicative calculus* paradigm introduced by Volterra. The two chief tools of multiplicative calculus are the *product-integral*, which generalizes integration into a cumulative product, and the *quotient-derivative*, which describes the rate of change of a function as the ratio of its values.

Product-integral. The product-integral is the continuous counterpart to the product of a sequence, much in the same way that the standard integral is the continuous counterpart to summation:

	Continuous	Discrete
Addition	$\int$ Integral	$\sum$ Sum
Multiplication	$\prod$ Product-integral	$\prod$ Product

Just as the standard integral is defined via the limit

$$F(b) - F(a) = \int_a^b u(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} u(x_k)\, \Delta x, \tag{1.xx}$$

with $x_k = k\Delta x$ and $\Delta x = \frac{b-a}{n}$, the product integral is

$$\frac{\mathcal{F}(b)}{\mathcal{F}(a)} = \prod_a^b (I + v(x)\, dx) = \lim_{n\to\infty} \prod_{k=1}^{n} (I + v(x_k)\, \Delta x), \tag{1.xxi}$$

with I the identity matrix (or equivalent identity element), and x_k and Δx the same as for the standard integral.

Derivatives. In the standard integral, u represents the *(difference-)derivative*, a differential quantity that is continuously being added to the total value and which can be calculated as the limit of the difference between the function values at adjacent points as the step-size between points goes to zero,

$$u(x) = \lim_{\Delta x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}. \tag{1.xxii}$$

In the product-integral, v represents the *quotient-derivative*, a differential quantity that is being continuously multiplied into the total, and is the limit of the *quotient* between consecutive function values, rather than their difference:

$$v(x) = \lim_{\Delta x\to 0} \frac{\big(F(x+\Delta x)/F(x)\big) - I}{\Delta x}. \tag{1.xxiii}$$

The inclusion of the identity element in the definition of the product-integral reflects this multiplicative nature – if v is a small change to the total, it must be incorporated as a small variation on the identity element.

continued...

Multiplicative Calculus *continued*

Left and Right Multiplications. If the order in which the multiplications are executed matters, (1.xxi) is specifically referred to as the *left product integral*,

$$\prod_a^b (I + v(x)\,dx) = \lim_{n\to\infty} (I + v(x_n)\,\Delta x) \times (I + v(x_{n-1})\,\Delta x) \times \ldots \times (I + v(x_0)\,\Delta x), \tag{1.xxiv}$$

in which each element added to the sequence left-multiplies the previous total, and $v(x)$ is calculated as

$$v(x) = \lim_{\Delta x\to 0} \frac{\big(F(x+\Delta x)\circ F^{-1}(x)\big) - I}{\Delta x}. \tag{1.xxv}$$

For many problems, such as when the integral represents a trajectory evolving over a Lie group, it is more appropriate to use the *right product integral*

$$(I + v(x)\,dx)\prod_a^b = \lim_{n\to\infty} (I + v(x_k)\,\Delta x)\prod_{k=1}^{n} \tag{1.xxvi}$$

$$= \lim_{n\to\infty} (I + v(x_0)\,\Delta x) \times (I + v(x_1)\,\Delta x) \times \ldots \times (I + v(x_n)\,\Delta x), \tag{1.xxvii}$$

which places new elements at the end of the sequence, thus representing small changes *from* what came before, rather than *applied to* the existing total. In this case, the quotient-derivative should be calculated as

$$v(x) = \lim_{\Delta x\to 0} \frac{\big(F^{-1}(x)\circ F(x+\Delta x)\big) - I}{\Delta x}. \tag{1.xxviii}$$

Group Derivatives and Parameter Derivatives In a multiplicative Lie group, where the group elements act on each other by multiplication, but are parameterized by elements of an (additive) manifold, it is useful to be able to relate multiplicative group derivatives to their corresponding additive derivatives through the manifold. Conveniently, we can convert the quotient-derivative expressions in (1.xxv) and (1.xxviii) into difference-derivatives by extracting an $F^{-1}(x)$ term from the limit contents,

$$v_L(x) = \left(\lim_{\Delta x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}\right)\circ F^{-1}(x) = \frac{dF}{dx}(x)\circ F^{-1}(x) \tag{1.xxix}$$

and

$$v_R(x) = F^{-1}(x)\circ\left(\lim_{\Delta x\to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}\right) = F^{-1}(x)\circ\frac{dF}{dx}(x). \tag{1.xxx}$$

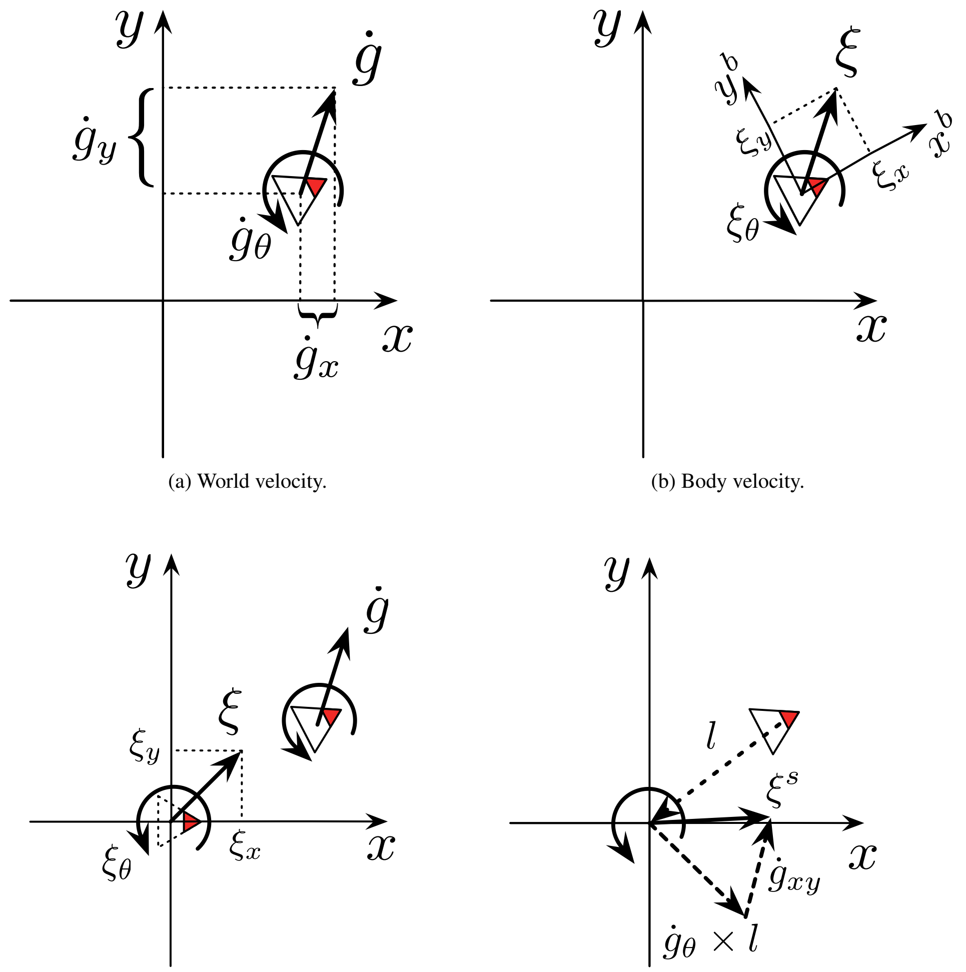
If we take f as the set of parameters of F, the multiplicative derivative then relates to the rates of change of the parameters as

$$v_L(x) = \frac{dF}{df}(f(x))\cdot\frac{df}{dx}(x)\circ F^{-1}(x) \tag{1.xxxi}$$

and

$$v_R(x) = F^{-1}(x)\circ\frac{dF}{df}(f(x))\cdot\frac{df}{dx}(x) \tag{1.xxxii}$$

These left and right group derivatives are thus the matrix forms of the right and left (note the order) lifted actions mapping parameter velocities back to the origin.



(a) World velocity.

(b) Body velocity.

(c) Alternate interpretation of body velocity

(d) Spatial velocity.

Figure 1.15 Four representations of the velocity of a robot. The robot, represented by the triangle, is translating up and to the right while spinning counterclockwise. In (a), the *world velocity*, $\dot{g}$, is measured with respect to the global frame, while in (b), the same vector is measured with respect to the triangle's body frame as the *body velocity* ξ. The body velocity is actually calculated by transporting the body back to the origin frame, as in (c), but by symmetry this is equivalent to bringing the world frame to the system. The *spatial velocity*, ξ^s in (d) is the velocity of a point rigidly attached to the object and overlapping the origin. Its translational component is the vector sum of $\dot{g}_{xy}$, the translational component of $\dot{g}$ and $\dot{g}_\theta \times l$, the rotational velocity crossed with the vector from the object to the origin; the rotational component is the same as for $\dot{g}$.

and $h = (u, v, \beta)$ introduced above, the lifted action takes the form

$$T_g L_h = \frac{\partial(hg)}{\partial g} \tag{1.29}$$

$$= \begin{bmatrix} \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial x} & \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial y} & \frac{\partial(x\cos\beta - y\sin\beta + u)}{\partial \theta} \\ \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial x} & \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial y} & \frac{\partial(x\sin\beta + y\cos\beta + v)}{\partial \theta} \\ \frac{\partial(\theta+\beta)}{\partial x} & \frac{\partial(\theta+\beta)}{\partial y} & \frac{\partial(\theta+\beta)}{\partial \theta} \end{bmatrix} \tag{1.30}$$

$$= \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.31}$$

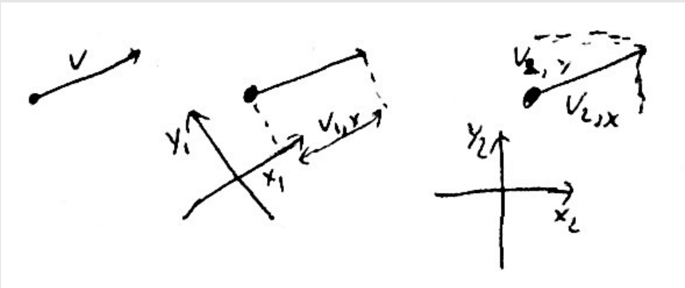
This lifted action bears a strong resemblance to the matrices in (1.26) and (1.28), and it is straightforward to show that $T_g L_h$ preserves body velocities: For any left action L_h that rotates the body frame by an angle β, the

Frames

The kinematics and mechanics approaches discussed in this book are heavily based on the interactions of *frames*. In such discussions, it is important to recognize two distinctions that are sometimes overlooked: the difference between a *reference frame* and a *coordinate frame*, and the difference between the *moving* and *instantaneous* body frames of a system.

Reference frames define a sense of relative motion with respect to an "observer". Whenever we describe a point as moving with a certain velocity, this is implicitly with respect to a chosen reference frame, such as a stationary inertial frame or a moving reference frame associated with a rigid body.

Coordinate frames parameterize reference frames, providing them with origins, scales, and orientations. Multiple coordinate frames may be associated with a single reference frame. For example, consider a point moving on this page, with a velocity vector as illustrated below. We can measure its position and velocity with respect to either coordinate frame shown, but the point and its velocity vector have a fundamental existence in the reference frame of the page, independent of the choice of parameterization.



Many problems in mechanics involve the interaction of several reference frames. Rigid body motion is fundamentally the relative motion between a reference frame in which points on the body have fixed locations and a second reference frame, such as the stationary inertial frame or that of another rigid body. This motion is parameterized by the choice of coordinate frames on the two reference frames, with the relative positions of these coordinate frames completely defining the relative states of points in the two frames.

In addition to representing the positions and velocities of points relative to various frames, it is often useful to represent how a given coordinate frame is moving along its own axes, *e.g.* when finding the longitudinal and lateral motion of a rigid body. In this case, the **body frame** (the chosen coordinate frame on the rigid body reference frame) must be distinguished from the **instantaneous body frame** of the system, which is instantaneously aligned with the body frame, but attached to the stationary inertial frame. The velocity of the body frame with respect to itself is by definition zero, and so not very interesting. Its velocity with respect to the instantaneous body frame, however, is its velocity relative to the stationary inertial frame, but represented in its own coordinates, producing the *body velocity* illustrated in Figure 1.15(b).

accompanying lifted action T_gL_h rotates the velocity vector by the same amount, leaving its expression in the body frame unchanged.

The left lifted action has some especially interesting properties when h is set equal to g^{-1}. The combination of $L_{g^{-1}}$ and $T_gL_{g^{-1}}$ take a rigid body from g and place it at the origin with equivalent body velocity, as illustrated in Figure 1.15(c). The alignment of the origin and body frames in this new configuration produces the equality

$$\xi = T_gL_{g^{-1}}\dot{g} \in T_eG, \tag{1.32}$$

meaning that we can *interchangeably treat the body velocity either in its standard interpretation or as a vector* *in the tangent space at the origin.*[11] The world velocity corresponding to a given body velocity is found by inverting (1.32),

$$\dot{g} = (T_g L_{g^{-1}})^{-1} \xi = T_e L_g \xi. \tag{1.33}$$

On Lie groups (including $SE(2)$), the tangent space at the origin (the identity element of the group) is a privileged space known as the *Lie algebra*. Several powerful mathematical techniques are available for working with elements of the Lie algebra, including the *exponential map* defined in §1.6.4 and the *Lie bracket* that underlies the gait analysis in Chapter **??**. The dual interpretations of the body velocity provide an important link between these abstract principles and the motion of physical systems.

1.5.2 Spatial Velocity

In §1.3.1, we observed that the right action $R_h g$ on $SE(2)$ finds the frame at position and orientation h with respect to g. When finding this frame, as in the examination of articulated systems in Chapter **??**, it is often useful to find its velocity as a function of $\dot{g}$. As above, we could mechanically calculate this velocity, this time using the standard cross-product of rotational velocity and change in position,

$$\dot{(gh)} = \begin{bmatrix} 1 & 0 & -(u \sin\theta + v \cos\theta) \\ 0 & 1 & u \cos\theta - v \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \dot{g}, \tag{1.34}$$

but once again we can gain significantly more insight by relating this operation to the structure of $SE(2)$.

Here, the right lifted action,

$$\begin{aligned} T_g R_h &= \frac{\partial (gh)}{\partial g} && (1.35) \\ &= \begin{bmatrix} \frac{\partial(x+u\cos\theta - v\sin\theta)}{\partial x} & \frac{\partial(x+u\cos\theta - v\sin\theta)}{\partial y} & \frac{\partial(x+u\cos\theta - v\sin\theta)}{\partial \theta} \\ \frac{\partial(y+u\sin\theta + v\cos\theta)}{\partial x} & \frac{\partial(y+u\sin\theta + v\cos\theta)}{\partial y} & \frac{\partial(y+u\sin\theta + v\cos\theta)}{\partial \theta} \\ \frac{\partial(\theta+\beta)}{\partial x} & \frac{\partial(\theta+\beta)}{\partial y} & \frac{\partial(\theta+\beta)}{\partial \theta} \end{bmatrix} && (1.36) \\ &= \begin{bmatrix} 1 & 0 & -(u \sin\theta + v \cos\theta) \\ 0 & 1 & u \cos\theta - v \sin\theta \\ 0 & 0 & 1 \end{bmatrix}, && (1.37) \end{aligned}$$

exactly matches the matrix in (1.34), demonstrating that when an action R_h maps between rigidly attached frames, the "equivalent" vector selected by the accompanying lifted action $T_g R_h$ ensures that the new frame's velocity is compatible with the rigid attachment.

In satisfying the rigid constraint between frames, the right lifted action preserves their *spatial velocity*. Distinct from the world velocity, the spatial velocity ξ^s of a body, illustrated in Figure 1.15(d), is the velocity of the (possibly imaginary) point on that body that is instantaneously over the origin, calculated as

$$\xi^s = T_g R_{g^{-1}} \dot{g}. \tag{1.38}$$

This velocity is rarely useful in and of itself, but, as the spatial velocity is a shared value across any set of rigidly attached frames, serves as a useful "common denominator" for identifying the motion of a rigid body. Additionally, as ξ^s is a vector in the tangent space at the identity, it, like the body velocity ξ, is an element of the Lie algebra and may be operated on by mathematical tools that work in that space, such as the *adjoint operator* discussed below. Inverting (1.38) maps the spatial velocity of a rigid body to the world velocity of any chosen frame attached to it,

$$\dot{g} = (T_g R_{g^{-1}})^{-1} \xi^s = T_e R_g \xi^s. \tag{1.39}$$

[11] Note that this interpretation is not restricted to any particular choice of origin – the value of g is with respect to whichever origin was previously selected, and a change in origin (in effect, a reparameterization of the configuration space) would modify g, $L_{g^{-1}}$, and $T_g L_{g^{-1}}$ accordingly.

1.5.3 Adjoint Operators

In the next chapter, we will find it useful to convert a frame's body velocity into its spatial velocity, or to find the body velocity corresponding to a given spatial velocity. These operations correspond to pairs of lifted actions, which, when assembled together, form the Lie group's *adjoint actions*. Specifically, the mapping from body to spatial velocity for a frame g is the adjoint action Ad_g,

$$\xi^s = \overbrace{(T_g R_{g^{-1}})(T_e L_g)}^{Ad_g}\,\xi, \tag{1.40}$$

and the map from spatial to body velocity is the *adjoint inverse action* $Ad_g^{-1} = Ad_{g^{-1}}$,

$$\xi = \overbrace{(T_g L_{g^{-1}})(T_e R_g)}^{Ad_g^{-1}}\,\xi^s. \tag{1.41}$$

On $SE(2)$, the adjoint action has the effect of mapping the body velocity of a rigid body to its spatial velocity,

$$\xi^s = \overbrace{\begin{bmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}}^{T_g R_{g^{-1}}} \underbrace{\overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_e L_g} \xi}_{\dot{g}} \tag{1.42}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & y \\ \sin\theta & \cos\theta & -x \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g} \xi. \tag{1.43}$$

Similarly, the adjoint inverse on $SE(2)$ maps the spatial velocity to the corresponding body velocity,

$$\xi = \overbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_g L_{g^{-1}}} \underbrace{\overbrace{\begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}}^{T_e R_g} \xi^s}_{\dot{g}} \tag{1.44}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & x\sin\theta - y\cos\theta \\ -\sin\theta & \cos\theta & x\cos\theta + y\sin\theta \\ 0 & 0 & 1 \end{bmatrix}}_{Ad_g^{-1}} \xi^s. \tag{1.45}$$

1.6 Geodesics

As an extension to considering equivalent velocities at different configurations, it is often useful to work with trajectories in which a system moves through its configuration space with a constant velocity. These trajectories form the *geodesics* of the configuration space, generalizing the notion of "straight lines" onto non-Euclidean spaces. On Lie groups, there are two principle classes of geodesics: those defined with respect to the manifold structure, which travel along the locally "shortest" paths through a manifold, and those defined with respect to the group action, which trace out curves of constant group velocity.

1.6.1 Geodesics Through a Manifold

The simplest manifold-based geodesics are constant-speed motions through $\mathbb{R}^1$ or around $\mathbb{S}^1$. On two-dimensional spaces, geodesics that are straight with respect to their manifolds include

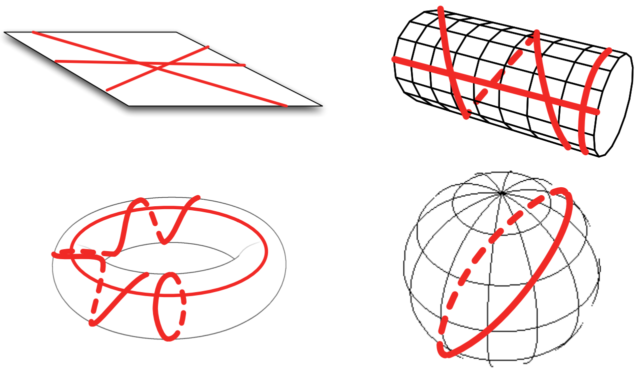


Figure 1.16 Examples of geodesics as “straightest lines” through basic manifolds.

- straight lines in the plane,
- loops, helices, or axial lines on a cylinder,
- loops and helices on a torus, and
- great circles on a sphere,

as illustrated in Figure 1.16. The formal definition of these curves as geodesics relies on a *metric structure* for defining lengths and speeds on a manifold. We will return to this topic in Chapter **??**; for now it is sufficient to note that each of these curves starts out pointing in a given direction and follows as “straight” a line as possible while remaining on its manifold, thus connecting points it passes through by the locally shortest path.

1.6.2 Geodesics on a Lie Group

Geodesics defined with respect the Lie group action are trajectories with constant group velocity. These trajectories correspond to flows on *left-invariant* or *right-invariant* vector fields, defined respectively as

$$\mathbf{X}_L(g) = T_e L_g \xi \tag{1.46}$$

and

$$\mathbf{X}_R(g) = T_e R_g \xi^s, \tag{1.47}$$

where the body and spatial velocities serve as the *generating vectors* of the fields. All vectors in these fields are equivalent under their respective left or right lifted actions, and thus flows along these fields have constant groupwise velocity.

For additive groups, where the lifted actions are simple identity mappings, the geodesics are flows along constant vector fields. Consequently, the group geodesics for addition on the plane, cylinder, and torus $((\mathbb{R}^2, +)$, $(\mathbb{R}^1 \times \mathbb{S}^1, +)$, and $(\mathbb{T}^2, +))$ are the same lines, loops, and helices as seen in Figure 1.16, traversed at constant speed. On multiplicative groups, the geodesics take different forms. Left and right invariant fields on $(\mathbb{R}^+, \times)$, for instance, are scaled by their distance from the identity element ($e = 1$). Flows along these fields are either constantly accelerating (for positive generating velocities), or constantly decelerating (for negative generating velocities), as depicted in Figure 1.17.

1.6.3 $SE(2)$ vs. $(\mathbb{R}^2 \times \mathbb{S}^1, +)$, revisited

In §1.3.3, we noted that the fundamental difference between $SE(2)$ and $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ lies in the way that sequences of group actions concatenate. On $(\mathbb{R}^2 \times \mathbb{S}^1, +)$, the group actions add linearly, and so ultimately specify displacements in world coordinates, while on $SE(2)$, actions are each specified with respect to the result of the previous action. The same difference between group structures appears in their geodesics.

As an additive group, $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ has a trivial lifted action, making equivalent velocities equal at all configurations. The geodesics for this group are thus straight lines through the underlying $\mathbb{R}^2 \times \mathbb{S}^1$ manifold (possibly

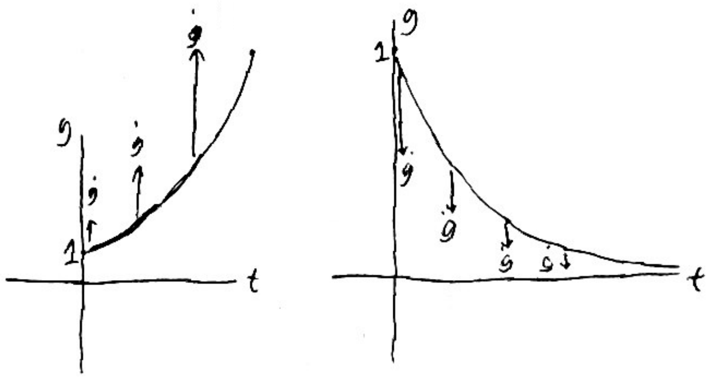


Figure 1.17 The geodesics of the multiplicative group are constantly accelerating (left) or decelerating (right).

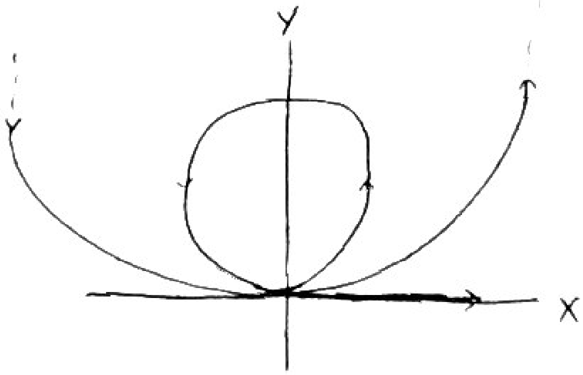


Figure 1.18 $SE(2)$ geodesics for body velocities $[1\ 0\ \dot{\theta}]^T$, with varying magnitudes of $\dot{\theta}$, form a set of helices that project down to tangent circles in the xy plane.

forming loops or helices if the manifold were embedded in a higher-dimensional space). In contrast, the left and right geodesics on $SE(2)$ (constant body or spatial velocities) form helices *within* the $\mathbb{R}^2 \times \mathbb{S}^1$ manifold. If projected into the xy plane, these helices are sets of circles passing through the origin with tangent vectors $[\xi^x\ \xi^y]^T$ and diameters based on the rate of rotation. An example set of these geodesics, for body velocities $[1\ 0\ \dot{\theta}]^T$, with varying magnitudes of $\dot{\theta}$, is shown in Figure 1.18

Comparing a pair of representative geodesics from $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ and $SE(2)$ shows the advantage of using $SE(2)$ to represent rigid body motion. An $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ system starting at the origin with velocity $\xi = (1, 1, 1)$ and maintaining a velocity $\dot{g} = T_e L_g \xi$ moves in a straight line while rotating, in the manner of a spinning hockey puck. An $SE(2)$ system moving with $\dot{g} = T_e L_g \xi$ instead maintains a constant body velocity, making its path curve into an arc as the orientation changes. These two paths are illustrated in Figure 1.19.

The $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ behavior initially appears attractive, as it mimics the expected path for a momentum-conserving system. $SE(2)$, however, has the advantage of following a *symmetric* path. Much in the same way that each $SE(2)$ displacement in §1.3.3 was "forward and to the left," the $SE(2)$ path curves "left of forward" from every point along it. The $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ path exhibits no such symmetry, and oscillates from "forward" to "backward" as the system spins.

1.6.4 Exponential Map

- Exponential map of a velocity is ending point of a unit-time flow along the corresponding geodesic.
- Relates (at least locally) structure of the manifold or group to tangent space at the identity.
- How does this relate to standard notion of exponential as $\exp(x) = e^x$? (note that e here is not the same as group-identity e)

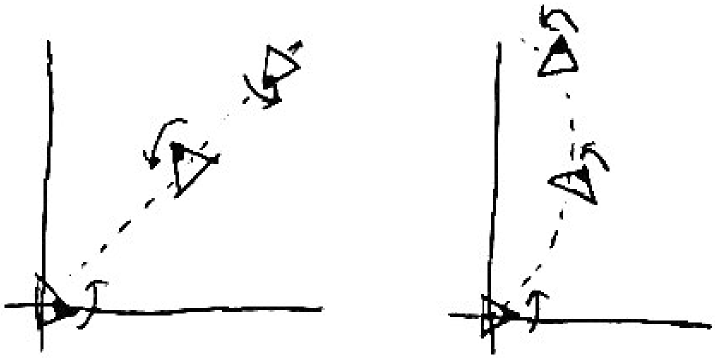


Figure 1.19 Constant-velocity paths on $(\mathbb{R}^2 \times \mathbb{S}^1, +)$ (left) and $SE(2)$ (right)

- $\exp(x)$ on multiplicative group is unit-time path along always-accelerating (or decelerating) trajectory.
- over any interval of a given length, ratio of start and end values will always be the same – this is the same basic definition for a function k^x
- e is by definition the value of $\exp(1)$ on the multiplicative group (unit flow along the exponential trajectory starting with a slope of 1)
- once we have $e^1 = \exp 1$, everything else follows. For example, increasing the magnitude of the argument just pushes the result further along the curve, and we then have identities like

$$\exp(2)/\exp(1) = \exp(1)/\exp(0) \rightarrow \exp(2) = e^2 \tag{1.48}$$

• How does this relate to power series expression for an exponential, $\exp(A) = \sum_{k=0}^{\infty}(A^k/k!)$?
  - express exponential as a product integral, and then in limit form:

$$\exp(A) = \prod_0^1 (I + A\,dx) = \lim_{n\to\infty}\left(I + \frac{A}{n}\right)^n \tag{1.49}$$

  - coefficients of a binomial raised to a power have a specific form:

$$(x+y)^n = \sum_{k=0}^{n}\left(\frac{n^k}{k!}\right) x^{n-k}y^k \tag{1.50}$$

  - apply this form , cancel terms, and power series appears

$$\lim_{n\to\infty}\left(I + \frac{A}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{n^k}{k!}\right) I^{n-k}\left(\frac{A}{n}\right)^k = \sum_{k=0}^{\infty}\frac{A^k}{k!} \tag{1.51}$$

  - Note that the power series is as applicable to matrix multiplication as it is to scalar multiplication. In general, the binomial expansion does not apply to matrices or other non-commutative binomials, but as multiplication with the identity commutes ($IA = AI$), it does apply here.

• Exponential maps on Lie groups are compatible either as flows along the left- and right-invariant fields, or as exponentiations of the corresponding group velocity.
• Useful relationships between group and body velocities (based on relationship between group and parameter velocities, evaluated at the identity):

$$v_R = g^{-1}\frac{dg^{\text{matrix}}}{dg^{\text{param}}}\frac{dg^{\text{param}}}{dt}\bigg|_{g=e} \tag{1.52}$$

  - On $(\mathbb{R}^+, \times)$:

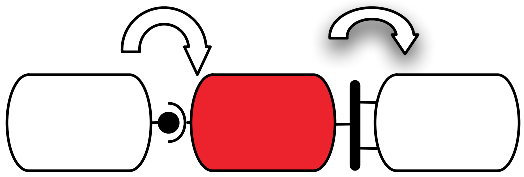
$$v_R = \xi \tag{1.53}$$

– On $SE(2)$

$$v_R = \hat{\xi} = \begin{bmatrix} 0 & -\xi^\theta & \xi^x \\ \xi^\theta & 0 & \xi^y \\ 0 & 0 & 0 \end{bmatrix} \tag{1.54}$$

Exercises

1.1 What are the configuration spaces of (a) a universal joint and (b) a similar joint where the rotational axes do not intersect, as depicted below?



1.2 Calculate the group and lifted actions for the set of scaling matrices

$$\left(\begin{bmatrix} \mathbb{R}^+ & 0 \\ 0 & \mathbb{R}^+ \end{bmatrix}, \times \right) \tag{1.55}$$

1.3 Demonstrate for both $(\mathbb{R}^+, \times)$ and $SE(2)$ that $T_g L_{g^{-1}} = (T_e L_g)^{-1}$ and $T_g R_{g^{-1}} = (T_e R_g)^{-1}$

1.4 Exponential map on $SE(2)$

a. Derive the left exponential map of $\xi = [\xi^x\, \xi^y\, \xi^\theta]^T$ on $SE(2)$ in closed form, by solving for the unit flow along a geodesic on the corresponding right-invariant vector field.

b. Derive the right exponential map of $\xi^s = [\xi^{s,x}\, \xi^{s,y}\, \xi^{s,\theta}]^T$ on $SE(2)$ in closed form, by solving for the unit flow along a geodesic on the corresponding left-invariant vector field.

c. Relate the pitch, diameter, and xy-projected arclength of the helical path followed during $SE(2)$ exponentiation to the components of the vector being exponentiated.

d. Compare the numerical convergence rates of the left and right exponential maps of $[1\ 0\ \dot{\theta}]^T$, with varying magnitudes of $\dot{\theta}$, using

- Euler integration of the geodesic flows on the invariant fields.
- Cumulative product evaluation of the exponential map, using the group velocity. representation

$$\hat{\xi} = \begin{bmatrix} 0 & -\xi^\theta & \xi^x \\ \xi^\theta & 0 & \xi^y \\ 0 & 0 & 0 \end{bmatrix}$$

- Taylor-series evaluation of the exponential map, using the group velocity representation above.

Show the convergence rate of the exponential map as a function of the number of terms used to evaluate each method, comparing against the closed form. Comment on any interpretation you can give to the intermediate values in the cumulative evaluations.

1.5 Explain how a slide rule makes use of the isomorphism between $(\mathbb{R}^+, \times)$ and $(\mathbb{R}, +)$ to facilitate calculation of products. Do you see anything interesting in the fact that the isomorphism is with respect to the exponential and logarithmic functions? We will return to this point in Chapter 4.

2

Articulated Systems

In the previous chapter, we laid out a framework for describing the configuration and velocity of points and rigid bodies. In this chapter, we extend the basic principles to articulated systems composed of multiple rigid bodies and then to flexible systems with continuous modes of deformation. A key aspect of this extension is the notion of *forward kinematics* relating the physical motion of the system to changes in its configuration. As we explore this subject, we introduce connections to fundamental mathematical concepts such as *holonomic constraints* and *Jacobians*.

2.1 Forward Kinematics

The full configuration space of a set of n planar rigid bodies is the direct product of their individual configuration spaces,

$$SE(2) \times SE(2) \times \ldots \times SE(2) = SE(2)^n. \tag{2.1}$$

If the bodies are linked, however, not all of these configurations are accessible. For example, the two bodies in Figure 2.1 are pinned together at the origins of their body frames, so the system as a whole can only achieve configurations in which $(x, y)_1 = (x, y)_2$.

One approach to working with such constraints is to incorporate them into the system's equations of motion. A more elegant approach is to recognize that the pin imposes a pair of *holonomic constraints*[1] on the system, reducing the effective configuration space from six dimensions ($SE(2)^2$) to four. The reduced space, the *accessible manifold*, is the configuration space of the constrained system (in this case $SE(2) \times \mathbb{S}^1$ for the full position of one body and the relative orientation of the second); if generalized coordinates on this space (such as joint angles) are used to represent the system's configuration, the holonomic constraints are automatically respected.

Despite the usefulness of this reduction, one thing that it does not do is to remove the dependence of the

[1] See the box on the next page.

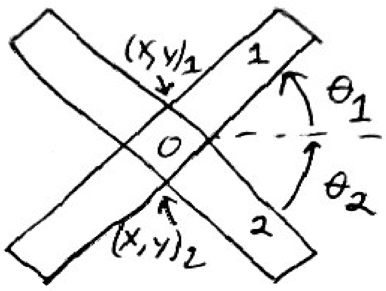


Figure 2.1 Articulated system are composed of multiple rigid bodies with constraints on their relative positions.

Holonomic Constraints

Holonomic constraints remove degrees of freedom from a system, reducing the dimensionality of its configuration space. Formally, a holonomic constraint is defined as a (possibly time-varying) constraint function f on the system's configuration space Q. The *zero set* of the function (set of points $Q_0 \subset Q$ satisfying $f(q_0, t) = 0$ for all $q_0 \in Q_0$) forms the *accessible manifold* of the constrained system, the set of configurations satisfying the constraint.

When a holonomic constraint is applied to an n-dimensional manifold, the resulting accessible manifold is $(n-1)$-dimensional. As it contains all of the admissible configurations of the system, this accessible manifold is itself the configuration space of the constrained system, and in many cases can be used in place of the full configuration space. Multiple holonomic constraints act in concert: the accessible manifold for a multi-constrained system is the intersection of the individual accessible manifolds, and each independent constraint reduces the manifold dimension by one.

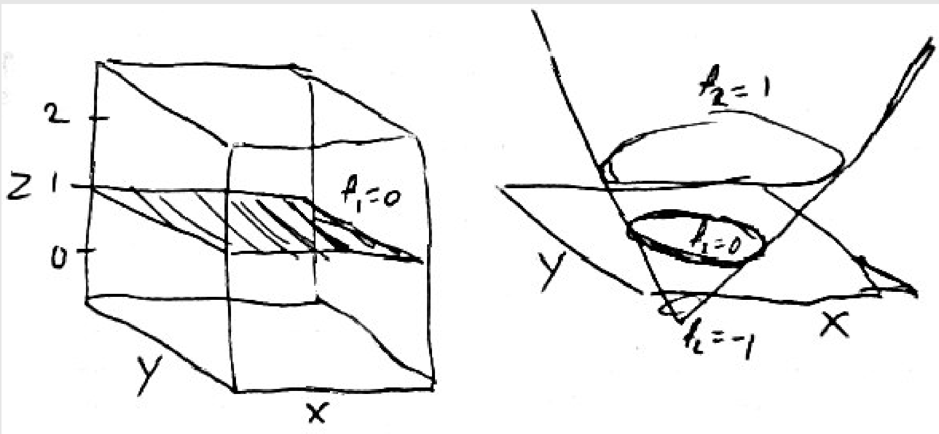
As an example, consider the problem of restricting a point $p = (x, y, z) \in \mathbb{R}^3$ to move only within a unit circle on the plane $z = 1$. The planar condition corresponds to the holonomic constraint function

$$f_1(p, t) = z - 1, \tag{2.i}$$

which has a zero set (and accessible manifold) at the $z = 1$ plane. Once this constraint is made, the configuration space of the system is reduced by one dimension and effectively becomes $p_1 = (x, y) \in \mathbb{R}^2$. Restricting the point to the unit circle is accomplished by the constraint function

$$f_2(p_1, t) = \sqrt{(x^2 + y^2)} - 1. \tag{2.ii}$$

with the final accessible manifold formed by the intersection of a cone representing f_2 with the xy plane. Note that points on the f_2 cone are *not* elements of the original $\mathbb{R}^3$ space.



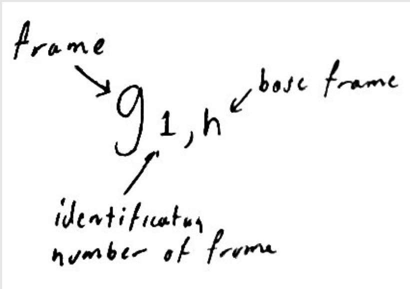
system dynamics on the actual physical positions of the component bodies. Similarly, tasks such as robotic manipulation are specified not in terms of the generalized coordinates, but rather with respect to the location and orientation of an end effector. *Forward kinematics* describe the relationship between these physical positions and the generalized coordinates.

2.2 Fixed-base Arms

A simple yet informative example of an articulated system is a single link attached to a fixed pivot, as illustrated at the left of Figure 2.2. Clearly, its configuration can be described by as a single generalized coordinate $\alpha \in \mathbb{S}^1$

A Note on Notation

When working with a kinematic chain or other collection of rigid bodies, it is convenient to use notation that concisely describes both the absolute positions of frames on the bodies and their positions relative to each other. In this book, we use the notation illustrated below:



In this notation, a frame and its position are designated by a letter with two optional subscripts. The first subscript is an identification number used to distinguish between frames that share the same letter, such as two frames at corresponding positions on different bodies. This subscript may be omitted if there is no such ambiguity (as in the single link arm at the beginning of §2.2) or to indicate a privileged frame (such as the system body frames in §2.3).

The second subscript indicates the base frame that the frame's coordinates are defined with respect to; for example, $g_{1,h}$ denotes the position of frame g_1 with respect to frame h,

$$g_{1,h} = h^{-1} g_1. \tag{2.iii}$$

If this subscript is absent, the frame position is with respect to the origin, $g_1 = g_{1,e}$; for such frames, we will only use the latter notation if a single subscript would introduce ambiguity as to its meaning.

This definition gives rise to a pair of simple rule for concatenating transforms together:

1. Frames on the left cancel with subscripted frames on the right,

$$g h_g = g g^{-1} h = h \tag{2.iv}$$

2. During this cancellation, base-frame subscripts on the left are transferred to the right:

$$g_{1,g_0} h_{g_1} = g_0^{-1}(g_1 g_1^{-1}) h = g_0^{-1} h = h_{g_0} \tag{2.v}$$

specifying the angle the link makes with respect to a reference line, as discussed in §1.1. From a broader perspective, however, we can view the link as being a rigid body at position and orientation $g \in SE(2)$ with respect to the pivot, as shown at the right of Figure 2.2. The pin joint imposes the two holonomic constraints

$$x = 0 \qquad \text{and} \qquad y = 0 \tag{2.2}$$

on the link, restricting the free variables in g to only the orientation component of $SE(2)$. As noted in 1.2, this orientation component is isomorphic to the circle,

$$g = \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ x,y=0} \equiv \overbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}^{SO(2)} \equiv \overbrace{\alpha}^{\mathbb{S}^1} \tag{2.3}$$

bringing the full rigid-body interpretation into agreement with the simple interpretation of the system.

Using the full rigid body interpretation systematizes the process of determining the positions and orientations of other frames attached to the rigid body, and from there the positions and orientations of other linked bodies.

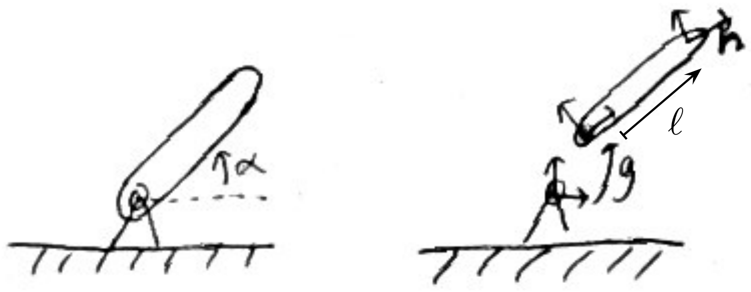


Figure 2.2 A pivoted link (left) can be viewed as a rigid body at position g with respect to the pivot and restrictions $x = 0$ and $y = 0$ on the possible values of g.

Continuing the single-link example, the world position of a second body-attached frame at the other end of the link and aligned with the first frame (*i.e.* at $h_g = (\ell, 0, 0)$ with respect to g as in Figure 2.2) is

$$h = gh_g = \begin{bmatrix} \cos\alpha & -\sin\alpha & \ell\cos\alpha \\ \sin\alpha & \cos\alpha & \ell\sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

This position can be interpreted as either the left action $L_g h_g$ transforming the frame position h_g by g, or as the right action $R_{h_g} g$ placing h_g into g. In §2.5, we use the right action interpretation to facilitate determination of systems' velocity kinematics.

Adding a second link to the system (creating the *kinematic chain* with *proximal* and *distal* members in Figure 2.3) is also a well-defined operation in $SE(2)$. Taking a second link with end frames g_2 and h_2 as in Figure 2.3, and relabeling the first link's frames g_1 and h_1, we can define the second body's base position with respect to the end of the first link as $g_{2,h_1} = (x_{2,h_1}, y_{2,h_1}, \theta_{2,h_1})$. Applying the pin constraint in (2.3) to this position attaches the new link to the first, and restricts the degrees of freedom of g_{2,h_1} to the joint angle $\theta_{2,h_1} = \alpha_2$ in the manner of (2.3). The world positions of the frames g_2 and h_2 then follow straightforwardly via $SE(2)$ actions,

$$g_2 = \overbrace{(g_1 h_{1,g_1})}^{h_1} g_{2,h_1} = \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1\cos\alpha_1 \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1\sin\alpha_1 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.5}$$

and

$$h_2 = g_2 h_{2,g_2} \tag{2.6}$$

$$= \begin{bmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) & \ell_1\cos\alpha_1 + \ell_2\cos(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) & \ell_1\sin\alpha_1 + \ell_2\sin(\alpha_1+\alpha_2) \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.7}$$

If the links were instead connected by a prismatic joint with displacement δ, as illustrated in Figure 2.4, the constraint functions in (2.2) would instead take the form

$$y = 0 \qquad \text{and} \qquad \theta = 0, \tag{2.8}$$

replacing the equivalency relationship in (2.3) with

$$g = \overbrace{\begin{bmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{SE(2),\ y,\theta=0} \equiv \overbrace{\begin{bmatrix} \delta \\ 0 \end{bmatrix}}^{\mathbb{R}^2,\ y=0} \equiv \overbrace{\delta}^{\mathbb{R}^1}. \tag{2.9}$$

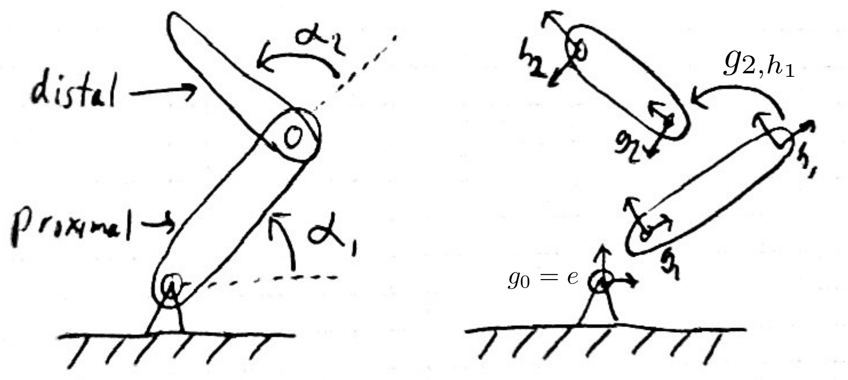
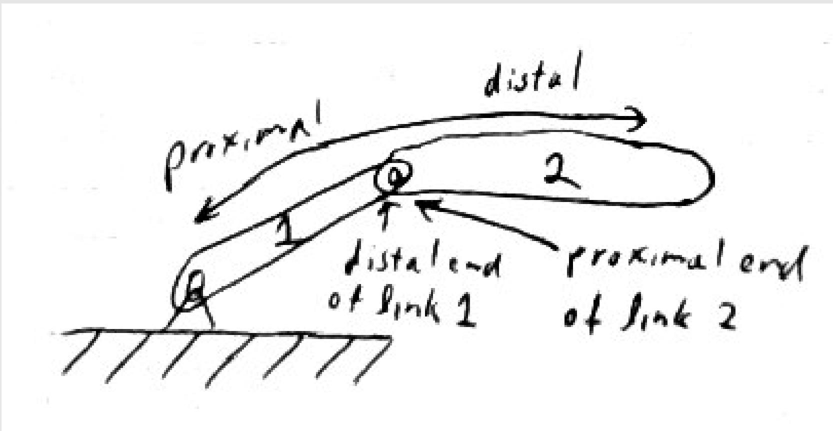


Figure 2.3 A two-link arm, with frames, transforms, and proximal/distal terminology illustrated.

Proximal, Medial and Distal

Elements of a kinematic chain are often described in terms of the local directions *proximal* (closer to the base of the chain), *medial* (at the center), and *distal* (further from the base). For individual links, these terms may describe a relationship between two links ("link 1 is proximal to link 2") or an absolute position in the chain ("link 2 is the distal link"). Other uses include identifying positions on a link ("the distal end of link 1") or more general regions along the chain ("the proximal portion of a chain" to refer to a set of links near the base).



The world positions of the second link frames would then be

$$g_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

and

$$h_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1+\delta+\ell_2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1+\delta+\ell_2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.11}$$

Chains of more than two links can naturally be constructed along the same pattern, with constrained elements of $SE(2)$ representing the relative positions of linked joints and fixed elements connecting different frames on the links. The expressions of the frame positions naturally increases in complexity with each link added, but,

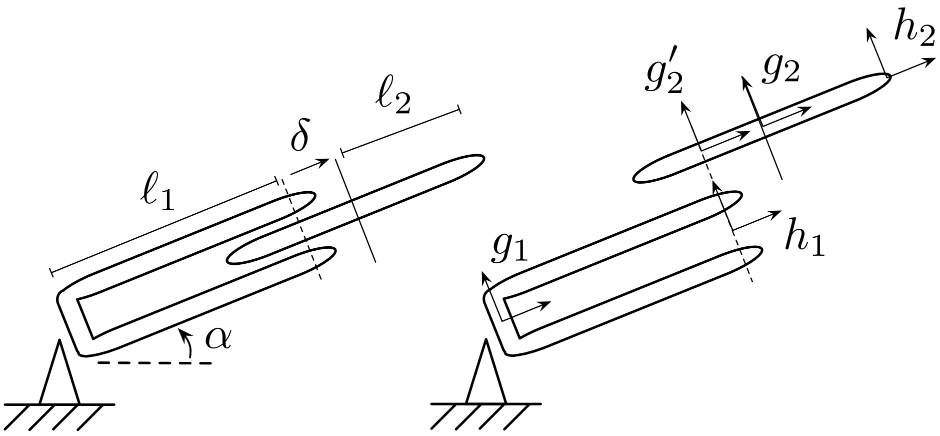


Figure 2.4 A two-link arm with one rotary and one prismatic joint. Note that g_2 is placed at the middle of the link to accommodate positive values of δ, and that we have introduced a new frame g_2' on the second body that is instantaneously aligned with h_1; this frame will be explicitly used in §2.5.2.

Frame Naming

In our examples using fixed-base systems with rotary joints, we use a naming convention of g for the proximal ends of links and h for their distal ends. This naming convention reflects the primary importance of these two frames in describing the motion of fixed-base chains. When working with mobile systems and those with prismatic joints, we alter this convention to place g at the center of the link. In the case of prismatic joints, this allows us to use a natural joint parameterization in which positive values of δ extend the joint. For mobile systems, the locomotion equations in Chapter 3 are conveniently described in terms of the motion of the center of each link, and so we directly incorporate these positions into our kinematic development and introduce f to denote frames at the proximal end of the link.

as they will almost certainly be handled by a computer, this is not a major concern; what matters is that the configuration of any given link can be easily represented as a sequence of the relative link positions leading up to it.

2.3 Mobile Articulated Systems

Forward kinematics for articulated systems that move freely in the plane combine aspects of both rigid-body and fixed-base analysis. As with a rigid bodies discussed in §1.3, the *position* of an articulated system is defined by the location and orientation of its body frame. Rather than each point on a system having a fixed location with respect to this body frame, however, the locations of these points depend on the placement of the component rigid bodies relative to the system body frame. These placements are specified by the system's *shape* variables, which correspond to the the configuration variables of fixed-base systems.

Together, the system's position and shape fully define its configuration. We denote the position as $g \in G$ (highlighting the group nature of the position space) and the shape as $r \in M$ (which, for historical reasons, does not follow the capital/lowercase convention we use elsewhere). For the overall configuration, we then have $q = (g, r)$, with a configuration space structured as $Q = G \times M$.

The simplest means of combining the shape and position terms into a kinematic structure is to choose a "base link" for the chain to define its body frame, then iteratively attach links to this first link following the procedure in §2.2. For example, the two-link system in Figure 2.5 has a configuration space $Q = SE(2) \times \mathbb{S}^1$, elements of which we can characterize as g, the position and orientation of link 1, and the shape $r = \alpha$, the angle between the links. Given these parameters, we can apply our general forward kinematics approach to find the position of

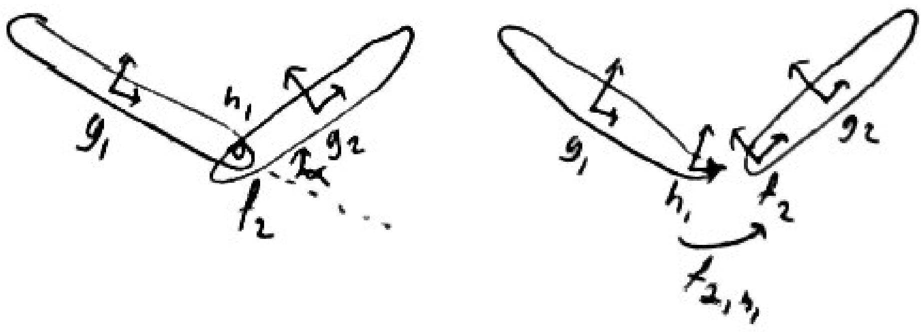


Figure 2.5 Kinematics of a mobile system with two links.

any other frame on the system, such as the center of link 2,

$$g_2 = \overbrace{g_1 \circ h_{1,g_1}}^{\text{link 1}} \circ \overbrace{f_{2,h_1} \circ g_{2,f_2}}^{\text{link 2}} \tag{2.12}$$

$$= g_1 \begin{bmatrix} \cos\alpha & -\sin\alpha & (\ell_1/2) + (\ell_2/2)\cos\alpha \\ \sin\alpha & \cos\alpha & (\ell_1/2) + (\ell_2/2)\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.13}$$

with the product of the final multiplication being too long to display here.

Subsequent links can then be added either to the distal end of link 2 or to the proximal end of link 1. As a practical matter, it is often useful (but by no means necessary) to build out evenly from a center link, rather than extending all the links in the same direction. The even approach minimizes the number of matrix multiplications required to find the link positions, simplifying their expression and improving the numerical stability of algorithms incorporating them, such as the coordinate optimizations in Chapter **??**.

During this construction process, the orientation convention for the joint angles should be chosen to meaningfully reflect the nature of the system. In the case of serial chains such as the three-link system examined in the next chapter, it is preferable to define the joint angles as based on the overall proximal-distal relationship of the links (so that positive values of α_1 correspond to positive rotations of link 2 with respect to link 1) as depicted at the left of Figure 2.6, even when using link 2 as the base for the system. First, this convention provides an unambiguous scheme for defining the joint angles that is independent of the base link. Second, it means that configurations in which the joint angles have "even" and "odd" symmetry as functions of distance along the chain correspond to configurations with even (bilateral) and odd (rotational) physical symmetries, as shown at the right of Figure 2.6. In Chapter **??**, we will re-examine this angle convention as it relates to the body's *curvature*.

2.4 Generalized Body Frames

There is a great deal of freedom in selecting the body frame for a given system. As was implicitly noted in our discussion of the choice of base links in §2.3, the position of any rigid body in the system may be used as the position of the system as a whole (and therefore any body frame rigidly attached to that link may be taken as the system's body frame). More generally, however, we may also select *any frame whose position with respect to the base link is a function of the shape variables*, as illustrated in Figure 2.7.

To see that this is the case, consider our definition of a body frame for an articulated system: *a frame in which the position of every component body (and, by extension, any point on those bodies) is a function of the shape r.* Clearly, frames on the base link meet this definition, as do frames on the other links (both by the "choose any

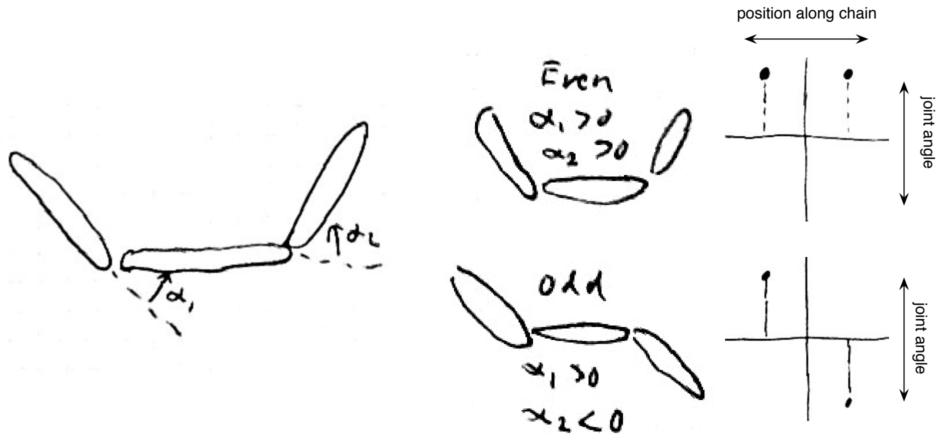


Figure 2.6 The sign convention for mobile articulated systems (left) and the correspondence between odd and even signs on the joint angles and physically odd and even shapes (right).

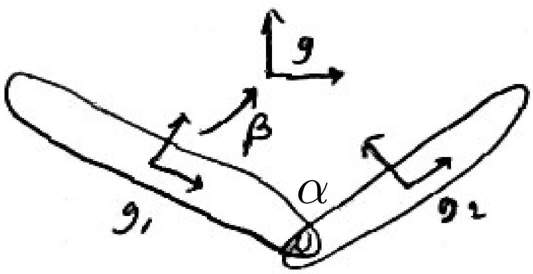


Figure 2.7 Any frame that is a shape-dependent transform away from a link of an articulated system is a valid body frame for that system.

base link" argument, and because $SE(2)$ elements are invertible). Now consider a frame g at $\beta \in SE(2)$ with respect to the base link. The positions of links 1 and 2 with respect to g are respectively

$$g_{1,g} = \beta^{-1} \tag{2.14}$$

and

$$g_{2,g} = \beta^{-1} g_{2,g_1}(\alpha). \tag{2.15}$$

If (and only if) we can express β as a function of α, then $g_{1,g}$ and $g_{2,g}$ are both functions of the system shape, meeting the necessary and sufficient conditions for g to serve as the system body frame.

Defining the valid body frames in this manner provides several tools for working with articulated systems that we make use of in the following chapters. Most importantly, our previous definition for a valid body frame describes a test for validity (for any given frame, we can evaluate whether the link positions are all functions of the shape), but does not provide a *space* of valid body frames. In contrast, the new definition identifies each valid body frame with a function $\beta(r)$, allowing us to, for example, choose an optimal β for locomotion analysis, as in Chapter **??**. Second, defining the valid frames with respect to an obviously valid frame (the base link) simplifies the process of evaluating the system kinematics with respect to a more abstract frame (such as at the center of mass). Rather than directly determining the position of each link relative to an abstract body frame, we can initially evaluate the system kinematics with respect to a base link, and then simply add a β^{-1} transform to change into the new frame.

2.5 Velocity Kinematics

Along with mapping the configuration of a system to its physical position, forward kinematics also relates the system's physical and configuration velocities. At each frame on the system, with position and orientation $g(q)$, the latter relationship is given by the *Jacobian* of the forward kinematic map,

$$\dot{g}(q,\dot{q}) = J_g\dot{q} = \frac{\partial g}{\partial q}\dot{q}. \tag{2.16}$$

Several methods for finding this Jacobian are available. In this section, we examine two: direct differentiation of the forward kinematic map and assembly of the Jacobian from individual joint kinematics.

2.5.1 Jacobians from Differentiation

The most straightforward option for finding this Jacobian is to explicitly evaluate the differential in (2.16). For example, the distal frame on the single-link arm in Figure 2.2 has a position $h = (\ell\cos\alpha, \ell\sin\alpha, \alpha)$. We can calculate its velocity by differentiating this relationship with respect to the configuration variable α,

$$\dot{h} = J_h\dot{\alpha} = \frac{\partial h}{\partial \alpha}\dot{\alpha} = \begin{bmatrix} -\ell\sin\alpha \\ \ell\cos\alpha \\ 1 \end{bmatrix}\dot{\alpha}. \tag{2.17}$$

Systems with more than one configuration variable have a corresponding number of columns in their Jacobians. The distal frame on the two-link chain in Figure 2.3,

$$h_2(q) \quad = \quad \Big((\ell_1\cos\alpha_1 \; + \; \ell_2\cos(\alpha_1+\alpha_2)), (\ell_1\sin\alpha_1 \; + \; \ell_2\sin(\alpha_1+\alpha_2)), (\alpha_1 \; + \; \alpha_2)\Big), \tag{2.18}$$

has a two-column Jacobian,

$$\dot{h}_2 = \begin{bmatrix} -(\ell_1\sin\alpha_1 + \ell_2\sin(\alpha_1+\alpha_2)) & -\ell_2\sin(\alpha_1+\alpha_2) \\ (\ell_1\cos\alpha_1 + \ell_2\cos(\alpha_1+\alpha_2)) & \ell_2\cos(\alpha_1+\alpha_2) \\ 1 & 1 \end{bmatrix}\begin{bmatrix}\dot{\alpha_1} \\ \dot{\alpha_2}\end{bmatrix}, \tag{2.19}$$

while the mobile three-link system from Figure 2.7 has five columns in its Jacobian, three corresponding to the velocity of the body frame and two to the shape velocity:

$$\dot{g}_2 = \begin{bmatrix} \frac{\partial g_2}{\partial g} & \frac{\partial g_2}{\partial r} \end{bmatrix}(\dot{g},\dot{r}). \tag{2.20}$$

2.5.2 Iterative Jacobian Assembly

An alternative approach to generating the Jacobian of an articulated system is to build it progressively along with the forward kinematic map. Methods based on this approach take advantage of the iterated $SE(2)$ structure of the system to "pre-differentiate" the relative motions of the links at the joints before placing them into the system structure.

Two basic principles underly this pre-differentiation. First, as we saw in §1.5.2, the velocities of any two frames on a rigid body are linked by the right lifted action. Second, at a joint the velocities of the interacting links are matched, modulo the relative motion allowed by the joint constraints.

Taking the two-link system in Figure 2.3 as an example, the fixed pivot point (at $g_0 = e$) is grounded, and so has velocity

$$\dot{g}_0 = (0,0,0). \tag{2.21}$$

The proximal frame of link 1 has equal translational velocity to the pivot and a relative velocity of $\dot{\alpha}_1$, and so is

$$\dot{g}_1 = \dot{g}_0 + (0,0,\dot{\alpha}_1) = (0,0,\dot{\alpha}_1) \tag{2.22}$$

At the distal end of the first link, h_1 has the same spatial velocity as g_1,

$$T_{h_1}R_{h_1^{-1}}\dot{h}_1 = T_{g_1}R_{g_1^{-1}}\dot{g}_1, \tag{2.23}$$

or, taking advantage of the equality in (1.39),

$$\dot{h}_1 = \overbrace{(T_e R_{h_1})(T_{g_1} R_{g_1^{-1}})}^{T_{g_1} R_{h_{1,g_1}}} \dot{g}_1. \tag{2.24}$$

Combining this relationship with $\dot{g}_1$ from (2.22) and the kinematic map for h_1 in (2.4) gives the velocity of the distal frame and its Jacobian with respect to the joint angle,

$$\dot{h}_1 = \overbrace{\begin{bmatrix} 1 & 0 & -\ell_1 \sin\alpha_1 \\ 0 & 1 & \ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_e R_{h_1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_1} R_{g_1^{-1}}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_1 \end{bmatrix}}^{\dot{g}_1} = \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 \\ \ell_1 \cos\alpha_1 \\ 1 \end{bmatrix}}^{J_{h_1}} \dot{\alpha}_1, \tag{2.25}$$

which matches the results of our direct differentiation in (2.17).

For the second link, the relative velocity of the proximal end of link 2 and the distal end of link 1 is given by the second joint's rotation, making its net velocity

$$\dot{g}_2 = \dot{h}_1 + (0, 0, \dot{\alpha}_2) \tag{2.26}$$

$$= ((-\ell_1 \sin\alpha_1)\dot{\alpha}_1, (\ell_1 \cos\alpha_1)\dot{\alpha}_1, (\dot{\alpha}_1 + \dot{\alpha}_2)), \tag{2.27}$$

which can be written as a Jacobian product as

$$\dot{g}_2 = \begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \tag{2.28}$$

The velocity of the distal end of link 2 is then

$$\dot{h}_2 = \overbrace{\begin{bmatrix} 1 & 0 & -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) \\ 0 & 1 & (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) \\ 0 & 0 & 1 \end{bmatrix}}^{T_e R_{h_2}} \overbrace{\begin{bmatrix} 1 & 0 & \ell_1 \sin\alpha_1 \\ 0 & 1 & -\ell_1 \cos\alpha_1 \\ 0 & 0 & 1 \end{bmatrix}}^{T_{g_2} R_{g_2^{-1}}} \overbrace{\begin{bmatrix} -\ell_1 \sin\alpha_1 & 0 \\ \ell_1 \cos\alpha_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}}^{\dot{g}_2}, \tag{2.29}$$

which, when the multiplications are carried out, resolves to

$$\dot{h}_2 = \overbrace{\begin{bmatrix} -(\ell_1 \sin\alpha_1 + \ell_2 \sin(\alpha_1 + \alpha_2)) & -\ell_2 \sin(\alpha_1 + \alpha_2) \\ (\ell_1 \cos\alpha_1 + \ell_2 \cos(\alpha_1 + \alpha_2)) & \ell_2 \cos(\alpha_1 + \alpha_2) \\ 1 & 1 \end{bmatrix}}^{J_{h_2}} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}, \tag{2.30}$$

once again equaling the results of direct differentiation (this time in (2.19)).

The overall pattern in this example is that the distal velocity of each link in a chain is defined recursively as

$$\dot{h}_i = (T_e R_{h_i})(T_{g_i} R_{g_i^{-1}}) \overbrace{(\dot{h}_{i-1} + v_i)}^{\dot{g}_i}, \tag{2.31}$$

where v_i is the velocity of body i with respect to body $i - 1$ at joint i.

2.5.3 Body Velocity Formulation of Iterative Jacobian

Depending on how the velocities and relative positions of the links are represented, it may be preferable to modify (2.31) to better accommodate this representation. For instance, in the locomotion problems we examine in Chapter 3, it is more useful to work with the link body velocities than their absolute velocities. A related concern appears in the case of prismatic joints (and, outside the scope of the present material, in the case of joints in three-dimensional systems) where the relative motion at the joint is more easily defined in the frame of the attached link than with respect to the world.

In this case, we can first insert two balanced sets of left lifted actions into (2.31) and regroup the terms into body velocities (employing (1.32)) and *adjoint actions* (as in (1.43)) to produce a map between the body velocities of any two frames on a rigid body:

$$\underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}\dot{h}_i}_{\xi_{h_i}} = \underbrace{\overbrace{(T_{h_i}L_{h_i^{-1}})}^{\text{new}}(T_eR_{h_i})}_{Ad_{h_i}^{-1}}\underbrace{(T_{g_i}R_{g_i^{-1}})\,\overbrace{(T_eL_{g_i})}^{\text{Identity}}}_{Ad_{g_i}}\underbrace{\overbrace{(T_{g_i}L_{g_i^{-1}})}\,\dot{g}_i}_{\xi_{g_i}}. \tag{2.32, 2.33}$$

At the joints, the relative velocity between the links is most easily considered by recognizing that there are in fact three frames involved: h_{i-1}, the distal frame on link $i-1$; g_i, the proximal frame on link i; and g_i', the frame on link i that is instantaneously aligned with h_{i-1}. For a joint with relative motion defined as $v_i = (\dot{\delta}_{x,i}, \dot{\delta}_{y,i}, \dot{\alpha}_i)$ relative to the proximal link, the body velocity of frame g_i' is

$$\xi_{g_i'} = \xi_{h_{i-1}} + v_i. \tag{2.34}$$

As frames g_i' and g_i are attached to the same rigid body, we can convert between their body velocities using the same approach as in (2.33),

$$\xi_{g_i} = Ad_{g_i}^{-1} Ad_{g_i'} \xi_{g_i'}. \tag{2.35}$$

Note that when the x and y components of g_i' and g_i are equal, this conversion reduces to rotation by $-\alpha_i$, as discussed in the box on page 39.

Together, (2.34), (2.33), and (2.35) provide a formulation equation for the body velocities of the distal ends of each link,

$$\xi_{h_i} = (Ad_{h_i}^{-1})(Ad_{g_i'})(\xi_{h_{i-1}} + v_i), \tag{2.36}$$

from which the adjoint terms relating to the frame g have been dropped, as they do not directly affect the calculation of ξ_{h_i}.

If the relative positions of frames on a link are more convenient to use their absolute positions, a further reduction is possible, by evaluating (2.36) with the origin temporarily placed at g_i'. This change of coordinates transforms the position of each link frame by $(g_i')^{-1}$, so that g_i' and h_i respectively become e and $h_{i,g_i'}$. As the adjoint action at the origin is an identity matrix, the body velocity of the distal frame simplifies to the inverse adjoint action of that frame relative to g_i',

$$\xi_{h_i} = (Ad_{h_{i,g_i'}}^{-1})(\xi_{h_{i-1}} + v_i). \tag{2.37}$$

2.5.4 The Rotary-Prsimatic Arm

The rotary-prismatic arm in Figure 2.4 provides an intuitive example of using the body-frame approach to iteratively calculating the Jacobian. As in the previous arm example, we have a fixed pivot at g_0, which thus has zero body velocity,

$$\xi_{g_0} = (0, 0, 0). \tag{2.38}$$

The body velocity of frame g_1', attached to the first link, but aligned with g_0 is then found by adding in its velocity relative to g_0,

$$\xi_{g_1'} = \xi_{g_0} + \overbrace{(0, 0, \dot{\alpha})}^{v_1} = (0, 0, \dot{\alpha}), \tag{2.39}$$

and can be converted into the body velocity of frame g_1 by applying the relative inverse adjoint action from (2.37),

$$\xi_{g_1} = Ad_{g_{1,g_1'}}^{-1}\xi_{g_1'} = \overbrace{\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad_{g_{1,g_1'}=(0,0,\alpha)}^{-1}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_1'}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{J_{g_1}^b} \dot{\alpha}, \tag{2.40}$$

Change of Basis

In treatments of rigid body motion, it is typical to encounter "change of basis" operations that move vectors between coordinate representations. On a casual level, these changes of basis are commonly identified with simple rotations of the coordinates. For example, consider a pair of frames g and h at the same location but with orientations differing by α, as in. A vector v at this location may be represented as v_g with respect to frame g, and as v_h with respect to frame h. The relationship between these to vectors would then be described by a rotation matrix of $-\alpha$,

$$v_h = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} v_g. \tag{2.vi}$$

While useful, this rotation is not well-defined in terms of $SE(2)$ operations. For instance, if we consider v as the velocity of frames g and h, it quickly becomes apparent that v in fact represents two separate vectors $\dot{g} \in T_g G$ and $\dot{h} \in T_h G$ in separate tangent spaces, and that a rigorous definition of change-of-basis operations must take this structure into account.

There are in fact at least two (mutually compatible) interpretations of $SE(2)$ that lead to the change of basis formulation in (2.vi). The first we have already seen in §1.5.1, where transforming the body to the origin was equivalent to bringing the coordinate frame to the body, and the associated lifted action corresponded to the change of basis. More generally, the change of basis from frame g to frame h is given by

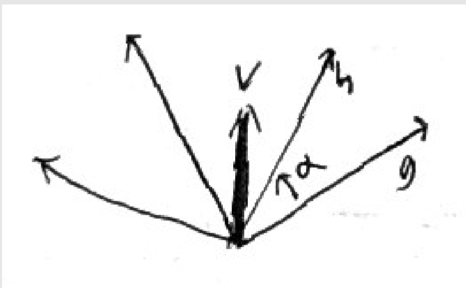
$$v_h = (T_g L_{h_g^{-1}}) v_g, \tag{2.vii}$$

which evaluates to the rotation matrix in (2.vi).

The second source for the change of basis equation is based on the observation that as locked frames, g and h have a common spatial velocity, and thus

$$v_h = Ad_h^{-1} Ad_g v_g = Ad_{h_g}^{-1} v_g. \tag{2.viii}$$

When the x and y values in g and h are the same, they nullify each other in the adjoint product, leaving only the rotation component.



where $J_{g_1}^b$ is the Jacobian for the proximal end of the link, expressed in its own body frame. Moving to the end of the first link, the distal frame's body velocity is found in the same manner,

$$\xi_{h_1} = Ad_{h_{1,g_1}}^{-1} \xi_{g_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_1 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad_{h_{1,g_1}=(\ell_1,0,0)}^{-1}} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}}^{\xi_{g_1}} = \begin{bmatrix} 0 \\ \ell_1\dot{\alpha} \\ \dot{\alpha} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 \\ \ell_1 \\ 1 \end{bmatrix}}^{J_{h_1}^b} \dot{\alpha}, \tag{2.41}$$

confirming our intuition that the tip of a rigid link should move laterally when pivoted around its base.

Up to this point in the example, working with the frames' body velocities has allowed us to build the system

Jacobian using only their relative positions, without incorporating the full forward kinematics of the system. This locality becomes even more useful when we apply it to prismatic joints, for which the relative motion of frames g_i' and h_{i-1} is typically represented in the local frame. While we can of course convert this velocity into a global frame and insert it into (2.31) when calculating the Jacobian, it is significantly more convenient to use it directly in (2.37). Continuing with our rotary-prismatic arm example, the relative velocity at the prismatic joint is

$$v_2 = (\dot{\delta}, 0, 0), \tag{2.42}$$

giving the velocity of the distal link at the joint as

$$\xi_{g_2'} = \xi_{h_1} + v_2 = (\dot{\delta}, \ell_1\dot{\alpha}, \dot{\alpha}). \tag{2.43}$$

The body-frame Jacobians for the midpoint and distal end of the second link then follow naturally,

$$\xi_{g_2} = Ad^{-1}_{g_{2,g_2'}}\xi_{g_2'} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \delta\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{g_{2,g_2'}=(\delta,0,0)}} \overbrace{\begin{bmatrix}\dot{\delta}\\ \ell_1\dot{\alpha}\\ \dot{\alpha}\end{bmatrix}}^{\xi_{g_2'}} = \begin{bmatrix}\dot{\delta}\\ (\ell_1+\delta)\dot{\alpha}\\ \dot{\alpha}\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1\\ \ell_1+\delta & 0\\ 1 & 0\end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix}\dot{\alpha}\\ \dot{\delta}\end{bmatrix}, \tag{2.44}$$

and

$$\xi_{h_2} = Ad^{-1}_{h_{2,g_2'}}\xi_{g_2'} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \delta+\ell_2\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{h_{2,g_2'}=(\delta+\ell_2,0,0)}} \overbrace{\begin{bmatrix}\dot{\delta}\\ \ell_1\dot{\alpha}\\ \dot{\alpha}\end{bmatrix}}^{\xi_{g_2'}} = \begin{bmatrix}\dot{\delta}\\ (\ell_1+\delta+\ell_2)\dot{\alpha}\\ \dot{\alpha}\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1\\ \ell_1+\delta+\ell_2 & 0\\ 1 & 0\end{bmatrix}}^{J^b_{h_2}} \begin{bmatrix}\dot{\alpha}\\ \dot{\delta}\end{bmatrix}. \tag{2.45}$$

2.5.5 Three-link Systems

The local approach to calculating the Jacobian further shows its worth in the analysis of mobile articulated systems. A common operation when working with such systems is to find the body velocity of each link as a function of the systems overall body velocity ξ and its shape velocity $\dot{r}$; the resulting Jacobians underly, for instance, the locomotive relationships explored in Chapter 3.

For a three-link system such as that shown in Figure 2.8, with link lengths ℓ_i and proximal, medial, and distal frames f_i, g_i. and h_i, we can start building the Jacobians by selecting the middle link to define the system's body frame.[2] This choice gives the middle link a trivial Jacobian,

$$\xi_{g_2} = \xi = \begin{bmatrix}I^{3\times3} & \mathbf{0}^{3\times2}\end{bmatrix} \begin{bmatrix}\xi\\ \dot{r}\end{bmatrix}, \tag{2.46}$$

and facilitates calculation of the velocity of the other links. To calculate the body velocity of link 1, we first find the body velocity of the proximal end of link 2,

$$\xi_{f_2} = Ad^{-1}_{f_{2,g_2}}\xi_{g_2} = \overbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\ell_2/2\\ 0 & 0 & 1\end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(-\ell_2/2,0,0)}} \overbrace{\begin{bmatrix}\xi^x\\ \xi^y\\ \xi^\theta\end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix}\xi^x\\ \xi^y - (\xi^\theta\ell_2)/2\\ \xi^\theta\end{bmatrix}. \tag{2.47}$$

and use it to calculate the body velocity of the distal end of link 1, which has rotational velocity of $-\dot{\alpha}_1$ with

[2] In §**??**, we consider the kinematics under different choices of body frame, with the computation of these kinematics for a center-of-mass frame provided as an exercise for the reader. Here, using the middle link simplifies our initial computations, without introducing any loss of generality.

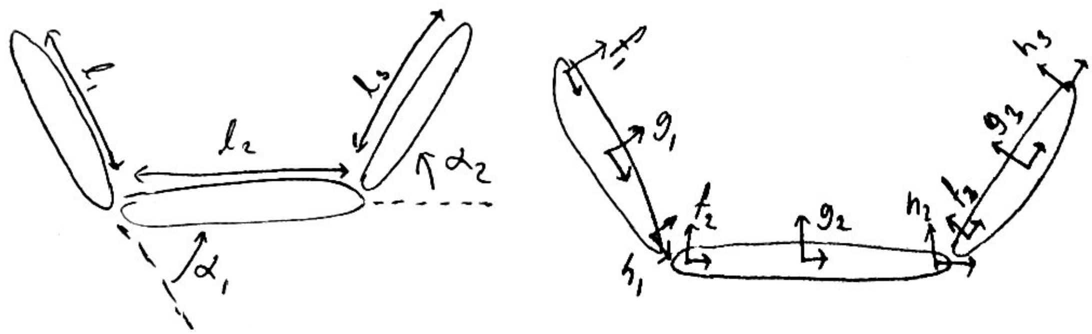


Figure 2.8 Parameterization and frame placement for a generic three-link mobile system.

respect to f_2,

$$\xi_{h_1} = Ad^{-1}_{h_{1,h_1'}}\xi_{h_1'} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{h_{1,h_1'}=(0,0,-\alpha_1)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y - (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{f_2}} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\alpha}_1 \end{bmatrix} \right)}^{\xi_{h_1'}} \tag{2.48}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}. \tag{2.49}$$

Finally, we arrive at the body velocity of g_1 by moving along the link by $\ell_1/2$,

$$\xi_{g_1} = Ad^{-1}_{g_{1,h_1}}\xi_{h_1} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\ell_1/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_{1,h_1}=(0,-\ell_1/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y - (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}}^{\xi_{h_1}} \tag{2.50}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot{\alpha}_1) \\ \xi^\theta - \dot{\alpha}_1 \end{bmatrix}. \tag{2.51}$$

Similarly, we calculate the body velocity of link 3 by first taking the body velocity of the distal end of link 2,

$$\xi_{h_2} = Ad^{-1}_{h_{2,g_2}}\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_2/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_{2,g_2}=(\ell_2/2,0,0)}} \overbrace{\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix}}^{\xi_{g_2}} = \begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}, \tag{2.52}$$

converting it into the proximal velocity of link 3,

$$\xi_{f_3} = Ad^{-1}_{f_3,f_3'}\xi_{f_3'} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{f_3,f_3'=(0,0,\alpha_2)}} \overbrace{\left(\underbrace{\begin{bmatrix} \xi^x \\ \xi^y + (\xi^\theta \ell_2)/2 \\ \xi^\theta \end{bmatrix}}_{\xi_{h_2}} + \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha}_2 \end{bmatrix} \right)}^{\xi_{f_3'}} \tag{2.53}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}, \tag{2.54}$$

and moving to the center of the link,

$$\xi_{g_3} = Ad^{-1}_{g_3,f_3}\xi_{f_3} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \ell_3/2 \\ 0 & 0 & 1 \end{bmatrix}}^{Ad^{-1}_{g_3,f_3=(0,\ell_3/2,0)}} \overbrace{\begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}}^{\xi_{f_3}} \tag{2.55}$$

$$= \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot{\alpha}_2) \\ \xi^\theta + \dot{\alpha}_2 \end{bmatrix}. \tag{2.56}$$

Expressed as Jacobian products, the body velocities of the links are

$$\xi_{g_1} = \overbrace{\begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & (\ell_2 \sin\alpha_1)/2 & 0 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & -(\ell_2\cos\alpha_1 + \ell_1)/2 & \ell_1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}}^{J^b_{g_1}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}, \tag{2.57}$$

$$\xi_{g_2} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}^{J^b_{g_2}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \tag{2.58}$$

$$\xi_{g_3} = \overbrace{\begin{bmatrix} \cos\alpha_2 & \sin\alpha_2 & (\ell_2 \sin\alpha_2)/2 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}^{J^b_{g_3}} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \tag{2.59}$$

2.5.6 Jacobians and Alternative Body Frames

What happens to the three-link system's Jacobians when we use a frame that is kinematically linked to the system, such as the alternative body frame options described in §2.4?

This is really two questions:

1. If we keep the middle link as the system body frame, what is the body velocity of frame g in Figure 2.9?
2. If we take frame g as the system body frame, what are the body velocities of the links?

Answers:

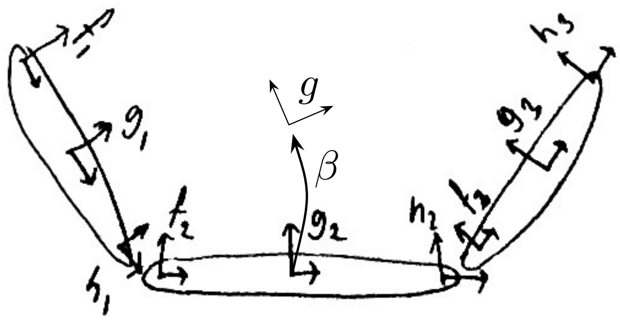


Figure 2.9 Three-link mobile system with a frame g a transformation β away from the middle link.

1. Taking our body-frame iterative Jacobian formula from (2.37), we get

$$\xi_g = Ad_{\beta}^{-1}(\xi_{g_2} + v_{\beta}). \tag{2.60}$$

This leaves the question, however, of calculating v_{β}. We don't just want $\dot{\beta} = (\partial\beta/\partial\alpha)\ \dot{\alpha}$, we want *the velocity with respect to g_2 of the frame rigidly attached to g and coincident with g_2*. Taking advantage of properties of the spatial velocity, we can use a right action to find this velocity,

$$v_{\beta} = T_{\beta}R_{\beta^{-1}}\dot{\beta} = T_{\beta}R_{\beta^{-1}}\tfrac{\partial\beta}{\partial\alpha}\dot{\alpha}, \tag{2.61}$$

and thus the body velocity of g,

$$\xi_g = Ad_{\beta}^{-1}(\xi_{g_2} + T_{\beta}R_{\beta^{-1}}\tfrac{\partial\beta}{\partial\alpha}\dot{\alpha}). \tag{2.62}$$

The Jacobian for this frame can then be found by expanding this expression to

$$\xi_g = \begin{bmatrix} \cos\beta^{\theta} & \sin\beta^{\theta} & \beta^x\sin\beta^{\theta} - \beta^y\cos\beta^{\theta} \\ -\sin\beta^{\theta} & \cos\beta^{\theta} & \beta^x\cos\beta^{\theta} + \beta^y\sin\beta^{\theta} \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^{\theta} \end{bmatrix} + \begin{bmatrix} 1 & 0 & \beta^y \\ 0 & 1 & -\beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial\beta^x}{\partial\alpha_1} & \frac{\partial\beta^x}{\partial\alpha_2} \\ \frac{\partial\beta^y}{\partial\alpha_1} & \frac{\partial\beta^y}{\partial\alpha_2} \\ \frac{\partial\beta^{\theta}}{\partial\alpha_1} & \frac{\partial\beta^{\theta}}{\partial\alpha_2} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \right), \tag{2.63}$$

evaluating the matrix multiplications, and collecting terms.

Alternatively, we can reform (2.62) to simplify the matrix algebra by separating the adjoint action into its components,

$$\xi_g = \overbrace{(T_{\beta}L_{\beta^{-1}})(T_eR_{\beta})}^{Ad_{\beta}^{-1}}(\xi_{g_2} + T_{\beta}R_{\beta^{-1}}\tfrac{\partial\beta}{\partial\alpha}\dot{\alpha}). \tag{2.64}$$

and distributing the right lifted action over the velocity terms,

$$\xi_g = T_{\beta}L_{\beta^{-1}}(T_eR_{\beta}\xi_{g_2} + \overbrace{(T_eR_{\beta})(T_{\beta}R_{\beta^{-1}})}^{I}\tfrac{\partial\beta}{\partial\alpha}\dot{\alpha}). \tag{2.65}$$

This form of the equation expands as

$$\xi_g = \begin{bmatrix} \cos\beta^{\theta} & \sin\beta^{\theta} & 0 \\ -\sin\beta^{\theta} & \cos\beta^{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & -\beta^y \\ 0 & 1 & \beta^x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{g_2}^x \\ \xi_{g_2}^y \\ \xi_{g_2}^{\theta} \end{bmatrix} + \begin{bmatrix} \frac{\partial\beta^x}{\partial\alpha_1} & \frac{\partial\beta^x}{\partial\alpha_2} \\ \frac{\partial\beta^y}{\partial\alpha_1} & \frac{\partial\beta^y}{\partial\alpha_2} \\ \frac{\partial\beta^{\theta}}{\partial\alpha_1} & \frac{\partial\beta^{\theta}}{\partial\alpha_2} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \right), \tag{2.66}$$

which, in addition to being more compact than (2.63) also brings us back to how we might have informally described the motion of g if we did not have the benefit of the more general adjoint formulation: ξ_g is a combination of the body velocity of the reference link (with the right lifted action providing the "cross product" term for motion away from the center of that link's rotation), the relative velocity of g with respect

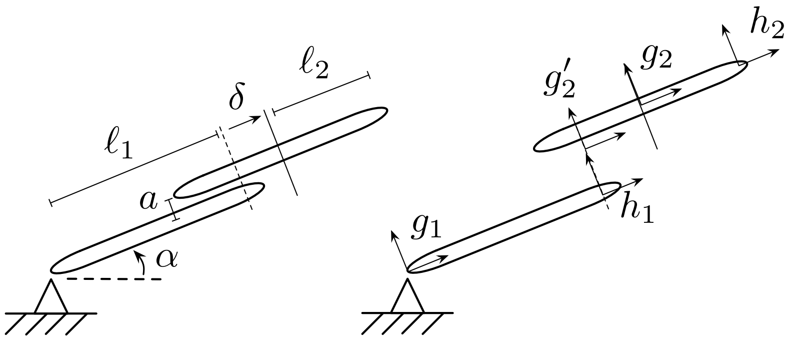


Figure 2.10 Rotary-prismatic arm with links laterally offset by a.

to the reference link, and a coordinate rotation (provided by the left lifted action) to bring the vector's components into the new body frame,

$$\xi_g = \overbrace{T_\beta L_{\beta^{-1}}}^{\text{coordinate transformation}}(\underbrace{T_e R_\beta \xi_{g_2}}_{\text{motion of original frame}} + \overbrace{\tfrac{\partial \beta}{\partial \alpha}\dot{\alpha}}^{\text{relative motion}}). \tag{2.67}$$

2. Building the link Jacobians based on the body velocity of g is accomplished by inverting (2.60) to provide ξ_{g_2} as a function of ξ_g,

$$\xi_{g_2} = Ad_\beta \xi_g - v_\beta, \tag{2.68}$$

then following an expansion similar to that presented above. The new expressions for ξ_{g_2} can then be inserted into the iterative development of the Jacobian as the velocity of the second link.

Exercises

2.1 Find a set of holonomic constraints that produce the same accessible manifold as that produced by the constraints in (2.i) and (2.ii), but for which the first constraint restricts the system to points that are a fixed distance from $(0, 0, 0) \in \mathbb{R}^3$.

2.2 Find the Jacobian for the end-effector of the offset rotary-prismatic arm shown in Figure 2.10, using

a. the absolute-position iterative Jacobian method
b. the relative-position iterative Jacobian method

2.3 Three-link systems:

a. For the three-link system shown in Figure 2.9, calculate the transform

$$\beta(\alpha) = g_{g_2} \tag{2.69}$$

that makes g the body frame defined by the mean center of mass position of the three links (assume uniform mass distribution along their lengths) and their mean orientations.

b. Calculate the (body-frame) Jacobian of this center-of-mass frame, taking the middle link as the system body frame.

c. Calculate the (body-frame) Jacobians for the three links, taking the center-of-mass frame as the system body frame.

2.4 Show (in coordinates) that $Ad_h^{-1} Ad_g$ where the x and y values of g and h are equal produces the rotational change of basis operation in $SE(2)$

3

Kinematic Locomotion

A fundamental difference between the motions of fixed-base and mobile articulated systems lies in the degree to which they are actuated. As a general principle, motors or muscles can be attached to an articulated system at the joints, and can then be controlled to specify the relative positions of the links. For a fixed-base system, in which mechanical ground acts as a link, controlling the joint angles completely specifies the configuration of the system, making it *fully actuated*. Mobile systems, in contrast, are almost exclusively *underactuated*—controlling their joint angles specifies their shapes, but does not directly influence their positions.[1]

Despite this underactuation, many mobile articulated systems *can* propel themselves through their environments, by exploiting physical constraints on their motions. As a system changes shape, it may be subject to restrictions on the velocities of points on the body, momentum conservation laws, or other effects, such as fluid or frictional drag. These interactions produce reaction forces on the system that ultimately dictate the motion of its body frame. The process of using these reaction forces to turn internal shape changes into external position changes is *locomotion*.

3.1 Kinematic locomotion

As a general rule, any configuration trajectory executed by a locomoting system over a time interval $[0, T]$ can be decomposed into a *shape trajectory* ψ in which each shape variable is defined as a function of time,

$$\begin{aligned} \psi : [0, T] &\to M \\ t &\mapsto r, \end{aligned} \tag{3.1}$$

and an associated *induced a position trajectory* g^ψ

$$\begin{aligned} g^\psi : [0, T] &\to G \\ t &\mapsto g, \end{aligned} \tag{3.2}$$

defined by the interaction between the shape change, the system constraints, and the initial state of the system.

Kinematic locomoting systems are those for which changing the *pacing*, or time parameterization, of a shape trajectory changes the pacing of its induced position trajectory in the same fashion, without affecting the path it traces out: given two shape changes ψ and ψ' defined such that

$$\psi(t) = \psi'(\tau(t)), \tag{3.3}$$

i.e. such that ψ' is a smooth reparameterization of ψ with respect to time, the induced position trajectories of a kinematic system have the same relationship,

$$g^\psi(t) = g^{\psi'}(\tau(t)), \tag{3.4}$$

so that $g^{\psi'}$ can be found equivalently by repacing g^ψ or by evaluating the system constraints during the execution of ψ', as indicated in the commutative diagram in Figure 3.1. This geometric reparameterizability then

[1] An articulated system may also be underactuated in the sense that one or more joints are uncontrolled, but we do not consider this case here.

$$
\begin{tikzcd}
\psi(t) \arrow[r, "\text{apply constraints}"] \arrow[d, "\tau = f(t)"', "\text{replace timing}"] & g^{\psi}(t) \arrow[d, "\tau = f(t)"', "\text{replace timing}"] \\
\psi'(\tau) \arrow[r, "\text{apply constraints}"] & g^{\psi'}(\tau)
\end{tikzcd}
$$

Figure 3.1 Commutative diagram showing relationship between shape changes, induced position changes, and time reparameterizations that defines a kinematic system.

means that the displacement induced by a given shape trajectory is entirely a function of the geometric path it follows, and not the rate with it is executed.

Kinematic systems both provide useful mathematical examples, embodying many fundamental concepts in geometric mechanics and differential geometry, and are important systems in their own rights, appearing independently across several areas of applied mechanics research. Here and in the following chapters, we focus our attention on a particularly interesting class of such systems, *symmetric linear-kinematic locomotors*, over a range of physical domains.

At a differential level, kinematic locomotion corresponds to a configuration-dependent *directionally linear*[2] relationship between a system's shape velocity and induced position velocity,

$$\dot{g} = f(q, \hat{\dot{r}})\|\dot{r}\|, \tag{3.5}$$

with the effect that scaling the shape velocity proportionally scales the position velocity, without altering its direction. In a *linear-kinematic* locomoting system, the mapping between the shape velocity and position velocity is fully linear at each configuration,

$$\dot{g} = f(q)\dot{r}. \tag{3.6}$$

Linear-kinematic effects feature in the locomotion of many engineered and biological systems, to the extent that the existence of the broader class of general kinematic systems is seldom mentioned, and the "linear" qualifier is consequently dropped in much of the literature. Following this section, we will adopt this convention, except where linearity or the lack thereof needs to be clarified or emphasized.

In a uniform environment, the dynamics of many locomoting systems are *symmetric*[3] with respect to the position and orientation of the body frame: if the equations of motion are expressed in the body frame, then they are independent of the position variables. For a symmetric system that is also linear-kinematic, the body velocity can thus be expressed as a shape-dependent linear function of the shape velocity,

$$\xi = f(r)\dot{r}. \tag{3.7}$$

Exploiting the symmetric nature of such systems offers several analytical advantages. First, removing the position variables from the equations of motion both reduces their dimensionality and provides an intuitive description of how changes in the system shape move the system "forward" or "sideways." Second the dual interpretation of the body velocity as a vector in the tangent space at the origin (§1.5.1) allows us to combine (3.7) with powerful differential geometric tools that operate on T_eG, such as the Lie bracket in Chapter **??**.

Two familiar, concrete examples of symmetric linear-kinematic locomotors are the differential drive and Ackerman-steered cars. As depicted in Figure 3.2, both of these systems move in the plane, with positions $g \in SE(2)$ defining body frames aligned with the vehicle chassis and located at the axle. In these frames, the

[2] See the box on the facing page.
[3] See the box on page 48.

Directional Linearity

A function

$$
\begin{aligned}
f : A &\to B \\
a &\mapsto b,
\end{aligned}
\tag{3.i}
$$

where A is a vector space, is *directionally linear* if each element of b is a linear function of a along each line passing through the origin, *i.e.* it can be written in the form

$$b_i = f_i(\hat{a})\|a\|, \tag{3.ii}$$

where $\hat{a}$ is the unit vector $a/\|a\|$, and satisfies the condition

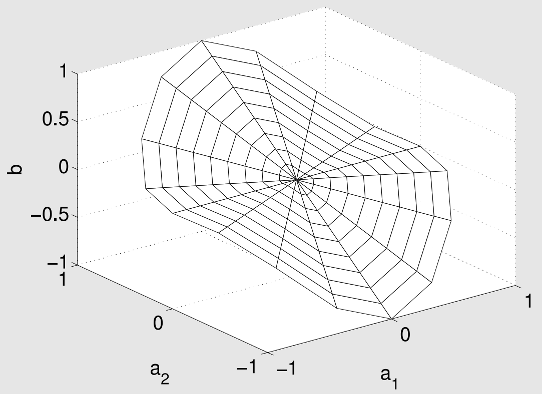
$$b(-a) = -b(a). \tag{3.iii}$$

The four functions shown below illustrate characteristics of functions that satisfy this definition, and how they relate to more commonly encountered fully-*linear* functions. The directionally linear function in (a) is linear along each azimuth (characterized by its unit vector $\hat{a}$), and the slope is maintained across the origin, satisfying $b(-a) = -b(a)$. Linear functions, as shown in (b), are a special case of directionally linear functions in which the slopes along the azimuths are linearly related, and (3.ii) takes the form

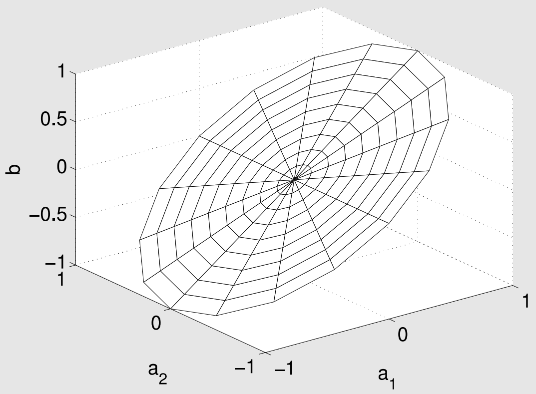
$$b = \overbrace{\mathcal{M}\hat{a}}^{f(\hat{a})}\|a\| = \mathcal{M}a, \tag{3.iv}$$

where $\mathcal{M}$ is a matrix that multiplies a to produce b.

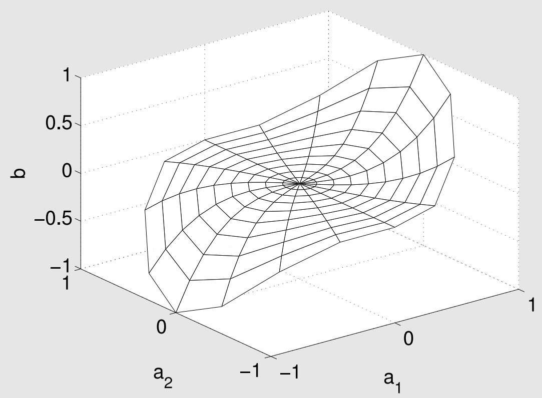
The functions in (c) and (d) are *not* directionally linear. The first function's magnitude increases nonlinearly along the azimuths, violating the condition in (3.ii). The second function is linear along the azimuths, but does not meet the second linearity condition in (3.iii).



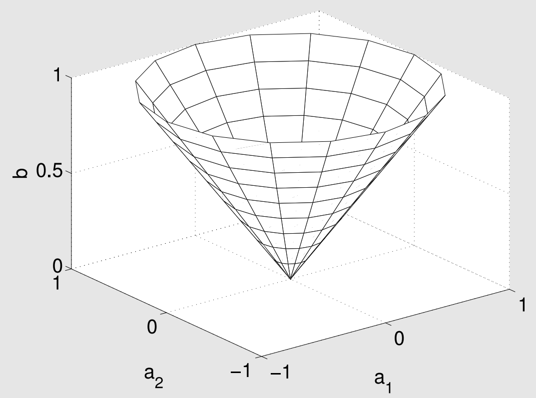
(a) Directionally linear function.



(b) Linear function.



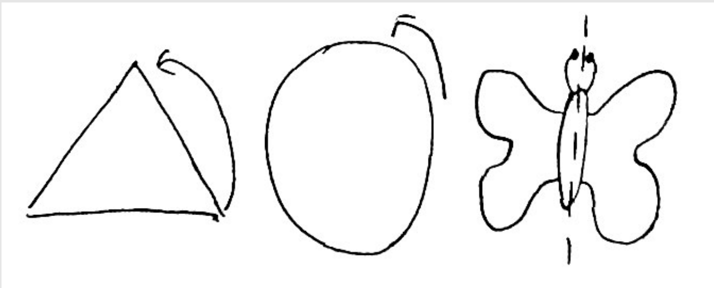
(c) Nonlinear function 1.



(d) Nonlinear function 2.

Symmetry and Groups

When we say that something has *symmetry*, we mean that there is some action we can perform on it so that its starting and ending configurations are indistinguishable. For example, an equilateral triangle is symmetric with respect to rotations by $120°$, a circle is symmetric with respect to any rotation, and a butterfly is symmetric with respect to reflections across its central axis.



Symmetry is closely linked to mathematical groups structure—the set of actions under which an object is symmetric form a group, satisfying the four basic properties outlined on on page 8:

1. **Closure**: Two symmetry-preserving actions can be concatenated into a third action that is also symmetry-preserving. Example: two rotations by $120°$ form a $240°$ rotation, which is also a symmetric action on the equilateral triangle.
2. **Associativity**: If a series of transformations is conducted, the order of operations does not matter (though left-right order may still be important). Example: rotating the triangle twice by $120°$ is equivalent to concatenating the rotations into a single $240°$ rotation and applying it to the triangle.
3. **Identity**: Objects are trivially symmetric under null actions, which can be incorporated as zero-magnitude members of the symmetry group. Example: the rotational symmetry group for the triangle is rotations by $k \cdot 120°$ for integer values of k, and the identity element (no rotation) is the element for whcih $k = 0$.
4. **Inverse**: Any symmetry-preserving transformation may be "undone" or reversed, and concatenating this reversal with the original action produces a null action. Example: the inverse of a $120°$ rotation is a $-120°$ rotation, and together these actions produce a $0°$ rotation.

In many cases, we can use symmetries in a system to simplify its representation. Several examples of this principle have implicitly appeared in previous chapters; for instance, when we describe the configuration space of a rotary joint as $\mathbb{S}^1$, we are recognizing that the configuration of the attached link is symmetric with respect to complete revolutions of the joint, and thus (unless the number of revolutions is important for other reasons) that we can treat such configurations as equivalent. Similarly, our definitions of rigid and articulated bodies are fundamentally based on their symmetry with respect to $SE(2)$—under any $SE(2)$ transformation of the system (which may be applied directly to the particles using (1.12)), the body-frame positions of particles in the system are unchanged, allowing us to characterize the system by its shape in the body frame.

More generally, link between symmetries and rigid bodies leads us to an important aspect of symme

components of the body velocity, ξ^x, ξ^y, and ξ^θ, correspond respectively to the vehicle's longitudinal, lateral, and rotational velocities. The "shape" of these systems describes their internal degrees of freedom. For the differential drive car, this shape is the rotation of each wheel around its axle (relative to a reference angle); for the Ackerman car (also known as the "kinematic bicycle"), the shape variables are the rotation of the drive axle and the steering angle.

The differential drive car moves forward when its wheels are turned together, and rotates when they are turned oppositely; for appropriately normalized wheel diameters and separations, we can encode these rules in

Figure 3.2 Differential drive and Ackerman-steered systems

an equation matching the form of (3.7),

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \tag{3.8}$$

Similarly, the Ackerman car moves forward when the drive axle is rotated, and simultaneously rotates at a rate dictated by the steering angle,

$$\begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \tan\alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \tag{3.9}$$

3.2 Pfaffian Constraints and the Local Connection

In geometric mechanics, the equations of motion for symmetric linear-kinematic systems (which, from now on we will refer to as "kinematic locomotors") are generally expressed as a *kinematic reconstruction equation*,

$$\xi = -\mathbf{A}(r)\dot{r}, \tag{3.10}$$

where $\mathbf{A}(r)$ is the *local connection*[4] associated with the system constraints, and the negative sign appears for historical reasons. As a general principle, a system's local connection corresponds to, and can be derived from, a *Pfaffian constraint*[5] on its configuration velocities,

$$\mathbf{0} = \omega(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}. \tag{3.11}$$

For an $SE(2)$ system with m individual Pfaffian constraints and n shape variables, the overall Pfaffian constraint on the system is encoded by an $m \times (3+n)$ element matrix,

$$\mathbf{0}^{m\times 1} = \omega^{m\times(3+n)} \begin{bmatrix} \xi^{3\times 1} \\ \dot{r}^{n\times 1} \end{bmatrix}, \tag{3.12}$$

each row of which represents a restriction on the motion of the system by mapping the configuration velocity (the body and shape velocities) to zero. Kinematic locomoting systems have Pfaffians composed of at least three independent constraints, one per degree of freedom in the position space. A system with fewer constraints gains the ability to *drift* through the position space without changing shape, and thus is not fully kinematic. With exactly as many constraints as position variables, specifying the system's shape velocity "uses up" all of the local degrees of freedom, so the body velocity becomes a (linear) function of the shape velocity, making

[4] For the origin of this term, see the discussion on fiber bundles in Chapter. **??**.
[5] See the box on the next page.

Nonholonomic Constraints

Nonholonomic constraints restrict the velocity with which a system can move, but *without* restricting the accessible configurations. Formally, a nonholonomic constraint is defined by a (possibly time-varying) function c on the system's configuration tangent bundle, TQ. The zero set of this function (the set of velocities $\dot{Q}_0 \in T_qQ$ at each point q in the configuration space such that $c(q, \dot{q}, t) = 0$) defines the system's allowable velocities at each configuration that satisfy the constraint.

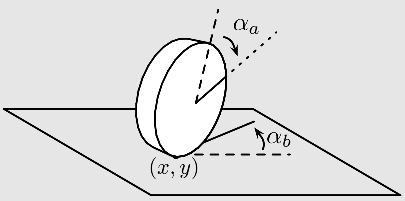
Rolling disk. The classic example of a nonholonomically constrained system is a rolling disk or coin. Viewed from the top as an $SE(2)$ system, the coin is free to roll forward and backward or turn, but is unable to slide sideways; this no-slide condition can be expressed as a nonholonomic constraint either in world coordinates, as

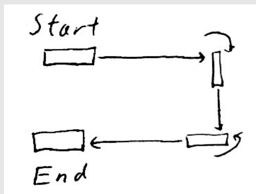
$$c_{\text{world}} = \dot{y}\cos\theta - \dot{x}\sin\theta, \tag{3.v}$$

or with respect to the body velocity of the disk,

$$c_{\text{body}} = \xi^y. \tag{3.vi}$$

This constraint prevents lateral motion of the disk, but does not prevent it from achieving arbitrary net lateral displacement. By combining rolling and turning actions into a "parallel parking" motion, the disk can access any point in the $SE(2)$ space despite the constraint on its velocity.





Pfaffian constraints. A particularly important class of nonholonomic constraints are linear *Pfaffian* constraints, which take the form

$$c(q, \dot{q}) = \omega(q)\dot{q}, \tag{3.vii}$$

where ω is a matrix with as many rows as there are independent constraints in the Pfaffian, and as many columns as there are dimensions in the configuration space. At each configuration, the allowable velocities for the system are in the nullspace of $\omega(q)$, forming a vector space. The collection of these vector spaces is the *distribution* of the system's allowable velocities.

An unconstrained system with an n-dimensional configuration manifold has an n-dimensional distribution at each point in the configuration space, corresponding to the whole of T_qq. Each independent Pfaffian constraint added to the system removes one dimension from the distribution, reducing the local degrees of freedom of the system in much the same way that multiple holonomic constraints reduce the global degrees of freedom.

Nonintegrablity. To qualify as a nonholonomic constraint function, c must not be integrable into a holonomic constraint $f(q, t) = 0$. For instance, the constraint function

$$c = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{x}x + \dot{y}y, \tag{3.viii}$$

applied to a point moving on the (x, y) plane dictates that the point cannot move radially with respect to the origin. Although written in the form of a Pfaffian constraint, it is not a nonholonomic constraint: the restriction against radial velocities means that we can rewrite the function as the holonomic constraint

$$f = \sqrt{x^2 + y^2} - R, \tag{3.ix}$$

where R is the initial distance of the point from the origin, and the corresponding accessible manifold is the circle of radius R.

the system kinematic. If there are more constraints present than positional degrees of freedom, the system is overconstrained from a controls perspective, and only certain shape trajectories can be executed.

The systems we consider below all have three independent constraints, and thus are kinematic without being overconstrained. In part, this simplifies our subsequent analysis, in that we are able to make unconstrained choices of shape trajectories when designing locomotive gaits for the system in Chapters **??**, **??**, and **??**. More saliently, however, the physical principles imposing the constraints often intrinsically match the number of constraints to the dimension of the position space. For example, the inertial systems in §3.4.3 are constrained to conserve linear and angular momentum; these quantities together have as many components as there are in G, and thus provide a corresponding number of constraints (in the case of SE(2), two for linear momentum and one for the angular term).

Given a three-constraint Pfaffian ($m = 3$), we can directly calculate the system's local connection by making the natural decomposition of ω into ω_ξ and $\omega_{\dot{r}}$, the submatrices that are respectively multiplied by the body and shape velocity terms,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_\xi^{3\times 3} & \omega_{\dot{r}}^{3\times n} \end{bmatrix} \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}, \tag{3.13}$$

then separating the constraint equation into the sum of two smaller matrix multiplications,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \omega_\xi \xi + \omega_{\dot{r}} \dot{r}. \tag{3.14}$$

Bringing the ξ terms to the left hand side of the equation,

$$\omega_\xi \xi = -\omega_{\dot{r}} \dot{r}, \tag{3.15}$$

and multiplying both sides by ω_ξ^{-1},

$$\xi = -\omega_\xi^{-1} \omega_{\dot{r}} \dot{r}, \tag{3.16}$$

puts the relationship between the body and shape velocities into the form of (3.10), revealing the local connection as

$$\mathbf{A} = \omega_\xi^{-1} \omega_{\dot{r}}. \tag{3.17}$$

As an example of this process, consider a differential-drive car modeled as a three-body planar system as in Figure 3.3. The body frame of the system, g is located at the center of the axle-line and fixed to the vehicle chassis. The two wheels, with body frames g_1 and g_2, have a radius R and are at a distance w from the center of the chassis. Under the conventional no-slip, no-slide condition for a rolling wheel, the wheel's point of contact with the ground has zero velocity; these conditions thus impose a pair of nonholonomic constraints on each wheel,

$$\xi_{g_i}^x - R\dot{\alpha}_i = 0 \qquad \text{(no-slip)} \tag{3.18}$$

$$\xi_{g_i}^y = 0 \qquad \text{(no-slide)} \tag{3.19}$$

To turn these constraints on the wheels into constraints on the system as a whole, we can employ the Jacobian generation techniques in §2.5.2 to express ξ_{g_1} and ξ_{g_2} in terms of the system shape and body velocities. Here, as the wheels have fixed positions with respect to the system body frame,

$$g_{1,g} = (0, w, 0) \qquad \text{and} \qquad g_{2,g} = (0, -w, 0), \tag{3.20}$$

finding these expressions reduces to applying a single inverse adjoint action to the body velocity to find the body velocity of each wheel,

$$\xi_{g_1} = Ad_{g_{1,g}}^{-1} \xi = \begin{bmatrix} \xi^x - w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix} \tag{3.21}$$

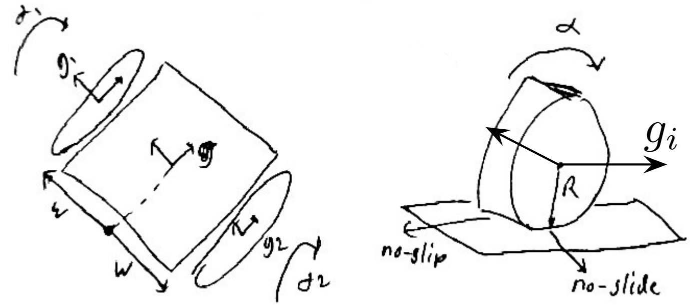


Figure 3.3 Parameterized planar model for a differential-drive car. At left, the (out-of-plane) no-slip, no-slide constraint on the wheels.

and

$$\xi_{g_2} = Ad^{-1}_{g_{2,g}}\xi = \begin{bmatrix} \xi^x + w\xi^\theta \\ \xi^y \\ \xi^\theta \end{bmatrix}. \tag{3.22}$$

Substituting the components of ξ_{g_1} and $\xi_{g_{g_2}}$ from (3.21) and (3.22) into the no-slip and no-slide constraints from (3.18) and (3.19) produces three independent nonholonomic constraints on the car,

$$\xi^x - w\xi^\theta - R\dot{\alpha}_1 = 0 \qquad \text{(no-slip wheel 1)} \tag{3.23}$$

$$\xi^x + w\xi^\theta - R\dot{\alpha}_2 = 0 \qquad \text{(no-slip wheel 2)} \tag{3.24}$$

$$\xi^y = 0 \qquad \text{(no-slide wheels 1 \& 2),} \tag{3.25}$$

with the no-slide constraints on the two wheels each representing the same constraint on the whole vehicle. These constraints are linear in the shape and body velocities, and so can be expressed in Pfaffian form as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 & -w \\ 1 & 0 & w \\ 0 & 1 & 0 \end{matrix}}^{\omega_\xi} & \overbrace{\begin{matrix} -R & 0 \\ 0 & -R \\ 0 & 0 \end{matrix}}^{\omega_{\dot{r}}} \end{bmatrix} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \xi \\ \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \dot{r} \end{matrix}. \tag{3.26}$$

The reconstruction equation and local connection for the system can then be directly calculated as

$$\xi = -\overbrace{\begin{bmatrix} 1 & 0 & -w \\ 1 & 0 & w \\ 0 & 1 & 0 \end{bmatrix}}^{\omega_\xi}{}^{-1} \overbrace{\begin{bmatrix} -R & 0 \\ 0 & -R \\ 0 & 0 \end{bmatrix}}^{\omega_{\dot{r}}} \dot{r} \tag{3.27}$$

$$= -\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ -1/(2w) & 1/(2w) & 0 \end{bmatrix} \begin{bmatrix} -R & 0 \\ 0 & -R \\ 0 & 0 \end{bmatrix} \dot{r} \tag{3.28}$$

$$= -\underbrace{\begin{bmatrix} -R/2 & -R/2 \\ 0 & 0 \\ R/(2w) & -R/(2w) \end{bmatrix}}_{\mathbf{A}} \dot{r}, \tag{3.29}$$

which, for $R = 2$ and $w = 1$, exactly matches (3.8).

3.3 Connection Vector Fields

How can we visualize the local connection?

- First, consider what the rows and columns mean.
- Columns: jth column is the body velocity the system will have if the jth joint is moving with unit velocity and all other joints are fixed.
- Rows: ith row is the *local gradient* of the position with respect to the shape – its dot product with the shape velocity givesif we plot it as a vector field
-

show examples of plane fields and local contour plots to demonstrate why they make poor visualizations

Will show diffdrive and ackerman connection vector fields here as examples

3.4 Full-body Locomotion

Wheeled vehicles, such as the differential-drive and Ackerman-steered cars, tend to have well-defined "drive" and "steering" inputs. In many other systems, the notions of drive and steering are less distinct. For example, snakes, fish, and microorganisms locomote by undulating their bodies in traveling-wave patterns. During these motions, sections of the body alternately take on roles as thrust elements or control surfaces guiding the motion of the system, and applying control abstractions like "drive forward" or "turn with a given radius" becomes less intuitive.

The local connection and other tools from differential geometry offer a systematic overview of the kinematics of such full-body locomotion modes. This overview is especially powerful in that it unifies the analysis of kinematic systems across a wide range of physical domains and body articulations. As an introduction to this analysis, in this section we consider the local connections of *three-link locomotors* in several physical regimes. Three-link systems play an important role in locomotion research; as discussed in §**??**, they are among the "simplest" systems capable of effective locomotion, and so readily lend themselves as minimal examples of locomotive principles. The physical regimes we consider, based on no-slide, inertial, and fluid constraints, are likewise representative of research in geometric mechanics, reflecting the communities that have driven development of these tools.

3.4.1 Three-link Locomotors

Three-link locomoting systems

§2.5.5

velocities from (2.46), (2.51), and (2.56)

$$\xi_{g_1} = \begin{bmatrix} \xi^x \cos\alpha_1 - (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_1 \\ \xi^x \sin\alpha_1 + (\xi^y - (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot\alpha_1) \\ \xi^\theta - \dot\alpha_1 \end{bmatrix} \tag{3.30}$$

$$\xi_{g_2} = \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \end{bmatrix} \tag{3.31}$$

$$\xi_{g_3} = \begin{bmatrix} \xi^x \cos\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\sin\alpha_2 \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \\ \xi^\theta + \dot\alpha_2 \end{bmatrix} \tag{3.32}$$

Vectors, Covectors, and One-forms

In vector calculus, the term "vector" applies to any directional quantity, such as the velocity of a point, a flow rate, or the gradient of a function. In differential geometry, there is a distinction between *tangent vectors* (often referred to simply as vectors), which are velocity-like terms that describe motion *through* the underlying space, and *cotangent vectors* (often shortened to *covectors*) that represent gradient-like terms describing how a quantity varies *across* the space.

Tangent and cotangent vectors share many characteristics. Just as (tangent) vectors are elements of a manifold's tangent bundle, TQ, covectors are elements of its *cotangent bundle*, T^*Q, which is *dual* to TQ. A *covector field* assigns a covector to each cotangent space T_q^*Q in a subspace of T^*Q, in the same manner as a vector field assigns vectors to a set of tangent spaces.

Notation. Components of a vector are identified with superscripts,

$$v = (v^1, v^2, \dots, v^n) \in T_qQ \tag{3.x}$$

and components of a covector with subscripts,

$$\omega = (\omega_1, \omega_2, \dots, \omega_n) \in T_q^*Q. \tag{3.xi}$$

When equivalent bases are used for the tangent and cotangent spaces, vectors and covectors with matched coefficients $\omega_i = v^i$ are *natural duals* to each other. To convert a vector into its natural dual covector, or vice versa, we use *musical notation* to "lower" or "raise" the indices,

$$v^\flat = \omega \quad \text{and} \quad \omega^\sharp = v. \tag{3.xii}$$

Products Vectors and covectors have a natural product corresponding to the vector-calculus notion of a dot product between two vectors,

$$\omega v = \langle \omega, v \rangle = \sum_i \omega_i v^i = \begin{bmatrix} \omega_1 & \dots & \omega_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \omega^\sharp \cdot v, \tag{3.xiii}$$

meaning that we can consider a covector $\omega \in T_q^*Q$ to be a mapping from vectors to real numbers,

$$\begin{aligned} \omega : T_qQ &\to \mathbb{R} \\ v &\mapsto \langle \omega, v \rangle. \end{aligned} \tag{3.xiv}$$

If ω is a function of the configuration (*i.e.* is a covector field, rather than an isolated covector), it becomes a map from configuration, velocity pairs to real numbers,

$$\begin{aligned} \omega : TQ &\to \mathbb{R} \\ (q, v) &\mapsto \langle \omega(q), v \rangle, \end{aligned} \tag{3.xv}$$

that is linear with respect to v, but may vary nonlinearly with q. Functions of this type are also known as *(differential) one-forms*. Differential forms play an important role in geometric mechanics, and we will examine them in greater depth in Chapter **??**.

continued...

Vectors, Covectors, and One-forms *continued*

Vector-valued output. Multiple covector fields may be combined to form a *vector-valued one-form*, in which each element of the output is the product of one of the component covectors with the input vector. A vector-valued one-form ω with m components, $(\omega^1, \dots, \omega^m)$,[a] can be encoded as an $m \times n$ matrix,

$$\omega(q)v = \begin{bmatrix} \omega_1^1(q) & \dots & \omega_n^1(q) \\ \vdots & \ddots & \vdots \\ \omega_1^m(q) & \dots & \omega_n^m(q) \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}. \tag{3.xvi}$$

Many familiar mathematical objects have this structure, including the Jacobians in §2.5 and the Pfaffians and local connections in §3.2. In the following chapters, we combine the physical nature of these objects with differential-geometric properties of differential forms to arrive at a rigorous mathematical description of locomotion.

Directional derivative. Above, we commented that covectors represent "gradient-like" terms that describe the rates of change of functions with respect to the underlying space. More specifically, any covector ω locally represents the derivative[b] of an implied function f with respect to the configuration space,

$$\omega = \mathbf{d}f = \left(\frac{\partial f}{\partial q^1}\ , \ \dots \ , \ \frac{\partial f}{\partial q^n} \right) \tag{3.xvii}$$

Under this interpretation, the product $\langle \omega, v \rangle$ represents the *directional derivative* of f along v,

$$D_v f = \langle \mathbf{d}f, v \rangle = \frac{\partial f}{\partial q} v, \tag{3.xviii}$$

i.e., the rate of change of f along v. In some sources, the directional derivative is normalized by the magnitude of v,

$$D_v f = \frac{\langle \mathbf{d}f, v \rangle}{\|v\|} = \frac{\partial f}{\partial q} \hat{v}. \tag{3.xix}$$

By removing the speed of v from the calculation, this definition better matches the notion of "directionality" than does the definition in (3.xviii), but is ill-defined when v is a zero-vector; consequently the first definition is preferred in differential-geometric contexts.

Note that the function f may be defined over the whole space, in which case

$$\nabla f = \omega^\sharp \tag{3.xx}$$

is its *gradient vector field*,[c] pointing in the direction in which f increases the most quickly. Alternately, it may be only locally definable – the local connection is the derivative of the system's position with respect to its shape, $\partial g / \partial r$, but the output is represented in the system's instantaneous body coordinates, which may change as the system moves. The ability to define f globally is closely related to the *conservativity* of ω and to the *integrability* of nonholonomic constraints. We will return to these subjects in Chapters **??** and **??**.

[a] While elements of a covector are called out by subscripts, here we are in effect dealing with a "vector of covectors," and so indicate individual covectors with a superscript.
[b] Formally, the *exterior derivative* discussed on on page ??.
[c] An alternate definition of the gradient additionally incorporates the *distance metrics* on a space, as discussed in Chapter **??**.

3.4.2 No-slide Constraints

For

$$\begin{bmatrix} \xi^y_{g_1} \\ \xi^y_{g_2} \\ \xi^y_{g_3} \end{bmatrix} = \begin{bmatrix} \xi^x \sin\alpha_1 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_1 - (\ell_1/2)(\xi^\theta - \dot\alpha_1) \\ \xi^y \\ -\xi^x \sin\alpha_2 + (\xi^y + (\xi^\theta \ell_2)/2)\cos\alpha_2 + (\ell_3/2)(\xi^\theta + \dot\alpha_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.33}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 & \ell_1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 & 0 & \ell_3/2 \end{bmatrix} \begin{bmatrix} \xi^x \\ \xi^y \\ \xi^\theta \\ \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix} \tag{3.34}$$

$$\xi = -\overbrace{\begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 & (\ell_2\cos\alpha_1 - \ell_1)/2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & \cos\alpha_2 & (\ell_2\cos\alpha_2 + \ell_3)/2 \end{bmatrix}}^{\omega_\xi}{}^{-1} \overbrace{\begin{bmatrix} -\ell_1/2 & 0 \\ 0 & 0 \\ 0 & \ell_3/2 \end{bmatrix}}^{\omega_{\dot r}} \dot\alpha \tag{3.35}$$

$$\tag{3.36}$$

$$\xi = -\overbrace{\frac{1}{D}\begin{bmatrix} -\ell_1(\ell_3 + \ell_2\cos\alpha_2)/2 & \ell_3(\ell_1 - \ell_2\cos\alpha_1)/2 \\ 0 & 0 \\ -\ell_1\sin\alpha_2 & \ell_3\sin\alpha_1 \end{bmatrix}}^{\mathbf{A}} \dot\alpha \tag{3.37}$$

where $D = \ell_2 \sin(\alpha_1 + \alpha_2) - \ell_1 \sin\alpha_2 + \ell_3 \sin\alpha_3$.

3.4.3 Inertial Constraints

The position and shape of the floating snake are chosen similarly to those for the kinematic snake, with $g = (x, y, \theta) \in SE(2)$ and $r = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. In this case, however, the center of mass is a more natural choice for the x and y coordinates; conservation of linear momentum ensures that the velocity of the center of mass does not change, so the first two rows of the local connection are correspondingly zero. The third row of the local connection identifies rotational velocities which preserve a net angular momentum of zero in response to specified joint velocities **?**. Taking the rows together, the reconstruction equation for a floating snake with equal link lengths and inertias is

$$\xi = -\frac{1}{D}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} \dot\alpha_1 \\ \dot\alpha_2 \end{bmatrix}, \tag{3.38}$$

where

$$\begin{aligned} a_{31} &= 5 + 3\cos(\alpha_2) + \cos(\alpha_1 - \alpha_2) \\ a_{32} &= 5 + 3\cos(\alpha_1) + \cos(\alpha_1 - \alpha_2) \\ D &= 19 + 6(\cos(\alpha_1) + \cos(\alpha_2)) + 2\cos(\alpha_1 - \alpha_2). \end{aligned}$$

The derivation of (3.38) corresponds naturally to a Pfaffian in which the output of ω is the net linear and angular momentum of the system. In some cases, calculation of this quantity is made easier by an alternate, energy-based

Noether's Theorem

Symmetry and Noether's Theorem, with momentum-conservation example

calculation ??. For a momentum-conserving system whose Lagrangian is equal to its kinetic energy (*i.e.* that has no means of storing potential energy), and whose kinetic energy can be expressed as

$$KE = \frac{1}{2} \begin{bmatrix} \xi & \dot{r} \end{bmatrix} \mathbb{M}(r) \begin{bmatrix} \xi \\ \dot{r} \end{bmatrix}, \tag{3.39}$$

the mass matrix $\mathbb{M}$ contains within itself the local connection. Specifically, $\mathbb{M}$ is of the form

$$\mathbb{M} = \begin{bmatrix} \mathbb{I}(r) & \mathbb{I}(r)\mathbf{A}(r) \\ (\mathbb{I}(r)\mathbf{A}(r))^T & m(r) \end{bmatrix}, \tag{3.40}$$

from which $\mathbf{A}$ is easily extracted.

3.4.4 Low Reynolds Number Swimming

At very low Reynolds numbers, viscous drag forces dominate the fluid dynamics of swimming and any inertial effects are immediately damped out. This effect has two consequences, which we can combine to find the local connection. First, the drag forces on the swimmer are linear functions of the body and shape velocities. Second, the net drag forces and moments on an isolated system interacting with the surrounding fluid go to zero.

For an illustration of the first consequence, consider a three-link swimmer with links modeled as slender members according to Cox theory (Cox, 1970). For simplicity here, we regard the flows around each link as independent, per resistive force theory, but the solution for coupled flows is of the same form. The drag forces and moments on the ith link are based on a principle of lateral drag forces being approximately twice the longitudinal forces (Cox, 1970), with the moment around the center of the link found by assuming the lateral drag forces to be linearly distributed along the link according to its rotational velocity, *i.e.*,

Note that L here is a link half-length. Correct the moment terms accordingly (this is a holdover from an old paper, and will be fixed in the book).

$$F_{i,x} = \int_{-L}^{L} \frac{1}{2} k\xi_{i,x} d\ell = kL\xi_{i,x} \tag{3.41}$$

$$F_{i,y} = \int_{-L}^{L} k\xi_{i,y} d\ell = 2kL\xi_{i,y} \tag{3.42}$$

$$M_i = \int_{-L}^{L} k\ell(\ell\xi_{i,\theta}) d\ell = \frac{2}{3} kL^3 \xi_{i,\theta}, . \tag{3.43}$$

where $F_{i,x}$ and $F_{i,y}$ are respectively the longitudinal and lateral forces, M_i the moment, k the differential viscous drag constant, and $\xi_i = [\xi_{i,x}, \xi_{i,y}, \xi_{i,\theta}]^T$ is the body velocity of the center of the ith link.[a] The link body velocities are readily calculated from the system body and shape velocities as

[a] Note that by "body velocity", we mean the longitudinal, lateral, and rotational velocity of the link, and not its velocity with respect to the body frame of the system.

By extension, the forces in (3.41)–(3.43), which are linearly dependent on the link body velocities, are also linear functions of ξ and $\dot{\alpha}$ and nonlinear functions of α. Summing these forces into the net force and moment on the system (as measured in the system's body frame),

Note that L here is a link half-length. Correct the moment terms accordingly (this is a holdover from an old paper, and will be fixed in the book).

$$\begin{bmatrix} F_x \\ F_y \\ M \end{bmatrix} = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ -\sin\alpha_1 & \cos\alpha_1 & 0 \\ L\sin\alpha_1 & -L(1+\cos\alpha_1) & 1 \end{bmatrix} \begin{bmatrix} F_{1,x} \\ F_{1,y} \\ M_1 \end{bmatrix} + \begin{bmatrix} F_{2,x} \\ F_{2,y} \\ M_2 \end{bmatrix} + \begin{bmatrix} \cos\alpha_2 & -\sin\alpha_2 & 0 \\ \sin\alpha_2 & \cos\alpha_2 & 0 \\ L\sin\alpha_2 & L(1+\cos\alpha_2) & 1 \end{bmatrix} \begin{bmatrix} F_{3,x} \\ F_{3,y} \\ M_3 \end{bmatrix}, \quad (3.44)$$

preserves the linear relationship with the velocity terms while only adding further nonlinear dependence on α, such that the net forces $F = [F_x, F_y, M]^T$ can be expressed with respect to the velocities as

$$F = \omega(\alpha) \begin{bmatrix} \xi \\ \dot{\alpha} \end{bmatrix}, \quad (3.45)$$

where ω is a 3×5 matrix.

We now turn to the second consequence of being at low Reynolds number, that the net forces and moments on an isolated system should be zero, *i.e.* $F = [0, 0, 0]^T$. Applying this rule turns (3.45) into a Pfaffian constraint equation, and thus provides the system's local connection.

3.4.5 High Reynolds Number Swimming

At very large Reynolds numbers, viscous drag is negligible and inertial effects dominate the swimming dynamics. While these conditions appear to be the direct opposite of those in the low Reynolds number case, they also result in the system equations of motion forming a kinematic reconstruction equation. This property can be demonstrated via several approaches of varying technical depth Kelly (n.d.); Melli et al. (2006), but to maximize the physical intuition associated with this derivation, we give here a novel presentation based on the that used for the floating snake.

As discussed in the case of the floating snake, for a momentum-conserving system whose Lagrangian is equal to its kinetic energy (*i.e.* it has no means of storing potential energy), the local connection can be easily extracted from the inertial matrix. To apply this principle to the high Reynolds number system, it just remains to be shown that the three-link swimmer at high Reynolds number meets the afore-mentioned conditions. The first condition, that momentum is conserved in the system, follows from the lack of dissipative forces in the high Reynolds number regime. The second condition, that the Lagrangian equal the kinetic energy, can be easily seen by observing that for a planar system with no gravity effects in the plane, there is no mechanism for storing potential energy, leaving only the kinetic term in the Lagrangian. The third condition is more subtle, and as above, we will use a hydrodynamically decoupled example while noting the existence of an equivalent coupled solution.

An object immersed in a fluid displaces this fluid as it moves. In an ideal inviscid fluid, the drag forces on the object are entirely due to this displacement, and act as directional *added masses* $\mathcal{M}$ on the object that sum with the actual inertia of the object to produce the effective inertia of the combined system. The added masses of single rigid bodies (and elements of articulated bodies when the inter-body fluid interactions are neglected) are solely functions of the geometries of the bodies. For example, the added mass tensor of an ellipse with semi-major axis a and semi-minor axis b in a fluid of density ρ is

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_x & 0 & 0 \\ 0 & \mathcal{M}_y & 0 \\ 0 & 0 & \mathcal{M}_\theta \end{bmatrix} = \begin{bmatrix} \rho\pi b^2 & 0 & 0 \\ 0 & \rho\pi a^2 & 0 \\ 0 & 0 & \rho(a^2 - b^2)^2 \end{bmatrix}, \tag{3.46}$$

with $\mathcal{M}_x$, $\mathcal{M}_y$, and $\mathcal{M}_\theta$ respectively corresponding to the added mass for longitudinal, lateral, and rotational motion.

Returning to the three-link swimmer, the kinetic energy associated with motion of the ith link through the fluid is

$$KE_i = \frac{1}{2}\xi_i^T (I_i + \mathcal{M}_i)\xi_i, \tag{3.47}$$

where I_i is the link's inertia tensor and ξ_i is its body velocity, as calculated in (**??**)–(**??**). Using the same linear dependence of ξ_i on ξ and $\dot{r}$ as we exploited in the low Reynolds number case, it is relatively easy to transform (3.47), and thus $KE = \sum KE_i$, into the form of (3.39), and from there to extract the local connection $\mathbf{A}$. The derivation for the hydrodynamically coupled case is essentially similar, with the chief difference being the additional dependence of $\mathcal{M}$ on r, which captures the distortion of the flow around each link caused by the proximity of the other links Kelly (n.d.).

Exercises

3.1 Show that the nonholonomic constraints on the differential drive car induce a *holonomic* constraint between its orientation and its wheel angles.

3.2 What would change in the calculation of the local connection if the Pfaffian had < 3 rows? How about > 3?

3.3 Find the Pfaffian constraints on an Ackerman car (modeled as a kinematic bicycle), and use them to calculate the system's local connection

a. For the standard rear-wheel drive model
b. For a front wheel drive with passive rear wheels

3.4 Find the local connection for two link swimmers at low and high Reynolds numbers. Assume

- links of equal length ℓ
- a body frame located at the hinge and oriented as the mean angle of the two links
- a $2:1$ drag force ratio for lateral/longitudinal drag on the low Re swimmer
- elliptical links with a $10:1$ aspect ratio for the high reynolds number system

Notes

References

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