Curvature

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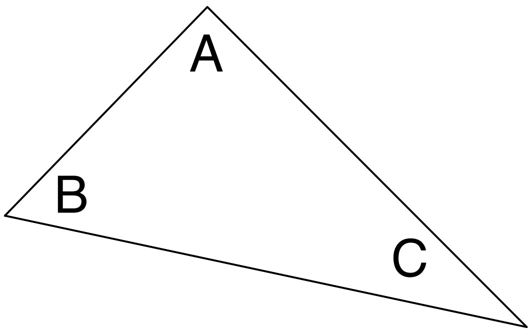
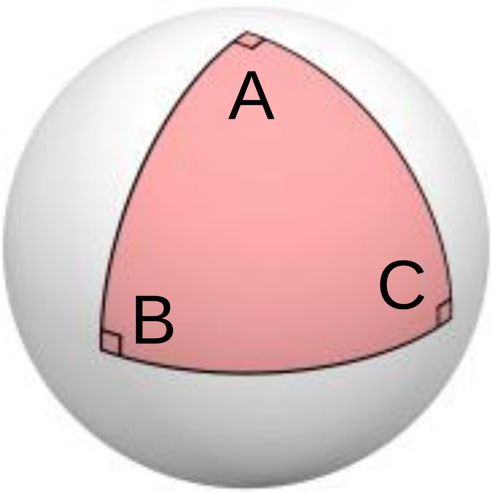
Total Angle in a Triangle



Triangle in the plane

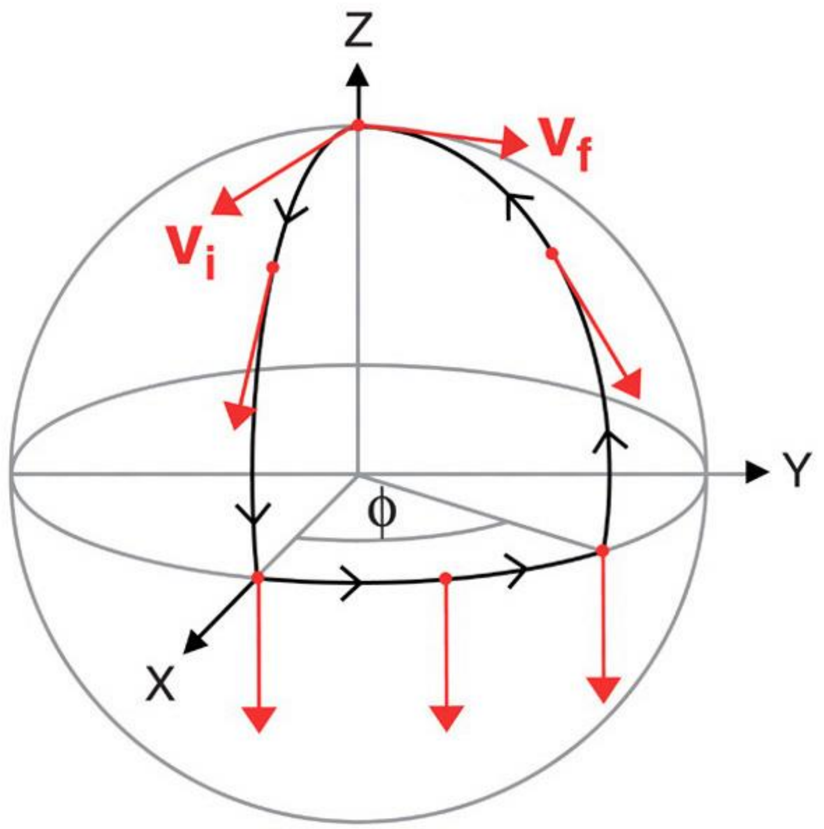
$$A+B+C=\pi$$



Triangle on a sphere

$$A+B+C>\pi$$

Changing Orientation on the Sphere



If we *parallel transport* a vector along the three sides of the triangle, the net change in direction from v_i to v_f is equal to the total interior angle minus pi

In *parallel transport,* a vector is moved along a curve, keeping the angle between it and the tangent vector of the curve constant

In this figure, the angle is 0 for the first curve, 90° for the second, and 180° for the third

Hairy Ball Theorem: Can not comb a tennis ball

Straight lines = paths whose tangent vectors are parallel transports of each others

Gauss-Bonnet Theorem

$$\int_M K \, dA + \int_{\partial M} k_g \, ds = 2\pi\chi(M)$$

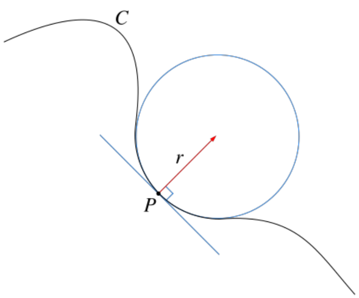
Total surface curvature (Gaussian)

Total boundary curvature (Geodesic)

Euler Characteristic of M. For a disk (2D section with boundary), this is 1

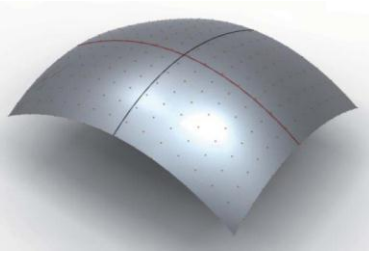
Interpreting the Gauss-Bonnet Theorem: Curvature

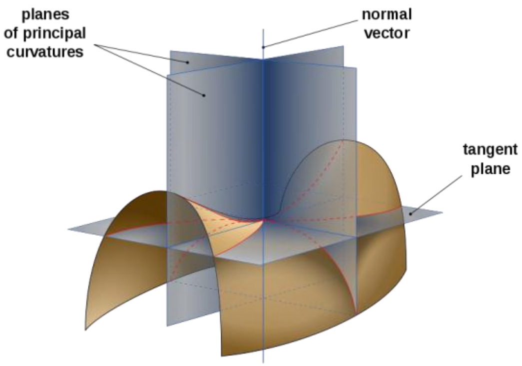
Curvature in general describes the circle locally approximated by a curve.



$$\kappa = \frac{1}{r}$$

In two dimensions, simple natural generalization is to think of the ellipsoid[1] locally approximated by a surface. Orientation and scaling of ellipsoid gives *principal curvatures*

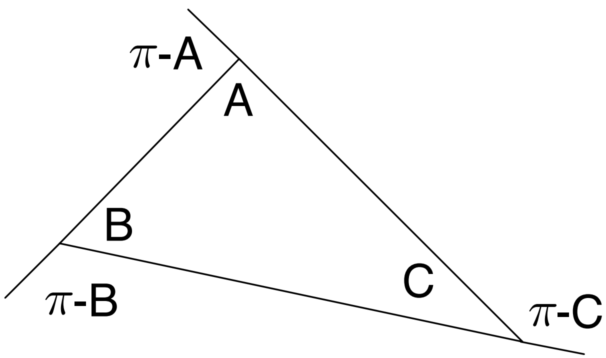




[1]Also recognize that curvatures may have different signs

[1]Ellipsoid if signs of curvature are same, otherwise a saddle

Interpreting the Gauss-Bonnet Theorem: Triangle on a Plane



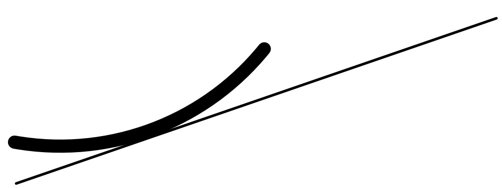
$$\int_M \cancel{K\,dA} + \int_{\partial M} k_g \, ds = 2\pi \overset{1}{\chi(M)}$$

No
Gaussian
Curvature

Perimeter curvature is the sum of the exterior angles

$$3\pi - (A + B + C) = 2\pi$$

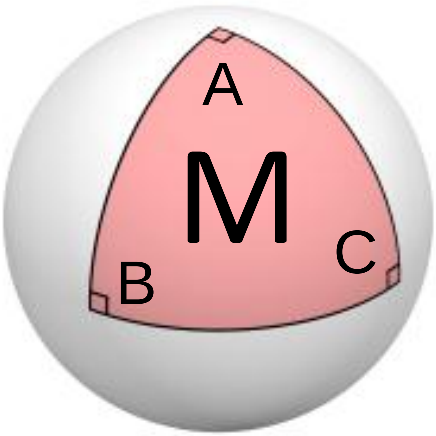
Curvature measures deviation of path from a straight line. Progressing CCW around the triangle, each vertex is an impulse (Dirac delta function) of curvature



Standard trigonometric rule for interior angles of a triangle

$$A + B + C = \pi$$

Interpreting the Gauss-Bonnet Theorem: Triangle on a Sphere



$$\int_M K \, dA + \int_{\partial M} k_g \, ds = 2\pi\chi(M)$$

1

For unit-radius sphere, K=1

Using the same interior angles formula, but adding in the Gaussian term,

$$\pi/2 + 3\pi - (A + B + C) = 2\pi$$

This triangle is 1/8th the surface of the sphere. Area integral of curvature over the triangle is thus

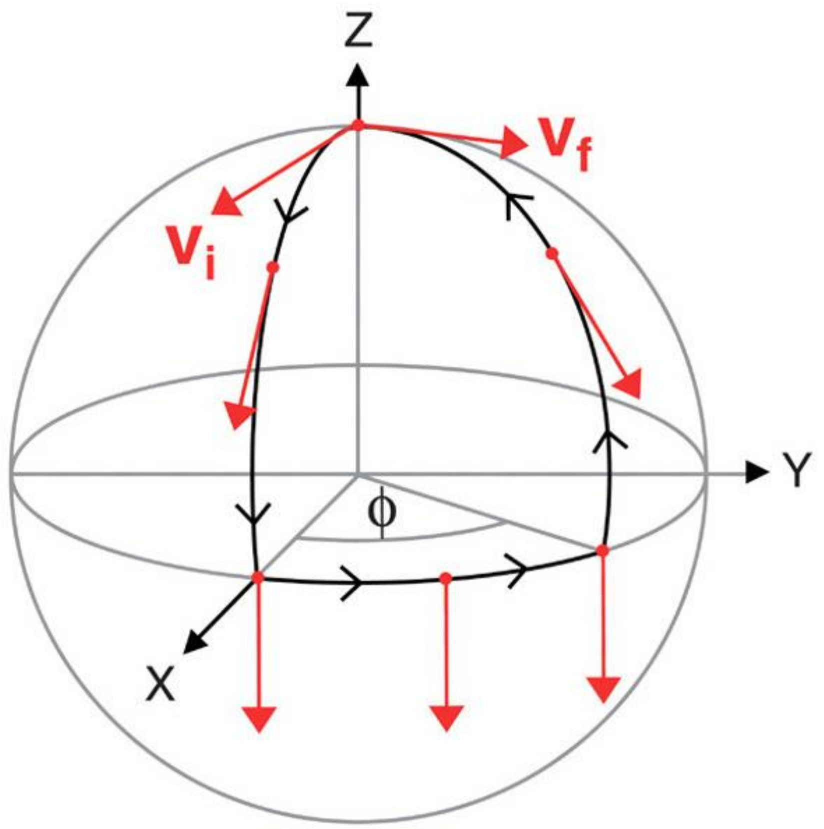
$$\int_M K \, dA = \frac{1}{8}\left(K 4\pi r^2\right) = K\frac{\pi}{2}r^2 = \frac{\pi}{2}$$

On a sphere $K = 1/r^2$

Total interior angles are three right angles

$$A + B + C = 3\pi/2$$

Geodesic Curvature



Note that on the sphere, we still treated the perimeter as straight lines with Dirac delta curvature, even though the lines on the sphere are inherently “curved.”

The relevant curvature here is *geodesic curvature* – how curved the lines are relative to a geodesic

For the triangle on the sphere, each line follows a great circle geodesic, so the only curvature is the impulse when switching geodesics

The net change of orientation between v_i and v_f corresponds to the curvature of the surface, which modifies the total accrued geodesic curvature for a closed loop

Curvature and Distance

$$ds_{\mathbb{S}}^2 = \begin{bmatrix} d\lambda & d\phi \end{bmatrix} \underbrace{\begin{bmatrix} \cos^2(\phi) & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{M}_{\mathbb{S}}} \begin{bmatrix} d\lambda \\ d\phi \end{bmatrix}$$

Brioschi formula converts derivatives of *M* to curvature. Shortest path (according to distance metric) become "straightest lines" (along the curved surface), thus satisfying dual properties of the geodesics

Geodesics were either

1. Shortest path
2. "Straightest" lines

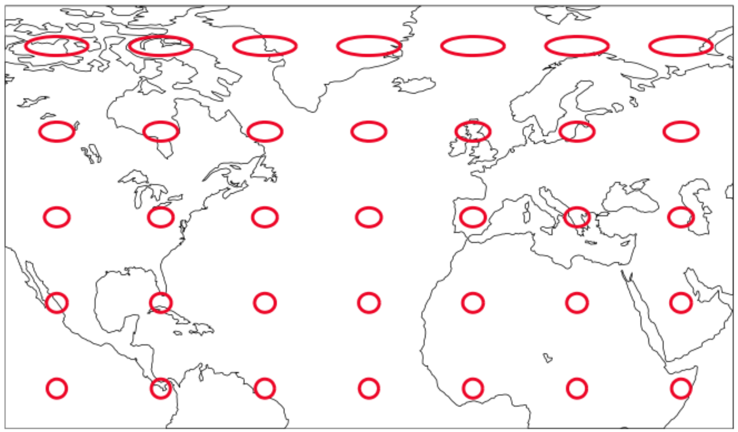
Curvature and distance are directly linked

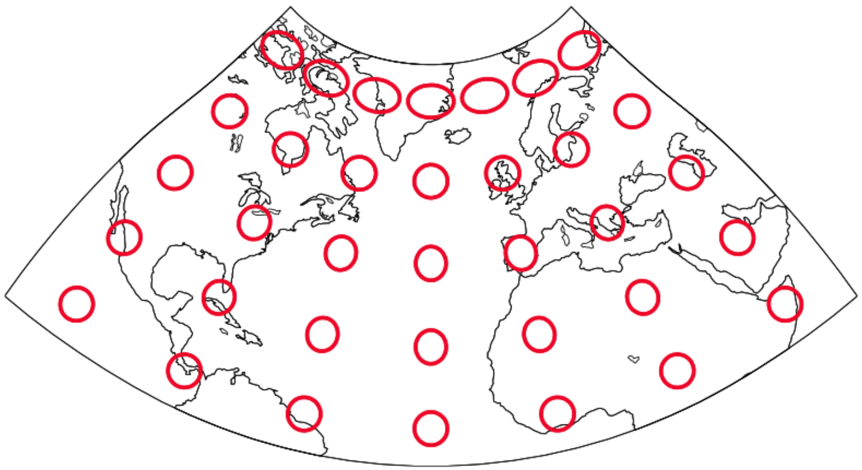
Curvature is what accommodates a rate of change in the distance metric

For example, with the longitude/latitude distance metric on a sphere, curvature allows longitude-distance to shrink as latitude increases

Curvature and Distance

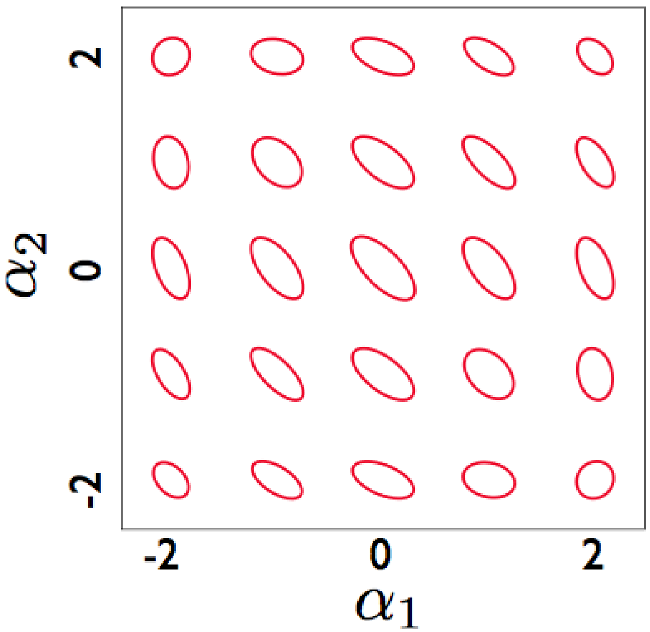
Corollary: Because curvature is intrinsically defined by distances, distances can *only* be directly transferred between surfaces with the same curvature. This means that no flat map (zero Gaussian curvature) can completely accurately represent the globe ($1/r^2$ Gaussian curvature)

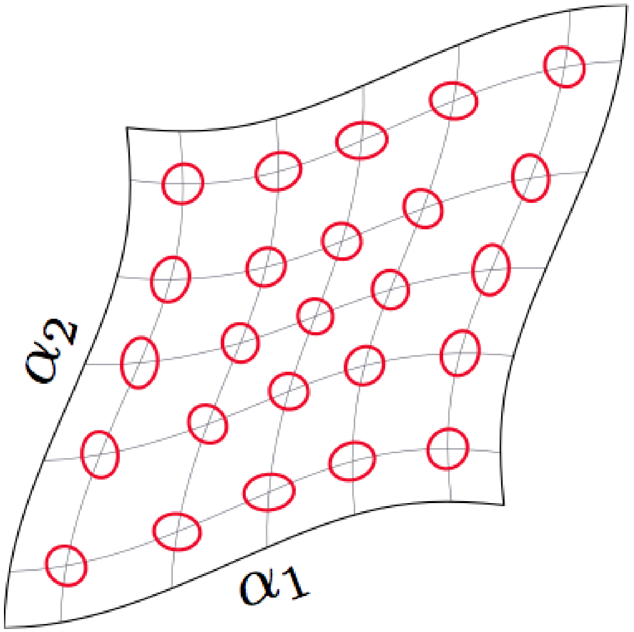




Some will do better than others, but there will always be distortion

Things to Think About





If we generate curvature from power-based distance metric, what does the shape space look like?

Curvature and Locomotion

$$z(\phi) = \iint_\phi \overbrace{\underbrace{-\mathrm{curl}\mathbf{A}}_{\text{nonconservativity}} + \underbrace{[\mathbf{A}_1, \mathbf{A}_2]}^{\text{CCF (Lie bracket)}} dr + \text{higher-order terms}}_{\text{noncommutativity}}$$

Why do we call the first two terms the *Constraint Curvature Function*?

Some intuition:

1. **A** is a (local) Jacobian from shape to position space, and hence acts something like a metric – it says how far you move in the position space for a given motion in shape space.

2. curl **A** measures how **A** changes across the shape space. This is like how the curvature of the globe captures the change in the distance metric with latitude.

3. [$\mathbf{A}_1$,$\mathbf{A}_2$] measures how much flowing backwards and forwards along geodesics fails to bring you back to your starting configuration. This is like our geodesic triangle on the sphere. If we had used 4 flows (a geodesic quadrilateral), we would have seen the interior angles not add up to 360°, or we could have constructed a path with equal positive and negative flows that ended up at a different point from the starting point