Chapter 4:
Gaits and Cyclic Motion

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Motion planning

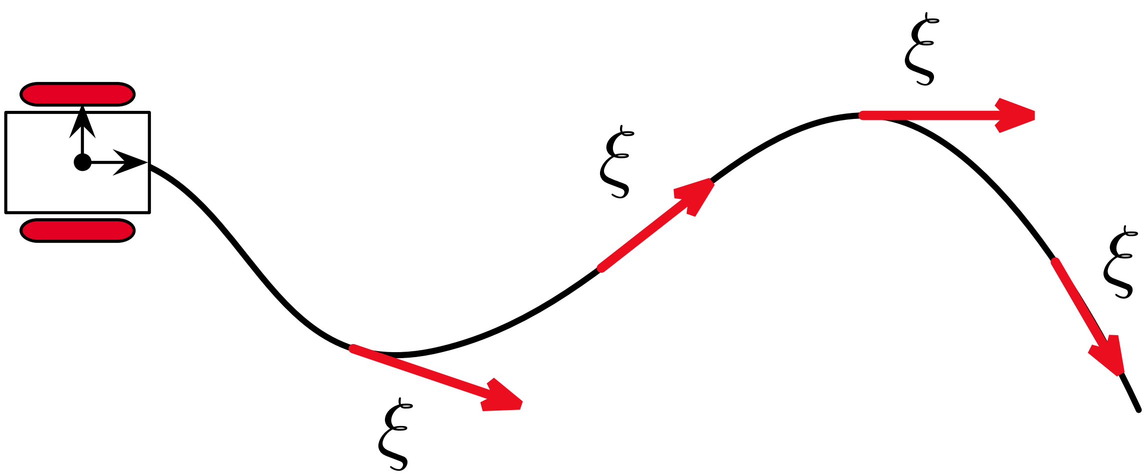
Find shape inputs to move system between two configurations

Motion Planning

Given the local connection, could we just define a path through the world, and then use

$$\dot{r} = \mathbf{A}^{-1}(r)\xi$$

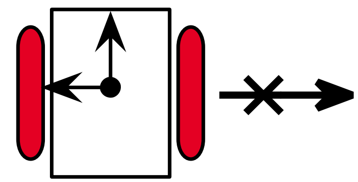
(inverse kinematics with a matrix pseudoinverse) to determine the shape trajectories?



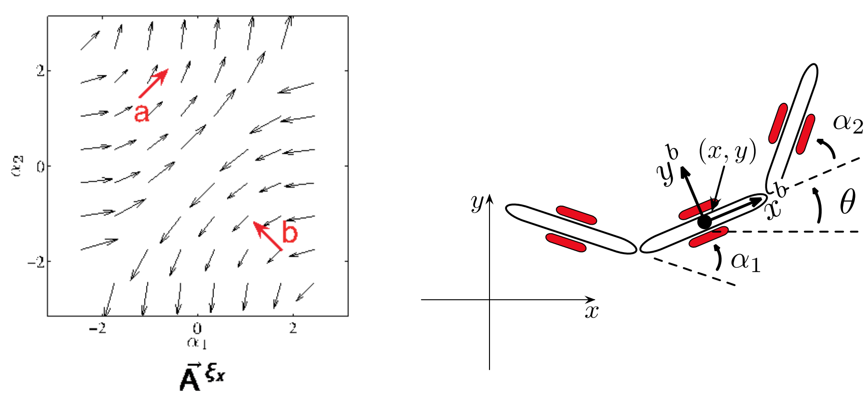
Motion Planning

Inverse kinematics not always sufficient:

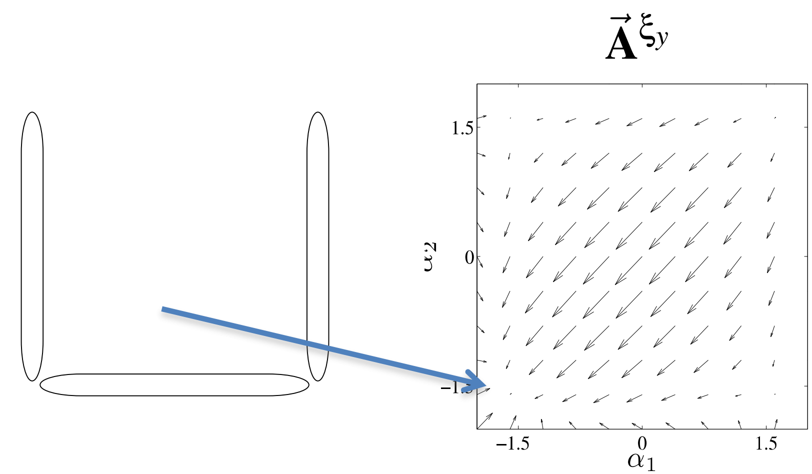
1. Constrained directions – only certain paths are admissible



2. Singularities, Sinks, and Joint limits: what's wrong with these



Kinematic snake can't pass through singularities at "C" shapes



Three-link swimmer can't move laterally from this shape

General motion planning needs "look ahead" to workaround these problems

Motion Planning

- Full optimal control requires solving whole path at once (respecting constraints, limits, etc.)

- Maneuver-based planning precalculates short paths that
  1. Avoid joint limits and singularities
  2. Displace the system in some “useful” manner
  3. Can be concatenated together to form a longer motion plan

Gaits

Gaits are *cyclic* motions in the shape space

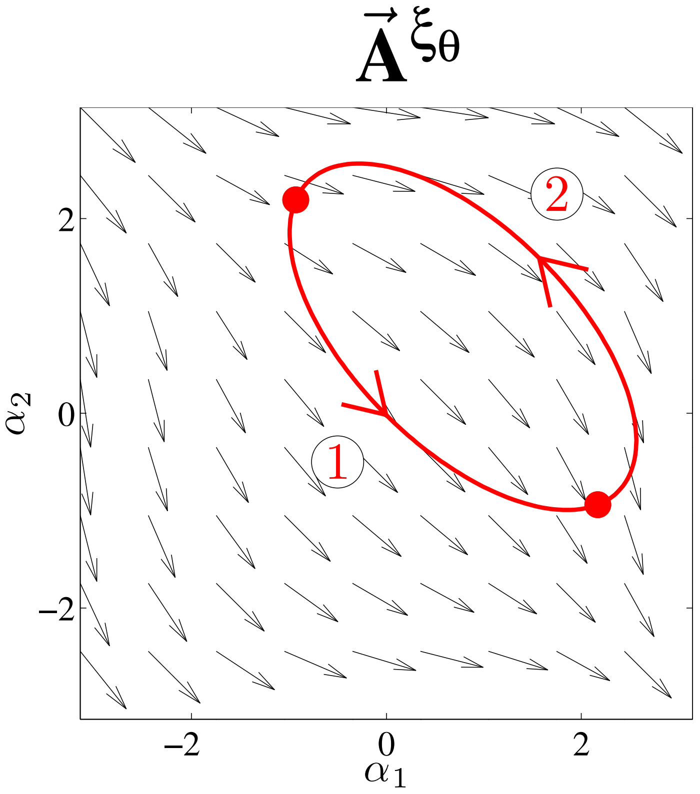
Cyclic nature makes them inherently composable

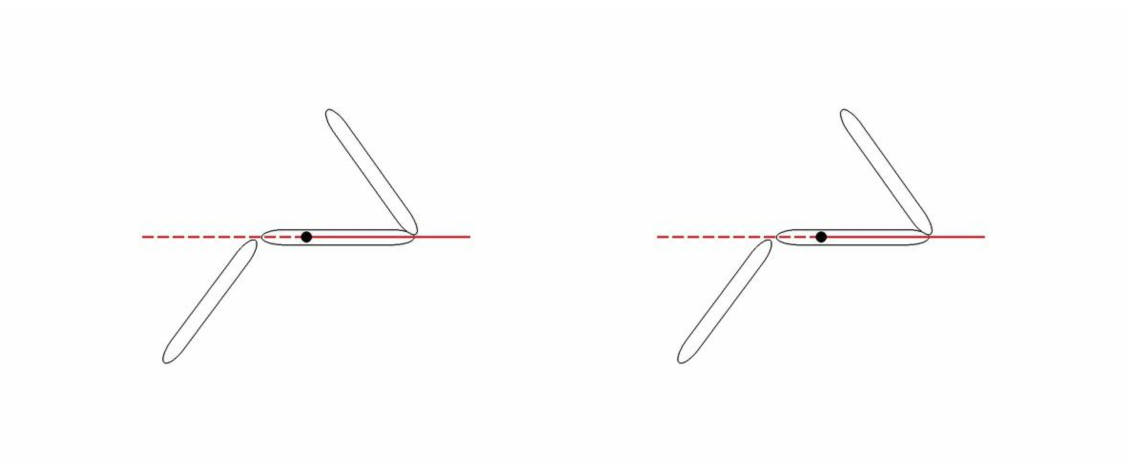
Nonconservativity and Noncommutativity

Kinematic gaits operate on two basic principles:

- <u>Nonconservatvity:</u> System pulls itself forward more than it pushes itself backward each cycle
- <u>Noncommutativity</u>: System rotates during the course of the gait, so forward and backward motions do not directly cancel

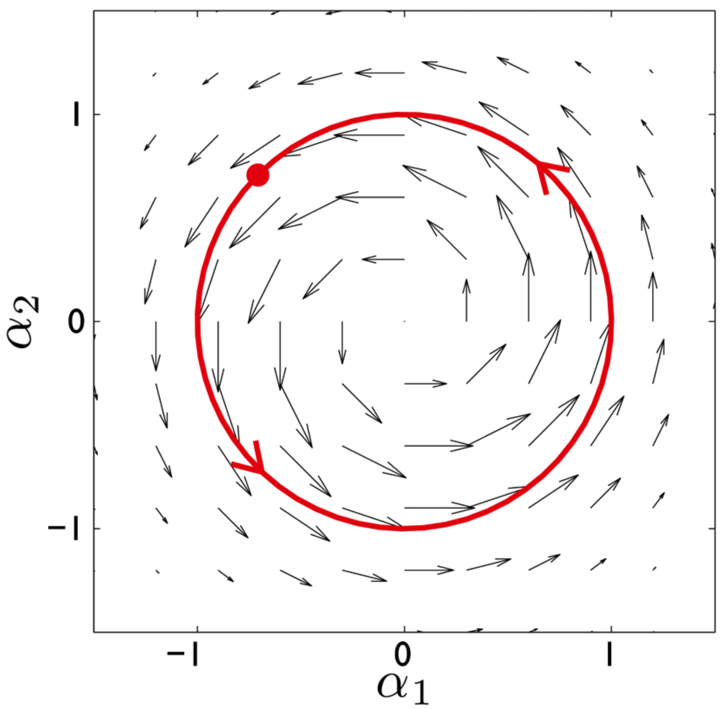
Nonconservativity Example

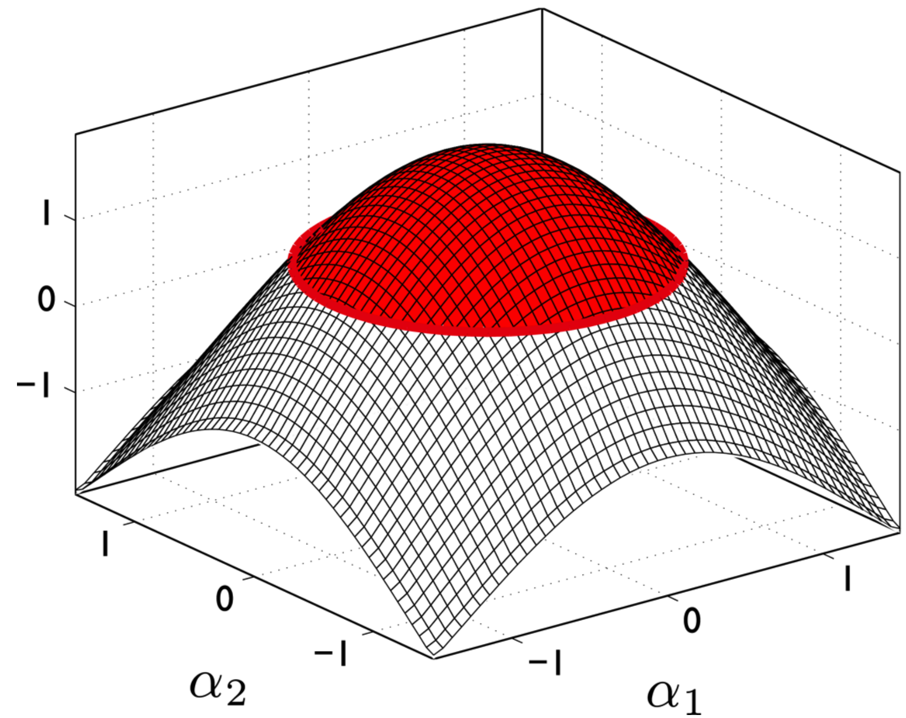




Gait “flows against” field more than it “flows with” it, and thus has net negative rotation.

Stokes's Theorem
(vector calculus version)





Line integral of a closed path on a vector field is equal to the area integral of the field's *curl*

Curl

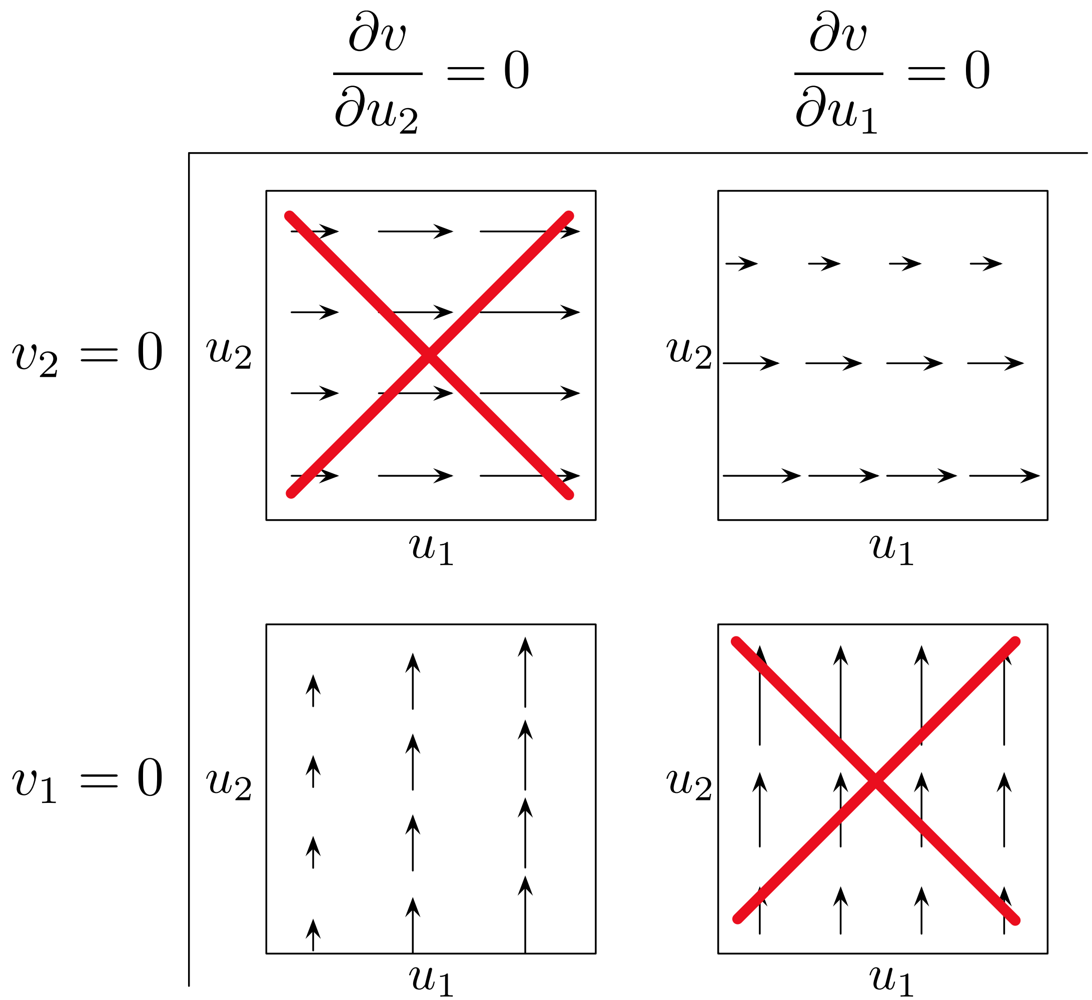
$$\text{curl } V = \frac{\partial v_2}{\partial u_1} - \frac{\partial v_1}{\partial u_2}$$

Curl is sometimes described as "how much the vector field is 'rotating' around a point"

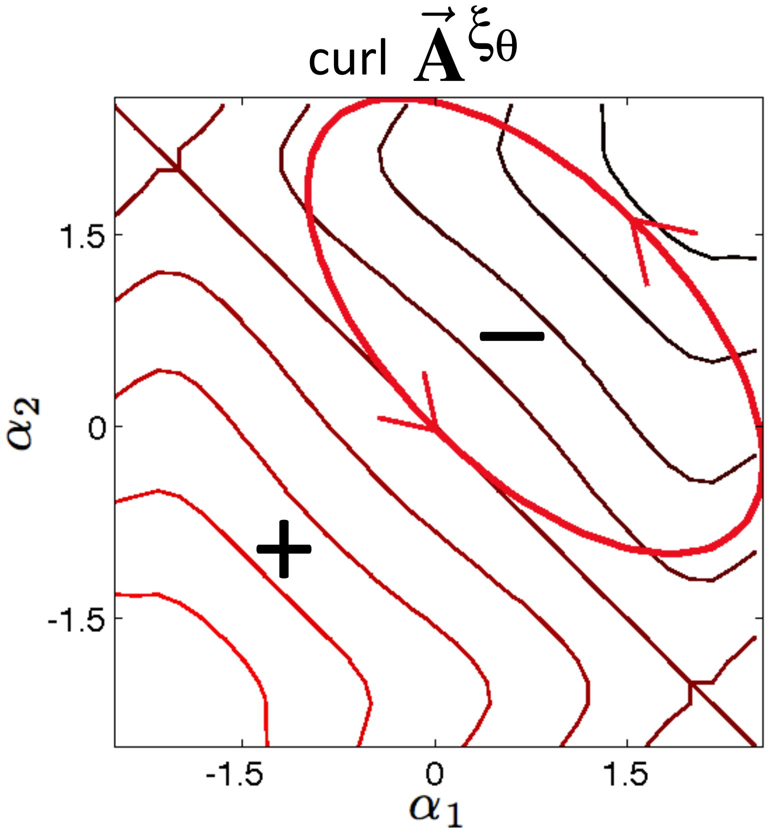
Magnitude also comes in here, so this description can be somewhat confusing

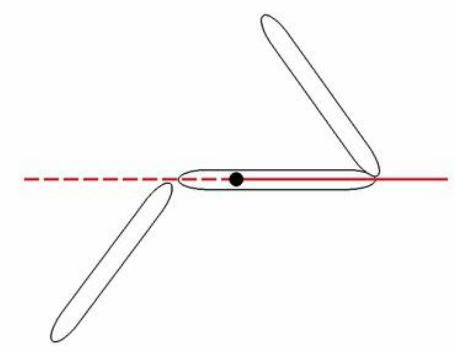
Another way to think of it is "the derivatives of the vector components along orthogonal directions in the space"

Around a cycle, line integrals on top right and bottom left fields will cancel, but those on top left and bottom right will not. Curl measures this non-canceling



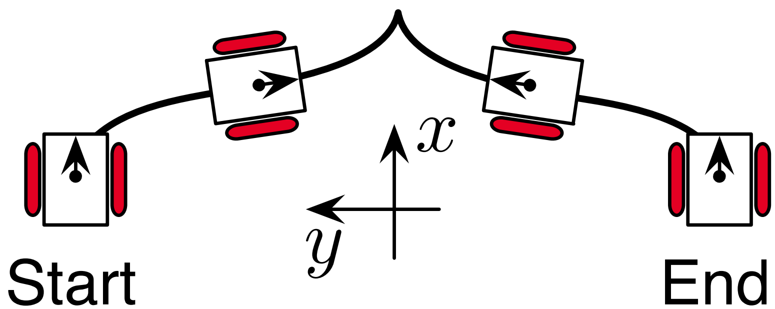
Curl and the Floating Snake





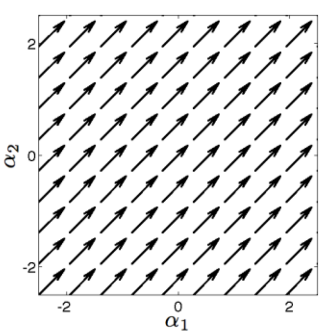
Encircling a region of negative curl means floating snake rotates negatively

Noncommutativity

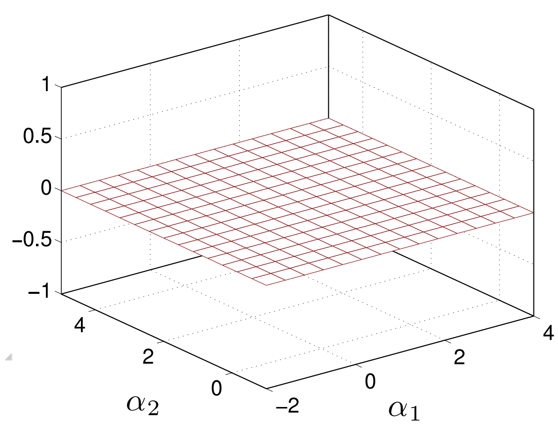


Car cannot move sideways, but, because translations and rotations do not commute, it can “parallel park” to get a net lateral motion

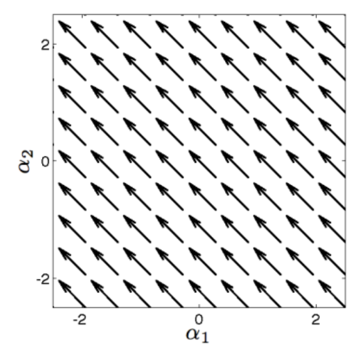
$\vec{\mathbf{A}}^{\xi_x}$



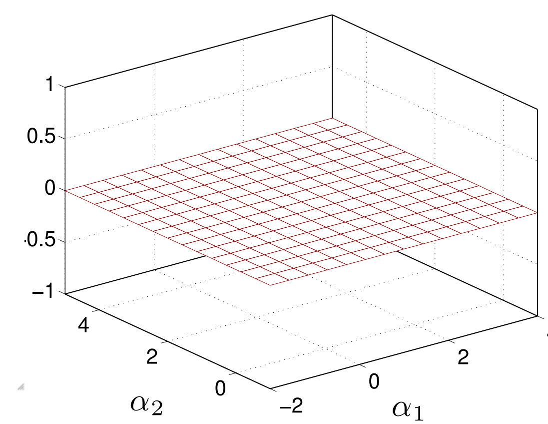
curl $\vec{\mathbf{A}}^{\xi_x}$



$\vec{\mathbf{A}}^{\xi_\theta}$



curl $\vec{\mathbf{A}}^{\xi_\theta}$



None of this shows up in the curl

Lie Brackets

Lie bracket measures how vector fields change along each other

Lie bracket | Directional derivative of Y along X | Directional derivative of X along Y

$$[X, Y] = (\nabla Y \cdot X) - (\nabla X \cdot Y)$$

Output of the Lie bracket of two vector fields is a vector field.

At each point, differentially flowing along X, Y, -X, and –Y (in order) is equivalent to differentially flowing along [X,Y]

When we have a Lie group, we can talk about the Lie bracket of two vectors *u,v* in T_eG as meaning the Lie bracket of their left-invariant fields, evaluated at the origin

$$[u, v] \equiv [T_e L_g u, T_e L_g v]\Big|_{g=\epsilon}$$

When we do this, we are treating T_eG as the *Lie algebra* of the group

Lie bracket on SE(2)

Lie group Lie bracket

$$[u, v] \equiv [T_e L_g u, T_e L_g v]\Big|_{g=\epsilon}$$

SE(2) Lie bracket

$$\left[\begin{pmatrix} u^x \\ u^y \\ u^\theta \end{pmatrix}, \begin{pmatrix} v^x \\ v^y \\ v^\theta \end{pmatrix} \right] = \begin{pmatrix} v^\theta u^y - u^\theta v^y \\ u^\theta v^x - v^\theta u^x \\ 0 \end{pmatrix}$$

Lie Bracket and the Car

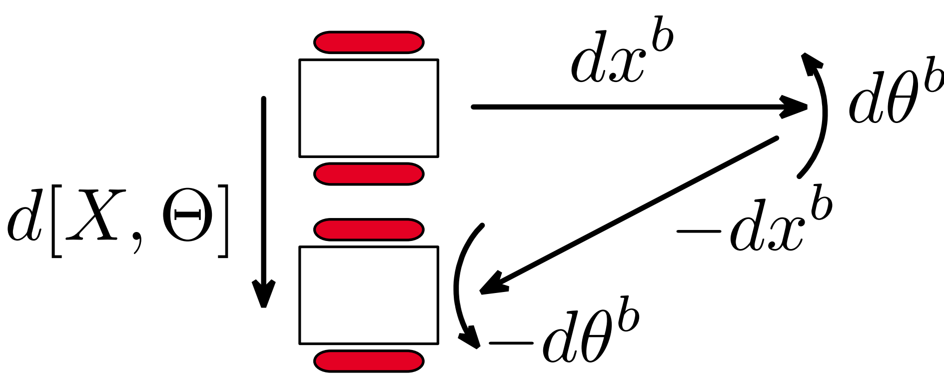
Differential drive car

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad [u, v] = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Drive forward

Turn in place

Drive+, turn+, drive-



$$\left[\begin{pmatrix} u^x \\ u^y \\ u^\theta \end{pmatrix}, \begin{pmatrix} v^x \\ v^y \\ v^\theta \end{pmatrix} \right] = \begin{pmatrix} v^\theta u^y - u^\theta v^y \\ u^\theta v^x - v^\theta u^x \\ 0 \end{pmatrix}$$

This is called *(Lie bracket) averaging* – if you alternate between differentially moving forward/backward and rotating cw/ccw, *on average* you move laterally

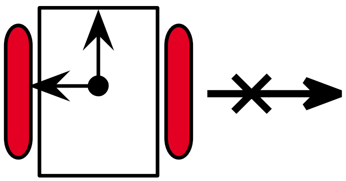
Reminder

- Controllability: we have a set of inputs that can move the system in any direction.

- Distribution: A set of vector fields along which a system can move

Importance of Lie brackets

1. Controlability: Nonholonomically-constrained systems by definition cannot instantaneously move in some directions: their distributions do not span their configuration spaces



If we take a distribution

$D=\{V_1,V_2\}$

and find that the Lie bracket $[V_1,V_2]$ is outside the span of D, we can combine D with the Lie bracket field to make a new distribution

$D'=\{V_1,V_{2,}\ [V_1,V_2]\}$

whose span is one dimension higher, and represents the directions the system can move either with a differential control input or with a first order oscillation of the controls. In some cases, we can further expand the distribution with higher-order Lie brackets, e.g.,

$D'' = \{V_1,V_{2,}\ [V_1,V_2], [V_1, [V_1,V_2]]\}$

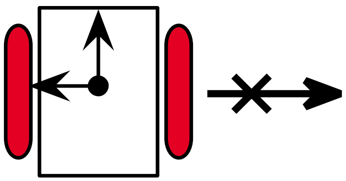
Importance of Lie brackets

Controlability (cont'd)

$D=\{V_1,V_2\}$

$D'=\{V_1,V_{2,}\ [V_1,V_2]\}$

$D'' = \{V_1,V_{2,}\ [V_1,V_2],\ [V_1,\ [V_1,V_2]]\}$



INVOLUTIVE DISTRIBUTION:

If we reach a point where we can't add dimensions by taking more Lie brackets, this is an *involutive distribution*: it describes the full set of directions in which the system can move differentially.

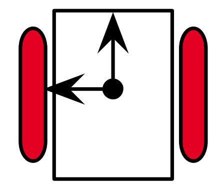
a distribution in which the Lie Bracket of any two elements is in the span of the distribution

If a system's involutive distribution spans the whole configuration space, then the system is said to be *controllable* – we have a set of inputs that can move the system in any direction. (Note that in general there will be more efficient/effective ways to move the system in non-primary directions, controllability is a proof of existence)

Importance of Lie brackets

2. Relationship to gaits.

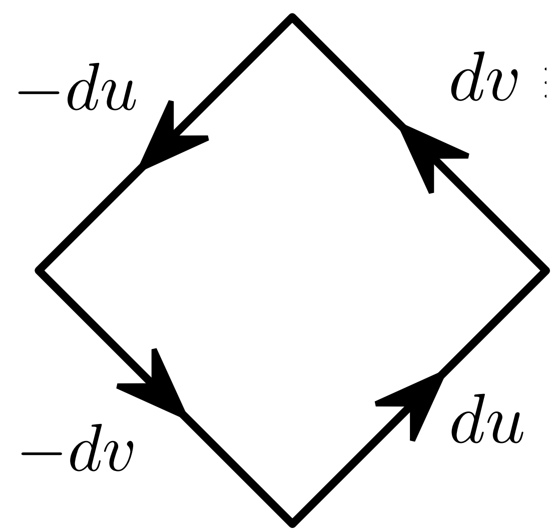
$$du \equiv d\alpha_1 + d\alpha_2$$
$$dv \equiv -d\alpha_1 + d\alpha_2$$



$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are associated with unit shape velocities, and so their Lie bracket corresponds to the motion produced by a differential oscillation in the shape space (i.e. a differential gait)

$$[u, v] = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$



Lie Brackets and Exponential Maps

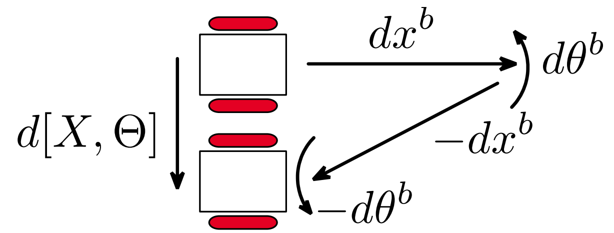
Lie bracket gives us **average** velocity of the system: moving for dt along each of *u, v*, -*u*, and -*v* is equivalent to moving for dt along [u,v]. (note that the total "time" in the cycle is different from the "flow time" along the Lie bracket field).

Note average velocity is a body-frame velocity (a direction and speed) and not a displacement

To get the *displacement* over the cycle, we can exponentiate this velocity:

$$\Delta g = \exp([u, v]dt)$$

Note: still making small angle approximations



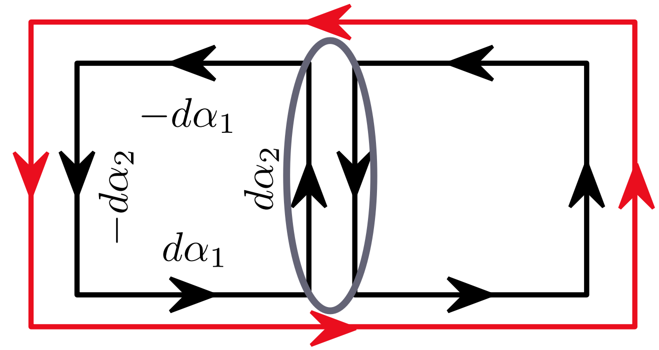
Here, [*u,v*]*dt* gives the *exponential coordinates* of the net displacement

we will denote such exponential coordinates as *z*.

Also note that if we scale *u* or *v*, we proportionally increase the displacement

Lie Bracket and Larger Motions

One of the nice things about thinking of Lie bracket motions as differential gaits is that it gives us intuition as to how the system will move over larger gaits: Two differential cycles are equivalent to their combined perimeter if the overlapping edges cancel out.

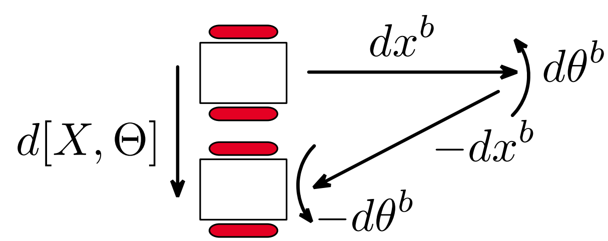


This gives us an area rule: over a larger gait, magnitude of average flow field is the area integral of the Lie bracket over the region enclosed by the gait

This rule is consistent with the Lie bracket as a *bilinear operator* – it scales linearly with *u* and *v.*

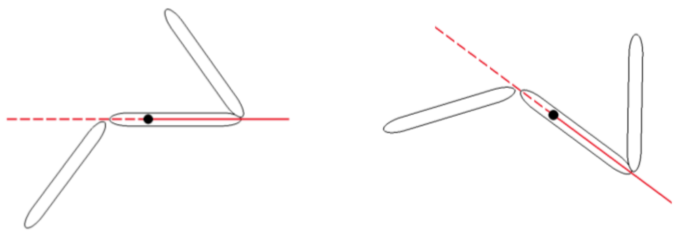
$$\left[\begin{pmatrix} u^x \\ u^y \\ u^\theta \end{pmatrix}, \begin{pmatrix} v^x \\ v^y \\ v^\theta \end{pmatrix}\right] = \begin{pmatrix} v^\theta u^y - u^\theta v^y \\ u^\theta v^x - v^\theta u^x \\ 0 \end{pmatrix}$$

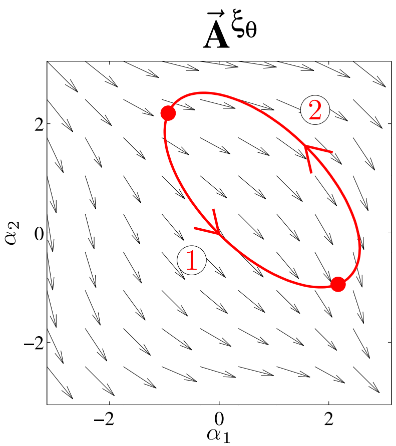
Example: Increasing forward translation and rotation each linearly increases the lateral translation

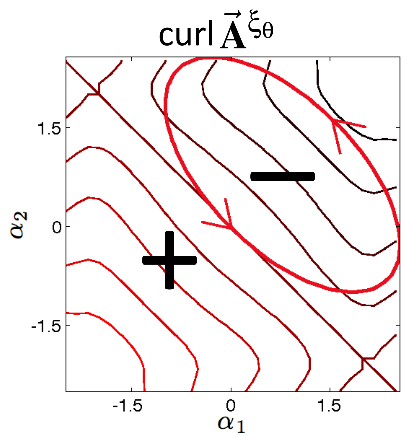


Note that here there is a small-angle condition on this assumption, and we are also assuming that local connection is constant (and thus that input fields are left invariant)

Nonconservativity

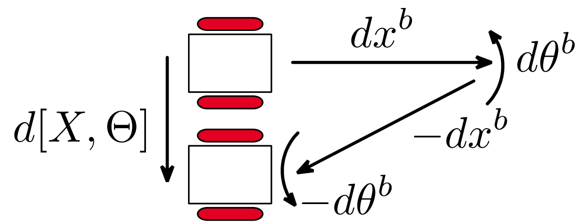


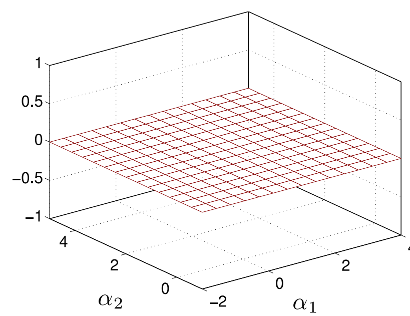




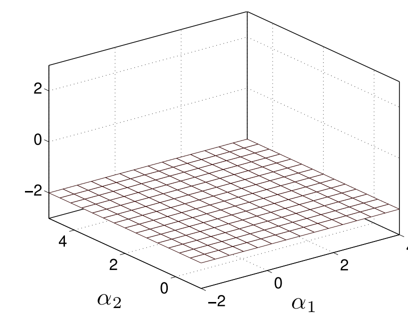
$$\text{curl } V = \frac{\partial v_2}{\partial u_1} - \frac{\partial v_1}{\partial u_2}$$

Noncommutatvity





x



y

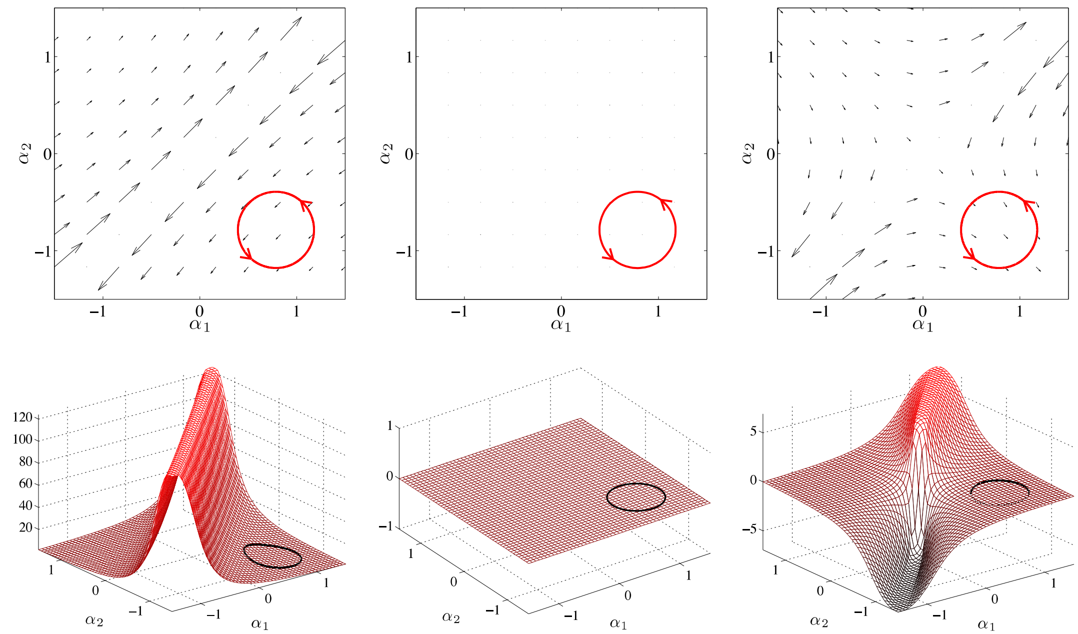
$$[X, Y] = (\nabla Y \cdot X) - (\nabla X \cdot Y)$$

Curl and Lie bracket look very similar. Up next, we combine them into a single concept

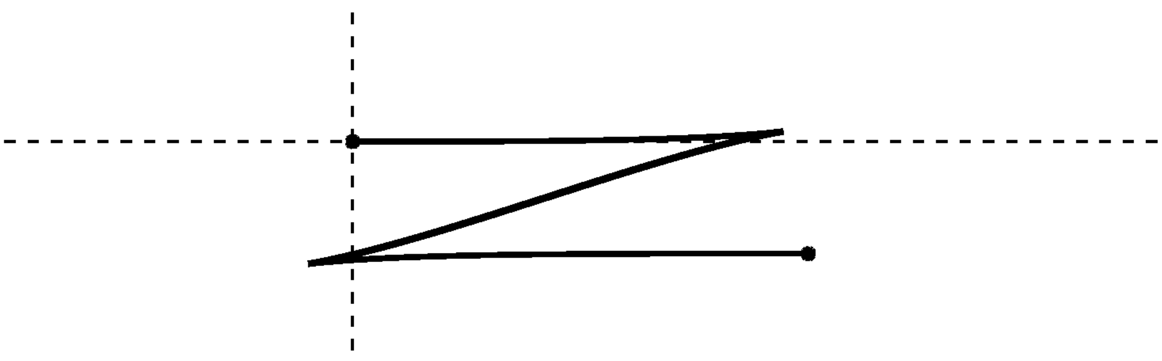
Combining Nonconservativity and Noncommutativity

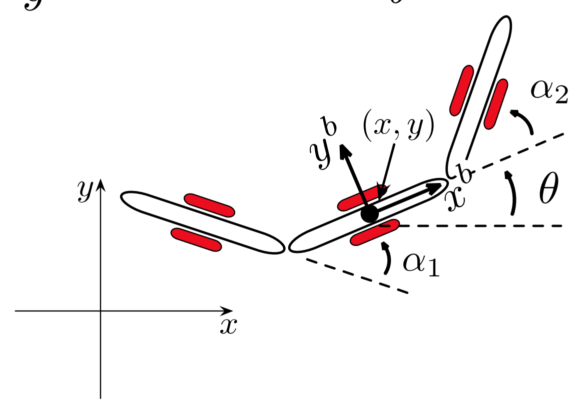
Example: Kinematic snake combines both effects

1. curl of the connection vector field means the system moves forward more than backward,
2. noncommutativity means it also moves laterally



y θ





Combining Nonconservativity and Noncommutativity

Is it legitimate to just add nonconservative and noncommutative elements together?

Yes (as an approximation)

Exponential coordinates (average velocity for normalized time) of net displacement over a cycle are in general

$[A_1,A_2]$ is the *local* Lie bracket, as if A did not change with *r*

$$z(\varphi) = \left(\iint_{\varphi_a} -\mathbf{d}A \, dr_1 \, dr_2 + \iint_{\varphi_a} [A_1, A_2] \, dr_1 \, dr_2 + \text{higher-order terms} \right)$$

d is the *exterior derivative*, which is like the curl, but more technically correct for talking about covectors

Higher order terms are linearizaton error in local Lie bracket, and noncommutativity interacting with nonconservativity

Derivation for this equivalence

Full Lie bracket on the configuration space

$$\left[\begin{pmatrix} \dot{r}_1 \\ -T_eL_gA(r)\dot{r}_1 \end{pmatrix}, \begin{pmatrix} \dot{r}_2 \\ -T_eL_gA(r)\dot{r}_2 \end{pmatrix}\right] = \begin{pmatrix} \left(\frac{\partial \dot{r}_2}{\partial r}\dot{r}_1 - \frac{\partial \dot{r}_1}{\partial r}\dot{r}_2\right) + \left(\frac{\partial \dot{r}_2}{\partial g}\dot{g}_1 - \frac{\partial \dot{r}_1}{\partial g}\dot{g}_2\right) \\ \left(\frac{\partial \dot{g}_2}{\partial r}\dot{r}_1 - \frac{\partial \dot{g}_1}{\partial r}\dot{r}_2\right) + \left(\frac{\partial \dot{g}_2}{\partial g}\dot{g}_1 - \frac{\partial \dot{g}_1}{\partial g}\dot{g}_2\right) \end{pmatrix}$$

Differential cycle in the shape space is $\dot{r}_1 = [1\ 0]^T$ and $\dot{r}_2 = [0\ 1]^T$

Conditions we can apply for these shape inputs:

$$\frac{\partial \dot{r}_i}{\partial r} = 0 \quad \text{and} \quad \frac{\partial \dot{r}_i}{\partial g} = 0$$

(shape input vector field components are constant)

Top half of full Lie bracket is thus zero, which makes sense: over a cycle in the shape space, there should be no residual change of shape

(T_eL_g is independent of r)

$$-\left.\frac{\partial \dot{g}_i}{\partial r}\right|_{(r_0,g_0)} = \left.\frac{\partial\, T_eL_gA_i(r)}{\partial r}\right|_{(r_0,g_0)} = \cancel{\frac{\partial\, T_eL_g}{\partial r}}A_i(r_0) + T_eL_{g_0}\frac{\partial A_i(r)}{\partial r}$$

$$-\left.\frac{\partial \dot{g}_i}{\partial g}\right|_{(r_0,g_0)} = \left.\frac{\partial\, T_eL_gA_i(r)}{\partial g}\right|_{(r_0,g_0)} = \frac{\partial\, T_eL_g}{\partial g}A_i(r_0) + \cancel{T_eL_{g_0}\frac{\partial A_i(r)}{\partial g}}$$

(A is independent of g)

$\dot{r}_1 = [1\ 0]^T$ and $\dot{r}_2 = [0\ 1]^T$

Derivation, continued

each derivative of $\dot{g}_i$ in the lower-left term of $\begin{pmatrix} \left(\frac{\partial \dot{r}_2}{\partial r}\dot{r}_1 - \frac{\partial \dot{r}_1}{\partial r}\dot{r}_2\right) + \left(\frac{\partial \dot{r}_2}{\partial g}\dot{g}_1 - \frac{\partial \dot{r}_1}{\partial g}\dot{g}_2\right) \\ \left(\frac{\partial \dot{g}_2}{\partial r}\dot{r}_1 - \frac{\partial \dot{g}_1}{\partial r}\dot{r}_2\right) + \left(\frac{\partial \dot{g}_2}{\partial g}\dot{g}_1 - \frac{\partial \dot{g}_1}{\partial g}\dot{g}_2\right) \end{pmatrix}$

is multiplied by the shape velocity
$\dot{r}_j$ with $i \neq j$, selecting out the jth derivative and giving these terms the form

$$-\frac{\partial \dot{g}_i}{\partial r}\dot{r}_j\bigg|_{(r_0,g_0)} = \frac{\partial \dot{g}_i}{\partial r^j}\bigg|_{(r_0,g_0)} = T_e L_{g_0}\frac{\partial A_i(r)}{\partial r^j}$$ (based condition from last slide)

Derivation, concluded

Collecting all terms gives us

$$\begin{pmatrix} 0 \\ T_eL_{g_0}\left(-\left(\frac{\partial A_2(r)}{\partial r^1}-\frac{\partial A_1(r)}{\partial r^2}\right)+\left(\frac{\partial\, T_eL_g\, A_2(r_0)}{\partial g}T_{g_0}L_gA_1(r_0) - \frac{\partial\, T_eL_g\, A_1(r_0)}{\partial g}T_{g_0}L_gA_2(r_0)\right)\right)\end{pmatrix}$$

(This is the curl/exterior derivative)

$$\left(\frac{\partial\, T_eL_g\, A_2(r_0)}{\partial g}T_eL_gA_1(r_0) - \frac{\partial\, T_eL_g\, A_1(r_0)}{\partial g}T_eL_gA_2(r_0)\right)$$ (Make $g_0{=}e$)

$$[T_eL_gA_1(r_0), T_eL_gA_2(r_0)] = [A_1, A_2](r_0)$$ (Recognize this as an SE(2) Lie bracket)

Net result: nonconservative and noncommutative terms both appear in total Lie bracket

$$\left[\begin{pmatrix} \dot r_1 \\ -T_eL_gA(r)\dot r_1\end{pmatrix}, \begin{pmatrix} \dot r_2 \\ -T_eL_gA(r)\dot r_2\end{pmatrix}\right]\Bigg|_{q_0} = \begin{pmatrix} 0 \\ \left(-dA+[A_1,A_2]\right)(r_0)\end{pmatrix}$$

This is the *curvature form* for the connection, ***DA***